ELSEVIER

Journal of Mathematical Economics 38 (2002) 393–410

JOURNAL OF Mathematical ECONOMICS

www.elsevier.com/locate/jmateco

Arbitrage and price revelation with asymmetric information and incomplete markets

Bernard Cornet [a,*], Lionel De Boisdeffre [a,b]

[a] *CERMSEM, Université de Paris 1, 106–112 boulevard de l'Hôpital, 75647 Paris Cedex 13, France*
[b] *Cambridge University, Cambridge, UK*

Received 5 January 2002; received in revised form 7 September 2002; accepted 10 September 2002

Abstract

This paper deals with the issue of arbitrage with differential information and incomplete financial markets, with a focus on information that no-arbitrage asset prices can reveal. Time and uncertainty are represented by two periods and a finite set S of states of nature, one of which will prevail at the second period. Agents may operate limited financial transfers across periods and states via finitely many nominal assets. Each agent i has a private information about which state will prevail at the second period; this information is represented by a subset S_i of S. Agents receive no wrong information in the sense that the "true state" belongs to the "pooled information" set $\cap_i S_i$, hence assumed to be non-empty.

Our analysis is two-fold. We first extend the classical symmetric information analysis to the asymmetric setting, via a concept of no-arbitrage price. Second, we study how such no-arbitrage prices convey information to agents in a decentralized way. The main difference between the symmetric and the asymmetric settings stems from the fact that a classical no-arbitrage asset price (common to every agent) always exists in the first case, but no longer in the asymmetric one, thus allowing arbitrage opportunities. This is the main reason why agents may need to refine their information up to an information structure which precludes arbitrage.

Keywords: Asymmetric information; Private information; No-arbitrage; Incomplete markets; Refinement; Information revealed by prices

1. Introduction

In economies subject to uncertainty and asymmetric information, agents seek to infer relevant information from market indicators, such as prices, to refine their strategies. This

* Corresponding author. Tel.: +33-1-55-43-4282; fax: +33-1-55-43-4301.
E-mail addresses: cornet@univ-paris1.fr (B. Cornet), lionel.de.boisdeffre@wanadoo.fr (L. De Boisdeffre).

0304-4068/02/$ – see front matter © 2002 Elsevier Science B.V. All rights reserved.
PII: S0304-4068(02)00075-7

issue is traditionally tackled by the so-called "rational expectations" models by assuming, quoting Radner (1979), that "agents have a "model" or "expectations" of how equilibrium prices are determined."

In this paper, agents learn from asset prices about partners' private information, by analyzing the arbitrage opportunities of the financial markets. They need not know the ex ante characteristics of the economy (preferences, endowments of other agents) or a defined relationship between prices and the collection of private information signals in the economy, as in the rational expectations' models. Thus, they are only required to know the market prices and their own characteristics. We define a notion of equilibrium, embedding the way agents infer information from asset prices, and its properties will be presented in a companion paper. For this purpose, however, we need first to study arbitrage theory with asymmetric information, which is the main aim of the present paper.

We consider the simplest tractable setting for the study of arbitrage. Time and uncertainty are represented by two periods ($t = 0$ and $t = 1$) and a finite set S of states of nature, one of which will prevail at the second period. Agents may operate limited financial transfers across periods and states via finitely many nominal assets. Each consumer receives a private information signal about which state will prevail at the second period. Asymmetric information is thus represented, for each agent i, by a subset S_i of S. Agents receive no wrong information in the sense that the "true state" belongs to the "pooled information" set $\cap_i S_i$, hence assumed to be non-empty. Similarly, when agents refine their information, i.e. when they infer a smaller set $\Sigma_i \subset S_i$, they also receive no wrong signal, so that $\cap_i \Sigma_i \neq \emptyset$. This is guaranteed, in particular, when the refinement (Σ_i) of the collection (S_i) preserves its pooled information, that is, $\cap_i S_i = \cap_i \Sigma_i$.

Our analysis is two-fold. We first extend the classical non-arbitrage analysis to the asymmetric setting, via a concept of no-arbitrage price, and second, we study how such no-arbitrage prices convey information to agents in a decentralized way. The main difference between the symmetric and asymmetric settings stems from the fact that a classical no-arbitrage asset price (common to every agent) always exists in the symmetric case, but no longer in the asymmetric one, thus allowing arbitrage opportunities. This is the main reason why agents may need to refine their information up to an information structure precluding arbitrage.

The paper is organized as follows. In Section 2, we present the framework and recall the basic concepts of arbitrage-free information structures, refinements, and no-arbitrage prices. We also define the notion of financial equilibrium in an asymmetric setting, which explicitly presents consumers' behavior and the need for a refinement of information when it is not arbitrage-free at the outset (Definition 2.2). In Section 3, we characterize arbitrage-free structures by the absence of future (i.e. at $t = 1$) arbitrage opportunities on the financial market, called the AFAO property (Proposition 3.1). We show that every information structure (S_i) has a unique coarsest arbitrage-free refinement, denoted by $(\bar{\boldsymbol{S}}_i)$, which does not contain any wrong signals since $\cap_i S_i = \cap_i \bar{\boldsymbol{S}}_i$ (Proposition 3.2). We end the section with the relationship between "fully-revealing" structures, i.e. such that $\bar{\boldsymbol{S}}_i$ coincides with agents' pooled information, market completeness (Proposition 3.3) and symmetric information (Proposition 3.4). In Section 4, we first define, for every agent i and every asset price q, the "revealed information set" $\boldsymbol{S}_i(q) \subset S_i$ (Definition 4.1). We then define an extended notion of no-arbitrage asset price (Definition 4.2), as the common no-arbitrage prices associated to all arbitrage-free refinements. This allows us to characterize no-arbitrage prices q as those which "reveal" an information structure, i.e. such that $\cap_i \boldsymbol{S}_i(q) \neq \emptyset$ (Definition 4.2 and Proposition 4.2). Finally, we show that the coarsest arbitrage-free refinement $(\bar{\boldsymbol{S}}_i)$ can always be revealed by some no-arbitrage price q, that is, $\bar{\boldsymbol{S}}_i = \boldsymbol{S}_i(q)$ for every i (Proposition 4.3).

Some conclusions may be drawn in terms of the financial equilibrium notion presented in the paper (Definition 2.3). When the initial information structure (S_i) is arbitrage-free, consumers may keep their initial information sets. Otherwise, they must refine their beliefs up to an arbitrage-free information structure, to be able to perform their maximization problem (Proposition 2.2). The coarsest arbitrage-free refinement $(\bar{\boldsymbol{S}}_i)$ allows to do it in such a way that it can always be revealed by some price q and it does not contain any wrong signal since $\cap_i S_i = \cap_i \bar{\boldsymbol{S}}_i$. Hence, agents may always update their beliefs through prices in a decentralized way: neither the presence of another agent, nor the knowledge of the other agents' characteristics is required.[1]

2. The model

2.1. The two-period model and financial markets

We consider the basic model of a two time-period economy with private information, and nominal assets: the simplest tractable model which allows us to present arbitrage. It is also assumed that there are finite sets I, S, and J, respectively, of agents, states of nature, and nominal assets.

In what follows, the first period will also be referred to as $t = 0$ and the second period, as $t = 1$. There is an a priori uncertainty at the first period $(t = 0)$ about which of the states of nature $s \in S$ will prevail at the second period $(t = 1)$. For the sake of unified notations of time and uncertainty, the non-random state at the first period is denoted by $s = 0$ and S' stands for the set $\{0\} \cup S$. Similarly, if $\Sigma \subset S$, Σ' will stand for $\{0\} \cup \Sigma$. Each agent $i \in I$ has a private information at the first period about the possible states of nature of the second period, that is, she knows that the true state will be in a subset S_i of S, or, equivalently, that the true state will not belong to the complementary set (in S) of S_i. Agents receive no wrong information in the sense that the "true state" belongs to the "pooled information" set $\cap_i S_i$, hence assumed to be non-empty throughout the paper.

[1] We shall use hereafter the following notations. If I and J are finite sets, the space $\mathbb{R}^I$ (identified to $\mathbb{R}^{\#I}$ whenever necessary) of functions $x : I \to \mathbb{R}$ (also denoted $x = (x(i))_{i\in I}$ or $x = (x_i)$) is endowed with the Euclidean product $x \cdot y := \sum_{i\in I} x(i)y(i)$, and we denote by $\|x\| := \sqrt{x \cdot x}$ the Euclidean norm. In $\mathbb{R}^I$, the notation $x \geq y$ (respectively $x \gg y$) means that $x(i) \geq y(i)$ (respectively $x(i) > y(i)$) for every i and we let $\mathbb{R}^I_+ = \{x \in \mathbb{R}^L | x \geq 0\}$, $\mathbb{R}^I_{++} = \{x \in \mathbb{R}^L | x \gg 0\}$. An $I \times J$-matrix $A = (a_i^j)_{i\in I, j\in J}$ (identified with a classical $(\#I) \times (\#J)$-matrix if necessary) is an element of $\mathbb{R}^{I\times J}$ whose rows are denoted $A[i]$ for $(a_i^j)_{j\in J} \in \mathbb{R}^J$ $(i \in I)$, and columns $A^j = (a_i^j)_{i\in I} \in \mathbb{R}^I$ $(j \in J)$. To the matrix A, we associate the linear mapping, from $\mathbb{R}^J$ to $\mathbb{R}^I$, also denoted by A, defined by $Ax = (A_{[i]} \cdot x)_{i\in I}$. The span of the matrix A, also called the image of A, is the set $\langle A\rangle := \{Ax | x \in \mathbb{R}^J\}$. The transpose matrix of A, denoted by ${}^{\mathrm{t}}A$, is the $J \times I$-matrix whose rows are the columns of A, or equivalently, is the unique linear mapping ${}^{\mathrm{t}}A : \mathbb{R}^I \to \mathbb{R}^J$, satisfying $(Ax) \cdot y = x \cdot ({}^{\mathrm{t}}Ay)$ for every $x \in \mathbb{R}^J$, $y \in \mathbb{R}^I$.

Agents may operate financial transfers across states in S' (i.e. across the two periods and across the states of the second period) by exchanging a finite number of nominal assets $j \in J$, which define the financial structure of the model. The nominal assets are traded at the first period ($t = 0$) and yield payoffs at the second period ($t = 1$), contingent on the realization of the state of nature. The payoff of the nominal asset $j \in J$, when state $s \in S$ is realized, is V_s^j, and we denote by V the $S \times J$-return matrix $V = (V_s^j)$, which does not depend upon the asset prices $q \in \mathbb{R}^J$ (and will not either depend upon the commodity prices p in the associated equilibrium model). A portfolio $z = (z_j) \in \mathbb{R}^J$ specifies the quantities $|z_j|$ ($j \in J$) of each asset j (with the convention that it is bought if $z_j > 0$ and sold if $z_j < 0$), thus Vz is its random financial return across states at time $t = 1$, and $V[s] \cdot z$ its return if state s prevails.

We summarize by $[(I, S, J), V, (S_i)_{i\in I}]$ the financial and information characteristics, referred to as the financial and information structure, or simply the structure. This structure, which is fixed throughout the paper, is sufficient to present the arbitrage theory of the paper, with only one exception, in Section 2.4, when we shall introduce the notion of no-arbitrage equilibrium. A real sector with spot markets for commodities will then be added together with preferences relation for the agents (then called consumers).

2.2. Information structures and refinements

At the first period, each agent $i \in I$ has some private information $S_i \subset S$ about which states of the world may occur at the next period. Either the private information S_i is kept by the agent i, or she can infer that the true state will be in a smaller set $\Sigma_i \subset S_i$. In the latter case, agents are assumed to receive no wrong information signal, that is, the true state always belongs to the set $\cap_{i\in I}\Sigma_i$, hence assumed to be non-empty. A collection $(\Sigma_i)_{i\in I}$ such that $\cap_{i\in I}\Sigma_i \neq \emptyset$ is called an *information structure* and (Σ_i) is said to be a *refinement* of (S_i). The following definition presents the basic notions on information structures used in the paper.

Definition 2.1. Let S be a given finite set. A collection $(\Sigma_i)_{i\in I}$ of subsets $\Sigma_i \subset S$, also denoted by (Σ_i), is said to be an information structure if $\cap_{i\in I}\Sigma_i \neq \emptyset$. It is said to be symmetric if all the Σ_i are equal.

The order relation $(\Sigma_i^1) \leq (\Sigma_i^2)$ on the set of information structures is defined by $\Sigma_i^1 \subset \Sigma_i^2$ for every i, and we say indifferently that (Σ_i^2) is coarser than (Σ_i^1), (Σ_i^1) is finer than (Σ_i^2), or (Σ_i^1) is a refinement of (Σ_i^2).

The non-empty subset $\cap_{j\in I}\Sigma_j$ is called the pooled information of the information structure (Σ_i); it is obtained by the agents when they decide to share their private information. The pooled refinement of (Σ_i) is then the symmetric information structure $(\boldsymbol{\Sigma}_i)$ defined by $\boldsymbol{\Sigma}_i := \cap_{j\in I}\Sigma_j$ for every i.

A refinement (Σ_i) of (S_i) is said to be self-attainable if it is coarser than the pooled refinement $(\mathbf{S}_i)$, i.e. if $\cap_{j\in I}S_j \subset \Sigma_i$ for every i, which is equivalent to $\cap_{j\in I}S_j = \cap_{j\in I}\Sigma_j$.

The above definitions need no special comment, apart from the notion of “self-attainable” refinement, which refers to the idea that it is performed endogenously, without the help of an information source outside the given set I of agents (auctioneer, ...); hence, it is coarser than what the agents can get by pooling their information. This is illustrated by the following example, which is encountered in contract or insurance models, where agents have a private knowledge regarding their own risk.

Remark 1. Consider an economy where the random state of nature $s = (s_0, (s_i)_{i\in I})$ is the product of a macro-economic component $s_0 \in \Sigma_0$, whose probability distribution is known and common to all agents, and of components $s_i \in \Sigma_i$, representing the individual risk of agent i ($i \in I$), whose realization $\bar{s}_i$ is known by each agent i (and by no other) at the first period ($t = 0$) and is revealed to the other agents at the second period ($t = 1$) (see, for example, Bisin and Gottardi, 1999). In that case, the total information set S and the private information sets S_i ($i \in I$) are

$$S := \Sigma_0 \times \Pi_{j\in I}\Sigma_j,$$

$$S_i := \{s = (s_0, (s_j)_{j\in I}) \in \Sigma_0 \times \Pi_{j\in I}\Sigma_j | s_i = \bar{s}_i\},$$

and one checks that

$$\cap_{i\in I} S_i = \Sigma_0 \times \Pi_{i\in I}\{\bar{s}_i\} \neq \emptyset.$$

2.3. The classical concept of no-arbitrage price

We recall the following standard definitions.

Definition 2.2. Given the return matrix V and a non-empty set $\Sigma \subset S$, the price $q \in \mathbb{R}^J$ is said to be a no-arbitrage price of the couple (V, Σ), or the couple (V, Σ) is said to be q-arbitrage-free, if one of the following equivalent assertions is satisfied:

(i) there is no portfolio $z \in \mathbb{R}^J$ such that $-q \cdot z \geq 0$ and $V[s] \cdot z \geq 0$ for every $s \in \Sigma$, with at least one strict inequality;
(ii) there exists $\lambda = (\lambda(s)) \in \mathbb{R}^{\Sigma}_{++}$, such that $q = \sum_{s\in\Sigma} \lambda(s)V[s]$.

We denote by $Q[V, \Sigma]$ the set of no-arbitrage prices associated to (V, Σ). By convention, when Σ is empty, we shall also say that the couple $(V, \emptyset)$ is q-arbitrage-free for every $q \in \mathbb{R}^J$, that is, we let $Q[V, \emptyset] = \mathbb{R}^J$.

Given the return matrix V and an information structure (Σ_i), the price $q \in \mathbb{R}^J$ is said to be a no-arbitrage price for agent i if it is a no-arbitrage price of the couple (V, Σ_i), that is, $q \in Q[V, \Sigma_i]$. The price $q \in \mathbb{R}^J$ is said to be a common no-arbitrage price of the structure $[V, (\Sigma_i)]$ if it is a no-arbitrage price for every agent $i \in I$ and we denote by $Q_c[V, (\Sigma_i)] := \cap_i Q[V, \Sigma_i]$ the set of common no-arbitrage prices. The structure $[V, (\Sigma_i)]$ is said to be arbitrage-free (respectively q-arbitrage-free) if it admits a common no-arbitrage price, that is, if $Q_c[V, (\Sigma_i)] \neq \emptyset$ (respectively $q \in Q_c[V, (\Sigma_i)]$).

When no confusion is possible, the reference to V will be omitted and we shall simply use the terms of arbitrage-free, q-arbitrage-free information structure, refinement or information set. We denote by $\mathcal{S}$ the set of arbitrage-free refinements (Σ_i) of (S_i), and by $\mathcal{S}(q)$ the set of q-arbitrage-free refinements (Σ_i) of (S_i).

The equivalence between the two assertions (i) and (ii) is standard and relies on the following version of Farkas' lemma (Lemma 1), letting $W := W(q, V, \Sigma)$ be the $\Sigma' \times J$-matrix, defined by $W(q, V, \Sigma)[0] = -q$, and $W(q, V, \Sigma)[s] = V[s]$ for every $s \in \Sigma$. We refer, for example, to Magill and Quinzii (1996) for the proof of the lemma, which is extensively used in arbitrage theory and will also be needed hereafter.

Lemma 1. *Let W be a $\Sigma' \times J$-matrix, then the following conditions are equivalent*:

(i) $\langle W \rangle \cap \mathbb{R}_+^{\Sigma'} = \{0\}$;
(ii) $\exists \lambda \in \mathbb{R}_{++}^{\Sigma'}, {}^{t}W\lambda = 0$;
(ii′) $\exists \lambda = (\lambda(s)) \in \mathbb{R}_{++}^{\Sigma'}, \sum_{s \in \Sigma'} \lambda(s) W[s] = 0$.

We state a simple but important result on symmetric structures which does not hold, in general, in the asymmetric setting (see the example below).

Proposition 2.1. *Let $[V, (S_i)]$ be a structure such that (S_i) is symmetric, then it is arbitrage-free, that is, the set $Q_c[V, (S_i)]$ is non-empty.*

Proof. It is a direct consequence of condition (ii) of Definition 2.1, which implies that, for every $\lambda = (\lambda(s)) \in \mathbb{R}_{++}^{S_1}$, $q := \sum_{s \in S_1} \lambda(s) V[s]$ belongs to $Q_c[V, (S_i)]$. Hence, the $\lambda(s)$ $(s \in S_1)$ need not depend on agent i. □

We now give an example, which will be used throughout the paper.

Example. Consider two agents ($I = \{1, 2\}$), five states ($S = \{1, 2, 3, 4, 5\}$), private information sets $S_1 = \{1, 2, 3\}$, $S_2 = \{1, 4, 5\}$, and the payoff matrix:

$$V = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- The structure $[V, S_1, S_2]$ is not arbitrage-free, i.e. $Q_c[V, S_1, S_2] = \emptyset$. Otherwise, there would exist some $q \in \mathbb{R}^3$, $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}_{++}^3$, $(\mu_1, \mu_4, \mu_5) \in \mathbb{R}_{++}^3$ such that $q = \lambda_1(-1, 0, 0) + \lambda_2(1, 1, 0) + \lambda_3(0, 0, 1) = \mu_1(-1, 0, 0) + \mu_4(0, 1, 0) + \mu_5(0, 0, 0)$. Hence, $q_3 = \lambda_3 = 0$, a contradiction with $\lambda_3 > 0$.

  The following structures are arbitrage-free refinements of (S_1, S_2):
- the pooled refinement: $\underline{S}_1 = \underline{S}_2 = \{1\}$, and $\underline{q} = (-1, 0, 0) \in Q_c[V, \underline{S}_1, \underline{S}_2]$,
- (Σ_1, Σ_2), defined by $\Sigma_1 = \{1\}$, $\Sigma_2 = \{1, 5\}$, and $\underline{q} = (-1, 0, 0) \in Q_c[V, \Sigma_1, \Sigma_2]$,
- $(\bar{S}_1, \bar{S}_2)$ defined by $\bar{S}_1 = \{1, 2\}$, $\bar{S}_2 = \{1, 4, 5\}$, and $\bar{q} = (-1, 1, 0) \in Q_c[V, \bar{S}_1, \bar{S}_2]$.

2.4. No-arbitrage financial equilibria in an asymmetric setting

We consider a two time-period finite pure exchange economy with private information, where time, uncertainty, and the financial structure are defined as previously. In addition, we now assume that there is a finite set, H, of commodities, which are available on spot markets at each period ($t \in \{0, 1\}$). For a better understanding, we specify interim periods as follows. Each consumer i receives the private information set $S_i \subset S$ at a first fictitious interim period $t = \epsilon$ ($\epsilon > 0$ arbitrarily close to zero) and this information may be refined to a smaller set $\Sigma_i \subset S_i$ at a second fictitious interim period $t = 2\epsilon$. So we assume that ex ante, i.e. at time $t = 0$, the consumption sets, endowments and preferences are defined conditionally to the information set $\Sigma_i \subset S$ that each agent may infer.

Formally, for every $\Sigma \subset S$ (recalling that $\Sigma' := \{0\} \cup \Sigma$), every consumer $i \in I$ is endowed with a conditional consumption set $X_i(\Sigma) := (\mathbb{R}^H_+)^{\Sigma'}$, a utility function $u_i(\cdot|\Sigma) : (\mathbb{R}^H_+)^{\Sigma'} \to \mathbb{R}$, and an endowment $e_i(\Sigma) \in (\mathbb{R}^H_+)^{\Sigma'}$. To illustrate this model we consider hereafter the particular case of Von Neumann–Morgenstern utility functions. A fixed utility index $v_i : (\mathbb{R}^H_+)^2 \to \mathbb{R}$ is given and we denote by $p_i(s|\Sigma)$ the subjective probability that agent i assigns to the realization of state $s \in S$, conditionally on the event $s \in \Sigma$. Then, the conditional utility is defined as follows:

$$u_i(x|\Sigma) = \sum_{s\in\Sigma} p_i(s|\Sigma)v_i(x(0), x(s)) \quad \text{for every } \Sigma \subset S \quad \text{and} \quad x \in (\mathbb{R}^H_+)^{\Sigma'}.$$

The economy that we have described can thus be summarized by the collection:

$$\mathcal{E} = [(I, H, S, J), V, (S_i, X_i, u_i, e_i)_{i\in I}].$$

Given her initial private information set $S_i \subset S$, consumer i may need to "infer" a better information set $\Sigma_i \subset S_i$, before maximizing her utility under her budget constraint, as explained below. For given commodity prices $p = (p(s)) \in (\mathbb{R}^H)^{S'}$ and asset prices $q \in \mathbb{R}^J$, agent i will then maximize her utility (for the known information set Σ_i) $u_i(\cdot|\Sigma_i)$ in her budget set $B_i(p, q, V, \Sigma_i)$, defined as follows:

$$\begin{aligned} B_i(p, q, V, \Sigma_i) :=\{(x_i, z_i) \in (\mathbb{R}^H_+)^{\Sigma_i'} \times \mathbb{R}^J | p(0) \cdot [x_i(0) - e_i(0)] \\ \le -q \cdot z_i, \quad \forall s \in \Sigma_i, p(s) \cdot [x_i(s) - e_i(\Sigma_i)(s)] \le V[s] \cdot z_i\}. \end{aligned}$$

We recall that $\mathcal{S}$ denotes the set of arbitrage-free refinements of (S_i).

Definition 2.3. A no-arbitrage financial equilibrium of the economy $\mathcal{E}$ is a collection $((S_i^*), (x_i^*), (z_i^*), p^*, q^*)$ in $\mathcal{S} \times \prod_{i\in I}(\mathbb{R}^H_+)^{S_i^{*'}} \times (\mathbb{R}^J)^I \times (\mathbb{R}^H_+)^{S'} \times \mathbb{R}^J$ such that:

(a) $q^* \in Q_c[V, (S_i^*)]$ and (S_i^*) is self-attainable, that is, $\cap_{i\in I} S_i = \cap_{i\in I} S_i^*$;
(b) for every $i \in I$, (x_i^*, z_i^*) maximizes the utility $u_i(\cdot|S_i^*)$ in the budget set $B_i(p, q, V, S_i^*)$;
(c) $\sum_{i\in I} x_i^*(s) = \sum_{i\in I} e_i(S_i^*)(s)$ for every $s \in \cap_{i\in I} S_i^{*'}$;
(d) $\sum_{i\in I} z_i^* = 0$.

The above definition coincides with the standard notion of a no-arbitrage financial equilibrium (see, for example, Magill and Quinzii, 1996) when $S_i = S$ for every i, that is, agents have no private information at time $t = \varepsilon$. Indeed, the only self-attainable refinement of (S_i) is itself. However, this is no longer the case, in general, in an asymmetric setting, and the example in Section 2.3 has exhibited several arbitrage-free self-attainable refinements of (S_i). Thus, the above definition of equilibrium needs to be further refined to specify which refinements (S_i^*) agents might infer and how they will be attained.

The purpose of this paper is to propose refinements (S_i^*) that agents can implement at equilibrium in a decentralized way through prices. Section 3 shows that there exists a unique coarsest arbitrage-free refinement of (S_i), denoted by $(\bar{S}_i)$, which is self-attainable (Proposition 3.2). Section 4 will show that $(\bar{S}_i)$ can be "revealed" by some asset price $\bar{q}$ and the only knowledge by each agent of her own characteristics. By observing the given asset price $\bar{q}$, each agent will infer the information set $\bar{S}_i$ as the outcome of a rational behavior consisting in inferring the largest q-arbitrage-free subset of S_i (Definition 4.1). Implementing this behavior, referred to as the "no-arbitrage principle", does not require any representation of how equilibrium prices are determined, as in REE models, or any particular knowledge of the ex ante characteristics of the economy (e.g. endowments, preferences of other agents). Only the observation of the asset price and the knowledge of her own characteristics are required. Section 4 will also discuss the existence of other refinements than $(\bar{S}_i)$ that can be "revealed" by prices.

We further point out that the equilibrium condition (a) requires that the revealed information structure (S_i^*) be self-attainable, that is, $\cap_{i\in I} S_i = \cap_{i\in I} S_i^*$. Hence, the assumption that the initial information sets S_i $(i \in I)$ convey no wrong information (i.e. always contain the state that will prevail at the second period) insures that agents will make no wrong inference at equilibrium.

We end this section by a proposition, showing that the arbitrage-free requirement on the equilibrium refinement (S_i^*) is in fact a consequence of the non-satiation of consumers, as for the standard notion of a financial equilibrium when $S_i = S$ for every i (see again Magill and Quinzii, 1996). We first introduce the following assumption.

Assumption (NSS) (Non-satiation of preferences at every state). $\forall i \in I, \forall \Sigma_i \subset S, \forall s_i \in \Sigma_i', \forall x \in (\mathbb{R}_+^H)^{\Sigma_i'}, \exists x' \in (\mathbb{R}_+^H)^{\Sigma_i'}, \forall s \in \Sigma_i' \backslash \{s_i\}, x'(s) = x(s), u_i(x'|\Sigma_i) > u_i(x|\Sigma_i)$.

Proposition 2.2. *Under Assumption (NSS), if, for every agent $i \in I$, the strategy (x_i^*, z_i^*) maximizes the utility $u_i(\cdot|S_i^*)$ in the budget set $B_i(p^*, q^*, V, S_i^*)$, then $q^* \in Q_c[V, (S_i^*)]$.*

Proof (By contraposition). If $q^* \notin Q_c[V, (S_i^*)]$, there exists $i \in I$, and $z \in \mathbb{R}^J$ such that $w(z)[0] := -q^* \cdot z \geq 0$ and $w(z)[s] := V[s] \cdot z \geq 0$, for every $s \in S_i^*$, with at least one strict inequality, say for $s_i \in S_i^{*'} := \{0\} \cup S_i^*$. From Assumption (NSS), there exists $x_i' \in (\mathbb{R}_+^H)^{S_i^{*'}}$ such that $x_i'(s) = x_i^*(s)$ for every $s \in S_i^{*'} \backslash \{s_i\}$ and $u_i(x_i'|S_i^*) > u_i(x_i^*|S_i^*)$. Let $\lambda = |p^*(s_i) \cdot [x_i'(s_i) - x_i^*(s_i)]|/w(z)[s_i]$ and $z_i' = z_i^* + \lambda z$. We let the reader check that $(x_i', z_i') \in B_i(p^*, q^*, V, S_i^*)$. But the conditions $(x_i', z_i') \in B_i(p^*, q^*, V, S_i^*)$ and $u_i(x_i'|S_i^*) > u_i(x_i^*|S_i^*)$ contradict the fact that (x_i^*, z_i^*) maximizes the utility of agent i in her budget set $B_i(p^*, q^*, V, S_i^*)$. □

3. No-arbitrage prices with asymmetric information

3.1. A characterization of arbitrage-free information structures

We provide the following characterization of arbitrage-free structures.

Proposition 3.1. *The structure* $[V, (S_i)]$ *is arbitrage-free if and only if it satisfies the following condition*: (*absence of future arbitrage opportunities, AFAO*) *there is no* $(z_i) \in (\mathbb{R}^J)^I$ *such that* $\sum_{i\in I} z_i = 0$ *and* $V[s_i] \cdot z_i \geq 0$ *for all* $i \in I$ *and all* $s_i \in S_i$, *with at least one strict inequality*.

Condition (AFAO) generalizes to any group of agents a property that could be better understood for bilateral barter of portfolios, namely (absence of bilateral future arbitrage opportunities, ABFAO) there exist no agents i, j in I, and no portfolios z_i, z_j in $\mathbb{R}^J$ satisfying the conditions $z_i + z_j = 0$ and $V[s_i] \cdot z_i \geq 0$ for every $s_i \in S_i$, $V[s_j] \cdot z_j \geq 0$ for every $s_j \in S_j$, with at least one strict inequality.

In other words, no two agents can barter a portfolio, with both getting non-negative returns at the second period ($t = 1$), and one agent having some positive return in a state, which is believed to be possible. We point out that no asset price is involved in this barter and only the second period is concerned, to the difference of the arbitrage-free condition in Definition 2.2. Condition (AFAO) clearly implies condition (ABFAO) but is not equivalent to it, as shown in Remark 2 below.

Proposition 3.1 is also related to the arbitrage cone literature, which shows the same type of equivalence property in a similar but different context, namely between Hart's and Werner's conditions (see Hart, 1975; Werner, 1987 and the ongoing work on this subject).

Proof of Proposition 3.1. Assume that condition (AFAO) holds, and define the linear mapping $W : (\mathbb{R}^J)^I \to \mathbb{R}^J \times \mathbb{R}^J \times \Pi_{i\in I}\mathbb{R}^{S_i}$ by:

$$Wz = \left(\sum_{i\in I} z_i, -\sum_{i\in I} z_i, [(V[s_i] \cdot z_i)_{s_i\in S_i}]_{i\in I}\right) \text{ for } z = (z_i)_{i\in I} \in (\mathbb{R}^J)^I.$$

Then, condition (AFAO) is equivalent to the following:

$$\langle W \rangle \cap [\mathbb{R}^J \times \mathbb{R}^J \times \Pi_{i\in I}\mathbb{R}^{S_i}]_+ = \{0\}. \tag{1}$$

A characterization of condition (1) is given by Lemma 1 and, for this purpose, we let the reader check that the transpose ${}^tW$ of the linear mapping W is the mapping from $\mathbb{R}^J \times \mathbb{R}^J \times \Pi_{i\in I}\mathbb{R}^{S_i}$ to $(\mathbb{R}^J)^I$ defined by:

$${}^tW(\alpha, \beta, (\lambda_i)_{i\in I}) = \left(\alpha - \beta + \sum_{s\in S_i} \lambda_i(s)V[s]\right)_{i\in I}.$$

Consequently, from Lemma 1, the condition (1) is equivalent to the existence of some α, β in $\mathbb{R}^J_{++}$, and some $\lambda_i = (\lambda_i(s)) \in \mathbb{R}^{S_i}_{++}$, such that

$$\text{for every } i \in I, \quad 0 = \alpha - \beta + \sum_{s\in S_i} \lambda_i(s)V[s].$$

But this latter condition, by Definition 2.2, is equivalent to saying that $q := \beta - \alpha$ (which is independent of i) is a common no-arbitrage price for the structure $[V, (S_i)]$, that is, the structure $[V, (S_i)]$ is arbitrage-free.

Conversely, let us assume that the structure $[V, (S_i)]$ is arbitrage-free and let $q \in Q_c[V, (S_i)]$. We prove that condition (AFAO) holds by contraposition. If it is not true, there exists a collection of portfolios $(z_i)_{i \in I} \in (\mathbb{R}^J)^I$ such that $\sum_{i \in I} z_i = 0$ and $V[s_i] \cdot z_i \geq 0$, for all $i \in I$, all $s_i \in S_i$, with at least one strict inequality. By Definition 2.2, for every $i \in I$, there exists $\lambda_i = (\lambda_i[s])_{s \in S_i} \in \mathbb{R}^{S_i}_{++}$ such that $q = \sum_{s \in S_i} \lambda_i[s]V[s]$. Consequently, $q \cdot z_i = (\sum_{s \in S_i} \lambda_i[s]V[s]) \cdot z_i \geq 0$ and one inequality is strict. Hence, $\sum_{i \in I} q \cdot z_i > 0$, which contradicts $\sum_{i \in I} z_i = 0$. □

We end the section with the relationship between conditions (AFAO) and (ABFAO).

Remark 2 (Conditions (AFAO) and (ABFAO) are not equivalent). Condition (AFAO) clearly implies condition (ABFAO) but is not equivalent to it, as shown by the following counterexample. Consider an economy with three agents ($I = \{1, 2, 3\}$), seven states ($S = \{1, 2, 3, 4, 5, 6, 7\}$) and assume that $S_1 = \{1, 2, 3\}$, $S_2 = \{1, 4, 5\}$, $S_3 = \{1, 6, 7\}$ and that the return matrix V is defined as follows:

$$V = \begin{pmatrix} 0 & 0 \\ -2 & 1 \\ 1 & -1 \\ 0 & -1 \\ 0 & 1 \\ -1 & 0 \\ 1 & 0 \end{pmatrix}.$$

The above structure $[V, (S_i)]$ yields no bilateral arbitrage opportunity but does not satisfy condition (AFAO): take $z_1 = (-1, -1)$, $z_2 = (1, 0)$, $z_3 = (0, 1)$.

3.2. *The coarsest arbitrage-free refinement*

The following proposition shows that there exists a unique *coarsest arbitrage-free refinement* of (S_i). We recall that $\mathcal{S}$ denotes the set of arbitrage-free refinements of (S_i).

Proposition 3.2. *Given the structure $[V, (S_i)]$, there is a unique coarsest element in $\mathcal{S}$, denoted by $(\bar{\mathbf{S}}_i[V, (S_i)])_i$ or simply $(\bar{\mathbf{S}}_i)$, when no confusion is possible, that is: $(\bar{\mathbf{S}}_i) \in \mathcal{S}$, and $(\Sigma_i) \in \mathcal{S}$ implies $(\Sigma_i) \leq (\bar{\mathbf{S}}_i)$, i.e. $\Sigma_i \subset \bar{\mathbf{S}}_i$ for every i. Moreover, $(\bar{\mathbf{S}}_i)$ is self-attainable, that is, $\cap_i S_i = \cap_i \bar{\mathbf{S}}_i$.*

We prepare the proof with a claim. The upper bound of a finite family of information structures $(\Sigma_i^1) \cdots (\Sigma_i^k)$, denoted by $(\Sigma_i) := \vee_{h=1}^k (\Sigma_i^h)$, is defined by the relations $\Sigma_i := \cup_{h=1}^k \Sigma_i^h$ for every $i \in I$.

Claim 1. *The upper bound of a finite family of arbitrage-free information structures is also an arbitrage-free information structure.*

We only need to prove the claim for $k = 2$. If (Σ_i^1) and (Σ_i^2) are information structures, then $(\Sigma_i) := (\Sigma_i^1) \vee (\Sigma_i^2)$ is also an information structure. Indeed, for all $i \in I$, $\emptyset \neq \cap_{j\in I}\Sigma_j^1 \subset \Sigma_i^1 \subset \Sigma_i \subset S_i$, hence $\emptyset \neq \cap_{i\in I}\Sigma_i$. Assume now that (Σ_i^1) and (Σ_i^2) are both arbitrage-free, but not $(\Sigma_i^1) \vee (\Sigma_i^2)$. Then, from the characterization property AFAO (in Proposition 3.1) there exists a collection of portfolios $(z_i)_{i\in I} \in (\mathbb{R}^J)^I$, which satisfies, for every $i \in I, \sum_i z_i = 0$ and $V[s_i] \cdot z_i \geq 0$, for all $s_i \in \Sigma_i^1 \cup \Sigma_i^2$, with at least one strict inequality. That strict inequality may be assumed, non-restrictively, to be met for some $j \in I$ and $s \in \Sigma_j^1$. Hence, the conditions $\sum_i z_i = 0$ and $V[s_i] \cdot z_i \geq 0$, for all $i \in I$ and all $s_i \in \Sigma_i^1$ hold, together with $V[s] \cdot z_j > 0$ for $j \in I$ and $s \in \Sigma_j^1$, which contradicts the assumption that (Σ_i^1) is arbitrage-free. This ends the proof of the claim.

Proof of Proposition 3.2. First, the set $\mathcal{S}$ is non-empty, since it contains the pooled refinement $(\underline{\boldsymbol{S}}_i)$ which is arbitrage-free, by Proposition 2.1. Second, the (non-empty) set $\mathcal{S}$ is finite and we can define the information structure $(\bar{\boldsymbol{S}}_i)$ as the upper bound of all the elements in $\mathcal{S}$, i.e. $(\bar{\boldsymbol{S}}_i) := \vee_{(\Sigma_i)\in\mathcal{S}}(\Sigma_i)$. From the above claim, $(\bar{\boldsymbol{S}}_i)$ is an arbitrage-free refinement of (S_i), and satisfies $(\Sigma_i) \leq (\bar{S}_i)$ for every $(\Sigma_i) \in \mathcal{S}$.

Finally, $(\bar{\boldsymbol{S}}_i)$ is self-attainable. Indeed, $(\bar{\boldsymbol{S}}_i)$ is coarser than the pooled refinement $(\underline{\boldsymbol{S}}_i)$ (which belongs to $\mathcal{S}$ from above). Hence, $\cap_i S_i \subset \bar{\boldsymbol{S}}_i \subset S_i$ for every i. Consequently, $\cap_i S_i = \cap_i \bar{\boldsymbol{S}}_i$. □

Example (Continued). Consider again the previous example, with two agents ($I = \{1, 2\}$), five states ($S = \{1, 2, 3, 4, 5\}$), private information sets $S_1 = \{1, 2, 3\}$, $S_2 = \{1, 4, 5\}$, and the payoff matrix:

$$V = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We recall that (V, S_1, S_2) is not arbitrage-free and we show that the coarsest arbitrage-free refinement is $(\bar{\boldsymbol{S}}_1, \bar{\boldsymbol{S}}_2)$, defined by $\bar{\boldsymbol{S}}_1 = \{1, 2\}$ and $\bar{\boldsymbol{S}}_2 = \{1, 4, 5\}$. Indeed, it is $\bar{q}$-arbitrage-free, with $\bar{q} = (-1, 1, 0)$, that is, $\bar{q} \in Q_c[V, (\bar{\boldsymbol{S}}_i)_i]$, and the only coarser refinement of (S_1, S_2) is itself, but (S_1, S_2) is not arbitrage-free, from above.

3.3. Complete markets, symmetric structures and full revelation

Given a structure $[V, (S_i)]$, we can define, in the set $\mathcal{S}$, the interval $[(\underline{\boldsymbol{S}}_i), (\bar{\boldsymbol{S}}_i)]$, that is, the set of arbitrage-free refinements of (S_i) which are coarser than the pooled refinement $(\underline{\boldsymbol{S}}_i)$ and finer than the coarsest arbitrage-free refinement $(\bar{\boldsymbol{S}}_i)$, defined previously. The structure $[V, (S_i)]$ is then said to be *fully-revealing* when this interval is reduced to a unique element, that is, the pooled refinement $(\underline{\boldsymbol{S}}_i)$, is the only self-attainable arbitrage-free refinement of

(S_i). In Section 4, we shall see that full revelation implies that $(\underline{\boldsymbol{S}}_i)$ can be "revealed" via arbitrage, in a precise sense.

Definition 3.1. The structure $[V, (S_i)]$ is said to be fully-revealing if it satisfies one of the following equivalent assertions:

(i) the pooled refinement and the coarsest arbitrage-free refinement coincide, i.e. $(\underline{\boldsymbol{S}}_i)_i = (\bar{\boldsymbol{S}}_i)_i$;
(ii) the coarsest arbitrage-free refinement $(\bar{\boldsymbol{S}}_i[V, (S_i)])_i$ is symmetric;
(iii) every self-attainable arbitrage-free refinement of (S_i) is symmetric.

The proof of the equivalence between (i), (ii) and (iii) is immediate. To prove the implication $[(\mathrm{i}) \Rightarrow (\mathrm{iii})]$, consider a self-attainable arbitrage-free refinement of (S_i), then it belongs to the interval $[(\underline{\boldsymbol{S}}_i), (\bar{\boldsymbol{S}}_i)]$, which by (i) is reduced to the pooled refinement $(\underline{\boldsymbol{S}}_i)$, which is symmetric. The implication $[(\mathrm{iii}) \Rightarrow (\mathrm{ii})]$ is a consequence of the fact that $(\bar{\boldsymbol{S}}_i[V, (S_i)])_i$ is arbitrage-free and self-attainable (by Proposition 3.2). The proof of the last implication $[(\mathrm{ii}) \Rightarrow (\mathrm{i})]$ is obvious.

The structure $[V, (S_i)]$ was shown to be fully-revealing when (S_i) is symmetric (see Proposition 2.1), and it is also the case when the financial markets are complete (see below). The following propositions provides a proof of these two results, together with converse assertions.

Proposition 3.3. *Let the return matrix V be given, and assume that $\#S > 1$. Then the following conditions are equivalent*:

(i) *the financial markets are complete, that is, rank $V = \#S$*;
(ii) *for every information structure (S_i), the structure $[V, (S_i)]$ is fully-revealing*;
(iii) *every arbitrage-free information structure (S_i) is symmetric*.

When $\#S = 1$, that is, there is no uncertainty at the second period, every structure $[V, (S_i)]$ is always fully-revealing.

Proof. $[(\mathrm{i}) \Rightarrow (\mathrm{iii})]$ Let V be a return matrix with full rank and let (S_i) be an arbitrage-free information structure, we want to show that it is symmetric. If it is not, there exist two agents i, j and some state $\bar{s} \in S_i \setminus S_j$. Consider the Arrow-security paying one in state $\bar{s}$ and zero in other states, and let $\bar{V} \in \mathbb{R}^S$ be the return of this asset, i.e. $\bar{V}[s] = 1$ if $s = \bar{s}$ and $\bar{V}[s] = 0$ otherwise. Since the matrix V has full rank, there exists $\bar{z} \in \mathbb{R}^J$ such that $\bar{V} = V\bar{z}$. Defining $z_i = \bar{z}$ and $z_j = -\bar{z}$, we check that $V[s] \cdot z_i \geq 0$, for every $s \in S_i$, $V[\bar{s}] \cdot z_i = 1$, for $\bar{s} \in S_i$ and $V[s] \cdot z_j \geq 0$, for all $s \in S_j$ (since $\bar{s} \notin S_j$). Thus, $[V, (S_i)]$ does not satisfy condition (AFAO) of Proposition 3.1, hence is not arbitrage-free, a contradiction.

$[(\mathrm{iii}) \Rightarrow (\mathrm{ii})]$ Let (S_i) be an information structure, then its coarsest arbitrage-free refinement $(\bar{\boldsymbol{S}}_i)$ is symmetric by assertion (iii). Hence, condition (ii) of full revelation holds.

$[(\mathrm{ii}) \Rightarrow (\mathrm{i})]$ (By contraposition). Assume that assertion (ii) holds and that rank $V < \#S$. We first consider the case where there exists some $s \in S$ such that $V[s] = 0$. Choose $\underline{s} \in S$, $\underline{s} \neq s$, and define the information structure (S_i) by $S_1 = \{\underline{s}, s\}$ and, for $i \neq 1$, $S_i = \{\underline{s}\}$. Then (S_i) is not symmetric, but is arbitrage-free (since $q := V[\underline{s}]$ is a common no-arbitrage price). Hence, $[V, (S_i)]$ is not fully-revealing, a contradiction with assertion (ii).

Assume now that $V[s] \neq 0$ for every $s \in S$. Since rank $V < \#S$ the vectors $\{V[s] | s \in S\}$ are linearly dependent, hence there exists $\alpha = (\alpha(s)) \in \mathbb{R}^S$, such that $\alpha(\bar{s}) > 0$ for some $\bar{s} \in S$, and $\sum_{s \in S} \alpha(s) V[s] = 0$. We let $S_+ = \{s \in S | \alpha(s) > 0\}$ and $S_- = \{s \in S | \alpha(s) < 0\}$, and we define the information sets $S_1 := S_+$ (which contains $\bar{s}$) and, for $i \neq 1$, $S_i := S_- \cup \{\bar{s}\}$. Then the family (S_i) defines an information structure (since $\cap_i S_i = \{\bar{s}\} \neq \emptyset$), which is arbitrage-free, since one can choose for common no-arbitrage price the vector

$$q := V[\bar{s}] + \sum_{s \in S_+} \alpha(s) V[s] = V[\bar{s}] + \sum_{s \in S_-} -\alpha(s) V[s].$$

Moreover, (S_i) is not symmetric; otherwise $S_+ = S_- \cup \{\bar{s}\}$, which implies that $S_+ = \{\bar{s}\}$ and $S_- = \emptyset$ (since $S_+ \cap S_- = \emptyset$), a contradiction with the fact that $V[\bar{s}] \neq 0$. Hence, $[V, (S_i)]$ is not fully-revealing, a contradiction with assertion (ii). □

Proposition 3.4. *Let the information structure (S_i) of S be given, then the following are equivalent*:

(i) *the information structure (S_i) is symmetric*;
(ii) *for every finite set J and every $S \times J$-financial return matrix V, $[V, (S_i)]$ is fully-revealing*;
(iii) *for every $S \times \{1\}$-financial return matrix V, $[V, (S_i)]$ is fully-revealing.*

Proof. [(i) $\Rightarrow$ (ii)] From (i) the information structure (S_i) is symmetric, hence is arbitrage-free, by Proposition 2.1. Consequently, $(\bar{\mathbf{S}}_i)_i = (S_i)$ and $(\bar{\mathbf{S}}_i)$ is thus symmetric. The condition (ii) of full-revelation is thus satisfied.

[(ii) $\Rightarrow$ (iii)] is obvious.

[(iii) $\Rightarrow$ (i)] Consider the $S \times \{1\}$-financial return matrix V associated to the riskless asset, that is, $V[s] = 1$ for every $s \in S$. We first show that $[V, (S_i)]$ is arbitrage-free. Indeed $q = 1$ is a common no-arbitrage price of the structure $[V, (S_i)]$, since, for every i, $1 = \sum_{s \in S_i} \lambda_i(s) V[s]$ with $\lambda_i(s) = 1/\#S_i$. Since $[V, (S_i)]$ is arbitrage-free, (S_i) coincides with $(\bar{\mathbf{S}}_i)$. But $[V, (S_i)]$ is fully-revealing, and by condition (ii) of Definition 3.1, the structure $(\bar{\mathbf{S}}_i)$ is symmetric. Hence, (S_i) is also symmetric and (i) holds. □

4. Information revealed by prices

This section shows that given the structure $[V, (S_i)]$, every no-arbitrage price $q \in \mathbb{R}^J$ "reveals" a uniquely defined information structure, denoted by $(\mathbf{S}_i(q))$, which is the coarsest q-arbitrage-free refinement of (S_i). The refinement process will be shown to be decentralized, in the sense that the price q conveys enough information for each agent to update her beliefs up to the refinement $(\mathbf{S}_i(q))$, without having any information from the other agents.

4.1. Individual information sets $\mathbf{S}_i(q)$ revealed by prices

Before defining the information set $\mathbf{S}_i(q)$ "revealed" by the price q, we need a lemma.

Lemma 2. *Given the return matrix V, a non-empty set $\Sigma \subset S$, and the price $q \in \mathbb{R}^J$, there exists a unique* (*possibly empty*) *subset of Σ, denoted by $\boldsymbol{S}(V, \Sigma, q)$, which is the greatest element* (*for the inclusion*) *in the set, $\mathcal{S}(V, \Sigma, q)$, of q-arbitrage-free subsets of Σ. Moreover, if $\boldsymbol{S}(V, \Sigma, q)$ is non-empty, one has*

$$q = \sum_{s \in \boldsymbol{S}(V,\Sigma,q)} \lambda(s) V[s] \quad \text{for some } \lambda(s) > 0 (s \in \boldsymbol{S}(V, \Sigma, q)).$$

Definition 4.1. Given the structure $[V, (S_i)]$, for every agent i we define the information set revealed by the price $q \in \mathbb{R}^J$ as the (possibly empty) set

$$\boldsymbol{S}_i(V, (S_i)_i, q) := \boldsymbol{S}(V, S_i, q),$$

simply denoted $\boldsymbol{S}_i(q)$ when the structure $[V, (S_i)]$ remains fixed and only price q varies.

As stressed in the definition, the revealed information set $\boldsymbol{S}_i(q)$ may be empty. Moreover, the family $(\boldsymbol{S}_i(q))_i$ may not be an information structure, that is, one may have $\cap_i \boldsymbol{S}_i(q) = \emptyset$. This situation is illustrated in the example below. In the next section, we shall characterize prices q which reveal an information structure. We now prove Lemma 2.

Proof of Lemma 2. The set $\mathcal{S}(V, \Sigma, q)$ is always non-empty; indeed, it always contains the empty set, from Definition 2.2. We now show that $\mathcal{S}(V, \Sigma, q)$ is stable for the inclusion, i.e. if Σ^1, Σ^2 belong to $\mathcal{S}(V, \Sigma, q)$, then $\Sigma^1 \cup \Sigma^2$ also belongs to $\mathcal{S}(V, \Sigma, q)$ and it is clearly sufficient to prove it when both sets Σ^1 and Σ^2 are non-empty. Indeed, if it is not true, there exists $z \in \mathbb{R}^J$ such that $-q \cdot z \geq 0$ and $V[s] \cdot z \geq 0$, for every $s \in \Sigma^1 \cup \Sigma^2$, with one strict inequality. Then, either $-q \cdot z > 0$ or $V[s] \cdot z > 0$, for some $s \in \Sigma^1 \cup \Sigma^2$, say in Σ^1. In both cases, this contradicts the fact that Σ^1 is q-arbitrage-free for agent i.

We now define the set $\boldsymbol{S}(V, \Sigma, q)$ as the union of all the sets in the non-empty finite set $\mathcal{S}(V, \Sigma, q)$. Hence, from above, we deduce that $\boldsymbol{S}(V, \Sigma, q)$ belongs to $\mathcal{S}(V, \Sigma, q)$ and is the greatest element in $\mathcal{S}(V, \Sigma, q)$ for the inclusion.

The last assertion of the lemma is a consequence of the fact that the non-empty set $\boldsymbol{S}(V, \Sigma, q)$ is q-arbitrage-free, using condition (ii) of Definition 2.2. □

Example (Continued). Consider again the previous example, with two agents ($I = \{1, 2\}$), five states ($S = \{1, 2, 3, 4, 5\}$), private information sets $S_1 = \{1, 2, 3\}$, $S_2 = \{1, 4, 5\}$, and the payoff matrix:

$$V = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- For $q = (1, 0, 0)$ the sets $\boldsymbol{S}_1(q)$ and $\boldsymbol{S}_2(q)$ are empty.
- For $q = (1, 1, 0)$, $\boldsymbol{S}_1(q) = \{2\}$ and $\boldsymbol{S}_2(q) = \emptyset$.
- For $q = (0, 1, 0)$, $\boldsymbol{S}_1(q) = \{1, 2\}$ and $\boldsymbol{S}_2(q) = \{4, 5\}$, hence $\boldsymbol{S}_1(q) \cap \boldsymbol{S}_2(q) = \emptyset$.

- For $\underline{q} = (-1, 0, 0)$, $\boldsymbol{S}_1(\underline{q}) = \{1\}$ and $\boldsymbol{S}_2(\underline{q}) = \{1, 5\}$, hence $\boldsymbol{S}_1(\underline{q}) \cap \boldsymbol{S}_2(\underline{q}) \neq \emptyset$, that is, $(\boldsymbol{S}_i(\underline{q}))$ is an information structure.
- For $\bar{q} = (-1, 1, 0)$, $\boldsymbol{S}_1(\bar{q}) = \{1, 2\}$ and $\boldsymbol{S}_2(\bar{q}) = \{1, 4, 5\}$, hence $\boldsymbol{S}_1(\bar{q}) \cap \boldsymbol{S}_2(\bar{q}) \neq \emptyset$, that is, $(\boldsymbol{S}_i(\bar{q}))$ is an information structure.

The next section will characterize prices, such as $\underline{q}$ and $\bar{q}$, which reveal information structures.

4.2. Information structures revealed by no-arbitrage prices

When the initial information structure (S_i) is not arbitrage-free, the agents may need to refine their information and reach an arbitrage-free structure. The common no-arbitrage prices associated to all the arbitrage-free refinements of (S_i) lead to the following concept of no-arbitrage price.

Definition 4.2. The price $q \in \mathbb{R}^J$ is said to be a no-arbitrage price of the structure $[V, (S_i)]$ if q is a common no-arbitrage price of some refinement (Σ_i) of (S_i), that is, there exists a refinement (Σ_i) of (S_i) such that $q \in Q_c[V, (\Sigma_i)]$.

We denote by $Q[V, (S_i)]$ the set of no-arbitrage prices of the structure $[V, (S_i)]$.

We point out the following simple, but important result.

Proposition 4.1. *Every structure $[V, (S_i)]$ has a no-arbitrage price, i.e. $Q[V, (S_i)] \neq \emptyset$.*

Proof. The pooled refinement $(\underline{S}_i)$ of (S_i) is always arbitrage-free, since it is symmetric (from Proposition 2.1). Hence, $\emptyset \neq Q_c[V, (\underline{S}_i)] \subset Q[V, (S_i)]$. □

The next result states that the family of sets $(\boldsymbol{S}_i(q))$ defines an information structure if and only if q is a no-arbitrage price. We recall that $\mathcal{S}(q)$ denotes the set of q-arbitrage-free refinements (Σ_i) of (S_i).

Proposition 4.2. *Let the structure $[V, (S_i)]$ and the price $q \in \mathbb{R}^J$ be given, the following conditions are equivalent*:

(i) *q is a no-arbitrage price of $[V, (S_i)]$, i.e. $q \in Q[V, (S_i)]$;*
(ii) *$\cap_{i \in I} \boldsymbol{S}_i(q) \neq \emptyset$, i.e. $(\boldsymbol{S}_i(q))$ is an information structure;*
(iii) *$(\boldsymbol{S}_i(q))$ is the coarsest q-arbitrage-free refinement of (S_i).*

Proof. [(i) $\Rightarrow$ (ii)] If $q \in Q[V, (S_i)]$, there exists a q-arbitrage-free refinement (Σ_i) of (S_i), that is, for every i, Σ_i is q-arbitrage-free. Thus, from the definition of $\boldsymbol{S}_i(q)$ we deduce that $\Sigma_i \subset \boldsymbol{S}_i(q)$. Consequently, $\emptyset \neq \cap_i \Sigma_i \subset \Sigma_i \subset \boldsymbol{S}_i(q) \subset S_i$, for every i, which implies condition (ii), that is, $\cap_i \boldsymbol{S}_i(q) \neq \emptyset$.

[(ii) $\Rightarrow$ (iii)] Assume that condition (ii) holds. We first show that $(\boldsymbol{S}_i(q))$ belongs to $\mathcal{S}(q)$. Indeed, $(\boldsymbol{S}_i(q))$ is a refinement of (S_i) (from (ii)) and it is arbitrage-free since $q \in Q_c[V, (\boldsymbol{S}_i(q))]$ from Lemma 2 and Definition 4.1. Now let $(\Sigma_i) \in \mathcal{S}(q)$ then, $q \in Q_c[V, (\Sigma)]$, which implies that, for every i, Σ_i is q-arbitrage-free. Consequently, from Lemma 2 and Definition 4.1, for every i, $\Sigma_i \subset \boldsymbol{S}_i(q)$, that is, $(\boldsymbol{S}_i(q))$ is coarser than (Σ_i).

[(iii) $\Rightarrow$ (i)] If condition (iii) holds, we deduce that $(\boldsymbol{S}_i(q)) \in \mathcal{S}(q)$, i.e. $(\boldsymbol{S}_i(q))$ is a refinement of (S_i), and $q \in Q_c[V, (\boldsymbol{S}_i(q))]$. Consequently, $q \in Q_c[V, (\boldsymbol{S}_i(q))] \subset Q[V, (S_i)]$. □

We can now introduce the following definition.

Definition 4.3. Given the structure $[V, (S_i)]$, we say that a refinement (Σ_i) of (S_i) is revealed by a price q if $\Sigma_i = \boldsymbol{S}_i(q)$ for every i. The refinement is said to be price-revealable if it is revealed by some price q.

If the refinement (Σ_i) of (S_i) is revealed by a price q, then (Σ_i) is arbitrage-free and q is a common no-arbitrage price of (Σ_i), hence a no-arbitrage price of $[V, (S_i)]$. However, the converse is not true, in general, that is, there exist arbitrage-free refinements of (S_i) that cannot be revealed by prices. Moreover, the pooled refinement $(\underline{\boldsymbol{S}}_i)$ may not be price-revealable.

Example (Continued). Consider again a setting two agents ($I = \{1, 2\}$), five states ($S = \{1, 2, 3, 4, 5\}$), private information sets $S_1 = \{1, 2, 3\}$, $S_2 = \{1, 4, 5\}$, and the payoff matrix:

$$V = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- We recall that V is not arbitrage-free
- The pooled refinement $\underline{\boldsymbol{S}}_1 = \underline{\boldsymbol{S}}_2 = \{1\}$ cannot be revealed by prices. Indeed, for every common no-arbitrage price $q \in Q_c(V, (\underline{\boldsymbol{S}}_i)_i) = \{\lambda(-1, 0, 0) | \lambda > 0\}$, we have $\boldsymbol{S}_2(q) = \{1, 5\}$.
- We recall that the coarsest arbitrage-free refinement is $(\bar{\boldsymbol{S}}_1, \bar{\boldsymbol{S}}_2)$, defined by $\bar{\boldsymbol{S}}_1 = \{1, 2\}$ and $\bar{\boldsymbol{S}}_2 = \{1, 4, 5\}$ (see Section 3.2) and it can be revealed by the price $q = (-1, 1, 0)$ (see Section 4.1).

The last property of the example, that the coarsest arbitrage-free refinement is price-revealable, is shown to be true, in general, in the next section.

Remark 3 (Equivalent no-arbitrage prices). Let q^1 and q^2 be two no-arbitrage prices, we say that q^1 is finer than q^2, or that q^2 is coarser than q^1, denoted $q^1 \preceq q^2$, if the information structure $(\boldsymbol{S}_i(q^1))$ is finer than $(\boldsymbol{S}_i(q^2))$, that is, $\boldsymbol{S}_i(q^1) \subset \boldsymbol{S}_i(q^2)$ for every i. The relation $\preceq$ clearly defines a preorder on the set of no-arbitrage prices $Q[V, (S_i)]$ and we can associate to it the strict preorder relation $\prec$ and the equivalence relation $\sim$, defined standardly by $q^1 \prec q^2$ if $[q^1 \preceq q^2$ and not $(q^2 \preceq q^1)]$, and by $q^1 \sim q^2$ if $[q^1 \preceq q^2$ and $q^2 \preceq q^1]$. In other words, the no-arbitrage prices q^1 and q^2 are equivalent if $\boldsymbol{S}_i(q^1) = \boldsymbol{S}_i(q^2)$ for every i, i.e. q^1 and q^2 reveal the same information structure. Since S is finite, there is a finite number of equivalent classes, denoted $\dot{q}$ (for the relation $\sim$) which define a finite partition of the set $Q[V, (S_i)]$.

4.3. A characterization of price-revealable information structures

We first show that the coarsest arbitrage-free refinement $(\bar{\boldsymbol{S}}_i)$ defined in Section 3.2 is always price-revealable.

Proposition 4.3.

(a) *Let* $[V, (S_i)]$ *be a given structure, then*

$$\emptyset \neq \{q \in \mathbb{R}^J | \forall i, \quad \bar{\boldsymbol{S}}_i = \boldsymbol{S}_i(q)\} = Q_c[V, (\bar{\boldsymbol{S}}_i)].$$

Hence, the information structure $(\bar{\boldsymbol{S}}_i)$ *can be revealed by every price* $q \in Q_c[V, (\bar{\boldsymbol{S}}_i)]$.

(b) *If the structure* $[V, (S_i)]$ *is arbitrage-free, then*

$$\emptyset \neq \{q \in \mathbb{R}^J | \forall i, \qquad S_i = \boldsymbol{S}_i(q)\} = Q_c[V, (S_i)].$$

Proof (Part (a)). We first notice that from Proposition 3.2, the set $Q_c[V, (\bar{\boldsymbol{S}}_i)]$ is non-empty. We now let $q \in \mathbb{R}^J$ be such that $(\bar{\boldsymbol{S}}_i) = (\boldsymbol{S}_i(q))$, then, from Proposition 4.2, $q \in Q_c[V, (\boldsymbol{S}_i(q))] = Q_c[V, (\bar{\boldsymbol{S}}_i)]$. Conversely, let $q \in Q_c[V, (\bar{\boldsymbol{S}}_i)]$, then from Proposition 4.2 (iii), for every i, $\bar{\boldsymbol{S}}_i \subset \boldsymbol{S}_i(q)$. This implies that $(\boldsymbol{S}_i(q))$ is an information structure (i.e. $\cap_{i \in I} \boldsymbol{S}_i(q) \neq \emptyset$) and it is clearly arbitrage-free (since q is a common no-arbitrage price). But $(\bar{\boldsymbol{S}}_i)$ is the coarsest arbitrage-free refinement, hence for every i, $\bar{\boldsymbol{S}}_i = \boldsymbol{S}_i(q)$. This ends the proof of the equality. The second part of the proposition is straightforward.

(Part (b)). If the structure $[V, (S_i)]$ is arbitrage-free, then $\bar{\boldsymbol{S}}_i = S_i$ for every i, from the definition of $(\bar{\boldsymbol{S}}_i)$. Thus, the desired conclusion follows from Part (a). □

The price-revealable property of $(\bar{\boldsymbol{S}}_i)$ is not shared by all arbitrage-free refinements, as shown by the example in Section 4.2. We recall that $\mathcal{S}(q)$ denotes the set of q-arbitrage-free refinements (Σ_i) of (S_i). The following proposition shows that an information structure is price-revealable if and only if it is the coarsest element in one of the sets $\mathcal{S}(q)$ (and q is a no-arbitrage price).

Proposition 4.4. *Let* $[V, (S_i)]$ *be a given structure and let* (Σ_i) *be a refinement of* (S_i), *then the three following conditions are equivalent*:

(i) *the information structure* (Σ_i) *is price-revealable*;
(ii) (Σ_i) *is the coarsest information structure in* $\mathcal{S}(q)$ *for some price* $q \in \mathbb{R}^J$;
(iii) $\emptyset \neq \{q \in \mathbb{R}^J | \forall i, \Sigma_i = \boldsymbol{S}_i(q)\} \subset Q_c[V, (\Sigma_i)]$.

Proof. [(i) ⇒ (iii)] If (Σ_i) is price-revealable, there exists $q \in \mathbb{R}^J$ such that $\Sigma_i = \boldsymbol{S}_i(q)$ for every i. Hence, $q \in Q_c(V, (\boldsymbol{S}_i(q))) = Q_c(V, (\Sigma_i))$ (from Lemma 2 and Definition 4.1).

[(iii) ⇒ (ii)] From (iii), there exists $q \in \mathbb{R}^J$ such that, for every i, $\Sigma_i = \boldsymbol{S}_i(q)$ and $q \in Q_c[V, (\Sigma_i)]$. From Proposition 4.2, we deduce that is the coarsest element in $\mathcal{S}(q)$.

[(ii) ⇒ (i)] From (ii), there exists $q \in Q_c(V, (\Sigma_i))$ and, from the definition of $(\boldsymbol{S}_i(q))$ (Definition 4.1) one has, for every i, $\emptyset \neq \Sigma_i \subset \boldsymbol{S}_i(q)$ and $q \in Q_c(V, (\boldsymbol{S}_i(q)))$. Consequently, $(\boldsymbol{S}_i(q))$ belongs to $\mathcal{S}(q)$, hence from condition (ii), it is finer than (Σ_i), that is, for every

i, $\boldsymbol{S}_i(q) \subset \Sigma_i$. We have thus shown that, for every i, $\boldsymbol{S}_i(q) = \Sigma_i$, that is, the information structure (Σ_i) can be revealed by the price q. □

We end the section with a characterization of (no-arbitrage) prices q which reveal self-attainable refinements of (S_i). Let $\mathcal{S}$ denote the set of self-attainable refinements (Σ_i) of (S_i), that is, $\cap_{i\in I} S_i = \cap_{i\in I} \Sigma_i$.

Proposition 4.5. *Let $q \in \mathbb{R}^J$, the following conditions are equivalent*:

(i) *q reveals a self-attainable refinement of (S_i), i.e. $q \in \cup_{(\Sigma_i)\in\mathcal{S}} Q_c(V, (\Sigma_i))$*;
(ii) *$\cap_{i\in I} S_i = \cap_{i\in I} \boldsymbol{S}_i(q)$, that is, $(\boldsymbol{S}_i(q))$is self-attainable*;
(iii) *$(\boldsymbol{S}_i(q))$ is the coarsest information structure in $\mathcal{S}(q) \cap \mathcal{S}$.*

Furthermore, there exists prices q satisfying one of the above assertions (i)–(iii).

Proof. The proof is a direct consequence of Proposition 4.2. We further notice that there clearly exist prices q, satisfying one of the above equivalent assertions (i)–(iv), since both information structures $(\underline{\boldsymbol{S}}_i)$ and $(\bar{\boldsymbol{S}}_i)$ belong to $\mathcal{S}$. □

Acknowledgements

The support from the Economic and Social Research Council (ESRC, UK) is gratefully acknowledged. The authors would also like to thank the anonymous referees for helpful comments and suggestions.

References

Bisin, A., Gottardi, P., 1999. Competitive equilibria with asymmetric information. Journal of Economic Theory 87 (1), 1–48.

Hart, O., 1975. On the optimality of equilibrium when the market structure is incomplete. Journal of Economic Theory 11, 418–443.

Magill, M., Quinzii, M., 1996. Theory of Incomplete Markets. MIT Press, Cambridge, MA.

Radner, R., 1979. Rational expectations equilibrium: generic existence and the information revealed by prices. Econometrica 47, 655–678.

Werner, J., 1987. Arbitrage and the existence of competitive equilibrium. Econometrica 55, 1403–1418.

Elimination of Arbitrage States in Asymmetric Information Models

Bernard CORNET, *Lionel DE BOISDEFFRE†

Abstract

In a financial economy with asymmetric information and incomplete markets, we study how agents, having no model of how equilibrium prices are determined, may still refine their information by eliminating sequentially "arbitrage state(s)", namely, the state(s) which would grant the agent an arbitrage, if realizable.

Key words: Arbitrage, incomplete markets, asymmetric information, information revealed by prices.

JEL classification: D52

*Depatment of Economics, University of Kansas and Université de Paris 1. E-mail: cornet@ku.edu

†CERMSEM, Maison des Sciences Economiques, Université de Paris 1, 106 − 112 boulevard de l'Hopital, 75647, Cedex 13, FRANCE. E-mail: lionel.de.boisdeffre@wanadoo.fr

1 Introduction

In a financial economy with asymmetric information and incomplete markets, will agents be able to learn from prices about their partners' private information when they have no prior "model" or "expectations" of how equilibrium prices are determined? This paper, which completes an earlier one on arbitrage and price revelation under asymmetric information [Cornet-De Boisdeffre (2002)], answers positively and introduces a sequential and decentralized process of inferences, where agents learn from prices by analyzing arbitrage opportunities on financial markets. Refinement of information is then achieved in a decentralized manner by each agent eliminating sequentially her "arbitrage state(s)", namely, the state(s) which would grant the agent an arbitrage, if realizable. The paper also shows that a coarse refinement of information, which precludes arbitrage, may always be attained, alternatively, in a decentralized way through prices, or by a similar sequential elimination process when no price is given.

Section 2 presents the framework and recalls the basic concepts of information structures, refinements and arbitrage with asymmetric information. Section 3 introduces the "no-arbitrage principle", by which agents who only know their own characteristics may refine their information in successive steps, by eliminating "arbitrage states". Section 4 defines a concept of no-arbitrage prices with asymmetric information and explains how such prices may reveal information to agents. Section 5 describes the refinement process in the absence of market prices.

2 The model

We consider the basic model of a two time-period economy with private information and nominal assets: the simplest tractable model which allows us to present arbitrage. The economy is finite, in the sense that there are finite sets I, S, and J, respectively, of consumers, states of nature, and nominal assets.

In what follows, the first period will also be referred to as $t = 0$ and the second period, as $t = 1$. There is an a priori uncertainty at the first period ($t = 0$) about which of the states of nature $s \in S$ will prevail at the second period ($t = 1$). The non-random state at the first period is denoted by $s = 0$ and if $\Sigma \subset S$, Σ' will stand for $\{0\} \cup \Sigma$.

Agents may operate financial transfers across states in S' (i.e., across the two periods and across the states of the second period) by exchanging a finite number of nominal assets $j \in J$, which define the financial structure of the model. The nominal assets are traded at the first period ($t = 0$) and yield payoffs at the second period ($t = 1$), contingent on the realization of the state of nature. We denote by V_s^j the payoff of asset $j \in J$, when state $s \in S$ is realized, by V the $S \times J$ -return matrix $V := (V_s^j)$, which does not depend upon the asset price $q \in \mathbb{R}^J$, and by $V[s]$ its row vector in state s (for each $s \in S$). A portfolio $z = (z_j) \in \mathbb{R}^J$ specifies the quantities $|z_j|$ ($j \in J$) of each asset j (with the convention that it is bought if $z_j > 0$ and sold if $z_j < 0$) and Vz is thus its financial return across states at time $t = 1$.

At the first period, each agent $i \in I$ has some private information $S_i \subset S$ about which states of the world may occur at the next period: either this information is kept, or it is possible to infer that the true state will be in a smaller set $\Sigma_i \subset S_i$. In both cases agents are assumed to receive no wrong information signal, that is, the true state always belongs to the set $\cap_{i\in I} S_i$ or $\cap_{i\in I}\Sigma_i$, hence assumed to be non-empty. A collection $(S_i)_{i\in I}$ such that $\cap_{i\in I} S_i \neq \emptyset$ is called an *information structure* and a collection (Σ_i) such that $\Sigma_i \subset S_i$ for every i is called a *refinement* of (S_i).

We summarise by $[(I, S, J), V, (S_i)_{i\in I}]$ the financial and information characteristics, which are fixed throughout the paper and referred to as the (financial and information) structure.

We recall the following standard definitions. Given the return matrix V and a nonempty set $\Sigma \subset S$, the price $q \in \mathbb{R}^J$ is said to be a no-arbitrage price for the couple (V, Σ), or the couple (V, Σ) to be q-arbitrage-free, if one of the following equivalent assertions holds:

(i) there is no portfolio $z \in \mathbb{R}^J$ such that $-q \cdot z \geq 0$ and $V[s] \cdot z \geq 0$ for every $s \in \Sigma$, with at least one strict inequality;

(ii) there exists $\lambda = (\lambda(s)) \in \mathbb{R}^{\Sigma}_{++}$, such that $q = \sum_{s\in\Sigma} \lambda(s)V[s]$.

We denote by $Q[V, \Sigma]$ the set of no-arbitrage prices associated to (V, Σ). By convention, we shall also say that the couple $(V, \emptyset)$ is q-arbitrage-free for every $q \in \mathbb{R}^J$, that is, we let $Q[V, \emptyset] = \mathbb{R}^J$.

3 Sequential elimination of arbitrage states

Given V, a nonempty subset Σ of S and $q \in \mathbb{R}^J$, we now define the sets:

$$\mathbf{A}(V,\Sigma,q) = \{\widetilde{s} \in \Sigma : \exists z \in \mathbb{R}^J, -q\cdot z \geq 0, V[\widetilde{s}]\cdot z > 0, V[s]\cdot z \geq 0, \forall s \in \Sigma\}$$
$$\mathbf{S}^1(V,\Sigma,q) := \Sigma \backslash \mathbf{A}(V,\Sigma,q),$$
with the convention that $\mathbf{A}(V,\emptyset,q) := \emptyset$ and $\mathbf{S}^1(V,\emptyset,q) := \emptyset$, for all $q \in \mathbb{R}^J$.

Given V and an information structure (S_i), the set $\mathbf{A}(V,S_i,q)$ consists in the so-called "arbitrage states" (of the second period $t=1$), that is, states which grant agent i an arbitrage when his beliefs are represented by the set S_i. The first stage of elimination of arbitrage states leads to the set $\mathbf{S}^1(V,S_i,q)$. However, the refined set $\mathbf{S}^1(V,S_i,q)$ may display new arbitrage states, that is, there may exist states $s \in \mathbf{A}(V,\mathbf{S}^1(V,S_i,q),q)$ such that $s \notin \mathbf{A}(V,S_i,q)$. Thus, the elimination process may need to carry on. It is defined sequentially, hereafter, in two slightly different ways, which will be shown to be equivalent.

Given V, an agent i with (nonempty) private information set $S_i \subset S$, for every $q \in \mathbb{R}^J$, we define, by induction on $k \in N$, the sets $S_i^k(q)$ as follows:

$S_i^0(q) = S_i$, and for $k \geq 1$

$S_i^{k+1}(q) = \mathbf{S}^1(V,S_i^k(q),q) := S_i^k(q)\backslash\mathbf{A}(V,S_i^k(q),q)$.

Similarly, we define by induction on $k \in N$, the sets $S_i'^k(q)$ as follows:

$S_i'^0(q) = S_i$, and for $k \geq 1$

$$S_i'^{k+1}(q) = \begin{cases} S_i'^k(q), & \text{if } \mathbf{A}(V,S_i'^k(q),q) = \emptyset, \\ S_i'^k(q)\backslash\{s^k\} \text{ for some } s^k \in \mathbf{A}(V,S_i'^k(q),q) & \text{if } \mathbf{A}(V,S_i'^k(q),q) \neq \emptyset. \end{cases}$$

The two sequences $(S_i^k(q))_{k\in N}$ and $(S_i'^k(q))_{k\in N}$ are clearly decreasing, that is, $S_i^{k+1}(q) \subset S_i^k(q)$ and $S_i'^{k+1}(q) \subset S_i'^k(q)$ for every k. Since both sequences are contained in the finite set S_i, each sequence must be constant for k large enough. We let

$S_i^*(q) := \cap_{k\in N}\ S_i^k(q)$ (in fact equal to $S_i^{k^*}(q)$ for some k^* large enough);

$S_i^{**}(q) := \cap_{k\in N}\ S_i'^k(q)$ (in fact equal to $S_i'^{k^{**}}(q)$ for some k^{**} large enough).

The following result shows that for every price q, the successive elimination of arbitrage states leads agents to infer the same information sets, whether they rule out the states of arbitrage one by one (and then, whatever the chronology of inferences), or in bundles.

Theorem 1 Let $[V, (S_i)]$ be a given structure and $q \in \mathbb{R}^J$. Then, for every $i \in I$, $S_i^*(q) = S_i^{**}(q)$, and this set is the (possibly empty) greatest subset Σ of S_i such that $\mathbf{A}(V, \Sigma, q) = \emptyset$.

The successive elimination of arbitrage states, may be interpreted as a rational behavior. This behavior, referred to as the "no-arbitrage principle", does not require any specific knowledge on the ex ante characteristics of the economy (endowments and preferences of the other consumers) or of a relationship between prices and the private information of other agents. This is the main difference between the model we consider and rational expectations models with differential information.

We prepare the proof of Theorem 1 with two claims.

Claim 1. *Let* $q \in \mathbb{R}^J$ *and* $\Sigma^1 \subset \Sigma^2 \subset S$, *then* $\mathbf{S}^1(V, \Sigma^1, q) \subset \mathbf{S}^1(V, \Sigma^2, q)$.

Proof of Claim 1. By contraposition. Suppose that there exists some $\tilde{s} \in \mathbf{S}^1(V, \Sigma^1, q) \subset \Sigma^1 \subset \Sigma^2$ and $\tilde{s} \notin \mathbf{S}^1(V, \Sigma^2, q)$. Then, $\tilde{s} \in \mathbf{A}(V, \Sigma^2, q)$, that is, there exists $z \in \mathbb{R}^J$ such that $-q \cdot z \geq 0$, $V[s] \cdot z \geq 0$, for every $s \in \Sigma^2$ and $V[\tilde{s}] \cdot z > 0$. Since $\tilde{s} \in \Sigma^1 \subset \Sigma^2$, we deduce that $\tilde{s} \in \mathbf{A}(V, \Sigma^1, q)$, which contradicts the fact that $\tilde{s} \in \mathbf{S}^1(V, \Sigma^1, q)$. ∎

Claim 2. For every $\Sigma \subset S_i$, such that $\mathbf{A}(V, \Sigma, q) = \emptyset$ one has:
$\Sigma \subset S_i^k(q) \subset S_i'^k(q)$ for every k.

Proof of Claim 2. By induction on k. The above inclusions are true for $k = 0$, since $S_i^0(q) = S_i'^0(q) := S_i$. Assume now that the inclusions hold up to rank k. Let $\Sigma \subset S_i$, such that $\mathbf{A}(V, \Sigma, q) = \emptyset$. From Claim 1, we get

$$\mathbf{S}^1(V, \Sigma, q) \subset \mathbf{S}^1(V, S_i^k(q), q) \subset \mathbf{S}^1(V, S_i'^k(q), q).$$

Since $\mathbf{A}(V, \Sigma, q) = \emptyset$, we deduce that $\mathbf{S}^1(V, \Sigma, q) = \Sigma$ and from the definitions of the sets $S_i^k(q)$ and $S_i'^k(q)$, we get $S_i^{k+1}(q) := \mathbf{S}^1(V, S_i^k(q), q)$, and $\mathbf{S}^1(V, S_i'^k(q), q) := S_i'^k(q) \backslash \mathbf{A}(V, S_i'^k(q), q) \subset S_i'^{k+1}(q)$. Consequently, $\Sigma \subset S_i^{k+1}(q) \subset S_i'^{k+1}(q)$. ∎

Proof of Theorem 1. Let $\Sigma \subset S_i$ such that $\mathbf{A}(V, \Sigma, q) = \emptyset$. From Claim 2, taking k large enough, we get $\Sigma \subset S_i^*(q) \subset S_i^{**}(q)$ for every i. But, from the definitions of $S_i^*(q)$ and $S_i^{**}(q)$ we deduce that $\mathbf{A}(V, S_i^*(q), q) = \mathbf{A}(V, S_i^{**}(q), q) = \emptyset$. We thus deduce first that $S_i^*(q) = S_i^{**}(q)$ (take above $\Sigma = S_i^{**}(q)$) and second that $S_i^*(q) = S_i^{**}(q)$ is the greatest element (for the inclusion) among the subsets Σ of S_i such that $\mathbf{A}(V, \Sigma, q) = \emptyset$. ∎

4 Sequential procedures and price revelation

Given the structure $[V, (\Sigma_i)]$, the above Theorem 1 shows the existence of a unique set, denoted $\tilde{\mathbf{S}}_i(q)$, which is the greatest subset Σ of S_i such that $\mathbf{A}(V, \Sigma, q) = \emptyset$. This section will compare the above definition of $\tilde{\mathbf{S}}_i(q)$ with the definition of the information set $\mathbf{S}_i(q)$ revealed to agent i by the price $q \in \mathbb{R}^J$ which was introduced in Cornet-De Boisdeffre (2002). We recall that $\mathbf{S}_i(q)$ is the unique (possibly empty) subset of S_i, which is the greatest subset of S_i that is q-arbitrage-free. It is immediate to see that $\mathbf{S}_i(q) \subset \tilde{\mathbf{S}}_i(q)$ and that both sets may be empty.

For arbitrary prices, the families $(\mathbf{S}_i(q))$, $(\tilde{\mathbf{S}}_i(q))$ may not be information structures, that is, one may have $\cap_i \mathbf{S}_i(q) = \emptyset$ or $\cap_i \tilde{\mathbf{S}}_i(q) = \emptyset$. To get information structures we now need to consider no-arbitrage prices, as in Cornet-De Boisdeffre (2002) and we recall the following definitions. Given the structure $[V, (\Sigma_i)]$, the price $q \in \mathbb{R}^J$ is said to be a no-arbitrage price for agent i if it is a no-arbitrage price for the couple (V, Σ_i), and the price $q \in \mathbb{R}^J$ is said to be a common no-arbitrage price of the structure $[V, (\Sigma_i)]$ if it is a no-arbitrage price for every agent $i \in I$, that is, if it belongs to the set $Q_c[V, (\Sigma_i)] = \cap_i Q[V, \Sigma_i]$. Alternatively, the structure $[V, (\Sigma_i)]$ is said to be arbitrage-free (resp. q-arbitrage-free) if it admits a common no-arbitrage price, that is, if $Q_c[V, (\Sigma_i)] \neq \emptyset$ (resp. $q \in Q_c[V, (\Sigma_i)]$). The price $q \in \mathbb{R}^J$ is said to be a no-arbitrage price of $[V, (S_i)]$ if q is a common no-arbitrage price for some information structure (Σ_i) refining (S_i). We denote by $Q[V, (S_i)]$ the set of no-arbitrage prices of the structure $[V, (S_i)]$.

Theorem 2 *Let $[V, (S_i)]$ be a given structure. Then for every $q \in Q[V, (S_i)]$ and every $i \in I$, $\mathbf{S}_i(q) = \tilde{\mathbf{S}}_i(q) = S_i^*(q) = S_i^{**}(q)$.*

The proof of Theorem 2 is a direct consequence of Theorem 1 and the following proposition, which is also of interest for itself.

Proposition 1 *Given a structure $[V, (S_i)]$ and a price $q \in \mathbb{R}^J$, then the following three conditions are equivalent:*

(i) q is a no-arbitrage price, that is, $q \in Q[V, (S_i)]$;

(ii) $(\mathbf{S}_i(q))$ is an information structure, (i.e., $\cap_i \mathbf{S}_i(q) \neq \emptyset$);

(iii) $(\tilde{\mathbf{S}}_i(q))$ is an information structure (i.e., $\cap_i \tilde{\mathbf{S}}_i(q) \neq \emptyset$), and $\tilde{\mathbf{S}}_i(q)$ is q-arbitrage-free for every agent i at the first period ($t = 0$), in the sense that there is no portfolio $z \in \mathbb{R}^J$ such that $-q \cdot z > 0, V[s] \cdot z \geq 0$ for every $s \in \tilde{\mathbf{S}}_i(q)$.

Moreover, if one of the above conditions (i), (ii) or (iii) holds, $\mathbf{S}_i(q) = \tilde{\mathbf{S}}_i(q)$ *for every* i*.*

Proof. The equivalence between (i) and (ii) is proved in Cornet-De Boisdeffre (2002).

$[(ii) \Rightarrow (iii)]$. From (ii) we first deduce that $\emptyset \neq \cap_i \mathbf{S}_i(q) \subset \cap_i \tilde{\mathbf{S}}_i(q)$. Since $\mathbf{S}_i(q) \neq \emptyset$, from its definition we get $q \in Q[V, \mathbf{S}_i(q)]$. This clearly implies that $q \in Q[V, \tilde{\mathbf{S}}_i(q)]$since $\mathbf{S}_i(q) \subset \tilde{\mathbf{S}}_i(q)$. Hence $\tilde{\mathbf{S}}_i(q)$ is q-arbitrage-free at $t = 0$.

$[(iii) \Rightarrow (ii)]$ It is clearly sufficient to show that $\mathbf{S}_i(q) = \tilde{\mathbf{S}}_i(q)$ for every i. By definition of $\tilde{\mathbf{S}}_i(q)$, for every $\tilde{s} \in \tilde{\mathbf{S}}_i(q)$, there is no $z \in \mathbb{R}^J$ such that $-q \cdot z \geq 0, V[s] \cdot z \geq 0$, for every $s \in \tilde{\mathbf{S}}_i(q)$, and $V[\tilde{s}] \cdot z > 0$. That condition, together with the fact that $\tilde{\mathbf{S}}_i(q)$ is q-arbitrage free for every agent i at the first period, implies that $\tilde{\mathbf{S}}_i(q)$ is q-arbitrage-free for agent i. Consequently, from the definition of the set $\mathbf{S}_i(q)$, one has $\tilde{\mathbf{S}}_i(q) \subset \mathbf{S}_i(q)$. The inclusion $\mathbf{S}_i(q) \subset \tilde{\mathbf{S}}_i(q)$ is immediate, hence the equality holds.

Finally the equality $\mathbf{S}_i(q) = \tilde{\mathbf{S}}_i(q)$ for every i has been shown above as a consequence of (iii). ■

We point out that the above assertions (ii) and (iii) of Proposition 1 may not be equivalent if we do not assume in (iii) that $\tilde{\mathbf{S}}_i(q)$ is q-arbitrage free for every agent i at the first period.

Example. Consider two agents, five states ($S = \{1, 2, 3, 4, 5\}$), private information sets $S_1 = \{1, 2, 3, 5\}$, $S_2 = \{1, 4, 5\}$, and the payoff matrix

$$V = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, for $q = (1, 1, 0)$, $\mathbf{S}_1(q) = \tilde{\mathbf{S}}_1(q) = \{2, 5\}$, $\mathbf{S}_2(q) = \emptyset$, while $\tilde{\mathbf{S}}_2(q) = \{4, 5\}$. Thus $\emptyset = \mathbf{S}_1(q) \cap \mathbf{S}_2(q) \subset \tilde{\mathbf{S}}_1(q) \cap \tilde{\mathbf{S}}_2(q) = \{5\}$. ▪

5 Reaching the coarsest arbitrage-free refinement

We denote by $\mathcal{S}$ the set of arbitrage-free refinements of (S_i). Given the structure $[V, (S_i)]$, we recall that there exists a unique coarsest element in $\mathcal{S}$, denoted by $(\overline{\mathbf{S}}_i[V,(S_i)])$ or simply $(\overline{\mathbf{S}}_i)$ when no confusion is possible, that is $(\overline{\mathbf{S}}_i) \in \mathcal{S}$ and $(\Sigma_i) \in \mathcal{S}$ implies $\Sigma_i \subset \overline{\mathbf{S}}_i$, for every i. We refer to Cornet-De Boisdeffre (2002) for this definition and also below for an alternative proof of the existence of $(\overline{\mathbf{S}}_i)$.

This section provides an alternative process of inferences leading to the coarsest refinement $(\overline{\mathbf{S}}_i)$ (without the use of prices). Given the structure $[V,(S_i)]$ and a refinement (Σ_i) of (S_i), for each $i \in I$, we let:

$$\begin{aligned}
\mathbf{A}_i(V,(\Sigma_i)) \quad &:= \quad \{\widetilde{s}_i \in \Sigma_i \mid \exists (z_j) \in (\mathbb{R}^J)^I, \sum_{j\in I} z_j = 0, \forall j \in I, \\
&\qquad \forall s_j \in \Sigma_j,\ V[s_j]\cdot z_j \geq 0 \text{ and } V[\widetilde{s}_i]\cdot z_i > 0\}, \\
\mathbf{S}_i^1(V,(\Sigma_i)) \quad &:= \quad \Sigma_i \backslash \mathbf{A}_i(V,(\Sigma_i)).
\end{aligned}$$

We then define, similarly as in the previous section, two alternative inference processes by induction on the integer k. For every $i \in I$ we let :

$S_i^0 := S_i$, and, for every $k \geq 0$,
$S_i^{k+1} := \mathbf{S}_i^1(V,(S_i^k))$.

Similarly:

$S_i'^0 := S_i$, and, for every $k \geq 0$,
$$S_i'^{k+1} := \left\{ \begin{array}{c} S_i'^k, \text{ if } \mathbf{A}_i(V,(S_i'^k)) = \emptyset \text{ and, otherwise} \\ S_i'^k \backslash \{s_i^k\}, \text{ for some arbitrary } s_i^k \in \mathbf{A}_i(V,(S_i'^k)) \neq \emptyset \end{array} \right\}$$

We note that both sequences $(S_i^k)_k$ and $(S_i'^k)_k$ are decreasing in the finite set S_i, hence they are constant for k large enough. We note (S_i^*) and (S_i^{**}) the limits of these sequences .

Theorem 3 *Let $[V,(S_i)]$ be a given structure. Then, for every $i \in I$* $S_i^* = S_i^{**}$ and (S_i^*) is the coarsest arbitrage-free refinement of (S_i).

The proof of Theorem 3 is similar the proof of Theorem 1 and is left to the reader.

Again, Theorem 3 shows that the chronology of all agents' inferences in the above processes will not change the outcome. Whatever the individual paths of inferences, they always lead to the same limit, namely the coarsest arbitrage-free refinement $(\overline{\mathbf{S}}_i)$.

References

[1] Cornet, B., and De Boisdeffre, L. (2002): "Arbitrage and Price Revelation with Asymmetric Information and Incomplete Markets", *Journal of Mathematical Economics* **38** 4, 393–410.