On the Direction of Innovation

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Abstract

How are resources allocated across different R&D areas? We show that under a plausible set of assumptions, the competitive market allocates excessive innovative efforts into high returns areas, meaning those with higher private (and social) payoffs. The underlying source of market failure is the absence of property rights on problems to be solved, which are the source of R&D value. The competitive bias towards high return areas comes three distortions: 1) The cannibalization of returns of competing innovators; 2) excessive turnover and duplication costs; 3) excessive entry into high return areas results as the market does not take into account the future value of an unsolved problem while a social planner does. Allocation of resources to problem solving leads to a stationary distribution over open problems. The distribution of the socially optimal solution stochastically dominates that of the competitive equilibrium. A severe form of rent dissipation occurs in the latter, where the total value of R&D activity equals the value of allocating all resources to the least valuable problem solved.

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1 Introduction

Innovation resources are quite unequally distributed across different research areas. This is true not only in the case of commercial innovations but even so in our own fields of research. Some areas become more fashionable (hot) than others and attract more attention. A quick look at the distribution of patenting by different classes since the 80's reveals significant changes in the distribution of patent applications: while early on the leading sector was the chemical industry followed closely by others, starting 1995 the areas of computing and electronics surpassed by an order of magnitude all other areas in patent filings. The so-called dot-com bubble is an example of what many considered excessive concentration in the related field of internet startups. This example suggests that innovation resources might be misallocated across different areas and perhaps too concentrated on some, yet to date almost no economic theory has been devoted to this question.

This shift in innovative activity is likely the result of technological, demographic and other changes that introduce new sets of opportunities and problems to solve. As new opportunities arise, firms compete by allocating innovation resources across these opportunities, solving new open problems and thus creating value. The process continues as new opportunities and problems arise over time and innovation resources get reallocated. We model this process and characterize the competitive equilibrium as well as the socially optimal allocations. Our main finding is that the market engages disproportionately in hot R&D lines, characterized by higher expected rates of return per unit of research input and there excessive turnover of researchers.

We model this process as follows. At any point in time there is a set of open problems (research opportunities) that upon being solved generate some social and private value v. This value is known at the time research inputs are allocated and is the main source of heterogeneity in the model. The research side of the economy is as follows. There is a fixed endowment -inelastically supplied- of a research input to be allocated across problems, that for simplicity we call researchers. The innovation technology specifies probabilities of discovery (i.e. problem solution) as a function of the number of researchers involved. Ex-ante the expected value of solving a problem is split equally among the researchers engaged, consistently with a winner-take-all, as in patent races, or an equal sharing rule. Once a problem is solved, the researchers involved are reallocated to other problems after paying an entry cost. We consider both, an environment where the set of problems is fixed as well as a steady state with exogenous arrival of new problems. Firms compete by allocating researchers to the alternative research opportunities to maximize the value per unit input. As there is a large number of firms, we can equally assume that each researcher maximizes its value by choosing research lines. As a result, the value of joining any active research line is equalized.

The key source driving market inefficiency is differential rent dissipation resulting from competitive entry into research. This is due to the pecuniary externality imposed by a marginal entrant to all others involved in this research line. It is useful to contrast our results to the standard models of patent races, where there is a perfectly elastic supply to enter at a cost a patent race, where competitive forces drive average value down to the entry cost. With a concave discovery function, this exceeds the marginal value of an entrant, thus resulting in excessive entry. The gap between average and marginal value is a reflection that part of the return to an entrant comes from a decrease in the expected returns of the remaining participants, the pecuniary externality.

In contrast, in our model we assume that the total research endowment to be allocated is inelastically supplied and entry costs are the same across all research lines, so there cannot be excessive entry overall. But as we find, there will be excessive entry in some areas and too little in others, as well as excessive turnover. It is useful to divide the sources of this misallocation between static and dynamic ones.

The static source of misallocation arises as the pecuniary externality changes with the number of researchers in a research line. To illustrate this, consider the case where the probability of innovation is linear up to a certain number of researchers $\bar{m}$ and constant thereafter and there are two research lines, a 'hot' one with high value and one with low value. Furthermore, suppose that given the total endowment of researchers, more than $\bar{m}$ enter into the former while less than $\bar{m}$ in the latter one, so the average values are equalized. It follows immediately that there is excessive entry into the hot area, where there are negative pecuniary externalities, and too little in the low value one, where there are none.

This example suggests that the extent of pecuniary externalities can vary with scale, and will do so in general. As total discovery probability is bounded, the effect described in the example will occur in some region and as a result there will be excessive entry into the higher value areas. As we show in the paper, the distortion holds globally (there is excessive entry above a value threshold and too little below) for the canonical model of innovation considered in the literature.

We now turn to the dynamic sources of misallocation that can be orders of magnitude more important as illustrated by our back-of-the-envelope calculations. The first one arises from the cost of reallocation. When a researcher joins a research line and succeeds, this generates a capital loss to the remaining researchers that must incur a new entry cost in order to switch to a new, equally valuable, research line. This externality grows with the number of researchers affected and thus with the value v of innovation, leading to excessive entry into hot areas. The second source is more subtle. As a consequence of rent dissipation the value of entering any innovation line is equalized in the competitive equilibrium. In the eyes of competitors, there is no distinction between different open problems in the future, as they all give the same value. In contrast, a planner recognizes that better problems (those with higher v) have higher residual value an so carry a higher future option value if they are not immediately solved. As a result, the planner is less rushed to solve these high value problems.

We analyze a steady state allocation with an exogenous arrival of new problems and endogenous exit of existing ones, resulting from the allocation of researchers. These two forces determine a stationary distribution for open problems. High value problems are solved faster in the competitive equilibrium, due to the biases indicated above, so the corresponding stationary distribution has a lower fraction of good open problems. In addition, as the distribution of innovators is more skewed than in the optimal allocation, turnover is higher and so are reallocation costs. This leads to a severe form of rent dissipation, where in a competitive equilibrium the total value of R&D activity equals the value of allocating all resources to the least valuable problem solved. The magnitude of this distortion can be extremely severe, leading to very large welfare effects as shown by our simple calculations.

Throughout our analysis we assume that the private and social value of innovations is the same across research lines, or likewise that the ratio of private to total value is identical. We do this to abstract from some other important but more obvious sources of misallocation. As patents attempt to align private incentives with social value, they are of no use in solving the distortions that we consider. The source of market failure in our model is the absence of property rights on problems to be solved, which are the source of R&D value. Patents and intellectual property are no direct solutions to this problem as they entitle innovators to value once problems have been solved. Our research suggests that there might be an important role for the allocation of property rights at an earlier stage.

The paper is organized as follows. The related literature is discussed in the next section. Section 3 provides a simple example to illustrate the main ideas in the paper. Section 4 Describe the model and analyzes the static forces of misallocation. Section 5 considers the reallocation of researchers and the dynamic sources of misallocation for a fixed set of problems. Section 6 considers the steady state with continuous arrival of new problems. Section

7 concludes.

2 Literature Review

We begin the literature review by clarifying how the market inefficiency we identify is distinct from the forms of market failure previously singled out in the R&D literature.

Early literature (e.g., Schumpeter, 1911; Arrow, 1962; and Nelson, 1959) pointed at limited appropriability of the innovations' social value by innovators, and at limited access to finances as the main distorting forces in R&D markets, both leading to the implication that market investment in R&D is insufficient relative to first best.[1] A large academic literature has developed to provide policy remedies, often advocating strong innovation protection rights,[2] and the subsidization of R&D. Wright (1983) compares patents, prizes, and procurement as three alternative mechanisms to fund R&D. Patents provide incentives so that they exert R&D effort efficiently, as they delegate R&D investment decisions to innovators, i.e. to the 'informed parties,' but they burden the market with the IP monopoly welfare loss. Kremer (1998) suggests an ingenious mechanism based on the idea of

[1]There appears to be significant empirical evidence for these forces. Since the classical work of Mansfield et al. (1977) estimates of the social return of innovations calculate that they may be twice as large as private returns to innovators; whereas evidence of a "funding gap" for investment innovation has been documented, for example, by Hall and Lerner (2010), especially in countries where public equity markets for 'venture capitalist exit' are not highly developed.

[2]However, others also underline the negative effects of patents on social welfare through monopoly pricing, and on the incentives for future innovations; and Boldrin and Levine (2008) have even provocatively challenged the views that patents are needed to remunerate R&D activity.

patent buyout, to design a prize system that provide efficient R&D investment incentives. Cornelli and Schankerman (1999) show that optimality can be achieved using either an up-front menu of patent lengths and fees or a renewal fee scheme. Hopenhayn and Mitchell (2001) show that if innovations differ both in terms of expected returns and in the likelihood to be replaced by follow-on invention, then the optimal contract involves a menu of patent lengths and breadths.

Another known source of market inefficiency is caused by the sequential, cumulative nature of innovations. This 'sequential spillover' problem arises when, without a 'first' innovation, the idea for 'follow-on' innovations cannot exist, and the follow-on innovators are distinct from the first innovator (see Horstmann, MacDonald and Slivinski, 1985, and Scotchmer, 1991).[3] Summarizing the conclusion of the literature that studies optimal patent length and breadth for this 'cumulative innovation case' (e.g., Green and Scotchmer, 1995, Scotchmer, 1996, O'Donoghue et al., 1998, O'Donoghue, 1998, Denicoló, 2000), there appears to be a strong argument for protection from literal imitation (large lagging breadth) if licensing is fully flexible and efficient, and a general strong argument for leading breadth, whereas strong patentability requirements receive some support when licensing does not function well. In terms of mechanism design, Hopenhayn et al. (2006) study a quality ladder model of cumulative innovations and find the optimal mechanism to be a mandatory buyout system.

[3]As well as distorting investment decisions, sequential innovations can also make the timing of innovation disclosure inefficient (see, for example, Matutes et al., 1996, and Hopenhayn and Squintani, 2016).

The issue of innovation spillovers may complicate policy design not only among sequential innovations, but because of 'horizontal' market value complementarities or substitutabilities among innovations (see Cardon and Sasaki, 1998, and Lemley and Shapiro, 2007, for example). This possibility is distinct from the market inefficiency we identify in this paper: our results hold also in the case of innovations whose market values are independent of each other. A possible policy remedy to inefficiencies caused by horizontal spillovers are patent pools: agreements among patent owners to license a set of their patents to one another or to third parties. Lerner and Tirole (2004) build a tractable model of a patent pool, and identify a simple condition to establish whether patent pools are welfare enhancing.[4]

A recent paper, Bryan and Lemus (2016), provides a valuable general framework on the direction of innovation that encompasses the models cited here, as well as models of horizontal spillovers and of sequential innovation. Building on the interaction across these different kinds of spillovers, they use their framework to assess when it is optimal to achieve incremental innovations versus large step innovations, and show that granting strong IP rights to 'pioneer patents' may lead to distortions in the direction of R&D. They also identify market distortions that are entirely independent of the market inefficiency identified here.[5] Relatedly, Acemoglu, Akcigit and Kerr (2016)

[4]Even further distantly related to our work, there is also a literature studying the welfare effects of complementarities and substitutabilities among different research approaches to achieve the same innovation (e.g., Bhattacharya and Mookherjee, 1986; Dasgupta and Maskin, 1987). Of course, this is very different from the analysis of this paper, which considers several innovations, without distinguishing different approaches to achieve any of them.

[5]These distortions are demonstrated in a model with costless switching of researchers across R&D lines, and without duplication of efforts in R&D races, so that it is optimal

compute a map of the U.S. innovation network using on 1.8 million U.S. patents and their citations.

Finally, it is interesting to note that our static misallocation force can be related to the study of the so-called ‘price of anarchy’ in the congestion games developed by Rosenthal (1973). These games model a traffic net, in which drivers can take different routes to reach a destination, and routes get easily congested. In the optimal outcome, the drivers coordinate in taking different routes, whereas in equilibrium they excessively take routes that would be faster if they were not congested by their suboptimal driving choices. The analysis of our extended model in section 4 identifies economic features of R&D competition that lead to the ‘congestion’ of hot R&D lines. That these features (costly switching of R&D resources across R&D lines and duplication of efforts in R&D lines) lead to congestion is a novel result for the literature on congestion games, and it may be used as an additional motivation for the study these games.

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to concentrate all R&D resources on a single, most valuable, R&D line, to then move on to the second most valuable one, after the first innovation is discovered, and so on and so forth. Under these assumptions, this paper's market inefficiency that innovators overinvest in the most valuable R&D line can not arise. Conversely, the distortions identifies in their paper do not occur in the simple model of section **??** in which we demonstrate our market distortion in its simplest form.

3 A simple example

There are two problems with private and social values $z_H > z_L$[6], and 2 researchers to be allocated to finding their solution. In any of the problems, the probability of success with one researcher is p and with two is $q > p$. We assume that $q - p < p$, capturing the idea that there is congestion or superfluous duplication of efforts. This holds in case of independence, where $q = 2p - p^2 < 2p$. We examine optimal and competitive allocations with one and two periods.

3.1 One Period Case

Consider first the optimal allocation. Both researchers are allocated to H iff $qz_H \geq p(z_L + z_H)$ or likewise:

$$(q - p)\, z_H \geq pz_L, \tag{1}$$

with a straightforward interpretation.

For the competitive case, we assume that if two researchers are allocated to H, then expected payoffs for each are $\frac{1}{2}qz_H$. This would happen for instance in a patent race where all value accrues to the first to solve the problem. The necessary and sufficient condition for both researchers to work on the H problem is that

$$\frac{1}{2}qz_H \geq pz_L. \tag{2}$$

[6]Likewise, we can assume that private values are an equal proportion of social values. Differences in the degree of appropriability are a a very relevant, yet well know source of inefficiency.

It is easy to verify that condition (1) implies (2) so the competitive allocation will always assign both researchers to H when it is optimal, but might do so also when it is not. The difference between these two conditions can be related to the pecuniary externality (market stealing effect) caused by entry into the H problem that equals $(p - q/2)\, z_H$. Note that here the externality is not present when entering into the L problem, since there will be at most one researcher there. In the more general setting that we examine below with multiple research inputs, this externality will occur for more than one research line and its relative strength is a key factor in determining the nature of the bias in the competitive allocation.

Another interpretation of this external effect is value burning. In the more general setup with many researchers that we examine below, the expected value of solving different research problems is equalized to the least attractive active one. All *differential rents* from solving more attractive problems are dissipated.

3.2 Two Period Case

As above, in each period researchers can be allocated to the unsolved problems. In case both succeed in the first period, then there are no more problems to solve. If one problem is solved in the first period, then in the optimal as well as in the competitive allocation both researchers are assigned to solve the remaining one.

To compare both alternatives, it is convenient to decompose total payoffs of the alternative strategies into first and second period payoffs. The second period problem is a static one. If only one problem is left, then the two researchers will be assigned to it. If the two problems remain to be solved, we will assume for simplicity that condition (1) holds, so that both researchers are assigned to the H problem. Denoting by w_{it} the total expected payoffs for each strategy $i \epsilon \{1,2\}$ in each period $t \epsilon \{1,2\}$ we can write

$$w_{21} = qz_H,\ w_{22} = q\left[(1-q)\,z_H + qz_L\right]$$
$$w_{11} = p\,(z_H + z_L)\,,\ w_{12} = q\left[(1-p)\,z_H + pz_L - p^2 z_L\right]$$

The difference in first period payoffs is identical to the calculation in the static case. Consider now second period payoffs. The terms in brackets represent the expected value of the *problems that remain to be solved.* We call this the option value effect: as the planner has the option of solving problems in the second period it recognizes that if not solved in the first there is a residual value. This is higher when the problem that remains to be solved is H. Ignoring the quadratic term (which becomes irrelevant in the continuous time Poisson specification that follows) $w_{12} > w_{22}$, so incentives for allocating initially both researchers to H will be weaker than in the static case.

Consider now the competitive allocation. Assuming one player chooses H and letting v_{2t} represent expected payoffs in period t for the other player when also choosing H and v_{1t} when choosing L, it follows that

$$v_{21} = \frac{1}{2}qz_H,\ v_{22} = \frac{1}{2}q\left[(1-q)\,z_H + qz_L\right] = \frac{1}{2}w_{22}$$
$$v_{11} = pz_L,\ w_{12} = q\left[(1-p)\,z_H + pz_L - p^2 z_L\right] = \frac{1}{2}w_{12}.$$

Again we are assuming here that in the second period if both problems remain, the two players will choose H. The difference $v_{21} - v_{11}$ is identical to the one for the one period allocation. As shown above and ignoring the quadratic term the difference $w_{22} - w_{12}$ is negative, mitigating the gain from choosing H in the first period as in the optimal allocation. However this difference here is divided by two. The reason is that the deviating agent does not internalize the value that leaving a better mix of problems to be solved for the second period has for the other researcher while the planner does. In the more general setting that follows, as the number of players gets large the dynamic effect vanishes from the competitive allocation condition, while it remains essentially unchanged in the planner's problem. The dynamic effect tilts the incentives in the competitive case towards the hot problem, relative to that in the optimal allocation.

It is straightforward to find find parameter values where: 1) In the static allocation it is optimal to allocate both researchers to the H problem; 2) In the two period case it is optimal to diversify while specialization occurs in the competitive allocation. As an example, this will happen when $z_H = 3$, $z_L = 1$, $p = 3/8$ and $q = 1/2$.

The static allocation problem considered in (3.1) can be reinterpreted as the multi-period one where researchers are fully specialized, so no reallocation takes place. Probabilities q and p should then be interpreted as those corresponding to final success.

4 Assignment without Reallocation

In this Section we lay out the basic model used in the rest of the paper and consider a general form of the static allocation problem discussed in Section 3.1.

There is a large number (in fact, a continuum) of problems or R&D lines, with one potential innovation each. Upon discovery, an innovation delivers value z that is distributed across research lines with cumulative distribution function F, which we assume to be twice differentiable. To isolate the findings of this paper from the well-known effects discussed earlier, we assume that the social value of an innovation coincides with the private value z.[7] There is a mass $M > 0$ of researchers, who are allocated to the different R&D lines according to a measurable function m. The technology of discovery is given by a function $P(m)$ that is strictly increasing, concave and identical for all problems and $P(0) = 0$. For each innovation of value z, we denote by $m(z)$ the mass of researchers competing for the discovery of that innovation. Hence, the resource constraint $\int_0^\infty m(z)\, dF(z) \leq M$ needs to be satisfied.

The expected payoff of participating in an R&D line with value z and a total of $m(z)$ researchers is given by $U(z, m(z)) = P(m(z))\, z/m(z)$. As we show below, these payoffs can be interpreted as a winner-take-all

[7] The expected present discounted value z_j of the patented innovation j does not necessarily equal the market profit for the patented product, net of development and marketing costs. It may also include the expected license fees paid by other firms which market improvements in the future, or the profit for innovations covered by continuation patents. Thus, our model is compatible with standard sequential models of innovation, both those assuming that new innovations do not displace earlier ones from the market, as in the models following Green and Scotchmer (1995), and those assuming the opposite, as in the quality ladder models that follow Aghion and Howitt (1992).

patent race where all participating researchers have equal probability of being first to innovate. Our model can thus be interpreted as an extension of the standard patent race to multiple lines. While that literature considers a single race with a perfectly elastic supply of researchers/firms with some entry/opportunity cost, we consider here the opposite extreme where a fixed supply of research inputs M must be allocated across multiple innovations. [8]

In a competitive equilibrium expected payoffs $P\left(m\left(z\right)\right)z/m\left(z\right)$ are equalized among all active lines, where $m\left(z\right) > 0$. We call this *differential rent dissipation* in analogy to absolute rent dissipation in the standard patent race literature. In contrast, in the optimal allocation $\tilde{m}\left(z\right)$ that maximizes $W(\tilde{m}) = \int_0^\infty zP\left(\tilde{m}\left(z\right)\right)dF\left(z\right)$, the marginal contributions $P'\left(\tilde{m}\left(z\right)\right)z$ are equalized for all active lines.

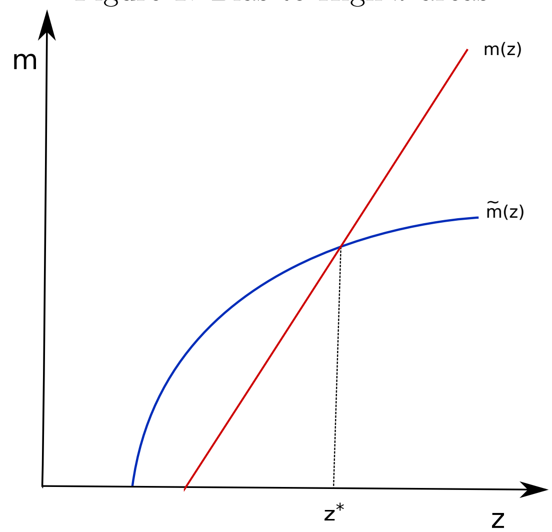
In a competitive equilibrium, a marginal researcher contributes $P'\left(m\right)z$ to total value but gets a return $P\left(m\right)z/m$ which is greater, as a result of the concavity of P. The difference $P\left(m\right)z/m - P'\left(m\right)z$ is the pecuniary externality inflicted on competing innovators. Relative to the value created, this externality is given by:

$$\frac{P\left(m\left(z\right)\right)}{P'\left(m\left(z\right)\right)m\left(z\right)} - 1 = \frac{1}{\varepsilon_{Pm}} - 1 \tag{3}$$

where the first term corresponds to the inverse of the elasticity of discovery with respect to the number of researchers. It is immediate to see that the

[8]Obviously with perfectly elastic supply the problem trivializes as there would be no connection between entry decisions into different research areas. We consider the opposite extreme to emphasize the tradeoff in allocating research inputs across different research lines, but our results should also hold for intermediate cases.

Figure 1: Bias to High z areas



competitive allocation will be optimal if and only if this external effect is the same across research lines. Given that differential rent dissipation implies an increasing function $m(z)$, this condition will hold only when the elasticity of discovery is independent of m, i.e. when the discovery function $P(m) = Am^{\theta}$ for some constant A.[9]

When this condition does not hold, the direction of the bias will depend on how this external effect varies with m. Intuitively, when it increases (i.e. the elasticity of the discovery function is decreasing in m) there will be excessive concentration in high z areas, as we show below. We say that the *competitive equilibrium is biased to higher* z ("hot") *research lines* when the competitive and optimal allocations $m(z)$ and $\tilde{m}(z)$ have the single crossing condition shown in Figure 1. Formally, there exists a threshold $\bar{z}$ such that $m(z) <$

[9]Note, however, that since P is bounded by 1 this function can only hold for a range where $m^{\theta} \leq 1/A$ and beyond this range the elasticity must be zero.

$\tilde{m}(z)$ for $z < \bar{z}$ and $m(z) > \tilde{m}(z)$ for $z > \bar{z}$. Further, when this condition holds, it is also the case that the smallest active R&D line innovation value is higher in equilibrium than in the first best; i.e., that $\tilde{z}_0 = \inf_z\{\tilde{m}(z) > 0\} \leq z_0 = \inf_z\{m(z) > 0\}$. The following Proposition gives conditions for this to hold.

Proposition 1. *In the absence of reallocation, the competitive equilibrium is biased to higher z areas when the elasticity of discovery is decreasing in m.*

Proof. See Appendix. □

While the condition in this Proposition might appear somewhat special, it holds in the canonical model of innovation used in the patent race literature, as we show below. Moreover, as $P(m)/m$ is bounded by 1, the elasticity must converge to zero as $m \to \infty$ so it must decrease in some region.

Stationary Innovation Process

The above setting can be inscribed in a dynamic environment as follows. Let t denote the random time of discovery and $p(t;m)$ the corresponding density when m researchers are assigned from time zero to this research line. The expected utility for each of them is given by

$$U(z_j;m_j) = \int_0^\infty \left(\frac{z}{m}\right) e^{-rt_j} p(t;m)\,dt.$$

Expected payoffs are divided by m since each innovator is equally likely to win the race and if m researchers are engaged in a particular research line, $p(t,m)$ denotes the density of discovery at time t. Letting $P(m) \equiv$

$E\left[e^{-rt};m\right]=\int_0^\infty e^{-rt}p\left(t;m\right)dt$, we can write $U\left(z;m\right)=zP(m)/m$, which is identical to the formulation given above. It is important to emphasize that while time is involved in the determination of payoffs, we are assuming here that once discovery takes place, the m researchers involved become idle and cannot be reallocated to other research lines. The following sections relax this assumption and considers explicitly the problem of reallocation.

We specialize now the setting to a stationary environment that is used in the remainder of the paper. Let $\lambda\left(m\right)m$ denote the hazard rate for discovery at any moment of time so that $p\left(t,m\right)=\lambda\left(m\right)me^{-\lambda(m)mt}$. Assume that $\lambda\left(m\right)m$ is increasing, concave and that $\lambda\left(0\right)=0$. It follows easily that

$$P\left(m\right)=\frac{\lambda\left(m\right)m}{r+\lambda\left(m\right)m}$$

and the elasticity

$$\epsilon_{Pm}=\left(\frac{r}{r+\lambda\left(m\right)m}\right)\left(1-\epsilon_{\lambda m}\right)$$

where where $\epsilon_{\lambda m}$ denotes the elasticity of λ with respect to m, $-\lambda'\left(m\right)m/\lambda\left(m\right)$ It immediately follows that

Proposition 2. *If the elasticity $\epsilon_{\lambda m}$ is weakly increasing in m the competitive allocation will be biased to high z lines.*

We can interpret the elasticity $\varepsilon_{\lambda m}$ as the market stealing externality per unit of value created: $\lambda\left(m\right)m/\lambda'(m)$. The condition given in the Proposition then states that this externality increases with z. Note also that this is a sufficient but not necessary condition, as the first term is decreasing in m. The Proposition applies to the canonical R&D models as the ones discussed in the literature, where $\lambda\left(m\right)=\lambda m$ (i.e. discovery is independent across participants in a patent race) and the elasticity $\epsilon_{\lambda m} = 1$. Each active research line can be thus interpreted as a patent race, where arrival rates given by independent Poisson processes with rate λ and the first to innovate gets the rights to the full payoff z. More generally, the result applies for the constant elasticity case where $\lambda(m) = \lambda m^{-\theta}$ for $0 \leq \theta < 1$. For the canonical model of patent races where $\theta = 0$, an explicit solution for the equilibrium and optimal allocations is given below.

Proposition 3. *Suppose that there is a continuum of R&D lines, whose innovation discoveries are independent events, equally likely among each engaged researcher, with time constant hazard rate λ. Then, the equilibrium and optimal allocation functions are*

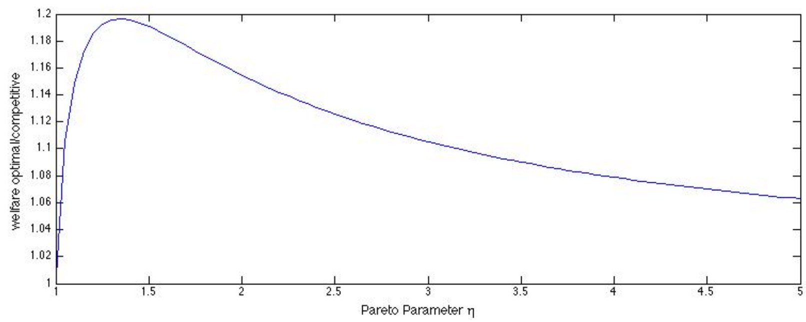
$$m(z) = \frac{z - z_0}{\pi}, \quad \textit{for all } z \geq z_0 = r\pi/\lambda \tag{4}$$

$$\tilde{m}(z) = \frac{r}{\lambda}\left(\sqrt{\frac{z}{\tilde{z}_0}} - 1\right), \quad \textit{for } z \geq \tilde{z}_0 = r\mu/\lambda, \tag{5}$$

where π is the equilibrium profit of each R&D line, and μ is the Lagrange multiplier of the resource constraint. In equilibrium, innovators over-invest in the hot R&D lines relative to the optimal allocation of researchers: there exists a threshold $\bar{z}$ such that $m(z) < \tilde{m}(z)$ for $z < \bar{z}$ and $m(z) > \tilde{m}(z)$ for $z > \bar{z}$.

Importantly, this result demonstrates the market bias that is the theme of this paper (that competing firms over-invest in hot R&D lines) within a canonical dynamic model directly comparable with the many R&D models since Loury (1979) and Reinganum (1981), that are built on the assumption of exponential arrival of innovation discoveries.

Figure 2: Plot of the welfare wedge $W(m)/W(\tilde{m})$.



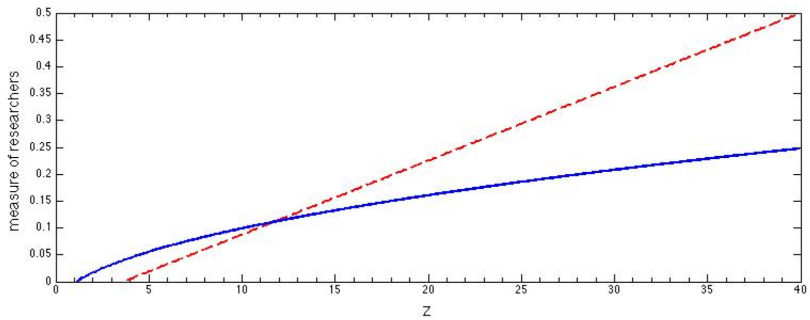
To get a sense of the possible size of this distortion, we perform a simple back-of-the-envelope calculation. Supposing that innovation values are distributed according to a Pareto distribution of parameter $\eta > 1$, so that $F(z) = 1 - z^{-\eta}$ for $z \geq 1$. As proved in Appendix B, when $\lambda(m) = \lambda$ and $\eta > 1$, in an interior allocation where $\tilde{z}_0 > 1$, the welfare gap is:

$$\frac{W(\tilde{m})}{W(m)} = \frac{\eta}{\eta - 1}\left(\frac{\eta - 1}{2\eta - 1}\right)^{1/\eta}.$$

This ratio is plotted in Figure 2. It is negligible for η close to 1, but quickly increases as η grows, so that $W(\tilde{m})/W(\tilde{m}) - 1$ reaches its maximum of about 20% for η close to 1.35 to then slowly decrease and disappear asymptotically as $\eta \to \infty$. Figure 3 gives the corresponding equilibrium and optimal allocations. The solid line corresponds to the optimal allocation and the dashed line to the equilibrium.

We have assumed here that arrival rates are the same for all research lines. Our results can be extended for heterogeneity where the attractiveness

Figure 3: Equilibrium and Optimal Allocation ($\eta = 1.35$)



of R&D lines is not determined only by the innovations' expected market values, but also by the ease of discovery.

[Francesco: Should we do more of that here: formula showing low hanging fruit?]

5 Dynamic Allocation

We extend our previous analysis by allowing the mobility of researchers once a research line is completed. As before, we assume there is a unit mass of research areas (the problems to be solved) with continuous distribution $F(z)$ and there is an inelastic supply M of researchers. While here the set of problems is fixed, Section 6 considers a steady state with entry of new problems. Throughout this section we assume that $\lambda(m)$ is twice continuously differentiable, $\lambda(m)\, m$ is strictly increasing and strictly concave and that $\lambda(0) = 0$. These assumptions imply that the arrival rate per researcher $\lambda(m)$ is decreasing, i.e. there is *instantaneous congestion.*[10] Researchers are free to move across different problems so the equilibrium and optimal allocations determine at any time t the number of researchers $m(t,z)$ assigned to each line of research z. This assignment, together with the results of discovery imply an evolution for the distribution of open problems $G(t,z)$, where

$$\partial G(t,z)/\partial z = -\int^{z} \lambda(m(t,s))\, m(t,s)\, G(t,ds) \tag{6}$$

with $G(0,z) = F(z)$. An allocation is feasible if at all times the resource constraint:

$$\int m(t,z)\, G(t,dz) \leq M \tag{7}$$

is satisfied.

Equilibrium

Because the set of undiscovered innovations shrinks over time, it is never the case that innovators choose to move across R&D lines in equilibrium, nor that it is optimal to do so, unless the R&D line in which the researchers are engaged is exhausted as a consequence of innovation discovery. Indeed, the mass of researchers assigned to a particular line or research z will increase over time. Because mobility is free, the value of participating in any research

[10]Strict concavity implies that the arrival rate does not scale linearly with innovation, which can also capture duplication of innovation effort. In the process of achieving a patentable innovation, competing innovators often need to go through the same intermediate steps (see, for example the models of Fudenberg et al., 1983, and Harris and Vickers, 1985), and this occurs independently of every other innovators' intermediate results, that are jealously kept secret. Hence, the arrival rate of an innovation usually does not double if twice as many innovators compete in the same R&D race.

line z at time t is equated to some value $w(t)$ whenever $m(t,z) > 0$. The value $v(t,z)$ of joining research line z at time t follows the Bellman equation:

$$rv(t,z) = \lambda(m)\, m\left(\frac{z}{m} + w(t) - v(t,z)\right) + v_t(t,z). \tag{8}$$

The first term represents the result of discovery which gives the researcher the value z with probability $1/m$ and the change in value $w(t) - v(t,z)$. The second term represents the change in value that occurs over time as the number of researchers allocated to every line increases. An *equilibrium* is given by an allocation $m(t,z)$ and distribution of open problems $G(t,z)$ together with values $v(t,z)$ and $w(t)$ such that:

1. The allocation m and distribution G satisfy equations (6) and (7)

2. The value function $v(t,z)$ satisfies the functional equation (8) and $v(t,z) \leq w(t)$ with strict equality when $m(t,z) > 0$.

Because the value of active research lines is equalized, $v(t,z) = w(t)$ and $v_t(t.z) = w'(t)$. As a result equation (8) simplifies to:

$$rv(t,z) = \lambda(m)\, m\left(\frac{z}{m}\right) + w'(t). \tag{9}$$

and since this value is equated across active research lines, it follows that $\lambda(m(t,z))\, z$ must be equal too. This corresponds to the *instantaneous* value of participating in research line z and because of free mobility it must be the same across all active research lines. Differenting this expression with respect to z, it follows that

$$m_z(t,z) = -\frac{\lambda(m(t,z))}{\lambda'(m(t,z))\, z} \tag{10}$$

This equation can be integrated starting at a value $z_0(t)$ where $m(t, z_0(t)) = 0$ and $z_0(t)$ is the unique threshold where the resource constraint

$$\int_{z_0(t)} m(t, z(t)) G(t, dz) = M \tag{11}$$

is satisfied. As the mass of G decreases over time, it also follows that the threshold $z_0(t)$ decreases.

Proposition 4. *The equilibrium allocation is the unique solution* $m(z)$ *that satisfies equation (10) and* $m(t, z) > 0$ *iff* $z > z_0(t)$, *where the threshold* $z_0(t)$ *satisfies equation (11).*

Optimal Allocation

Consider an allocation $\tilde{m}(t, z)$. At time t this gives a flow of value $\lambda(\tilde{m}(t, z)) z$. Integrated over all active research lines and time periods, gives the objective:

$$U = \max_{\tilde{m}(t,z)} \int e^{-rt} \int \lambda(\tilde{m}(t, z)) \tilde{m}(t, z) zG(t, dz) dt \tag{12}$$

The optimal allocation maximizes (12) subject to the resource constraint (7) and the law of motion (6). The latter is more conveniently expressed by the change in the density

$$\partial g(t, z) / \partial t = -\lambda(m(t, z)) m(t, z).$$

The formal expressions for the Hamiltonian are given in Appendix C. Letting $u(t)$ denote the multiplier of the resource constraint and $v(t, z)$ the one corresponding to this law of motion, we can write the functional equation:

$$r\tilde{v}(t, z) = \max_{\tilde{m}} \lambda(\tilde{m}) \tilde{m} [z - \tilde{v}(t, z)] - u(t) \tilde{m} + \tilde{v}_t(t, z) \tag{13}$$

for all $z \geq \tilde{z}_0(t)$.

Equation (13) represents the value of an unsolved problem of type z at time t. It emphasizes that problems are indeed an input to innovation, and as can be easily shown the value of an open problem increases with z. Note the contrast to the private value $v(t,z)$ that is equal for all z as a result of the differential rent dissipation: in the eyes of competing innovators, all problems become equally attractive and valuable. The value function defined by (13) can also be interpreted as part of a decentralization scheme where property rights are assigned for each problem z and the owner of each open problem chooses the number of researchers to hire at a rental price $u(t)$. This interpretation highlights the source of market failure in our model precisely due to the lack of such property rights.

The solution to the maximization problem in equation (13) gives:

$$[\lambda'(\tilde{m}(t,z))\,\tilde{m}(t,z) + \lambda(\tilde{m}(t,z))]\,[z - \tilde{v}(t,z)] = u(t) \tag{14}$$

Comparing to the equilibrium condition where $\lambda(m(t,z))\,z$ is equalized, reveals the two key sources of market failure that were illustrated in our simple example in Section 3. First is that the planner internalizes congestion (i.e. the market stealing effect) that is why payoffs are multiplied and not just by the arrival rate λ. It is useful to rewrite the term in brackets as:

$$\lambda(\tilde{m}(t,z))\,[1 - \varepsilon_{\lambda m}] \tag{15}$$

Note the parallel to the results in the case without reallocation considered in Section 4, where the term in brackets corresponds to the wedge between the optimal and competitive allocation. The second term in brackets in equation (14) captures the fact that the payoff for discovery is smaller for the social planner as it internalizes the fact that a valuable problem is lost as a consequence, as we found in our simple example. Taking the ratio of the two conditions gives:

$$\frac{\lambda\left(m\left(t,z\right)\right)}{\lambda\left(\tilde{m}\left(t,z\right)\right)} = \left(1-\varepsilon_{\lambda m}\left(\tilde{m}\left(t,z\right)\right)\right)\left(\frac{z-v\left(t,z\right)}{z}\right)\frac{v\left(t\right)}{u\left(t\right)}$$

For fixed t, the allocation functions cross when $\lambda\left(\tilde{m}\left(t,z\right)\right) = \lambda\left(m\left(t,z\right)\right)$. Since the λ function is decreasing, a sufficient condition for $m\left(t,z\right)$ to remain higher after crossing is that this ratio decreases with z. This is the composition of two effects, represented by the two terms in brackets above. The first term is decreasing if the elasticity is increasing in z, i.e. the market stealing effect increasing. Because the value function is convex in z, the second term decreases in z. This corresponds to the option value effect that we found in our simple example. We have proved:

Proposition 5. *Consider the model with free research mobility with individual arrival rate $\lambda\left(m\right)$. Suppose that the elasticity $\epsilon_{\lambda m}$ is weakly increasing in m. Then, in equilibrium, innovators over-invest in the hot R&D lines: there exists a twice differentiable threshold function $\bar{z}$ such that $m\left(t,z\right) < \tilde{m}\left(t,z\right)$ for $z < \bar{z}(t)$ and $m\left(t,z\right) > \tilde{m}\left(t,z\right)$ for $z > \bar{z}(t)$.*

Note that even abstracting from the first effect (e.g. when the elasticity is constant) the competitive bias to hot areas still holds.

A borderline case occurs in the absence of congestion, i.e. where $\lambda\left(m\right) = \lambda$ and total arrival is linear in λ. Since there is no force to equalize rents in the competitive case, the solution is extreme and all researchers join the highest remaining payoff line at any point in time. The same turns out to be true in the optimal allocation, so in this knife-edge case the equilibrium is efficient. Reallocation costs provide an alternative rent equalizing force that can lead to a non-degenerate equilibrium. These are examined in the following section.

Costly Reallocation

Assume that in the stationary dynamic model discussed above $\lambda(m) = \lambda$. Suppose that, at any point in time, each researcher can be moved across research lines by paying an entry cost $c > 0$. For every innovation of value z and time t, we denote the mass of engaged researchers as $m(t, z)$, and let $z_0(t)$ to be the smallest active R&D line innovation value at time t; i.e., $z_0(t) = \inf_z\{m(t, z) > 0\}$. An equilibrium is defined in the same way as done in the previous section.

Because the set of undiscovered innovations shrinks over time, there is positive entry into any active line of research, so it is never the case that innovators choose to move resources across R&D lines in equilibrium, nor that it is optimal to do so, unless the R&D line in which the researchers are engaged is exhausted as a consequence of innovation discovery.[11] So, we can approach again the problem using standard dynamic programming techniques. We express the equilibrium value $v(t, z)$ of a researcher engaged

[11]Further, as we show in Proposition 6 below, there exists a time T after which researchers are not redeployed into other R&D lines, even when their research line is exhausted due innovation discovery.

in a R&D line of innovation value z at t through the Bellman equation:

$$rv(t,z) = \lambda m(t,z)\left[\frac{z}{m(t,z)} + w(t) - v(t,z)\right] + \frac{d}{dt}v(t,z). \qquad (16)$$

The flow equilibrium value $rv(t,z)$ includes two terms. The first one is the expected net benefit due to the possibility of innovation discovery. The hazard rate of this event is $\lambda m(t,z)$; if it happens, each researcher gains z with probability $1/m(t,z)$ and experiences a change in value $w(t) - v(t,z)$ where $w(t)$ represents the value of being unmatched. The second term, $\frac{d}{dt}v(t,z)$, is the time value change due to the redeployment of researchers into the considered R&D line from exhausted research lines with discovered innovations.

For any time t, both the equilibrium value $v(t,z)$ and its derivative $\frac{d}{dt}v(t,z)$ are constant across all active R&D lines of innovation value $z \geq z_0(t)$.[12] Let $v(t)$ and $v'(t)$ denote these values. In addition, note that since an unmatched researcher can join any research line at cost c, it follows that $v(t) = u(t) + c$. Substituting in (16) we obtain the no-arbitrage equilibrium condition:

$$\lambda [z - m(t,z)c] = rv(t) - v'(t), \text{ for all } z \geq z_0(t). \qquad (17)$$

implying that $z - m(t,z)c$ is equated across all active research lines. Given that $m(t,z) \downarrow 0$ as $z \downarrow z_0$, it follows that payoffs of all active lines $z - m(t,z)$

[12]These conditions are akin to value matching and smooth pasting conditions in stopping problems (for example, see Dixit and Pindyck, 1994). Because R&D firms are competitive, and labor is a continuous factor, the equilibrium dissipates all value differences from discovery of different innovations, through congestion and costly redeployment of researchers. This is similar to the phenomenon of rent dissipation in models of patent races with costly entry.

are equated to z_0 : differential rents are dissipated through higher entry rates in higher return areas, being all equated to the lowest active value line. Notice the parallel to the results in the patent race literature, where all rents are dissipated through entry. It follows that the flow value in the economy at time t is $\lambda M z_0(t)$.

Solving for $m(t,z)$ using the above gives

$$m(t,z) = [z - z_0(t)]/c, \text{ for all } z \geq z_0(t) \tag{18}$$

When the resource constraint $\int_{z_0(t)}^{\infty} m(t,z)\, dG(t,z) \leq M$ binds, the initial condition $z_0(t)$ is pinned down by the equation:

$$cM = c\int_{z_0(t)}^{\infty} m(t,z)\, dG(t,z) = \int_{z_0(t)}^{\infty} (z - z_0(t))\, dG(t,z), \tag{19}$$

where $G(t,z)$ is again the cumulative distribution function of innovations not discovered yet at time t.

We also note that, because active R&D lines with innovation value $z \geq z_0(t)$ get exhausted over time, more researchers engage in the remaining lines, i.e., $m_t(t,z) > 0$ for all $z \geq z_0(t)$; less valuable lines becoming active, i.e., $z_0'(t) < 0$, and each active research line becomes less valuable, i.e., $v'(t) < 0$. Indeed, the value $v(t)$ decreases over time until the time T such that $v(T) = c$. At that time, redeployment of researchers stops at the end of the R&D race in which they are engaged. Active research lines have become so crowded that their value is not sufficient to recover the entry cost c any longer.[13]

[13]The characterization of the allocation $m(t,z)$ of researchers on undiscovered R&D lines at any time $t \geq T$ is covered by the earlier analysis of the canonical dynamic model without redeployment of researchers (cf. Proposition 3). In our set up with a continuum of R&D lines distributed according to the twice differentiable function G, arguments invoking 'laws of large number' suggest that the allocation $m(t,z)$ would smoothly converge to the allocation $m(t)$ described in Proposition 3.

The next Proposition summarizes the above equilibrium analysis.

Proposition 6. *Assume $\lambda(m) = \lambda$ and that researchers can be moved across R&D lines at cost $c > 0$. The equilibrium allocation:*

$$m(t,z) = \frac{z - z_0(t)}{c}, \text{ for all } z \geq z_0(t), \tag{20}$$

where the boundary $z_0(t)$ solves equation (19). Researchers are redeployed into different active R&D lines only until the time T such that $z_0(T) = rc/\lambda$, and only if their research line is exhausted due to innovation discovery. The flow value in the economy at time t is $\lambda M z_0(t)$.

We now consider the optimal allocation that is defined as in the previous section, after subtracting total entry costs. Following the same approach, we solve for the optimal allocation $\tilde{m}$ using the Bellman equation defined by the co-state dynamic condition in the Hamiltonian. The details are provided in Appendix C. The value of a research line z at time t satisfies

$$r\tilde{v}(t,z,0) = \max_{\hat{m}\in\mathbb{R}} \lambda\hat{m}\left[z - \tilde{v}(t,z,\tilde{m})\right] - r\tilde{m}c - u(t)\tilde{m} + \tilde{v}_t(t,z,\tilde{m}), \text{ for all } z \geq \tilde{z}_0(t). \tag{21}$$

There are several comments to make about this equation. First, note that due to the irreversible entry cost c, the value function has as an additional argument the number of researchers $\tilde{m}$. On the left hand side we consider the flow value of an empty research line with $\tilde{m} = 0$. On the right hand side we consider the optimal choice of $\tilde{m}$. The entry cost $\tilde{m}c$ is expressed in flow terms consistent with the formulation of the value function. As before $u(t)$ is the multiplier for the resource constraint. Finally note that at the optimal choice $\tilde{v}(t,z,\tilde{m}(t,z)) = \tilde{v}(t,z,0) + m(t,z)c$. This also implies that

$v_t(t,z,m)$ is independent of m, which is used below. Substituting in (21) we obtain:

$$r\tilde{v}(t,z,0) = \max_{\hat{m}\in\mathbb{R}} \lambda\hat{m}\left[z - \tilde{v}(t,z,0) - mc\right] - r\tilde{m}c - u(t)\,\tilde{m} + \tilde{v}_t(t,z,\tilde{m}), \text{ for all } z \geq \tilde{z}_0(t). \tag{22}$$

This can be also interpreted as the Bellman equation for a firm that is assigned the property rights to a problem z at time t and needs to choose the initial amount of researchers to hire, paying the initial entry cost $\tilde{m}c$ and rental price $u(t)$. The solution of program (21) leads to the first order conditions:

$$\lambda\left[z - \tilde{v}(t,z,0) - 2\tilde{m}(t,z)c\right] = rc + u(t), \text{ for every } z \geq \tilde{z}_0(t). \tag{23}$$

Equating these first order conditions leads to the differential equation

$$\tilde{m}_z(t,z) = \frac{1 - \tilde{v}_z(t,z,0)}{2c}. \tag{24}$$

By comparison, the differential equation for the equilibrium allocation obtained by differentiating (18) gives:

$$m_z(t,z) = \frac{1}{c}. \tag{25}$$

It follows immediately that the derivative $\tilde{m}_z$ of the optimal allocation function $\tilde{m}$ is smaller than m_z, the derivative of the equilibrium allocation function m.[14] Because both functions m and $\tilde{m}$ need to satisfy the same resource allocation constraint, this implies that the competitive equilibrium is biased towards high return areas.

[14] We prove in the appendix that $0 < \tilde{v}_z(t,z) < 1$.

Further, the comparison of equations (24) and (25) allows us to single out two separate effects that lead to this result. First, is the option value effect described earlier, where the marginal value of a better research line is $1 - v_z$ which is less than one, the marginal value in the competitive equilibrium. The derivative v_z captures the fact that a better problem has also more value in the future, in contrast to equalization due to rent dissipation that occurs in the competitive case. As a result, when engaging in a R&D line of value z, competing firms do not internalize the negative externality $\tilde{v}_z(t, z)$, the change in the continuation value due to the reduced likelihood of discovering the innovation later. This leads the competing firms to sub-optimally anticipate investment in the hot R&D lines, leading to over-investment at every time t.

At the same time, the additional social marginal cost for engaging an additional researcher in a marginally more profitable line, $2c \cdot \tilde{m}_z(t, z)$, is twice the private additional expected cost $c \cdot m_z(t, z)$ incurred by the individual researcher. On top of this private cost, the society suffers also an additional redeployment cost. This cost is incurred in expectation by all researchers already engaged in the more profitable R&D line, in case the additional researcher wins the R&D race. This additional redeployment cost is not internalized by the competing firms, and it also pushes towards equilibrium over-investment in the hot R&D lines.[15]

[15] The result that R&D firms overinvest in hot R&D lines fails to hold only when $c = 0$ (the case of perfectly costless redeployment of researchers). In this case, assuming that the innovation value distribution has bounded support, all researchers will be first engaged in the most valuable R&D lines. When these innovation are discovered, they will all be redeployed to marginally less valuable ones, until also these innovations are discovere, and so on and so forth. This unique equilibrium outcome is also socially optimal.

Proposition 7. *Assume* $\lambda(m) = \lambda$ *and there is a cost of entry* $c > 0$ *to start solving any new problem. Then, in the competitive equilibrium innovators over-invest in high return R&D lines at every time* t*: there exists a threshold function* $\bar{z}(t)$ *such that* $m(t, z) < \tilde{m}(t, z)$ *for* $z < \bar{z}(t)$ *and* $m(z) > \tilde{m}(t, z)$ *for* $z > \bar{z}(t)$.

The following section extends this model by considering the arrival of new R&D lines.

6 Steady State Economy

Consider the model analyzed in the last section and suppose in addition new problems arrive with Poisson intensity α and returns z distributed according to an exogenous distribution $F(z)$. We focus on R&D line replacement that keeps the economy in steady state.[16]

In steady state, the equilibrium allocation m is independent of time t, and thus calculated with obvious modifications of the analysis presented earlier in this section. The expression $\lambda[z - m(z)c]$ is constant for all $z \geq z_0$, so that the equilibrium solves the differential equation $m'(z) = 1/c$, which gives the solution

$$m(z) = (z - z_0)/c, \quad \text{for every } z \geq z_0. \tag{26}$$

Likewise, obvious modifications of the Bellman equations (21) shows that the

[16] For simplicity, we assume that when an innovation is discovered, the cost c for re-deploying researchers is the same for all R&D lines, including the follow up lines of the innovation discovery. Our results would extend to a more complicated model in which the redeployment cost is smaller for these lines as long as they are not exactly equal to zero.

social planner problem takes the following form, in steady state:

$$r\tilde{v}(z) = \max_{\tilde{m}\in\mathbb{R}} \lambda\tilde{m}[z - \tilde{v}(z) - \tilde{m}c] - r\tilde{m}c - u\tilde{m}, \quad \text{for all } z \geq z_0, \tag{27}$$

under the constraint that u satisfies the resource constraint. The associated first-order conditions are:

$$\lambda[z - \tilde{v}(z) - 2\tilde{m}(z)c] = rc + u, \quad \text{for every } z \geq z_0. \tag{28}$$

Inspection of the equilibrium condition and equation (27) reveal the same two forces identified earlier leading to excessive equilibrium investment in hot areas, so there exists a threshold $\bar{z}$ such that $m(z) < \tilde{m}(z)$ for $z < \bar{z}$, and $m(z) > \tilde{m}(z)$ for $z > \bar{z}$.

With simple manipulations presented in appendix B, we obtain:

$$\lambda\left[z - c\frac{\lambda\tilde{m}(z)^2}{r} - 2c\cdot\tilde{m}(z)\right] = rc + u, \quad \text{for every } z \geq z_0. \tag{29}$$

These equations are analogous to the ones obtained in the first-order conditions (23) for the model redeployment that we solved earlier. The only difference is that the term $\tilde{v}(z)$ takes the constant form $c\cdot\lambda\tilde{m}(z)^2/r$, here, which is the discounted cost of all future redeployment of the mass $\tilde{m}(z)$ of researchers engaged in the considered R&D line —the term $c\cdot\lambda\tilde{m}(z)/r$ is the individual discounted cost. So, we can identify as $c\cdot\lambda\tilde{m}(z)^2/r + c\cdot\tilde{m}(z)$, the 'redeployment cost externality' that an additional researcher imposes on the $\tilde{m}(z)$ researchers engaged in the R&D line. As shown in appendix B, equation (29) can be used to obtain an explicit solution for the optimal allocation as a function of z_0 :

$$\tilde{m}(z) = \frac{r}{\lambda}\left(\sqrt{\lambda\frac{z - \tilde{z}_0}{rc} + 1} - 1\right), \text{ for all } z \geq z_0. \tag{30}$$

17

To further the comparison between the equilibrium and first best allocation functions m and $\tilde{m}$, we continue the analysis under the assumption that, in the steady state economy, the value distribution of the new R&D lines is independent of the values of the R&D lines that they replace. Under this assumption, both allocation functions m and $\tilde{m}$, satisfy the simple steady state conditions:[18]

$$\lambda m(z)\, g(z) = \alpha f(z)\,, \text{ for all } z \geq z_0, \tag{31}$$

$$\lambda \tilde{m}(z)\, \tilde{g}(z) = \alpha f(z)\,, \text{ for all } z \geq \tilde{z}_0. \tag{32}$$

where g and $\tilde{g}$ denote the stationary equilibrium densities of undiscovered innovation values associated with m and $\tilde{m}$ respectively, f the density of the innovation values of the new R&D lines, and $\alpha \leq \lambda M$ the flow arrival rate of R&D lines. These densities are defined for values of z above the respective thresholds. Those below the threshold become untouched and thus grow unboundedly. Conditions (31) and (32) imply that, for any innovation value z with active R&D lines, the total mass of researchers allocated in

[17] The net benefit of an additional researcher in the R&D line equals this researcher's discovery hazard rate λ, multiplied by the innovation value z, minus the current and future discounted switching costs $c\tilde{m}(c)+c\lambda\tilde{m}(z)^2/r$ borne by the other $\tilde{m}(z)$ researchers, minus the cost of $\tilde{m}(z)\,c$ of redeploying this marginal researcher. The latter, grouped with z, gives the expression $\lambda[z-\tilde{m}(z)]c$ which is the private marginal net benefit of researchers in the R&D line, as reported earlier.

[18] In appendix A, we consider the general case in which the values distribution of new R&D lines is not independent of the values of the discovered innovations. Unless also the R&D line value distribution support changes with the values of the discovered innovations, there exists an equilibrium that also satisfies equations (31) and (32). In the extreme case in which each discovery leads to a R&D line with the same innovation value, the option value effect identified comparing program (21) with equation (16) disappears, but our main result that competing researchers overinvest in the hot R&D lines persists. In every other case, both the option value effect and our main result persist.

the steady state equilibrium and optimal allocations, respectively $m(z)\, g(z)$ and $\tilde{m}(z)\, \tilde{g}(z)$, are both equal to $(\alpha/\lambda)\, f(z)$, the net inflow of R&D lines of innovation value z. Of course, this does not mean that also the mass of researchers engaged in each R&D line is the same: it need not be that $m(z) = \tilde{m}(z)$ for any active R&D line of innovation value z since the stationary distributions g and $\tilde{g}$ will differ.

We now turn to the determination of the thresholds. Assuming the resource feasibility constraint $\int_{z_0}^{+\infty} m(z)\, g(z)\, dz \leq M$ binds in both allocations, the threshold z_0 is pinned down by plugging the stationarity condition (31) into the binding resource constraint, so as to obtain the equation

$$\alpha\,(1 - F(z_0)) = \lambda M. \tag{33}$$

Again λM is the outflow of solved problems which must equal in steady state the inflow of new *relevant* problems. Remarkably, this implies that the threshold z_0 is determined independently of the allocation function m, so in particular $\tilde{z}_0 = z_0$. Equations (26), (30) and (33) can be used to solve for the equilibrium and optimal allocations. [19]

Because the thresholds z_0 and $\tilde{z}_0$ coincide, $m_z > \tilde{m}_z$ and $\lim_{z \downarrow z_0} m(z) = \lim_{z \downarrow z_0} \tilde{m}(z) = 0$, it follows that $m(z) > \tilde{m}(z)$ for all $z > z_0$. This is consistent with both allocations integrating to total resources M precisely because the stationary distribution of open problems $\tilde{G}$ in the optimal allocation stochastically dominates G, the one in the stationary competitive equilibrium.

[19] If the resource constraint is satisfied with a strict inequality, $\int_{z_0}^{\infty} m(z)\, dG(z) < M$, then the economy cannot support entry by all firms, the participation constraint $\bar{v} \geq c$ binds and pins down z_0 through the equality $c = \bar{v} = (\lambda/r) z_0$.

In words, the density of the R&D lines with undiscovered innovations is very large for small innovation values, very few researchers are engaged on these R&D lines, and hence innovation discoveries arrive with a very low rate. As the innovation value grows larger, the density of R&D lines with undiscovered innovations decreases. The rate of decrease is larger for the competitive equilibrium than for the optimal allocation function. So, the market sub-optimally exhausts too many high value R&D lines too early, and leaves too few for future discovery. As a consequence of this, the number of researchers per project is always higher in equilibrium than in the social optimum, because the there are more high-value R&D lines in the social optimum, and these high-value R&D lines take up more researchers than low value R&D lines.

Welfare

In any allocation $m(z)$ the flow of value

$$
\begin{aligned}
rV &= \lambda \int_{z_0} m(z)(z - m(z)c)\, g(z)\, dz \\
&= \alpha \int_{z_0} (z - m(z)c)\, f(z)\, dz. \\
&= \alpha \int_{z_0} z f(z)\, dz - \alpha c \int_{z_0} m(z)\, f(z)\, dz
\end{aligned}
\tag{34}
$$

The first term is the same in any allocation and it is precisely the value of the outflow of problems solved that in a stationary equilibrium equals the corresponding inflow. Since the latter is independent of the allocation, so is the former. The second term corresponds to the total flow costs of redeployment, that differs across the two allocations. In the competitive allocation $cm(z) = (z - z_0)$. Substituting in the equation above gives:

$$\begin{aligned} rV &= \alpha \int_{z_0} zf(z)\, dz - \alpha \int_{z_0} (z - z_0) f(z)\, dz \\ &= \alpha z_0 (1 - F(z_0)) = \lambda M z_0. \end{aligned}$$

This represents a value equivalent to the flow of all innovations equalized to the lowest value one, reflecting again differential rent dissipation. Note that as z_0 is independent of c, this value is the same for all c, within a range where all researchers are employed in the steady state. In particular, it holds surprisingly even as $c \downarrow 0$ due to tan unboundedly increasing concentration in high return areas.

Consider now the flow value of the optimal allocation. Using (30), it follows that

$$\alpha \tilde{m}(z) f(z) c = \frac{r\alpha}{\lambda} \left(\sqrt{c\lambda \frac{(z - \tilde{z}_0)}{r} + c^2} - c \right) f(z).$$

Substituting in (34) and using our previous result proves:

Proposition 8. *When $\lambda(m) = \lambda$, the cost of entry is $c > 0$ and the flow of entry of new problems α with density f, aggregate equilibrium and optimal welfare are given, respectively by:*

$$W(m) = (\lambda/r) z_0 M \tag{35}$$

and

$$W(\tilde{m}) = \frac{\alpha}{r} \int_{z_0}^{\infty} zf(z)\, dz + cM - \frac{\alpha}{\lambda} \int_{z_0}^{\infty} \sqrt{\frac{c\lambda}{r}(z - z_0) + c^2} \cdot f(z)\, dz. \tag{36}$$

These closed-form expressions make welfare assessments simple and precise. Welfare is dissipated in the equilibrium allocation with excess researcher turnover to equal the flow of the lowest active research area. The reason why the welfare is not dissipated in the optimal solution is because the social planner spreads out researchers more evenly and leaves a larger number of hot R&D lines for later, so that the society does not pay as much in terms of relocation costs.

When the switching costs are small, the optimal welfare expression simplifies further:

$$\begin{aligned}\lim_{c\to 0^+} W(\tilde{m}) &= (\alpha/r)\int_{z_0}^{\infty} zf\left(z\right)dz = (\alpha/r)E\left(z|z \geq z_0\right)\left[1 - F(z_0)\right] \\ &= (\lambda/r)E\left(z|z \geq z_0\right)M.\end{aligned}$$

Again, this result makes transparent the comparison between rent dissipation in the competitive equilibrium that is not present in the planner's solution. Thus, for small switching cost c, the welfare ratio $W(m)/W(\tilde{m})$ takes the form:

$$\lim_{c\to 0^+} \frac{W(m)}{W(\tilde{m})} = \frac{z_0}{E\left(z|z \geq z_0\right)}.$$

In words, the welfare ratio converges to the smallest active R&D line innovation value z_0, divided by the expected active R&D line innovation value. It is intuitive that this quantity can be significantly small, for standard cumulative distributions F.[20]

[20]We performed a back-of-the-envelope calculation of the welfare ratio $W(m)/W(\tilde{m})$, under the assumption that the distribution F is lognormal with mean equal to 7 and standard deviation equal to 1.5, consistently with the estimates provided by Schankerman (1998). With cost $c = 1$ million, the welfare ratio $W(m)/W(\tilde{m})$ is approximately 0.28. As the cost c vanishes, the ratio $W(m)/W(\tilde{m})$ converges to 0.17 approximately.

7 Conclusion

Research on the efficiency of innovation markets is usually concerned on whether the level of innovator investment is socially optimal. This paper has asked a distinct, important question: Does R&D go in the right direction? In a simple dynamic model, we have demonstrated that R&D competition pushes firms to disproportionately engage in the hot research lines, characterized by higher expected rates of return. The identification of this form of market failure is a novel result, as we explained in details in the introduction and literature review.

After demonstrating this result within a simple model, we have extended our framework in different directions, so as to include various features of economic interest. The canonical dynamic model we have formulated is populated with a continuum of innovation R&D lines, whose discovery is an independent random event, equally likely across all engaged researchers, and with time-constant hazard rate. We have allowed for both possibilities that researchers may or may not be moved across R&D lines costly. We have studied the case in which the innovation discovery hazard rate grows with the number of engaged researchers proportionally, and the case in which it grows less than proportionally because of duplicative R&D effort. We have worked both under the assumption that R&D lines that are exhausted due to innovation discovery may not be replaced by successive innovation vintages, and under the assumption that they are replaced, so as to keep the economy in steady state.

We have recovered the result that competing innovators overinvest in the hot research areas in all cases, with the exception of the knife-hedge case without duplicative R&D effort, without R&D line replacement, and in which researchers can be switched across R&D line costlessly. Importantly, this analysis allowed us to clarify which specific features of R&D competition lead to the market failure that is the theme of this paper. The analysis for the case in which the economy is in steady state has also allowed us calculate the equilibrium and optimal welfare expressions in simple closed forms.

To offset the distortion that competing innovators overinvest in the hot research lines in equilibrium, a simple policy recommendation is that R&D lines with less profitable or less feasible innovations are subsidized, so as to rebalance remuneration across R&D lines. Importantly, our case for R&D subsidization is based on a framework without any market frictions. We argue that even frictionless markets and competition forces cannot solve the form of market failure identified here. And in fact, we have shown that increasing market competition can only make the equilibrium distortion worse.

Details of the existing forms and mechanisms of non-market R&D funding have been discussed in appendix A. The main sources of funding are research grants and fiscal incentives, in the form of subsidies or tax breaks. Prizes, procurements and the funding of academia also serve to subsidize R&D, but they do not seem plausibly effective in alleviating the form of market inefficiency identified here (overinvestment in the hot R&D areas). Indeed, it is even possible that prizes, procurement and career concerns in academia exacerbate the market inefficiency singled out in this paper. Plausibly, they may bias incentives of individual researchers so that they disproportionately compete on a small set of high-profile breakthroughs, instead of spreading their efforts more evenly across valuable innovations.[21]

Returning to comparing the main forms of R&D funding, grants and fiscal incentives, the extant literature identifies as the main advantage of fiscal incentives, the fact that they leave the choice of the direction of R&D to the informed parties: the competing innovators. Unfortunately, this is exactly the fundamental source of the market inefficiency that we have identified in this paper. It is therefore possible that fiscal incentives are ineffective to offset the distortion we identified, and that direct State intervention through grants and procurement would be a preferable mechanism. Specifically, it may be useful that research grants are spread across a large variety of R&D lines, rather than financing only a few high profile projects. The verification of this concluding conjecture will require extensive work, both in terms of formal modelling and data-based quantitative assessments, and is beyond the boundaries of this paper.

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Appendix A: Omitted Extensions and Discussions

Heterogenous Arrival Rates Here, we allow for the possibility that the arrival rates of innovations differ across R&D lines. Letting $\lambda(z)$ be the discovery arrival rate of innovations with value z, we see that, now, $P(z, m(z)) = m(z)\lambda(z)/[r + m(z)\lambda(z)]$. The same arguments that lead to Proposition 3 imply that again, innovators over-invest in the hot, most attractive, research lines, in equilibrium. Here, however, the attractiveness of an R&D line is not determined by its innovation value z alone, but by the expected flow value $z\lambda(z)$ of engaging in the R&D line. So, we can reformulate and extend Proposition 3 as follows.

Proposition A.3. *Consider the canonical dynamic model in which the discovery arrival rate of innovations with value z is time-constant and equal to $\lambda(z)$. In equilibrium, firms over-invest in the R&D lines with the highest expected flow value $z\lambda(z)$: there exists a threshold ζ such that $m(z) < \tilde{m}(z)$ for $z\lambda(z) < \zeta$ and $m(z) > \tilde{m}(z)$ for $z\lambda(z) > \zeta$.*

Proof. Here, the equilibrium arbitrage conditions require that the expression $\frac{P(z,m(z))}{m(z)} = \frac{z\lambda(z)}{r+m(z)\lambda(z)}$ is constant in z. Likewise, equating the first-order conditions to find the optimal allocation $\tilde{m}$ implies that the expression $P_m(z, \tilde{m}(z))\, z = \frac{rz\lambda(z)}{[r+\tilde{m}(z)\lambda(z)]^2}$ is constant in z.

Take any two values of z, z_1 and z_2 and say without loss of generality that $z_2\lambda(z_2) > z_1\lambda(z_1)$. Because the function $\frac{r}{(r+\tilde{m}\lambda)^2}$ decreases in $\tilde{m}\lambda$, it follows that $\tilde{m}_2\lambda_2 > \tilde{m}_1\lambda_1$, where we write $\tilde{m}_j$ instead of $\tilde{m}(z_j)$ and λ_j instead of $\lambda(z_j)$ for brevity. Dividing the no arbitrage condition

$$\frac{P_1(m_1)}{m_1} z_1 = z_1\lambda_1 \frac{1}{r + m_1\lambda_1} = z_2\lambda_2 \frac{1}{r + m_2\lambda_2} = \frac{P_2(m_2)}{m_2} z_2,$$

where we write m_j instead of $m(z_j)$, by the social planner's solution condition

$$P_1'(\tilde{m}_1) z_1 = z_1\lambda_1 \frac{r}{(r + \tilde{m}_1\lambda_1)^2} = z_2\lambda_2 \frac{r}{(r + \tilde{m}_2\lambda_2)^2} = P_2'(\tilde{m}_2) z_2,$$

we obtain:

$$\frac{(r + \tilde{m}_1\lambda_1)^2}{r(r + m_1\lambda_1)} = \frac{(r + \tilde{m}_2\lambda_2)^2}{r(r + m_2\lambda_2)}.$$

Now suppose, by contradiction, that $m_2 \leq \tilde{m}_2$ and that $m_1 \geq \tilde{m}_1$. Then, we obtain the contradiction:

$$\frac{(r + \tilde{m}_1\lambda_1)^2}{r(r + m_1\lambda_1)} \leq \frac{(r + \tilde{m}_1\lambda_1)^2}{r(r + \tilde{m}_1\lambda_1)} < \frac{(r + \tilde{m}_2\lambda_2)^2}{r(r + \tilde{m}_2\lambda_2)} \leq \frac{(r + \tilde{m}_2\lambda_2)^2}{r(r + m_2\lambda_2)},$$

using the fact that the function $\frac{(r+\tilde{m}\lambda)^2}{r(r+\tilde{m}\lambda)}$ increases in $\tilde{m}\lambda$. □

This result provides a useful generalization of our finding that competing firms overinvest in hot R&D lines. In most applications, R&D lines differ both in terms of the expected rate of returns and the expected feasibility of innovations. Because of the canonical nature of exponential arrivals, this generalized result can be easily taken to industry datasets.

Proposition A.3 can be further generalized to broader classes of arrival densities $p(t, z, m(z))$, beyond the canonical exponential class in which $p(t, z, m(z)) = m(z)\lambda(z)e^{-m(z)\lambda(z)t}$, whenever an appropriate parametrization is suitable.

Steady State Economy with Innovation Value Dependence Across Vintages To further the comparison between the steady state equilibrium and first best allocation functions m and $\tilde{m}$, we now describe the steady state conditions. We stipulate that, when an innovation of value z' is discovered, it generates R&D lines whose value z is distributed according to the probability density $f(z|z')$. Denoting by α the flow arrival rate of R&D lines, the steady state conditions (31) and (32) take the form:

$$\lambda m(z)\, g(z) = \int_{z_0}^{\infty} \lambda \alpha f(z|z')\, m(z')\, g(z')\, dz', \text{ for all } z \geq z_0,$$
$$\lambda \tilde{m}(z)\, \tilde{g}(z) = \int_{\tilde{z}_0}^{\infty} \lambda \alpha f(z|z')\, \tilde{m}(z')\, \tilde{g}(z')\, dz', \text{ for all } z \geq \tilde{z}_0.$$

When it is the case that $z_0 = \tilde{z}_0$ and that $m(z)\, g(z) = \tilde{m}(z)\, \tilde{g}(z)$ for any $z \geq z_0$, it is easy to extend all our earlier results on the comparison of m and $\tilde{m}$, derived for the case in which the density $f(z|z')$ is independent of z'. Indeed, letting the overall density of new innovation opportunity values be $f(z) \equiv \int_{z_0}^{\infty} \lambda f(z|z')\, m(z')\, g(z')\, dz' = \int_{\tilde{z}_0}^{\infty} \lambda \alpha f(z|z')\, \tilde{m}(z')\, \tilde{g}(z')\, dz'$, and substituting in the above steady state conditions, we recover the precise expressions (31) and (32). Hence, we have concluded that the steady state economy with innovation value dependence across vintages has a solution pair m, $\tilde{m}$ for which all our earlier results extend.

Appendix B: Omitted Proofs

Proof. [Proof of Proposition 1] In a competitive equilibrium, the average payoff $P(m(z))z/m(z)$ per researcher is constant for all $m(z) > 0$. Differentiation with respect to z yields $[P'(m(z))m'(z)z + P(m(z))]m(z) - P(m(z))zm'(z) = 0$ for all $m(z) > 0$. Rearranging, the slope of the competitive allocation m is:

$$m'(z) = \frac{P(m(z))m(z)}{z[P(m(z)) - m(z)P'(m(z))]}.$$

This derivative is strictly positive because P is positive and concave, so the set of z over which $m(z) > 0$ is connected.

In the optimal allocation $\tilde{m}$, the marginal payoff $P'(\tilde{m}(z))z$ is constant for all $\tilde{m}(z) > 0$. Differentiating with respect to z and rearranging,

$$\tilde{m}'(z) = -\frac{P'(\tilde{m}(z))}{zP''(\tilde{m}(z))} > 0.$$

By concavity of P, this derivative is positive: also the set of z over which $m(z) > 0$ is connected.

If the allocation functions m and $\tilde{m}$ cross at any point $\bar{z}$, then, letting $\bar{m} = m(\bar{z}) = \tilde{m}(\bar{z})$,

$$m'(\bar{z}) - \tilde{m}'(\bar{z}) \propto P(\bar{m})\bar{m}P''(\bar{m}) + P'(\bar{m})P(\bar{m}) - \bar{m}[P'(\bar{m})]^2.$$

This quantity is proportional to the derivative of the elasticity of discovery $\varepsilon_{P(\bar{m})}$ with respect to m calculated at $P(\bar{m})$, and it is thus negative if the elasticity of discovery decreases in m. Hence, there is at most a single crossing point $\bar{z}$ for the allocation functions m and $\tilde{m}$, and at such a $\bar{z}$, the competitive allocation function m is steeper than $\tilde{m}$.

Because both functions m and $\tilde{m}$ are subject to the same resource constraint $\int_0^\infty m(z)\,dF(z) \leq M$, the functions m and $\tilde{m}$ cross exactly once, at $\bar{z}$. It is then the case that $m(z) < \tilde{m}(z)$ for all $z < \bar{z}$, that $m(z) > \tilde{m}(z)$ for all $z > \bar{z}$, and that $\tilde{z}_0 = \inf_z\{\tilde{m}(z) > 0\} \leq z_0 = \inf_z\{m(z) > 0\}$. □

Proof. [Proof of Proposition 3]For all innovation values z with active R&D lines, the equilibrium no-arbitrage conditions are

$$z\frac{P(m)}{m} = z\frac{\lambda}{r + m\lambda} = \pi.$$

Solving out, we obtain

$$m(z) = \max\{0, z/\pi - r/\lambda\} = r/\lambda \max\{0, z/z_0 - 1\},$$

where $z_0 = r\pi/\lambda$ is the smallest-value R&D line z such that $m(z) > 0$, thus obtaining expression (4).

A social planner chooses $\tilde{m}(\cdot)$ to maximize the social welfare:

$$W(\tilde{m}(\cdot)) = \int_0^\infty z \frac{\tilde{m}(z)\lambda}{r + \tilde{m}(z)\lambda} f(z)\, dz \quad \text{s.t.} \quad \int_0^\infty \tilde{m}(z) f(z) dz = M.$$

Hence, the Euler conditions are that, for all z,

$$z \frac{r\lambda}{(r + \tilde{m}(z))\lambda)^2} = \mu,$$

where μ is the Lagrange multiplier of the resource constraint. Solving out, we get

$$\tilde{m}(z) = \max\left\{0, \sqrt{(z/\mu)(r/\lambda)} - r/\lambda\right\} = \max\left\{0, \sqrt{z/\tilde{z}_0} - 1\right\} r/\lambda,$$

where $\tilde{z}_0 = r\mu/\lambda$ is the smallest-value R&D line z such that $\tilde{m}(z) > 0$, thus obtaining expression (5).

The proof that there exists a threshold $\bar{z}$ such that $m(z) < \tilde{m}(z)$ for $z < \bar{z}$ and $m(z) > \tilde{m}(z)$ for $z > \bar{z}$ is analogous to the proof of Proposition 1 above, once realized that the generalized inverse hazard rate $\Gamma(\hat{m}) = r/(r + \hat{m}\lambda)$ strictly decreases in $\hat{m}$. □

Derivation of Welfare Ratio for Pareto Distribution

Returning to expression (4) for the equilibrium allocation m, we note that, here, z_0 is pinned down by the resource constraint:

$$M = \int_0^\infty m(z) f(z) dz = \frac{r}{\lambda} \int_{z_0}^\infty \left(\frac{z}{z_0} - 1\right) \frac{1}{z^{\eta+1}} \eta dz = \frac{r}{\lambda} \frac{z_0^{-\eta}}{\eta - 1}.$$

Hence, the equilibrium welfare is simply:

$$W(m) = M\pi = \frac{z_0^{1-\eta}}{\eta - 1},$$

as each researcher earns the value π, and there is a continuum of mass M of researchers.

Returning to expression (5) for the optimal allocation $\tilde{m}$, we see that $\tilde{z}_0$ is pinned down by the resource constraint

$$M = \int_1^{\infty} \tilde{m}(z)\, dF(z) = \frac{r}{\lambda} \int_{\tilde{z}_0}^{\infty} \left(\sqrt{\frac{z}{\tilde{z}_0}} - 1 \right) \frac{1}{z^{\eta+1}} \eta dz = \frac{r}{\lambda} \frac{\tilde{z}_0^{-\eta}}{2\eta - 1}.$$

Because the expected social value of employing $\tilde{m}(z)$ researchers in any R&D line of value z is

$$\begin{aligned} zP(\tilde{m}(z)) = z\frac{\tilde{m}(z)\,\lambda}{r + \tilde{m}(z)\,\lambda} &= z\frac{\frac{r}{\lambda}\left(\sqrt{\frac{z}{\tilde{z}_0}} - 1\right)\lambda}{r + \frac{r}{\lambda}\left(\sqrt{\frac{z}{\tilde{z}_0}} - 1\right)\lambda} \\ &= z - \sqrt{z\tilde{z}_0}, \end{aligned}$$

integrating over z, we obtain that the optimal welfare is:

$$\begin{aligned} W(\tilde{m}) = \int_{\tilde{z}_0}^{\infty} zP(\tilde{m}(z))dF(z) &= \int_{\tilde{z}_0}^{\infty} \left(z - \sqrt{z\tilde{z}_0}\right) \frac{1}{z^{\eta+1}} \eta dz \\ &= \eta \left[\frac{\tilde{z}_0^{1-\eta}}{(\eta - 1)(2\eta - 1)} \right]. \end{aligned}$$

Dividing $W(m)$ by $W(\tilde{m})$, we obtain expression given in Section 4.

Proof. [Completion of Proof of Proposition 6]From equation 19 and since the mass of G is decreasing at all times above point $z_0(t)$ it follows that z_0 must decrease over time.

Once concluded that $z_0'(t) < 0$, it immediately follows that $m_t(t, z) > 0$ for all $z \geq z_0(t)$, and this implies that $v'(t) < 0$. Now, let $\bar{v}(z, \hat{m}) = \frac{\lambda}{r + \lambda \hat{m}} z$ be the per-researcher expected discounted value of an innovation of value z, when a mass $\hat{m}$ of researchers are permanently engaged on the R&D line. When $c < \bar{v}(z, \hat{m})$, The cost of deploying an additional researcher on the considered R&D line cannot be recovered. Because $m_t(t, z) > 0$, it follows that the value $\bar{v}(z, m(t, z))$ decreases in t. Hence, there exists a time $T(z)$, after which innovators do not engage researchers in any R&D line of innovation value z any longer. Solving the equation $c = \frac{\lambda}{r+\lambda m(t,z)} z$, with $m(t, z) = \frac{z - z_0(t)}{c}$, we obtain the expression $z_0(T) = rc/\lambda$, reported in the statement of the Proposition. At that time T, it is also the case that $v(T) = \bar{v}(z, m(t, z))$: the equilibrium value function v is smoothly pasted with the function $\bar{v}(z, m(z, \cdot))$ for any innovation value $z \geq z_0(T)$. □

Proof. [Proof of Proposition 7] To complete the proof, we only need to show that $\tilde{v}_z(t, z) > 0$, i.e., that the optimal social value of researching undiscovered innovations increases in their value z. Written in 'forward form,' the optimal social value is:

$$\tilde{v}(t, z) = \max_{m(\cdot,z)} \int_t^\infty \lambda e^{\lambda(s-t)} \left[e^{-rs}(z - m(s, z)) - \int_0^s e^{-r\tau} m(\tau, z)\, u(\tau)\, d\tau \right],$$

where, again, $u(\tau)$ is the equilibrium researcher wage at time τ, in the decentralized implementation of the first best. it is immediate to verify that this expression of $\tilde{v}$ is linear and increasing in z. □

Proof. [Proof that $\bar{v} = (\lambda/r)\, z_0$, and derivation of expressions (29) and (30)]In order to prove that $\bar{v} = (\lambda/r)\, z_0$, we consider the expression:

$$\bar{v} = \int_0^\infty e^{-rt} \left[\frac{z}{m(z)} + \bar{v} - c \right] m(z)\, \lambda e^{-m(z)\lambda t} dt = \frac{\lambda m(z)}{r + \lambda m(z)} \left(\frac{z}{m(z)} + \bar{v} - c \right),$$

where $\bar{v} - c$ is the value for redeploying researchers once the innovation is discovered. Simplifying, we obtain: $\bar{v} = (\lambda/r)\,[z - m(z)\, c] = (\lambda/r)\, z_0$, for all $z \geq z_0$.

Plugging the solution $\tilde{m}(z)$ in the program (27), we solving for the optimal value $\tilde{v}(z)$, and obtain:

$$\tilde{v}(z) = \frac{\lambda \tilde{m}(z)\,[z - \tilde{m}(z)c] - \tilde{m}(z)u}{r + \lambda \tilde{m}(z)}. \tag{38}$$

Substituting this optimal value $\tilde{v}(z)$ into the first-order conditions (28), we obtain

$$\lambda \left[z - \frac{\lambda \tilde{m}(z)\,[z - \tilde{m}(z)c] - \tilde{m}(z)u}{r + \lambda \tilde{m}(z)} - 2\tilde{m}(z)c \right] = u.$$

Solving for u and simplifying, we derive expression (29).

Expression (30) follows by first solving equation (29) for $\tilde{m}(z)$ to obtain that:

$$\tilde{m}(z) = (r/\lambda)\left(\sqrt{\frac{\lambda z - u}{rc} + 1} - 1\right), \text{ for all } z \geq z_0,$$

and then by noting that researchers are a perfectly divisible factor in our model, so that $\tilde{m}(z) = 0$ at $z = z_0$, and, hence, $u = \lambda z_0$. □

Proof. [Proof of Proposition 8]We begin by calculating the aggregated welfare $W(m)$ associated with any allocation function m and associated density g.

The flow of aggregate welfare is expressed as:

$$rW(m) = \int_{z_0}^{\infty} \lambda m(z)[z - m(z)c]\, g(z)\, dz.$$

Each innovation of value z is discovered at arrival rate $\lambda m(z)$, upon discovery it accrues value z to the aggregate welfare but induces the aggregate cost $m(z)c$ as $m(z)$ researchers need to be allocated to different R&D lines.

Substituting in the expression $m(z)\, g(z) = (\alpha/\lambda) f(z)$, and rearranging, we obtain:

$$rW(m) = \alpha \int_{z_0}^{\infty} z f(z)\, dz - \alpha c \int_{z_0}^{\infty} m(z) f(z)\, dz. \tag{39}$$

Now, we consider the equilibrium allocation function $m(z) = (z - z_0)/c$, so the second term in the expression (39) takes the form:

$$\alpha c \int_{z_0}^{\infty} m(z) f(z)\, dz = \alpha \int_{z_0}^{\infty} (z - z_0) f(z)\, dz,$$

which, substituted back into the expression (39), gives

$$rW(m) = \alpha z_0 [1 - F(z_0)]. \tag{40}$$

Integrating condition (31) across z, we obtain the expression $\lambda M = \alpha[1 - F(z_0)]$, that we substitute into expression (40), so as to obtain expression (35) for the aggregate equilibrium welfare.

We now consider the aggregate welfare $W(\tilde{m})$ associated with the optimal allocation $\tilde{m}$. Substituting the expression (30) of $\tilde{m}$ in the second term expression (39), and using $\lambda M = \alpha[1 - F(z_0)]$, we obtain:

$$\alpha c \int_{z_0}^{\infty} m(z) f(z)\, dz = (\alpha/\lambda) \int_{z_0}^{\infty} \sqrt{c\lambda r (z - z_0) + c^2 r^2} \cdot f(z)\, dz - rcM.$$

Further simplification leads to the aggregate optimal welfare expression (36).

□

Appendix C.

Hamiltonian for Dynamic Problem - no entry cost

The objective is

$$\int e^{-rt} \int \lambda(m(t,z))\, zg(t,dz)\, dzdt$$

The law of motion for $g(t,z)$ is:

$$\dot{g}(t,z) = -\lambda(m(t,z))\, g(t,z)$$

and the constraint for the controls $m(t,z)$ are given by:

$$\int m(t,z)\, g(t,z)\, dz \leq M$$

The Hamiltonian is

$$H = \int e^{-rt} \int \lambda(m(t,z))\, zg(t,dz)\, dzdt - \int w(t,z)\, (\lambda(m(t,z))\, g(t,z))\, dz$$

and the Lagrangian:

$$L(t,m,g,w,\mu) = H(t,m,g,w) - \mu(t)\, (m(t,z)\, g(t,z)\, dz - M)$$

Using the Pontryagin principle of optimal control we first find:

$$\dot{w}(t,z) = \frac{\partial L}{\partial g(t,z)} = \lambda(m(t,z)) \left(e^{-rt} z - w(t,z)\right) - \mu(t)\, m(t,z)$$

Letting $v(t,z) = e^{rt} w(t,z)$ so that $\dot{w}(t.z) = -re^{-rt} v(t,z) + e^{-rt} v_t(t,z)$ and also letting $u(t) = e^{rt}\mu(t)$ wet get our equation for the value of a problem:

$$rv(t,z) = \lambda(m(t,z))(z - v(t,z)) - u(t)m(t,z) + v_t(t,z)$$

Using now the principle for the optimal control $m(t,z)$ by setting $\partial L/\partial m(z,t) = 0$ gives:

$$\lambda'(m(t,z))\left(e^{-rt}z - w(t,z)\right) - \mu(t) = 0$$

and substituting this gives

$$\lambda'(m(t,z))(z - v(t,z)) = u(t)$$

which is also the expression we derived in the paper.

Hamiltonian for Dynamic Problem - entry cost

Here we define the control variable $x(t,z) = \partial m(t,z)/\partial t$ taking c to be a cost of entry. Now we have two state variable-functions $g(t,z)$ and $m(t,z)$. The objective is

$$\int e^{-rt} \int (\lambda m(t,z) z - x(t,z) c) g(t,dz) dzdt$$

The law of motion for $g(t,z)$ is:

$$\dot{g}(t,z) = -\lambda m(t,z) g(t,z),$$

the law of motion for $m(t,z)$

$$\dot{m}(t,z) = x(t,z)$$

and the resource constraint given by:

$$\int m(t,z) g(t,z) dz \leq M$$

$$\dot{w}(t,z,m) = \frac{\partial L}{\partial g(t,z)} = e^{-rt}(\lambda m(t,z)(z - w(t,z,m)) - x(t,z)c) - \mu(t)m(t,z) \quad (41)$$

$$\dot{\rho}(t,z,g) = \frac{\partial L}{\partial m(t,z)} = e^{-rt}\lambda z g(t,z) - \lambda w(t,z,m)g(t,z) - \mu(t)g(t,z) \quad (42)$$

$$\left(-e^{-rt}cg(t,z) - \rho(t,z)\right) = 0$$

Using the last equation, it follows that

$$\begin{aligned}\dot{\rho}(t,z,g) &= re^{-rt}cg(t,z) - e^{-rt}c\dot{g}(t,z) \\ &= re^{-rt}cg(t,z) + e^{-rt}c\lambda m(t,z)g(t,z)\end{aligned}$$

Substituting (42) we get:

$$(r + \lambda m)e^{-rt}c = e^{-rt}\lambda z - \lambda w(t,z,m) - \mu(t)$$

and with our previous definitions of v and u this simplifies to:

$$\lambda(z - v(t,z,m) - mc) = u(t) + rc \quad (43)$$

Rewriting the co-state condition (41) by using our definitions of v and u gives:

$$rv(t,z,m) = \lambda m(t,z)(z - v(t,z,m)) - \dot{m}(t,z)c - u(t)m(t,z) + v_t(t,z,m) + v_m(t,z,m)\dot{m(t,z)}$$

Here I am using $\dot{v}(t,z,m(z,t)) = v_t(t,z,m) + v_m(t,z,m)\dot{m}$. Note that at the optimal $m(t,z)$ it must be the case that $v_m(t,z,m) = c$, this equation simplifies to:

$$rv(t,z,m) = \lambda m(t,z)(z - v(t,z,m)) - u(t)m(t,z) + v_t(t,z,m).$$

Moreover, given the linear entry technology, $v(t,z,m) = v(t,z,0) + cm$, we can substitute in the above equation:

$$r(v(t,z,0) + m(t,z)c) = \lambda m(t,z)(z - v(t,z,0) - m(t,z)c) - u(t)m(t,z) + v_t(t,z,m)$$

or

$$rv(t,z,0) = \lambda m(t,z)(z - v(t,z,0) - m(t,z)c) - u(t)m(t,z) - rm(t,z)c + v_t(t,z,0)$$

The last term is justified given that $v(t,z,m) = v(t,z,0) + mc$ whenever $m \leq m(t,c)$ (the optimal). Redefining $u(t)$ to include also the term rc gives the exact equation in the paper and the same first order conditions for the choice of m. This can be seen doing the substitution $v(t,z,m) = v(t,z,0) + mc$ in (43) which gives:

$$\lambda(z - v(t,z,0) - 2mc) = u(t) + rc.$$