DISCOUNT FACTORS EX POST AND EX ANTE, AND DISCOUNTED UTILITY ANOMALIES

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Abstract. The real options approach is used to explain discounted utility anomalies as artifacts of the optimizing behavior of an individual with standard preferences, who perceives the utility from consumption in the future as uncertain. For this individual, waiting is valuable because uncertainty is revealed over time. The fair price (or compensation) that the individual agrees to pay (or accept) today is the expected value of utility of the future gain (or loss) multiplied by a certain non-exponential factor which we interpret as a *discount factor ex ante*. The factors ex ante are different for gains and losses, and depend on the utility function and underlying uncertainty. After the decision of exchange had been made, *valuation ex post* reduces to calculation of the standard expected present value. We provide analytic expressions and numerical examples for discount factors assuming different utility functions and models of uncertainty, and demonstrate that our explanation of discounted utility anomalies is robust.

Key words: real options, time preference, discounted utility anomalies

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1. Introduction

The main lesson of the real options theory is that in an uncertain environment, waiting has value. Various economic and non economic applications of the real options theory were outlined by Dixit and Pindyck (1994). However, both Dixit and Pindyck (1994) and the majority of papers on real options concentrated mainly on various capital (dis)investment problems, be it capital budgeting, exploitation of natural resources,

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strategic interactions of investors, and many other. Grenadier and Wang (2005) extend the real options framework to investment timing with exogenously given time-inconsistent preferences. We would like to remind the reader that real options are ubiquitous in the sense that in the real life, agents have to make decisions under uncertainty, which are either irreversible or (partially) reversible at a cost. The theory that explains how to make such decisions optimally is the real options theory. The key ingredients of the real options theory are uncertainty, partial or complete irreversibility of decisions, and the opportunity to choose the time of the decisions. We use the insights of this theory to explain time dependent discounting and other departures from the standard discounted utility assumptions.

1.1. **History of discounted utility theory.** In the first three decades of the twentieth century, "time preference" was analyzed mainly qualitatively, as interaction among different factors which may influence intertemporal decisions. In 1933, Paul Samuelson invented the discounted utility theory (DU theory), which compressed the influence of many factors into one number: the discount rate. In continuous time models, an individual with the time-separable utility u calculates the value of consumption of a stream c_t over time interval $(0, T)$ according to the formula

$$U = \int_0^T e^{-rt} u(c_t) dt, \tag{1.1}$$

where $r > 0$ is the discount rate. In discrete time models, the counterpart of equation (1.1) is

$$U = \sum_{t=0}^{T} \delta^t u(c_t) dt, \tag{1.2}$$

where $\delta \in (0, 1)$. Due to the analytical simplicity (and probably, similarity to the compound interest formula), the exponential discounted utility model was almost instantly adopted as a standard tool in intertemporal models, although Samuelson suggested the DU model as a convenient tool only, and explicitly disavowed an idea that individuals really optimize an integral of the form (1.1). Almost 30 years later, Koopmans (1960) constructed an axiomatic theory of time preference which lead to the exponential discount factor in Samuelson's model. As a result, a general feeling emerged that the DU model was justified. However, later, in many empirical studies, it was shown that the real behavior of individuals did not agree with the exponential discounting model.

1.2. **Observed anomalies.** We will consider the following anomalies of discounted utility model (*DU anomalies*):

1. *hyperbolic discounting*, which means that the instantaneous discount rate for gains *decreases* with time (in the DU model, it is constant);
2. the *sign effect* (gains are discounted more than losses);

3. the *delay-speedup* asymmetry: if the change of the delivery time of an outcome is perceived as an acceleration from a reference point, then the imputed discount rate is larger than if the change is perceived as a delay;
4. the *negative discounting for losses*: an individual may prefer to expedite a payment;
5. the *magnitude effect* (small outcomes are discounted more than large ones).

All these anomalies have been well-documented. For the discussion of DU anomalies and references to the literature on each type of DU anomalies, see the excellent review Frederick et al. (2002).

1.3. **Quasi-hyperbolic discounting and other resolutions.** To account for DU anomalies, several alternative (types of) models have been developed. In the (β, δ)-model of quasi-hyperbolic discounting introduced first by Phelps and Pollak (1968), equation (1.2) is replaced by

$$U = u(c_0) + \sum_{t=1}^{T} \delta\beta^t u(c_t)dt, \tag{1.3}$$

where $\beta, \delta \in (0, 1)$. Equation (1.3) is analytically simple, and captures many qualitative features of hyperbolic discounting. Thus, as in Samuelson (1933), the discount factors are postulated. Another strand of literature initiated by Koopmans (1960) deals with the axiomatic systems for time preferences, which are consistent with DU anomalies - see Ok and Masatlioglu (2003) and the bibliography therein. Fudenberg and Levine (2004) suggested a "dual-self" model as a unified explanation for several empirical regularities. Habit formation models, reference point models and a number of other models incorporate non-standard features into the utility function. Still other alternative models depart from the DU model even further (once again, we refer the reader to the review Frederick et al. (2002) for more details and extensive bibliography).

In this paper, we neither postulate the non-standard dependence of the discount factor on time as in the quasi-hyperbolic discounted utility models nor deduce it from time preference axioms. Instead, we show that the DU anomalies result from optimizing behavior of an individual facing a menu of options in an uncertain environment. To be more specific, we derive general explicit formulas for the discount factors for gains and losses from two simple general assumptions.

1.4. **Assumption 1.** Our starting point is that an individual perceives the future – hence the utility of consumption – as uncertain. The uncertainty may be caused by changes both in the anticipated consumption level and utility function per se: obviously, the satisfaction from possession of a certain widget may change (and typically, changes) in a not completely predictable fashion. Similar ideas are used in Gul and Pesendorfer (2004) ("changing tastes") and Manzini and Mariotti (2005) ("the perception of future events becomes increasingly "blurred" as the events are pushed further in time"), among the others.

We believe that this assumption can serve as an alternative to (or another interpretation of) the "multiple selves" hypothesis widely used in the literature on the discount rate anomalies. Instead of assuming that an individual has to play against "future selves", we simply presume that an individual cannot know for sure what her state of mind will be tomorrow, and how her preferences may change.

The importance of uncertainty is well-understood, and, as Frederick et al. (2002, p. 384) notice, once uncertainty and other confounding factors are accounted for, there may be no place left for "pure time preference". Dasgupta and Maskin (2005) show that if the "average" situation entails some uncertainty about the time when payoffs are realized, the corresponding preferences may well entail hyperbolic discounting.

Even if the future payoff is a certain amount of money, there is still uncertainty: \$100 for a person with a good job and \$100 for an unemployed person have different values; \$100 may have different values if a loved one is healthy or deathly ill (see, e.g., Karni (2005)); in addition, there is the inflation uncertainty. In combination, these and other factors make the consumption stream stochastic. Recall that in experiments on DU anomalies, people are typically asked to compare dollars "today" ($t = 0$) vs. dollars "tomorrow" ($t = 1$), and dollars at $t = \tau$ vs. dollars at $t = \tau + 1$. While the value of dollars "today" is certain, the future values are uncertain, so people are asked to compare a certain value vs. uncertain value, and one uncertain value vs. another uncertain value. No wonder that they treat them differently[1].

1.5. **Assumption 2.** The second (and crucial) assumption that we make is that the individual regards any exchange between current and future consumption gains or losses (expressed, say, in monetary terms) as a right but not an obligation, in other words, as an option. The rational individual chooses optimally the timing of the decision to exchange. Whenever the individual agrees to pay for the future consumption, she suffers not only the cost of this consumption, but also an opportunity cost: she loses the right to make the decision later. By postponing the decision till a later time period the individual can make the decision having observed partial revelation of the uncertainty. The individual agrees to buy the future consumption gain only if the present value of the utility of the gain covers not only the amount of money to be paid, but the opportunity cost as well.

In other words, the maximal amount of money that the individual agrees to pay for the future gain (we call it the *fair price*) is less than the present value of the utility of the future gain. Similarly, the minimal amount of money which the individual agrees to accept as a compensation for the future consumption losses (we call it the *fair compensation*) is larger than the present value of utility of losses, because the agent wants to be compensated both for losses and for the foregone opportunity of making the decision later.

[1]This observation explains, in the nutshell, the *quasi-hyperbolic* discounting; to account for more interesting shapes of the discount rate curves, more subtle arguments are needed.

1.6. **American options.** Recall that an American call (respectively, put) option is the right to sell (respectively, buy) an asset with a stochastic payoff at a given strike price. The option holder has to determine when (if ever) it is optimal to exercise the option. Here, we ask a sort of an "opposite" question: what strike price would make it optimal to exercise the option now? In other words, the fair price defined above is the maximal strike price that makes it optimal to exercise the American call option now. Similarly, the fair compensation is the minimal strike price which makes it optimal to exercise the American put option now.

Apart from providing a robust explanation of DU anomalies without resorting to exotic time preference, this approach may potentially have serious policy implications. For example, in asking how the population could or should be compensated in order to favor a proposed social security reform, a correct answer must take into account that the population has an option to wait. The correct compensation must be higher than the naive present value approach presumes.

Contingent valuation of environmental goods is another potential application of our results[2]. The contingent valuation method involves the use of sample surveys to elicit the willingness of respondents to pay for environmental programs or projects. For the history of the contingent valuation method and contingent valuation debate see Portney (1994), and Hanemann (1994). According to Portney (1994), one of the most influential papers in natural resource and environmental economics was "Conservation reconsidered" by Krutilla (1967). That paper identified the importance of the essentially irreversible nature of the development of natural resources and suggested that the difference between willingness-to-pay and willingness-to-accept compensation for "grand scenic wonders" may be large indeed. Hanemann (1992) presented a deterministic model that demonstrates that the differences in the willingness-to-pay and willingness-to-accept are due to the lack of substitutes for a public good. According to our results, compensation for losses requested by individuals is higher than the price the same individuals agree to pay for gains due to the presence of uncertainty and option-like nature of decisions. Thus, when facing a question of the sort "How much should the government pay for the damage to an endangered species", the same individual will name a greater price than when asked a question of the sort "How much should the government pay to preserve an endangered species."

1.7. **Results.** We demonstrate that the individual contemplating a decision leading to a gain or loss of consumption in the future and regarding such a gain or loss as a payoff of an option, uses the *ex ante* effective discount factors determined endogenously in our model, and these discount factors exhibit the aforementioned DU anomalies. Factors *ex ante* are different in these two situations, which naturally leads to the asymmetry of valuation of gains and losses in the future. After the decision has been made, the naive calculation of the EPV's becomes appropriate, and this calculation uses exponential discounting with *ex post* discount factors (also determined endogenously). We cannot

[2]We are thankful to Don Fullerton for a discussion of this point.

avoid the use of an exogenous background discount factor, killing rate[3], for valuations of losses if we keep the standard assumption that the instantaneous utility function is concave. In this case, if the exogenous discount rate is zero, the endogenous *ex post* discount rate is negative. So the standard assumption that the individual discounts the future at rate $r > 0$ is used. We show that for valuation of gains, the last assumption is unnecessary, because the endogenous *ex post* discount rate is positive.

1.8. **Preferences and uncertainty.** To demonstrate that under our assumptions, the optimizing behavior of the individual in an uncertain environment leads to a consistent explanation of the DU anomalies (certainly, other factors considered in the literature may contribute to the DU anomalies as well), we make standard assumptions about properties of the utility function (time and state additive separability) and underlying uncertainty. To account for preferences over gains and losses separately we assume that the individual assesses the utility of gains and disutility from losses as departures from some exogenous consumption path, $\{e_t\}_{t\geq 0}$, so that at time t, the instantaneous utility function is of the form $u(c_t) = \tilde{u}(c_t + e_t)$. For example, e_t may be viewed as the agent's endowment and c_t is the excess demand or supply at date t.

Notice that with our approach, the loss aversion of u is not required. To be clear, the instantaneous utility function for gains, $u_G(c) = u(c)$, $c > 0$, need not to be smaller than the (dis)utility function, $u_L(c) = -u(-c)$, $c > 0$, for losses in order to explain the delay-speedup asymmetry and sign effects (cf. Loewenstein (1988) and Loewenstein and Prelec (1992) who used the loss aversion assumption). Also, we do not impose the restriction that the utility function over losses is convex, as in the latter citation.

1.9. **Outline.** In Section 2 we present the main idea of our approach using the simplest models for utility/disutility function and the underlying uncertainty. The optimizing agent is risk-neutral, and uncertainty is resolved in one period. We explain why the fair price is always less, and the fair compensation is always larger than the expected present value of the utility of the future gain or loss, respectively. The sign effect, negative discounting, delay-speedup asymmetry, and quasi-hyperbolic discounting follow immediately. In Section 3, for simplest concave utility functions (CRRA) and uncertainty modelled as a Brownian motion, we provide analytical formulas for *ex ante* and *ex post* discount factors that demonstrate quasi-hyperbolic discounting and explain all of the other DU anomalies listed earlier except the magnitude effect. We observe that the sign effect and the delay-speedup asymmetry may be reproduced even with the naive (in the terminology of the real options theory) computation of the EPV if the individual is loss averse as in Loewenstein (1988) and Loewenstein and Prelec (1992). Essentially the same results are valid if the uncertainty is modelled as a Lévy process or random walk; we consider these cases in Appendix A. In Section 4, we examine more general models

[3]As Voland put it: "Of course man is mortal, but that's only half the problem. The trouble is that mortality sometimes comes to him so suddenly! And he cannot even say what he will be doing this evening." M. Bulgakov, *The Master and Margarita*, transl. by Michael Glenny, 1967, Hamper and Row, New York.

for utility/disutility functions; the uncertainty is modelled as the (geometric) Brownian motion (BM). We derive relatively simple analytical expressions for discount factors, and reproduce all of the DU anomalies listed earlier. In particular, we obtain more general form of the hyperbolic discounting than the quasi-hyperbolic one, and explain the magnitude effect. Sections 3–4 and Appendix A deal with the instantaneous consumption of a perishable good, and Section 5 with consumption of a durable good. In Appendix B, we model the dynamics of the underlying stochastic factor as a mean-reverting process. In all these sections, we study the most tractable case when an individual believes that she can wait as long as she pleases for an offer of a fair price. The case of a deadline (deterministic or random), when an individual believes that she can make her decision till moment T_1 in the future, is considered in some detail in Appendix C. All of the effects that are present with no deadline are also present when there is a deadline, but at this level of generality, the arguments are more difficult. The size of effects will be smaller but not negligible even if the deadline is imminent. Section 6 concludes. Some further technical details are relegated to Appendix D.

For the readers' convenience, we summarize the results of the paper in Table 1. Each of the rows in Table 1 represents a basic model. The key characteristics of each model are utility function, model uncertainty, and methodology (real options approach vs. naive approach). Columns of the table are the DU anomalies. Check marks in Table 1 indicate that a given model reproduces a given DU anomaly.

2. Value of waiting in a two-period model

This section demonstrates quasi-hyperbolic discounting, negative discounting, the sign effect, and the delay-speedup asymmetry in the simplest possible model.

2.1. **Decision problem.** A risk-neutral consumer facing a random sequence c_t of possible consumptions must choose the optimal time, t^*, to pay K dollars for the T-delayed random consumption, c_{t^*+T}. Assume that it is prohibitively costly to re-sell the consumption good so that the purchase is irreversible. Qualitative result will be the same if the resale price is smaller than the price of the new consumption good, rather than zero. To make things simple, let us assume that the consumer anticipates that at time $t = 1$, the consumption value may go up or down from the current level c with probabilities 1/2, and will remain at its new level forever (i.e., all the uncertainty is resolved at $t = 1$). To be more specific, for all $t \geq 1$, $c_t = c_1 \in \{(1-d)c; (1+d)c\}$, where $0 < d < 1$. This captures nominally fixed c with random marginal utility of consumption arising from a stochastic environment.

2.2. **Optimal strategy.** Let $\beta \in (0, 1)$ be an exogenous discount factor. Waiting with the decision to buy or not beyond $t = 1$ is non-optimal, because it reduces the present value of the potential gain without adding new information. Suppose that the consumer observes $c_0 = c$ and does not buy the good now. Then, at time $t = 1$, she makes a decision of the kind "now or never". If $K > \beta^T c_1$, she will not buy the good and her

TABLE 1

Model description	Hyperbolic disct'g	Sign effect	Negative disct'g	Delay-speedup asym.	Magnitude effect
Linear utility, simple uncertainty resolved in one period, real options approach (§ 2.3, 2.4)	√	√	√	√	
CRRA with loss aversion, BM, naive approach (§ 3.1)		√		√	
CRRA, BM, real options approach (§ 3.2)	√	√	√	√	
Non-homogeneous* utility functions, BM, real options approach (Sections 4 and 5)	√	√	√	√	√
CRRA, mean reversion, real options approach (Appendix B)	√	√	√	√	√

* Only rules out $u(c) = c^\gamma$

gain is 0, otherwise she will buy (as usual, we presume that she buys if she is indifferent between buying or not) and receive the payoff $\beta^T c_1 - K$. Thus we may conclude that the expected present value at $t = 0$ of buying the good at $t = 1$ given the current level of consumption, c, is

$$V^1(c) = \beta E\left[(\beta^T c_1 - K)_+ |\, c_0 = c\right] = \frac{1}{2}\beta\left(\beta^T(1+d)c - K\right)_+ + \frac{1}{2}\beta\left(\beta^T(1-d)c - K\right)_+$$

(here we use the standard notation $a_+ = \max\{a, 0\}$). We see that if $c < K\beta^{-T}/(1+d)$, then it is optimal to buy the consumption good in neither state at $t = 1$. If $K\beta^{-T}/(1+d) \le c < K\beta^{-T}/(1-d)$, then it is optimal to buy the good at $t = 1$ only if at date $t = 1$ consumption value goes up. For $c \ge K\beta^{-T}/(1-d)$, it is optimal to buy the good in

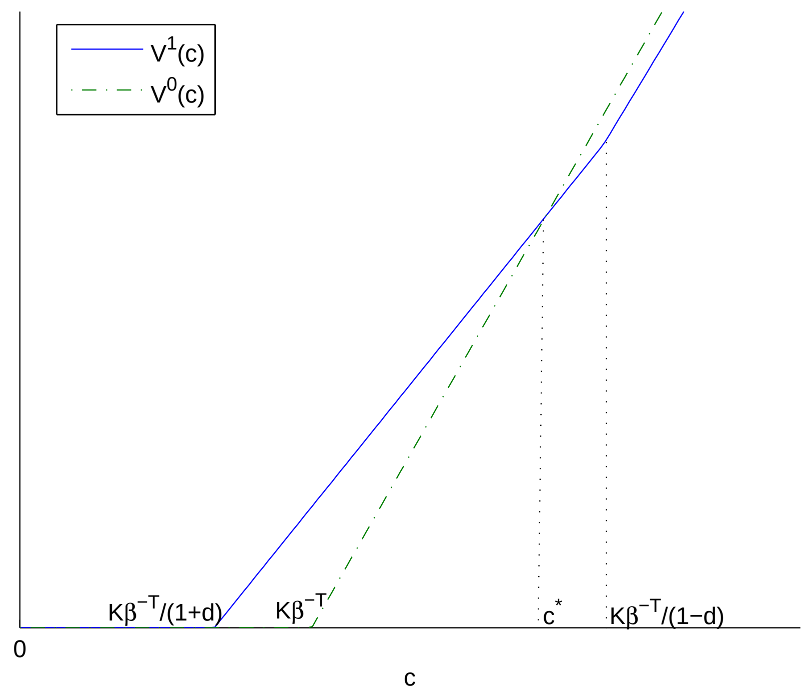

FIGURE 1. The value of the option to buy. Parameters: $T = 3$, $K = 3$, $d = 0.5$, $\beta = 0.95$

either state at $t = 1$. Hence (see Figure 1)

$$V^1(c) = \begin{cases} 0, & \text{if } c < K\beta^{-T}/(1+d), \\ \frac{\beta}{2}\left(\beta^T(1+d)c - K\right), & \text{if } K\beta^{-T}/(1+d) \leq c < K\beta^{-T}/(1-d), \\ \beta(\beta^T c - K), & \text{if } c \geq K\beta^{-T}/(1-d). \end{cases}$$

Consider the value of buying at $t = 0$:

$$V^0(c) = \frac{\beta^T}{2}(1+d)c + \frac{\beta^T}{2}(1-d)c - K = \beta^T c - K,$$

if $c \geq K\beta^{-T}$, and $V^0(c) = 0$ otherwise (see Figure 1).

The rational consumer chooses $\max\{V^0(c), V^1(c)\}$. Since $K\beta^{-T}/(1+d) < K\beta^{-T} < K\beta^{-T}/(1-d)$, and $\beta(1+d)/2 < 1$, there exists $c^* \in \left(K\beta^{-T}, K\beta^{-T}/(1-d)\right)$ such that $V^0(c^*) = V^1(c^*)$, i.e., at the spot value of consumption c^*, the consumer is indifferent between buying at $t = 0$ and $t = 1$ (see Figure 1). For all $c > c^*$, $V^0(c) > V^1(c)$, hence it is optimal to purchase c_T immediately. If $K\beta^{-T}/(1+d) < c < c^*$, then it is optimal to wait till $t = 1$ and buy only if the consumption goes up, otherwise, it is optimal not to buy at all.

2.3. **Emergence of quasi-hyperbolic discounting.** It is easy to derive $\beta^T c^* = K/\delta_G$, where $\delta_G = (1-\beta(1+d)/2)/(1-\beta/2) \in (0,1)$. For all c such that $\beta^T c \geq K/\delta_G$, it is optimal to buy at $t=0$. Notice that $\beta^T c = \beta^T E[c_T|\, c_0 = c]$, which is the expected present value (EPV) of the future consumption gain c_T. We conclude that in order it would be optimal to purchase c_T at $t=0$, the EPV of the future gain should exceed K by the factor $1/\delta_G > 1$. This is an analog of the correction factor in Dixit and Pindyck (1994). Let $K_G = K_G(c;T)$ be the maximal amount of money that the consumer would agree to pay today for the gain c_T. Evidently, $K_G(c;T) = \delta_G \beta^T c$, for any $T > 0$. Now we see that the rate of substitution between consumption at $t=0$ and $t=T$ is $\delta_G \beta^T$. At the same time, the rate of substitution between consumption a date $t > 0$ and $t+T > t$, as seen from time 0, is β^T. Thus we obtained the (β, δ)-model with $\delta = \delta_G < 1$.

2.4. **Sign effect, negative discounting, and delay-speedup asymmetry.** Similarly, one may consider a risk-neutral agent who will lose the consumption T periods after she gets the compensation K. Let the uncertainty be modelled as above. Then the agent has to decide whether to accept the compensation immediately, or at $t=1$, if ever. It is straightforward to show that there exists c_* such that the agent is indifferent between accepting the compensation at $t=0$ and waiting till $t=1$. For all $c \leq c_*$, it is optimal to accept the compensation immediately. The value of c_* is given by $\beta^T c_* = K/\delta_L$, where $\delta_L = (1-\beta(1-d)/2)/(1-\beta/2) > 1$. Let $K_L = K_L(c;T)$ be the minimal amount of money that the consumer would agree to accept today for the loss c_T. Evidently, $K_L(c;T) = \delta_L \beta^T c$, for any $T > 0$. This means that the fair compensation always exceeds the EPV of the utility of the loss by the factor $\delta_L > 1$. Now we see that the rate of substitution between consumption at $t=0$ and $t=T$ is $\delta_L \beta^T$. At the same time, the rate of substitution between consumption a date $t > 0$ and $t+T > 0$, as seen from time 0, is β^T. Thus we obtained the (β,δ)-model with $\delta = \delta_L > 1$. For $T < \log \delta_L / (-\log \beta)$ the negative discounting for losses is observed. Since $\delta_G < 1 < \delta_L$, gains are discounted more than losses. The delay-speedup asymmetry evidently follows.

3. Basic examples

If the underlying uncertainty is less trivial than in the previous Section, then it becomes possible to separate corrections to the exogenous discount factors that are due to the presence of uncertainty only from those that emerge from timing decisions. The discount factors that are used for the calculation of the standard EPV's of the future payoffs, and therefore account for uncertainty only, are called *ex post* discount factors. The discount factors that are used to evaluate the EPV's of the future payoffs together with their option value, are called discount factors *ex ante*.

3.1. **Uncertainty corrections to the background discount factor.** In this subsection, we analyze how the observed discount factors should change if only uncertainty is taken into account but no decision-making is involved. Assume that the (instantaneous) utility of future gains $u_G(c_t)$ and disutility of future losses $u_L(c_t)$ are random variables.

In our considerations, only the dynamics of $u_G(c_t)$ and $u_L(c_t)$ matter, and the same dynamics can be observed for different pairs: utility/disutility function and consumption c_t. For simplicity of presentation, in this paper, we assume that the utility/disutility functions do not change over time but consumption, c_t, is stochastic. To ensure that c_t is positive, we model it as $c_t = e^{X_t}$. In this section, we make the simplest assumptions:

(i) $u_G(c) = c^{\gamma_G}$ and $u_L(c) = c^{\gamma_L}$, with, possibly, $\gamma_G \neq \gamma_L$; for the utility function over gains, $\gamma = \gamma_G \in (0,1)$ but for disutility function over losses, we admit any $\gamma = \gamma_L > 0$;
(ii) X_t is the Brownian motion with drift μ and variance σ^2.

Let us first consider gains, for which $u_G(c_t) = e^{\gamma X_t}$. The discounted expected utility of consumption T periods from now is

$$e^{-rT}E^x[u_G(c_T)] = e^{-rT}E^x[e^{\gamma X_T}], \tag{3.1}$$

where $x = \log c$ is the current value of the stochastic factor, $E^x[f(X_t)] := E[f(X_t)|\, X_0 = x]$ is the conditional expectation operator. Since the moment generating function of a random variable $y \sim N(\mu, \sigma^2)$ is

$$E[e^{\gamma y}] = e^{\Psi(\gamma)},$$

where $\Psi(\gamma) = \frac{\sigma^2}{2}\gamma^2 + \mu\gamma$, and X_T, conditioned on $X_0 = x$, is distributed as a normal variable with the mean $\mu T + x$ and variance $\sigma^2 T$, we derive

$$e^{-rT}E^x[e^{\gamma X_T}] = e^{-(r-\Psi(\gamma))T}e^{\gamma x} = e^{-(r-\Psi(\gamma))T}u(c). \tag{3.2}$$

Formula (3.2) can be interpreted as follows: an individual discounts the future consumption at rate $r^n_G = r - \Psi(\gamma)$, or the discount factor is not e^{-rT} but $e^{-(r-\Psi(\gamma))T}$.

Several simple observations are in order. First, if we presume that an individual anticipates that the future consumption level c_t will be the same as today, on average, i.e., $c = e^x = E^x[c_T] = e^{\Psi(1)T}e^x$, then we have $\Psi(1) = 0$. Since Ψ is convex and $\Psi(0) = 0$, we have $\Psi(\gamma) < 0$ for $\gamma \in (0,1)$. Therefore, future gains are discounted more than in the standard exponential discounting model: $r^n_G = r - \Psi(\gamma) > r$, but the discounting remains exponential. We will call $\mathcal{D}^n_G := \mathcal{D}^n_G(T) = e^{\Psi(\gamma)T}$ the *uncertainty correction* to the *background discount factor* e^{-rT}.

Second, we observe the exponential discounting even if the *background discount rate*, r, is zero[4]. The observed discount rate $-\Psi(\gamma)$ will be positive. We may say that if $r = 0$ then $\mathcal{D}^n_G(T)$ is the *endogenous* discount factor, and $-\Psi(\gamma) > 0$ is the *endogenous* discount rate.

Third, the same conclusions hold if we consider the disutility of losses $u_L(c) = c^\gamma$ with $\gamma \in (0,1)$. However, if we assume that for losses, $\gamma > 1$, then $\Psi(\gamma) > 0$, and the correction factor $\mathcal{D}^n_L(T)$ to the discount factor for losses is larger than 1. Therefore, in this case, we need to have the positive background rate r, and it must be larger than $\Psi(\gamma)$ (otherwise, we get arbitrary large negative discounting for losses in the distant

[4]That is, the individual totally ignores the possibility of dying.

future; the losses at the infinity are felt as infinite ones). Further, if $0.5 < \gamma_G < \gamma_L < 1$, we conclude that the gains are discounted more than losses. Hence, the sign effect can be reproduced, and the delay-speedup asymmetry naturally follows. Notice that here we need to use different u_G and u_L as in Loewenstein (1988) and Loewenstein and Prelec (1992) who explained the delay-speedup asymmetry using the loss aversion assumption $u_L(c) > u_G(c)$, for all $c > 0$.

Notice also that the naive correction factors $\mathcal{D}^n_G(T)$ and $\mathcal{D}^n_L(T)$ to the exponential discounting formula cannot explain the hyperbolic discounting, negative discounting and magnitude effect. In other words, the naive pricing formulas

$$K^n_G = K^n_G(u;x) = e^{-rT}E^x\left[u_G(c_T)\right] \tag{3.3}$$

and

$$K^n_L = K^n_L(u;x) = e^{-rT}E^x\left[u_L(c_T)\right] \tag{3.4}$$

for the gain or loss in consumption c_T at time T, respectively, do not lead to the hyperbolic discounting, negative discounting and magnitude effects. However, as experiments show, behavior of people in real life exhibit these effects. The reason is that an individual, who is under no obligation to buy or sell the right for the payoff c_T instantly, and can wait for a more favorable realization of the underlying uncertainty, prices the future gains or losses not as prescribed by (3.3) and (3.4). This fact is well-known in the theory of financial and real options. One says that there exists the *option value of waiting*.

3.2. **Real options approach and DU anomalies.** We now deduce formulas for the discount factors $\mathcal{D}_G(T)$ and $\mathcal{D}_L(T)$, which take into account not the uncertainty only as the factors $\mathcal{D}^n_G(T)$ and $\mathcal{D}^n_L(T)$ above, but the decision-making aspect as well. We will call the former factors discount factors *ex ante*, and the latter - discount factors *ex post* (for gains and losses, respectively). From these formulas, the discounted utility anomalies will follow.

Our crucial assumption is as follows: when contemplating an acquisition of an instantaneous payoff c_T, the individual regards this action as an American call option, either a perpetual one or with a finite (possibly random) maturity date, and prices this option accordingly. As a result, in an experiment, the individual agrees to pay, at most, the fair price for this future payoff, i.e., the maximal strike price that makes it optimal to exercise the option at the date specified in the experiment. Similarly, when contemplating losses, the individual views this action as an American put option. In this case, the individual agrees to get the compensation, which is at least fair from her point of view, i.e., the minimal strike price that makes it optimal to exercise the put option at the date specified in the experiment. Notice that unlike traders in financial markets, our individual anticipates neither the presence of arbitrageurs to exploit the arising "natural bid-ask spread," nor appearance of competing consumers willing to offer a higher price for the good to be delivered in the future. Hence, our assumption does not contradict basic principles of the rational behavior unless we apply it to traders in efficient financial markets.

3.2.1. *Valuation of future gains and perpetual American call options.* We now calculate the fair strike price, K_G, that makes it optimal to exercise now a perpetual American option to acquire an instantaneous payoff c_T (in dollar terms) T periods from now. Suppose that the individual discounts the future at rate $r > 0$. Given the current realization of the stochastic factor, X_t, is equal to x, the expected present value of her utility of consumption is $e^{-rT}E^x[u_G(c_T)]$. So the payoff function is $g(x) = e^{-rT}E^x[u_G(c_T)] - K_G$. Recall that in this section, we model X_t as the Brownian motion. We assume that the *no-bubble* condition $r^n_G = r - \Psi(\gamma) > 0$ holds, equivalently, the expected present value of the utility of consumption $e^{-rT}E^x[u(c_T)]$ decreases as T increases.

Whenever X_t is at or above certain level $h^* = h^*(K_G, T)$, it is optimal to exercise the option. The optimal exercise boundary, h^*, for the perpetual American call option was derived by McKean (1964). In our case, h^* solves the equation

$$e^{\gamma h^*} = \frac{\beta^+}{\beta^+ - \gamma} e^{T(r-\Psi(\gamma))} K_G, \tag{3.5}$$

where $\beta^+ > \gamma$ is the positive root of the "fundamental quadratic" $r - \Psi(\beta) = 0$. The inequality $\beta^+ > \gamma$ follows from the no-bubble condition.

Suppose that the current realization of the stochastic factor, x, is such that the individual finds it optimal to exercise the option right now, i.e., $x = h^*(K_G, T)$. Then (3.5) defines the highest strike price that makes such an exercise optimal. This strike price is

$$\begin{aligned} K_G &= (1 - \gamma/\beta^+)e^{\gamma x-(r-\Psi(\gamma))T} = (1 - \gamma/\beta^+)e^{-(r-\Psi(\gamma))T}u(c) \\ &= (1 - \gamma/\beta^+)e^{-rT}E^x[u_G(c_T)]. \end{aligned} \tag{3.6}$$

Recall that the naive fair price K^n_G is given by (3.3), therefore $K_G = (1 - \gamma/\beta^+)K^n_G$. According to our earlier definition, the price K_G defined by (3.6) is the fair strike price of the perpetual American call option. Of course, the individual will agree to pay for c_T any price that is smaller than K_G as well if she faces a menu of options with different strikes.

Observe that $1 - \gamma/\beta^+ \in (0, 1)$, therefore the fair price for the delivery of a payoff in the future, K_G, is always less than the naive fair price K^n_G. Introduce $\mathcal{D}^c_G = (1 - \gamma/\beta^+)$ - the correction factor to the naive exponential discount factor $\mathcal{D}^n_G(T)$. Thus, we recover the quasi-hyperbolic discounting model for gains:

$$\mathcal{D}_G(T) = \delta e^{-(r-\Psi(\gamma))T}$$

with $\delta = \mathcal{D}^c_G$. To conclude this section, we notice that if the individual has to name the fair price she would agree to pay at time $t > 0$ for the payoff c_{t+T} that she will get T periods after date t, then the fair strike price is

$$K_G(t) = (1 - \gamma/\beta^+)e^{-rT}E_t[u_G(c_{t+T})],$$

where E_t is the expectation operator conditioned on the information available at date t.

3.2.2. *Valuation of future losses and perpetual American put options.* A similar argument applies if the individual can get a compensation, K_L, now in exchange for the loss of a consumption c_T T periods from now. Assuming that the instantaneous disutility of losses is measured by a function u_L (so that $-u_L(c_T)$ is the utility of a negative consumption $-c_T$: $-u_L(c_T) = u(-c_T)$), the present value of the loss is $e^{-rT}E^x[u_L(c_T)]$. Now the individual regards herself as the holder of the American put option with the payoff function $g(x) = K_L - e^{\gamma x-(r-\Psi(\gamma))T}$. Assume that the option is perpetual.

The optimal exercise boundary, $h_* = h_*(K_L, T)$, for the perpetual American put option was derived by Merton (1973). In our case, h_* solves the equation

$$e^{\gamma h_*} = \frac{\beta^-}{\beta^- - \gamma} e^{T(r-\Psi(\gamma))}K_L, \tag{3.7}$$

where $\beta^- < 0$ is the negative root of the "fundamental quadratic" $r - \Psi(\beta) = 0$. Let the current realization of the stochastic factor, x, be such that the individual finds it optimal to exercise the option right now, i.e. $x = h_*(K_L, T)$. Then (3.7) defines the lowest strike price that makes such an exercise optimal. This strike price is

$$\begin{aligned} K_L &= (1-\gamma/\beta^-)e^{\gamma x - T(r-\Psi(\gamma))} = (1-\gamma/\beta^-)e^{-(r-\Psi(\gamma))T}u(c) \\ &= (1-\gamma/\beta^-)e^{-rT}E^x[u_L(c_T)] = (1-\gamma/\beta^-)K_L^n. \end{aligned} \tag{3.8}$$

The last equality holds on the strength of formula (3.4) for the naive fair compensation, K_L^n. Clearly, the individual will agree to accept any compensation that is higher than or equal to K_L, therefore K_L is the fair strike price of the American put option that is exercised on the spot.

Introduce the correction factor to the naive exponential discount factor $\mathcal{D}_L^n(T)$ as $\mathcal{D}_L^c = 1 - \gamma/\beta^- > 1$. Thus, we derived the quasi-hyperbolic discounting model for losses

$$\mathcal{D}_L(T) = \delta e^{-(r-\Psi(\gamma))T}$$

with $\delta = \mathcal{D}_L^c$, and for $T < (\log \mathcal{D}_L^c)/(r - \Psi(\gamma))$, the negative discounting will be observed. Also, from (3.8) it is evident that $K_L > K_L^n$, i.e., the fair compensation for the loss of a payoff in the future is always higher than the naive fair compensation.

Lastly, if the individual has to specify the minimal compensation she would agree to accept at date t for the loss she would incur in T periods from date t, the compensation would be

$$K_L(t) = (1-\gamma/\beta^-)e^{-rT}E_t[u_L(c_{t+T})].$$

We have demonstrated the quasi-hyperbolic discounting and negative discounting effects. The following simple general arguments explain the sign effect and delay-speedup asymmetry effect.

3.2.3. *Sign effect.* In Subsection 3.1, we showed that the sign effect is observed iff $\gamma_G < \gamma_L$, i.e., if the preferences exhibit loss aversion. Let $\gamma_G = \gamma_L = \gamma$, then we have

$$\mathcal{D}_G(T) = (1-\gamma/\beta^+)e^{-(r-\Psi(\gamma))T} < (1-\gamma/\beta^-)e^{-(r-\Psi(\gamma))T} = \mathcal{D}_L(T).$$

Thus, our model shows that even if the disutility function for losses is identical to the utility function for gains, the discount factor *ex ante* for gains is smaller than the discount factor *ex ante* for losses. Thus, gains are discounted more than losses.

3.2.4. *Delay-speedup asymmetry.* Assume that the individual is asked whether she is willing to delay the delivery of a widget (say, a DVD–player) and receive it at date T instead of the present date. She is offered the compensation K_{delay} for the expected disutility of losses during the period $[0, T]$. Assuming that the decision to suffer losses has been made, the individual evaluates the disutility of losses using the discount factor *ex post* for losses; and she finds

$$E^x\left[\int_0^T e^{-rt}u_L(c_t)dt\right] = \int_0^T e^{-rt}E^x[u_L(c_t)]dt = e^{\gamma x}\int_0^T e^{-rt+\Psi(\gamma)t}dt = e^{\gamma x}\frac{1-e^{-T(r-\Psi(\gamma))}}{r-\Psi(\gamma)}.$$

(Here $\gamma = \gamma_L$). The same argument as is Subsection 3.2.2 demonstrates that she does not consider an offer fair unless K_{delay} exceeds $E^x\left[\int_0^T e^{-rT}u_L(c_t)dt\right]$ by factor $1-\gamma/\beta^- > 1$. Similarly, if she expects the delivery at time $T > 0$, then the instant delivery provides an additional utility stream over the period $[0, T]$, $E^x\left[\int_0^T e^{-rT}u_L(c_t)dt\right]$. If the individual is asked to pay K_{exp} to expedite the delivery, she is offered the American call option with the payoff

$$E^x\left[\int_0^T e^{-rt}u_L(c_t)dt\right] - K_{\text{exp}} = e^{\gamma x}\frac{1-e^{-T(r-\Psi(\gamma))}}{r-\Psi(\gamma)} - K_{\text{exp}}.$$

(Here $\gamma = \gamma_G$). As the argument is Subsection 3.2.1 demonstrates, she should be willing to pay K_{exp} which is not larger than $(1-\gamma/\beta^+)E^x\left[\int_0^T e^{-rT}u_L(c_t)dt\right]$. Even if $u_L(c) = u_G(c)$, for all $c > 0$ (the disutility function for losses equals the utility function for gains), equivalently, $\gamma_L = \gamma_G$, we observe the delay-speedup asymmetry:

$$K_{\text{exp}} \le (1-\gamma/\beta^+)(1-\gamma/\beta^-)^{-1}K_{\text{delay}},$$

whence $K_{\text{exp}} < K_{\text{delay}}$. The individual asks more as a compensation for the delay than she is willing to pay to expedite the payment. Notice that Loewenstein (1988) and Loewenstein and Prelec (1992) explained the delay-speedup asymmetry using the loss aversion assumption $u_L(c) > u_G(c)$, for all $c > 0$.

4. Hyperbolic discounting and magnitude effect

In order to obtain richer patterns of time dependence of discount factors that distinguish hyperbolic from quasi-hyperbolic discounting and reproduce the magnitude effect, one has to consider either more general than CRRA utility functions or to work with mean-reverting processes instead of processes with i.i.d. increments. In this section, we consider an increasing concave utility function u and Brownian motion uncertainty. The case of mean-reverting processes is relegated to Appendix B.

4.1. **General utility function, Brownian motion case.** As in the previous section, we define the utility function over gains, $u_G(c) = u(c), c > 0$, and disutility function over losses, $u_L(c) = -u(-c), c > 0$, and we model the stochastic consumption level as the Geometric Brownian motion: $c_t = e^{X_t}$. We assume that the expected present value of consumption $e^{rt}E^c[c_t]$ is bounded as $t \to +\infty$; equivalently, $\beta^+ > 1$.

Theorem 4.1. *The correction factors for gains and losses to the naive exponential discount factor are*

$$\mathcal{D}_G^c(c,T) = \frac{E[u(c_T) - (1/\beta^+)c_T u'(c_T) \mid c_0 = c]}{E[u(c_T) \mid c_0 = c]} \tag{4.1}$$

and

$$\mathcal{D}_L^c(c,T) = \frac{E[u(c_T) - (1/\beta^-)c_T u'(c_T) \mid c_0 = c]}{E[u(c_T) \mid c_0 = c]} \tag{4.2}$$

Proof. Set $v_G(x) = u_G(c(x))$. If the individual is offered the instantaneous consumption c_T at time T for K dollars, she is offered the perpetual American call option with the strike K and payoff function $g(x) = e^{-rT}E^x[v_G(X_T)] - K$. Direct calculations show that

$$\text{function } v_G - (1/\beta^+)v'_G \text{ is increasing.} \tag{4.3}$$

Indeed, assuming for simplicity that u''_G exists (and is negative) and using $\beta^+ > 1$ and $u'_G > 0$, we find that the derivative of $v_G(x) - (1/\beta^+)v'_G(x) = u_G(c) - (1/\beta^+)cu'_G(c)$ w.r.t. $c (= e^x)$ is positive:

$$(u_G(c) - (1/\beta^+)cu'_G(c))' = u'_G(c) - (1/\beta^+)u'_G(c) - (1/\beta^+)cu''_G(c) > 0.$$

In Subsection D.1, we prove the following lemma.

Lemma 4.2. *Assume that condition (4.3) hold, and*

$$(v_G - (1/\beta^+)v'_G)(-\infty) < Ke^{rT} < (v_G - (1/\beta^+)v'_G)(+\infty) \tag{4.4}$$

Then the optimal exercise boundary of the American call with the payoff function g is the unique solution of the equation

$$K = e^{-rT}E^{h^*}[v_G(X_T) - (1/\beta^+)v'_G(X_T)]. \tag{4.5}$$

Assuming that the current level is $X_0 = x$, and the instantaneous consumption c_T at time T is offered for K_G dollars, the individual considers as fair the price given by (4.5) with $h^* = x$ and $K = K_G$; hence,

$$K_G = e^{-rT}E[u_G(c_T) - (1/\beta^+)c_T u'_G(c_T) \mid c_0 = c].$$

Since the naive pricing formula is $K_G^n = e^{-rT}E[u_G(c_T) \mid c_0 = c]$, we obtain (4.1).

To prove (4.2), we set $v_L(x) = u_L(e^x)$. Assuming for simplicity that u'' exists and is negative, we conclude that u''_L exists, and it is positive. Using $\beta^- < 0$ and $u'_L > 0$, we find that

$$(u_L(c) - (1/\beta^-)cu'_L(c))' = u'_L(c) - (1/\beta^-)u'_L(c) - (1/\beta^-)cu''_L(c) > 0.$$

Hence,

$$\text{function } v_L - (1/\beta^-)v'_L \text{ is increasing.} \tag{4.6}$$

If the individual may get a compensation K dollars now for the loss of instantaneous consumption c_T at time T, she holds the perpetual American put option with the strike K and payoff function $g(x) = K - e^{-rT}E^x[v(X_T)]$. In Subsection D.2, we prove the following lemma.

Lemma 4.3. *Assume that condition (4.6) and the following condition hold:*

$$(v_L - (1/\beta^-)v'_L)(-\infty) < Ke^{rT} < (v_L - (1/\beta^-)v'_L)(+\infty) \tag{4.7}$$

Then the optimal exercise boundary of the European put with the payoff function g is the unique solution of the equation

$$K = e^{-rT}E^{h_*}[v_L(X_T) - (1/\beta^-)v'_L(X_T)]. \tag{4.8}$$

Assuming that the current level is $X_0 = x$, and a compensation K dollars is offered now for the loss of instantaneous consumption, the individual considers as fair the compensation given by (4.8) with $h_* = x$ and $K = K_L$; hence,

$$K_L = e^{-rT}E[u_L(c_T) - (1/\beta^+)c_T u'_L(c_T) \mid c_0 = c].$$

Since the naive pricing formula is $K^n_L = e^{-rT}E[u_L(c_T) \mid c_0 = c]$, we obtain (4.2). $\square$

4.2. **Sign effect, delay-speedup asymmetry and negative discounting.** Since v' is positive and $-1/\beta^- > 0 > -1/\beta^+ > -1$, we see that the numerator in (4.1) (respectively, (4.2)) is smaller (respectively, larger) than the denominator. Hence, the correction factor for gains is smaller than 1, and the correction factor for losses is larger than 1. All three effect are immediate.

4.3. **Magnitude effect and hyperbolic discounting: small consumption levels.** Assume that the current level $c = c_0$ is small. Then, if the variance $T\sigma_T^2$ of X_T is not large, c_T will be small with high probability. Assuming that u_G is twice continuously differentiable at 0, increasing and concave, we may approximate $u(c_T)$ and $c_T u'(c)$:

$$\begin{aligned} u(c_T) &= \alpha_0 c_T + \alpha_1 c_T^2 + \cdots, \\ c_T u'(c_T) &= \alpha_0 c_T + 2\alpha_1 c_T^2 + \cdots, \end{aligned}$$

where $\alpha_0 > 0$, $\alpha_1 < 0$. Using $E^c[c_T^k] = E^x[e^{kX_T}] = c^k e^{T\Psi(k)}$ and assuming that c_t is a martingale: $E^c[c_t] = c$, we obtain from (4.1) an approximate equality

$$\begin{aligned} \mathcal{D}^c_G(c,T) &= \frac{\alpha_0(1-1/\beta^+)c + \alpha_1(1-2/\beta^+)e^{T\Psi(2)}c^2}{\alpha_0 c + \alpha_1 e^{T\Psi(2)}c^2} + O(c^2) \\ &= (1-1/\beta^+)\left(1 + \left(\frac{\alpha_1(1-2/\beta^+)}{\alpha_0(1-1/\beta^+)} - \frac{\alpha_1}{\alpha_0}\right)e^{T\Psi(2)}c\right) + O(c^2), \end{aligned}$$

and after simplification,

$$\mathcal{D}^c_G(c,T) = (1-1/\beta^+)\left(1+\frac{-\alpha_1}{\alpha_0(\beta^+-1)}e^{T\Psi(2)}c\right) \tag{4.9}$$

Since $\beta^+ > 1$, $\Psi(2) > 0$ and $-\alpha_1, \alpha_0 > 0$, we conclude from (4.9) that the discount factor increases both in c and T (while c remains small). Hence, the magnitude effect is observed (large gains are discounted less than small ones), and the discount rate ex ante for gains decreases as T increases.

Since $u_L(c) = -u(-c) = \alpha_0 c - \alpha_1 c^2 + \cdots$, the analog of (4.9) for losses is

$$\mathcal{D}^c_L(c,T) = (1-1/\beta^-)\left(1+\frac{\alpha_1}{\alpha_0(\beta^--1)}e^{T\Psi(2)}c\right) \tag{4.10}$$

Since $\beta^- < 0$, $\Psi(2) > 0$ and $-\alpha_1, \alpha_0 > 0$, we conclude from (4.9) that the discount factor increases both in c and T (while c remains small). Hence, the magnitude effect is observed (large losses are discounted less than small ones), and the discount rate ex ante for gains decreases as T increases.

In Subsection D.3, we derive the magnitude effect and hyperbolic discounting at moderate consumption levels.

5. Consumption of a durable good

Suppose now that the good is durable, and hence, an individual contemplates the gain of a consumption stream c_t, which she expects to consume from the moment $T > 0$ till infinity or during T' years, say. We will demonstrate the same effects as in the case of instantaneous consumption. Below we consider the case when uncertainty is modelled as a general Lévy process. The case of the Ornstein-Uhlenbeck process is relegated to Appendix D.

5.1. **Utility and disutility functions of the form** $u_G(c) = c^\gamma, u_L(c) = c^\gamma$**.** The expected present value of consumption

$$E^x\left[\int_T^{T+T'} e^{-rt}u_G(c_t)dt\right] = E^x\left[\int_T^{T+T'} e^{-rt+\gamma X_t}dt\right] = e^{\gamma x}\int_T^{T+T'} e^{-t(r-\Psi(\gamma))}dt$$

can be represented as

$$e^{\gamma x - T(r-\Psi(\gamma))}A(T,T') = A(T,T')e^{-rT}E^x[u_G(c_T)],$$

where the Lévy exponent $\Psi(\gamma)$ is defined from $E^x[e^{\gamma X_t}] = e^{t\Psi(\gamma)+\gamma x}$, and

$$A(T,T') = \frac{1-e^{-T'(r-\Psi(\gamma))}}{r-\Psi(\gamma)}.$$

This value differs from the expected present value of the instantaneous consumption at date T by a constant factor $A(T,T')$, and therefore the arguments for the instantaneous consumption case apply. The results are the quasi-hyperbolic discounting models for gains and losses with the correction factors $\delta \in (0,1)$ and $\delta > 1$, respectively, and the negative discounting for losses.

5.2. **Utility and disutility functions of the general form.** In this subsection, X_t is a Lévy process, and the durable good will be consumed since moment T till infinity; the present is $t = 0$. Set $v(x) = u(c(x)) = u(e^x)$ and $w(T,x) = e^{-rT}E^x[v(X_T)]$. The expected present value of consumption of this stream can be represented in the form

$$e^{-rT}E^x\left[E^{X_T}\left[\int_T^{+\infty} e^{-rt}v(X_t)dt\right]\right] = E^x\left[\int_0^{+\infty} e^{-rt}w(T;X_t)dt\right].$$

Since u is an increasing function of c, both v and w are increasing functions of x. Therefore, an individual who is offered to pay K dollars for this stream, is offered the payoff which can be represented as the right to the perpetual stream $g(T;X_t) = w(T;X_t) - rK$. It has been proved in Boyarchenko (2004) for payoffs of the form $Ae^{X_t} - B$ and general Lévy processes, and in Boyarchenko and Levendorskiĭ (2005, 2004a,b) for general increasing payoffs g and wide classes of Lévy processes and random walks, that the perpetual American option on the stream g must be exercised the first time the expected present value of the stream g under the infimum process $\underline{X}_t = \inf_{0\le s\le t} X_s$ becomes non-negative. Hence, the optimal exercise boundary h^* is the unique solution of the equation

$$E^{h^*}\left[\int_0^{+\infty} e^{-rt}g(\underline{X}_t)dt\right] = 0 \tag{5.1}$$

(this is the *record-setting bad news principle* developed in Boyarchenko (2004) as a modification and generalization of Bernanke's (1983) bad news principle). For the stream $g(T;X_t) = w(T;X_t) - rK$, (5.1) can be reformulated as follows:

$$E^{h^*}\left[\int_0^{+\infty} e^{-rt}w(T;\underline{X}_t)dt\right] = rK. \tag{5.2}$$

The naive NPV equation for the threshold is

$$E^{h^*}\left[\int_0^{+\infty} e^{-rt}w(T;X_t)dt\right] = rK. \tag{5.3}$$

If at the current level $x = X_0$, the individual finds it optimal to exercise the option, she must regard the price

$$K = r^{-1}E^x\left[\int_0^{+\infty} e^{-rt}w(T;\underline{X}_t)dt\right] \tag{5.4}$$

as fair, whereas the standard expected exponential discounting rule gives the same formula but with X_t instead of $\underline{X}_t$. We conclude that the correction factor for gains is

$$\mathcal{D}^c_G(u_G;T,x) = \frac{E^x\left[\int_0^{+\infty} e^{-rt}w(T;\underline{X}_t)dt\right]}{E^x\left[\int_0^{+\infty} e^{-rt}w(T;X_t)dt\right]}. \tag{5.5}$$

Excluding cases when the trajectories of the process X_t are non-decreasing a.s., we find that $\mathcal{D}^c_G(u_L; T, x) < 1$, and the hyperbolic discounting is observed.

Similarly, if the individual can get a compensation of K dollars for the loss of a perpetual consumption stream which she expects to enjoy from moment T from now, then she finds the following price fair:

$$K = r^{-1}E^x\left[\int_0^{+\infty} e^{-rt}w(T; \bar{X}_t)dt\right] \tag{5.6}$$

(the corollary of the *record-setting good news principle* in Boyarchenko (2004) and Boyarchenko and Levendorskiǐ (2004a,b, 2005b)). We conclude that the correction factor for gains is

$$\mathcal{D}^c_L(u_L; T, x) = \frac{E^x\left[\int_0^{+\infty} e^{-rt}w(T; \bar{X}_t)dt\right]}{E^x\left[\int_0^{+\infty} e^{-rt}w(T; X_t)dt\right]}. \tag{5.7}$$

Excluding cases when the trajectories of the process X_t are non-increasing a.s., we find that $\mathcal{D}^c_L(u_L; T, x) > 1$, and the negative discounting is observed. Assuming that the trajectories move up and down with non-zero probability, we find that $\mathcal{D}^c_G(u_G; T, x) < \mathcal{D}^c_L(u_L; T, x)$ even if $u_G = u_L$. The sign effect and the delay-speedup asymmetry effects follow but we cannot prove the magnitude effect; however, we demonstrated it numerically for many parameter values.

6. Conclusion

In the paper, we explained the observed discount utility anomalies as resulting from the optimizing behavior of an individual who believes that she can wait for some time for an offer of a fair price, and regards the future consumption as uncertain. We define the fair price as the strike price of the American option for which the current level of the stochastic factor is at the optimal exercise boundary. This is the call option if an individual may pay now for delivery of a consumption good or a certain amount of money in the future, and the put option, if she may get a compensation now for the loss of consumption in the future. Using the results for the American options, we showed that for this individual, the fair price for gains in the future is always smaller than the price which the standard expected exponential discounting model predicts, and the fair compensation for losses is larger than in the standard model. From this observation, the negative discounting for losses, sign effect and delay-speedup asymmetry follow. Modelling the logarithm of consumption $X_t = \log c_t$ as the Brownian motion, other processes with i.i.d. increments, and Ornstein-Uhlenbeck process, we derived explicit formulas for the discount factors which the individual uses to evaluate gains and losses (discount factors ex ante as opposed to discount factors ex post in the standard model which she uses to evaluate the consumption when no decision is involved), and demonstrated the hyperbolic discounting. We identified the quasi-hyperbolic (β, δ) model (and its natural continuous time analog) as the model for an individual who

(i) has the utility function for gains and the disutility function for losses of the form $u_G(c) = c^\gamma$, $u_L(c) = c^\gamma$ (possibly, with different γ's);
(ii) anticipates the uncertainty as a process with i.i.d. increments.

For this individual, the discount factors ex ante are independent of the consumption level, and so the magnitude effect cannot be demonstrated.

For more involved utility/disutility functions or uncertainty modelled as the Ornstein-Uhlenbeck process, larger gains/losses are discounted less than small ones. For the Ornstein-Uhlenbeck process, this fact is proved analytically, and for the Brownian motion, it is proved analytically at small levels of consumption and at moderate levels of consumption if the uncertainty is small. Numerical examples support these claims.

We explained that if the deadline for making a choice of a fair price is introduced, and hence the individual can be regarded as a holder of an American option with finite maturity T_1 instead of perpetual American option, then the discount utility anomalies become smaller but for many specifications of uncertainty, they do not vanish even in the limit $T_1 \to +0$. We concluded that the discount utility anomalies should be observed in the case of a random deadline. We showed that if the deadline is imminent then the quasi-hyperbolic discounting is a good approximation in all cases.

The qualitative and quantitative results obtained in the paper admit the empirical verification. The following hypotheses can be tested in experiments:

(1) *Competition decreases the size of anomalies.* Experiment # 1 (standard one): participants are asked to choose the fair price from a wide menu of prices. Experiment # 2 (the second price sealed-bid auction): participants are asked to do the same but are told that the future payoff will be given to a person who offers the highest bid. We expect that in Experiment #2, the DU anomalies will be smaller because there may be no value of waiting for more information if an object becomes unavailable (goes to the bidder with the highest bid). At the same time, if bidders do have hyperbolic time preferences, the presence of competitors should not affect their time preference.

More generally, we expect that many strategic situations will have different equilibria if peoples' preferences demonstrate DU anomalies or if they are described by our discount factors *ex ante.*

(2) *The deadline decreases the size of anomalies, and the closer the deadline, the smaller the sizes of anomalies are.* Experiment # 1: participants are asked to choose the fair price from a narrow menu of prices. It is explained that if they do not make a choice today they will be offered a different menu tomorrow, which can be more favorable or less favorable than the menu offered today (on average, roughly the same), and the experiment will be repeated several times, say, each Monday during a month; after that, they will lose the right to choose. Experiment # 2: the same experiment, but with a different time scale, say, each day of a week.

(3) *Anomalies do not vanish when the deadline is very close.* Presumably, this experiment requires a preliminary stage, when the participants can infer (on a subconscious

level) the dynamics of the menu they are offered at different dates; it is not clear whether this preliminary stage is needed to test Hypothesis (2). On the second stage, the participants are offered a menu of choices and informed that if they do not make a choice now, then they will be able to make a choice the next time, which will be the last one.

Appendix A. Processes with i.i.d. increments: instantaneous consumption

A.1. **Lévy processes.** For a general process with i.i.d. increments, in continuous time (a Lévy process), the optimal exercise boundary for the perpetual American call option with the payoff $g(x) = e^{\gamma x - T(r-\Psi(\gamma))} - K$ is given by

$$e^{\gamma h^*} = \kappa_+(\gamma) e^{T(r-\Psi(\gamma))} K, \tag{A.1}$$

where $\Psi(\gamma)$ is definable from $E^x[e^{\gamma X_t}] = e^{\gamma x + t\Psi(\gamma)}$, and $\kappa_+(\gamma) = E[e^{\gamma \bar{X}_\tau}]$, where τ is an exponential random variable with mean $1/r$, independent of process X_t, and $\bar{X}_t = \sup_{0\le s\le t} X_s$ is the supremum process. See Boyarchenko and Levendorskiĭ (2000, 2002, 2005b), Mordecki (2002) and Alili and Kyprianou (2004) for the results of the increasing degree of generality and completeness of the proofs, and further references. Therefore, at the spot level x, the fair price for the instantaneous gain $u(c_T)$ in the future is

$$K = \kappa_+(\gamma)^{-1} e^{-rT} E^x[u_G(c_T)].$$

Once again we obtain the quasi-hyperbolic discounting model with the correction factor $\delta = \mathcal{D}^c_G = \kappa_+(\gamma)^{-1}$. Trajectories of the supremum process are non-decreasing, hence $\delta \le 1$; and for a Lévy process whose upward movements have positive probability, they are increasing with non-zero probability, hence $\delta = \mathcal{D}^c_G < 1$.

Similarly, the optimal exercise boundary for the perpetual American put option with the payoff $g(x) = K - e^{\gamma x - T(r-\Psi(\gamma))}$ is given by

$$e^{\gamma h_*} = \kappa_-(\gamma) e^{T(r-\Psi(\gamma))} K, \tag{A.2}$$

where $\kappa_-(\gamma) = E[e^{\gamma \underline{X}_\tau}]$, τ is an exponential random variable with mean $1/r$, independent of the process X_t, and $\underline{X}_t = \inf_{0\le s\le t} X_s$ is the infimum process. Therefore, at the spot level x, the fair price for the instantaneous loss $u(c_T)$ in the future is

$$K = \kappa_-(\gamma)^{-1} e^{-rT} E^x[u_L(c_T)].$$

Once again we obtain the quasi-hyperbolic discounting model with the correction factor $\delta = \mathcal{D}^c_L = \kappa_-(\gamma)^{-1}$. Trajectories of the infimum process are non-increasing, hence $\delta \ge 1$; and for a Lévy process whose downward movements have positive probability, they are decreasing with non-zero probability, hence $\delta = \mathcal{D}^c_L > 1$. For $T < -\log \kappa_-(\gamma)/(r - \Psi(\gamma))$, we observe the negative discounting. Comparing the correction factors for gains and losses, we can explain the delay-speedup asymmetry and sign effect but not magnitude effect.

A.2. **Random walks.** Consider a discrete time model. Let $X_t = X_0 + Y_1 + Y_2 + \cdots Y_t$ be a random walk, that is, $X_0, Y_1, Y_2, \ldots$ are pairwise independent, and $Y_1, Y_2, \ldots$ are i.i.d. Suppose that an individual discounts future using the discount factor $\beta \in (0,1)$. Set $M(\gamma) = E[e^{\gamma Y_1}]$, and to rule out bubbles, assume that $1 - \beta M(\gamma) > 0$. Let τ be the exponential random variable with mean $\beta/(1-\beta)$, and $\bar{X}_t$ and $\underline{X}_t$ the supremum and infimum processes for the random walk X_t. Define $\kappa_+(\gamma)$ and $\kappa_-(\gamma)$ as above. Then it follows from the general results Darling et al. (1972) and Mordecki (2002) that the optimal exercise price for the perpetual American put option with the payoff $K - \beta^T E^x[e^{\gamma X_T}] = K - (\beta M(\gamma))^T e^{\gamma x}$ is given by

$$e^{\gamma h_*} = \kappa_-(\gamma)(\beta M(\gamma))^{-T} K,$$

and the optimal exercise price for the perpetual American call with the payoff

$$\beta^T E^x[e^{\gamma X_T}] - K = (\beta M(\gamma))^T e^{\gamma x} - K$$

is given by

$$e^{\gamma h^*} = \kappa_+(\gamma)(\beta M(\gamma))^{-T} K.$$

Therefore, if the individual may pay K dollars now for the gain c_T in the future, and the spot value of the stochastic factor is x, she considers the price

$$K = \kappa_+(\gamma)^{-1}(\beta M(\gamma))^T e^{\gamma x} = \kappa_+(\gamma)^{-1}\beta^T E^x[u_G(c_T)]$$

as fair. We obtain the correction factor for gains $\mathcal{D}^c_G = \kappa_+(\gamma)^{-1}$, which is less than 1 if upward movements happen with positive probability, and recover the (β,δ)-model (with $\beta M(\gamma)$ in place of β, and $\delta = \mathcal{D}^c_G$). Similarly, we obtain the correction factor for losses $\mathcal{D}^c_L = \kappa_-(\gamma)^{-1}$, which is greater than 1 if downward movements happen with positive probability, and obtain the (β,δ)–model for losses, with $\delta = \mathcal{D}^c_L > 1$. The negative discounting follows for $T < \log \kappa_-(\gamma)/\log(\beta M(\gamma))$. Comparing the correction factors for gains and losses, we can explain the delay-speedup asymmetry and sign effect but not magnitude effect.

A.3. **Finite deadline and random deadline.** The case of a finite deadline corresponds to American options with finite time horizon. For random walks, the early exercise boundary is separated by non-zero margin from the strike, and the same holds for many Lévy processes (see Boyarchenko and Levendorskiǐ (2002), Levendorskiǐ (2004a,b)). Hence, the analysis in Section 3 applies, and we conclude that all the discounted utility anomalies discussed above should be observed in the case of a finite deadline and random deadline even if the deadline is imminent.

Appendix B. Ornstein-Uhlenbeck processes

B.1. **Instantaneous consumption.** In this section, we model the uncertainty of consumption $c_t = \exp X_t$ as the exponential Ornstein-Uhlenbeck process; this is a mean-reverting model, which is more appropriate for modelling of inflation uncertainty than the geometric Brownian motion model. We assume that X_t is the solution of the stochastic differential equation

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t, \tag{B.1}$$

where $\kappa > 0$ is the speed of reversion, θ is the "normal" level of X_t, and dW_t is the increment of the standard Brownian motion. For simplicity, we confine ourselves to the simplest utility function $u_G(c) = c^\gamma$ over gains and disutility function $u_L(c) = c^\gamma$ over losses although the case of more general u_G and u_L can be studied as well. Since the law of X_T conditioned on $X_0 = x$ is normal with mean $\theta + (x - \theta)e^{-\kappa T}$ and variance $(\sigma^2/2\kappa)(1 - e^{-2\kappa T})$ (see, for example, Dixit and Pindyck (1996)), we have

$$E^x[e^{\gamma X_T}] = \exp\left[\gamma e^{-\kappa T}x + \frac{\sigma^2\gamma^2}{4\kappa}(1 - e^{-2\kappa T}) + \theta\gamma(1 - e^{-\kappa T})\right]. \tag{B.2}$$

An individual, who can pay K dollars for delivery of consumption, c_T, T periods after the payment, can be regarded as a holder of the perpetual American call option with the payoff $g(x) = e^{B(T)-rT+A(T)x} - K$, where $A(T) = \gamma e^{-\kappa T}$ and $B(T) = \sigma^2\gamma^2(1 - e^{-2\kappa T})/(4\kappa) + \theta\gamma(1 - e^{-\kappa T})$. Set $\nu = r/\kappa$, $\bar\sigma = \sigma/\sqrt{2\kappa}$, $\bar h = (h - \theta)/\bar\sigma$, and recall the notation $D_{-\nu}$ for the Weber-Hermite parabolic cylinder functions. For the definition and basic properties, see e.g. Borodin and Salminen (2002), A 2.9, p. 639, and Buchholz (1969). Values of $D_{-\nu}$ can be calculated using the standard packages, MAPLE, for instance.

Theorem B.1. *The optimal exercise boundary, h, is the unique solution of the equation*

$$\frac{\nu}{\bar\sigma}\frac{D_{-\nu-1}(-\bar h)}{D_{-\nu}(-\bar h)} = \frac{g'(h)}{g(h)}, \tag{B.3}$$

on interval $((\log K - B(T) + rT)/A(T), +\infty)$.

Equation (B.3) is of the same form as equation (3.11) in Levendorskiĭ (2005); the proof is similar to the proof in the Gaussian case. For the sake of completeness, we reproduce the proof in Subsection B.2.

Clearly, the RHS decreases on the interval $((\log K - B(T) + rT)/A(T), +\infty)$ from $+\infty$ (this is an interval, where $g(x)$ is positive), and it can be shown that the LHS in (B.3) increases on $\mathbb{R}$ to $+\infty$ – see Section 2.1 in Levendorskiĭ (2005). Therefore, a unique solution of (B.3) exists, and it can be found numerically quite easily. Suppose, that at the current level $x = X_0$, the individual considers the price K as fair; then, using $g'(x) = A(T)e^{B(T)-rT+A(T)x}$, we find that

$$K = e^{B(T)-rT+A(T)x}\left(1 - \gamma e^{-\kappa T}\frac{\bar\sigma}{\nu}\frac{D_{-\nu}(-(x-\theta)/\bar\sigma)}{D_{-\nu-1}(-(x-\theta)/\bar\sigma)}\right). \tag{B.4}$$

Comparing with (B.2), we find the correction factor for gains:

$$\mathcal{D}^c_G(u_G; T, x) = 1 - \gamma e^{-\kappa T}\frac{\bar\sigma}{\nu}\frac{D_{-\nu}(-(x-\theta)/\bar\sigma)}{D_{-\nu-1}(-(x-\theta)/\bar\sigma)}. \tag{B.5}$$

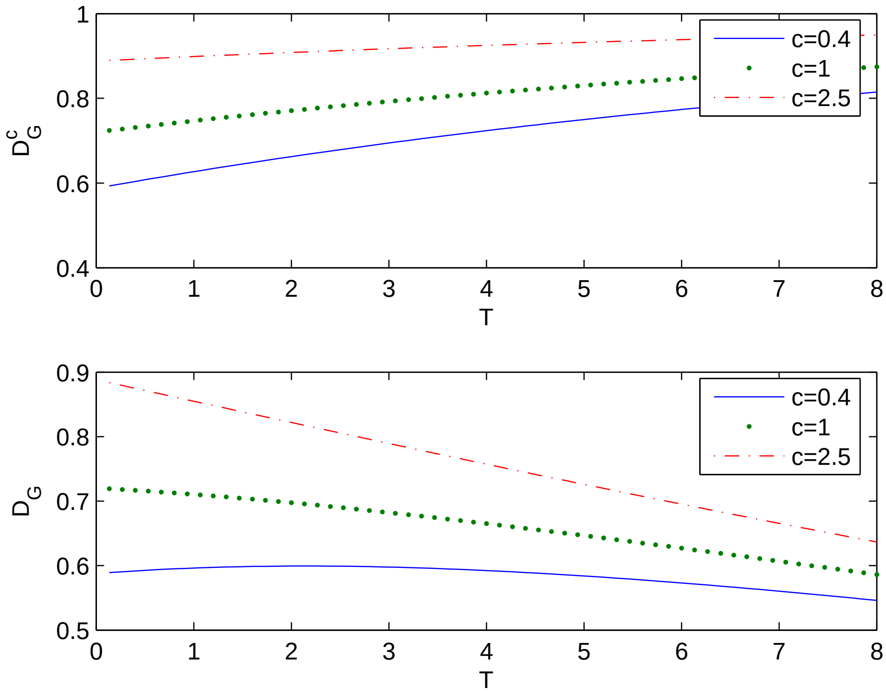

FIGURE 2. Ornstein-Uhlenbeck model. Correction factors and discount factors for gains: hyperbolic effect and magnitude effect. Utility function over gains: $u_G(c) = c^\gamma$, $\gamma = 0.5$. Parameters: $\theta = 0, \kappa = 0.1, \sigma^2 = 0.2$; $r = 0.05$.

We see that the correction factor for gains is less than 1, and it is bounded away from 1 for $T \in (0, +\infty)$. Moreover, $\gamma e^{-\kappa T}\bar{\sigma}/\nu > 0$, and $D_{-\nu}(-(x-\theta)/\bar{\sigma})/D_{-\nu-1}(-(x-\theta)/\bar{\sigma})$ decreases as $x \to +\infty$ (see Section 2.1 in Levendorskiĭ (2005)), therefore we observe both the hyperbolic effect and magnitude effect. See Fig. 2 for a numerical example. Notice the slight hump of the discount curve for $c = 0.4$, which indicates that in some cases, the discount factor ex ante for gains may *increase* as T increases - but up to a certain limit only. This effect can be regarded as an analog of the negative discounting for losses.

Now consider an individual who assesses the fair compensation K for the loss of consumption, c_T, T periods later. Using the equation (3.13) in Levendorskiĭ (2005) for the optimal exercise boundary for the perpetual American put option with the payoff function $g(x) = K - e^{-rT}E^x[e^{\gamma X_T}]$:

$$\frac{\nu}{\bar{\sigma}} \frac{D_{-\nu-1}(\bar{h})}{D_{-\nu}(\bar{h})} = \frac{g'(h)}{g(h)}, \tag{B.6}$$

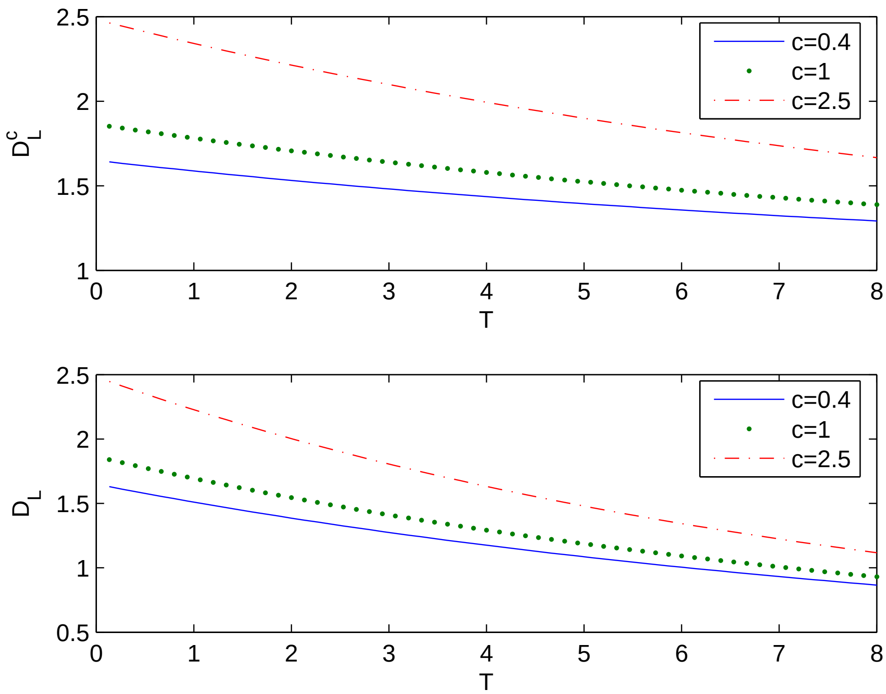

FIGURE 3. Ornstein-Uhlenbeck model. Correction factors and discount factors for losses: negative discounting and magnitude effect. Disutility function for losses: $u_L(c) = c^\gamma$, $\gamma = 0.5$. Parameters: $\theta = 0, \kappa = 0.1, \sigma^2 = 0.2$; $r = 0.05$.

we conclude that at the current value $x = X_0$, the individual considers the following compensation as fair:

$$K = e^{B(T)-rT+A(T)x}\left(1 + \gamma e^{-\kappa T}\frac{\bar\sigma}{\nu}\frac{D_{-\nu}((x-\theta)/\bar\sigma)}{D_{-\nu-1}((x-\theta)/\bar\sigma)}\right) \tag{B.7}$$

Comparing with (B.2), we find the correction factor for losses:

$$\mathcal{D}^c_L(u_L; T, x) = 1 + \gamma e^{-\kappa T}\frac{\bar\sigma}{\nu}\frac{D_{-\nu}((x-\theta)/\bar\sigma)}{D_{-\nu-1}((x-\theta)/\bar\sigma)}. \tag{B.8}$$

This correction factor is greater than 1, hence for moderate values of T, we observe the negative discounting. As $x \to +\infty$, $D_{-\nu}((x-\theta)/\bar\sigma)/D_{-\nu-1}((x-\theta)/\bar\sigma)$ decreases, therefore we observe the magnitude effect of the correct sign; this time, for any $\gamma > 0$. See Fig. 3 for a numerical example.

B.2. **Proof of Theorem B.1.** The free boundary problem for the rational price $V(x)$ of the option is

$$\begin{aligned}(r-\frac{\sigma^2}{2}\partial^2-\kappa(\theta-x)\partial)V(x) &= 0, \quad x<h, &\text{(B.9)}\\ V(h) &= g(h), &\text{(B.10)}\\ V'(h) &= g'(h), &\text{(B.11)}\end{aligned}$$

where $\partial=\partial_x$, so that $\partial V(x)=V'(x)$. Naturally, we also impose the condition

$$V(x)\to 0, \quad \text{as } x\to-\infty. \tag{B.12}$$

To solve the free boundary problem (B.9)–(B.12), we fix h, a candidate for the exercise log-price, set $\bar\sigma=\sigma/\sqrt{2\kappa}$, and change the variable $z=(x-\theta)/\bar\sigma$ and unknown function $V(x)=e^{z^2/4}w(z)$. Equation (B.9) becomes

$$(r-\kappa\partial_z^2+\kappa z\partial_z)e^{z^2/4}w(z)=0, \quad z<\bar h, \tag{B.13}$$

where $\bar h=(h-\theta)/\bar\sigma$. Set $\nu=r/\kappa$, divide (B.13) by $-\kappa e^{z^2/4}$, and use the commutation relation $e^{-z^2/4}\partial_z e^{z^2/4}=\partial_z+z/2$. We obtain

$$(\partial_z^2+\frac{1}{2}-\nu-\frac{z^2}{4})w(z)=0, \quad z<\bar h. \tag{B.14}$$

Since $-\nu$ is not a positive integer, the general solution of (B.14) can be represented in the form

$$w(z)=AD_{-\nu}(-z)+BD_{-\nu}(z), \tag{B.15}$$

where $D_{-\nu}(\pm z)$ are the *parabolic cylinder functions* or *Weber-Hermite functions*. For the representations of $D_{-\nu}$ as a series or integral, see, e.g., Buchholz (1969) or Borodin and Salminen (2002), A 2.9, p. 639. For completeness, we give the series representation:

$$\begin{aligned}D_{-\nu}(z) \; := \; & e^{-z^2/4}\sqrt{\frac{\pi}{2^\nu}}\left\{\frac{1}{\Gamma((\nu+1)/2)}\left(1+\sum_{k=1}^{\infty}\frac{\nu(\nu+2)\cdots(\nu+2k-2)}{(2k)!}z^{2k}\right)\right.\\ & \left.-\frac{z\sqrt{2}}{\Gamma(\nu/2)}\left(1+\sum_{k=1}^{\infty}\frac{(\nu+1)(\nu+3)\cdots(\nu+2k-1)}{(2k+1)!}z^{2k}\right)\right\},\end{aligned}$$

although in numerical examples, we will use the built-in procedures in the standard packages. We will need the formula for the derivative

$$\left(e^{z^2/4}D_{-\nu}(z)\right)'=-\nu e^{z^2/4}D_{-\nu-1}(z) \tag{B.16}$$

(see e.g. Borodin and Salminen (2002), A 2.9, p. 639), and asymptotic formulas, as $z\to+\infty$ (see equations (5a) and (5b) on p. 92 and (25) on p.40 in Buchholz (1969)):

$$D_{-\nu}(z)=z^{-\nu}e^{-z^2/4}(1+O(z^{-2})), \tag{B.17}$$

$$D_{-\nu}(-z)=\frac{\sqrt{2\pi}}{\Gamma(\nu)}e^{z^2/4}|z|^{\nu-1}(1+O(z^{-2})). \tag{B.18}$$

Notice also that for positive ν, $D_{-\nu}$ has no zeroes on the real line. Hence, from (B.16), we see that $D_{-\nu}$ is decreasing.

From (B.15), $V(x)$ can be represented in the form

$$V(x) = e^{z^2/4}\left(AD_{-\nu}(-z) + BD_{-\nu}(z)\right), \tag{B.19}$$

and we see from (B.17) and (B.18), that $V(x)$ satisfies (B.12) if and only if $B = 0$. We set $B = 0$, substitute (B.19) into (D.2) and (D.3) and use(B.16):

$$Ae^{\bar{h}^2/4}D_{-\nu}(-\bar{h}) = g(h), \tag{B.20}$$

and

$$A\frac{\nu}{\bar{\sigma}}e^{\bar{h}^2/4}D_{-\nu-1}(-\bar{h}) = g'(h). \tag{B.21}$$

Now we can exclude A, and obtain equation (B.3) for the optimal exercise price.

B.3. **Consumption of durable goods.** Consider utility and disutility functions of the form $u_G(c) = c^\gamma, u_L(c) = c^\gamma$. Using (B.2), we can represent the expected present value of the utility stream $u_G(c_t) = e^{\gamma X_t}$ from moment T till $T + T'$ as

$$\int_T^{T+T'} e^{-rt}E^x[e^{\gamma X_t}]dt = v(T, T'; x), \tag{B.22}$$

where

$$v(T, T'; x) = \int_T^{T+T'} \exp\left[-rt + \gamma e^{-\kappa t}x + \frac{\sigma^2\gamma^2}{4\kappa}(1 - e^{-2\kappa t}) + \theta\gamma(1 - e^{-\kappa t})\right] dt. \tag{B.23}$$

An individual, who can pay K dollars for this utility stream, can be viewed as a holder of the American call option with the payoff $g(x) = v(T, T'; x) - K$. It can be shown that $v'(T, T'; x)/(v(T, T'; x) - K)$ is decreasing on the interval $\{x \mid v(x) > K\}$ from $+\infty$ (see Section 3.1 in Levendorskiĭ (2005)), therefore equation (3.11) in Levendorskiĭ (2005) gives the equation (B.3) for the optimal exercise threshold. Here and below, v' denotes the derivative w.r.t. x. Suppose, that at the current level $x = X_0$, the individual considers the price K as fair; then, using

$$g'(x) = v'(T, T'; x) = \int_T^{T+T'} A(t)e^{B(t)-rt+A(t)x}dt, \tag{B.24}$$

where $A(t) = \gamma e^{-\kappa t}$ and $B(t) = \frac{\sigma^2\gamma^2}{4\kappa}(1 - e^{-2\kappa t}) + \theta\gamma(1 - e^{-\kappa t})$, we find that

$$K = v(T, T'; x) - v'(T, T'; x)\frac{\bar{\sigma}}{\nu}\frac{D_{-\nu}(-(x-\theta)/\bar{\sigma})}{D_{-\nu-1}(-(x-\theta)/\bar{\sigma})}. \tag{B.25}$$

Comparing with (B.22), we find the correction factor for gains:

$$\mathcal{D}^c_G(u_G; T, T', x) = 1 - \frac{v'(T, T'; x)}{v(T, T'; x)} \cdot \frac{\bar{\sigma}}{\nu}\frac{D_{-\nu}(-(x-\theta)/\bar{\sigma})}{D_{-\nu-1}(-(x-\theta)/\bar{\sigma})}. \tag{B.26}$$

We see that the correction factor for gains is less than 1, and it depends on x. Thus, we can reproduce the hyperbolic discounting. It is difficult to prove analytically that the magnitude effect is observed; however, numerical examples support this claim.

The correction factor for losses can be derived similarly:

$$\mathcal{D}_L^c(u_L; T, T', x) = 1 + \frac{v'(T,T';x)}{v(T,T';x)} \cdot \frac{\bar{\sigma}}{\nu} \frac{D_{-\nu}((x-\theta)/\bar{\sigma})}{D_{-\nu-1}((x-\theta)/\bar{\sigma})}, \tag{B.27}$$

and we observe the negative discounting. Once again, the magnitude effect can be shown numerically.

Appendix C. Options of finite maturity

C.1. **Valuation of future gains and American call options of finite maturity.** An individual who believes that her option to acquire payoff c_T (T periods after the decision had been made) will never expire is too optimistic. Suppose, that she expects that there is a deadline T_1 after which she will lose the right to choose the fair price for the gain c_T. Then she is a holder of the American call option with the maturity date T_1, and the payoff

$$g(X_t) = e^{-rT} E_t[u_G(c_{t+T})] - K = e^{-(r-\Psi(\gamma))T} e^{\gamma X_t} - K.$$

We may regard $S_t := e^{-(r-\Psi(\gamma))T} e^{\gamma X_t}$ as the spot price of a stock at time t. The optimal exercise rule is of the form: exercise the option the first time S_t reaches the early exercise boundary $S = H^*(K, T_1, t)$ from below. It is well-known (see e.g. Duffie (1996) and Hull (2000)) that for fixed K and T_1, the curve $S = H^*(K, T_1, t)$ is downward sloping, and it is above the line $S = K$ for all $t < T_1$. This implies that at the spot level $x = X_t$, at time $t < T_1$, the individual perceives the price K as fair if K satisfies

$$e^{-(r-\Psi(\gamma))T} e^{\gamma X_t} = H^*(K, T, T_1, t).$$

Unfortunately, an explicit analytical expression for $H^*(K, T, T_1, t)$ is not available, and therefore, we cannot find analytically $K = K(x, T, T_1, t)$ (there exist many numerical methods, though). However, since $H^*(K, T, T_1, t) > K$, we conclude that the fair price which the individual should be willing to pay, is smaller than the naive present value $e^{-(r-\Psi(\gamma))T} e^{\gamma X_t}$ by factor $K/H^*(K, T, T_1, t)$. Hence, the correction discount factor for gains is $\mathcal{D}_G^c(T_1, t; T, x) = K/H^*(K, T, T_1, t) < 1$, and the discount factor *ex ante* for gains is

$$\mathcal{D}_G(T_1, t; T, x) = \frac{K}{H^*(K, T, T_1, t)} e^{-(r-\Psi(\gamma))T}. \tag{C.1}$$

(Here we have suppressed the dependence of the early exercise boundary, hence of $H^*(K, T, T_1, t)$, $\mathcal{D}_G^c$ and $\mathcal{D}_G$ on u_G, equivalently, on $\gamma = \gamma_G$, and the fact that K is determined by T, T_1, t, x). Note that now the discount factor depends not on the delay period T but on time to deadline $\tau = T_1 - t$ as well, so it seems reasonable to talk about the *term structure of discount factors ex ante*. For a fixed $t < T_1$, equation (C.1) demonstrates the quasi-hyperbolic discounting.

Since the correction factor increases as the deadline is getting closer, one may expect that in the limit $t \to T_1 - 0$, the correction factor becomes 1, and the discount factor *ex ante* becomes the discount factor *ex post*. If $\Psi(\gamma) \le 0$, then, indeed, the limit is 1 because $H^*(K, T, T_1, t) \to K$ as $t \to T_1 - 0$, but if $\Psi(\gamma) > 0$, then the limit of the correction factor is $\delta = (r - \Psi(\gamma))/r = 1 - \Psi(\gamma)/r < 1$ because $H^*(K, T, T_1, t) \to \frac{r}{r-\Psi(\gamma)} K$ as $t \to T_1 - 0$. Condition $\Psi(\gamma) > 0$ may seem too stringent. However, if the process X_t has jumps, then the limit $H^*(K, T, T_1, T_1 - 0) = \lim_{t \to T_1 - 0} H^*(K, T, T_1, t)$ can be larger than K even if $\Psi(\gamma) \le 0$ (See Levendorskiĭ (2004b)). We conclude that a non-negligible quasi-hyperbolic discounting can be observed even if the deadline for making a decision is imminent.

C.2. **Valuation of future losses and American put options of finite maturity.** Now our individual can be regarded as a holder of the American put option with the maturity date T_1, and the payoff

$$g(X_t) = K - e^{-rT} E_t[u_G(c_{t+T})] = K - e^{-(r-\Psi(\gamma))T} e^{\gamma X_t}.$$

We may view $S_t := e^{-(r-\Psi(\gamma))T} e^{\gamma X_t}$ as the spot price of a stock at time t. The optimal exercise rule is of the form: exercise the option the first time S_t reaches the early exercise boundary $S = H_*(K, T_1, t)$ from above. It is well-known (see e.g. Duffie (1996) and Hull (2000)) that for fixed K and T_1, the curve $S = H_*(K, T_1, t)$ is upward sloping, and it is below the line $S = K$ for all $t < T_1$. This implies that at the spot level $x = X_t$, at time t prior to the deadline T_1, the individual perceives the price K as fair if K solves the equation

$$e^{-(r-\Psi(\gamma))T} e^{\gamma X_t} = H_*(K, T, T_1, t).$$

Unfortunately, an explicit analytical expression for $H_*(K, T, T_1, t)$ is not available. However, since $H_*(K, T, T_1, t) < K$, we conclude that the fair compensation, which the individual should be willing to accept for the loss in the future, is higher than the naive present value $e^{-(r-\Psi(\gamma))T} e^{\gamma X_t}$ by factor $K/H_*(K, T, T_1, t)$. Hence, the correction discount factor for gains is $\mathcal{D}^c_L(T_1, t; T, x) = K/H_*(K, T, T_1, t) > 1$, and the discount factor ex ante for gains is

$$\mathcal{D}_L(T_1, t; T, x) = \frac{K}{H_*(K, T, T_1, t)} e^{-(r-\Psi(\gamma))T}. \tag{C.2}$$

(We have suppressed the dependence of the early exercise boundary, hence of $H_*(K, T, T_1, t)$, $\mathcal{D}^c_L$ and $\mathcal{D}_L$ on u_L, equivalently, on $\gamma = \gamma_L$, and the fact that K is determined by T, T_1, t, x). For a fixed $t < T_1$, equation (C.2) demonstrates the quasi-hyperbolic discounting for losses, with $\delta = \mathcal{D}^c_L > 1$, and for small T, the negative discounting results.

Since the correction factor decreases as the deadline is getting closer, one may expect that in the limit $t \to T_1 - 0$, the correction factor becomes 1, and the discount factor ex ante becomes the discount factor ex post. If $\Psi(\gamma) \ge 0$, then, indeed, the limit is 1 because $H_*(K, T, T_1, t) \to K$ as $t \to T_1 - 0$, but if $\Psi(\gamma) < 0$, then the limit of the correction factor is $\delta = (r - \Psi(\gamma))/r = 1 - \Psi(\gamma)/r > 1$ because $H_*(K, T, T_1, t) \to \frac{r}{r-\Psi(\gamma)} K$ as $t \to T_1 - 0$.

Condition $\Psi(\gamma) < 0$ may seem too stringent. However, if the process X_t has jumps, then the limit $H_*(K, T, T_1, T_1 - 0)$ can be smaller than K even if $\Psi(\gamma) \geq 0$ (See Levendorskiǐ (2004a,b)). We conclude that a non-negligible quasi-hyperbolic discounting and negative discounting can be observed even if the deadline for making a decision is imminent.

C.3. **The sign effect and the delay-speedup asymmetry.** The same arguments as in the case of no deadline apply; essentially, we only need to know that if $u_L = u_G$, then the correction factor for losses is larger than the correction factor for gains.

C.4. **The case of a random deadline.** In our opinion, intuitively, the most appealing approach would be when the individual realizes that the option to exchange future gains/losses for current ones cannot be perpetual, and at the same time, the individual does not have any specific maturity date in mind. In other words, she perceives the option at hand as an American option with the random maturity date. Whatever a random maturity date is, the early exercise boundary for the American call with a random maturity date is higher than the limit $H^*(K, T, T_1, T_1 - 0)$, and the early exercise boundary for the American put with a random maturity date is lower than the limit $H_*(K, T, T_1, T_1 - 0)$. If the former limit is higher than K, and the latter is lower than K, then the arguments above apply and all the discounted utility anomalies but the magnitude effect can be reproduced. Note that using the explicit formulas for the case of a random maturity date in Carr (1998), Levendorskiǐ (2004a,b) and Boyarchenko and Levendorskiǐ (2005), one can derive explicit formulas for the correction factors; these formulas are rather involved, and so we omit them.

Appendix D. Technical proofs

D.1. **Proof of Lemma 4.2.** Let $V(x)$ be the value of the perpetual American option with the payoff function $g(x) = e^{-rT}E^x[v(X_T)] - K$. Function g is increasing, and it satisfies conditions (4.3)-(4.4) since v does. The unknown function V and boundary h solve the following free boundary problem

$$\begin{aligned} rV(x) - \mu V'(x) - \frac{\sigma^2}{2}V''(x) &= 0, \quad x < h, & \text{(D.1)}\\ V(h) &= g(h), & \text{(D.2)}\\ V'(h) &= g'(h). & \text{(D.3)} \end{aligned}$$

(These are the stationary Black-Scholes equation, and the value matching and smooth pasting conditions). The general solution of (D.1) is of the form

$$V(x) = Ae^{\beta^+ x} + Be^{\beta^- x}, \tag{D.4}$$

where $\beta^- < 0 < \beta^+$ are negative and positive roots of the "fundamental quadratic" $r - \Psi(\beta) = 0$. Since the option value decreases as $x \to -\infty$, we must have $B = 0$. Substituting $V(x) = Ae^{\beta^+ x}$ into (D.2)-(D.3), we can write (D.3) as $\beta^+ V(h) = g'(h)$, and dividing (D.3) by (D.2), obtain $g(h) = g'(h)/\beta^+$. Since $g(x) = e^{-rT}E^x[v(X_T)] - K$, and

$g'(x) = e^{-rT}E^x[v'(X_T)]$, the equation for the optimal exercise boundary can be written as (4.5). On the strength of (4.3) and (4.4), the RHS of (4.5) increases, a solution of equation (4.5) exists, and it is unique.

D.2. **Proof of Lemma 4.3.** Let $V(x)$ be the value of the perpetual American option with the payoff function $g(x) = K - e^{-rT}E^x[v(X_T)]$. Function g is decreasing, and $-g$ satisfies conditions (4.6)-(4.7) since v does. The unknown function V and boundary h solve the following free boundary problem

$$rV(x) - \mu V'(x) - \frac{\sigma^2}{2}V''(x) = 0, \quad x > h, \tag{D.5}$$

$$V(h) = g(h), \tag{D.6}$$

$$V'(h) = g'(h). \tag{D.7}$$

The general solution of (D.5) is of the form (D.4). Since the option value decreases as $x \to +\infty$, we must have $A = 0$. Substituting $V(x) = Be^{\beta^- x}$ into (D.6)-(D.7), we can write (D.3) as $\beta^- V(h) = g'(h)$, and dividing (D.7) by (D.6), obtain $g'(h)/\beta^- = g(h)$. Since $g(x) = K - e^{-rT}E^x[v(X_T)]$, and $g'(x) = -e^{-rT}E^x[v'(X_T)]$, the equation for the optimal exercise boundary can be written as (4.8). On the strength of (4.6) and (4.7), the RHS of (4.8) increases, a solution of equation (4.8) exists, and it is unique.

D.3. **Magnitude effect and hyperbolic effect: moderate consumption levels.** Assume that c_t is a martingale: $E[c_t] = c_0$, and $T\sigma_T^2$ is small, so that we can use approximations

$$\begin{aligned}
u(c_T) &= u(c_0) + u'(c_0)(c_T - c_0) + \frac{1}{2}u''(c_0)(c_T - c_0)^2 + \cdots, \\
u'(c_T) &= u'(c_0) + u''(c_0)(c_T - c_0) + \frac{1}{2}u'''(c_0)(c_T - c_0)^2 + \cdots, \\
c_T u'(c_T) &= c_0 u'(c_0) + c_0 u''(c_0)(c_T - c_0) + \frac{1}{2}c_0 u'''(c_0)(c_T - c_0)^2 \\
&\quad + u'(c_0)(c_T - c_0) + u''(c_0)(c_T - c_0)^2 + \cdots,
\end{aligned}$$

to calculate expectations in (4.1):

$$\begin{aligned}
E^c[u(c_T)] &= u(c) + \frac{1}{2}u''(c)c^2\sigma_T^2 + \cdots, \\
E^c[c_T u'(c_T)] &= cu'(c) + c^2[u''(c) + \frac{1}{2}cu'''(c)]\sigma_T^2 + \cdots
\end{aligned}$$

Substituting into (4.1), we obtain an approximation

$$\mathcal{D}_G^c(c,T) = \frac{u_G(c) - (1/\beta^+)cu_G'(c) - c^2[(1/\beta^+ - 1/2)u_G''(c) + cu_G'''(c)/(2\beta^+)]\sigma_T^2}{u_G(c) + u_G''(c)c^2\sigma_T^2/2}, \tag{D.8}$$

and even more rough approximation

$$\mathcal{D}^c_G(c,T) = \frac{u_G(c) - (1/\beta^+)cu'_G(c)}{u_G(c)} = 1 - \frac{1}{\beta^+}\cdot\frac{cu'_G(c)}{u_G(c)}. \tag{D.9}$$

Similarly to (D.8)-(D.9), we obtain

$$\mathcal{D}^c_L(c,T) = \frac{u_L(c) - (1/\beta^-)cu'_L(c) - c^2[(1/\beta^- - 1/2)u''_L(c) + cu'''_L(c)/(2\beta^-)]\sigma^2_T}{u_L(c) + u''_L(c)c^2\sigma^2_T/2}, \tag{D.10}$$

and even more rough approximation

$$\mathcal{D}^c_L(c,T) = \frac{u_L(c) - (1/\beta^-)cu'_L(c)}{u_L(c)} = 1 - \frac{1}{\beta^-}\cdot\frac{cu'_L(c)}{u_L(c)}. \tag{D.11}$$

Since $\beta^- < 0 < \beta^+$ and $cu'_L(c)/u_L(c)$ is increasing iff $cu'(c)/u(c)$ is decreasing, the following theorem is immediate from (D.9) and (D.11).

Theorem D.1. *If $cu'(c)/u(c)$ is decreasing, then large outcomes are discounted less than small ones.*

The derivative of the RHS in (D.8) w.r.t. σ^2_T is equal to $F_G/u^2_G(c)$, where
$F_G := (u_G(c) - (1/\beta^+)cu'_G(c))(-u''_G(c)c^2/2) - c^2[(1/\beta^+ - 1/2)u''_G(c) + cu'''_G(c)/(2\beta^+)]u_G(c)$.
Simplifying, we find

$$\frac{\mathcal{D}^c_G(c,T)}{d\sigma^2_T} = \frac{-u''_G(c)}{2\beta^+u_G(c)}\left\{2 + c\left(\frac{u'''_G(c)}{u''_G(c)} - \frac{u'_G(c)}{u_G(c)}\right)\right\}. \tag{D.12}$$

Therefore, the correction factor for gains depends on T unless

$$2 + c\left(\frac{u'''_G(c)}{u''_G(c)} - \frac{u'_G(c)}{u_G(c)}\right) = 0, \tag{D.13}$$

and it increases as T increases, that is, the ex ante discount rate for gains decreases, if

$$2 + c\left(\frac{u'''_G(c)}{u''_G(c)} - \frac{u'_G(c)}{u_G(c)}\right) > 0. \tag{D.14}$$

Similarly, we derive from (D.10)

$$\frac{\mathcal{D}^c_L(c,T)}{d\sigma^2_T} = \frac{-u''_L(c)}{2\beta^-u_L(c)}\left\{2 + c\left(\frac{u'''_L(c)}{u''_L(c)} - \frac{u'_L(c)}{u_L(c)}\right)\right\}. \tag{D.15}$$

The utility function u is concave, therefore u_L is convex, and since $\beta^- < 0$, we conclude that the correction factor for losses depends on T unless

$$2 + c\left(\frac{u'''_L(c)}{u''_L(c)} - \frac{u'_L(c)}{u_L(c)}\right) = 0, \tag{D.16}$$

and it increases as T increases, that is, the ex ante discount rate for losses decreases, if

$$2 + c\left(\frac{u'''_L(c)}{u''_L(c)} - \frac{u'_L(c)}{u_L(c)}\right) > 0. \tag{D.17}$$

Remark. a) For $u_G(c) = c^\gamma$ the exact formulas (4.1) and (4.2) give the correction factors $\mathcal{D}^c_G(c,T) = 1 - \gamma/\beta^+$ and $\mathcal{D}^c_L(c,T) = 1 - \gamma/\beta^-$, and an approximate conditions (D.13) and (D.16) hold as well.

b) The approximate conditions (D.14) and (D.17) cannot be used in the region of small values of c. For instance, the asymptotic calculations for $u_G(c) = (1+c)^{\gamma_G} - 1$, $0 < \gamma_G < 1$, in the region of small and large values of c, and numerical calculations in the region of moderate levels of c show that the discount rate ex-ante decreases as T increases. However, for small c, the RHS in (D.14) is negative.

References

[1] Alili, L., Kyprianou, A.E. (2004), Some remarks on first passage of Lévy processes, the American put and pasting principles, *Annals of Applied Probability* 15 (2005), 2062-2080.

[2] B.S. Bernanke (1983), Irreversibility, uncertainty, and cyclical investment, *Quarterly Journal of Economics*, 98, 85–106.

[3] A.N. Borodin, P. Salminen (2002), *Handbook of Brownian Motion – Facts and Formulae*, 2d edition, Birkhäser Verlag, Basel Boston Berlin

[4] S.I. Boyarchenko (2004), Irreversible decisions and record setting news principles. *American Economic Review*, 94, 557–568.

[5] S.I. Boyarchenko and S.Z. Levendorskiĭ (2000), Option pricing for truncated Lévy processes, *International Journal of Theoretical and Applied Finance*, 3, 549–552.

[6] S.I. Boyarchenko and S.Z. Levendorskiĭ (2002), *Non-Gaussian Merton-Black-Scholes theory*, World Scientific, New Jercey, London, Singapore, Hong Kong

[7] S.I. Boyarchenko and S.Z. Levendorskiĭ (2004a), *Practical Guide to Real Options in Discrete Time*, (February 24, 2004). http://ssrn.com/abstract=510324

[8] S.I. Boyarchenko and S.Z. Levendorskiĭ (2004b), *Optimal stopping made easy* (October 26, 2004). http://ssrn.com/abstract=610621

[9] S.I. Boyarchenko and S.Z. Levendorskiĭ (2005), American options: the EPV pricing model, *Annals of Finance*, 1:3, 267–292.

[10] H. Buchholz (1969) *The confluent hypergeometric function*, Springer-Verlag, Berlin heidelberg New York

[11] P. Carr (1998), Randomization and the American put, *Review of Financial Studies*, 11, 597-626.

[12] P. Dasgupta and E. Maskin (2005), Uncertainty and Hyperbolic Discounting, *American Economic Review*, 95:4, 1290–1299.

[13] Dixit, A.K., and R.S. Pindyck (1994) *Investment under Uncertainty*. Princeton: Princeton University Press.

[14] D. Duffie (1996) *Dynamic Asset Pricing Theory*, Princeton University Press, Princeton, NJ

[15] S. Frederick, G. Loewenstein, and T. O'Donoghue (2002), Time discounting and time preference: a critical review. *J. Econ. Literature*, 40:2, 351–401.

[16] D. Fudenberg, D.K. Levine (2004), A dual self model of impulse control, Working paper.

[17] S.R. Grenadier, N. Wang (2005), Investment under unceratinty and time-inconsistent preferences, Research Paper No. 1899, Research Paper Series, Stanford Graduate School of Business

[18] F. Gul, W. Pesendorfer (2004), *The revealed preference theory of changing tastes*, Working paper

[19] M.W. Hanemann (1991), Willingness to pay and willingness to accept: how much can they differ? *American Economic Review*, 81:3, 635–647

[20] M.W. Hanemann (1994), Valuing the environment through contingent valuation, *Journal of Economic Perspectives*, 8:4, 19–43

[21] J.C. Hull (2000), *Options, Futures, & Other Derivatives*, Prentice Hall, Upper Saddle River, NJ
[22] E. Karni (2005), Subjective Expected Utility Theory without States of the World, The Johns Hopkins University,Department of Economics, Economics Working Paper Archive http://www.econ.jhu.edu/pdf/papers/WP523karni.pdf
[23] T.C. Koopmans (1960), Stationary Ordinal Utility and Impatience, *Econometrica*, 28, 287–309.
[24] J. Krutilla (1967), Conservation reconsidered, *American Economic Review*, 56, 77–86
[25] S.Z. Levendorskiĭ (2004a), Pricing of the American put under Lévy processes, *International Journal of Theoretical and Applied Finance*, 7, 303–335.
[26] S.Z. Levendorskiĭ (2004b), Early exercise boundary and option prices in Lévy driven models, *Quantitative Finance*, 4, 525–547.
[27] S.Z. Levendorskiĭ (2005), *Perpetual American Options and Real Options under Mean-Reverting Processes.* http://ssrn.com/abstract=714321
[28] G. Loewenstein (1988), Frames of mind in intertemporal choice, *Management Science*, 34, 200–214.
[29] G. Loewenstein, and D. Prelec (1992), Anomalies in intertemporal choice: Evidence and interpretation, *Quart. J. Econ.*, 107:2, 573–597.
[30] P. Manzini, and M. Mariotti (2005), *A vague theory of choice over time*, Working paper.
[31] Mc Kean, Jr H.P. (1964), Appendix: A free boundary problem for heat equation arising from a problem in Mathematical Economics, *Industrial Manag. Rev.*, 6, 32–39.
[32] E. Mordecki (2002), Optimal stopping and perpetual options under Lévy processes. Finance and Stochastics **6**, 473–493
[33] R.C. Merton (1973), Theory of rational option pricing, *Bell J. Econom. Manag. Sci.,* 4, 141–183.
[34] E.A. Ok, and Y. Masatlioglu (2003), *A general theory of time preference*, Working paper
[35] E.S. Phelps, and R. Pollack (1968), On second-best national saving and game-equilibrium growth, *Rev. Econ. Stud.,* 35, 186–199.
[36] P.R. Portney (1994), The contingent valuation debate: why economists should care, *Journal of Economic Perspectives*, 8:4, 3–17
[37] P. Samuelson (1933), A note on measurement of utility, *Rev. Econ. Stud.*, 4, 155-161