On the Robustness of the Coase Conjecture*

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Abstract

We revisit the classic dynamic durable goods monopoly and introduce a possibility of monopoly necessarily insisting on a price. We demonstrate the Coase conjecture is very robust with this perturbation. Though the effect is amplified by the normal monopoly's mimicking behavior, the monopoly can't get more than the competitive profit unless a price-insistent type is very likely. In particular, if the price-insistent type is rational (in the sense that she optimally chooses her own price), the full-blown Coase conjecture is strongly robust.

Keywords : Coase conjecture, reputational bargaining, rational commitment

1 Introduction

How does a monopoly selling durable goods price dynamically? Does the monopoly necessarily ask a price higher than the competitive one, so that there's an accompanying social inefficiency? There are two prominent answers to this problem. If the monopoly can make credible commitment, she commits to the static monopoly price and gets the static monopoly profit, which yields a socially inefficient outcome. If the monopoly doesn't possess a commitment device and it's common knowledge that there's always a positive surplus in trading, the *Coase conjecture* holds. When the buyers are patient enough or the monopoly can adjust price frequently enough, the monopoly asks the competitive price from the start and all buyers purchase immediately, which leads to social efficiency.

We revisit this classic problem and introduce a possibility of monopoly necessarily insisting on a price. The motivation comes from the observation that the driving force of the difference between two famous answers is buyers' perception on the pricing schedule of the monopoly. When the monopoly has commitment power, buyers believe that the seller will never lower price. On the contrary, facing a monopoly without commitment buyers never believe the seller keeps price higher than the

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competitive one. Having a possibility for the monopoly to necessarily commit to a price creates a non-trivial problem. Even if the chance is very small, the effect is amplified (reputation effect). The normal monopoly pretends be a price-insistent type, and so buyers' perception on the price is greatly distorted. The purpose of this paper is to investigate the interaction between the monopoly and buyers in this environment and evaluate whether and to what extent the Coase conjecture is robust.

In our model, with some exogenously given probability the monopoly is restricted not to adjust price. This type of seller is called a commitment type, because if her type is known, buyers know that the price does not change as the commitment monopoly. There are two kinds of commitment type: A behavioral commitment type insists a price which is exogenously given, while the rational commitment type optimally chooses her own price, knowing that she can't adjust price later. Both types will be considered in this paper.

Our equilibrium characterization strategy is "divide and conquer". We first solve a game in which the seller asks a price p and the buyer believes that the seller is a commitment type (whether she is the rational commitment type or a behavioral commitment type) with probability μ. If the buyer's valuation is binary (either low valuation or high valuation), this game essentially becomes a two-sided reputational bargaining game in the sense of Abreu and Gul (2000) in which there exists a unique equilibrium in a closed form with a war-of-attrition feature. With a continuum of the buyer's type, we demonstrate that there still exists a unique equilibrium in which almost closed-form solution is available. This enables us to derive an indirect utility function of the seller, based on which we resolve the problem how equilibrium price and equilibrium belief are determined in the first place.

The main result is that the Coase conjecture is very robust. If the probability of commitment types is not big enough, both the rational commitment type and the normal type get the competitive profit. The rational commitment type always asks the competitive price, and the normal type does with high probability. In the limit where the probability of behavioral commitment types (not that of the rational commitment type) vanishes, the normal type also asks the competitive price for sure. The main reasoning is twofold.

First and foremost, there's an inherent bargaining power for the buyer which stems from the fact that buyers can pretend to have lower valuation. Hence this problem is not a one-sided (though it looks), but a two-sided reputation model. Even if the effect of price insistence is magnified by the normal type, it is effective only when it (the seller's bargaining power) outweighs the natural reluctance of the buyer (the buyer's bargaining power). When it's unlikely that the seller is a commitment type, buyers are very reluctant to accept high price, which makes the seller get only the competitive profit.

Second, whenever the rational commitment type asks price higher than the competitive one, the normal type has incentive to follow, which creates an incentive for the buyer to wait until the seller adjusts price. This in turn provides an incentive for the rational commitment type to avoid delay cost by lowering price. If buyers strongly believe that the seller is the normal type, the consequent delay cost could be so huge that it might be optimal for the rational commitment type to not exert her commitment power at all (ask the competitive price), which leaves the normal type no choice but to charge the competitive price as well.

Apart from robustness, we identify an interesting phenomenon when the probability of behavioral commitment types is relatively high. The normal type may charge price higher than the competitive one and get more than the static monopoly profit. This is intriguing because no payoff-concerned seller asks price higher than the static monopoly price, and the static monopoly profit is the maximum any seller can achieve without uncertainty. The reason is that it's impossible for the normal type to behave opportunistically without uncertainty, while it's not with uncertainty. Suppose the monopoly is a behavioral commitment type almost surely and there are sufficiently rich set of behavioral commitment types. Then no matter what price was asked by the seller, a buyer should purchase immediately as long as his valuation is higher than the price. The only reason a buyer waits is an expectation of lower price later. When the seller is unlikely to be a normal type, that possibility is too low compared to the associated delay cost. Under this scenario the normal type asks a price higher than the static monopoly one. When the static monopoly (or the rational commitment type when her type is known) increases price, the marginal benefit is an additional exploitation from the existing consumers, while the marginal cost is a loss of some consumers (who were willing to buy before, but decided not to with higher price). With uncertainty the marginal benefit to the normal type is essentially the same as that of the static monopoly, while the marginal cost becomes smaller because the monopoly can lower price and sell to those lost consumers later.

The remainder of the paper is organized as follows. The basic setup is introduced in the next section. In Section 3, we characterize an equilibrium in a game in which the initial price the seller would ask and the buyer's belief over the seller's type were already determined. Indirect utility functions and isoprofit curves are derived. In Section 4 and 5, we analyze a game without the rational commitment types and without behavioral commitment types, respectively. The full-fledged model is examined in Section 6. We conclude by discussing several relevant issues in section 7.

2 The Model

A seller and a buyer bargain over the trade of one unit of a good. The seller's production cost is known and normalized to 0. The buyer has valuation v for the good, which is private information and known to be drawn from a distribution function F with support $[\underline{v}, \overline{v}]$, $0 < \underline{v} < \overline{v} < \infty$. To get sharper results, we assume F has a positive and continuously differentiable density f. At each date $t \in \mathcal{R}_+$, the seller makes an offer and the buyer decides whether to accept or not. Since we are always interested in the limit where offer interval Δ is arbitrarily small, time is essentially continuous. The buyer's time preference is represented by discount rate $r > 0$.

The seller is either a payoff type (Ξ_1) or a behavioral commitment type (Ξ_2). If the seller is behavioral, her strategy is exogenously given. More specifically, her type $\xi \in \Xi_2 \subseteq [\underline{v}, \overline{v}]$ represents a price which she insists independent of time and history[1]. There are two payoff types, the normal type ξ_0 and the rational commitment type ξ_1. Both share the same expected utility over timing of agreement and price, with the same time preference r as the buyer. The two payoff types differ each other in that they have different sets of feasible strategies. The normal type can adjust price at any time without cost, while the rational commitment type can never. It's commonly believed that the seller is the normal type, the rational commitment type, and one of behavioral commitment types with probability μ_0, μ_1 and $\mu_2(= 1 - \mu_0 - \mu_1)$. Conditional on that the seller is behavioral, the probability measure over a Borel σ-field, $\mathcal{F}$, over Ξ_2 is given by λ.

The two classic results on the dynamic durable goods monopoly concern with two degenerate cases, $\mu_0 = 0$ or $\mu_0 = 1$. If $\mu_0 = 0$, the (rational commitment) monopoly chooses the static monopoly price that maximizes $p(1 - F(p))$ and all consumers with valuation higher than the price purchase immediately. When $\mu_0 = 1$, the Coase conjecture holds. Buyers rationally expect the seller to lower price in the future, and so the monopoly should ask a reasonable price to induce early purchase. When the buyers are patient enough or the monopoly can adjust price frequently enough, she can't help but to ask the competitive price $\underline{v}$[2] at date 0, at which all consumers buy immediately.

Relying on the Coase conjecture ($\mu_0 = 1$), we restrict the normal type seller's strategy so that at each time and each history, she can continue to ask the initial price (unless she adjusted before), or she should lower price to $\underline{v}$. Whenever she changes price, it reveals her type to be normal, after which the Coasian dynamics works. So any price change is essentially equivalent to immediately lowering to $\underline{v}$, which suggests that our restriction has no loss of generality.

The assumption that $\underline{v}> 0$ is called the "Gap" case in the sequential bargaining literature and plays an important role in our analysis. First it enables us to pin down equilibrium behavior in case the seller's type is revealed to be normal. With "No Gap" there are multiple equilibria, without a clear equilibrium selection argument. In addition it makes the normal type seller exhaust her reputation when necessary. If there's nothing to gain, she has no reason to reveal her type in any situation. In the last section we briefly introduce the equilibrium in the limit as $\underline{v}$ approaches 0.

$\{F, (\mu_0, \mu_1, \mu_2), (\Xi_2, \lambda)\}$ constitutes a sequential bargaining game with two-sided incomplete information. We maintain the following assumption during our discussion. It basically tells that two functions $p(1 - F(p))$ and $p(1 - F(p)) + F(p)\underline{v}$ are monotonely single-peaked. It will turn out that the first function is related to the rational commitment type's expected payoff, while the second one

[1] This type was first introduced by Myerson (1991).

[2] In fact, any price in the range $[0, \underline{v}]$ constitutes a competitive price. We refer to $\underline{v}$ as the competitive price in our model, because any payoff type seller never asks price lower than $\underline{v}$.

to that of the normal type. In Section 7, we discuss what results are robust with weaker assumption.

Assumption 1 *There exist* $p_0^*, p_1^* \in (\underline{v}, \overline{v})$ *such that*

$$\frac{1-F(p)}{f(p)} \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} p \text{ } \textit{if } p \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} p_0^*$$

$$\underline{v} + \frac{1-F(p)}{f(p)} \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} p \text{ } \textit{if } p \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} p_1^*$$

A sufficient condition for this assumption is that $(1-F(p))/f(p)$ is strictly decreasing and $1 - \underline{v}f(\underline{v}) > 0$. The condition that inverse hazard ratio is strictly decreasing has been widely used in screening models and more generally in mechanism design literature. The requirement $p_0^* \in (\underline{v}, \overline{v})$ excludes the trivial case where $\underline{v}$ is the static monopoly price, and $1 - \underline{v}f(\underline{v}) > 0$ is a sufficient condition for that.

3 $(p, \mu)-$Concession Game: A Building Block

3.1 Characterization

Suppose the seller asks a price p and the buyer believes the seller is a commitment type (whether the rational or a behavioral commitment type) with probability μ. The strategic issue of each player at this level is finding an optimal concession time. The normal type seller decides the time to lower price to $\underline{v}$, while each buyer decides the time to buy the object at price p. Once one party concedes the game ends, and each player prefers the opponent to concede earlier. As noticed by Abreu and Gul (2000) in the context of a variant of a bilateral bargaining game á la Rubinstein, this game structure is almost the same as that of the standard war-of-attrition.

If the buyer's valuation follows a binary distribution, the analysis is essentially equivalent to that of Abreu and Gul. Given a price greater than the lower valuation, the normal type seller and the high valuation buyer mix between conceding and waiting over some time interval, and their concession behavior is determined so that the opponent is indifferent between conceding and waiting at each time.

In our problem with a continuum of the buyer's type, an equilibrium is characterized by (G_S, τ) where $G_S : R_+ \rightarrow [0,1]$ is the seller's unconditional concession probability distribution and $\tau : [\underline{v}, \overline{v}] \rightarrow R_+$ is the buyer's concession function. $G_S(t)$ is the cumulative probability that the seller concedes (lowers price) up to time t conditional on that the buyer hasn't purchased yet, and $\tau(v)$

is the optimal concession time of the buyer with valuation v conditional on that the seller hasn't lowered price yet. Since a commitment type seller and the buyer with valuation lower than p never concede, $G_S(t) \leq 1-\mu, \forall t$ and $\tau(v) = \infty, \forall v < p$. In characterizing an equilibrium, it's convenient to define the buyer's unconditional concession probability distribution $G_B : R_+ \to [0,1]$ by $G_B(t) = \Pr\{v : \tau(v) \leq t\}$. Though each type of the buyer has a deterministic concession time, it looks like a random variable from the seller's viewpoint and only G_B is relevant to the seller's payoff.

To state the main proposition in this section, we present one measure which can be interpreted as the buyer's bargaining power in this (p,μ)-concession game. Given a distribution function F and $p \in [\underline{v}, \overline{v}]$, define

$$\phi(F,p) = \exp\left(-\frac{1}{\underline{v}}\int_p^{\overline{v}} \frac{(v-p)f(v)}{F(v)}dv\right)$$

This is one way to measure how left-skewed the distribution is relative to p. Intuitively, the more likely the buyer has valuation lower than p, the easier he can pretend not being able to accept price p and so the stronger his bargaining power becomes. $\phi(F,p)$ is strictly increasing in p, which is a natural consequence from the fact that facing a higher price the buyer has more incentive to wait, which relatively increases the buyer's bargaining power.

Proposition 1 *In the (p,μ)-concession game, there exists a unique equilibrium (G_S,τ), which is (1) if $\mu \leq \phi(F,p)$, then*

$$\begin{aligned}
\tau(v) &= \begin{cases} -\frac{p-v}{\underline{v}r}\ln\left(F(t)\right) & \text{if } v \geq p \\ \infty & \text{if } v < p \end{cases} \\
G_B(t) &= \begin{cases} 1-\exp\left(-\frac{\underline{v}}{p-\underline{v}}rt\right) & \text{if } t \leq -\frac{p-\underline{v}}{\underline{v}r}\ln(F(p)) \\ 1-F(p) & \text{otherwise} \end{cases} \\
G_S(t) &= \min\left\{1-\exp\left(\frac{1}{\underline{v}}\int_p^{F^{-1}\left(\exp\{-\frac{\underline{v}}{p}rt\}\right)} \frac{(v-p)f(v)}{F(v)}dv\right), 1-\mu\right\}
\end{aligned}$$

(2) If $\mu > \phi(F,p)$, then

$$\begin{aligned}
\tau(v) &= \begin{cases} 0 & \text{if } v \geq F^{-1}(c_B) \\ -\frac{p-\underline{v}}{\underline{v}r}\ln\left(\frac{F(t)}{c_B}\right) & \text{if } p \leq v < F^{-1}(c_B) \\ \infty & \text{if } v < p \end{cases} \\
G_B(t) &= \begin{cases} 1-c_B\exp\left(-\frac{\underline{v}}{p-\underline{v}}rt\right) & \text{if } t \leq -\frac{p-\underline{v}}{\underline{v}r}\ln(\frac{F(p)}{c_B}) \\ 1-F(p) & \text{otherwise} \end{cases} \\
G_S(t) &= \min\left\{1-\exp\left(\frac{1}{\underline{v}}\int_p^{F^{-1}\left(c_B\exp\{-\frac{\underline{v}}{p}rt\}\right)} \frac{(v-p)f(v)}{F(v)}dv\right), 1-\mu\right\}
\end{aligned}$$

where

$$\frac{1}{\underline{v}}\int_p^{F^{-1}(c_B)} \frac{(v-p)f(v)}{F(v)}dv + \ln\mu = 0$$

Proof : See Appendix.

Though the proof is rather involved, we can provide an essential idea. The indifference of the normal type between conceding and waiting at t such that $0 < G_S(t) < 1-\mu$ yields

$$r\underline{v} = \frac{dG_B(t)/dt}{1-G_B(t)}(p-\underline{v})$$

The left-hand side is the marginal cost of waiting instant more, while the right-hand side is the marginal benefit, which consists of a conditional concession rate of the buyer at time t and the seller's additional benefit by selling at p instead of $\underline{v}$. This first-order ordinary differential equation has a nice solution $G_B(t) = 1 - c_B \exp\left(-\underline{v}rt/(p-\underline{v})\right)$ where $c_B \in [F(p), 1]$ is unknown.

The optimality of $\tau(v)$ for each v produces

$$r(v-p) = \frac{dG_S(\tau(v))/dt}{1-G_S(\tau(v))}(p-\underline{v})$$

because the marginal cost (left) and the marginal benefit (right) of the buyer with valuation v are equated at his own optimal concession time $\tau(v)$. This equation is not readily solvable as before, but we can invoke the skimming property in a sequential bargaining game, which is $v > v' \Rightarrow \tau(v) < \tau(v')$. Intuitively, the marginal benefit of the buyer waiting more is independent of the buyer's valuation and decreases as time goes by (the probability for the seller to be the normal type decreases), while the marginal cost is strictly increasing in the buyer's valuation and is constant across time. Hence the higher valuation the buyer has, the faster his optimal concession time is. Using this property, the equation is essentially equivalent to

$$r(\tau^{-1}(t)-p) = \frac{dG_S(t)/dt}{1-G_S(t)}(p-\underline{v})$$

whose solution is

$$G_S(t) = 1 - c_S \exp\left(-\int_0^t \frac{r(\tau^{-1}(s)-p)}{p-\underline{v}}ds\right)$$

with another unknown $c_S \in [\mu, 1]$. In order to remove an endogenous object τ in the equation, the skimming property is invoked once again, which gives

$$1 - F(\tau^{-1}(t)) = G_B(t) = 1 - c_B \exp\left(-\frac{\underline{v}}{p-\underline{v}}rt\right)$$

Rearranging terms, we finally get

$$G_S(t) = 1 - c_S \exp\left(-\frac{1}{\underline{v}} \int_{F^{-1}\left(c_B \exp\{-\frac{\underline{v}}{p} rt\}\right)}^{F^{-1}(c_B)} \frac{(v-p)f(v)}{F(v)} dv\right)$$

The two unknowns c_B and c_S are found using the following two facts : (1) Either $c_B = 1$ or $c_S = 1$ and (2) $G_S(t^*) = 1 - \mu \Leftrightarrow G_B(t^*) = 1 - F(p)$. The former one states that a player strictly prefers waiting an instant to conceding immediately, if the opponent concedes with a positive probability at date 0. The latter one reflects the fact that once the opponent turns out to be a strong type (commitment type or buyer with valuation lower than p), a player concedes immediately. During this discussion, it's revealed that $\phi(F, p)$ can be interpreted as the buyer's bargaining power.

From Proposition 1, it's immediate to calculate each type's indirect expected utility, $U_i(p, \mu), i = 0, 1$.

Corollary 1 *If $\mu \leq \phi(F, p)$, then*

$$\begin{aligned} U_0(p, \mu) &= \underline{v} \\ U_1(p, \mu) &= \underline{v}(1 - F(p)^{p/\underline{v}}) \text{ if } p \in [\underline{v}, \overline{v}] \end{aligned}$$

If $\mu > \phi(F, p)$, then

$$\begin{aligned} U_0(p, \mu) &= p(1 - c_B) + c_B \underline{v} \\ U_1(p, \mu) &= p(1 - c_B) + c_B \underline{v} \left(1 - \left(\frac{F(p)}{c_B}\right)^{p/\underline{v}}\right) \end{aligned}$$

where

$$\frac{1}{\underline{v}} \int_p^{F^{-1}(c_B)} \frac{(v-p)f(v)}{F(v)} dv + \ln \mu = 0$$

3.2 Isoprofit Curves

In this subsection, we develop basic properties of indirect utility functions for future use. For each $\mu \in [0, 1]$, let

$$\begin{aligned} p_0^*(\mu) &\in \arg\max_{p \in [\underline{v}, \overline{v}]} U_0(p, \mu) \\ p_1^*(\mu) &\in \arg\max_{p \in [\underline{v}, \overline{v}]} U_1(p, \mu) \end{aligned}$$

In words, $p_i^*(\mu)$ is the price which maximizes ξ_i type seller's expected payoff when the buyer's belief is fixed by μ. Notice that $p_i^*(1) = p_i^*$, $i = 0, 1$ since $c_B \to F(p)$ as $\mu \to 1$. First we present comparative statics results that are immediate.

Lemma 1 *Suppose $\mu > \phi(F, p)$, Then*

(1) $\lim_{\mu \to 1} c_B(p, \mu) = F(p)$.

(2) $c_B(\underline{v}, \mu) = \lim_{p \to \underline{v}} c_B(p, \mu) > 0$

(3) $\partial c_B(p, \mu)/\partial p > 0$ *and* $\partial c_B(p, \mu)/\partial \mu < 0$.

(4) $\partial U_i(p, \mu)/\partial \mu > 0$

For each $p' \in [\underline{v}, p_i^*(1)]$, let $\rho_i(p') \in [p_i^*(1), \overline{v}]$ be the value such that $U_i(\rho_i(p'), 1) = U_i(p', 1)$ and

$$I_i^{p'} = \{(p, \mu) \in [\underline{v}, \underline{v}] \times (0, 1] : U_i(p, \mu) = U_i(p', 1)\}$$

That is, $I_i^{p'}$ is the level set on which ξ_i type is indifferent.

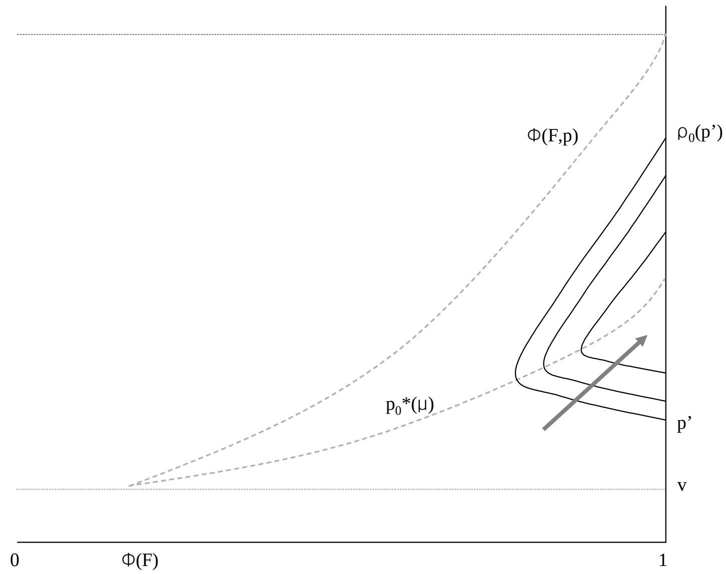
Lemma 2 *For each $p \in (p', \rho_i(p'))$, there exists a unique $\mu_i \in (\phi(F, p), 1)$ such that $(p, \mu_i) \in I_i^{p'}$*

Proof : Suppose $p \in (p', p_i^*(1)]$. Since $\underline{v} \leq U_i(p, \phi(F, p)) < U_i(p', 1) < U_i(p, 1)$ and $U_i(p, \cdot)$ is continuous, by the Intermediate Value Theorem, there exists $\mu_i \in (\phi(F, p), 1)$ which makes $(p, \mu_i) \in I_i^{p'}$. Also it's unique because $U_i(p, \cdot)$ is strictly increasing when bigger than $\underline{v}$. Analogous proof applies for $p \in [p_i^*(1), \rho_{i+}(p'))$ **Q.E.D.**

Because of lemma 2, for each $p' \in (\underline{v}, p_i^*(1))$, we can define an associated isoprofit curve as a function $\psi_i^{p'}$ from $(p', \rho_i(p'))$ to $(\phi(F), 1)$. In other words, $\psi_i^{p'}$ is defined so that $(p, \psi_i^{p'}(p)) \in I_i^{p'}, \forall p \in (p', \rho_i(p'))$. The continuity of $\psi_i^{p'}$ is implied by the continuity of $U_i(\cdot, \cdot)$, and

Lemma 3 *$\psi_i^{p'}$ is strictly decreasing at p if $p < p_i^*(\psi_i^{p'}(p))$, and strictly increasing at p if $p > p_i^*(\psi_i^{p'}(p))$.*

Proof : Suppose $p < p_i^*(\psi_i^{p'}(p))$, but $\psi_i^{p'}$ is not strictly decreasing at p. Then there exists $p'' < p_i^*(\psi_i^{p'}(p))$ such that $U_i(p'', \mu^1) = U_i(p'', \mu^2) > \underline{v}$ for distinct $\mu^1, \mu^2 \in (\phi(F), 1)$. This is not possible because $\partial U_i(p'', \mu)/\partial \mu > 0$ when $U_i(p'', \mu) > \underline{v}$. The proof of the second part is analogous. **Q.E.D.**



Isoprofit curves of the normal type

The following lemma tells that the monotone single-peakedness of $U_i(\cdot, 1)$ implies the same property of $U_i(\cdot, \mu)$. Also, combined with (4) in Lemma 1, it implies that $p_i^*(\cdot)$ is strictly increasing.

Lemma 4 *If $U_i(p, \mu) > \underline{v}$, $U_i(p, \mu)$ is strictly increasing in p for $p < p_i^*(\mu)$ and strictly decreasing in p for $p > p_i^*(\mu)$*

Proof : Suppose $p^1 < p^2 < p_i^*(\mu)$. Take p and p' such that $(p^1, \psi_i^p(p^1)) \in I_i^p$ and $(p^2, \psi_i^{p'}(p^2)) \in I_i^{p'}$. If $U_i(p^1, \mu) > U_i(p^2, \mu) > \underline{v}$ then ψ_i^p and $\psi_i^{p'}$ should intersect at least once, which is not possible. If $U_i(p^1, \mu) = U_i(p^2, \mu)$ then there exists p^3 such that $U_i(p^3, \mu^1) = U_i(p^3, \mu^2) > \underline{v}$ for distinct $\mu^1, \mu^2 \in (\phi(F, p^3), 1)$, which is not possible again. **Q.E.D.**

4 Behavioral Commitment Types vs. Normal Type

From this section, we analyze how the initial price is determined in equilibrium. The availability of indirect utility functions simplifies analysis a lot. Both the buyer's and the seller's behaviors which occur after (p, μ) was pinned down are already reflected in the indirect utility functions. We focus on determining the initial price and the buyer's conditional belief.

We start with a game without the rational commitment type ($\mu_1 = 0$). This setup is very similar to the standard reputation model, and the issue is what behavioral commitment type the normal type pretends to be. Different from the standard reputation model in repeated games (in which a patient long-lived player can increase credibility by repeatedly playing the same action), however, the normal type suffers from the fact that choosing a price necessarily worsens the credibility of that price, which in turn decreases her own expected payoff. Hence the normal type necessarily randomizes over multiple prices in equilibrium. We represent the normal type seller's mixed strategy by a cumulative distribution function H_0 over $[\underline{v}, \overline{v}]$. In addition, let $\mu : [\underline{v}, \overline{v}] \rightarrow [0,1]$ be the buyer's belief over the seller's type where $\mu(p)$ is the probability (the buyer's belief) that the seller is a commitment type conditional on that she asks a price p.

Definition 1 *A pair (H_0^*, μ^*) constitutes an equilibrium in a game without the rational commitment type if (i) given μ^*, for any $p' \in \Xi_0^*$,*

$$U_0(p', \mu^*(p')) = \max_{p \in [\underline{v}, \overline{v}]} U_0(p, \mu^*(p))$$

where $\Xi_0^* = cl\{p \in [\underline{v}, \overline{v}] : G_0^*$ is strictly increasing at $p\}$ and

(ii) given H_0^* (and λ), μ^* is a conditional probability that the seller is a commitment type

In general, a conditional probability is defined up to measure zero sets, which implies that we may end up with possibly many consistent μ^*. But requirement (i) helps effectively remove the possible multiplicity of consistent beliefs. Since the normal type can choose any specific price, in equilibrium we should have one specific μ^* on Ξ_0^*. Though μ^* is still defined up to measure zero sets outside Ξ_0^*, $\mu^*(P) = 1$ for any positive measure set P, so it's without loss of generality assuming $\mu^*(p) = 1$ for any $p \notin \Xi_0^*$.

To achieve our main purpose in this section (and in section 6), we assume for any price there exists a corresponding behavioral commitment type. No matter what price was asked by the seller, there's a chance that the seller is a commitment type. Formally, we focus on the game in which $\lambda([\underline{v}, p])$ is strictly increasing in $p \in (\underline{v}, \overline{v})$. One important implication of this assumption is that once either μ^* or H_0^* is determined, the other is also settled. Since there's no off-the-equilibrium path, a consistent belief μ^* is completely determined by H_0^* (and λ) and conversely for a strategy and belief profile to constitute an equilibrium, H_0^* should be adjusted so that μ^* is a consistent belief. We work with μ^* instead of H_0^* and will not explicitly deal with H_0^*, relying on this observation.

In addition, we assume that λ doesn't put positive mass on $\underline{v}$ ($\lambda(\{\underline{v}\}) = 0$). This is mainly for notational simplicity, and the analysis itself is not more complicated in a general case. λ may have a positive mass at any other price, though it's most conceivable that λ has a density.

We first provide a useful lemma, whose results are immediate from the continuity of indirect utility functions.

Lemma 5 *Ξ_0^* can be represented by $[p', \rho_0(p')]$ for some $p' \in [\underline{v}, p_0^*(1)]$. In addition, if $p' > \underline{v}$, then $\mu^*(p') = \mu^*(\rho_0(p')) = 1$.*

Proof : Suppose $p' \in \Xi_0^*$. Then $p \in \Xi_0^*, \forall p \in (p', \rho_0(p'))$, because $U_0(p,1) > U_0(p',1) \geq$

$U_0(p', \mu^*(p'))$. Hence $\Xi_0^* = [p', \rho_0(p')]$ for some $p' \in [\underline{v}, p_0^*(1)]$. This establishes the first result. Now suppose $\Xi_0^* = [p', \rho_0(p')], p' > \underline{v}$ but $\mu^*(p') < 1$. Then for $\varepsilon > 0$ sufficiently small, $U_0(p' - \varepsilon, 1) > U_0(p', \mu^*(p'))$, which can't be the case in equilibrium. Analogous proof applies to $\rho_0(p')$. **Q.E.D.**

Suppose in equilibrium $\Xi_0^* = [p', \rho_0(p')]$ for some $p' \in (\underline{v}, p_0^*(1)]$. One immediate, but critical consequence is that $\mu^*(p) = \psi_0^{p'}(p)$ on Ξ_0^*. In words, for Ξ_0^* to be an equilibrium support the equilibrium belief should follow an isoprofit curve of the normal type starting from $(p', 1)$. This is simply a restatement of an obvious requirement that in equilibrium there shouldn't exist a profitable deviation. Given p', the equilibrium payoff is also determined as $U_0(p', 1)$. Therefore the equilibrium characterization shrinks to a problem of finding an appropriate p'.

The algorithm of finding p' is simple. Given μ_2, we start from $p_0^*(1)$, and continuously lower price. As p' decreases, a corresponding isoprofit curve expands to the left, encompassing previous isoprofit curves. In order to make μ^* follow $\psi_0^{p'}$, the normal type should play each price in $[p', \rho_0(p')]$ with a specific probability, and we can take integration over probabilities the normal type consumed for each price. The integration is continuously and strictly increasing as p' decreases. We reach a proper p' if the integration becomes equal to 1 (the normal type should exhaust probability 1) and the uniqueness of such p' also follows from the observation that isoprofit curve expands to the left in a monotone fashion.

If there doesn't exist such $p' > \underline{v}$, this means that $\Xi_0^* = [\underline{v}, \overline{v}]$. The equilibrium belief is also unique in this case as $\mu^*(p) = \phi(F, p), \forall p > \underline{v}$. If we focus on the choice of price, it seems to be possible that $\mu^*(p) < \phi(F, p)$ for some p. But this can't happen in a real game. Remembering the analysis in Section 3, the buyer's belief should be equal to $\phi(F, p)$ conditional on that the seller actually asks p. The normal type might plan to play p with a probability with which $\mu^*(p) < \phi(F, p)$, but she should concede with positive probability at date 0 so that the buyer's belief is equal to $\phi(F, p)$. Hence conditional on that the seller asks p, $\mu^*(p) = \phi(F, p), \forall p > \underline{v}$.

We introduce a quantity that helps state the main proposition in this section. Given λ and F, let $\varphi(F, \lambda)$ so that

$$\frac{1 - \varphi(F, \lambda)}{\varphi(F, \lambda)} = \int_{(\underline{v}, \overline{v})} \frac{1 - \phi(F, p)}{\phi(F, p)} d\lambda$$

Roughly $\varphi(F, \lambda)$ is an average value of $\phi(F, p)$ with respect to λ, and has a similar role to $\phi(F, p)$. As shown below, it can be interpreted as the bargaining power of the buyer facing behavioral commitment types distributed according to λ. If λ first-order stochastically dominates λ', $\varphi(F, \lambda) > \varphi(F, \lambda')$ because $\phi(F, p)$ is strictly increasing in p. It is a tempting thought that the seller would be better off if there are more behavioral commitment types asking high prices. However, high price makes the buyer more willing to endure the war of attrition, which strengthens the buyer's relative bargaining position.

Proposition 2 *(1) If $\mu_2 \leq \varphi(F,\lambda)$ then there exist a unique equilibrium in which $\Xi_0^* = \Xi_2, \mu^*(p) = \phi(F,p), \forall p > \underline{v}$, and the normal type's expected payoff is $\underline{v}$.*

(2) If $\mu_2 > \varphi(F,\lambda)$ then there exist a unique equilibrium in which for some $p' \in (\underline{v}, p_1^*(1))$, $\Xi_0^* = (p', \rho_1(p'))$,

$$\mu^*(p) = \begin{cases} \psi_1^{p'}(p) \text{ if } p \in \Xi_0^* \\ 1 \text{ otherwise} \end{cases}$$

and the normal type's payoff is equal to $U_0(p', 1)$

Proof of Proposition 3.

We present a property of a mixed strategy that we will continuously use. For simplicity, suppose λ and H_0 have densities f_λ and h_0 respectively (the result holds more generally). Then

$$\mu^*(p) = \frac{\mu_2 f_\lambda(p)}{(1-\mu_2)h_0(p) + \mu_2 f_\lambda(p)} \Leftrightarrow \frac{1-\mu^*(p)}{\mu^*(p)} f_\lambda(p) = \frac{(1-\mu_2)h_0(p)}{\mu_2}$$

For H_0 to be an equilibrium,

$$\begin{aligned} \frac{1-\mu_2}{\mu_2} &= \frac{1-\mu_2}{\mu_2}\int_{\Xi_0^*} h_0(p)dp = \frac{1-\mu_2}{\mu_2}\int_{\Xi_2} h_0(p)dp = \int_{\Xi_2} \frac{1-\mu^*(p)}{\mu^*(p)} f_\lambda(p)dp \\ \Leftrightarrow \quad \frac{1-\mu_2}{\mu_2} &= \int_{\Xi_0^*} \frac{1-\mu^*(p)}{\mu^*(p)} d\lambda = \int_{\Xi_2} \frac{1-\mu^*(p)}{\mu^*(p)} d\lambda \end{aligned}$$

Notice that $1-\mu^*(p)$ is the equilibrium belief of the buyer that the seller is normal conditional on p is asked. Hence $(1-\mu^*(p))/\mu^*(p)$ is the relative ratio between the normal type and commitment type conditional on p. The above inequality simply says that the aggregate relative ratio between the normal type and behavioral commitment types should be equal to the expectation of a conditional relative ratio between the normal type and a behavioral commitment type.

Suppose $\mu^*(p) > \phi(F,p)$ for some $p > \underline{v}$. Since $U_0(p, \mu^*(p)) > \underline{v}$, $\mu^*(p) > \phi(F,p), \forall p \in \Xi_0^*$ and $\underline{v} \notin \Xi_0^*$. But for $\mu^*(p)$ to be an equilibrium belief,

$$\frac{1-\mu_2}{\mu_2} = \int_{\Xi_0^*} \frac{1-\mu^*(p)}{\mu^*(p)} d\lambda < \int_{(\underline{v},\overline{v})} \frac{1-\phi(F,p)}{\phi(F,p)} d\lambda = \frac{1-\varphi(F,\lambda)}{\varphi(F,\lambda)}$$

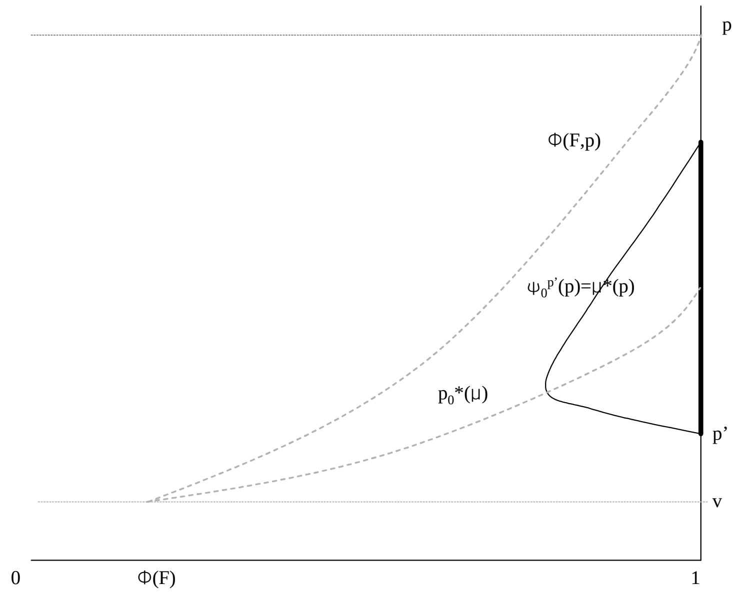
Hence if $\mu_2 \leq \varphi(F,\lambda)$ then $\mu^*(p) = \phi(F,p), \forall p > \underline{v}$. Also, for this to be true, $\Xi_0^* = [\underline{v}, \overline{v}]$.

Now suppose $\mu_2 > \varphi(F,\lambda)$. Define $J : (\underline{v}, p_0^*(1)] \rightarrow R_+$ by

$$J(p') = \int_{(p',\rho_0(p'))} \frac{1-\psi_0^{p'}(p)}{\psi_0^{p'}(p)} d\lambda$$

From comparative static results in Section 3.2, J is strictly decreasing (because if $p' < p''$ then

$\psi_0^{p'}(p) < \psi_0^{p''}(p)$) and the image of J is $[0, (1-\varphi(F,\lambda))/\varphi(F,\lambda)\ )$. Therefore for each $\mu_2 > \varphi(F,\lambda)$, there exists a unique $p' \in (\underline{v}, p_0^*(1)]$ such that $(1-\mu_2)/\mu_2 = J(p')$. Based on the previous Lemma, it's immediate that this establishes the unique equilibrium. **Q.E.D.**



A typical equilibrium

The first part in Proposition 3 says that the implication on the monopoly's expected payoff in the Coase conjecture is strongly robust with the introduction of behavioral commitment types. Though the effect of uncertainty over the seller's type is amplified, buyers are very reluctant to purchase at a price higher than the competitive one. The monopoly becomes better off only when μ_2 is sufficiently high. The efficiency implication is weakly robust. It's true that the delay disappears as μ_2 tends to 0, but there does exist delay as soon as $\mu_2 > 0$.

One interesting observation is that the normal type asks price higher than the static monopoly one, if μ_2 is high enough. Formally, if

$$\frac{1-\mu_2}{\mu_2} < C(F,\lambda) \equiv \int_{(p_1^*(1), \rho_0(p_1^*(1)))} \frac{1-\psi_0^{p_1^*(1)}(p)}{\psi_0^{p_1^*(1)}(p)} d\lambda$$

then any price the normal type asks is higher than the static monopoly one. For exposition, suppose μ_2 is arbitrarily close to 1. Then no matter which price was asked, the buyer believes that the seller is almost surely a commitment type, so he purchases immediately at high price as long as his valuation is greater. Instant later, the normal type can lower price and gets additional profit. Hence the effective objective function of the normal type in this scenario is $p(1-F(p)) + F(p)\underline{v}$, whose maximum $p_0^*(1)$ is strictly greater than $p_1^*(1)$. Without uncertainty, her flexibility is not a blessing but a curse. With uncertainty, however, it may become a blessing, and the benefit becomes larger, the more likely the seller is believed to be a commitment type. Of course, the normal type's expected profit is higher than the static monopoly profit with μ_2 high enough.

5 Rational Commitment Type vs. Normal Type

In this section, we study a game with the normal type and the rational commitment type. There are two main reasons why we study the rational commitment type. First it provides a way to endogenize the behavior of commitment type. We are interested in the effect on the robustness result and equilibrium behavior when the distribution of commitment types is not given arbitrarily. Second the choice of the rational commitment type is itself interesting. The rational commitment type is the seller, if her type is known to the buyer, who achieves the static monopoly profit by asking the static monopoly price. Our setup can be interpreted as the one in which the seller is typically the rational commitment type (who has commitment power), but the buyer has some doubt that the seller may be opportunistic. So by looking at the choice of the rational commitment type, we can examine how the behavior of the monopoly with commitment power changes, responding to the buyer's uncertainty on her credibility. In this respect, it's worthy of studying the case with μ_1 big enough, as well as that with μ_1 closed to 0.

Suppose the rational commitment type chooses a single price $p(> \underline{v})$ in equilibrium. Then the normal type undoubtedly follows the rational commitment type. Asking a different price reveals her type to be normal, after which she suffers from the Coasian dynamics. Therefore the analysis in Section 3 is enough for our purpose. Given μ_1, we only need to determine what price can be supported as an equilibrium price.

The immediate, but very important observation is that if $\mu_1 \leq \phi(F)(\equiv \phi(F, \underline{v}))$ then there exists a unique sequential equilibrium in which both types ask $\underline{v}$. In equilibrium, both types should have payoff at least as much as $\underline{v}$, because they have option to ask $\underline{v}$. Since obviously $p_1^*(\mu) = \underline{v}$ for $\mu_1 \leq \phi(F)$, the rational commitment type chooses $\underline{v}$ and consequently the normal type should ask $\underline{v}$ as well. Hence the full-blown Coase conjecture is strongly robust in this setup. Not only the monopoly gets the competitive profit, but also the competitive price is asked from the beginning, which leads to social efficiency.

In fact, the Coase conjecture is even more robust. As soon as $\mu_1 > \phi(F)$ (the buyer starts to concede with positive probability at date 0 for some price), the normal type strictly prefers price higher than $\underline{v}$. On the contrary, the rational commitment type may not be better off by asking higher price because the buyer may concede with too small probability at date 0. The difference arises from the fact that the normal type is flexible in adjusting price, which ensures her at least as much expected utility as $\underline{v}$ under any circumstance, while the rational commitment type should endure all delay cost coming from a war of attrition. The following Lemma pins down the exact critical value of μ_1 on which the rational commitment type is indifferent between choosing $\underline{v}$ and marginally increasing price.

Lemma 6 *Suppose* $\mu_1 > \phi(F)$. *Then* $\lim_{p \to \underline{v}} \partial U_1(p, \mu_1)/\partial p = 1 - c_B(\underline{v}, \mu_1) - \underline{v} f(\underline{v})$

Proof :

$$\begin{aligned} \frac{\partial U_1(p,\mu_1)}{\partial p} &= 1 - c_B(p,\mu_1) \\ &\quad - \left(\frac{F(p)}{c_B(p,\mu_1)}\right)^{p/\underline{v}} \left(c_B(p,\mu_1)\left(\ln\left(\frac{F(p)}{c_B(p,\mu_1)}\right) + p\frac{f(p)}{F(p)}\right) - \frac{\partial c_B(p,\mu_1)}{\partial p}(p - \underline{v})\right) \end{aligned}$$

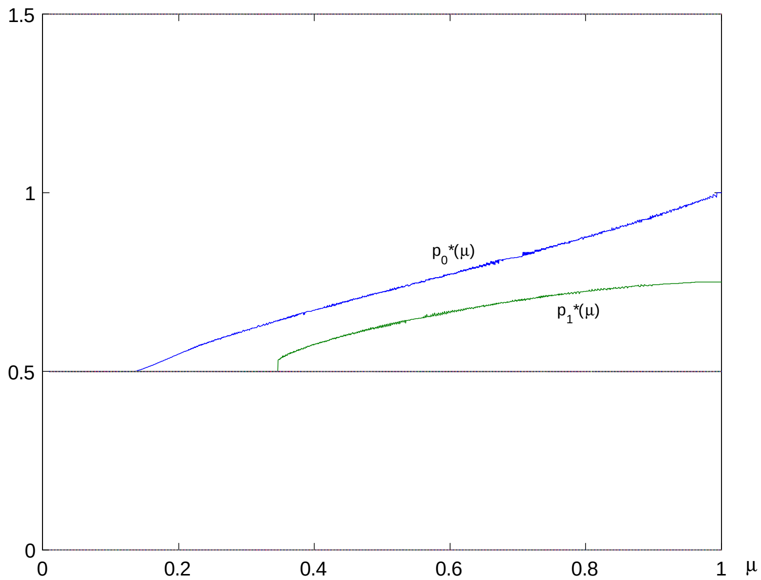
Applying Lemma 1, we get the result. **Q.E.D.**

From Lemma 1, we know that $c_B(\underline{v}, \mu_1)$ continuously and strictly decreases from 1 to 0 as μ_1 increases from $\phi(F)$ to 1. Hence there exists a unique $\widetilde{\phi}(F)$ such that $c_B(\underline{v}, \widetilde{\phi}(F)) = 1 - \underline{v} f(\underline{v})$. Combining Lemma 4 and Lemma 5, we conclude that $p_1^*(\underline{v}) = \underline{v}$ if and only if $\mu_1 \leq \widetilde{\phi}(F)$, and the full-blown Coase conjecture holds up to $\widetilde{\phi}(F)$.

We provide a numerical example to help understand the result. Suppose F is uniformly distributed over $[1/2, 3/2]$. Then

$$\begin{aligned} \phi(F) &= \exp\left(-2\int_{1/2}^{3/2} \frac{v - 1/2}{v - 1/2} dv\right) = e^{-2} \approx 0.1353 \\ c_B(0.5, \widetilde{\phi}(F)) &= -\frac{\ln \widetilde{\phi}(F)}{2} = \frac{1}{2} \Rightarrow \widetilde{\phi}(F) = e^{-1} \end{aligned}$$

By the previous results, $p_1^*(\mu_1) = \underline{v}$ if $\mu_1 \leq \widetilde{\phi}(F) = e^{-1}$. The following graph shows numerical results, which match the theoretical ones.



A Numerical Example

Now suppose $\mu_1 > \widetilde{\phi}(F)$. Define $\overline{p}_1(\mu) \in (\underline{v}, \overline{v}]$ so that $U_1(\overline{p}_1(\mu), \mu) = \underline{v}$ and $p^m(\mu) = \min\{p_0^*(\mu), \overline{p}_1(\mu)\}$. Fix some $p \in [\underline{v}, \overline{p}_1(\mu)]$ and consider the following belief of the buyer: the commitment type chooses p in equilibrium and the deviator is the normal type for sure. Given this belief, the commitment type has no choice but to ask p (the normal type's choice is obvious), which makes the buyer's belief consistent. Hence for any $p \in [\underline{v}, \overline{p}_1(\mu)]$ there exists a corresponding sequential equilibrium. We summarize all findings.

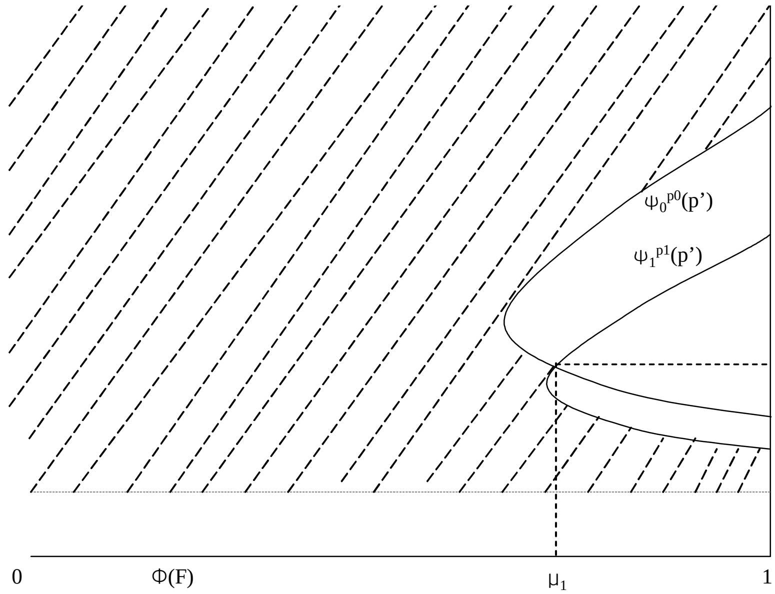
Proposition 3 *When $\mu_2 = 0$ there exists $\widetilde{\phi}(F) > \phi(F)$ such that*

(1) if $\mu_1 \leq \widetilde{\phi}(F)$ then there exists a unique sequential equilibrium in which both the rational commitment type and the normal type choose $\underline{v}$, and

(2) if $\mu_1 > \widetilde{\phi}(F)$ there exist multiple sequential equilibria in which some price in $[\underline{v}, \overline{p}_1(\mu)]$ is asked by both types.

The worst off-the-equilibrium-path belief was used in order to support an equilibrium. Since introducing a small perturbation (for example, the seller charges a different price than what she intended with very small probability) may destroy the equilibrium, this is not quite compelling. However, this doesn't have to be the case. Given $\mu_1 > \widetilde{\phi}(F)$, fix some $p \in (\underline{v}, \overline{p}_1(\mu))$. Then we can find $p_0, p_1 \in (\underline{v}, p_1^*(1))$ such that $\mu_1 = \psi_1^{p_1}(p) = \psi_0^{p_0}(p)$. In fact, every off-the-equilibrium-path belief $\mu(p')$ which satisfies $\mu(p') \leq \min\{\psi_1^{p_1}(p'), \psi_0^{p_0}(p')\}$ can be used to support a sequential equilibrium with p. In particular, we can make μ be continuous, for example, with off-the-equilibrium belief

$\mu(p') = \min\{\psi_1^{p_1}(p'), \psi_0^{p_0}(p')\}$.



Possible off-the-equilibrium beliefs

6 The Full-fledged Model

Finally we study a full-fledged model in which $\mu_0, \mu_1, \mu_2 > 0$. Though more complicated, this game provides a way to overcome drawbacks of the previous setups. Introducing the rational commitment type fixes the problem that equilibrium is affected by the distribution of behavioral commitment type which is exogenously and arbitrarily given, but invites multiplicity of equilibria in the meantime. In the full-fledged model the number of equilibria reduces significantly (up to generic finiteness) and equilibrium belief is uniquely determined. Furthermore the economic interpretation for each equilibrium is more clear.

The mixed strategy of the normal type and the rational commitment type is represented by a cumulative distribution function H_i over $[\underline{v}, \overline{v}], i = 0, 1$.

Definition 2 *A triplet* (H_0^*, H_1^*, μ^*) *constitutes an equilibrium in a full-fledged game if (i) given* μ^**, for any* $p' \in \Xi_i^*, i = 0, 1,$

$$U_i(p', \mu^*(p')) = \max_{p \in [\underline{v}, \overline{v}]} U_i(p, \mu^*(p))$$

where $\Xi_i^* = cl\{p \in [\underline{v}, \overline{v}] : H_i^*$ *is strictly increasing at* $p\}$ *and*

(ii) given H_0^* *(and* λ*),* μ^* *is a conditional probability that the seller is a commitment type*

The same remarks on μ^* and, importantly, Lemma 6 apply here as in Section 4. Once Ξ_i^* is settled, μ^* is uniquely identified, and H_1^* has one of the following two possibilities.

Lemma 7 *Either* $\Xi_1^* \subseteq \Xi_0^*$ *or* $\Xi_1^* = \{p_1^*(1)\}$

Proof : Suppose there exists $p \in \Xi_1^* - \Xi_0^*$. As long as Ξ_2 is sufficiently rich and $p \neq p_1^*(1)$, for ε small enough, there exists a profitable deviation to either $p - \varepsilon$ or $p + \varepsilon$, because $U_1(p, 1)$ is single-peaked. **Q.E.D.**

There are three possibilities of equilibrium according to Ξ_0^*. (1) $\Xi_0^* = [\underline{v}, \overline{v}]$, (2) $p_1^*(1) \notin \Xi_0^*$ and (3) $\Xi_1^* \subseteq \Xi_0^* \neq [\underline{v}, \overline{v}]$. The first two cases are relatively straightforward, because there's no meaningful strategic interaction between the normal type and the rational commitment type. For $\Xi_0^* = [\underline{v}, \overline{v}]$, $\mu^*(p) = \phi(F, p), \forall p > \underline{v}$, and then the rational commitment type obviously chooses $\underline{v}$, which in turn doesn't affect the normal type's choice. As long as the probability of behavioral commitment types is relatively smaller than that of the normal type, this can be an equilibrium. On the contrary, (from the analysis in Section 4) $p_1^*(1) \notin \Xi_0^*$ happens if and only if the relative ratio between the normal type and behavioral commitment types is small, and then the choice of the rational commitment type is obviously $p_1^*(1)$.

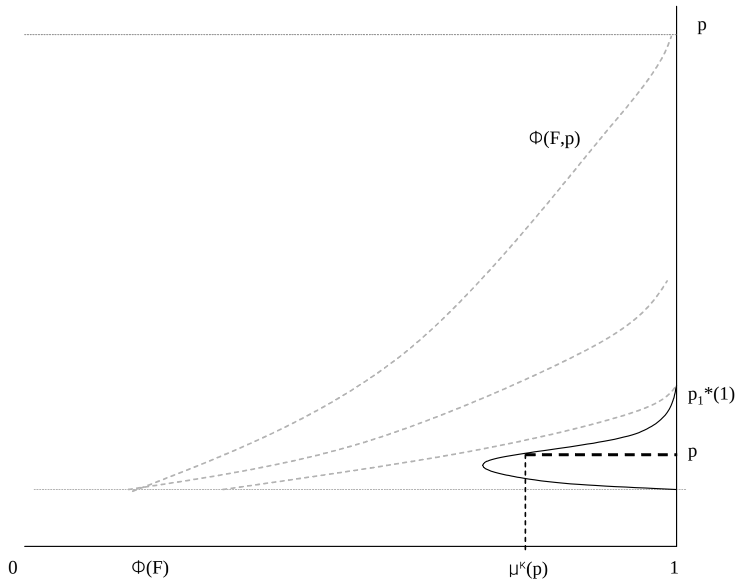
The case with $\Xi_1^* \subseteq \Xi_0^*$ is much more involved. Suppose $\Xi_0^* = [p', \rho_0(p')]$ for some $p' \in (\underline{v}, p_1^*(1))$. Since $\mu^*(p) = \psi_0^{p'}(p)$, we should have

$$int\Xi_1^* \subset \arg\max_p U_1(p, \mu^*(p)) = U_1(p, \psi_0^{p'}(p))$$

Hence the points at which U_0 and U_1 are tangent are potentially very important.

Lemma 8 *For each* $p' \in (\underline{v}, p_1^*(1))$*, there exists a unique* $\mu^\gamma(p') \in (\phi(F, p'), 1)$ *at which* U_0 *and* U_1 *are tangent. At such* $(p', \mu^\gamma(p'))$*,* U_1 *envelops* U_0*. In addition,* $p' < p_1^*(\mu^\gamma)$ *and* $\lim_{p' \to p_1^*(1)} \mu^\gamma(p') = \lim_{p' \to \underline{v}} \mu^\gamma(p') = 1$*.*

Proof : See Appendix



A typical shape of contract curve

In general, we can't guarantee that given $p' \in (\underline{v}, p_1^*(1))$, there exists a single price which maximizes $U_1(p, \psi_0^{p'}(p))$. If μ^γ was increasing, it would be always unique, but we necessarily have decreasing region because $\lim_{p' \to p_1^*(1)} \mu^\gamma(p') = \lim_{p' \to \underline{v}} \mu^\gamma(p') = 1$ and $\mu^\gamma(p') < 1$ for $p' \in (\underline{v}, p_1^*(1))$. For each $p' \in (\underline{v}, p_1^*(1))$, let

$$\Upsilon(p') = \arg\max U_1(p, \psi_0^{p'}(p))$$

If $p_1, p_2 \in \Upsilon(p')$ and $p_1 > p_2$, then $\psi_0^{p'}(p_1) > \psi_0^{p'}(p_2)$, because $\psi_0^{p'}$ is strictly increasing below $p_0^*(\mu)$. Let $p^{\Upsilon(p')} = \max\{p \in \Upsilon(p')\}$ and $p_{\Upsilon(p')} = \min\{p \in \Upsilon(p')\}$.

Proposition 4 *(1) If $\mu_0 > M^{\mu_1,\mu_2}$, then $\Xi_0^* = \Xi_2, \mu^*(p) = \phi(F,p), \forall p > \underline{v}$ and $\Xi_1^* = \{\underline{v}\}$ where*

$$M^{\mu_1,\mu_2} = \sup\left\{p' \in (\underline{v}, p_1^*(1)] : \mu_1 \frac{1 - \psi_0^{p'}(p^{\Upsilon(p')})}{\psi_0^{p'}(p^{\Upsilon(p')})} + \mu_2 \int_{(p',\rho_0(p'))} \frac{1 - \psi_0^{p'}(p)}{\psi_0^{p'}(p)} d\lambda\right\}$$

(2) If $\mu_2 C(F,\lambda) < \mu_0 < \mu_2(1 - \varphi(F,\lambda))/\varphi(F,\lambda)$, then $\Xi_0^* = [p', \rho_0(p')], \Xi_1^* \subset \Upsilon(p')$ for some $p' \in (\underline{v}, p_1^*(1))$,

$$\mu^*(p) = \begin{cases} 1 \text{ if } p \notin \Xi_0^* \\ \psi_0^{p'}(p) \text{ if } p \in \Xi_0^* \end{cases}$$

(3) If $\mu_2(1 - \varphi(F,\lambda))/\varphi(F,\lambda) \le \mu_0 < M^{\mu_1,\mu_2}$, then there are at least three equilibria. One is the like in (1) and two are in (2)

(4) If $\mu_0 = M^{\mu_1,\mu_2}$ then the equilibrium like in (1) always exists. Additionally, if M^{μ_1,μ_2} is achieved in $p' \in (\underline{v}, p_1^*(1))$, then there exists at least one more equilibrium belonging to (2).

(5) If $\mu_0 \leq \mu_2 C(F,\lambda)$*, then* $\Xi_0^* = [p', \rho_0(p')], \Xi_1^* = \{p_1^*(1)\}$ *for some* $p' \in [p_1^*(1), p_0^*(1))$*,*

$$\mu^*(p) = \begin{cases} 1 \text{ if } p \notin \Xi_0^* \\ \psi_0^{p'}(p) \text{ if } p \in \Xi_0^* \end{cases}$$

, the rational commitment type achieves the static monopoly profit $(U_1(p_1^*(1), 1))$ *and the normal type does more than the static monopoly profit*

Proof of Proposition 4 : Suppose $\Xi_0^* = \Xi_2$. From the analysis in Section 3, this can be an equilibrium if and only if

$$\frac{\mu_0}{\mu_2} \geq \int_{(\underline{v},\overline{v})} \frac{1-\phi(F,p)}{\phi(F,p)} d\lambda = \frac{1-\varphi(F,\lambda)}{\varphi(F,\lambda)}$$

Now suppose $p_1^*(1) \notin \Xi_0^*$. Again from the analysis in Section 3, this can be an equilibrium if and only if

$$\frac{\mu_0}{\mu_2} < C(F,\lambda) = \int_{(p_1^*(1),\rho_0(p_1^*(1)))} \frac{1-\psi_0^{p_1^*(1)}(p)}{\psi_0^{p_1^*(1)}(p)} d\lambda$$

Finally, suppose $\Xi_0^* = [p', \rho_0(p')]$ for some $p' \in (\underline{v}, p_0^*(1))$. For each $p' \in (\underline{v}, p_1^*(1)]$, define $\overline{\lambda}^{p'} : \mathcal{F} \to [0,1]$ and $\underline{\lambda}^{p'} : \mathcal{F} \to [0,1]$ by

$$\overline{\lambda}^{p'}(P) = \frac{\mu_1 1_{\{p^{\Upsilon(p')} \in P\}} + \mu_2 \lambda(P)}{\mu_1 + \mu_2}$$

and

$$\underline{\lambda}^{p'}(P) = \frac{\mu_1 1_{\{p_{\Upsilon(p')} \in P\}} + \mu_2 \lambda(P)}{\mu_1 + \mu_2}$$

Given μ_1 and μ_2, define $\overline{J}^{\mu_1,\mu_2}, \underline{J}^{\mu_1,\mu_2} : (\underline{v}, p_1^*(1)] \to R_+$ by

$$\begin{aligned} \overline{J}^{\mu_1,\mu_2}(p') &= \int_{(p',\rho_0(p'))} \frac{1-\psi_0^{p'}(p)}{\psi_0^{p'}(p)} d\overline{\lambda}^{p'} = \frac{\mu_1}{\mu_1+\mu_2} \frac{1-\psi_0^{p'}(p^{\Upsilon(p')})}{\psi_0^{p'}(p^{\Upsilon(p')})} + \frac{\mu_2}{\mu_1+\mu_2} \int_{(p',\rho_0(p'))} \frac{1-\psi_0^{p'}(p)}{\psi_0^{p'}(p)} d\lambda^{p'} \\ \underline{J}^{\mu_1,\mu_2}(p') &= \int_{(p',\rho_0(p'))} \frac{1-\psi_0^{p'}(p)}{\psi_0^{p'}(p)} d\underline{\lambda}^{p'} = \frac{\mu_1}{\mu_1+\mu_2} \frac{1-\psi_0^{p'}(p_{\Upsilon(p')})}{\psi_0^{p'}(p_{\Upsilon(p')})} + \frac{\mu_2}{\mu_1+\mu_2} \int_{(p',\rho_0(p'))} \frac{1-\psi_0^{p'}(p)}{\psi_0^{p'}(p)} d\lambda^{p'} \end{aligned}$$

Since $\psi_0^{p'}(p^{\Upsilon(p')}) \leq \psi_0^{p'}(p_{\Upsilon(p')})$, $\overline{J}^{\mu_1,\mu_2}(p') \geq \underline{J}^{\mu_1,\mu_2}(p')$. First the minimum (of both $\overline{J}^{\mu_1,\mu_2}(p')$ and $\underline{J}^{\mu_1,\mu_2}(p')$) is achieved at $p_1^*(1)$ because

$$\frac{1-\psi_0^{p_1^*(1)}(p_1^*(1))}{\psi_0^{p_1^*(1)}(p_1^*(1))} = 0 \leq \frac{1-\psi_0^{p'}(p^{\Upsilon(p')})}{\psi_0^{p'}(p^{\Upsilon(p')})} \leq \frac{1-\psi_0^{p'}(p_{\Upsilon(p')})}{\psi_0^{p'}(p_{\Upsilon(p')})}$$

and $\psi_0^{p'}(p)$ is strictly increasing in p', and it's value is $\mu_2 C(F,\lambda)/(\mu_1+\mu_2)$. Now define a correspondence $J^{\mu_1,\mu_2} : (\underline{v}, p_1^*(1)] \to R_+$ by

$$J^{\mu_1,\mu 2}(p') = [\underline{J}^{\mu_1,\mu_2}(p'), \overline{J}^{\mu_1,\mu_2}(p')]$$

Note that every value in $J^{\mu_1,\mu 2}(p')$ can be achieved by putting appropriate weights to $p^{\Upsilon(p')}$ and $p_{\Upsilon(p')}$. By the Theorem of Maximum, $J^{\mu_1,\mu 2}$ is upper hemi-continuous, and by definition,

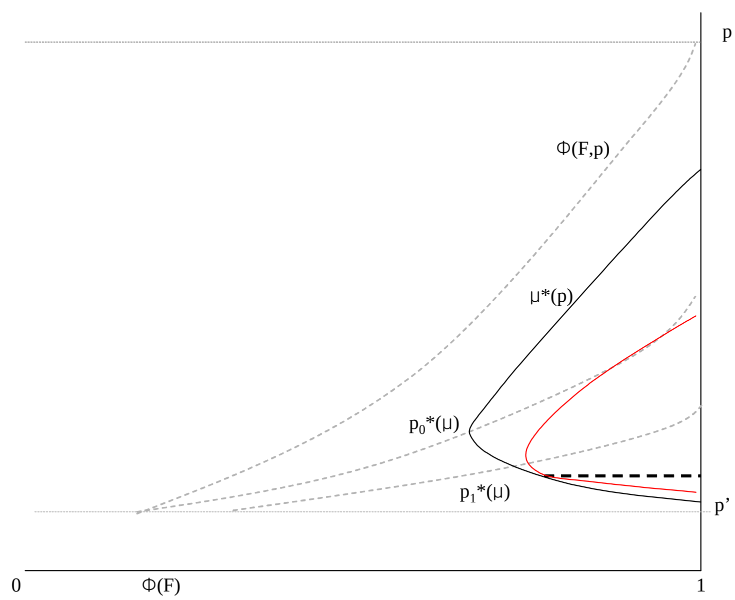
$$M^{\mu_1,\mu_2} = \sup\{p' \in (\underline{v}, p_1^*(1)] : (\mu_1+\mu_2)\overline{J}^{\mu_1,\mu_2}(p')\}$$

Applying the same logic as Proposition 3, it's clear that if

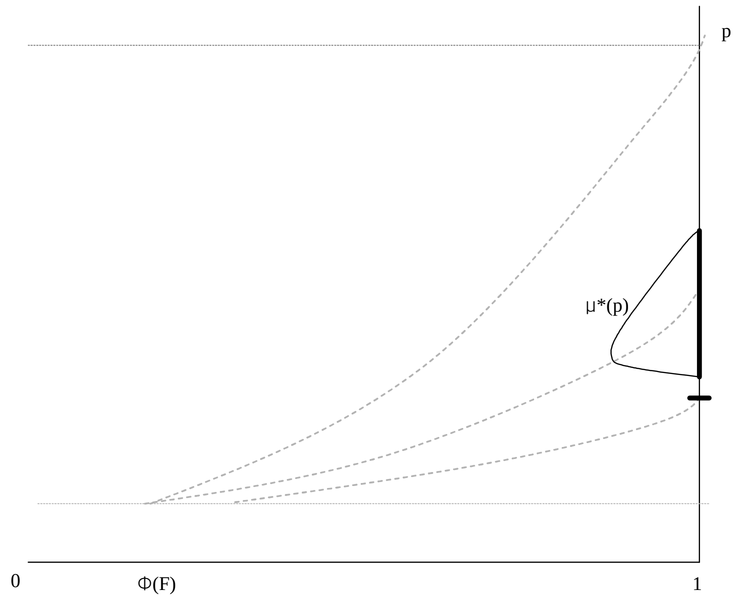
$$\mu_2 C(F,\lambda) < \mu_0 < M^{\mu_1,\mu_2}$$

then there exists $p' \in (\underline{v}, p_1^*(1)]$ and a subset of $\Upsilon(p')$ such that $\Xi_0^* = [p', \rho_0(p')]$.

Q.E.D.



A typical equilibrium in case (2)



A typical equilibrium in case (5)

To understand the statement of Proposition 4, let's make some assumptions which help clarify the argument. We assume $\Upsilon(p') = \gamma(p')$ is a singleton for all $p' \in (\underline{v}, p_1^*(1))$ and J^{μ_1,μ_2} is strictly concave for any (μ_1, μ_2) and is maximized at $p^{\mu_1,\mu_2} > \underline{v}$. Then

$$\begin{aligned} J^{\mu_1,\mu 2}([p^{\mu_1,\mu_2}, p_1^*(1))) &= \left(\frac{\mu_2}{\mu_1+\mu_2}C(F,\lambda), J^{\mu_1,\mu 2}(p^{\mu_1,\mu_2})\right] \\ J^{\mu_1,\mu 2}((\underline{v}, p^{\mu_1,\mu_1}]) &= \left(\frac{\mu_2}{\mu_1+\mu_2}\frac{1-\varphi(F,\lambda)}{\varphi(F,\lambda)}, J^{\mu_1,\mu 2}(p^{\mu_1,\mu_2})\right] \end{aligned}$$

and

$$M^{\mu_1,\mu_2} = (\mu_1+\mu_2)J^{\mu_1,\mu 2}(p^{\mu_1,\mu_2})$$

From the above calculation, we know there exists a unique equilibrium if

$$\mu_2 C(F,\lambda) < \mu_0 \leq \mu_2 \frac{1-\varphi(F,\lambda)}{\varphi(F,\lambda)}$$

with associated $p' > p^{\mu_1,\mu_2}$ such that $\mu_0 = (\mu_1+\mu_2)J^{\mu_1,\mu 2}(p')$. If

$$\mu_2 \frac{1-\varphi(F,\lambda)}{\varphi(F,\lambda)} < \mu_0 \leq (\mu_1+\mu_2)J^{\mu_1,\mu 2}(p^{\mu_1,\mu_2})$$

then there are three equilibria. Each one is associated with $\underline{v}$, a price lower than p^{μ_1,μ_2} but higher than $\underline{v}$, and a price higher than p^{μ_1,μ_2}. These equilibria correspond to three different, but possibly

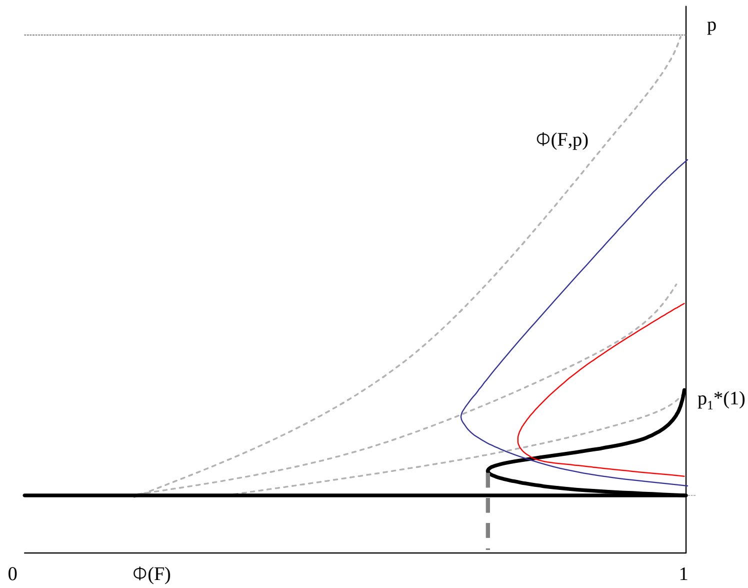


Figure 1: Equilibria when the probability for the seller to be one of behavioral commitment types vanishes.

consistent beliefs of the seller. When the rational commitment type is optimistic on the buyer's belief, she asks a sufficiently high price, which in turn enables the normal type to ask only high prices. With rather pessimism, she asks price low, but higher than $\underline{v}$, at which the normal type effectively pools. If the rational commitment type is too pessimistic, she believe the buyer will never concede at date 0, and simply asks $\underline{v}$, with which the normal type has no choice but to randomize over $[\underline{v}, \overline{v}]$

In the same spirit as in the previous section, the special interest is given to the case where μ_2 is arbitrarily small. In this way, not only the distribution of commitment types is fully endogenized, but also rather implausible beliefs are excluded. The following analysis is immediate from Proposition 4. Let

$$\widehat{\phi}(F) = \min_{p \in (\underline{v}, p_1^*(1))} \mu^{\gamma}(p)$$

For any $\mu_1 < 1$, it is an equilibrium that the rational commitment type and the normal type ask $\underline{v}$. The equilibrium belief is unique as $\mu^*(p) = \phi(F, p), \forall p > \underline{v}$. If $\mu_1 < \widehat{\phi}(F)$, this is a unique equilibrium. If $\mu_1 > \widehat{\phi}(F)$, then there exists at least two more equilibria. One is associated with a high price, and the other is associated with a low price. If the equilibrium price p^* is greater than $\underline{v}$, then the equilibrium belief is uniquely determined so as to follow the isoprofit curve of the normal type which crosses (p^*, μ_1).

The last interesting fact is that the equilibrium when $\mu_1 = 1$ is lower hemi-continuous, but not upper hemi-continuous. As $\mu_0 \to 0$ there always exists a sequence of equilibria which converges to the equilibrium in which $p_1^*(1)$ is asked. But when μ_2 is not always relatively larger than μ_0, there's another sequence (in fact two sequences) of equilibria which converges to the one with $\underline{v}$. This result stems from the fact that the buyer's belief over the seller's type across different prices doesn't matter in a degenerate problem, while it does a lot even with very small perturbation.

7 Discussion

(1) Extension

Are the results robust under weaker assumption on F? In particular, we relax the assumption that both $p(1-F(p))$ and $p(1-F(p))+\underline{v}F(p)$ are monotonely singled-peaked. Now they may have flat interval and several peaks.

Let $\overline{u}_0 = \max_p U_0(p,1)$ and $Z_0 = [\underline{v}, \overline{u}_0]$. For each $z \in Z_0$, define $\Gamma(z) = \{p' \in [\underline{v}, \overline{v}] : U_i(p',1) > z\}$. Then by similar analysis to that in Lemma 2, we know $\Gamma(z)$ replaces the role of $(p', \rho_0(p'))$. Given $\Gamma(z)$, equilibrium belief should follow isoprofit curve of the normal type. Without the rational commitment type, the equilibrium is again unique. With the rational commitment type and arbitrarily small probability of behavioral commitment types, a pair (p, μ_1) can be supported as an equilibrium only when the isoprofit curves of the normal type and the rational commitment type are tangent as before. Therefore there's no qualitative change in equilibrium under weaker assumption on F. The only difficulty is to characterize the set of points at which two isoprofit curves are tangent. In particular, Lemma 8 applies no longer.

(2) The emergence of the rational commitment type

It was assumed that some rational sellers are exogenously doomed not to change price (the rational commitment type). Ideally we should be able to suggest a mechanism that naturally generates the rational commitment type and ask with what probability an agent is the type. But it's another research question and we suggest only two tentative explanations here.

One possibility is a contractual arrangement within a selling firm. If a firm is suffering from a principal-agent problem, then the principal may restrict the agent's action to sell only at a pre-specified price, performs bargaining for himself, or sells the firm to the agent. The latter two cases may correspond to the normal type in our model, while the first to the commitment type. If firms are heterogenous with respect to their technology dealing with a principal-agent problem and each firm's technology is not observable to the buyer, this potentially gives rise to the bargaining problem we analyzed. Ultimately, no one tries to bargain over prices in a Wal-Mart, while it is different if you visit a small local shop. Of course, what kind of contract is optimal between the principal and the agent in our environment is another question to pursue.

The second possibility relates to the institutional feature of the selling side. In a wage bargaining, a firm may be uncertain over a union leader's level of discretion from the rank and file, or the union's responsiveness. In an international trade negotiation, a bargainer may be uncertain over the opponent's level of delegation from the government or the voters. Hence some institutional feature which is not observable to the opponent may explain the emergence of the commitment type and produce the bargaining problem we analyzed.

(3) Acquired stubbornness

Kambe (1999) introduced two possible scenarios on the rise of stubbornness. One is "inborn" stubbornness, in which agents know their types before deciding their demands. The other is "acquired" stubbornness, in which the seller doesn't know whether she will be stuck to her initial demand, but knows she may be with some probability. He argued that acquired stubbornness could arise either psychologically or economically (reputationally).

Inborn stubbornness was used in this paper. Adopting acquired stubbornness in our model is an immediate task. The only difference is that the relevant (indirect) expected utility function of the rational seller is a weighted sum of two indirect utility functions. One distinguishing feature of acquired stubbornness is that there always exists a unique equilibrium, which is a trivial consequence from the fact that there's only a single agent who determines the initial price.

(4) The sufficient condition for $\phi(F) > 0$[3].

Our result is not sensitive to whether $\phi(F) > 0$. But it's still interesting to ask under what conditions $\phi(F) = 0$, because it can be interpreted as the buyer's absolute bargaining power, as well as has a nice, but new functional form.

First if $F(v) = f(v)(v - \underline{v}) + o(v - \underline{v})$ around $\underline{v}$, then $\phi(F)$ is certainly strictly positive, because the term inside the integral is bounded. More generally, as long as $(v - \underline{v})f(v)/F(v) \leq K/(v - \underline{v})^{1-\delta}$ around $\underline{v}$ for some K and $\delta > 0$, $\phi(F) > 0$. Conversely, if $(v - \underline{v})f(v)/F(v) \geq K/(v - \underline{v})^{1+\delta}$ around $\underline{v}$ for some $K > 0$ and $\delta \geq 0$, $\phi(F) = 0$. The following example shows a concrete case in which $\phi(F) = 0$.

$$F(v) = \begin{cases} 0 & \text{if } v \leq \underline{v} \\ \exp\left(\frac{1}{\overline{v}-\underline{v}} - \frac{1}{v-\underline{v}}\right) & \text{if } \underline{v} \leq v \leq \overline{v} \\ 1 & \text{if } v > \overline{v} \end{cases}$$

Roughly if F decreases arithmetically fast, then the density function f decreases more slowly than F by one-order and the factor $(v - \underline{v})$ completely cancels out the explosion of F/f. In case F converges to 0 exponentially fast enough, the explosion of F may outweigh the offsetting effect of $(v - \underline{v})f(v)$, and hence $\phi(F)$ could be zero.

(5) The No Gap case

The difficulty with the No Gap case is that we can't pin down the bargaining outcome when the seller is known to be normal. In fact, Ausubel and Deneckere (1989) proved a folk theorem for durable

[3] I thank Aureo de Paula, Jan Eeckhout, and Phillip Kircher for pointing out this problem.

goods monopoly problem with No Gap, that all seller payoffs between zero and static monopoly profits can be supported by a sequential equilibrium. For now, let's suppose a Weak-Markov equilibrium is played in case the seller's type is revealed to be normal[4]. We take this equilibrium, not because there's a natural advantage in this equilibrium[5] but simply because it is the analogue and limit of the unique equilibrium in the Gap case, and therefore we can also use a limit argument for our problem.

Given p and μ, as $\underline{v} \to 0$, $\phi(F) \to 0$ and $c_B \to F^{-1}(p)$. Hence the sequential equilibrium converges to

$$\begin{aligned}\tau(v) &= \begin{cases} 0 \text{ if } v < p \\ \infty \text{ otherwise} \end{cases} \\ G_S(t) &= 0, \forall t\end{aligned}$$

In the limit, the buyer concedes immediately as long as his valuation is higher than p and the seller never concede. The driving force of this result is that the normal type becomes more reluctant to concede as the gain from conceding ($\underline{v}$) becomes smaller, and in the limit she has no reason to reveal her type. She is indifferent between conceding and waiting at time $t > 0$, but for the equilibrium to be established, she should never concede.

In a discrete time version, the dynamics will be very different. Even in a Weak-Markov equilibrium, there's a positive gain by the seller revealing her type to be normal, and so the normal type should adjust price if she is confident that the buyer has valuation lower than p. To put it another way, we have assumed that offer interval Δ is always arbitrarily close to 0, and characterized equilibrium based on that. Alternatively, when studying an equilibrium in the No Gap case, we can let $\underline{v}$ converge to 0 first and make Δ arbitrarily small later. Under the second scenario, the equilibrium behavior will differ a lot from the one we derived here.

(6) Different discount factors - Relation to the standard reputation result

What happens if the seller is more patient than the buyer? Suppose the buyer and the seller have discount factors r_B and r_S respectively. Then for all p,

$$\phi(F,p) = \exp\left(-\frac{1}{\underline{v}}\frac{r_B}{r_S}\int_p^{\overline{v}}\frac{(v-p)f(v)}{F(v)}dv\right)$$

All other results are immediate extension. As r_S becomes lower for fixed r_B (the seller becomes relatively more patient), the buyer's bargaining power $\phi(F,p)$ becomes lower, which makes the buyer concede faster. The robustness results of the Coase conjecture are still true, but the robustness gets

[4] A weak-Markov equilibrium is a sequential equilibrium in which the buyer's decision depends only on the current price.

[5] It is simple and intuitive, but not compelling enough. Why would the monopoly choose to play the worst equilibrium to her? For more discussion on this issue, see Ausubel and Deneckere (1989).

weaker, which is predictable.

This analysis shows that the standard reputation result still holds in our problem in terms of discount rates. More formally, in a model with behavioral commitment types, for fixed $\mu_2 > 0$, there exists $\bar{r}_S > 0$ such that if $r_S > \bar{r}_S$ then the expected payoff of the normal type is $\varepsilon-$close to her maximum expected payoff, which is $p(1-F(p))+\underline{v}F(p)$ in our problem. One difference is that there's no natural meaning of the "Stackelberg type" in our model, and so the statement of the reputation result should be adjusted in terms of the maximum possible payoff rather than the specific behavior of some type.

8 Appendix

Proof of Proposition 1. (Many steps in this proof are borrowed from the proof of Proposition 1 in Abreu and Gul (2000))

Suppose (G_S, τ) is a sequential equilibrium given p and μ. Let $u_0(t)$ and $u_v(t)$ be the normal type seller's and type v buyer's expected payoffs by conceding at time t given (G_S, τ). From τ, define $G_B : \mathcal{R}_+ \to [0,1]$ by $G_B(t) = \Pr\{v : \tau(v) \le t\}$. Assume we flip a fair coin in case of simultaneous concessions. Later it will be clear that this assumption can be replaced by any tie-breaking rule.

$$\begin{aligned} u_0(t) &= p\int_0^t e^{-rs}dG_B + \frac{p+\underline{v}}{2}(G_B(t) - \lim_{s\nearrow t} G_B(s)) + \underline{v}(1-G_B(t))e^{-rt} \\ u_v(t) &= (v-\underline{v})\int_0^t e^{-rs}dG_S(s) + \frac{(v-p)+(v-\underline{v})}{2}(G_S(t) - \lim_{s\nearrow t} G_S(s)) + (1-G_S(t))(v-p)e^{-rt} \end{aligned}$$

1) G_S is not constant.

Suppose $G_S(t) = c, \forall t$ for some $0 \le c < 1-\mu$. Then $\tau(v) = \lim_{t\searrow 0} t, \forall v \ge p$ and $\tau(v) = \infty, \forall v < p$. Since the normal type seller knows that the remaining buyer has valuation lower than p an instant later, she will immediately change her price offer. Hence we can't have $G_S(t) = c, \forall t$ for some $0 \le c < 1-\mu$.

Now suppose $G_S(t) = 1-\mu, \forall t$. Then $\tau(v) = \lim_{t\searrow 0} t, \forall v \ge p$ and $\tau(v) = \infty, \forall v < p$. If the normal type waits an instant, then her expected payoff is $(1-F(p))p + F(p)\underline{v} > \underline{v}$ because the buyer believes that the seller is commitment type for sure. Therefore we can't have $G_S(t) = 1-\mu, \forall t$.

2) Let $t_S^* = \inf\{t \in \mathcal{R}_+ | G_S(t) = 1-\mu\}$ and $t_B^* = \inf\{t \in \mathcal{R}_+ | G_B(t) = 1-F(p)\}$. Then $t^* \equiv t_S^* = t_B^*$.

At t_S^*, it's certain that the seller is a commitment type. Therefore the buyer has no reason to wait as long as his valuation is higher than p. Also at t_B^*, it's known that the buyer has valuation lower than p. Hence the normal type seller waits no longer.

3) If G_S has jump at t, then G_B should not have jump at t. Similarly, if G_B has jump at t, then G_S should not have jump at t.

If G_S has jump at t, then all buyers get strictly prefer waiting an instant more to conceding at t. Hence G_B can't have jump at t. The similar reasoning applies when G_B has jump.

4) There doesn't exist (t_1, t_2) such that $t_2 < t^*$ and both $G_S(t)$ and $G_B(t)$ are constant on (t_1, t_2).

Suppose not. Without loss of generality, suppose t_2 is the supremum of t_2 for which (t_1, t_2) satisfies the above property. Suppose G_S has no jump at t_2 and fix $t \in (t_1, t_2)$. Then by definition u_v is continuous at t_2 and

$$u_v(t) - u_v(t_2) = (1 - G_S(t))(v - p)(e^{-rt} - e^{-rt_2}) > 0, v > p$$

Hence there exists $\eta > 0$ such that $G_B(t) = G_B(t')$ for all $t' \in (t_2, t_2 + \eta)$. Then for $t \in (t_1, t_2)$ and $t' \in (t_2, t_2 + \eta)$, $u_0(t) - u_0(t') = \underline{v}(1 - G_B(t))(e^{-rt} - e^{-rt'}) > 0$, $\forall v > p$, which implies $G_S(t) = G_S(t')$. This contradicts the assumption that t_2 is the supremum.

Now suppose G_S has jump at t_2. Then by (3), G_B has no jump at t_2, which in turn implies that u_0 is continuous at t_2. Fix $t \in (t_1, t_2)$ and then

$$u_0(t) - u_0(t_2) = \underline{v}(1 - G_B(t))(e^{-rt} - e^{-rt_2}) > 0$$

The continuity of u_0 at t_2 implies that there exists $\eta > 0$ such that $u_0(t) = u_0(t')$ for all $t' \in (t_2, t_2 + \eta)$. Hence $G_S(t) = G_S(t')$. This contradicts the supposition that G_S has jump at t_2.

5) If $t < t' < t^*$, then $G_S(t) < G_S(t')$ and $G_B(t) < G_B(t')$

From the proof of 4), we know that if G_S is constant on some interval, then G_B is also constant on the same interval and vice versa, which can't be the case by 4).

6) Both G_S and G_B are continuous at $t > 0$.

If $G_S(G_B)$ has a jump at t, then $G_B(G_S)$ should be constant on $(t - \epsilon, t)$ for small enough $\epsilon > 0$. By 5), it can't be the case.

7) $G_B(t) = 1 - c_B \exp\{-\frac{\underline{v}}{p - \underline{v}} rt\}$ for some $c_B \in (0, 1]$ and $t^* < \infty$.

From the above results, both G_S and G_B should have positive density on $(0, t^*)$ that we will denote by g_S and g_B respectively. Also we know the normal type seller should be indifferent between conceding at all times $t < t^*$. Now

$$u_0(t) = p \int_0^t e^{-rs} dG_B + \underline{v}(1 - G_B(t))e^{-rt}$$

Since $du_0(t)/dt = 0$, for $t < t^*$,

$$0 = pe^{-rt} g_B(t) - \underline{v} g_B(t) e^{-rt} - \underline{v} r(1 - G_B(t))e^{-rt}$$

which implies

$$\frac{g_B(t)}{1 - G_B(t)} = \frac{\underline{v} r}{p - \underline{v}}$$

Hence

$$G_B(t) = 1 - c_B \exp\{-\frac{\underline{v}}{p - \underline{v}} rt\}$$

for some $c_B \in (0, 1]$. From the functional form of G_B, we know that $t^* < \infty$.

8) $\tau(v) = -p/(\underline{v} r) \ln{(F(v)/c_B)}$ for $v \in (p, \overline{v}]$ such that $F(v) \le c_B$ and $\tau(v) = 0$ for $v \in (p, \overline{v}]$ such that $F(v) > c_B$.

Given G_S, for all $v \geq p$, we should have

$$\tau(v) \in \arg\max_{0\leq t\leq t^*} \int_0^t e^{-rs}(v-\underline{v})dG_S + e^{-rt}(1-G_S(t))(v-p)$$

Hence for $\tau(v) \in (0, t^*)$,

$$\frac{g_S(\tau(v))}{(1-G_S(\tau(v)))} = \frac{r(v-p)}{(p-\underline{v})}$$

Observe that for $v \in (p, 2p-\underline{v})$, $g_S(\tau(v))/(1-G_S(\tau(v)) < r$. Suppose τ is strictly increasing $[v_1, v_2] \in (p, \overline{v}]$. Letting $\tau(v_1) = t_1$ and $\tau(v) = t$ for $v \in (v_1, v_2)$, we should have

$$(v-v_1)\int_t^{t_1} e^{-rt}dG_S + (v-v_1)[e^{-r_1 t}(1-G_S(t_1)) - e^{-rt}(1-G_S(t))] \leq 0$$

Since $(v-v_1) > 0$, for all $v > v_1$,

$$\int_t^{t_1} e^{-rt}dG_S + [e^{-r_1 t}(1-G_S(t_1)) - e^{-rt}(1-G_S(t))] \leq 0$$

$$\begin{aligned}
& \lim_{t\to t_1} \frac{1}{t-t_1}\int_t^{t_1} e^{-rt}dG_S + \frac{1}{t-t_1}[e^{-r_1 t}(1-G_S(t_1)) - e^{-rt}(1-G_S(t))] \\
= \quad & -e^{-rt_1}g_S(t_1) + re^{-rt_1}(1-G_S(t_1) \leq 0 \\
\Leftrightarrow \quad & \frac{g_S(t_1)}{1-G_S(t_1)} \geq r
\end{aligned}$$

Hence for $v \in (p, 2p-\underline{v})$, τ cannot be strictly increasing. Furthermore, it should be strictly decreasing, because G_B can't have a mass except at $t = 0$. Now suppose $\tau(\cdot)$ is strictly increasing at some interval. Let v be the infimum value from which $\tau(\cdot)$ is strictly increasing. Since $\tau(\cdot)$ is strictly decreasing at least on $(p, 2p-\underline{v})$, there should exist v' and v'' such that $v' < v < v''$ and $\tau(v') = \tau(v'')$. However, this is a contradiction because

$$\frac{r(v''-p)}{(p-\underline{v})} > \frac{r(v'-p)}{(p-\underline{v})} = \frac{g_S(\tau(v'))}{(1-G_S(\tau(v'))} = \frac{g_S(\tau(v''))}{(1-G_S(\tau(v''))} = \frac{r(v''-p)}{(p-\underline{v})}$$

Therefore $\tau(\cdot)$ should be strictly decreasing when positive and $v > p$. From now on, denote the inverse of τ by τ^{-1} on the domain $(0, t^*)$. Since $G_B(t) = 1 - c_B\exp\{-\frac{\underline{v}}{p-\underline{v}}rt\} = 1 - F(\tau^{-1}(t))$ for $t \in (0, t^*)$,

$$\tau^{-1}(t) = F^{-1}\left(c_B \exp\{-\frac{\underline{v}}{p-\underline{v}}rt\}\right) \Leftrightarrow \tau(v) = -\frac{p-\underline{v}}{\underline{v}r}\ln\left(\frac{F(t)}{c_B}\right)$$

9) For some $c_S \in (0, 1]$,

$$G_S(t) = 1 - c_S\exp\left(-\frac{1}{\underline{v}}\int_{F^{-1}\left(c_B\exp\{-\frac{\underline{v}}{p}rt\}\right)}^{F^{-1}(c_B)} \frac{(v-p)f(v)}{F(v)}dv\right)$$

Now we know for $t \in (0, t^*)$,

$$g_S(t)(p - \underline{v}) = r(1 - G_S(t))(\tau^{-1}(t) - p)$$

The solution to this problem is

$$G_S(t) = 1 - c_S \exp\left(-\int_0^t \frac{r(\tau^{-1}(s) - p)}{p - \underline{v}} ds\right)$$

for some c_S. Replacing $\tau^{-1}(s)$ by $F^{-1}\left(c_B \exp\{-\frac{\underline{v}}{p-\underline{v}} rs\}\right)$ and rearranging, we get

$$G_S(t) = 1 - c_S \exp\left(-\frac{1}{\underline{v}} \int_{F^{-1}\left(c_B \exp\{-\frac{\underline{v}}{p} rt\}\right)}^{F^{-1}(c_B)} \frac{(v - p)f(v)}{F(v)} dv\right)$$

10) Since $G_S(t^*) = 1 - \mu$,

$$c_S = \mu \exp\left(\frac{1}{\underline{v}} \int_p^{F^{-1}(c_B)} \frac{(v - p)f(v)}{F(v)} dv\right)$$

11) By 3), either $c_B = 1$ or $c_S = 1$. If $c_B = 1$, then

$$\begin{aligned} t^* &= -\frac{p - \underline{v}}{\underline{v} r} \ln(F(p)) \\ c_S &= \mu \exp\left(\frac{1}{\underline{v}} \int_p^{\overline{v}} \frac{(v - p)f(v)}{F(v)} dv\right) \end{aligned}$$

If $c_S = 1$, then

$$\begin{aligned} \frac{1}{\underline{v}} \int_p^{F^{-1}(c_B)} \frac{(v - p)f(v)}{F(v)} dv + \ln \mu &= 0 \\ t^* &= -\frac{p - \underline{v}}{\underline{v} r} \ln\left(\frac{F(p)}{c_B}\right) \end{aligned}$$

Notice that $c_B = c_S = 1$ if $\phi(F, p) = \mu$. Hence $c_B = 1$ and $c_S < 1$ if $\phi(F, p) > \mu$, and $c_B < 1$ and $c_S = 1$ if $\phi(F, p) > \mu$. **Q.E.D.**

Proof of Lemma 8
Let

$$\begin{aligned} K(p, \mu) &= \frac{\partial U_1(p, \mu)}{\partial p} \frac{\partial U_0(p, \mu)}{\partial \mu} - \frac{\partial U_1(p, \mu)}{\partial \mu} \frac{\partial U_0(p, \mu)}{\partial p} \\ &= \left(\frac{F(p)}{c_B}\right)^{p/\underline{v}} \frac{\partial c_B}{\partial \mu}(p - \underline{v}) \left(c_B \left(1 + \ln\left(\frac{F(p)}{c_B}\right) + p\frac{f(p)}{F(p)}\right) - 1\right) \end{aligned}$$

$K(p,\mu)=0$ if and only if

$$k(p,\mu)=c_B\left(1+\ln\left(\frac{F(p)}{c_B}\right)+p\frac{f(p)}{F(p)}\right)-1$$

Since $c_B<1$ when $p>\underline{v}$ and $\mu<1$, for $(p,\mu)\in W$, we should have

$$\ln\left(\frac{F(p)}{c_B}\right)+p\frac{f(p)}{F(p)}>0$$

Moreover, on (p,μ) which satisfies the above inequality, for fixed $p>\underline{v}$,

$$\frac{\partial k(p,\mu)}{\partial\mu}=\frac{\partial c_B}{\partial\mu}\left(\ln\left(\frac{F(p)}{c_B}\right)+p\frac{f(p)}{F(p)}\right)<0$$

These imply that for each p, if exists, there is a unique μ such that $(p,\mu)\in W$. Now notice that

$$\lim_{\mu\to 1}k(p,\mu)=pf(p)-(1-F(p))\left\{\begin{array}{c}<\\=\\>\end{array}\right\}0\text{ if }p\left\{\begin{array}{c}<\\=\\>\end{array}\right\}p_1^*(1)$$

Therefore only for $p\le p_1^*(1)$, we may have μ^γ at which U_0 and U_1 are tangent. $p\le p_1^*(1)$ is also sufficient because letting $\mu^p\in(\phi(F,p),1)$ be the value which makes $p=p_1^*(\mu^p)$, $k(p,\mu^p)>0$. Hence there exists such a unique μ^γ if and ONLY IF $p\le p_1^*(1)$. The second statement follows from $K(p,\mu)<0$ if $\mu<\mu^\gamma$ and $K(p,\mu)>0$ if $\mu>\mu^\gamma$ for all $p\in(\underline{v},p_1^*(1))$.

The fact that $p'<p_1^*(\mu^\gamma)$ simply comes from the fact that as soon as $p>p_1^*(\mu)$,

$$K(p,\mu)<0\Rightarrow p\frac{f(p)}{F(p)}+\ln F(p)>\frac{1-c_B(p,\mu)}{c_B(p,\mu)}+\ln c_B(p,\mu)$$

As $p'\to p_1^*(1)$, the left-hand side converges to $(1-F(p)/F(p)+\ln F(p)$, so $c_B(p',\mu^\gamma(p'))\to F(p_1^*(1))$, which happens only when $\mu^\gamma(p')\to 1$. When $p'\to\underline{v}$, the left-hand side becomes arbitrarily large, which can be matched only when $c_B(p',\mu)\to 0$, which occurs only when $\mu\to 1$, combined with $F(p')\to 0$ (We can't arbitrarily lower μ, because $p'<p_1^*(\mu^\gamma)$ implies that $\mu^\gamma>\phi(F)$). **Q.E.D.**

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