IMPLEMENTATION WITH EVIDENCE: COMPLETE INFORMATION

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Abstract. This paper studies full-implementation in Nash equilibrium. We generalize the canonical model (Maskin, 1977) by allowing agents to send *evidence* or discriminatory signals. We first study settings where evidence is hard information that proves something about the state of the world. In such environments, social choice rules that are not Maskin-monotonic can be implementable. We formulate a more general property, *evidence-monotonicity*, and show that this is a necessary condition for implementation. Evidence-monotonicity is also sufficient for implementation if there are three or more agents and the social choice rule satisfies *no veto power*. We illustrate how evidence-monotonicity constrains what is implementable even in the presence of rich evidence, and also derive conditions under which various evidentiary structures yield permissive results. The latter part of the paper generalizes these themes to settings where evidence take the form of costly signaling.

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1. Introduction

This paper is about Nash (full-)implementation in complete information settings, where all agents know the state of the world that is unknown to a social planner. A maintained assumption in most of the literature following Maskin (1999) is that agents can and will manipulate information about the state without restraint. Put differently, all messages that are available to an agent in any mechanism are assumed to be state independent, and moreover, directly payoff irrelevant. In this sense, all messages are "cheap talk", and only matter insofar as they affect the outcome determined by the mechanism. We focus attention on this aspect of the implementation problem, and generalize the set of environments under consideration. To motivate our treatment, here are two examples:

(1) A principal is concerned with dividing a fixed sum of money between agents as some function of their individual output. If only asked to send cheap-talk messages about their output, agents could claim anything they want. But the principal could also request agents to provide physical verification of their output. In this case, an agent would be unable to claim that his output is larger than it in fact is, but he could claim that it is less, just by not providing all of it. If it is costless to provide the physical verification, the setting is one of certifiability or hard information. If instead agents bear costs as a function of how much output they carry to the principal's court (so to speak), then we have a more complicated costly signaling instrument.

(2) Consider an income taxation problem, where the planner cannot observe agents' income. Agents who are legally employed have a document that stipulates their income. If requested by a mechanism to submit such a document, a legally employed agent can either costlessly submit his true document, or fabricate a false document at some cost. The cost may take the form of a fixed cost or possibly vary with the degree of falsification. Illegally employed agents must fabricate a document, or admit to not possessing one. In this example, the informational signal of a submitted document is heterogeneous across agents.

The key element in these examples is that some messages are only feasible in some states of the world, or messages have differential costs in different states. Since this theme naturally arises in numerous settings, we believe it is important to study implementation in a framework that accommodates this feature. Accordingly, this paper adds "*evidence*" to an otherwise standard implementation environment. The crucial feature that distinguishes evidence from standard (cheap-talk) messages is that a piece of evidence is a *discriminatory signal* about the state of the world. A mechanism in this setting can not only request agents to send cheap-talk messages as usual, but also to submit evidence. Naturally, the planner will generally benefit from the availability of evidence; our interest is in understanding exactly when and to what extent. In particular, which SCRs are implementable given some evidence structure, and what evidence structure is needed to make a particular SCR implementable?

We begin in Section 2 by considering environments where evidence takes the form of hard or non-manipulable information: an agent cannot produce evidence that she does not in fact possess.

Formally, in each state θ, each agent i has a set of evidence, E_i^θ, and can submit evidence e_i if and only if $e_i \in E_i^\theta$. Providing any feasible evidence is costless. We place no restriction on the evidence structure, $\{E_i^\theta\}_{i,\theta}$, so that the standard environment is a special case where a player's evidence set does not vary with the state.

A simple but significant observation is that SCRs that are not Maskin-monotonic can be implementable under some evidence structures. We identify a necessary condition, *evidence-monotonicity*, that a SCR must satisfy for it to be implementable. Evidence-monotonicity is a joint condition on the SCR, agents' preferences, and the evidence structure $\{E_i^\theta\}_{i,\theta}$. It is strictly weaker that Maskin-monotonicity, reducing to the latter if, and generally only if, there is no evidence available. To get a sense of what the condition entails, observe that if all agents' evidence sets are identical in two different states, then an implementable SCR must respect Maskin-monotonicity over these two states. Put differently, if a SCR is not Maskin-monotonic over two states, then it is implementable only if some agent possesses different evidence in the two states. Evidence-monotonicity demands this and more: loosely speaking, (i) if state θ' must be distinguished from both θ' and θ'' by evidence, there has to be a single profile of evidence for the agents that permits both distinctions simultaneously; (ii) it should be possible to reward agents for providing evidence when needed.

We then provide a converse: any evidence-monotonic SCR is implementable if there are three or more players and the SCR also satisfies *no veto power*. No veto power was introduced by Maskin (1999) and requires that no single agent be able to "veto" an outcome that is top ranked by all the other agents. To prove sufficiency, we explicitly construct a canonical implementing mechanism that uses an "integer game", building upon the canonical mechanism used in the literature. While such mechanisms are unappealing in certain respects, the result is useful for at least two reasons. First, it shows that evidence-monotonicity is a fairly tight condition for implementability. Second, it suggests the possibility of finding positive results with better-behaved mechanisms if one restricts attention to a particular environment (whereas the construction here works for all environments).

That implementability of a SCR turns on evidence-monotonicity produces a number of corollaries. We introduce a notion of when a state and an event (i.e., a set of states) are *distinguishable* under a given evidence structure. In some domains such as economic environments, a SCR satisfies evidence-monotonicity if and only if appropriate state-event pairs are distinguishable. This yields a partial order on evidence structures in terms of how *informative* they are, so that a "more informative" evidence structure implies that a larger set of SCRs are evidence-monotonic. There is (an equivalence class of) maximally informative evidence structures, under any of which every SCR is evidence-monotonic, hence implementable under the aforementioned conditions. We also use the notion of distinguishability to create a partial ranking of SCRs, so that a SCR is "easier" to implement if it is evidence-monotonic for a larger set of evidence structures. Naturally, Maskin-monotonic SCRs are the easiest to implement.

In Section 3, we generalize the environment so that a player can fabricate any piece of evidence in any state of the world. We posit a cost function $c_i(e_i, \theta)$ which specifies the cost for player i of producing evidence e_i when the true state is θ. A player's preferences are represented by $u_i(a, \theta) - c_i(e_i, \theta)$, where a is the outcome. The case of hard information essentially corresponds to the special case where $c_i(\cdot,\cdot) \in \{0, \infty\}$, with $c_i(e_i, \theta) = 0$ if and only if $e_i \in E_i^\theta$, whereas now we consider arbitrary cost functions. Our notion of implementation here is that in equilibrium there should be no costly fabrication of evidence. We formulate an appropriate notion of monotonicity called *cost-monotonicity*, and show that our previous results also generalize: an implementable SCR must be cost monotonic, and any cost-monotonic SCR satisfying no veto power is implementable when there are three or more players. In many settings, the planner may have the power to design the cost structure (or part of it), perhaps at social cost.[1] From the perspective that influencing evidence fabrication costs is an additional instrument for the planner, our analysis yields insight into how useful such an instrument can be.

Under some cost structures, every SCR is cost monotonic. A striking case is where at least one player has a "small preference for honesty." Roughly, this is an assumption on the cost structure so that an honest player would prefer to tell the truth (in terms of a direct message about the state) rather than lying if he knows that the outcome is not much affected by this decision. A corollary of our results is that in such a setting with three or more players, any SCR satisfying no veto power can be implemented. Dutta and Sen (2009) and Matsushima (2008a,b) have shown related results focussing specifically on preferences for honesty.

Before turning to a discussion of related literature, let us address one potential concern that some readers may have: why study Nash implementation in a setting with evidence when earlier literature has already shown that permissive results can be obtained without evidence by either using refinements of Nash equilibrium such as subgame perfect equilibrium (e.g. Moore and Repullo, 1988) or undominated Nash equilibrium (e.g. Jackson et al., 1994), or even still, approximate or virtual implementation (Abreu and Sen, 1991; Matsushima, 1988)? There are multiple reasons: first, we believe it is important to understand the role of evidence in implementation by studying such environments directly, and indeed our necessary conditions identify constraints on how evidence can be used; second, the aforementioned permissive results are not without limits;[2] and third, these results have been questioned from various perspectives.[3]

This paper contributes to the burgeoning literature on mechanism design with evidence.[4] Most studies concern partial- or weak-implementation and hard information. An early reference is Green and Laffont (1986), and a sample of more recent work is Bull (2008), Bull and Watson (2004, 2007),

[1] For instance, the degree of difficulty or technological cost of falsifying the income document in the second motivating example above could be modified by the planner.

[2] For instance, none of them have bite when players' preferences don't vary across states, as emphasized by Ben-Porath and Lipman (2009), whereas evidence can be useful in this regard.

[3] A well-known weakness of virtual implementation is that—even if it only occurs with small probability—the mechanism may provide an outcome arbitrarily inefficient, unfair, or, in any meaningful sense, far from the desired alternative. Implementation with refinements of Nash equilibrium has recently been critiqued in terms of robustness to the introduction of small amounts of incomplete information. In particular, if one requires these mechanisms to implement in environments with "almost" complete information, Maskin monotonicity is again a necessary condition (Chung and Ely, 2003; Aghion, Fudenberg, Holden, Kunimoto, and Tercieux, 2009).

[4] Beyond mechanism design, there are other literatures where evidence plays an important role. The introduction of hard evidence into implementation may be considered analogous to moving from communication games of cheap talk (Crawford and Sobel, 1982) to those of verfiable/certifiable information (Milgrom, 1981; Okuno-Fujiwara et al., 1990; Lipman and Seppi, 1995; Forges and Koessler, 2005). Costly evidence fabrication has been studied in communication games by Kartik, Ottaviani, and Squintani (2007) and Kartik (2009), and in contract settings by, for example, Maggi and Rodriguez-Clare (1995).

Deneckere and Severinov (2008), Glazer and Rubinstein (2004, 2006), Sher (2008), and Singh and Wittman (2001).

In the full-implementation literature, there is a small set of papers that study feasible implementation, where the set of feasible allocations is unknown to the planner. It is typically assumed that the planner can partially verify players' claims in particular ways. For example, in a Walrasian economy setting, Hurwicz, Maskin, and Postlewaite (1995) and Postlewaite and Wettstein (1989) assume that a player can claim to have any subset of his true endowment but not exaggerate; in a taxation problem with unknown incomes, Dagan, Volij, and Serrano (1999) make a similar assumption. In our model, the set of allocations is constant and known to the planner, instead it is the set of messages for players that either varies or has varying costs with the state.

Closest to our work is a recent paper by Ben-Porath and Lipman (2009), who also tackle complete information full-implementation with evidence. While our results were derived independently, we have benefitted from reading their treatment. The motivations for their work and ours are similar—particularly with respect to advancing the prior literature—but the analytical focus is different. In terms of setting, they restrict attention to hard evidence and assume that the evidence structure satisfies *normality* (Bull and Watson, 2007) or *full reports* (Lipman and Seppi, 1995). We do not impose this structural assumption when studying hard evidence and we also consider costly evidence fabrication. Moreover, they assume that the planner can augment monetary transfers off the equilibrium path. We show that some ability to reward players is in fact *necessary* to exploit hard evidence. Finally, Ben-Porath and Lipman (2009) are primarily interested in subgame perfect implementation, whereas we are entirely concerned with Nash implementation.[5]

2. Hard Evidence

2.1. **The Model.**

There is a non-empty set of *agents* or players, $I = \{1, \ldots, n\}$, a set of allocations or *outcomes*, A, a set of *states* of the world, Θ, and a vector of payoff functions, $\{u_i\}_{i=1}^n$, where each $u_i : A \times \Theta \to \mathbb{R}$.[6] To avoid trivialities, $|A| > 1$ and $|\Theta| > 1$. All agents are assumed to know the value of the state, whereas the planner does not. The planner's objectives are given by a *social choice rule* (SCR), which is a correspondence $f : \Theta \rightrightarrows A$; a social choice function is a single-valued SCR.

So far, the setting is standard. We now describe how evidence enters the model. In any state of the world, θ, agent i is endowed with a set of evidence, E_i^θ, which we assume to be non-empty without loss of generality.[7] Denote $E_i := \bigcup_\theta E_i^\theta$, $E^\theta := E_1^\theta \times \cdots \times E_n^\theta$ and $E := E_1 \times \cdots \times E_n$.

[5]Ben-Porath and Lipman's Theorem 2, derived contemporaneously with our work, provides sufficient conditions for one-stage subgame perfect implementation (hence, Nash implementation). It is strictly subsumed by our results. See Remark 6 for additional details.

[6]It is common to focus on just ordinal preferences in each state. We could do with ordinal preferences insofar as only pure strategy Nash equilibria are considered. For the extension to mixed Nash equilibria (see Remark 4), we would require that in any state θ, $u_i(\cdot, \theta)$ is an expected utility representation of player i's preferences over lotteries on outcomes. Moreover, our formulation allows the possibility of agents having cardinal valuations over outcomes.

[7]This is without loss of generality because we will permit the planner to make available cheap-talk messages, hence one can always assume that for each i, there is some $e_i \in \bigcap_\theta E_i^\theta$, which is just a cheap-talk evidence.

We refer to $\mathcal{E} := \left\{E_i^\theta\right\}_{i,\theta}$ as the *evidence structure*. The interpretation is that any $e_i \in E_i$ is a document, piece of evidence, non-falsifiable claim, etc., that is only available to agent i in the set of states $\{\theta : e_i \in E_i^\theta\}$. Note that evidence e_i is proof of the event $\{\theta : e_i \in E_i^\theta\}$.[8] The traditional setting is the special case where for all i, $E_i^\theta = E_i$ for all θ. For convenience, we will often refer to this case as a setting with "no evidence", even though E_i is non-empty for all i.

In standard implementation theory, a mechanism consists of a (cheap-talk) message space and an outcome function which specifies an outcome for every profile of messages. In the current setting, a mechanism can also take advantage of the evidence that agents might possess by conditioning the outcome on the evidence that is submitted. Formally, a *mechanism* is a pair (M, g), where $M = M_1 \times \cdots \times M_n$ is a *message space*, and $g : M \times E \to A$ is an *outcome function* that specifies an outcome for every profile of messages and evidence.[9]

A mechanism (M, g) induces a strategic game form in each state of the world, where a pure strategy for player i in state θ is $s_i \in M_i \times E_i^\theta$. Let $NE\,(M, g, \theta)$ be the set of pure strategy Nash equilibria (NE hereafter) of this game and $O\,(M, g, \theta) := \{a \in A : \exists (m, e) \in NE\,(M, g, \theta) \text{ s.t. } g\,(m, e) = a\}$ be the set of equilibrium allocations. For expositional simplicity, we restrict attention to pure strategy equilibria in the paper; our results can be extended to mixed strategy Nash equilibria as discussed in Remark 4 following Theorem 2.

We are interested in (full-)implementation in Nash equilibria, defined as follows.

Definition 1 (Implementation)**.** *A mechanism (M, g) implements the SCR f if $\forall\theta$, $f\,(\theta) = O\,(M, g, \theta)$. A SCR is implementable if there exists a mechanism that implements it.*

Remark 1. Our definition of a mechanism entails that every agent must submit some evidence. This is without loss of generality: if one wants to allow agent i to have the choice of not submitting evidence (as is often reasonable), we would just add a new element, say e_i, to E_i^θ for all θ, and interpret this e_i as submitting no evidence. While the model allows for this, we do not impose it because in some applications "silence" may not be an option in all states of the world.

Remark 2. There are two implicit restrictions in the class of mechanisms we are considering:

(1) A mechanism is *static*, i.e. the induced game entails one round of simultaneous moves by the players. In the standard environment without evidence, sequential or dynamic mechanisms (specifying a potentially complicated extensive-form game) cannot improve on static mechanisms for Nash implementation. However, perhaps surprisingly, this is not always the case with hard evidence, even though we are study Nash equilibrium.[10] We discuss the issue in detail in Appendix B, showing that the restriction to static mechanisms is without loss of generality under one of two conditions. The first is a *cost interpretation* of hard evidence, which is that in any state, θ, player i can produce every $e_i \in E_i$, but $e_i \in E_i^\theta$

[8] Indeed, for this section, one could work more directly with the "proof structure" induced by the evidentiary structure; we don't do so with a view towards costly fabrication of evidence in Section 3.

[9] Strictly speaking, letting the domain of g be $M \times E$ is excessive in the sense that some $e \in E$ may not be feasible in any state. It will be irrelevant what the outcome $g(\cdot, e)$ is for any such e. But, anticipating the analysis in Section 3, it is convenient to let the entire evidence structure be included in the domain of g.

[10] This was noted by Bull and Watson (2007) in the context of partial-implementation.

is costless to produce whereas $e_i \notin E_i^\theta$ is infinitely costly to produce; this contrasts with the *feasibility interpretation* we introduced initially, which is that $e_i \notin E_i^\theta$ is just not available to player i in state θ.[11] This interpretational distinction is irrelevant for static mechanisms. The second case where static mechanisms are without loss of generality is when the evidence structure satisfies a property called *normality*, which we formally introduce in Section 2.4.1.

(2) In some applications, it may be plausible that the planner can prohibit agents from submitting certain evidence. Our definition of a mechanism does not allow this. While forbidding evidence can sometimes be useful for implementation, permit it would not affect our main results. Again, details are in Appendix B.

Before turning to the analysis, we wish to emphasize that the current framework allows for different states to have identical profiles of agents' preferences. In contrast, in standard implementation theory, it is common to identify a state of the world with a profile of preferences. This is because in the standard model the planner cannot implement different outcomes unless some agent's preference ranking changes. Introducing evidence into the picture changes this perspective quite dramatically (as will be seen), so that a state can reflect anything relevant to the planner beyond just the agents' preferences. For example, consider legal settings where neither a plaintiff's nor defendant's preferences over outcomes may vary with the state, but the state of world reflects the degree of damages that has been caused by the defendant and is thus relevant to the planner.[12]

2.2. **Evidence-Monotonicity: Necessity.** Maskin (1999) showed that in settings without evidence, SCRs must satisfy a monotonicity condition to be implementable. We will refer to his now classic condition as *Maskin-monotonicity.*[13] This condition is not necessary for implementation with evidence, illustrated starkly in the following example.

Example 1. *Let $E_i = \Theta$ and $E_i^\theta = \{\theta\}$ for all i, θ. This can be interpreted as agents never being able to misrepresent the state of the world. In this case, no matter the specification of agents' preferences, any social choice function f can be implemented by a mechanism with an arbitrary message space, M, and the outcome function $g(m, (\theta, \ldots, \theta)) = f(\theta)$.*

The key to our analysis is the following generalization of Maskin-monotonicity, which we will show is necessary for implementability.

Definition 2 (Evidence-monotonicity)**.** *A SCR f is evidence-monotonic provided that for all θ and all $a \in f(\theta)$, there exists $e^*_{\theta,a} \in E^\theta$ such that for any θ', if*

$$\forall i \in I, \forall b \in A\colon \left[u_i\left(a,\theta\right) \geq u_i\left(b,\theta\right)\right] \implies \left[u_i\left(a,\theta'\right) \geq u_i\left(b,\theta'\right)\right] \tag{*}$$

[11] Naturally, this prompts the question of more general cost structures, which is the topic of Section 3. That section also makes precise what we mean by "infinitely costly to produce".

[12] Even if two states differ *only* in agents' preferences (and all other relevant aspects are identical), it still may be the case that an agent's evidence set is different across the two states. For instance, firms may have evidence about costs or revenues.

[13] A SCR is Maskin-monotonic provided that for all θ, θ' and $a \in f(\theta)$, if

$$\forall i \in I, \forall b \in A\colon \left[u_i\left(a,\theta\right) \geq u_i\left(b,\theta\right)\right] \implies \left[u_i\left(a,\theta'\right) \geq u_i\left(b,\theta'\right)\right]$$

then $a \in f(\theta')$.

and

$$\left(e^*_{\theta,a} \in E^{\theta'}\right) \text{ and } \left(\forall i : E_i^{\theta'} \subseteq E_i^{\theta} \text{ or } a \in \arg\max_b u_i(b,\theta')\right), \tag{**}$$

then $a \in f\left(\theta'\right)$.

To get some intuition for the definition, note that without condition (**), it is just Maskin-monotonicity. The presence of (**) makes evidence-monotonicity a weaker requirement: any Maskin-monotonic SCR is evidence-monotonic regardless of the evidence structure (any choice of $\left\{e^*_{\theta,a}\right\}_{\theta,a}$ would do). In a setting without evidence (i.e., for all i, $E_i^\theta = E_i$ for all θ), evidence-monotonicity reduces to Maskin-monotonicity because condition (**) will be satisfied no matter the choice of $\left\{e^*_{\theta,a}\right\}_{\theta,a}$. Generally, (**) will make it possible for some non-Maskin-monotonic SCRs to be evidence-monotonic: for instance, in Example 1, (**) is not satisfied for any $\theta \neq \theta'$ (since $E^\theta = \{(\theta, \ldots, \theta)\}$ for all θ), hence *every* SCR is evidence-monotonic, regardless of agents' preferences.

For any θ and $a \in f(\theta)$, $e^*_{\theta,a}$ can be thought of as the evidence profile that agents are supposed to provide the planner to indicate that state θ has occurred and a should be the outcome. To make a SCR f evidence-monotonic, one must find $\left\{e^*_{\theta,a}\right\}_{\theta,a}$ so that Condition (**) is *falsified* for every pair of states over which f violates Maskin-monotonicity. Plainly, this is not possible unless some player's evidence set varies across such a pair of states. Fix some $\left\{e^*_{\theta,a}\right\}_{\theta,a}$, a pair of states θ and θ', and some $a \in f(\theta)$ such that $a \notin f(\theta')$. If a does not go down in an agent's preference ordering when the state changes from θ to θ' (i.e. condition (*) is satisfied), then we know from Maskin (1999) that to implement f, a mechanism has to exploit evidence. Condition (**) says that for this to be possible, either: (i) the evidence profile being submitted at θ, $e^*_{\theta,a}$, is not available at θ' (negating the first part of (**)), or (ii) some player must have evidence at θ' that is not available at θ' *and* outcome a should not be this player's most-preferred outcome at θ' (together, negating the second part of (**)). The preference requirement is essential because otherwise a player cannot be given incentives to disprove θ when the true state is θ' and he has the ability to submit evidence supporting θ; this is why the requirement only enters in the second part of (**) and not the first part.

The existential quantifier over $\left\{e^*_{\theta,a}\right\}_{\theta,a}$ in Definition 2 raises the possibility of it being tedious to verify whether a given SCF is evidence-monotonic. In Section 2.4, we discuss how the task is often very simple. For the moment, we note that loosely speaking, one should choose $e^*_{\theta,a}$ to be an evidence profile that is "most informative" about state θ with respect to the other states that cause a problem for Maskin-monotonicity. In particular, if there is an evidence profile in state θ that proves more about the state than any other evidence profile available at θ, then for any $a \in f(\theta)$, one can take $e^*_{\theta,a}$ to be this evidence profile. This is illustrated in the following example.

Example 2. *Consider the first motivating example in the introduction. A principal is concerned with dividing a fixed sum of money, say $M > 0$, to agents as some function of their individual production. The outcome space is $A = \{(a_1, \ldots, a_n) \in \mathbb{R}^n_+ : \sum_i a_i \leq M\}$. A state of nature θ is a profile of units of output, i.e. $\theta = (\theta_1, ..., \theta_n) \in \Theta = \mathbb{R}^n_+$. Each agent can show his true output* *or some subset of it, hence an agent is unable to claim that his output is larger that it in fact is, but he can claim that it is less. Formally,* $E_i = \Theta$ *and* $E_i^\theta = [0, \theta_i]$ *for all* i, θ*. Assume that* $u_i((a_1, \ldots, a_n), \theta)$ *is strictly increasing in* a_i*.*

In this setting, any SCF f *where* $f(\theta) := (f_1(\theta), \ldots, f_n(\theta))$ *satisfies* $\forall i, \theta : f_i(\theta) < M$ *is evidence-monotonic. To see this, fix any such* f*. For any* θ *and* $a \in f(\theta)$*, let* $e^*_{\theta,a} = \theta$*. It suffices to argue that for any* $\theta' \neq \theta$*, condition (**) is violated. If* $\theta' \neq \theta$*, there exists an agent* i *such that* $\theta'_i \neq \theta_i$*. First, if* $\theta'_i < \theta_i$ *then* $e^*_{i,\theta,a} = \theta_i \notin [0, \theta'_i] = E_i^{\theta'}$ *and so the first part of (**) is violated. Second, if* $\theta'_i > \theta_i$ *then* $\theta'_i \in E_i^{\theta'}$ *but* $\theta'_i \notin [0, \theta_i] = E_i^\theta$*, hence* $E_i^{\theta'} \not\subseteq E_i^\theta$*. Moreover,* $f(\theta) \notin \arg\max_b u_i(b, \theta)$ *because* $f_i(\theta) < M$*. Therefore, the second part of (**) is violated.*

Our first main result is:

Theorem 1. *If a SCR is implementable, it is evidence-monotonic.*

Proof. Assume f is implemented by a mechanism (M, g). Then for each θ and $a \in f(\theta)$, there exists $\left(m^*_{\theta,a}, e^*_{\theta,a}\right) \in M \times E^\theta$ that is a NE at θ such that $g\left(m^*_{\theta,a}, e^*_{\theta,a}\right) = a$. Consider any θ' satisfying (*) and (**). Fix a player i. Since $\left(m^*_{\theta,a}, e^*_{\theta,a}\right)$ is a NE at θ,

$$u_i\left(g\left(m^*_{\theta,a}, e^*_{\theta,a}\right), \theta\right) \geq u_i\left(g\left(m'_i, m_{-i,\theta,a}, e'_{i,\theta}, e^*_{-i,\theta,a}\right), \theta\right)$$

for all $m'_i \in M_i$, $e'_{i,\theta} \in E_i^\theta$. By (*),

$$u_i\left(g\left(m^*_{\theta,a}, e^*_{\theta,a}\right), \theta'\right) \geq u_i\left(g\left(m'_i, m_{-i,\theta,a}, e'_{i,\theta}, e^*_{-i,\theta,a}\right), \theta'\right)$$

for all $m'_i \in M_i$, $e'_{i,\theta} \in E_i^\theta$. Now the last condition in (**) stipulates that either $E_i^{\theta'} \subseteq E_i^\theta$ or $a \in \arg\max_b u_i(b, \theta')$. In each of these cases, we have

$$u_i\left(g\left(m^*_{\theta,a}, e^*_{\theta,a}\right), \theta'\right) \geq u_i\left(g\left(m'_i, m_{-i,\theta,a}, e'_{i,\theta}, e^*_{-i,\theta,a}\right), \theta'\right)$$

for all i, $m'_i \in M_i$, $e'_{i,\theta} \in E_i^{\theta'}$. Since (**) stipulates $e^*_{\theta,a} \in E^{\theta'}$, $\left(m_\theta, e^*_{\theta,a}\right)$ is a NE at θ', and $g\left(m^*_{\theta,a}, e^*_{\theta,a}\right) = a \in f\left(\theta'\right)$. □

The example below illustrates how the Theorem constrains which SCRs are implementable and also provides more insight into the different elements of Condition (**).

Example 3. *Let* $n = 4$*,* $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$*,* $A = \{w, x, y, z\}$*, and* $E_1^{\theta_1} = \{\alpha, \beta\}, E_1^{\theta_2} = \{\alpha, \beta\}, E_1^{\theta_3} = \{\alpha\}$*, and* $E_1^{\theta_4} = \{\alpha\}$*, while for* $i > 1$*,* $E_i^\theta = \{\gamma\}$ *for all* θ*. So agent 1 is the only player with any evidence. Preferences are as follows: for all* i*,* $u_i(x, \theta_1) > u_i(y, \theta_1)$*,* $u_i(y, \theta_2) > u_i(x, \theta_2)$*,* $u_i(x, \theta_3) > u_i(y, \theta_3)$*, and* $u_i(x, \theta_4) > u_i(y, \theta_4)$*. In addition, for agents* 1 *and* 2*, in all states,* w *is the unique top-ranked alternative while* z *is the unique bottom-ranked alternative; analogously, for agents 3 and 4, in all states,* z *is the unique top-ranked alternative while* w *is uniquely bottom-ranked.*

(1) *Consider the SCR* f *where* $f(\theta_1) = f(\theta_4) = x$ *and* $f(\theta_2) = f(\theta_3) = y$*. Since* $E^{\theta_3} = E^{\theta_4} = \{(\alpha, \gamma, \gamma, \gamma)\}$*, we have to set* $e^*_{\theta_3,y} = e^*_{\theta_4,x}$*. Since no player's preferences change between*

states θ_3 and θ_4, both (*) and (**) are satisfied for $\theta = \theta_3$, $\theta' = \theta_4$ and $a = y$. Hence f is not evidence-monotonic, and by Theorem 1, not implementable.

(2) *Next consider the SCR f^* where $f^*(\theta_1) = x$ and $f^*(\theta_2) = f^*(\theta_3) = f^*(\theta_4) = y$. Even though f^* is not Maskin-monotonic (because $f^*(\theta_1) \neq f^*(\theta_3)$ while preferences don't change between states θ_1 and θ_3), one can check that f^* is evidence-monotonic by using $e^*_{\theta_1,x} = e^*_{\theta_2,y} = (\beta, \gamma, \gamma, \gamma)$ and $e^*_{\theta_3,y} = e^*_{\theta_4,y} = (\alpha, \gamma, \gamma, \gamma)$ in Definition 2. Note that $f(\theta_1) = x$ is not top-ranked for any agent in state θ_3 and similarly $f(\theta_3) = y$ is not top-ranked for any agent in state θ_1; this is essential to the violation of (**) for $(\theta = \theta_3, \theta' = \theta_1)$. We will show subsequently that this f^* is in fact implementable.*

(3) *Finally, consider the SCR $\tilde{f}$ where $\tilde{f}(\theta_1) = x$ and $\tilde{f}(\theta_2) = \tilde{f}(\theta_3) = \tilde{f}(\theta_4) = w$. Consider $\theta = \theta_3$, $\theta' = \theta_1$, and $a = w$. Since no player's preferences change between θ_1 and θ_3, (*) is satisfied. As we must set $e^*_{\theta_3,w} = (\alpha, \gamma, \gamma, \gamma)$ (this is the only evidence profile at state θ_3) and hence $e^*_{\theta_3,w} \in E^{\theta_1}$, the first part of (**) is also satisfied. The second part of (**) is trivially satisfied for all $i > 1$ (they have no evidence); it is also satisfied for player 1 because w is top-ranked for him in all states. Thus, (**) is satisfied and f is not evidence-monotonic, hence not implementable. The problem here is that even though $E_1^{\theta_3} \subsetneq E_1^{\theta_1}$, it is not possible to reward player 1 in state θ_1 for disproving θ_3, because $\tilde{f}(\theta_3) = w$ is 1's most preferred outcome in state θ_1.*

2.3. **Evidence-Monotonicity: Sufficiency.** This section shows that evidence-monotonicity is also sufficient for implementation when $n \geq 3$ and the (standard) additional condition of *no veto power* holds. Observe that evidence-monotonicity by itself cannot be generally sufficient, since it reduces to Maskin-monotonicity without evidence, and it is well known that Maskin-monotonicity is not sufficient for implementation in the standard setting (even with $n \geq 3$).

Definition 3 (No veto Power)**.** *A SCR f satisfies No veto power (NVP) if for all $\theta \in \Theta$ and $a \in A$,*

$$\left[\left| \{ i : a \in \arg\max_{b \in A} u_i(b, \theta) \} \right| \geq n - 1 \right] \implies a \in f(\theta).$$

NVP, introduced by Maskin (1999), says that if at state θ an outcome is top ranked by $n-1$ individuals, then the last individual cannot prevent this outcome from being in the socially desired set $f(\theta)$, i.e. it cannot be "vetoed".

Theorem 2. *Assume $n \geq 3$ and let f be a SCR satisfying no veto power. If f is evidence-monotonic, it is implementable.*

Proof. For each θ and $a \in f(\theta)$, let $e^*_{\theta,a}$ be the evidence profile per Definition 2. We will construct a canonical mechanism that implements f. Fix, for all i, $M_i = \Theta \times A \times \mathbb{N}$, and define g via the following three rules:

(1) If $m_1 = \cdots = m_n = (\theta, a, k)$ with $a \in f(\theta)$, and $e = e^*_{\theta,a}$, then $g(m, e) = a$.
(2) If $\exists i$ s.t. $\forall j \neq i$, $e_j = e^*_{j,\theta,a}$ and $m_j = (\theta, a, k)$ with $a \in f(\theta)$, and either $m_i = (\theta', b, l) \neq (\theta, a, k)$ or $e_i \neq e^*_{i,\theta,a}$ then

a) if $e_i \in E_i^{\theta}$, then $g(m,e) = \begin{cases} b \text{ if } u_i(a,\theta) \geq u_i(b,\theta) \\ a \text{ if } u_i(b,\theta) > u_i(a,\theta). \end{cases}$

b) if $e_i \notin E_i^{\theta}$, then $g(m,e) = b$.

(3) For any other (m,e), letting $m_i = (\theta_i, a_i, k_i)$ and $i^* = \min_{i \in I} \arg\max_{j \in I} k_j$, $g(m,e) = a_{i^*}$.

Step 1. We first show that for any θ and any $a \in f(\theta)$, $a \in O(M,g,\theta)$. It suffices to show that at state θ, for any $k \in \mathbb{N}$, each agent i playing $m_i = (\theta, a, k)$ with $a \in f(\theta)$ and sending $e_i = e^*_{i,\theta,a}$ is a NE, since by rule (1) of the mechanism, this results in outcome a. If some agent deviates from this strategy profile, rule (2) of the mechanism applies. There is no profitable for any player to rule (2a) since any such deviation yields a weakly worse outcome. A deviation to rule (2b) is not feasible.

For the remainder of the proof, suppose the true state is θ' and (m,e) is a NE. We must show that $g(m,e) \in f(\theta')$.

Step 2. Assume (m,e) falls into rule (2). Note that any player $j \neq i$ can deviate and by announcing an integer large enough get any outcome he wants. Since (m,e) is a NE, $g(m,e)$ must be every player j's ($j \neq i$) most-preferred outcome in state θ'. NVP then implies that $g(m,e) \in f(\theta')$. A similar argument applies if (m,e) falls into rule (3) as well.

Step 3. It remains to consider (m,e) falling into rule (1). Here $m_1 = \cdots = m_n = (\theta, a, k)$ with $g(m,e) = a \in f(\theta)$ and for all i, $e_i = e^*_{i,\theta,a}$. Observe that any player i can deviate into rule (2a) while producing the same evidence (say, by changing his integer announcement), and hence can induce any outcome b such that $u_i(a,\theta) \geq u_i(b,\theta)$. Since (m,e) is a NE, this implies that for any agent i and outcome b, $u_i(a,\theta') \geq u_i(b,\theta')$, and consequently condition (*) is satisfied. If (**) also holds, evidence-monotonicity implies that $a \in f(\theta')$, and we are done. To see that (**) must hold, suppose to contradiction that it does not. Since $e = e^*_{\theta,a}$ and the true state is θ', we have $e^*_{\theta,a} \in E^{\theta'}$, and consequently there is some player i such that $E_i^{\theta'} \nsubseteq E_i^{\theta}$ and $a \notin \arg\max_b u_i(b,\theta')$. This implies that there is some $\tilde{e}_i \in E_i^{\theta'}$ such that $\tilde{e}_i \notin E_i^{\theta}$, and some outcome b that is strictly prefered by player i to a at state θ', hence, agent i can profitably deviate into rule (2b), contradicting (m,e) being a NE. □

To see an illustration of the Theorem, return to Example 3. The SCR f^* defined there satisfies NVP because no alternative is top-ranked in any state by more than two players. Since we already showed that f^* is evidence-monotonic, it is implementable by Theorem 2.

Remark 3. As is well known, NVP is not necessary for implementability. Variations of Theorem 1 can be obtained with different assumptions along the lines that have been explored in the earlier literature without evidence. For example, instead of NVP, Theorem 2 would go through verbatim with Moore and Repullo's (1990) weaker condition of *restricted veto power*.[14] Alternatively, one can also appeal to stochastic mechanisms (Bochet, 2007; Benoît and Ok, 2008) to close the gap between necessity and sufficiency.

[14]Indeed, NVP is demanding when there are only two agents; Section 2.7 addresses implementability when $n = 2$ using restricted veto power.

Remark 4. The mechanism used in the proof of Theorem 2 does not work when mixed Nash equilibria are considered. In the the standard setting without evidence, any Maskin-monotonic SCR can be implemented in mixed Nash equilibria (under NVP and $n \geq 3$) so long as for each player i and state θ, $u_i(\cdot, \theta)$ is bounded from below (Kartik and Tercieux, 2009; Maskin and Sjöström, 2002, Section 4.3). These arguments can be adapted to the current setting with evidence, extending Theorem 2 to mixed Nash equilibria so long as utilities are bounded in each state.

2.4. **Weak Evidence-monotonicity and Distinguishability.** Theorems 1 and 2 place no restriction on the evidence structure. The key condition they identify for implementation, evidence-monotonicity, is therefore somewhat involved. In this section, we discuss how it can be considerably simplified. To this end, it is useful to separate properties of the evidence structure from properties of preferences. In general, Condition (**) shows that these are inextricably linked for implementability. But separation is possible under a domain restriction.

Definition 4 (Non-satiation)**.** *A SCR f satisfies non-satiation if for all $i \in I$, $\theta \in \Theta$, and $a \in \bigcup_{\theta' \in \Theta} f(\theta')$, there exists $\tilde{a} \in A$ such that $u_i(\tilde{a}, \theta) > u_i(a, \theta)$.*

In words, a SCR satisfies non-satiation if it never chooses an outcome that is any agent's ideal outcome in some state of the world. Note that non-satiation does not preclude Pareto efficiency. Although weaker versions of this condition would suffice for the analysis (discussed in previous versions of this paper), for expositional simplicity it is convenient to use the above statement.

Non-satiation (and no veto power) are automatically satisfied in so-called "economic environments" where there is a divisible private good which is positively valued by all agents, and the SCR does not ever allocate all of the private good to a single agent (cf. Moore and Repullo, 1988, condition EE1). Both conditions are also satisfied if the planner can augment outcomes with arbitrarily small transfers that are never to be used in equilibrium, even if he must maintain off-the-equilibrium-path budget balance (cf. Benoît and Ok, 2008; Ben-Porath and Lipman, 2009; Sanver, 2006).[15] There are also interesting pure public goods problems without transfers where both conditions hold.[16] On the other hand, without additional instruments like (small) transfers, there are important environments that would not satisfy non-satiation, such as classic voting problems.

[15]To be more precise: consider an underlying outcome space, $\tilde{A}$, with each agent having a payoff function $\tilde{u}_i : \tilde{A} \times \Theta \to \mathbb{R}$, and a SCR $\tilde{f}$. (Note that $\tilde{A}$ itself may include transfers.) Suppose that the planner can impose an additional vector of transfers $(t_1, \ldots, t_n) \in X \subseteq \mathbb{R}^n$, and each agent values his personal transfer quasi-linearly. Assume that the space of possible transfers satisfies three weak properties: i) $(0, \ldots, 0) \in X$; ii) for all i, there exists $(t_1, \ldots, t_i, \ldots, t_n) \in X$ with $t_i > 0$; iii) for all $(t_1, \ldots, t_n) \in X$, there exists $\left(\tilde{t}_1, \ldots, \tilde{t}_n\right) \in X$ and $i \neq j$ such that $\tilde{t}_i > t_i$ and $\tilde{t}_j > t_j$. An obvious example would be $X = \left\{(t_1, \ldots, t_i, \ldots, t_n) \in \mathbb{R}^n : \sum_i t_i = 0, |t_i| \leq k\right\}$ for some $k > 0$, i.e. the planner must balance his budget and cannot reward or punish any player by more than k utility units. We can then define an extended outcome space $A = \tilde{A} \times \mathbb{R}^n$, an extended payoff function for each agent $u_i : A \times \Theta \to \mathbb{R}$ where $u_i(\tilde{a}, t_1, \theta) = u_i(a, \theta) + x$, and an extended SCR f derived from $\tilde{f}$ by setting $f(\theta) = \left(\tilde{f}(\theta), 0, \ldots, 0\right)$. Assuming that $n \geq 3$, this extended environment satisfies both non-satiation and no veto power.

[16]For example, the decision concerns how much of a public good to produce, $a \in [0, 1]$. Agents are of one of two types, either they want $a = 1$ or $a = 0$. But, motivated by fairness, the planner's objective is to choose the mean of the desired levels. So long as $n \geq 4$ and there is always at least two agents of each preference type, both non-satiation and no veto power are satisfied.

Why is non-satiation a useful domain restriction? Intuitively, it ensures that it is always possible to reward players. More precisely, it allows us to drop the preference requirement in Condition (**) because the requirement is satisfied by any SCR satisfying non-satiation, independent of the evidence structure. Accordingly, the following weaker version of evidence-monotonicity will be useful.

Definition 5 (Weak Evidence-monotonicity). *A SCR f is* weak *evidence-monotonic provided that for all θ and all $a \in f(\theta)$, there exists $e^*_{\theta,a} \in E^\theta$ such that for any θ', if* (*) *and*

$$e^*_{\theta,a} \in E^{\theta'} \text{ and } E^{\theta'} \subseteq E^\theta, \tag{***}$$

then $a \in f\left(\theta'\right)$.

Plainly, the reason this is weaker than evidence-monotonicity is that (**) has been replaced with the stronger (***).

Remark 5. For any f satisfying non-satiation, evidence-monotonicity is equivalent to weak evidence-monotonicity.

In the absence of non-satiation, however, weak evidence-monotonicity is not generally sufficient for implementation (even when $n \geq 3$ and no veto power holds). This can seen by returning to the SCR $\tilde{f}$ in Example 3. Condition (***) is falsified for $\theta' = \theta_1$ and $\theta = \theta_3$ because $E^{\theta_1} \nsubseteq E^{\theta_3}$. Hence, $\tilde{f}$ is weak evidence-monotonic. But, as argued in the example, $\tilde{f}$ is not evidence-monotonic and thus not implementable. The reason is precisely that $\tilde{f}$ does not satisfy non-satiation.

Since weak evidence-monotonicity allows us to separate preferences and evidence structure (respectively captured by (*) and (***)), we now use this to derive a number of simplifications and implications of evidence-monotonicity for the restricted domain of SCRs satisfying non-satiation.

2.4.1. *Normal Evidence Structures.* A widely-used class of evidence structures has the following property.

Definition 6 (Normality). *The evidence structure is normal or satisfies normality if for all i and θ, there is some $\bar{e}_{i,\theta} \in E_i^\theta$ such that $\left[\bar{e}_{i,\theta} \in E_i^{\theta'} \implies E_i^\theta \subseteq E_i^{\theta'}\right]$.*

The formulation of normality above follows Bull and Watson (2007). It says that for any player i and state θ, there is some evidence $\bar{e}_{i,\theta}$ that can be interpreted as a maximal or summary evidence because it proves by itself what agent i could prove by jointly sending all his available evidence. Thus it is equivalent to the *full reports condition* of Lipman and Seppi (1995), and is somewhat weaker than Green and Laffont's (1986) *nested range condition* in their "direct mechanism" setting.[17] The literature on strategic communication with hard information following Milgrom (1981) and Grossman (1981) typically assumes much stronger conditions than normality.

[17]Green and Laffont (1986) take $E_i = \Theta$ and assume that $\theta \in E_i^\theta$. The nested range condition says: if $\theta' \in E_i^\theta$ and $\theta'' \in E_i^{\theta'}$, then $\theta'' \in E_i^\theta$. This implies normality because one can set, for all i and θ, $\bar{e}_{i,\theta} = \theta$. To see that normality is strictly weaker, consider the following example: $\Theta = \{\theta_1, \theta_2, \theta_3\}$; for all i, $E_i^{\theta_1} = \{\theta_1, \theta_2\}$ and $E_i^{\theta_2} = E_i^{\theta_3} = \{\theta_2, \theta_3\}$. Normality holds by choosing for all i: $\bar{e}_{i,\theta_1} = \theta_1$ and $\bar{e}_{i,\theta_2} = \bar{e}_{i,\theta_3} = \theta_3$. On the other hand, the nested range condition is violated because for all i, $\theta_2 \in E_i^{\theta_1}$ yet $\theta_3 \in E_i^{\theta_2}$ and $\theta_3 \notin E_i^{\theta_1}$.

To illustrate the property, consider Example 2 again, where $\Theta = \mathbb{R}^n_+$, $E_i = \Theta$ and $E_i^\theta = [0, \theta_i]$ for all i, θ. This evidence structure is seen to be normal by setting $\bar{e}_{i,\theta} = \theta_i$ for all i, θ: if $\bar{e}_{i,\theta} = \theta_i \in E_i^{\theta'} = [0, \theta'_i]$ then it must be that $\theta_i \leq \theta'_i$ and so $E_i^\theta = [0, \theta_i] \subseteq [0, \theta'_i] = E_i^{\theta'}$, as required. It is also straightforward to check that Example 3 satisfies normality: for all i, set $\bar{e}_{i,\theta_1} = \bar{e}_{i,\theta_2} = \beta$ and $\bar{e}_{i,\theta_3} = \bar{e}_{i,\theta_4} = \alpha$.

Under normality, the definition of weak evidence-monotonicity can be dramatically simplified by replacing condition (***) with

$$E^\theta = E^{\theta'}. \tag{1}$$

The reason is that under normality, one can take $e^*_{i,\theta,a}$ in Definition 5 to equal $\bar{e}_{i,\theta}$ of Definition 6, because normality implies that for any $e_i \in E_i^\theta$, if $\bar{e}_{i,\theta} \in E_i^{\theta'}$, then $e_i \in E_i^{\theta'}$. Thus the existential qualifier in the definition of weak evidence-monotonicity (or evidence-monotonicity for a SCR satisfying non-satiation) can be dropped altogether.

This observation leads to an alternative characterization of weak evidence-monotonicity under normality. Say that a triplet (θ, a, θ') where $a \in f(\theta)$ *violates Maskin-monotonicity* if $a \notin f(\theta')$ and (*) is satisfied. For any θ and $a \in f(\theta)$, we define

$$T^f(\theta, a) := \left\{\theta' \in \Theta : (\theta, a, \theta') \text{ violates Maskin-monotonicity}\right\},$$

and for any θ,

$$T^f(\theta) := \bigcup_{a \in f(\theta)} T^f(\theta, a).$$

In other words, given that one wishes to achieve the outcomes $f(\theta)$ in state θ, $T^f(\theta)$ are precisely the states that cause a problem for implementation of f in the absence of evidence. In particular, a SCR f is implementable without evidence only if $\bigcup_{\theta \in \Theta} T^f(\theta) = \emptyset$.

Proposition 1. *Assume that the evidence structure is normal. Then a SCR f is weak evidence-monotonic if and only if*

$$\forall \theta : \theta' \in T^f(\theta) \implies E^\theta \neq E^{\theta'}. \tag{2}$$

Proof. Assume normality. For the "only if" part of the result, suppose that (2) fails. Then there is some θ, θ', and a such that $E^\theta = E^{\theta'}$, $a \in f(\theta)$, $a \notin f(\theta')$, and (*). From Definition 2 and the simplification of (***) to $E^\theta = E^{\theta'}$ under normality, f is not weak evidence-monotonic. The proof of the converse is similar. □

We illustrate an application of this result by returning to an earlier Example:

Example 4 (Example 3 continued.)**.** *Consider the SCR f from Example 3, where the evidence structure is normal. Plainly, f satisfies non-satiation. One can check that $T^f(\theta_1) = T^f(\theta_4) = \theta_3$, $T^f(\theta_2) = \emptyset$, and $T^f(\theta_3) = \{\theta_1, \theta_4\}$. Since $E^{\theta_3} = E^{\theta_4}$, Proposition 1 confirms that f is not weak evidence-monotonic (as argued directly before). On the other, consider the SCR f^* from the example; it also satisfies non-satiation. We have $T^{f^*}(\theta_1) = \{\theta_3, \theta_4\}$, $T^{f^*}(\theta_2) = \emptyset$, $T^{f^*}(\theta_3) = T^{f^*}(\theta_4) = \theta_1$. Since $E^{\theta_1} \neq E^{\theta_3} = E^{\theta_4}$, (2) is satisfied and hence f^* is (weak) evidence-monotonic (as argued directly before).*

Proposition 1 highlights what evidence structure is needed to implement non-Maskin-monotonic SCRs. It can be combined with our earlier results to yield useful corollaries.

Corollary 1. *Assume $n \geq 3$ and that the evidence structure is normal. A SCR f that satisfies both no veto power and non-satiation is implementable if*

$$\forall \theta, \theta' : \quad \left[E^{\theta} = E^{\theta'} \implies f(\theta) = f(\theta')\right]. \tag{3}$$

Proof. Plainly, (3) implies (2). The result follows from Proposition 1 and Theorem 2. □

Remark 6. Ben-Porath and Lipman (2009) study implementation with hard evidence under three assumptions: (i) the SCR is a function; (ii) the evidence structure is normal; and (iii) players have state independent preferences.[18] They refer to condition (3) as "measurability" and show in their Proposition 1 that under their maintained assumptions, a SCR is implementable only if it satisfies (3). Our Theorem 1 in fact implies a stronger necessary condition for implementation under their assumptions:

$$\forall \theta, \theta' : \quad \left[\left(\forall i, \text{ either } E_i^{\theta} = E_i^{\theta'} \text{ or } \left(f(\theta) \in \arg\max_a u_i(a, \theta') \text{ and } E_i^{\theta} \subseteq E_i^{\theta'}\right)\right) \implies f(\theta) = f(\theta')\right]. \tag{4}$$

To verify this, observe first that under state-independent preferences, (*) is satisfied for all a, θ, and θ'; second, if the antecedent in (4) holds, then state-independent preferences, normality, and the SCR being a function combine to imply that (**) will be satisfied no matter the choice of $\left\{e^*_{\theta, f(\theta)}\right\}_{\theta, f(\theta)}$ (in particular, when $e^*_{i,\theta,f(\theta)} = \bar{e}_{i,\theta}$ for all i, θ).

Condition (4) is obviously stronger than (3) and emphasizes the necessity of being able to reward players for evidence submission. Under non-satiation, the two conditions are equivalent. Ben-Porath and Lipman (2009) have also independently proved a result similar to Corollary 1. Note that even under normality, non-satiation and no veto power, (3) is not necessary for implementation when preferences are not state-independent, for instance the SCR f^* in Example 3. The reason is that when hard evidence is the same between two states, the planner can still exploit preference reversals to implement different outcomes, just as in the standard environment without evidence.

Motivated by the previous result, consider the following condition on the evidence structure:

$$\forall \theta : \theta' \neq \theta \implies E^{\theta'} \neq E^{\theta}. \tag{UPD}$$

Condition (UPD), short for *Universal Pairwise Distinguishability*, requires that any pair of distinct states be distinguishable via evidence, i.e. there must be at least one player whose available evidence differs in the two states. Obviously, if two states θ and θ' cannot be distinguished in this sense, implementation requires that no triplet (θ, a, θ') violate Maskin-monotonicity. On the other hand, since (UPD) implies that (3) is trivially satisfied, we also have:

Corollary 2. *Assume $n \geq 3$ and that the evidence structure is normal.* Any *SCR that satisfies both no veto power and non-satiation can be implemented if* (UPD) *holds.*

[18] I.e., $\forall i \in I, \forall \theta, \theta' \in \Theta, \forall a, b \in A, u_i(a, \theta) \geq u_i(b, \theta) \implies u_i(a, \theta') \geq u_i(b, \theta')$.

The above sufficient condition is tight: if (UPD) is violated, we can specify a profile of utility functions and a SCR (satisfying no veto power and non-satiation) such that this SCR is not weak evidence-monotonic and therefore not implementable.

2.4.2. *Non-Normal Evidence Structures.* Despite being a prevalent assumption, normality is a fairly demanding requirement.[19] One interpretation is that it assumes that there is no constraint on time, effort, space, etc., in providing evidence. The next example introduces some of the issues that arise with non-normal evidence structures.[20]

Example 5. *There are two propositions: a and b. Each member of a group of three or more experts knows which of the two propositions are true, if any. Due to time or space limitations, however, each one can provide a proof of at most one proposition. This problem can be represented by $\Theta = \{\phi, a, b, ab\}$, and for all i, $E_i^\phi = \{\phi\}$, $E_i^a = \{\phi, a\}$, $E_i^b = \{\phi, b\}$, and $E_i^{ab} = \{\phi, a, b\}$, where ϕ represents "neither proposition is true" or "no proof provided." Although this evidence structure satisfies (UPD), it is not normal, because for any i and $x \in E_i^{ab}$, there exists $\theta' \in \{a, b\}$ such that $x \in E_i^\theta$ but $E_i^{ab} \nsubseteq E_i^\theta$. Note that if each expert could prove both a and b when both are true, then we would augment ab to E_i^{ab}, and normality would be satisfied.*

Suppose now that the preferences of the experts over outcomes are state-independent, so that (*) is always satisfied. Since any choice of $\{e_{i,ab}^*\}_{i=1}^n$ will create some $\theta' \in \{a, b\}$ such that (**) is satisfied with $\theta = ab$, it follows from Theorem 1 that not every SCR is implementable. In particular, if the SCR is a function, then implementability requires $f(ab) \in \{f(a), f(b)\}$. On the other hand, by choosing $e_{i,\phi}^* = \phi, e_{i,a}^* = a$, and $e_{i,b}^* = b$, Theorem 2 implies that $f(ab) \in \{f(a), f(b)\}$ is also sufficient for the social choice function to be implementable under no veto power and non-satiation.

This motivates the question of what evidence structures permit non-Maskin-monotonic SCRs to be implemented even when the evidence structure is not normal. A sharp answer can be provided by using a general notion of distinguishability.

Definition 7. *For any θ and $\Omega \subseteq \Theta$, θ and Ω are distinguishable if for any $\Omega' \subseteq \Omega$, $E^\theta \neq \bigcup_{\theta' \in \Omega'} E^{\theta'}$.*

Thus, a state θ is distinguishable from an event or set of states Ω if for *every* subset Ω' of Ω, either some player can disprove Ω' when θ is the true state (which requires $E_i^\theta \nsubseteq \bigcup_{\theta' \in \Omega'} E_i^{\theta'}$) or some player can disprove θ when some state in Ω' is the true state (which requires $E_i^\theta \nsupseteq \bigcup_{\theta' \in \Omega'} E_i^{\theta'}$). Notice that if θ is distinguishable from Ω then θ is distinguishable from any subset of Ω. Consequently, if θ and Ω are distinguishable, then θ must be pairwise distinguishable from every $\theta' \in \Omega$ (in particular, $\theta \notin \Omega$). The following result establishes the converse for normal evidence structures.

Proposition 2. *Assume the evidence structure is normal. For any $\theta \in \Theta$ and $\Omega \subseteq \Theta$, if θ is distinguishable from each $\theta' \in \Omega$ then θ is distinguishable from Ω.*

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[19] Bull and Watson (2007), Glazer and Rubinstein (2001, 2004, 2006), Lipman and Seppi (1995), and Sher (2008) study problems where normality does not hold.

[20] As mentioned in Remark 2, non-normal evidence structures also make it possible for dynamic mechanisms to improve upon static mechanisms under the feasibility interpretation of hard evidence. See Appendix B.

Proof. Fix θ and Ω and assume that $\forall\theta' \in \Omega : E^{\theta} \neq E^{\theta'}$. Suppose, per contra, that for some $\Omega' \subseteq \Omega : E^{\theta} = \bigcup_{\theta' \in \Omega'} E^{\theta'}$. Then for all $\theta' \in \Omega'$, $\bar{e}_{\theta'} \in E^{\theta}$ (where $\bar{e}$ is from the definition of normality), and moreover for some $\tilde{\theta} \in \Omega'$, $\bar{e}_{\theta} \in E^{\tilde{\theta}}$. By normality, $E^{\tilde{\theta}} = E^{\theta}$, a contradiction. □

Example 5 shows why normality is key to the above result: in the example, state ab is pairwise distinguishable from every other state but not distinguishable from the event $\{\phi, a, b\}$.

We can now state a useful general characterization of weak evidence-monotonicity in terms of distinguishability.

Proposition 3. *A SCR f is weak evidence-monotonic if and only if for all θ and $a \in f(\theta)$, θ and $T^f(\theta, a)$ are distinguishable.*

Proof. See Appendix. □

Proposition 1 can be seen as a corollary of Proposition 3, because when the evidence structure is normal, distinguishability of any θ and Ω reduces to distinguishability of θ from each $\theta' \in \Omega$ (Proposition 2). An immediate implication of Proposition 3 is that the following condition on the evidence structure guarantees that every SCR is weak evidence-monotonic, no matter what the agents' preferences are:

$$\forall\theta : \ \Omega \subseteq \Theta \diagdown \{\theta\} \implies \theta \text{ is distinguishable from } \Omega. \tag{UD}$$

Condition (UD), short for *Universal Distinguishability*, requires that each state must be distinguishable from any event that does not contain it. In general, this will be a stronger requirement than (UPD), but Proposition 2 implies that they are equivalent under normality.

Corollary 3. *If $n \geq 3$. Any SCR that satisfies both no veto power and non-satiation can be implemented if (UD) holds.*

Proof. Let f be an arbitrary SCR satisfying non-satiation and no veto power. Pick any θ and $a \in f(\theta)$. Since $\theta \notin T^f(\theta, a)$, (UD) implies that θ is distinguishable from $T^f(\theta, a)$. By Proposition 3, f is weak evidence-monotonic and so evidence-monotonic because of non-satiation. Theorem 2 yields the desired conclusion. □

The Corollary is tight in the sense that if (UD) is violated, we can specify a profile of utility functions and a SCR (satisfying no veto power and non-satiation) such that the SCR is not weak evidence-monotonic and hence not implementable. We illustrate the Corollary with the following example.

Example 6. *$\Theta = \{\theta_1, \theta_2, \theta_3\}$; $n = 3$; $E_1^{\theta_1} = \{x, y\}$, $E_1^{\theta_2} = \{x\}$, $E_1^{\theta_3} = \{y\}$, $E_2^{\theta_1} = \{x, y\}$, $E_2^{\theta_2} = \{y\}$, $E_2^{\theta_3} = \{x\}$, and $E_3^{\theta} = \{z\}$ for all θ. It is easy to see that normality does not hold.* (UD) *holds because $E^{\theta_1} = \{(x, x, z), (x, y, z), (y, x, z), (y, y, z)\}, E^{\theta_2} = \{(x, y, z)\}$, and $E^{\theta_3} = \{(y, x, z)\}$. Hence, by Corollary 3, any SCR satisfying no veto power and non-satiation is implementable.*

2.5. **Ranking Evidence Structures.** Since the structure of evidence crucially affects the possibility of implementation, it is natural to ask whether some evidence structures are "more informative" than others from a planner's perspective because they allow for a larger set of implementable SCRs. Not surprisingly, the notion of distinguishability is key.

Definition 8. *Evidence structure $\tilde{\mathcal{E}}$ is more informative than $\mathcal{E}$, denoted $\tilde{\mathcal{E}} \blacktriangleright \mathcal{E}$, if any $\theta \in \Theta$ and $\Omega \subseteq \Theta$ that are distinguishable under $\mathcal{E}$ are also distinguishable under $\tilde{\mathcal{E}}$.*

Thus, $\blacktriangleright$ is partial order on the set of possible evidence structures. At one end, if $\mathcal{E}$ satisfies (UD), then for any $\tilde{\mathcal{E}}$, $\mathcal{E} \blacktriangleright \tilde{\mathcal{E}}$. At the other end, if $\mathcal{E}$ represents no evidence ($\forall i, \theta : E_i^\theta = E_i$), then for any $\tilde{\mathcal{E}}$, $\tilde{\mathcal{E}} \blacktriangleright \mathcal{E}$.

Remark 7. If $\tilde{\mathcal{E}}$ is normal, then $\tilde{\mathcal{E}} \blacktriangleright \mathcal{E}$ if and only if

$$\{(\theta, \theta') : E^\theta \neq E^{\theta'}\} \subseteq \{(\theta, \theta') : \tilde{E}^\theta \neq \tilde{E}^{\theta'}\}.$$

Proof. See Appendix. □

The following result shows why $\blacktriangleright$ appropriately captures when one evidence structure is "better" than another.

Proposition 4. *Assume that $\tilde{\mathcal{E}} \blacktriangleright \mathcal{E}$. If a SCR is weak evidence-monotonic under $\mathcal{E}$ it is also weak evidence-monotonic under $\tilde{\mathcal{E}}$.*

Proof. Pick any SCR f that is weak evidence-monotonic under $\mathcal{E}$, and any θ and $a \in f(\theta)$. By the "only if" part of Proposition 3, θ is distinguishable from $T^f(\theta, a)$. Since $\tilde{\mathcal{E}} \blacktriangleright \mathcal{E}$, θ and $T^f(\theta, a)$ are also distinguishable under $\tilde{\mathcal{E}}$. Since θ and $a \in f(\theta)$ are arbitrary, the "if" part of Proposition 3 implies that f is weak evidence-monotonic under $\tilde{\mathcal{E}}$. □

Remark 8. The above result is tight in the sense that if $\tilde{\mathcal{E}}$ is not more informative than $\mathcal{E}$, then there exist preferences for agents and a SCR such that the SCR is weak evidence-monotonic under $\mathcal{E}$ but not under $\tilde{\mathcal{E}}$. See the Appendix for an explicit construction.

Proposition 4 and Theorem 2 imply the following.

Corollary 4. *Assume that $\tilde{\mathcal{E}} \blacktriangleright \mathcal{E}$ and $n \geq 3$. Let f be a SCR satisfying no veto power and non-satiation. If f is implementable under $\mathcal{E}$ then f is also implementable under $\tilde{\mathcal{E}}$.*

2.6. **Ranking Social Choice Rules.** Just as the notion of distinguishability permits a partial order on evidence structures, it also allows a partial order on SCRs in terms of how easy they are to implement. We begin with a definition capturing when a SCR is "more Maskin-monotonic".

Definition 9. *For SCRs f and h, f is more Maskin-monotonic than h, denoted $f \trianglerighteq h$, if*

$$\forall \theta, a \in f(\theta) : \exists a' \in h(\theta) \text{ s.t. } T^f(\theta, a) \subseteq T^h(\theta, a'). \tag{5}$$

Note that this definition is independent of any evidence structure. If f is Maskin-monotonic, then $f \trianglerighteq h$ for any h, and moreover, if h is not Maskin-monotonic, then [not $h \trianglerighteq f$]. The next result shows why the relation of more Maskin-monotonic is useful.

Proposition 5. *If $f \trianglerighteq h$, then if h is weak evidence-monotonic under evidence structure $\mathcal{E}$, f is also weak evidence-monotonic under $\mathcal{E}$.*

Proof. Suppose h is weak evidence-monotonic under $\mathcal{E}$. Pick any θ and $a \in f(\theta)$; by the "if" part Proposition 3, it suffices to show that θ is distinguishable from $T^f(\theta, a)$. By (5), there exists $a' \in h(\theta)$ such that $T^f(\theta, a) \subseteq T^h(\theta, a')$. By the "only if" part of Proposition 3, θ is distinguishable from $T^h(\theta, a')$, and hence distinguishable from every subset of $T^h(\theta, a')$, including $T^f(\theta, a)$. □

Proposition 5 and Theorem 2 imply the following corollary, which justifies why $\trianglerighteq$ ranks SCRs in terms of their implementability.

Corollary 5. *Assume $n \geq 3$ and f and h are SCRs satisfying NVP and non-satiation such that $f \trianglerighteq h$. If h is implementable under evidence structure $\mathcal{E}$, then f is also implementable under $\mathcal{E}$.*

2.7. **Two Agents.** As noted earlier, no veto power is a very strong condition when there are only two agents, since it requires that each agent's top-ranked alternative be chosen by the SCR in each state. Moreover, Theorem 2 does not extend to $n = 2$: it is not hard to construct two-player examples of evidence-monotonic SCRs that are not implementable despite no veto power being satisfied. Since the two agent case is quite important in problems of contracting, bargaining, and dispute resolutions, we give it a brief separate treatment in this section.[21]

In the standard setting without evidence, implementation with only two players is known to be more demanding than with many players. Maskin (1999) shows that with two players, a Pareto-efficient SCR defined over an unrestricted domain of preferences is implementable if and only if it is dictatorial.[22] The major complication with two-player implementation is that the planner faces a severe challenge in identifying which player has deviated when a non-equilibrium message profile is observed. Dutta and Sen (1991) and Moore and Repullo (1990) provide a full characterization of implementable SCRs for the two-player case without evidence by introducing a condition known as *condition-β*. In particular, Moore and Repullo (1990, Corollary 4) provide conditions under which Maskin-monotonicity is necessary and sufficient for implementation in settings without evidence. In this section, we derive an analogous result in our setting with evidence.[23]

Let $f(\Theta) := \bigcup_\theta f(\theta)$ denote the range of f. The following condition of *restricted veto power* is due to Moore and Repullo (1990):

[21] A previous version of this paper also contained a characterization for the one agent case, which is relevant for principal-agent problems.

[22] In our framework, unrestricted domain of preferences means that for any profile of complete and transitive binary relations over outcomes, $(\precsim_1, ..., \precsim_n)$, there exists a state of the world, θ, in which for each player i, $x \precsim_i y \Leftrightarrow u_i(x, \theta) \leq u_i(y, \theta)$. A SCR f is dictatorial if there is some i such that for all a and θ, $a \in f(\theta) \Leftrightarrow a \in \arg\max_b u_i(b, \theta)$.

[23] In a previous version of this paper, we extended *condition-β* to *condition evidence-β* and showed that this property is necessary and sufficient for implementation of a SCR satisfying the condition of restricted veto-power discussed below. This characterization is available upon request.

Definition 10 (Restricted Veto Power)**.** *A SCR f satisfies restricted veto power (RVP) if for all i, θ, a and $b \in f(\Theta)$,*

$$\left[\left(a \in \bigcap_{j \neq i} \arg\max_{c \in A} u_j(c,\theta)\right) \text{ and } (u_i(a,\theta) \geq u_i(b,\theta))\right] \implies a \in f(\theta).$$

In words, RVP says that if an outcome a is top-ranked at state θ by all players $j \neq i$, and if there is an outcome b in the range of f such that i weakly prefers a to b at state θ, then a must belong to $f(\theta)$. Thus, agent i does not have the power to veto outcome a unless it is strictly worse for him than every outcome in the range of f. Plainly, RVP is a weakening of NVP, trivially coinciding with NVP when $f(\Theta) = A$. Intuitively, the smaller the range of f, the less demanding is RVP.

We say that there is a *bad outcome* (relative to a SCR f) if there is some outcome $z \notin f(\Theta)$ such that for all $i, \theta : u_i(z,\theta) < u_i(a,\theta)$ for all $a \in f(\Theta)$. In other words, in any state, outcome z is strictly worse for all players than any outcome in the the range of the SCR. We are now in position to generalize a result of Moore and Repullo (1990, Corollary 4) to settings with evidence.

Theorem 3. *Assume $n = 2$. Let f be a SCR satisfying RVP, and suppose there is a bad outcome. f is implementable if and only if it is weak evidence-monotonic.*

Proof. See Appendix. □

Thus, when $n = 2$, there is a bad outcome, and RVP holds, weak evidence-monotonicity characterizes what is implementable. Note that we could also use evidence-monotonicity in place of weak evidence-monotonicity in the statement of Theorem 3, since it is a stronger condition that is necessary by Theorem 1. While substantive domain restrictions, both RVP and existence of a bad outcome are naturally satisfied in some economic environments where there is sufficient conflict between agents.[24] Example 7 below provides a non-economic environment satisfying both assumptions.

Combining Proposition 3 and Theorem 3 yields a two-agent counterpart of Corollary 3.

Corollary 6. *Assume $n = 2$. Let f be a SCR satisfying RVP, and suppose there is a bad outcome. f is implementable if* (UD) *holds.*

The following example illustrates an application of this result.

Example 7. *Suppose the outcome is how much of some public good to produce, so that $A = \mathbb{R}_+$. There are two agents, each of whom has single-peaked preferences with a bliss point. Specifically, if an agent has bliss point x, his payoff from amount a of the public good is $-(x-a)^2$. Agent 1's possible bliss points are $[2,3]$ whereas agent 2's possible bliss points are $[5,6]$. A state of the world is a pair of bliss points, i.e. $\Theta = [2,3] \times [5,6]$. The SCR f is given by the midpoint of the two agents' bliss points.*

[24] For example, if the outcome is how much of a divisible private good each agent gets and the SCR provides a strictly positive amount to each agent, then under usual preferences, $(0,0)$ is a bad outcome and RVP holds if there is a feasibility constraint on the total amount of the private good that can be provided.

Observe that $f(\Theta) = [3.5, 4.5]$. Consequently, 0 plays the role of a bad outcome. RVP is also satisfied because in any state, the bliss point for one agent is strictly worse for the other agent than any outcome in $f(\Theta)$. Note that neither NVP nor non-satiation are satisfied, however.

Now let the evidence structure be such that in any state $\theta = (\theta_1, \theta_2)$, $E_i^\theta = \{[x, y] : x \leq \theta_i \leq y\}$; in other words, each agent submit evidence that his bliss point lies in any interval containing his bliss point. (UD) *is satisfied because $\theta \in E^{\theta'}$ if and only if $\theta' = \theta$.*

By Corollary 6, f is implementable, even though f is not Maskin-monotonic.

Remark 9. Both RVP and existence of a bad outcome are essential to Theorem 3 and Corollary 6. Examples 10 and 11 in Appendix A show that without either of the two conditions, the results need not hold.

3. Costly Evidence Fabrication

In this section, we study a more general setting where agents can fabricate any evidence in any state of the world, but potentially bear differential costs of fabrication across states. Specifically, in each state of the world, θ, agent i can create evidence $e_i \in E_i$ but thereby incurs a disutility of $c_i(e_i, \theta) \in \mathbb{R}_+ \cup \{+\infty\}$. We refer to $(c_i(\cdot, \cdot))_{i \in I}$ as the *cost structure* and denote $E_i^\theta := \{e_i \in E_i : c_i(e_i, \theta) = 0\}$. Without loss of generality, assume that for all θ, i, $E_i^\theta \neq \emptyset$.[25] This defines a general costly signaling where an agent's payoff in state θ from outcome a when he sends evidence e_i is $u_i(a, \theta) - c_i(e_i, \theta)$. The separability assumption here in payoffs is for ease of exposition.[26] Note that this setting reduces to the hard evidence model (with cost interpretation) if the range of each $c_i(\cdot, \cdot)$ is $\{0, +\infty\}$.

In this setting, there are multiple possible notions of implementation. We assume the planner's objective is to avoid any signaling costs, i.e. in equilibrium only costless messages can be used. The definition of a mechanism is the same as earlier.

Definition 11 (Implementation with costly evidence)**.** *A mechanism (M, g) implements the SCR f if*

1. $\forall \theta,\ f(\theta) = O(M, g, \tau, \theta)$, *and*
2. $(m, e) \in NE(M, g, \tau, \theta) \implies e \in E^\theta$.

Clearly, the second requirement in the definition is what ensures that every equilibrium of an implementing mechanism must be costless. It is potentially interesting to weaken the implementation requirement by allowing agents to send costly evidence at equilibrium. However, with this approach, the comparison with the previous setting of hard evidence becomes less transparent, and in addition, new issues appear. For instance, if one thinks of a SCR as obtaining from the maximization of a welfare criterion, then there will often be a trade-off between maximizing this

[25] This is without loss of generality because we will allow the planner to add costless (cheap-talk) messages.

[26] For each i and θ, we could more generally define an extension of $u_i(\cdot, \theta)$ as $\tilde{u}_i(\cdot, \cdot, \theta) : A \times E_i \to \mathbb{R}$ satisfying the property that for any player i, evidence e_i, and outcome a, $\tilde{u}_i(a, e_i, \theta) \leq u_i(a, \theta)$. We would then set $E_i^\theta(a) := \{e_i \in E_i : u_i(a, \theta) = \tilde{u}_i(a, e_i, \theta)\}$ and assume this set is non-empty. See also footnote 27.

criterion and the welfare losses incurred by agents when they provide costly evidence. While interesting in their own right, these considerations are beyond the scope of the current paper.

We now define the relevant notion of monotonicity in this setting, which we call *cost-monotonicity.*

Definition 12 (Cost-monotonicity). *A SCF f is cost-monotonic provided that for all θ and all $a \in f(\theta)$, there exists $e^*_{\theta,a} \in E^\theta$ such that for all θ', if*

$$\left[e^*_{\theta,a} \in E^{\theta'} \text{ and } \left\{\forall i : [u_i(a,\theta) \geq u_i(b,\theta) - c_i(e_i,\theta)] \Rightarrow \left[u_i\left(a,\theta'\right) \geq u_i\left(b,\theta'\right) - c_i(e_i,\theta')\right]\right\}\right] \quad \text{(CM)}$$

then $a \in f(\theta')$.

Similar to the case of hard evidence, the $e^*_{\theta,a}$ in the definition above should be interpreted as a costless evidence profile that is to be sent by the agents at state θ in support of outcome a. Condition (CM) says that if an outcome a is socially desired at θ but not at θ', then one of two cases must holds: either (i) the profile $e^*_{\theta,a}$ is not costless at θ'; or (ii) there is some agent who strictly prefers obtaining an outcome b with evidence e_i to obtaining a costlessly at state θ' but prefers the latter over the former at state θ. Differently put, the latter condition requires that there must be some agent i for whom $(a, e^*_{i,\theta,a})$ "goes down" in preference ranking over the joint space of outcomes and evidence when the state changes from θ to θ'.[27]

Example 8. *Consider a setting where players have a (potentially small) "preference for honesty." Formally, we assume that for each player i, $E_i = \Theta$ and the cost structure is given by:*

$$c_i(\theta,\theta') = \begin{cases} 0 & \text{if } \theta = \theta' \\ \varepsilon & \text{if } \theta \neq \theta' \end{cases}$$

where $\varepsilon > 0$ can be arbitrarily small. This cost structure implies that for any θ and i, $E^\theta_i = \{\theta\}$. It follows that any SCR in this setting is cost-monotonic, as condition (CM) is violated for any θ, θ' no matter the SCR (because for any f, θ and $a \in f(\theta)$, one must set $e^*_{\theta,a} = (\theta, \dots, \theta)$). In fact, it is not needed that all players have such a preference for honesty, only that in each state, there be some player who does (the identity of the player could vary with the state).[28]

Plainly, cost-monotonicity is weaker than Maskin-monotonicity. For an arbitrary cost structure, cost-monotonicity neither implies nor is implied by evidence-monotonicity. However, they are equivalent in the setting of hard evidence (with cost interpretation).

Proposition 6. *If $c_i(e_i,\theta) \in \{0,\infty\}$ $\forall i, e_i, \theta$, then a SCR is cost-monotonic if and only if it is evidence-monotonic.*

[27] If preferences are not separable between outcomes and evidence costs, as described in footnote 26, a SCR f would be cost-monotonic provided that for all θ and all $a \in f(\theta)$, there exists $e^*_{\theta,a} \in E^\theta$ such that for all θ', if

$$\left[e^*_{\theta,a} \in E^{\theta'}(a) \text{ and } \left\{\forall i : [u_i(a,\theta) \geq \tilde{u}_i(b,e_i,\theta)] \Rightarrow \left[u_i\left(a,\theta'\right) \geq \tilde{u}_i\left(b,e_i,\theta'\right)\right]\right\}\right]$$

then $a \in f(\theta')$.

[28] It may be worth emphasizing that it is without loss of generality to assume that the planner knows the identity of the player with honesty preferences in any state: if the planner has uncertainty about which player it is, this only requires extending the state space.

Proof. See Appendix. □

The following necessity result is analogous to Theorem 1 and establishes that cost-monotonicity is necessary for implementability with costly evidence fabrication.

Theorem 4. *If f is implementable then f is cost-monotonic.*

Proof. Since f is implementable, there exists a mechanism (M, g) that implements f. Hence, for each θ, for each $a \in f(\theta)$, there exists $(m_{\theta,a}, e_{\theta,a}) \in M \times E^\theta$ that is a costless Nash equilibrium at θ such that $g(m_{\theta,a}, e_{\theta,a}) = a$. For each θ, set $e^*_{\theta,a} = e_{\theta,a}$. Now consider any θ and θ' satisfying (CM). Since $(m_{\theta,a}, e_{\theta,a}) \in M \times E^\theta$ is a costless Nash equilibrium at θ,

$$u_i(g(m_{\theta,a}, e_{\theta,a}), \theta) \geq u_i(g(m'_i, e'_i, m_{-i,\theta,a}, e_{-i,\theta,a}), \theta) - c_i(e'_i, \theta)$$

for any $(m'_i, e'_i) \in M_i \times E_i$. The second part of (CM) implies that

$$u_i(g(m_{\theta,a}, e_{\theta,a}), \theta') \geq u_i(g(m'_i, e'_i, m_{-i,\theta,a}, e_{-i,\theta,a}), \theta') - c_i(e'_i, \theta')$$

for any $(m'_i, e'_i) \in M_i \times E_i$. Consequently, $(m_{\theta,a}, e_{\theta,a})$ is a Nash equilibrium at θ'. Hence $a \in f(\theta')$ as claimed. □

To see how Theorem 4 has bite, consider the following example of education signaling.

Example 9. *There are n workers, each with an ability level measuring her marginal productivity. The state of the world is a vector of abilities. There is one job that must be allocated to a single worker along with a wage, so that $A = \{1, \ldots, n\} \times \mathbb{R}_+$. The SCR is $f(\theta) = (i^*, \theta_{i^*})$ where $\{i^*\} = \max(\arg\max_i \theta_i)$, i.e. allocate the job to the most-able worker and pay her marginal product (ties in ability are broken in favor of workers with higher indices, which is convenient but inessential). Agent i's utility from an outcome $a = (a_1, a_2)$ is state-independent: $u_i(a, \theta) = 0$ if $a_1 \neq i$ (he does not get the job), and $u_i(a, \theta) = a_2$ (the wage) otherwise. Suppose that agents can signal their ability through a choice of education, $e_i \in \mathbb{R}_+$. The cost of education for a worker only depends on his individual type; we abuse notation and write $c_i(e_i, \theta_i)$. Assume the cost of education satisfies two reasonable properties: for all i and θ_i, $c_i(e_i, \theta_i) = 0$ if and only if $e_i = 0$; for all i, $c_i(e_i, \theta_i)$ is strictly increasing in e_i and strictly decreasing in θ_i.*

Then $E_i^\theta = \{0\}$ for all i, θ, and since $u_i(a, \theta) - u_i(b, \theta) = u_i(a, \theta') - u_i(b, \theta')$ for any a, b, θ, θ', it follows that (CM) is satisfied if and only if $\theta' \leq \theta$ (in the sense of the usual vector order).[29] *Since $\theta' \leq \theta$ does not imply $f(\theta') = f(\theta)$, f is not cost-monotonic, and by Theorem 4, not implementable.*

The next result shows that if there are three or more players, cost-monotonicity is also sufficient for implementation when f satisfies NVP. This is analogous to Theorem 2.

Theorem 5. *Assume $n \geq 3$. If f is cost-monotonic and satisfies NVP then f is implementable.*

[29] To see the "only if", consider any $\theta < \theta'$. The latter part of (CM) is violated by considering, for some $\varepsilon > 0$, $a = f(\theta) = (i^*, \theta_{i^*})$, $b = (i^*, \theta_{i^*} + \varepsilon)$, and e_{i^*} s.t. $c_{i^*}(e_{i^*}, \theta_{i^*}) = \varepsilon$.

Proof. The proof is by construction of a canonical mechanism that implements any cost-monotonic SCR f when $n \geq 3$. Fix, for all i, $M_i = \Theta \times A \times \mathbb{N}$. Define $g(m,e)$ according to the following rules:

(1) If $m_1 = \cdots = m_n = (\theta, a, k)$ with $a \in f(\theta)$, and $e = e^*_{\theta,a}$, then $g(m,e) = a$.
(2) If $\exists i$ s.t. $\forall j \neq i$, $e_j = e^*_{j,\theta,a}$ and $m_j = (\theta, a, k)$ with $a \in f(\theta)$, and either $m_i = (\theta', b, l) \neq (\theta, a, k)$ or $e_i \neq e^*_{i,\theta,a}$ then
   a) if $u_i(a,\theta) \geq u_i(b,\theta) - c_i(e_i,\theta)$, then $g(m,e) = b$.
   b) if $u_i(a,\theta) < u_i(b,\theta) - c_i(e_i,\theta)$, then $g(m,e) = a$.
(3) For any other (m,e), letting $m_i = (\theta_i, a_i, k_i)$ and $i^* = \min_{i \in I} \arg\max_{j \in I} k_j$, $g(m,e) = a_{i^*}$.

Step 1. It is routine to verify that in any state θ, there is a "truthful" NE where for some $k \in \mathbb{N}$, each agent i plays $m_i = (\theta, a, k)$ with $a \in f(\theta)$ and $e_i = e^*_{i,\theta,a}$. This NE results in outcome a and, moreover, since $e^*_{i,\theta,a} \in E_i^\theta$, it is costless.

For the remainder of the proof, suppose the true state is θ', and (m,e) is a NE. We will first show that $g(m,e) \in f(\theta')$.

Step 2. Using NVP, standard arguments as in the proof of Theorem 2 show that if Rules (2) or (3) apply to (m,e), then $g(m,e) \in f(\theta')$.

Step 3. It remains to consider the case where (m,e) falls into rule (1). Here $g(m,e) = a \in f(\theta)$. It suffices to show that (CM) is satisfied, because cost-monotonicity then implies that $a \in f(\theta')$. The latter part of (CM) must hold because a player can always deviate into rule (2a) and get any outcome b such that $u_i(a,\theta) \geq u_i(b,\theta) - c_i(e_i,\theta)$ by producing the evidence e_i. To see that the first part of (CM) ($e^*_{\theta,a} \in E^{\theta'}$) must also hold, observe that if it were not satisfied, then there would exist a player i such that $e^*_{i,\theta,a} \notin E_i^{\theta'}$ and so $c_i(e^*_{i,\theta,a}, \theta') > 0$. This player i can deviate to $m_i = (\theta, a, k)$ with any evidence $\tilde{e}_i \in E_i^{\theta'}$. Since $\tilde{e}_i \neq e^*_{i,\theta'}$, this deviation would lead to rule (2a), with the outcome still being a while player i's gets utility $u_i(a,\theta') > u_i(a,\theta') - c_i(e^*_{i,\theta}, \theta')$. Therefore, this is a profitable deviation, contradicting (m,e) being a NE.

Step 4. Finally, we must show that no evidence cost is incurred in equilibrium, i.e. $e \in E^{\theta'}$. We have already argued this in Step 3 if (m,e) falls into Rule (1). If (m,e) falls into Rules (2) or (3), then $e \in E^{\theta'}$, because otherwise any player i for whom $e_i \notin E_i^{\theta'}$ can proftiably deviate by announcing a larger integer and requesting outcome $g(m,e)$ with costless evidence. □

As an application of this result, we return to the example of preferences for honesty.

Corollary 7. *Assume that in each state, at least one player has a preference for honesty as formalized in Example 8. By Theorem 5, if there are three or more players, any SCR satisfying no veto power is implementable.*

Independently, Dutta and Sen (2009) have obtained a very similar result to Corollary 7. Their paper is entirely about implementation with preferences for honesty, and thus uses a simpler mechanism than the one we use to prove Theorem 5. See also Matsushima (2008a,b) for related results in similar but not identical settings.

4. Conclusion

This paper has generalized the implementation problem to incorporate agents' ability to signal something about the state of the world. The central idea is that the planner can use either agents' preferences over outcomes or their signaling technology to discriminate between states of the world. We have studied both hard evidence, when players can prove that the state lies in some subset of all possible states, and the costly fabrication of evidence, where the costs of fabrication can vary across states. The results we have obtained may be useful in terms of both necessary conditions—in particular, the finding that some instrument to reward players is needed in appropriate situations—and sufficient conditions demonstrating how a wide class of social choice rules are implementable as a function of the evidence structure. We have formulated an appropriate generalization and weakening of Maskin-monotonicity—evidence-monotonicity in the case of hard evidence and cost-monotonicity in the case of costly evidence fabrication—and shown that this is the key to implementation in our framework. Natural evidence structures yield quite permissive results for implementation.

There are a number of directions that this research can be taken in. Our analysis here substantially exploits the complete information setting, and it is obviously important to understand how the arguments can be extended when agents have private information. In ongoing work, we are studying implementation with hard evidence in Bayesian Nash equilibrium in incomplete information environments. We extend the notion of evidence-monotonicity to *Bayesian evidence-monotonicity.* Bayesian evidence-monotonicity generalizes Jackson's (1991) Bayesian monotonicity condition, and together with the usual incentive compatibility condition is the key for Bayesian implementation with hard evidence.

Within the complete information framework, it would also be useful to understand how evidence changes the implementation problem when one restricts attention to "nice" mechanisms, for example, "bounded mechanisms" (Jackson, 1992). We are optimistic that based on the current paper and Ben-Porath and Lipman's (2009) interesting results for subgame perfect implementation with hard evidence, this will be a fruitful avenue. In a related vein, the presence of evidence can also generally allow greater scope for implementing with weaker solutions concepts, such as in dominant strategies. This too would be a useful direction for future work.

Appendix A. Omitted Proofs

Proof of Proposition 3. We first note that f is weak evidence-monotonic if and only if

$$\forall \theta, a \in f(\theta),\ \exists e^*_{\theta,a} \in E^\theta \text{ s.t. } \forall \theta' \in T^f(\theta,a),\ \exists i \in N : \left(e^*_{i,\theta,a} \notin E^{\theta'}_i\right) \text{ or } \left(E^{\theta'}_i \nsubseteq E^\theta_i\right). \tag{6}$$

In the sequel, we will use this equivalent formulation. For the "if part" of the result, assume that for each θ, $a \in f(\theta)$ and $\Omega \subseteq T^f(\theta,a) : E^\theta \neq \bigcup_{\theta' \in \Omega} E^{\theta'}$. Proceed by contradiction and assume that (6) is false. This implies that there exists θ and $a \in f(\theta)$ s.t. for all $e \in E^\theta$ there exists $\theta'(e) \in T^f(\theta,a)$, for which we have that for all $i \in N$: $\left(e_i \in E^{\theta'(e)}_i\right)$ and $\left(E^{\theta'(e)}_i \subseteq E^\theta_i\right)$. Set $\Omega := \bigcup_{e \in E^\theta} \theta'(e)$. Note that $\Omega \subseteq T^f(\theta,a)$. Since for each $e \in E^\theta$ it is the case that $e \in E^{\theta'(e)}$, this shows that $E^\theta \subseteq \bigcup_{\theta' \in \Omega} E^{\theta'}$. Finally, for each $e \in E^\theta : E^{\theta'(e)} \subseteq E^\theta$, hence we have $\bigcup_{\theta' \in \Omega} E^{\theta'} \subseteq E^\theta$, and so $E^\theta = \bigcup_{\theta' \in \Omega} E^{\theta'}$, a contradiction with the assumption. For the "only if part", assume that f satisfies (6) and proceed again by contradiction assuming that for some θ, some $a \in f(\theta)$ and some $\Omega \subseteq T^f(\theta,a) : E^\theta = \bigcup_{\theta' \in \Omega} E^{\theta'}$. This implies that for some θ and some $a \in f(\theta)$: (i) for all $e \in E^\theta$, there exists $\theta'(e) \in \Omega \subseteq T^f(\theta,a)$ s.t. $e \in E^{\theta'(e)}$ and (ii) $E^{\theta'(e)} \subseteq E^\theta$, which contradicts the assumption that (6) is true. □

Proof of Remark 7. Necessity is immediate, so we prove sufficiency. Assume

$$\{(\theta,\theta') : E^\theta \neq E^{\theta'}\} \subseteq \{(\theta,\theta') : \tilde{E}^\theta \neq \tilde{E}^{\theta'}\}. \tag{7}$$

Suppose θ and $\Omega \subseteq \Theta$ are distinguishable under $\mathcal{E}$. Then θ is distinguishable from any $\theta' \in \Omega$ under $\mathcal{E}$, any by (7), θ is distinguishable from any $\theta' \in \Omega$ under $\tilde{\mathcal{E}}$. Since $\tilde{\mathcal{E}}$ is normal, Proposition 2 implies that θ and Ω are distinguishable under $\tilde{\mathcal{E}}$. □

Proof of Remark 8. Suppose [not $\tilde{\mathcal{E}} \blacktriangleright \mathcal{E}$]. Then there exists a state θ^* and an event $\Omega^* \subseteq \Theta \setminus \{\theta^*\}$ that are distinguishable under $\mathcal{E}$ but not under $\tilde{\mathcal{E}}$. Let outcomes $x, y \in A$. Consider state independent preferences for all players except player 1, whose preferences are as follows: for any $\theta \in \Omega^* \cup \{\theta^*\}$, $u_1(y,\theta) > u_1(x,\theta) > u_1(a,\theta)$ for all $a \notin \{x,y\}$; for any $\theta \in \Omega \setminus \Omega^* \setminus \{\theta^*\}$, $u_1(x,\theta) > u_1(y,\theta) > u_1(a,\theta)$ for all $a \notin \{x,y\}$. Consider the SCR f given by $f(\theta) = \{y\}$ for all $\theta \in \Omega^*$ and $f(\theta) = \{x,y\}$ for all $\theta \notin \Omega^*$. It can be verified that $T^f(\theta^*,x) = \Omega^*$ and $T^f(\cdot,\cdot) = \emptyset$ otherwise. By Proposition 3, f is weak evidence-monotonic under $\mathcal{E}$ but not under $\tilde{\mathcal{E}}$. □

Proof of Theorem 3. (Necessity.) This is implied by Theorem 1.

(Sufficiency.) For each θ and $a \in f(\theta)$, let $e^*_{\theta,a}$ be the evidence profile per Definition 2. We will construct a mechanism that implements f. Fix, for all i,

$$M_i = \{(\theta, a, b, r, k) \in \Theta \times A \times A \times \{F, NF\} \times \mathbb{N} : a \in f(\theta)\},$$

where $\{F, NF\}$ is the set comprising the 2 elements "flag" and "no flag". Define g via the following four rules:

(1) If $m_1 = (\theta, a, b_1, NF, k_1)$ and $m_2 = (\theta, a, b_2, NF, k_2)$ and $e = e^*_{\theta,a}$ then $g(m,e) = a$.
(2) If $m_1 = (\theta_1, a_1, b_1, NF, k_1)$ and $m_2 = (\theta_2, a_2, b_2, NF, k_2)$ and Rule 1 does not apply then $g(m,e) = z$ where z is the worst outcome.
(3) If $\exists i$ s.t. $m_i = (\theta_i, a_i, b_i, F, k_i)$ and for $j \neq i : m_j = (\theta_j, a_j, b_j, NF, k_j)$ then

a) If $e_i \in E_i^{\theta_j}$ then $g(m,e) = \begin{cases} a_j \text{ if } u_i(a_i,\theta_j) \geq u_i(a_j,\theta_j) \\ a_i \text{ if } u_i(a_j,\theta_j) > u_i(a_i,\theta_j) \end{cases}$.

b) if $e_i \notin E_i^{\theta_j}$, then $g(m,e) = b_i$.

(4) If $m_1 = (\theta_1, a_1, b_1, F, k_1)$ and $m_2 = (\theta_2, a_2, b_2, F, k_2)$, then $g(m,e) = a_{i^*}$ where $i^* = \min_{i\in\{1,2\}} \arg\max_{j\in\{1,2\}} k_j$.

Step 1. We first show that for any θ and any $a \in f(\theta)$, $a \in O(M,g,\theta)$. It suffices to show that at state θ, for any $k \in \mathbb{N}$, each agent i playing $m_i = (\theta, a, a, NF, k)$ and sending $e_i = e^*_{i,\theta,a}$ is a NE, since by rule (1) of the mechanism, this results in outcome a. If some agent i deviates from this strategy profile, then only cases (2) or (3a) of the mechanism can apply. There is no profitable deviation for any player to rule (2) or (3a) since any such deviation yields a (weakly) worse outcome.

For the remainder of the proof, suppose the true state is θ', and (m,e) is a NE. We must show that $g(m,e) \in f(\theta')$.

Step 2. Any (m,e) that falls into Rule (2) is not a NE. To see this, recall that $g(m,e) = z$ and note that from (m,e), each player i can deviate by raising a flag and reach Rule (3). So consider a deviation for player 1 to $\tilde{m}_1 = (\cdot, a_2, a_2, F, \cdot)$. This leads to outcome $a_2 \in f(\theta_2)$, which player 1 strictly prefers to z since z is a bad outcome.

Step 3. Assume (m,e) falls into rule (3a). Without loss of generality, assume that $m_1 = (\theta_1, a_1, b_1, F, k_1)$ and $m_2 = (\theta_2, a_2, b_2, NF, k_2)$. Note that player 2 can raise a flag inducing rule (4). Thus, announcing an integer strictly higher than the one of player 1, he will get any outcome he wants. Hence, $g(m,e) \in \arg\max_{c\in A} u_2(c,\theta')$. In addition, player 1 can deviate to $(\theta_1, a_2, b_1, F, k_1)$ while producing the same evidence and so get outcome $a_2 \in f(\theta_2)$. Hence, we get $u_1(g(m,e),\theta') \geq u_1(a_2,\theta')$ and so RVP ensures that $g(m,e) \in f(\theta')$. Now, if (m,e) falls into rule (3b), player i can get any outcome he wants, and his opponent get any outcome he wants by raising a flag and announcing an integer strictly higher than player i. Hence, $g(m,e) \in \bigcap_{i\in\{1,2\}} \arg\max_{c\in A} u_i(c,\theta')$. Restricted no veto power here again completes the proof. A similar argument applies if (m,e) falls into rule (4) as well.

Step 4. It remains to consider (m,e) falling into rule (1), i.e. $m_1 = (\theta, a, b_1, NF, k_1)$ and $m_2 = (\theta, a, b_2, NF, k_2)$ while $e = e^*_{\theta,a}$. The outcome is $g(m,e) = a$. Observe that any player i can raise a flag and deviate into rule (3a) while producing the same evidence, and hence can induce any outcome in the set $L(a,\theta) := \{b : u_i(a,\theta) \geq u_i(b,\theta)\}$. Since (m,e) is a NE, this implies that for any agent i and outcome $b \in L(a,\theta)$, $u_i(a,\theta') \geq u_i(b,\theta')$, and consequently condition (*) is satisfied. If (***) also holds, weak evidence-monotonicity implies that $a \in f(\theta')$, and we are done. Now, in case (***) does not hold, there is some agent i such that $\left(e^*_{i,\theta,a} \notin E_i^{\theta'}\right)$ or $\left(E_i^{\theta'} \nsubseteq E_i^{\theta}\right)$.

Since $e_i = e^*_{i,\theta,a}$ and the true state is θ', we have $e^*_{i,\theta,a} \in E_i^{\theta'}$, and consequently $E_i^{\theta'} \nsubseteq E_i^{\theta}$. This implies that there is some $\tilde{e}_i \in E_i^{\theta'}$ such that $\tilde{e}_i \notin E_i^{\theta}$, and agent i can deviate into rule (3b), and get any outcome he desires. Hence, in this case, $a \in \arg\max_{c \in A} u_i(c, \theta')$. Note also that $a \in f(\theta)$. Hence, RVP applies and we get that $g(m, e) \in f(\theta')$. □

Proof of Remark 9. We provide two examples to show that if either existence of a bad outcome or RVP fails, then an evidence-monotonic SCR may not be implementable when $n = 2$. The examples also show that condition (UD) is not sufficient to implement a SCR if either there is no bad outcome or RVP fails.

Example 10. $\Theta = \{\theta_1, \theta_2\}$. *The evidence structure is as follows:* $E_1^{\theta_1} = \{\theta_1, \theta_2\}$ *and* $E_1^{\theta_2} = \{\theta_2\}$*; symmetrically,* $E_2^{\theta_1} = \{\theta_1\}$ *and* $E_2^{\theta_2} = \{\theta_1, \theta_2\}$. *The set of outcomes is* $A = \{a, b, c, d\}$. *Preferences are state independent: for all* θ, $u_1(c, \theta) > u_1(b, \theta) > u_1(a, \theta) > u_1(d, \theta)$ *and* $u_2(d, \theta) > u_2(a, \theta) > u_2(b, \theta) > u_2(c, \theta)$. *The SCR is given by* $f(\theta_1) = a$ *and* $f(\theta_2) = b$. *This SCR satisfies RVP (and non-satiation), but there is no bad outcome. Note that the evidence structure satisfies (UD), so that* f *is weak evidence-monotonic by Proposition 3, hence evidence-monotonic by non-satiation.*

However, f *is not implementable. To see this, assume to contradiction that* f *is implementable. Let* $(s^*_{1,\theta_1}, s^*_{2,\theta_1})$ *and* $(s^*_{1,\theta_2}, s^*_{2,\theta_2})$ *be NE at state* θ_1 *and state* θ_2 *respectively. We must have* $g(s^*_{1,\theta_1}, s^*_{2,\theta_1}) = a$ *and* $g(s^*_{1,\theta_2}, s^*_{2,\theta_2}) = b$. *Since* $s^*_{1,\theta_2} \in M_1 \times E_1^{\theta_2} \subseteq M_1 \times E_1^{\theta_1}$ *and* $(s^*_{1,\theta_1}, s^*_{2,\theta_1})$ *is a NE at state* θ_1*, we must have* $g(s^*_{1,\theta_2}, s^*_{2,\theta_1}) \in \{a, d\}$*; otherwise at state* θ_1*, player* 1 *would have an incentive to deviate from* $(s^*_{1,\theta_1}, s^*_{2,\theta_1})$ *playing* s^*_{1,θ_2}. *But note now that* $s^*_{2,\theta_1} \in M_2 \times E_2^{\theta_1} \subseteq M_2 \times E_2^{\theta_2}$ *and so at state* θ_2*, player* 2 *can deviate from* $(s^*_{1,\theta_2}, s^*_{2,\theta_2})$ *and get an outcome in* $\{a, d\}$ *by playing* s^*_{2,θ_1}*, which is strictly better for him than the equilibrium outcome* b*, a contradiction.*

Example 11. $n = 2$*;* $\Theta = \{\theta_1, \theta_2\}$*;* $A = \{a, b, c, d\}$*;* $f(\theta_1) = a$ *and* $f(\theta_2) = b$*; for* $i = \{1, 2\}$*:* $u_i(d, \theta_1) > u_i(b, \theta_1) > u_i(a, \theta_1) > u_i(c, \theta_1)$ *and* $u_i(d, \theta_1) > u_i(a, \theta_2) > u_i(b, \theta_2) > u_i(c, \theta_2)$*; for* $i = \{1, 2\}$*:* $E_i^{\theta_1} = \{\theta_1\}$ *and* $E_i^{\theta_2} = \{\theta_1, \theta_2\}$. *The evidence structure satisfies (UD), hence* f *is weak evidence-monotonic by Proposition 3, and therefore evidence-monotonic since non-satiation also holds. Outcome* c *is a bad outcome, since it is not in the range of the SCR and is the bottom-ranked outcome for both agents in both states.*

The SCR f *is not implementable by the following argument: if mechanism* (M, g) *implements* f*, there must exist some* $m^* \in M$ *such that* $g(m^*, (\theta_1, \theta_1)) = a$*, since* $f(\theta_1) = a$. *But then,* $(m^*, (\theta_1, \theta_1))$ *is a Nash equilibrium at state* θ_2 *since* a *is top-ranked by both agents in state* θ_2*, contradicting* $f(\theta_2) = b$. □

Proof of Proposition 6. Assume that $c_i(e_i, \theta) \in \{0, \infty\}$ $\forall i, e_i, \theta$.

First we show that evidence-monotonicity implies cost-monotonicity. For this it suffices to show that (CM) $\implies$ (*) and (**). Obviously (CM) implies the first part of (**). By considering $e_i = e^*_{i,\theta,a}$ in (CM), and noting that $e^*_{i,\theta,a} \in E_i^{\theta'}$, (*) follows because we have $c_i(e^*_{i,\theta,a}, \theta) = c_i(e^*_{i,\theta,a}, \theta') = 0$.

Now observe that the second part of (**) is equivalent to

$$\forall i : \left\{ E_i^{\theta'} \not\subseteq E_i^{\theta} \implies \left[\forall b : u_i(a,\theta') \geq u_i(b,\theta') \right] \right\}. \tag{8}$$

For any $e_i \in E_i^{\theta'} \setminus E_i^{\theta}$, it is straightforward to check that the latter part of (CM) implies that the consequent of (8) holds, since the antecedent of the latter part of (CM) is satisfied for any b (because $c_i(e_i,\theta) = \infty$).

Next we argue the converse, that cost-monotonicity implies evidence-monotonicity. For this it suffices to show that (*) and (**) $\implies$ (CM). Obviously the first part of (CM) is implied by the first part of (**). Next, (*) implies the latter part of (CM) for all $e_i \in E_i^{\theta} \cap E_i^{\theta'}$. The latter part of (CM) is trivially satisfied for all $e_i \notin E_i^{\theta'}$ (since the consequent always holds). Finally, for all $e_i \in E_i^{\theta'} \setminus E_i^{\theta}$, (8) (which, recall, is equivalent to the latter part of (**)) implies that the latter part of (CM) is also satisfied. □

Appendix B. Broader Class of Mechanisms for Hard Evidence

In this appendix, we address two restrictions in the class of mechanisms we allowed in Section 2 with hard evidence.

B.1. **Forbidding Evidence.** We assumed that a mechanism cannot constrain the evidence that agents can submit. In some applications, this assumption may be a priori restrictive. For example, certain kinds of evidence, such as hearsay, could be prohibited in legal proceedings. Accordingly, a more general definition of a mechanism would be to specify $(M, \hat{E}, g)$, where $\hat{E} = \hat{E}_i \times \cdots \times \hat{E}_n$ with for each i, $\hat{E}_i \subseteq E_i$, and $g : M \times \hat{E} \to A$. Here, agent i is only allowed to submit evidence in the set $\hat{E}_i$, i.e. the evidence player i can send when the state is θ is $\hat{E}_i^{\theta} = \hat{E}_i \cap E_i^{\theta}$. If for some i, $\hat{E}_i \neq E_i$, then the mechanism is said to be *forbidding evidence*.

The following example shows that forbidding evidence can be useful.

Example 12. *Suppose $\Theta = \{\theta_1, \theta_2\}$, $A = \{b, c\}$, and for all i: $E_i^{\theta_1} = \{x, y\}$, $E_i^{\theta_2} = \{x, z\}$, and preferences are represented by $u_i(b,\theta_1) > u(c,\theta_1)$ and $u_i(c,\theta_2) > u(b,\theta_2)$. Consider the SCR $f(\theta_1) = c$ and $f(\theta_2) = b$. If a mechanism (M, g) does not forbid evidence, it cannot implement f, by the following logic: we must have $g(m,(x,\ldots,x)) = c$ for all $m \in M$, otherwise $b \notin f(\theta_1)$ would be a Nash equilibrium outcome in state θ_1; but then $(m,(x,\ldots,x))$ is a Nash equilibrium at state θ_2 which leads to outcome c, hence f is not implemented. Intuitively, the problem is that since agents can produce evidence x in both states, it is impossible to achieve the outcome they like less in both states. But if a mechanism can forbid evidence, it is straightforward to implement f: let agents only submit evidence in $\hat{E}_i = \{y, z\}$, and let the outcome be c if they each submit y, and b otherwise.*[30]

Nevertheless, one can verify that Theorems 1 and 2 remain valid even when mechanisms that forbid some evidence are considered. Thus, even if evidence can be forbid, evidence-monotonicity

[30] In this example, it is useful to forbid uninformative evidence. One can construct richer examples where it is useful to forbid informative evidence.

remains necessary for implementation and is also sufficient when $n \geq 3$ and no veto power. Note that Example 12 satisfies evidence-monotonicity but violates no veto power.

B.2. **Dynamic Mechanisms.** Now we discuss the possibility of dynamic mechanisms. We will first illustrate how such mechanisms can improve upon static mechanisms, and then show a general result that considering only static mechanisms does not lose any generality if hard evidence has a cost interpretation or the evidentiary structure is normal.

B.2.1. *Dynamic mechanisms can help.* Consider the following example borrowed from Bull and Watson (2007):

Example 13. *There are $n = 2$ players, $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and $A = \{a, b\}$. Preferences are state-independent: for each θ, $u_1(b, \theta) > u_1(a, \theta)$ whereas $u_2(a, \theta) > u_2(b, \theta)$. Player 1 has only trivial evidence available in any state, i.e. $E_1^{\theta_1} = E_1^{\theta_2} = E_1^{\theta_3} = E_1$. Player 2's evidence sets are given by $E_2^{\theta_1} = \{x\}$, $E_2^{\theta_2} = \{x, y\}$ and $E_2^{\theta_3} = \{y\}$. This evidence structure is not normal. The SCR is given by $f(\theta_1) = f(\theta_3) = \{b\}$ and $f(\theta_2) = \{a\}$. This SCR is not weak evidence-monotonic, because $T^f(\theta_2) = \{\theta_1, \theta_3\}$ and $E^{\theta_2} = E_1 \times \{x, y\} = E^{\theta_1} \cup E^{\theta_3}$, violating the condition for weak evidence-monotonicity in Proposition 3. Thus, by Theorem 1, f is not implementable.*[31]

Whether a dynamic mechanisms can be useful to implement the SCR in the example above depends critically on the interpretation one has of evidence "not being available" at some state. As previously discussed, the *feasibility interpretation* is that any $e_i \notin E_i^\theta$ is physically unavailable and cannot be produced by agent i at state θ. An alternative is the *cost interpretation*, is that agents can always fabricate or produce any evidence in any state, but some evidence is more costly than others, with hard evidence being an extreme case where any $e_i \in E_i^\theta$ is costless, whereas any $e_i \notin E_i^\theta$ is infinitely costly. Insofar as static mechanisms are concerned, the distinction is irrelevant, since in equilibrium, a player would never produce evidence that is either unavailable or infinitely costly, and there are no out-of-equilibrium considerations.

With dynamic mechanisms, the issue is more subtle. Broadly, the reason that the analysis with dynamic mechanisms can depend on the feasibility versus cost interpretation is as follows: under the cost interpretation, at state θ any player i *will* be able to send any $e_i \notin E_i^\theta$ off the equilibrium path, and this will be a standard incredible threat; on the other hand, such a behavior is precluded under the feasibility interpretation. Let us illustrate by returning to the example.

Example 14 (Example 13 continued)**.** *Consider the following dynamic mechanism:*

1st Stage: *player 1 can announce any state $\theta \in \Theta$. Denote this (cheap-talk) announcement m_1.*

2nd Stage: *after observing m_1, player 2 has to submit evidence, e_2.*

[31]This has already been shown by Bull and Watson (2007) using a different argument.

Outcomes: *$g(m_1, e_2) = a$ if $m_1 = \theta_2$; else*

$$g(m_1, e_2) = \begin{cases} b \text{ if } e_2 \in E_2^{m_1} \\ a \text{ otherwise.} \end{cases}$$

Thus the mechanism chooses outcome a if player 1 claims (via cheap talk) that the true state is θ_2; if he claims anything else, the mechanism chooses a if player 2 submits evidence contradicting player 1's claim, otherwise the mechanism chooses b.

Start with the feasibility interpretation, i.e. assume that when the true state is θ, any $e_i \notin E_i^\theta$ is not available to player i (even at an infinite cost). Then, if the true state is θ_1, player 2 can only send evidence x, hence player 1 can ensure that his preferred outcome b is chosen by announcing $m_1 = \theta_2$, and this is the unique Nash equilibrium outcome. If the true state is θ_3, a similar reasoning shows that the only NE outcome is again b. Now assume the true state is θ_2, in which case player 2 has evidence x and y available. By playing the strategy of submitting evidence x if $m_1 \neq \theta_1$ and submitting evidence y if $m_1 = \theta_1$, player 2 guarantees that the outcome is a no matter what player 1 announces. Since a is player 2's preferred outcome, every Nash equilibrium must yield outcome a (and a subgame perfect NE exists). Therefore, under the feasibility interpretation, the SCR is implementable with a dynamic mechanism when it is not with any static mechanism.

Now consider the cost interpretation, where in any state θ, player 2 can send any $e_i \in E_2 = \{x, y\}$, but incurs an infinite cost if $e_i \notin E_2^\theta$ and no cost otherwise. We will argue that the SCR is no longer implementable by the given dynamic mechanism. If the true state is θ_1, then the following is a Nash equilibrium: player 1 announces θ_2; player 2 submits evidence y if $m_1 = \theta_1$ and submits evidence x otherwise. This NE yields outcome $a \neq f(\theta_1)$. Note that this NE is not subgame perfect.

We have thus proved:

Proposition 7. *Under the feasibility interpretation, a non-normal evidence structure can permit implementation with a dynamic mechanism even if weak evidence-monotonicity fails.*

However, we next show that the necessity of evidence-monotonicity for implementation is preserved under either: (i) the cost interpretation; or (ii) the feasibility interpretation but with a normal evidence structure. It should be emphasized that the results are proved assuming that extensive forms are restricted to those in which each player can only send evidence once in the game, as in Bull and Watson (2007). This avoids artificially "transforming" a non-normal evidence structure into a normal one, which would happen if an agent could send evidence an unlimited number of times.

B.2.2. *When are static mechanisms sufficient?* Let us formally define a dynamic mechanism, which we refer to as just "mechanism" hereafter. A mechanism is a tuple (M, H, g). For each player i, there is a set of cheap talk messages M_i and as before $M = M_1 \times \cdots \times M_n$. We introduce a new message available to the agent at each state $e_i^\emptyset$ that is interpreted as "player i sends no evidence". We will now define H the set of possible histories. Before doing so, let us define recursively the following set $\bar{H}$. First, $\emptyset \in \bar{H}$ and second if $h = (h_1, h_2, ..., h_k) \in \bar{H}$ then for any $X \subseteq I$ and $h_{k+1} \in \prod_{i \in X} \left(\left(E_i \cup \{e_i^{\emptyset}\} \right) \times M_i \right)$, $(h_1, h_2, ..., h_k, h_{k+1}) \in \bar{H}$. Given an history $h = (h_1, ..., h_k, h_{k+1}) \in \bar{H}$, we know that $h_{k+1} \in \prod_{i \in X} \left(\left(E_i \cup \{e_i^{\emptyset}\} \right) \times M_i \right)$ for some $X \subseteq I$. In line with this, we will write $h_{k+1} = \left\{ (e_i^{k+1}, m_i^{k+1}) \right\}_{i \in X}$.

Now the set of possible histories H is a subset of $\bar{H}$ satisfying the following properties. First if $h \in H$ it is of finite lenght i.e. there is some k such that $h = (h_1, ..., h_k)$. Now denote H_T the set of terminal histories in H i.e. the set of histories $h = (h_1, ..., h_k) \in H$ such that there is no h_{k+1} such that $(h_1, h_2, ..., h_k, h_{k+1}) \in H$. We assume, as in Bull and Watson (2007), that extensive forms are restricted to those in which each player sends an evidence exactly once in every path through the tree[32] i.e. for any $h = (h_1, h_2, ..., h_k) \in H_T$, and for any i, there exists a unique $\tilde{k}$ such that $e_i^{\tilde{k}} \neq e_i^{\emptyset}$ (note that this implies that each player plays at least once along any history). We will also impose the following consistency condition that if $(h_1, h_2, ..., h_k, h_{k+1}) \in H$, then $(h_1, h_2, ..., h_k) \in H$. Finally, our assumption that players cannot be compelled or forbidden to present evidence takes the following form. Let $(h_1, h_2, ..., h_k, h_{k+1}) \in H$ and denote X the subset of I such that $h_{k+1} \in \prod_{i \in X} \left(\left(E_i \cup \{e_i^{\emptyset}\} \right) \times M_i \right)$. Pick any $\hat{h}_{k+1} \in \prod_{i \in X} \left(\left(E_i \cup \{e_i^{\emptyset}\} \right) \times M_i \right)$ consistent with the assumption that each player sends an evidence exactly once along any history, i.e. such that if there is $i \in X$ and $k' \leq k$ such that $e_i^{k'} \neq e_i^{\emptyset}$ then $\hat{e}_i^{k+1} = e_i^{\emptyset}$. We make the assumption that in this case, $(h_1, h_2, ..., h_k, \hat{h}_{k+1}) \in H$.

H implicitly defines a function $P : H \diagdown H_T \to 2^I$ where $P(h)$ denotes the set of players who choose messages at history $h \in H \diagdown H_T$ where the interpretation is that players in $P(h)$ move simultaneously after having observed history h. We also denote $\hat{P}(h)$ for the set of players in $P(h)$ who have not sent their piece of evidence yet.

In addition, we have a function $g : H_T \to A$ specifying the outcome selected by the mechanism for each terminal node. We now define two notions of Nash equilibrium each corresponding to one interpretation, i.e. feasibility or cost interpretation. Now for each player i, let $H_i = \{h \in H : i \in P(h)\}$ be the set of histories where player i plays. Define player i's strategy as a mapping $\sigma_i : H_i \to \left(E_i \cup \{e_i^{\emptyset}\} \right) \times M_i$. For consistency, we assume that a strategy must satisfy that for any $h \in H$ and $i \in P(h) \diagdown \hat{P}(h) : \sigma_i(h) \in \{e_i^{\emptyset}\} \times M_i$. Each profile of strategy $\sigma = (\sigma_i)_{i \in I}$ generates a terminal history $h(\sigma) \in H_T$ where $h(\sigma) = (h_1(\sigma), ..., h_k(\sigma))$ for some integer k. Recall that an history $h' = (h'_1, ..., h'_{k'}) \in H$ is a truncation of history $h = (h_1, ..., h_k) \in H$ if $k' \leq k$ and $h' = (h_1, ..., h_{k'})$.

Definition 13. *Given a mechanism (M, H, g), a profile of strategy $\sigma = (\sigma_i)_{i \in I}$ is a NE under the cost interpretation at state θ if*

[32] There is a sense in which this assumption is made for consistency since we may have that evidence $\{x\}$ and evidence $\{y\}$ are available at θ but $\{x, y\}$ is not. If a player could submit $\{x\}$ first and then submit $\{y\}$ at a subsequent point, it is effectively as though we have enlarged the set of evidence available to a player, which we do not wish to do. Indeed, if we assumed that players can submit evidence an arbitrary number of times in a dynamic mechanism, then this would artificially make the evidence structure normal and thereby change the problem fundamentally.

(1) *$u_i(g(h(\sigma),\theta) \geq u_i(g(h(\sigma_i', \sigma_{-i}),\theta))$ for any strategy σ_i' s.t. for any $h \in H_i : \sigma_i'(h) \in \left(\left(E_i^\theta \cup \{e_i^\emptyset\}\right) \times M_i\right)$, and*
(2) *for any truncation h' of $h(\sigma)$, $\sigma_i(h') \in \left(\left(E_i^\theta \cup \{e_i^\emptyset\}\right) \times M_i\right)$ for each $i \in P(h')$.*

The set of NE under the cost interpretation at θ is denoted $NE\left(\left(M,H,g\right),\theta\right)$. Condition (1) of the above definition says that one cannot have profitable deviations, in particular to strategies where the evidence sent is not in E_i^θ. This is consistent with the interpretation that sending such a message is infinitely costly. Condition (2) imposes no restrictions on histories that are not truncations of $h(\sigma)$, i.e. on out-of-equilibrium histories. In particular, for such an history, h, it may be the case that a player i in $P(h)$ plays $\sigma_i(h) \notin \left(E_i^\theta \times M_i\right)$. Again, this definition is consistent with the interpretation that evidence $e_i \notin E_i^\theta$ is always available but is infinitely costly.

Let $O\left(\left(M,H,g\right),\theta\right) := \{a \in A : \exists \sigma \in NE\left(\left(M,H,g\right),\theta\right) \text{ s.t. } g\left(h(\sigma)\right) = a\}$ and say that a mechanism (M,H,g) implements a SCR f under the cost interpretation if for all $\theta : f(\theta) = O\left(\left(H,g,M\right),\theta\right)$. A SCR is implementable under the cost interpretation if there exists a mechanism that implements it under the cost interpretation.

Proposition 8. *If a SCR is implementable (in a dynamic mechanism) under the cost interpretation, it is evidence-monotonic.*

Proof. Assume f is implemented under the cost interpretation by a mechanism (M,H,g). Then for each θ and $a \in f(\theta)$, there exists $\sigma^* \in NE\left(\left(M,H,g\right),\theta\right)$ s.t. $g\left(h(\sigma^*)\right) = a$. By definition, for each player i, there is exactly one piece of evidence that is sent along $h(\sigma^*)$, call the profile of such evidences $e^*_{\theta,a}$ and note that by (2) in the definition of a NE under the cost interpretation, $e^*_{\theta,a} \in E^\theta$. Consider any θ' satisfying (*) and (**). Since σ^* is a NE at θ,

$$u_i\left(g\left(h(\sigma^*)\right),\theta\right) \geq u_i\left(g\left(h(\sigma_i', \sigma^*_{-i})\right),\theta\right)$$

for all i, σ_i' s.t. for any $h \in H_i : \sigma_i'(h) \in \left(\left(E_i^\theta \cup \{e_i^\emptyset\}\right) \times M_i\right)$. By (*),

$$u_i\left(g\left(h(\sigma^*)\right),\theta'\right) \geq u_i\left(g\left(h(\sigma_i', \sigma^*_{-i})\right),\theta'\right)$$

for all i, σ_i' s.t. for any $h \in H_i : \sigma_i'(h) \in \left(\left(E_i^\theta \cup \{e_i^\emptyset\}\right) \times M_i\right)$. By (**) either $E_i^{\theta'} \subseteq E_i^\theta$ or $a \in \arg\max_b u_i(b,\theta')$, in each case we have

$$u_i\left(g\left(h(\sigma^*)\right),\theta'\right) \geq u_i\left(g\left(h(\sigma_i', \sigma^*_{-i})\right),\theta'\right)$$

for all i, σ_i' s.t. for any $h \in H_i : \sigma_i'(h) \in \left(\left(E_i^{\theta'} \cup \{e_i^\emptyset\}\right) \times M_i\right)$. In addition, (**) stipulates that $e^*_{\theta,a} \in E^{\theta'}$ and so for any truncation h' of $h(\sigma)$, $\sigma_i^*(h') \in \left(\left(E_i^{\theta'} \cup \{e_i^\emptyset\}\right) \times M_i\right)$ for each $i \in P(h')$. Therefore, σ^* is a NE (under the cost interpretation) at θ' and since $g\left(h(\sigma^*)\right) = a$, we get that $a \in f\left(\theta'\right)$. □

By Theorem 2, evidence-monotonicity is sufficient for implementation in the above sense whenever there are three or more players and no veto power is satisfied. Indeed we do not need to go beyond static games in this case. Thus we get exactly the same type of characterization as in the main text when dynamic mechanisms are allowed but the cost interpretation is adopted.

Let us turn now to the feasibility interpretation. Here, at each state θ, it is impossible for any player i to send any evidence not in E_i^θ, even off-the-equilibrium path.

Definition 14. *Given a mechanism (M,H,g), a profile of strategy $\sigma=(\sigma_i)_{i\in I}$ is a NE under the feasibility interpretation at state θ if*

(1) $u_i(g(h(\sigma),\theta)\geq u_i(g(h(\sigma_i',\sigma_{-i}),\theta)$ *for any strategy σ_i' s.t. for any $h\in H_i:\sigma_i'(h)\in\left(\left(E_i^\theta\cup\{e_i^\emptyset\}\right)\times M_i\right)$, and*
(2) *for any history $h'\in H$, $\sigma_i(h')\in\left(\left(E_i^\theta\cup\{e_i^\emptyset\}\right)\times M_i\right)$ for each $i\in P(h')$.*

The set of NE under the feasibility interpretation at θ is denoted $NE^{\mathbb{F}}\left(\left(M,H,g\right),\theta\right)$. Define $O^{\mathbb{F}}\left(\left(M,H,g\right),\theta\right):=\left\{a\in A:\exists\sigma\in NE^{\mathbb{F}}\left(\left(M,H,g\right),\theta\right)\text{ s.t. }g\left(h(\sigma)\right)=a\right\}$, and say that a mechanism (H,g,M) implements a SCR f under the feasibility interpretation if for all $\theta:f(\theta)=O^{\mathbb{F}}\left(\left(M,H,g\right),\theta\right)$. A SCR is implementable under the feasibility interpretation if there exists a mechanism that implements it under the feasibility interpretation.

As already shown, if we use this notion then dynamic mechanisms may be useful to achieve implementation of a SCR that is not evidence-monotonic. However, we prove next that if the evidence structure is normal, then again our central analysis remains unchanged.

Proposition 9. *Assume the evidence structure is normal. If a SCR is implementable (in a dynamic mechanism) under the feasibility interpretation, it is evidence-monotonic.*

Proof. It is easily checked that in the case of normality, a SCR f is evidence-monotonic provided that for all θ and all $a\in f(\theta)$, for any θ' satisfying (*) and

$$\forall i,\ \text{either } E_i^\theta=E_i^{\theta'}\text{ or }\left(a\in\arg\max_b u_i(b,\theta')\text{ and }E_i^\theta\subseteq E_i^{\theta'}\right),$$

we have $a\in f\left(\theta'\right)$. Assume f is implemented under the feasibility interpretation by a mechanism (M,H,g). Pick θ and $a\in f\left(\theta\right)$, and an equilibrium $\sigma^*\in NE^{\mathbb{F}}\left(\left(M,H,g\right),\theta\right)$ s.t. $g\left(h(\sigma^*)\right)=a$. Consider any θ' satisfying (*) and

$$\forall i,\ \text{either } E_i^\theta=E_i^{\theta'}\text{ or }\left(f(\theta)\in\arg\max_b u_i(b,\theta')\text{ and }E_i^\theta\subseteq E_i^{\theta'}\right).$$

Since σ^* is a NE at θ,

$$u_i\left(g\left(h(\sigma^*)\right),\theta\right)\geq u_i\left(g\left(h(\sigma_i',\sigma_{-i}^*)\right),\theta\right)$$

for all i, σ_i' s.t. for any $h\in H_i:\sigma_i'(h)\in\left(\left(E_i^\theta\cup\{e_i^\emptyset\}\right)\times M_i\right)$. By (*),

$$u_i\left(g\left(h(\sigma^*)\right),\theta'\right)\geq u_i\left(g\left(h(\sigma_i',\sigma_{-i}^*)\right),\theta'\right)$$

for all i, σ_i' s.t. for any $h\in H_i:\sigma_i'(h)\in\left(\left(E_i^\theta\cup\{e_i^\emptyset\}\right)\times M_i\right)$. We will have to consider the two cases where $E_i^\theta=E_i^{\theta'}$ and where $[a\in\arg\max_b u_i(b,\theta')$ and $E_i^\theta\subseteq E_i^{\theta'}]$. Now, if $E_i^\theta=E_i^{\theta'}$ or if

$a \in \arg\max_b u_i(b, \theta')$, we have

$$u_i\left(g\left(h(\sigma^*)\right), \theta'\right) \geq u_i\left(g\left(h(\sigma'_i, \sigma^*_{-i})\right), \theta'\right)$$

for all i, σ'_i s.t. for any $h \in H_i : \sigma'_i(h) \in \left(\left(E_i^{\theta'} \cup \{e_i^{\emptyset}\}\right) \times M_i\right)$. In addition, if $E_i^{\theta} \subseteq E_i^{\theta'}$, we have that for any history $h' \in H$, $\sigma^*_i(h') \in \left(\left(E_i^{\theta'} \cup \{e_i^{\emptyset}\}\right) \times M_i\right)$ for each $i \in P(h')$. Therefore, in each case, σ^* is a NE (under the feasibility interpretation) at θ' and since $g\left(h(\sigma^*)\right) = a$, we get that $a \in f\left(\theta'\right)$. $\square$

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