MULTIDIMENSIONAL INCOME TAXATION AND ELECTORAL COMPETITION: AN EQUILIBRIUM ANALYSIS*

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Abstract

One of the fundamental problems of the positive theory of income taxation is explaining why the statutory income tax schedules in all industrialized democracies are marginal-rate progressive. While it is commonly believed that this is but a simple consequence of the fact that the number of relatively poor voters exceeds that of richer voters in such societies, putting this contention in a voting equilibrium context proves to be a nontrivial task. We study the Downsian model in the context of nonlinear taxation and inquire about the possibility of providing a formal argument in support of the aforementioned intuition. We first show existence of mixed strategy equilibria and then ask qualitative questions about the nature of these equilibria. Our positive results show that there are cases where marginal-rate progressive taxes are chosen with probability one by the political parties.

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Our negative results show that, if the tax policy space is not artificially constrained, equilibria exist whose support does not lie within the set of all marginal-rate progressive taxes.

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1 Introduction

One of the well-documented empirical regularities concerning income taxation is that all industrialized democracies (all OECD countries, in particular) implement statutory income tax schedules the marginal tax rates (slopes) of which are increasing in income.[1] Given that his/her views about the income tax policy is one of the most important traits of a political candidate, it is natural to expect that this stylized fact reflects (however indirectly) the preferences of the majority of the constituents of these societies. In fact, this way of thinking seems to suggest a straightforward explanation of the empirically observed popularity of marginal-rate progressivity, provided that one subscribes to the one-man one-vote rule. Since the income distributions of these countries are globally right-skewed (in the sense that the median income is strictly smaller than the mean income for any right truncation of the income distribution), the number of the poorer voters always exceeds that of the richer voters, regardless of how one defines the cutoff that separates the poor from the rich. Since poorer voters are typically the supporters of progressive policies, so the argument goes, there would then be a natural tendency for the marginal-rate progressive tax policies to be favored by the majority. Even though the actual political processes are far more complex than the scenario in which people vote directly over policies, this argument appears to suggest a convincing reason for why progressive tax policies are so widely adopted.

Put succinctly, the objective of the present paper is to understand what makes and breaks this heuristic argument. Recently there appeared sev-

[1]Such tax functions are called *marginal-rate progressive* in public finance; they are simply those tax functions that are convex. Concave tax schedules are called *marginal-rate regressive.*

eral papers in the related literature, which suggested certain formalizations.[2] However, somewhat surprisingly, a canonical representative democracy voting model that is suitable for an immediate examination of the said argument is not yet available. In this paper, we will try to remedy this situation by focusing on a simple (but general) model that is appropriate for this purpose. Our model consists of a two-party voting game in which each party (whose objective is to win the elections) proposes tax functions from a given set $\mathcal{A}$ of feasible tax functions (that raise the same revenue), and voters vote selfishly for the tax function that taxes them less (Section 3). Assuming that the income distribution is right-skewed, we shall ask if there is any reason to suspect that only the marginal-rate progressive tax policies will be proposed in the equilibrium of this game. Such a setup seems abundantly natural for the task at hand.

It is of course not sensible to view this simple model as novel; some of the existing literature can be thought of as studying precisely this sort of a game. But this literature functions by considering very restrictive choices of policy spaces $\mathcal{A}$ (to the extent of confining attention to only linear and quadratic tax functions). This is because, for large $\mathcal{A}$, the voting game becomes one of very large (possibly infinite) dimension, and it is long known that such games lack pure strategy equilibria; our setup is no exception (Subsection 4.1). In fact, this is perhaps a very good argument against the heuristic claim about the popular support of marginal-rate progressivity; due to voting cycles, one may not really be sure to which direction the majority demand may shift. However, this counter argument is not readily convincing, precisely because it ignores the possibility of mixed strategy equilibria. It is indeed surprising that the common game theoretical methodology of looking for mixed strategy equilibria in games that lack pure strategy equilibria seems not at all considered for the type of voting games we consider here.[3]

[2] See Cukierman and Meltzer (1991), Gouveia and Oliver (1996), Marhuenda and Ortuño-Ortín (1995, 1998), Roemer (1999), and Carbonell-Nicolau and Klor (2003).

[3] While one may find mixed strategies as conceptually more problematic than pure strategies, there is no reason for them to be particularly inappropriate in the present context. All the standard interpretations (limits of pure strategy equilibria of sequences of slightly perturbed games, direct randomization, stability in beliefs, or summary of average behavior) apply to the present setup without modification. In fact, mixed strategy equilibria of voting games of redistribution are commonly studied in the literature, cf. Myerson (1993), Lizzeri and Persico (2000, 2001), Laslier and Picard (2002), and Banks et al. (2002). Finally, we note that Laslier (2000) provides an interpretation of electoral mixed strategies.

The contribution of this paper is then two-fold. First, we study here the problem of existence of equilibrium in mixed strategies. This is not a trivial matter, for the payoffs of the game feature marked discontinuities, and hence the standard argument based on the Glicksberg existence theorem does not apply to this setting. Nevertheless, we show here that the recent existence results of Reny (1999) can be utilized to establish (in Subsection 4.2) the existence of mixed strategy equilibria. This shows that the heuristic claim about the majority support for marginal-rate progressive taxation cannot be dismissed simply by saying that the associated voting game lacks an equilibrium.

Second, we ask what a mixed strategy equilibrium would look like in this setup. Again as one may expect, the answer depends on which tax policies are feasible and which are not. If we wish to run, for example, the marginal-rate progressive policies against the marginal-rate regressive ones, then it is true that the policy that will be implemented in equilibrium will be marginal-rate progressive *with probability one* (Subsection 5.1). This is an interesting and not entirely obvious observation, and it amounts to embedding the results of Marhuenda and Ortuño-Ortín (1995, 1998) into an equilibrium context. Unfortunately, allowing for tax functions that are neither convex nor concave changes the picture completely. In this case marginal-rate progressivity loses its privileged position. As we shall demonstrate by means of a simple example, the probability of observing a marginal-rate regressive tax in equilibrium may *exceed* that of observing a marginal-rate progressive tax (Subsection 5.2). We find this observation quite puzzling. Nevertheless, our example is only suggestive. A complete analysis requires studying the structure of the supports of the mixed strategy equilibria in general. We show that the *ad hoc* constraints on the tax policy space introduced in Section 5 (and in virtually all the related literature) are essential. In particular, for a fairly general set of pre-tax income distributions, the supports of the mixed strategy equilibria need not lie within the set of all marginal-rate progressive tax policies (Subsection 5.2). Our results indicate that one should look somewhere else for the "explanation" of the commonly observed marginal-rate progressive policies; on a closer inspection, there is a fundamental difficulty in providing formal support of the claim that "there is a natural tendency for the tax policies to be progressive in societies with right-skewed income distributions." At the very least, this argument, which we formalize below for a suitable restriction of the tax policies that can be proposed in the elections, needs to be supplemented with another approach that explains the otherwise artificial constraints on the policy space.

2 Admissible Tax Schedules

The income taxation framework we adopt here is largely standard. We consider an endowment economy with continuum many individuals. Let $\mathcal{F}$ be the set of all distribution functions $F : \mathbf{R} \to [0,1]$ that are continuous and strictly increasing on $[0,1]$ and satisfy $F(0) = 0$ and $F(1) = 1$. The incomes of the constituents of the population are distributed on $[0,1]$ according to an element of $\mathcal{F}$. While each $F \in \mathcal{F}$ is completely arbitrary, as is standard (and duly realistic), we shall sometimes assume that it is right-skewed, or more generally, that the median income is strictly less than the average income according to F:

$$m_F := F^{-1}\left(\frac{1}{2}\right) < \int_0^1 x dF(x) =: \mu_F.$$

This weak assumption is but a straightforward formalization of the heuristic statement that "the number of the poor people in the society is strictly less than that of the rich people."

A map $t : [0,1] \to [0,1]$ is said to be a ***tax function*** if it is continuous and satisfies the following two properties:

- $0 \leq t(x) \leq x$ for all x,
- $x \mapsto t(x)$ and $x \mapsto x - t(x)$ are increasing maps on $[0,1]$.

The first property is a feasibility condition that disallows the presence of negative taxation. The first part of the second requirement is an unexceptionable fairness condition, while the second part guarantees (as all real-world tax policies do) that the before-tax and after-tax income rankings of taxpayers are identical.

In what follows, we will be interested in the aggregation of individual preferences about "how" the tax should be collected assuming that the question of "how much" should be collected is somehow answered outside the present model. (So, even though we do not allow for negative taxation, our model is, in effect, one of *pure redistribution.*) Specifically, we shall assume here that, given F, tax policies are designed to collect at least an exogenously given amount of revenue $R_F \in (0, \mu_F)$. Thus we impose the following *revenue* *requirement* for any tax function under consideration:

$$\int_0^1 t dF \geq R_F.$$

Any tax function that satisfies this property will be referred to as an ***admissible tax function***. We shall denote the set of all admissible tax functions relative to F by $\mathcal{T}(F)$. It is obvious that this set depends on both the income distribution F and the target tax revenue R_F. However, since we will keep each R_F as (arbitrarily) fixed throughout the sequel, adopting a notation that does not make the dependence between $\mathcal{T}(F)$ and R_F explicit will not cause any confusion.

It is important to note that $\mathcal{T}(F)$ includes essentially all nonlinear tax functions, and this class is simply too large to yield sharp insights relative to the majority support for alternative tax policies. Indeed, the literature on voting over income taxes is for the most part couched in terms of much smaller subclasses of $\mathcal{T}(F)$. For instance, to be able to make use of the median voter theorem, the seminal papers of Romer (1975), Roberts (1977), and Meltzer and Richards (1981), along with a large fraction of the recent literature on the relation between income inequality and growth (see Bénabou (1996) for a survey), consider only linear tax schemes.[4] Since the linearity assumption is obviously overly restrictive, many authors have tried to study the basic voting problem in terms of larger classes of tax functions. For instance, Gouveia and Oliver (1995) examine the issue for two-bracket piecewise linear tax functions, and Cukierman and Meltzer (1991) and Roemer (1999) study the quadratic tax functions. More generally, the work of Marhuenda and Ortuño-Ortín (1995, 1998) allows for the class of all concave (*marginal-rate regressive*) or convex (*marginal-rate progressive*) tax functions.[5]

Following the work of Marhuenda and Ortuño-Ortín, we shall also conduct a part of our analysis by using this latter class. So let $\mathcal{T}_{conv}(F)$ and $\mathcal{N}_{conv}(F)$ stand for the set of all convex and nonlinear convex tax functions in $\mathcal{T}(F)$ respectively, and denote the set of all concave tax functions in $\mathcal{T}(F)$ as $\mathcal{T}_{conc}(F)$. The set of all marginal-rate progressive and regressive taxes is in turn denoted as $\mathcal{C}(F)$, that is,

[4]Some of these papers allow for negative taxation, so strictly speaking, do not fit in the basic model of this paper. However, this is a minor point; the present model can be adjusted easily to incorporate negative taxation.

[5]There are recent papers that provide exceptions. For instance, Carbonell-Nicolau and Klor (2003) consider a representative democracy model that allows for the class of all piecewise linear tax functions.

$$\mathcal{C}(F) := \mathcal{T}_{conc}(F) \cup \mathcal{T}_{conv}(F).$$

The set $\mathcal{C}(F)$ is indeed an interesting subclass of $\mathcal{T}(F)$, which has the advantage of containing all the other sets of tax functions mentioned in the previous paragraph. Moreover, as shown by Marhuenda and Ortuño-Ortín (1995, 1998), within this class, the problem of demonstrating the popular support for marginal-rate progressive tax schedules becomes tractable. In fact, one of the main results of this paper will provide such a demonstration within a voting equilibrium analysis.

3 The Voting Game

This section introduces the basic voting game that we shall investigate in what follows. This game can be viewed as the simplest possible model of political competition that takes place in terms of income tax policies. It thus provides a natural framework for examining the validity of the statement "if the majority of a society is relatively poor, then there would be a majority support for progressive policies."

Take any nonempty subset $\mathcal{A}$ of $\mathcal{T}(F)$, and consider two political parties who are engaged in competition to hold office. Each party advocates an income tax policy in $\mathcal{A}$ which is to be put in effect in case this party obtains the support of the majority. Citizens evaluate proposals selfishly, that is, an individual with income x regards the tax function t as more desirable than the tax function τ if $t(x) < \tau(x)$. If party 1 proposes tax policy t and party 2 proposes tax policy τ, the share of individuals that strictly prefer t over τ is determined as

$$w(t, \tau) := p_F\Big\{x \in [0,1] : t(x) < \tau(x)\Big\} \equiv \int_{\{t<\tau\}} dF,$$

where p_F is the Lebesgue-Stieltjes probability measure induced by F on $[0, 1]$. Of course, in this case the share of individuals who strictly prefer party 2's victory is $w(\tau, t)$. Parties are not ideological, we posit that their objective is to maximize the net plurality defined as the difference between the vote shares obtained by the candidates.[6] We next provide a formal description of the electoral game.

[6]We assume that indifferent voters toss a fair coin to determine their vote or abstain.

Viewing $\mathcal{A} \subseteq \mathcal{T}(F)$ as a metric subspace of $\mathbf{C}[0,1]$, we formalize the scenario described above by means of a two-person zero-sum symmetric strategic game

$$G(\mathcal{A}) := (\mathcal{A}, (u_1, u_2)),$$

where $\mathcal{A}$ corresponds to the action space of either party, that is, the *feasible tax policy space*, and $(u_1, u_2) : \mathcal{A}^2 \to \mathbf{R}^2$ models the payoff functions of the players. In this paper, we will model the parties as maximizers of their net plurality, that is, we suppose that

$$u_i(t,\tau) := \begin{cases} w(t,\tau) - w(\tau,t) & \text{if } i = 1, \\ w(\tau,t) - w(t,\tau) & \text{if } i = 2. \end{cases} \tag{1}$$

This formulation is also used, for instance, by Kramer (1978) and Laslier and Picard (2002). Alternatively, one can model the parties as maximizing their vote shares with the proviso that indifferent individuals vote by tossing a fair coin. In this case, we would have

$$u_1(t,\tau) := w(t,\tau) + \frac{1}{2} p_F\{t = \tau\} \text{ and } u_2 = 1 - u_1. \tag{2}$$

We will adopt the formulation given in (1) throughout this paper, yet our entire development would remain unaltered (almost *verbatim*) if we used instead the formulation in (2).[7]

It is important to note that $G(\mathcal{A})$ depends on the action space $\mathcal{A} \subseteq \mathcal{T}(F)$ of the players. We do not make explicit the dependence of each u_i on $\mathcal{A}$, hoping the domain of reference will be clear from the context. The choice of $\mathcal{A}$ is crucial, for any particular choice (other than $\mathcal{T}(F)$) would really amount to limiting the possibilities for the tax designers in an *ad hoc* manner. The best case scenario is of course to take $\mathcal{A} = \mathcal{T}(F)$. We shall demonstrate shortly that it is possible to get results for various interesting choices for $\mathcal{A}$ (including $\mathcal{T}(F)$) and hence subgames of $G(\mathcal{T}(F))$.[8]

[7] The reader may wonder about the possibility that candidates maximize their respective probabilities of winning rather than vote shares. This would be equivalent to maximizing a function which takes on value 1 for positive net pluralities and 0 otherwise. This objective can be uniformly approximated by a continuous, strictly increasing, and symmetric around zero transformation $g(w(t,\tau)-w(\tau,t))$ of the net plurality $w(t,\tau)-w(\tau,t)$. The treatment of the present work is easily adapted to accommodate g.

[8] By a *subgame* of a game in strategic form, we understand a game in which the action spaces of the individuals are subsets of their corresponding action spaces in the mother game, and in which the payoff functions are obtained by restricting the original payoff functions to the resulting outcome space.

4 Existence of Equilibrium for $G(\mathcal{A})$

A fundamental difficulty about general voting problems is that they often fail to possess an equilibrium. It is presumably for this reason that the games of the form $G(\mathcal{A})$ have not been studied thoroughly in the related literature. In this section we show that while existence of pure strategy equilibrium is indeed a problem that should be taken seriously, switching attention to mixed strategies provides a way out of this problem.

4.1 Negative Results: Pure Strategy Equilibrium

While there is reason to view $G(\mathcal{T}(F))$ as a voting game of fundamental importance for the positive theory of income taxation, things get icy when one looks for its Nash equilibria in pure strategies. Indeed, the (potential) multi-dimensionality of the action spaces of the parties makes it impossible to utilize single dimensional voting equilibrium theorems like the median voter theorem. While this is no guarantee that $G(\mathcal{T}(F))$ does not possess equilibria, this is unfortunately the case. The existence of Condorcet-type cycles leads to the non-existence of a pure strategy Nash equilibrium for this game.

Proposition 4.1. $G(\mathcal{T}(F))$ *does not have a pure strategy Nash equilibrium.*

While its formal proof is somewhat tedious, this result is clearly a folk theorem the intuition of which is quite simple. Given any admissible tax function t with $t(0+) > 0$ (the no tax exemption case), one can always find another tax function τ which is below t over an interval of p_F-measure greater than 1/2. A similar trick applies to those tax functions with exemption as well, and hence the result. (We shall omit formalizing this elementary argument here for brevity.) As noted by Marhuenda and Ortuño-Ortín (1998), the situation is analogous to the problem of dividing a cake of a fixed size among three agents. The core of the induced (coalitional) game is empty, since for any division of the cake, there is another division which is preferred by exactly two of the individuals. This observation is the main culprit behind Proposition 4.1.

Of course, one can escape this result by suitably restricting the involved policy space. For instance, if $\mathcal{A}$ stands for the set of all two-bracket piecewise linear admissible tax functions and F is globally right-skewed, then there is a unique Nash equilibrium of $G(\mathcal{A})$, where the parties adopt the unique admissible tax function that exempts everyone below the maximum possible level of income (Gouveia and Oliver, 1996). However, the set of two-bracket piecewise linear admissible tax functions is certainly unduly restrictive. For instance, the result disappears if one allows for three-bracket taxes. Furthermore, there is no pure strategy equilibrium for other natural choices for $\mathcal{A}$. For instance, the games $G(\mathcal{C}(F))$ and $G(\mathcal{T}_{conv}(F))$ do not possess a pure strategy equilibrium (Klor, 2003).

Given the importance of the (large) subgames of $G(\mathcal{T}(F))$ for the positive theory of income taxation, however, we contend that Proposition 4.1 does not provide enough reason to lose interest. A natural next question concerns the existence of mixed strategy equilibria of $G(\mathcal{T}(F))$ and its subgames. These issues are discussed formally below.[9]

4.2 Positive Results: Mixed Strategy Equilibrium

Technically speaking, the difficulty with establishing the existence of equilibria for $G(\mathcal{T}(F))$ is the discontinuity of the objective functions u_i. In principle, these can be vast enough to yield even the non-measurability of the objective function, which would in turn disallow one to talk about the mixed strategies for this game in the standard way. Fortunately, however, this is not the case here due to the following useful observation.

Lemma 4.2. *The maps $(t,\tau) \mapsto w(t,\tau)$ and $(t,\tau) \mapsto w(\tau,t)$ are lower semicontinuous on $\mathcal{T}(F)^2$.*

Proof. Take any sequence (t^n,τ^n) in $\mathcal{T}(F)^2$ such that $t^n \to t$ and $\tau^n \to \tau$. By Fatou's lemma,

$$\liminf w(t^n,\tau^n) = \liminf \int_0^1 \mathbf{1}_{\{t^n<\tau^n\}} dF \geq \int_0^1 \liminf \mathbf{1}_{\{t^n<\tau^n\}} dF.$$

[9] There are other instances in the literature where a basic strategic model lacks an equilibrium in pure strategies, and yet the mixed strategy extension of the model admits and equilibrium, and reveals quite a bit about the structure of the game at hand. For instance, it is well-known that the Bertrand duopoly model with constant marginal costs and suitable capacity constraints does not have a pure strategy equilibrium—this is called the *Edgeworth paradox*. Maskin (1986) and Osborne and Pitchik (1986), however, have shown that this model possesses interesting mixed strategy equilibria. The definitive work of Kreps and Sheinkman (1983) on this issue also requires the use of mixed strategies.

But we have

$$\liminf \mathbf{1}_{\{t^n<\tau^n\}} \geq \mathbf{1}_{\{\lim t^n<\lim \tau^n\}}.$$

For, if the left hand side takes value 0 at some $y \in [0,1]$, then $t^n(y) \geq \tau^n(y)$ for infinitely many n, and this means that the right hand side cannot take value 1 at y. Combining this observation with the previous inequality, we get

$$\liminf w(t^n, \tau^n) \geq \int_0^1 \mathbf{1}_{\{\lim t^n<\lim \tau^n\}} dF = w(t, \tau),$$

proving that w is lower semicontinuous on $\mathcal{T}(F)$. The second claim is proved similarly. ‖

Let $\mathcal{A}$ be a nonempty subset of $\mathcal{T}(F)$. It follows from Lemma 4.2 that the map $w(t, \tau) - w(\tau, t)$ is Borel measurable. We may then conclude that u_i is a Borel measurable function for each i. This allows us to well-define the mixed strategy extension of our voting game.

A ***mixed strategy*** for the game $G(\mathcal{A})$ is defined as any Borel probability measure on $\mathcal{A}$. We extend the payoff functions of the players to the domain of mixed strategy profiles in the usual way:

$$U_i(\mu_1, \mu_2) := \int_{\mathcal{A}^2} u_i d(\mu_1 \times \mu_2), \quad \mu_i \in \mathbf{P}(\mathcal{A}),\ i = 1, 2,$$

where $\mathbf{P}(\mathcal{A})$ represents the set of all Borel probability measures on $\mathcal{A}$. Once again, the dependence of each U_i on $\mathcal{A}$ is not made explicit, hoping the domain of reference will be clear from the context. Each $U_i : \mathbf{P}(\mathcal{A})^2 \to \mathbf{R}$ is well-defined since any Borel measurable function on $\mathcal{A}^2$ is measurable in the associated product measure space.[10] As usual, by a ***mixed strategy equilibrium*** of $G(\mathcal{A})$ we mean a Nash equilibrium of the mixed extension $(\mathbf{P}(\mathcal{A}), (U_1, U_2))$.

The problem that we now pose is the existence of mixed strategy equilibria for some subgames of $G(\mathcal{T}(F))$. There is one case in which we do not have to work hard: the case of finite policy spaces. Indeed, Nash's classic existence theorem immediately yields the following observation.

Proposition 4.3. *For any nonempty finite subset $\mathcal{A}$ of $\mathcal{T}(F)$, $G(\mathcal{A})$ has a mixed strategy equilibrium.*

[10] Given that $\mathcal{T}(F)$ is a separable metric space (in fact a compact one, see Lemma 7.1), the Borel σ-algebra of $\mathcal{T}(F)^2$ is identical to the product Borel σ-algebra on $\mathcal{T}(F)^2$.

Since any choice of a finite subset of $\mathcal{T}(F)$ may be regarded as arbitrary, any characterization of the equilibria of finite versions of the electoral game at hand will be unconvincing, unless, perhaps, one provides results that are valid *for any* finite subset of the corresponding infinite strategy space. Rather than dealing with finite action spaces, one may attempt a characterization of the equilibria of $G(\mathcal{T}(F))$ and other infinite-action subgames of it. To do so, however, one needs to establish existence of equilibria for these games first. This is not a trivial matter, for one is required to study infinite-action games whose payoff functions are highly discontinuous.

To illustrate how badly behaved a game like $G(\mathcal{T}(F))$ may be, consider the tax functions τ, τ', and t depicted in Figure 1. Define the sequence (t^n) in $\mathcal{T}(F)$ by $t^n := \left(1 - \frac{1}{n}\right)\tau + \frac{1}{n}\tau'$ for each n. Observe that $w(t^n, t) - w(t, t^n)$ is positive and bounded away from zero for all n, yet we have $\|t^n - \tau\|_\infty \to 0$ and $w(\tau, t) - w(t, \tau) = -0.1$. Thus, it is possible that every member of a uniformly convergent sequence of tax policies yields a positive (and bounded away from zero) net plurality against a feasible tax function, whereas the limit of the sequence does a relatively bad job (in terms of net plurality) against this tax function. This example explains why the expected payoff function U_i is not lower semicontinuous, thereby demonstrating that the standard mixed strategy equilibrium existence results (such as those of Glicksberg (1952) and Tan, Jian, and Yuan (1995)) do not apply to the game $G(\mathcal{T}(F))$ and other subgames.

In this paper, we shall be interested in $G(\mathcal{T}(F))$ and $G(\mathcal{C}(F))$. It turns out that both $G(\mathcal{T}(F))$ and $G(\mathcal{C}(F))$ can be shown to have mixed strategy equilibria. This is, in fact, a main result of this paper.[11]

Theorem 4.4. *The games $G(\mathcal{T}(F))$ and $G(\mathcal{C}(F))$ possess a mixed strategy Nash equilibrium.*

The proof of Theorem 4.4 is given in Section 7, and is based on an existence theorem proved recently by Reny (1999). We note, for the record, that

[11]The techniques used elsewhere to study mixed electoral equilibria are not useful to establish existence of equilibrium in the present setting. On the one hand, Myerson (1993), Lizzeri and Persico (2000, 2001), and Laslier and Picard (2002) resort to constructive methods that do not apply to the games $G(\mathcal{T}(F))$ and $G(\mathcal{C}(F))$. On the other hand, Kramer (1978) provides a mixed strategy equilibrium existence result for an electoral game in which policies can be represented as points in $\mathbf{R}^n$. Since $\mathcal{T}(F)$ and $\mathcal{C}(F)$ differ from $\mathbf{R}^n$, Kramer's result does not apply to our framework.

a similar proof could be furnished using instead the main existence theorem of Baye, Tian, and Zhou (1993).[12]

In sum, we may conclude that the problem of existence of equilibrium can be resolved in terms of mixed strategies. Thus, it is not a futile exercise to ask qualitative questions about the nature of these equilibria, especially with regards to the majority support of marginal-rate progressive tax schedules. This issue will be addressed in the next section.

5 Popular Support for Progressive Taxation

Due to the well-known problem of pure strategy equilibrium existence, the implications of a model like $G(\mathcal{A})$ for the popular support of marginal progressivity are studied in the literature only for very restrictive classes of tax policies. To break free from the straightjacket of this existence problem, Marhuenda and Ortuño-Ortín (1995, 1998) have recently studied the question through a pairwise majority voting model, and they proved the interesting result that a nonlinear marginal-rate progressive tax scheme (a nonlinear member of $\mathcal{T}_{conv}(F)$) beats any marginal-rate regressive tax function (a member of $\mathcal{T}_{conc}(F)$) under pairwise majority voting. While promising, there are two major problems with this formulation.

The first difficulty is that it is not clear how one may be able to put this observation in an equilibrium context. Without doing this, the economic interpretation of the result is clearly suspect. The second difficulty is that, even if one may be able to give an equilibrium "outlook" to this result by adopting a weaker solution concept (such as Condorcet stability), this result alone is not informative about the majority support of a marginal-rate progressive tax against a "wiggling" tax function that lies in the exterior of $\mathcal{C}$. In the following subsection, we shall offer a solution to the first problem by using the model we have developed so far. Subsequently, we shall examine

[12]In passing, we would like to point out that the existence theorem of Simon and Zame (1990) is not useful for our analysis. The problem is that natural formulations of the game with an endogenous sharing rule corresponding to $G(\mathcal{A})$ give a payoff correspondence that fails to satisfy upper hemi-continuity. Thus, our game provides an example that violates Simon and Zame's conditions (for some versions of the game with an endogenous sharing rule) and, nonetheless, satisfies not only Reny's better reply security but also payoff security. By contrast, the auction setting of Jackson and Swinkels (2003) violates better reply security (for standard tie-breaking rules) and satisfies the conditions of an extension of the Simon-Zame theorem to Bayesian games.

the second difficulty, and show that the basic message of Marhuenda and Ortuño-Ortín (1995, 1998) should be taken with a grain of caution.

5.1 Positive Results: Majority Support for Progressivity

Let us denote the set of all nonlinear convex tax schedules in $\mathcal{T}(F)$ by $\mathcal{N}_{conv}(F)$. The Marhuenda–Ortuño-Ortín theorem states that $w(t,\tau) > 1/2$ for all $(t,\tau) \in \mathcal{N}_{conv}(F) \times \mathcal{T}_{conc}(F)$ with $\int_0^1 t dF = \int_0^1 \tau dF$.[13] Some authors have interpreted this result as "formalizing" the intuition that "in a society in which the numbers of poorer people exceed those of the richer, there would be a majority support for marginal-rate progressive taxation." Indeed, an obvious implication of this result is that, had $G(\mathcal{C}(F))$ possessed a pure strategy equilibrium, in this equilibrium both parties would have proposed marginal-rate progressive tax schedules. Unfortunately, $G(\mathcal{C}(F))$ does not have a pure strategy equilibrium (Klor, 2003), and hence the said "formulation" is incomplete.

However, we can complete the picture by switching attention to mixed strategies. Indeed, as shown in the previous section, $G(\mathcal{C}(F))$ possesses a mixed strategy equilibrium. We may then ask if the Marhuenda–Ortuño-Ortín result is powerful enough to predict that marginal-rate regressive taxes would never be played with positive probability in an equilibrium of such game (and subgames of $G(\mathcal{C}(F))$). The answer is yes.

We can actually prove a more general fact here.

Proposition 5.1. *Suppose that $m_F < \mu_F$. Let $\mathcal{A}$ be any subset of $\mathcal{T}(F)$ such that $\mathcal{N}_{conv}(F) \cap \mathcal{A} \neq \emptyset$ and $\frac{R_F}{\int_0^1 tdF} t \in \mathcal{A}$ whenever $t \in \mathcal{A}$ and $\int_0^1 tdF > R_F$. If, for all $t \in \mathcal{N}_{conv}(F) \cap \mathcal{A}$ and $\tau \in \mathcal{A} \setminus \mathcal{N}_{conv}(F)$, $t - \tau$ is a nonlinear convex function, $t(x) = \tau(x)$ holds for at most one $x \in (0,1]$, and $t < \tau$ on some open neighborhood of 0, then any mixed strategy Nash equilibrium (μ_1, μ_2) of $G(\mathcal{A})$ satisfies $\mu_1(\mathcal{N}_{conv}(F) \cap \mathcal{A}) = \mu_2(\mathcal{N}_{conv}(F) \cap \mathcal{A}) = 1$.*

Since it is easy to check that $\mathcal{C}(F)$ satisfies the conditions of this proposition, we obtain the following corollary that gives a formal answer to the question posed above.

[13] Lemma 7.5 provides a generalization of this result.

Corollary 5.2. *Suppose that $m_F < \mu_F$. Any mixed strategy Nash equilibrium (μ_1, μ_2) of $G(\mathcal{C}(F))$ satisfies $\mu_1(\mathcal{N}_{conv}(F)) = \mu_2(\mathcal{N}_{conv}(F)) = 1$.*

There is reason to view this result as the equilibrium version of the popular support theorem of Marhuenda and Ortuño-Ortín. It says that the parties may have to randomize over their choices (or, what is equivalent, they may have multiple best responses to their equilibrium beliefs about the proposals of their opponents), but when it comes to the observable outcomes, we are bound to see that all proposed tax policies are marginal-rate progressive. Put this way, this result sounds like the formalization of the claim that "there is a natural tendency for the tax policies to be marginal-rate progressive in societies with right-skewed income distributions."

It may be useful to outline the simple intuition behind Corollary 5.2. Assume that, at an equilibrium of the game $G(\mathcal{C}(F))$, both players play some concave taxes with positive probability. It is easily seen that the fact that the players are best responding to each other's strategy implies that only tax functions that collect exactly R_F are assigned positive probability at the equilibrium. But then at least one of these players may improve her payoff by imitating her opponent's strategy restricted to the part of the support in $\mathcal{N}_{conv}(F)$ (if no tax function in $\mathcal{N}_{conv}(F)$ lies in the support of the opponent's strategy, any pure strategy from $\mathcal{N}_{conv}(F)$ improves the payoffs of the player). By doing so, this player obtains, on the one hand, a positive net plurality against those regressive tax policies that are assigned positive probability by her opponent (this is guaranteed by the Marhuenda–Ortuño-Ortín theorem). On the other hand, her strategy ties against all progressive taxes that are played with positive probability by the other candidate. Thus, the deviation from the original equilibrium yields a positive payoff. Since the game is symmetric and zero-sum, no player can obtain a positive payoff at an equilibrium, and so the deviation is profitable.

5.2 Negative Results: Majority Support for Nonprogressivity

The obvious shortcoming of Corollary 5.2 is its *ad hoc* restriction of the feasible tax policies to be either marginal-rate progressive or marginal-rate regressive. It is difficult to think of a reason why a tax designer would not consider those tax schedules that are neither convex nor concave.[14] So, it is natural to ask how these results would modify if we included some such "wiggling" tax functions within the action spaces of the parties. It turns out that the basic theme of the previous subsection may be substantially altered in this case. Not only may then marginal-rate progressive tax policies be played with probability less than one, but it is also possible that the presence of "wiggling" taxes may act as a balancing factor that allows the marginal-rate regressive taxes to be chosen more frequently than the marginal-rate progressive taxes. This surprising possibility is demonstrated next.[15]

Example 5.3. Suppose that the distribution of income is represented by an F in $\mathcal{F}$ whose corresponding probability density function is

$$f(x) = 2 - 2x, \quad 0 \leq x \leq 1.$$

It is easy to verify that the median income is below the mean income here; in fact, the associated income distribution is globally right-skewed. We shall consider the following tax functions:

$$\begin{aligned}
t_1(x) &= \begin{cases} x/4 & \text{if } 0 \leq x \leq 1/4, \\ \alpha(x - \frac{1}{4}) + \frac{1}{16} & \text{if } 1/4 < x \leq 1, \end{cases} \\
t_2(x) &= \begin{cases} (4x)^{\beta}/8 & \text{if } 0 \leq x \leq 1/4, \\ x/2 & \text{if } 1/4 < x \leq 1, \end{cases} \\
t_3(x) &= \begin{cases} x/2 & \text{if } 0 \leq x \leq 1/4, \\ \gamma(x - \frac{1}{4}) + \frac{1}{8} & \text{if } 1/4 < x \leq 1, \end{cases}
\end{aligned}$$

where we set $\alpha = \frac{391}{540}$, $\beta = \frac{4\sqrt{6}}{3} + 1$, and $\gamma = \frac{103}{270}$. (As Figure 2 makes it transparent, t_2 can be taken to be a three-bracket piecewise linear tax function; these tax functions are hardly contrived.)

It is easily seen that all of these taxes collect revenue $R_F = 0.15$, and we have $t_1 \in \mathcal{N}_{conv}(F)$, $t_2 \in \mathcal{T}(F) \setminus \mathcal{C}(F)$, and $t_3 \in \mathcal{T}_{conc}(F)$. Furthermore,

[14] In fact, after the 1986 Tax Reform Act, there was a period of three years when the statutory federal income tax schedule in the US was neither convex nor concave due to the non-monotonicity of the top three brackets (see Mitra and Ok, 1996). So, it is fair to say that such tax policies are considered by politicians/tax designers, but only rarely they see the daylight.

[15] The following example is based on a modification of an example kindly communicated to us by Tapan Mitra.

routine calculations show that

$$
\begin{aligned}
w(t_1, t_2) &= 0.41,\ w(t_2, t_1) = 0.59, \\
w(t_1, t_3) &= 0.67,\ w(t_3, t_1) = 0.33, \\
w(t_2, t_3) &= 0.43,\ w(t_3, t_2) = 0.57
\end{aligned}
$$

(see Figure 2). Now let $\mathcal{A} := \{t_1, t_2, t_3\}$. The game $G(\mathcal{A})$ may then be represented by the following payoff bimatrix:

	t_1	t_2	t_3
t_1	$0, 0$	$-0.18, 0.18$	$0.34, -0.34$
t_2	$0.18, -0.18$	$0, 0$	$-0.14, 0.14$
t_3	$-0.34, 0.34$	$0.14, -0.14$	$0, 0$

While this game has no pure strategy equilibrium, it possesses a unique mixed strategy equilibrium in which both candidates randomize over the three tax policies according to the vector of probabilities $(7/33, 17/33, 3/11)$. Observe that, in equilibrium, the lowest probability is placed on the marginal-rate progressive tax function. $\|$

This example suggests that there is a major difficulty with viewing the Marhuenda–Ortuño-Ortín theorem as providing a basis for a positive theory of progressive taxation even in the case of endowment economies. True, this theorem can be put in an equilibrium format (as we did in Proposition 5.1 and in Corollary 5.2), but doing this *and* allowing for non-convex, non-concave tax schedules to be feasible may lead to a scenario in which the superiority of convex taxes over concave taxes vanishes.

Admittedly, Example 5.3 is only suggestive. A fuller analysis requires one to study the structure of the supports of the mixed strategy equilibria of the game $G(\mathcal{T}(F))$.[16] In particular, it is of interest to determine whether one

[16]In related contexts, this sort of a question was studied by Dutta and Laslier (1999) and Banks et al. (2002). These works, however, determine supersets of the supports of the mixed strategy equilibria of certain voting games. For example, the main result of Banks et al. (2002) entails in the present setup that the support of any mixed strategy equilibrium of $G(\mathcal{T}(F))$ is contained in the McKelvey uncovered set of $\mathcal{T}(F)$ (McKelvey, 1986). Unfortunately, this result does not reveal much here, for the uncovered set of $\mathcal{T}(F)$ equals $\mathcal{T}(F)$. In this regard, the present query is rather different than that of Banks et al. (2002). In contrast to these authors, here we prove the existence of mixed strategy equilibria for $G(\mathcal{T}(F))$ and $G(\mathcal{C}(F))$ and are interested in what is contained *within* the supports of these equilibria.

can replace $\mathcal{C}(F)$ with $\mathcal{T}(F)$ in Corollary 5.2. A negative result in this regard would yield the surprising conclusion that the link between the proportion of the poor voters in a society and the marginal-rate progressivity of tax functions is far weaker than commonly presumed.

What the following proposition will show is that the *ad hoc* constraints on the tax policy space introduced in this paper (and in virtually all the related literature) are essential. In fact, for a fairly general set of pre-tax income distributions, the supports of the mixed strategy equilibria of a slight perturbation of the game $G(\mathcal{T}(F))$ need not lie within the set of all marginal-rate progressive tax policies. To state this result formally, some preliminaries are needed.

Let $\mathcal{H}$ be the set of all distribution functions $H : \mathbf{R} \to [0,1]$ with $H(0) = 0$ and $H(1) = 1$. By an open neighborhood of $F \in \mathcal{F}$ we mean an open subset of $\mathcal{H}$ containing F.[17] Given $F \in \mathcal{F}$, we define x_F as the point from $(0,1)$ such that

$$\int_{x_F}^{1} (x - x_F) dF = R_F,$$

and, for a slightly perturbed version H of F,

$$\mathcal{T}(H) := \left\{ t : t \text{ is a tax function and } \int_0^1 t dH \geq R_F \right\};$$

here, the perturbation is small enough to ensure that $R_F \in \left(0, \int_0^1 x dH(x)\right)$.

We may now define $G(\mathcal{T}(H))$ as we defined $G(\mathcal{T}(F))$, with H replacing F.[18] With this terminology, our result can be stated as follows.

Theorem 5.4. *Suppose that $F \in \mathcal{F}$ satisfies $x_F \leq m_F$. Then any open neighborhood of F contains $H \in \mathcal{H}$ such that there exists a Nash equilibrium (μ_1, μ_2) of $G(\mathcal{T}(H))$ with $\mu_i(\mathcal{T}_{conv}(H)) < 1$ for each i.*

The condition on the pre-tax income distribution says that it is not possible to collect the target revenue by taxing at rate 0 below, at, and slightly above the median. As the following example illustrates, there exist, among those pre-tax income distributions for which Theorem 5.4 is valid, distributions that concentrate a relatively large share of the population around low

[17] Here the relevant metric may be taken to be the sup metric.

[18] Since Lemma 4.2 may also be stated in terms of H, the mixed extension of $G(\mathcal{T}(H))$ is well-defined.

income levels. Intuition suggests that, for such distributions, an electoral process in which candidates compete for votes and citizens cast their ballots to minimize their tax burden guarantees the implementation of marginal-rate progressive tax schemes. Theorem 5.4 demonstrates that this argument is flawed.

Example 5.5. Suppose that the distribution of income is represented by

$$F(x) = \begin{cases} 0.7ax & \text{if } 0 \leq x < 0.1, \\ 0.001(x-0.1) + 0.07a & \text{if } 0.1 \leq x < 0.9+\varepsilon, \\ 0.001(0.8+\varepsilon) + 0.07a + 0.299(x-0.9-\varepsilon) & \text{if } 0.9+\varepsilon \leq x \leq 1, \end{cases}$$

where $\varepsilon = 0.0999$ and $a = \frac{5000000}{354649} \simeq 14.1$. This distribution concentrates about 98.69% of the population between 0 and 0.1, 1.26% between 0.1 and $0.9+\varepsilon$, and 0.04% between $0.9+\varepsilon$ and 1, and so it is clearly right-skewed. It is easily verified that $x_F \leq m_F$ whenever $R_F \in \left(0.0188, \int_0^1 x dF\right)$ (i.e., the target revenue is at least 33.14% of total income). ||

6 Conclusion

In this paper we have tried to understand the apparently innocuous heuristic claim that "majority of the poorer voters (a right-skewed income distribution) is a central force behind the commonly observed popularity of marginal-rate progressive tax schemes." To this end, we have introduced a class of voting games that seem to be rather natural for the analysis of the problem at hand. The games considered here possess mixed strategy equilibria. Once existence is guaranteed, the characterization of the supports of the mixed strategy equilibria is not vacuous. We have demonstrated that it is possible to get some support out of our model for the said heuristic claim: the equilibria of certain classes may even have it that the probability of observing marginal-rate progressive taxes is one. Nonetheless, our positive results can only be sustained if the set of admissible tax schemes is sufficiently constrained. If one wishes to allow for policy spaces that are not artificially constrained, one must accept that it is not possible to provide a formal argument in support of the aforementioned claim. Our results indicate that one should either look somewhere else for the "explanation" of the link between right-skewedness of income distributions and the observed marginal-rate progressivity of income tax functions or else supplement our positive results with another approach that explains the constraints on the policy space.

7 Proofs

Notation and Definitions. Given $F \in \mathcal{F}$, define x_F as the point from $(0,1)$ such that

$$\int_{x_F}^{1} (x - x_F) dF = R_F.$$

Define $\mathcal{S}(F)$ as the family of all $H \in \mathcal{H}$ such that H is discrete on $[0, x_F)$, $H = F$ on $[0,1] \setminus [0, x_F)$, and H is sufficiently close to F to ensure that $R_F \in \Big(0, \int_0^1 x dH(x)\Big)$. For $H \in \mathcal{H}$, let p_H be the Lebesgue-Stieltjes measure induced by H on $[0,1]$.

For $F \in \mathcal{F}$ and $H \in \mathcal{S}(F)$, $\mathcal{T}(H)$ is compact in $\mathbf{C}[0,1]$ (Lemma 7.1), metric, and hence separable, so we may select a countable subset $\mathcal{T}^o(H)$ of $\mathcal{T}(H)$ that is dense in $\mathcal{T}(H)$. Let $\{q_1, q_2, q_3, ...\}$ be the set of all rational numbers in $[0,1]$, and define

$$\mathcal{T}_{q_n}(H) := \Big\{ t \in \mathcal{T}(H) : t = 0 \text{ on } [0, q_n] \text{ and } t > 0 \text{ elsewhere} \Big\}.$$

Each $\mathcal{T}_{q_n}(H)$ is a subset of a separable metric space and thus a separable space itself, so to each $\mathcal{T}_{q_n}(H)$ there corresponds a countable subset $\mathcal{T}^o_{q_n}(H)$ of $\mathcal{T}_{q_n}(H)$ that is dense in $\mathcal{T}_{q_n}(H)$. Put

$$\mathcal{T}^*(H) := \mathcal{T}^o(H) \cup \bigcup_n \mathcal{T}^o_{q_n}(H).$$

Being the union of countably many countable sets, $\mathcal{T}^*(H)$ is countable, and so we may write

$$\mathcal{T}^*(H) := \Big\{ t_1(H), t_2(H), t_3(H), ..., t_n(H), ... \Big\}.$$

Take a sequence of positive reals (ε_n) converging to 0. Define the sequences

$$\begin{aligned}
\mathcal{T}_1(H) &:= \Big\{ \tau_1^{\varepsilon_1}(H) \Big\}, \\
\mathcal{T}_2(H) &:= \Big\{ \tau_1^{\varepsilon_2}(H), \tau_2^{\varepsilon_1}(H) \Big\}, \\
\mathcal{T}_3(H) &:= \Big\{ \tau_1^{\varepsilon_3}(H), \tau_2^{\varepsilon_2}(H), \tau_3^{\varepsilon_1}(H) \Big\}, \\
&\vdots \\
\mathcal{T}_n(H) &:= \Big\{ \tau_1^{\varepsilon_n}(H), \tau_2^{\varepsilon_{n-1}}(H), \tau_3^{\varepsilon_{n-2}}(H), ..., \tau_n^{\varepsilon_1}(H) \Big\}, \\
&\vdots
\end{aligned}$$

as follows:

- $\tau_1^{\varepsilon_1}(H)$ is some element of $N_{\varepsilon_1}(t_1(H))$ such that $\tau_1^{\varepsilon_1}(H) = 0$ on $[0, a-\varepsilon_1]$ whenever $t_1(H) = 0$ on $[0, a]$;[19]
- for $n \in \{2, 3, 4, ...\}$ and $k \in \{1, 2, ..., n\}$, $\tau_k^{\varepsilon_{n-k+1}}(H)$ is some element of $N_{\varepsilon_{n-k+1}}(t_k(H))$ such that
  - $\tau_k^{\varepsilon_{n-k+1}}(H) = 0$ on $[0, a-\varepsilon_{n-k+1}]$ whenever $t_k(H) = 0$ on $[0, a]$ and
  - $\tau_k^{\varepsilon_{n-k+1}}(H)(x) \neq \tau_l^{\varepsilon_{m-l+1}}(H)(x)$ if $\tau_k^{\varepsilon_{n-k+1}}(H)(x) > 0$ for each $x \in \text{supp}\{p_H\} \cap D$, every $l \in \{1, 2, ..., k-1\}$, and any $m \in \{1, 2, ..., n-1\}$.

Set

$$\widetilde{\mathcal{T}}(H) := \bigcup_{n=1}^{\infty} \mathcal{T}_n(H).$$

We then define, for each $F \in \mathcal{F}$, $\mathcal{S}^*(F)$ to be the family of all $H \in \mathcal{S}(F)$ satisfying $p_H\{x\} < p_H\{(x, 1]\}$ for every $x \in \text{supp}\{p_H\} \cap [0, x_F)$.

The set of all tax functions from $\mathcal{T}(H)$ that are convex when restricted to the sub-domain $\text{supp}\{p_H\}$ is represented by $\mathcal{T}_{conv}(H)$. Finally, given $H \in \mathcal{H}$, $\mathbf{P}(\mathcal{T}(H))$ designates the set of all Borel probability measures on $\mathcal{T}(H)$. ‖

Lemma 7.1. *For $F \in \mathcal{F}$ and $H \in \mathcal{S}(F) \cup \mathcal{F}$, the sets $\mathcal{T}(H)$ and $\mathcal{T}_{conv}(H)$ are compact subsets of $\mathbf{C}[0, 1]$.*

Proof. Fix $F \in \mathcal{F}$ and $H \in \mathcal{S}(F) \cup \mathcal{F}$. Let us first show that $\mathcal{T}(H)$ is bounded and closed in $\mathbf{C}[0, 1]$. For any $t, \tau \in \mathcal{T}(H)$, we have $\|t - \tau\|_\infty \leq 1$ since the ranges of both t and τ are contained in $[0, 1]$. Thus $\text{diam}\big(\mathcal{T}(H)\big) \leq 1$ and hence $\mathcal{T}(F)$ is bounded. To prove the closedness claim, take any sequence (t_n) in $\mathcal{T}(F)$ and assume that $\|t - t_n\|_\infty \to 0$ (as $n \to \infty$) for some $t \in \mathbf{C}[0, 1]$. Then (t_n) converges uniformly to t, thereby guaranteeing that t is a tax function. Due to uniform convergence, we also have

$$\int_0^1 t dF = \int_0^1 \lim t_n dF = \lim \int_0^1 t_n dF = R_F,$$

and hence we may conclude that $t \in \mathcal{T}(F)$. Thus $\mathcal{T}(H)$ is closed in $\mathbf{C}[0, 1]$.

We next claim that $\mathcal{T}(H)$ is equicontinuous. To see this, pick an arbitrary $t \in \mathcal{T}(H)$ and take any $x, y \in [0, 1]$ with $x > y$. By monotonicity of the post-tax function, we have $x - t(x) \geq y - t(y)$ so that $t(x) - t(y) \leq x - y$.

[19] In what follows, for any $t \in \mathcal{T}(F)$ and $\delta > 0$, $N_\delta(t)$ stands for the open δ-neighborhood of t in $\mathcal{T}(F)$ or $\mathcal{T}(H)$, depending on the context.

Interchanging the roles of x and y, we may thus conclude that $|t(x)-t(y)| \leq |x-y|$ for all $x, y \in [0,1]$. So, for any $x \in [0,1]$ and any $\varepsilon > 0$, we have $|t(x) - t(y)| < \varepsilon$ whenever $|x-y| < \varepsilon$. This proves that $\mathcal{T}(H)$ is equicontinuous.

Given the observations noted in the previous two paragraphs, Arzelà-Ascoli theorem entails that $\mathcal{T}(H)$ is a compact subset of $\mathbf{C}[0,1]$. Since $\mathcal{T}_{conv}(H)$ is a closed subset of $\mathcal{T}(H)$, this observation also establishes the compactness of $\mathcal{T}_{conv}(H)$. ‖

Lemma 7.2. *Suppose that $p \in \mathbf{P}(\mathcal{A})$, $\varepsilon > 0$, and $\mathcal{A} \in \left\{\mathcal{T}(F), \mathcal{T}_{conv}(F)\right\}$. Then, for any $\tau \in \mathcal{A}$, there exists $\tau^* \in \mathcal{A}$ such that*

$$\liminf \int_{\mathcal{A}} (w(\tau^*, t) - w(t, \tau^*)) p_n(dt) \geq \int_{\mathcal{A}} (w(\tau, t) - w(t, \tau)) p(dt) - \varepsilon$$

for every sequence (p_n) converging weakly to p.

Proof. We first suppose that $\mathcal{A} = \mathcal{T}(F)$. Pick $\varepsilon > 0$ and $\tau \in \mathcal{T}(F)$. We suppose that $\int_0^1 \tau dF = R_F$, for the case where $\int_0^1 \tau dF > R_F$ is handled similarly. Define $x^* := \inf\left\{x : \tau(y) - \tau(x) = y - x \text{ whenever } x \leq y \leq 1\right\}$. Note that either $0 < x^* < 1$ or $x^* = 1$.

Suppose first that $0 < x^* < 1$. For $\epsilon > 0$ with $\left[x^* - \epsilon, x^* + \epsilon\right] \subseteq [0,1]$, let

$$\underline{t}_\epsilon(x) := \begin{cases} 0 & \text{if } 0 \leq x \leq x^* - \epsilon - \tau(x^* - \epsilon), \\ \tau(x^* + \epsilon) & \text{if } x^* - \epsilon + \tau(x^* + \epsilon) - \tau(x^* - \epsilon) < x \leq 1, \\ \tau(x^* - \epsilon) - (x^* - \epsilon) + x & \text{elsewhere,} \end{cases}$$

and put $\overline{t}_\epsilon := \max\{\tau, \underline{t}_\epsilon\}$. For each ϵ and every real α, $\int_0^1 [\alpha \overline{t}_\epsilon + (1-\alpha)\underline{t}_\epsilon] dF$ is continuous in α. Further, there exists ϵ° such that $\int_0^1 \underline{t}_\epsilon dF < R_F$ and $\int_0^1 \overline{t}_\epsilon dF > R_F$ for all $\epsilon \in (0, \epsilon^\circ)$. It follows from the intermediate value theorem that an $\alpha_\epsilon \in (0,1)$ exists associated to each $\epsilon \in (0, \epsilon^\circ)$ such that $\int_0^1 [\alpha_\epsilon \overline{t}_\epsilon + (1-\alpha_\epsilon)\underline{t}_\epsilon] dF = R_F$.

Suppose next that $x^* = 1$. Define, for each $\epsilon \in (0,1)$,

$$\underline{\tau}_\epsilon(x) := \begin{cases} 0 & \text{if } 0 \leq x \leq 1 - \epsilon - \tau(1-\epsilon), \\ \tau(1-\epsilon) - (1-\epsilon) + x & \text{if } 1 - \epsilon - \tau(1-\epsilon) < x \leq 1, \end{cases}$$

and $\overline{\tau}_\epsilon := \max\{\tau, \underline{\tau}_\epsilon\}$. Reasoning as before, one may establish the existence of $\epsilon^\bullet$ such that to each $\epsilon \in (0, \epsilon^\bullet)$ there corresponds a $\beta_\epsilon \in (0,1)$ such that $\int_0^1 [\beta_\epsilon \overline{\tau}_\epsilon + (1-\beta_\epsilon)\underline{\tau}_\epsilon] dF = R_F$.

Put

$$t_\epsilon := \begin{cases} \alpha_\epsilon \bar{t}_\epsilon + (1-\alpha_\epsilon)\underline{t}_\epsilon & \text{if } 0 < x^* < 1, \\ \beta_\epsilon \overline{\tau}_\epsilon + (1-\beta_\epsilon)\underline{\tau}_\epsilon & \text{if } x^* = 1. \end{cases}$$

Observe that each t_ϵ is an admissible tax function. Further, there is a sequence (ϵ_n) of positive real numbers converging to 0 such that $t_{\epsilon_n} < \tau$ on $\{\tau > 0\} \setminus \Big[x^* - \epsilon_n, x^* + \epsilon_n\Big]$ for every n.[20] Thus, we may fix $\epsilon^* \in \Big(0, \min\Big\{1 - x^* - \epsilon^*, x^* - \epsilon^*, \frac{\varepsilon}{4}\Big\}\Big)$ with $0 < x^* - \epsilon^*$, $x^* + \epsilon^* < 1$ if $x^* < 1$, and $t_{\epsilon^*} < \tau$ on $\{\tau > 0\} \setminus \Big[x^* - \epsilon^*, x^* + \epsilon^*\Big]$. Then to each $t \in \mathcal{T}(F)$ there corresponds a $\delta > 0$ such that every $f \in N_\delta(t)$ satisfies

$$w(t_{\epsilon^*}, f) - w(f, t_{\epsilon^*}) > w(\tau, t) - w(t, \tau) - \varepsilon. \tag{3}$$

We shall prove this assertion assuming that $0 < x^* < 1$. The case where $x^* = 1$ is dealt with in a similar fashion.

Choose $\eta \in \Big(\epsilon^*, \min\Big\{1 - x^* - \epsilon^*, x^* - \epsilon^*, \frac{\varepsilon}{4}\Big\}\Big)$ and define

$$d := \begin{cases} \min_{x \in [s_o + \eta, x^* - \eta] \cup [x^* + \eta, 1]} \tau(x) - t_{\epsilon^*}(x) & \text{if } s_o + \eta \le x^* - \eta, \\ \min_{x \in [x^* + \eta, 1]} \tau(x) - t_{\epsilon^*}(x) & \text{if } s_o + \eta > x^* - \eta, \end{cases}$$

where $s_o := \sup\{x : \tau(x) = 0\}$. Let t be arbitrary in $\mathcal{T}(F)$, and choose any $f \in N_\delta(t)$, where $0 < \delta < \min\Big\{d, t\Big(\sup\{x : t(x) = 0\} + \eta\Big)\Big\}/2$. Let S^c be the complement of $S := \Big[\sup\Big\{x : t(x) = 0\Big\}, \sup\Big\{x : t(x) = 0\Big\} + \eta\Big] \cup \Big[s_o, s_o + \eta\Big] \cup \Big[x^* - \eta, x^* + \eta\Big]$ in $[0,1]$. Then

$$\begin{aligned} w(\tau, t) - w(t, \tau) &= 2w(\tau, t) + p_F\{\tau = t\} - 1 \\ &= 2\Big(p_F\Big\{\{\tau < t\} \cap S\Big\} + p_F\Big\{\{\tau < t\} \cap S^c\Big\}\Big) + p_F\Big\{\{\tau = t\} \cap S\Big\} \\ &\quad + p_F\Big\{\{\tau = t\} \cap S^c\Big\} - 1 \\ &< \varepsilon + 2p_F\Big\{\{\tau < t\} \cap S^c\Big\} + p_F\Big\{\{\tau = t\} \cap S^c\Big\} - 1 \end{aligned}$$

[20] To see this, it suffices to show that there is a sequence (ϵ_n) of positive real numbers converging to 0 such that the left-hand derivative τ'_- of τ at $x^* - \epsilon_n$ is less than 1 for each n. If τ is constant on (a, x^*) for some a there is nothing to prove, so let τ be nonconstant on any (a, x^*). For each ϵ with $x^* \ge \epsilon > 0$, consider the function $f_\epsilon(x) := \tau(x^*) - \tau(x) - \frac{x^* - x}{\epsilon}[\tau(x^*) - \tau(x^* - \epsilon)]$, which vanishes when $x = x^* - \epsilon$ and $x = x^*$. Because f_ϵ is continuous and not nil on $[x^* - \epsilon, x^*]$, there exists $y \in (x^* - \epsilon, x^*)$ such that the left-hand derivative of f_ϵ at y is positive, whereby $\frac{\tau(x^*) - \tau(x^* - \epsilon)}{\epsilon} > \tau'_-(y)$. Since $\tau \in \mathcal{T}(F)$, the left-hand side of this inequality is less than 1, whence $\tau'_-(y) < 1$, as desired.

$$
\begin{aligned}
&= \varepsilon + 2\Big(p_F\Big\{\{t_{\epsilon^*} < \tau < t\} \cap S^c\Big\} + p_F\Big\{\{t_{\epsilon^*} \geq \tau < t\} \cap S^c\Big\}\Big) \\
&\qquad + p_F\Big\{\{t_{\epsilon^*} < \tau = t\} \cap S^c\Big\} + p_F\Big\{\{t_{\epsilon^*} \geq \tau = t\} \cap S^c\Big\} - 1 \\
&= \varepsilon + 2\Big(p_F\Big\{\{f \leq t_{\epsilon^*} < \tau < t\} \cap S^c\Big\} + p_F\Big\{\{f > t_{\epsilon^*} < \tau < t\} \cap S^c\Big\} \\
&\qquad + p_F\Big\{\{f \leq t_{\epsilon^*} \geq \tau < t\} \cap S^c\Big\} + p_F\Big\{\{f > t_{\epsilon^*} \geq \tau < t\} \cap S^c\Big\}\Big) \\
&\qquad + p_F\Big\{\{f \leq t_{\epsilon^*} < \tau = t\} \cap S^c\Big\} + p_F\Big\{\{f > t_{\epsilon^*} < \tau = t\} \cap S^c\Big\} \\
&\qquad + p_F\Big\{\{f \leq t_{\epsilon^*} \geq \tau = t\} \cap S^c\Big\} + p_F\Big\{\{f > t_{\epsilon^*} \geq \tau = t\} \cap S^c\Big\} - 1.
\end{aligned}
$$

Observe that, because $f \in N_\delta(t)$,

$$
\begin{aligned}
p_F\Big\{\{f \leq t_{\epsilon^*} < \tau < t\} \cap S^c\Big\} &= p_F\Big\{\{f \leq t_{\epsilon^*} \geq \tau < t\} \cap S^c\Big\} \\
&= p_F\Big\{\{f \leq t_{\epsilon^*} < \tau = t\} \cap S^c\Big\} = 0
\end{aligned}
$$

and $p_F\Big\{\{f \leq t_{\epsilon^*} \geq \tau = t\} \cap S^c\Big\} \leq p_F\{f = t_{\epsilon^*}\}$. Therefore,

$$
\begin{aligned}
&w(\tau, t) - w(t, \tau) \\
&\quad < \varepsilon + 2\Big(p_F\Big\{\{f > t_{\epsilon^*} < \tau < t\} \cap S^c\Big\} + p_F\Big\{\{f > t_{\epsilon^*} \geq \tau < t\} \cap S^c\Big\}\Big) \\
&\qquad\quad + p_F\Big\{\{f > t_{\epsilon^*} < \tau = t\} \cap S^c\Big\} + p_F\{f = t_{\epsilon^*}\} \\
&\qquad\quad + p_F\Big\{\{f > t_{\epsilon^*} \geq \tau = t\} \cap S^c\Big\} - 1 \\
&\quad \leq \varepsilon + 2w(t_{\epsilon^*}, f) + p_F\{t_{\epsilon^*} = f\} - 1 \\
&\quad = \varepsilon + w(t_{\epsilon^*}, f) - w(f, t_{\epsilon^*}).
\end{aligned}
$$

This establishes (3).

Pick any $p \in \mathbf{P}(\mathcal{T}(F))$. The proof the lemma (for the case where $\mathcal{A} = \mathcal{T}(F)$) is complete if we show that every sequence (p_n) converging weakly to p satisfies

$$
\begin{aligned}
\liminf \textstyle\int_{\mathcal{T}(F)} &(w(t_{\epsilon^*}, t) - w(t, t_{\epsilon^*}))p_n(dt) \\
&\geq \textstyle\int_{\mathcal{T}(F)} (w(\tau, t) - w(t, \tau))p(dt) - \varepsilon.
\end{aligned}
\tag{4}
$$

To this end, let $\varphi : \mathcal{T}(F) \to \mathbf{R}$ be the lim inf of the map $t \mapsto w(t_{\epsilon^*}, t) - w(t, t_{\epsilon^*})$ on $\mathcal{T}(F)$, that is,

$$
\varphi(t) := \lim_{\delta \downarrow 0} \inf \Big\{w(t_{\epsilon^*}, f) - w(f, t_{\epsilon^*}) : f \in N_\delta(t)\Big\}.
$$

Because φ is lower semicontinuous,

$$\liminf \int_{\mathcal{T}(F)} \varphi dp_n \geq \int_{\mathcal{T}(F)} \varphi dp \tag{5}$$

whenever (p_n) converges weakly to p (see, e.g., Aliprantis and Border (1999), Theorem 14.5). Because to each $t \in \mathcal{T}(F)$ there corresponds a $\delta > 0$ such that every $f \in N_\delta(t)$ satisfies (3), we must have $\varphi(t) \geq w(\tau, t) - w(t, \tau) - \varepsilon$ for all $t \in \mathcal{T}(F)$. Thus, we may write

$$w(t_{\epsilon^*}, t) - w(t, t_{\epsilon^*}) \geq \varphi(t) \geq w(\tau, t) - w(t, \tau) - \varepsilon \quad \text{for all } t \in \mathcal{T}(F).$$

These inequalities, along with (5), imply (4) for every sequence (p_n) converging weakly to p, as desired.

It remains to prove the lemma for $\mathcal{A} = \mathcal{T}_{conv}(F)$. This can be done by means of the previous argument after the replacement of $\mathcal{T}(F)$ by $\mathcal{T}_{conv}(F)$ and the following redefinition of $\underline{t}_\epsilon$:

$$\underline{t}_\epsilon(x) := \begin{cases} 0 & \text{if } 0 \leq x \leq x_\epsilon, \\ \frac{[t_o(x^*+\epsilon) - t_o(x^*-\epsilon)](x - x^* + \epsilon)}{2\epsilon} + t_o(x^* - \epsilon) & \text{if } x_\epsilon < x \leq 1, \end{cases}$$

where

$$x_\epsilon := \frac{(x^* - \epsilon) t_o(x^* + \epsilon) - (x^* + \epsilon) t_o(x^* - \epsilon)}{t_o(x^* + \epsilon) - t_o(x^* - \epsilon)}. \quad \|$$

Definition 7.3 (Reny, 1999)**.** Suppose that $\mathcal{A} \subseteq \mathcal{T}(F)$. The mixed extension of $G(\mathcal{A})$ is ***payoff secure*** if for every $i \in \{1, 2\}$, $\mu = (\mu_1, \mu_2) \in \mathbf{P}(\mathcal{A})^2$, and $\varepsilon > 0$, there exists $\nu_i \in \mathbf{P}(\mathcal{A})$ such that $U_i(\nu_i, \tilde{\mu}_{-i}) \geq U_i(\mu) - \varepsilon$ for all $\tilde{\mu}_{-i}$ in some open neighborhood of μ_{-i}.

Lemma 7.4. *The mixed extensions of $G(\mathcal{T}_{conv}(F))$ and $G(\mathcal{T}(F))$ are payoff secure.*

Proof. We prove the lemma for $G(\mathcal{T}(F))$. The argument for $G(\mathcal{T}_{conv}(F))$ is identical. Fix $\mu = (\mu_1, \mu_2) \in \mathbf{P}(\mathcal{T}(F))^2$, $\varepsilon > 0$, and $i \in \{1, 2\}$. We have to show that there exists $\nu_i \in \mathbf{P}(\mathcal{T}(F))$ such that $U_i(\nu_i, \tilde{\mu}_{-i}) \geq U_i(\mu) - \varepsilon$ for all $\tilde{\mu}_{-i}$ in some open neighborhood of μ_{-i}.

It is clear that there exists $\tau \in \mathcal{T}(F)$ such that

$$U_i(\tau, \mu_{-i}) > U_i(\mu) - \frac{\varepsilon}{4}. \tag{6}$$

By Lemma 7.2, there exists $\tau^* \in \mathcal{T}(F)$ such that every sequence $\{\mu_n\}$ converging weakly to μ_{-i} satisfies

$$\liminf \int_{\mathcal{T}(F)} (w(\tau^*, t) - w(t, \tau^*))\mu_n(dt) \geq \int_{\mathcal{T}(F)} (w(\tau, t) - w(t, \tau))\mu_{-i}(dt) - \frac{\varepsilon}{2}.$$

Hence, $U_i(\tau^*, \tilde{\mu}_{-i}) \geq U_i(\tau, \mu_{-i}) - \frac{3\varepsilon}{4}$ for every $\tilde{\mu}_{-i}$ in some open neighborhood of μ_{-i}. From these inequalities and (6), we see that $U_i(\tau^*, \tilde{\mu}_{-i}) \geq U_i(\mu) - \varepsilon$ for every $\tilde{\mu}_{-i}$ in some open neighborhood of μ_{-i}, as we sought. ‖

The following lemma is a generalization of the theorem of Marhuenda and Ortuño-Ortín (1995).

Lemma 7.5. *Suppose that $m_F < \mu_F$. If, for $t \in \mathcal{N}_{conv}(F)$ and $\tau \in \mathcal{T}(F) \setminus \mathcal{N}_{conv}(F)$ with $\int_0^1 t dF \leq \int_0^1 \tau dF$,*

(i) *$t - \tau$ is a nonlinear convex function,*

(ii) *$t(x) = \tau(x)$ holds for at most one $x \in (0, 1]$, and*

(iii) *$t < \tau$ on some open neighborhood of 0,*

then $w(t, \tau) > 1/2$.

Proof. Suppose that $m_F < \mu_F$ and fix $t \in \mathcal{N}_{conv}(F)$ and $\tau \in \mathcal{T}(F) \setminus \mathcal{N}_{conv}(F)$ satisfying

$$\int_0^1 t dF \leq \int_0^1 \tau dF, \tag{7}$$

(*i*), (*ii*), and (*iii*). Either $t \neq \tau$ everywhere on $(0, 1]$ or $t = \tau$ at a unique point x^* in $(0, 1]$. In the former case, (7) gives $w(t, \tau) > 1/2$, as desired. If $t = \tau$ at a unique point x^* in $(0, 1]$, (*iii*) ensures that $t|_{(0,x^*)} < \tau|_{(0,x^*)}$ and $t|_{(x^*,1]} > \tau|_{(x^*,1]}$. On the other hand, since $t - \tau$ is a non-affine convex function on $[0, 1]$, Jensen's inequality and (7) give

$$(t - \tau)(\mu_F) < \int_0^1 (t - \tau) dF \leq 0.$$

Therefore, $m_F < \mu_F < x^*$, and hence $w(t, \tau) > F(m_F) = 1/2$. ‖

Lemma 7.6. *If $G(\mathcal{T}_{conv}(F))$ has a mixed strategy Nash equilibrium, so does $G(\mathcal{C}(F))$.*

Proof. Let $\nu = (\nu_1, \nu_2)$ be a mixed strategy Nash equilibrium of $G(\mathcal{T}_{conv}(F))$. We shall show that ν is also a mixed strategy Nash equilibrium of $G(\mathcal{C}(F))$. Take $i \in \{1, 2\}$. Say $i = 1$. Fix any $\mu_1 \in \mathbf{P}(\mathcal{C}(F))$. Consider the case where any $t \in \text{supp}\{\mu_1\}$ has $\int_0^1 t dF = R_F$. If $\mu_1(\mathcal{T}_{conv}(F)) = 0$, we have

$$U_1(\mu_1, \nu_2) = \int_{\mathcal{T}_{conv}(F)} \int_{\mathcal{N}_{conc}(F)} u_1(t, \tau)\mu_1(dt)\nu_2(d\tau).$$

It is easily seen that any $t \in \text{supp}\{\nu_2\}$ satisfies $\int_0^1 t dF = R_F$ because ν is a Nash equilibrium of $G(\mathcal{T}_{conv}(F))$. Since, by Lemma 7.5, $u_1(t, \tau) \leq 0$ whenever $(t, \tau) \in \mathcal{N}_{conc}(F) \times \mathcal{T}_{conv}(F)$ and $\int_0^1 t dF = \int_0^1 \tau dF = R_F$, $U_1(\mu_1, \nu_2) \leq 0$. Further, $U_1(\nu) = 0$ because ν is a Nash equilibrium. Hence $U_1(\mu_1, \nu_2) \leq U_1(\nu)$. If, on the other hand, $\mu_1(\mathcal{T}_{conv}(F)) > 0$, define the probability measure $\tilde{\mu}_1$ on the Borel subsets of $\mathcal{T}_{conv}(F)$ as follows:

$$\tilde{\mu}_1(\mathcal{B}) := \mu_1(\mathcal{B})\Big(1 + \frac{\mu_1(\mathcal{N}_{conc}(F))}{\mu_1(\mathcal{T}_{conv}(F))}\Big).$$

Then,

$$\begin{aligned}
U_1(\mu_1, \nu_2) &= \int_{\mathcal{T}_{conv}(F)} \int_{\mathcal{N}_{conc}(F)} u_1(t, \tau)\mu_1(dt)\nu_2(d\tau) \\
&\qquad + \int_{\mathcal{T}_{conv}(F)} \int_{\mathcal{T}_{conv}(F)} u_1(t, \tau)\mu_1(dt)\nu_2(d\tau) \\
&\leq \int_{\mathcal{T}_{conv}(F)} \int_{\mathcal{T}_{conv}(F)} u_1(t, \tau)\mu_1(dt)\nu_2(d\tau) \\
&\leq \Big(1 + \frac{\mu_1(\mathcal{N}_{conc}(F))}{\mu_1(\mathcal{T}_{conv}(F))}\Big) \int_{\mathcal{T}_{conv}(F)} \int_{\mathcal{T}_{conv}(F)} u_1(t, \tau)\mu_1(dt)\nu_2(d\tau) \\
&= \int_{\mathcal{T}_{conv}(F)} \int_{\mathcal{T}_{conv}(F)} u_1(t, \tau)\tilde{\mu}_1(dt)\nu_2(d\tau) \\
&= U_1(\tilde{\mu}_1, \nu_2),
\end{aligned}$$

where the first inequality holds because

$$\int_{\mathcal{T}_{conv}(F)} \int_{\mathcal{N}_{conc}(F)} u_1(t, \tau)\mu_1(dt)\nu_2(d\tau) \leq 0$$

by Lemma 7.5. Because ν is a mixed strategy Nash equilibrium of $G(\mathcal{T}_{conv}(F))$, we have $U_1(\tilde{\mu}_1, \nu_2) \leq U_1(\nu)$. Combine this inequality with the previous equation to obtain $U_1(\mu_1, \nu_2) \leq U_1(\nu)$. It is clear that if $U_1(\mu_1, \nu_2) \leq U_1(\nu)$

for every $\mu_1 \in \mathbf{P}(\mathcal{C}(F))$ with $\int_0^1 t dF = R_F$ for any $t \in \text{supp}\{\mu_1\}$, then $U_1(\mu_1, \nu_2) \leq U_1(\nu)$ for every $\mu_1 \in \mathbf{P}(\mathcal{C}(F))$. We conclude that ν is also a mixed strategy Nash equilibrium of $G(\mathcal{C}(F))$. ‖

Proof of Theorem 4.4. The game $G(\mathcal{T}(F))$ is compact (Lemma 7.1). Further, its mixed extension is payoff secure (Lemma 7.4) and $u_1 + u_2$, being constant on $\mathcal{T}(F)^2$, is upper semicontinuous on $\mathcal{T}(F)^2$. It follows from Corollary 5.2 and Proposition 5.1 of Reny (1999) that $G(\mathcal{T}(F))$ possesses a mixed strategy Nash equilibrium. A similar argument guarantees the existence of a mixed strategy Nash equilibrium of $G(\mathcal{T}_{conv}(F))$. Lemma 7.6 then ensures that $G(\mathcal{C}(F))$ also possesses a mixed strategy Nash equilibrium. ‖

Lemma 7.7. *Suppose that $m_F < \mu_F$. Let $\mathcal{A}$ be any subset of $\mathcal{T}(F)$ such that $\mathcal{T}_{conc}(F) \cap \mathcal{A} \neq \emptyset$, $\mathcal{N}_{conv}(F) \cap \mathcal{A} \neq \emptyset$, and $\frac{R_F}{\int_0^1 tdF} t \in \mathcal{A}$ whenever $t \in \mathcal{A}$ and $\int_0^1 tdF > R_F$. Suppose that, for all $t \in \mathcal{N}_{conv}(F) \cap \mathcal{A}$ and $\tau \in \mathcal{A} \setminus \mathcal{N}_{conv}(F)$, the conditions (i)-(iii) of Lemma 7.5 are fulfilled. Then,*

$$\int_{\mathcal{A} \cap \mathcal{T}_{conc}(F)} \int_{\mathcal{A} \cap \mathcal{N}_{conv}(F)} u_1(t, \tau) \mu_1(dt) \mu_2(d\tau) > 0$$

for any $(\mu_1, \mu_2) \in \mathbf{P}(\mathcal{A})^2$ with $\int_0^1 tdF = R_F$ whenever $t \in \text{supp}\{\mu_i\}$ and $i \in \{1, 2\}$.

Proof. Assume the antecedent. Choose any $(\mu_1, \mu_2) \in \mathbf{P}(\mathcal{A})^2$ with $\int_0^1 tdF = R_F$ for each $t \in \text{supp}\{\mu_i\}$ and every i. Fix (t_o, τ_o) in $(\mathcal{A} \cap \mathcal{N}_{conv}(F)) \times (\mathcal{A} \cap \mathcal{T}_{conc}(F))$ and assume that (t_o, τ_o) is in the support of $\mu_1 \times \mu_2$. By Lemma 7.5, we have $w(t_o, \tau_o) > 1/2$. Since w is lower semicontinuous (Lemma 4.2), one can find $\epsilon > 0$ and an open neighborhood O of (t_o, τ_o) on which $w(t, \tau) > \frac{1}{2} + \epsilon$. Let $\mathcal{A}_1 := ((\mathcal{A} \cap \mathcal{T}_{conc}(F)) \times (\mathcal{A} \cap \mathcal{N}_{conv}(F))) \setminus O$ and $\mathcal{A}_2 := ((\mathcal{A} \cap \mathcal{T}_{conc}(F)) \times (\mathcal{A} \cap \mathcal{N}_{conv}(F))) \cap O$. Observe that

$$\begin{aligned}
&\int_{(\mathcal{A} \cap \mathcal{N}_{conv}(F)) \times (\mathcal{A} \cap \mathcal{T}_{conc}(F))} w d(\mu_1 \times \mu_2) \\
&\quad \geq \inf_{(t,\tau) \in \mathcal{A}_1} w(t, \tau)(\mu_1 \times \mu_2)(\mathcal{A}_1) + \inf_{(t,\tau) \in \mathcal{A}_2} w(t, \tau)(\mu_1 \times \mu_2)(\mathcal{A}_2) \\
&\quad \geq (\mu_1 \times \mu_2)(\mathcal{A}_1)\frac{1}{2} + (\mu_1 \times \mu_2)(\mathcal{A}_2)\Big(\frac{1}{2} + \epsilon\Big) \\
&\quad > 1/2.
\end{aligned}$$

This implies $\int_{(\mathcal{A}\cap\mathcal{T}_{conc}(F))\times(\mathcal{A}\cap\mathcal{N}_{conv}(F))} wd(\mu_1 \times \mu_2) < 1/2$. We may therefore write

$$\begin{aligned}
&\int_{\mathcal{A}\cap\mathcal{T}_{conc}(F)} \int_{\mathcal{A}\cap\mathcal{N}_{conv}(F)} u_1(t,\tau)\mu_1(dt)\mu_2(d\tau) \\
&\quad = \int_{(\mathcal{A}\cap\mathcal{N}_{conv}(F))\times(\mathcal{A}\cap\mathcal{T}_{conc}(F))} wd(\mu_1 \times \mu_2) - \int_{(\mathcal{A}\cap\mathcal{T}_{conc}(F))\times(\mathcal{A}\cap\mathcal{N}_{conv}(F))} wd(\mu_1 \times \mu_2) \\
&\quad > 0,
\end{aligned}$$

as we sought. ‖

Proof of Proposition 5.1. Suppose that $m_F < \mu_F$. Let $\mathcal{A}$ be any subset of $\mathcal{T}(F)$ such that $\mathcal{N}_{conv}(F) \cap \mathcal{A} \neq \emptyset$ and $\frac{R_F}{\int_0^1 tdF} t \in \mathcal{A}$ whenever $t \in \mathcal{A}$ and $\int_0^1 tdF > R_F$. Suppose that, for all $t \in \mathcal{N}_{conv}(F) \cap \mathcal{A}$ and $\tau \in \mathcal{A} \setminus \mathcal{N}_{conv}(F)$,

- $t - \tau$ is a nonlinear convex function, and
- $t(x) = \tau(x)$ holds for at most one $x \in (0, 1]$.

Take $\mu = (\mu_1, \mu_2) \in \mathbf{P}(\mathcal{A})^2$. Suppose that $\mu_2(\mathcal{A} \cap \mathcal{T}_{conc}(F)) > 0$. We shall show that μ is not a mixed strategy Nash equilibrium of $G(\mathcal{A})$. It is easily seen that, because $\frac{R_F}{\int_0^1 tdF} t \in \mathcal{A}$ whenever $t \in \mathcal{A}$ and $\int_0^1 tdF > R_F$, μ is not a Nash equilibrium of $G(\mathcal{A})$ if $\int_0^1 tdF > R_F$ for some $t \in \mathrm{supp}\{\mu_i\}$ and some i. Therefore, we consider only the case where $\int_0^1 tdF = R_F$ for each $t \in \mathrm{supp}\{\mu_i\}$ and every i.

Suppose that $U_1(\mu) > 0$. Then $U_2(\mu) < 0$. Since $U_2(\mu_1, \mu_1) = 0 > U_2(\mu)$, μ is not a Nash equilibrium. Next, suppose that $U_1(\mu) \leq 0$ and $\mu_2(\mathcal{A} \cap \mathcal{N}_{conv}(F)) = 0$. Pick $t \in \mathcal{N}_{conv}(F) \cap \mathcal{A}$. By assumption, $t^* := \frac{R_F}{\int_0^1 tdF} t \in \mathcal{N}_{conv}(F) \cap \mathcal{A}$. Since $\mu_2(\mathcal{A} \cap \mathcal{N}_{conv}(F)) = 0$, Lemma 7.7 implies $U_1(t^*, \mu_2) > 0$. Hence $U_1(t^*, \mu_2) > U_1(\mu)$, and so μ is not a Nash equilibrium.

We now turn to the case where $U_1(\mu) \leq 0$ and $\mu_2(\mathcal{A} \cap \mathcal{N}_{conv}(F)) > 0$. Define the probability measure $\tilde{\mu}_1$ on the Borel subsets of $\mathcal{A}$ as follows:

$$\tilde{\mu}_1(\mathcal{B}) := \frac{\mu_2(\mathcal{B} \cap \mathcal{N}_{conv}(F))}{\mu_2(\mathcal{A} \cap \mathcal{N}_{conv}(F))}.$$

Observe that $\tilde{\mu}_1(\mathcal{B}) = \mu_2(\mathcal{B})\Big(1 + \frac{\mu_2(\mathcal{A}\cap\mathcal{T}_{conc}(F))}{\mu_2(\mathcal{A}\cap\mathcal{N}_{conv}(F))}\Big)$ whenever $\mathcal{B} \subseteq \mathcal{N}_{conv}(F)$. Consequently, we have

$U_1(\tilde{\mu}_1, \mu_2)$

$$
\begin{aligned}
&= \int_{\mathcal{A}\cap\mathcal{T}_{conc}(F)} \int_{\mathcal{A}\cap\mathcal{N}_{conv}(F)} u_1(t,\tau)\tilde{\mu}_1(dt)\mu_2(d\tau) \\
&\qquad + \int_{\mathcal{A}\cap\mathcal{N}_{conv}(F)} \int_{\mathcal{A}\cap\mathcal{N}_{conv}(F)} u_1(t,\tau)\tilde{\mu}_1(dt)\mu_2(d\tau) \\
&> \int_{\mathcal{A}\cap\mathcal{N}_{conv}(F)} \int_{\mathcal{A}\cap\mathcal{N}_{conv}(F)} u_1(t,\tau)\tilde{\mu}_1(dt)\mu_2(d\tau) \\
&= \left(1+\frac{\mu_2(\mathcal{A}\cap\mathcal{T}_{conc}(F))}{\mu_2(\mathcal{A}\cap\mathcal{N}_{conv}(F))}\right) \int_{\mathcal{A}\cap\mathcal{N}_{conv}(F)} \int_{\mathcal{A}\cap\mathcal{N}_{conv}(F)} u_1(t,\tau)\mu_2(dt)\mu_2(d\tau) \\
&= 0 \\
&\geq U_1(\mu),
\end{aligned}
$$

where the first inequality holds by Lemma 7.7. Thus, μ is not a Nash equilibrium. ‖

Lemma 7.8. *For each $H \in \bigcup_{F\in\mathcal{F}}\mathcal{S}(F)$, $t \in \mathcal{T}(H)$, $\varepsilon > 0$, and every neighborhood $\mathcal{K}$ of t, there exists $\tau \in \widetilde{\mathcal{T}}(H)\cap\mathcal{K}$ such that $\tau = 0$ on $[0, a-\varepsilon]$ whenever $t = 0$ on $[0, a]$.*

Proof. Fix $F \in \mathcal{F}$, $H \in \mathcal{S}(F)$, $t \in \mathcal{T}(H)$, $\varepsilon > 0$, and a neighborhood $\mathcal{K}$ of t. We only consider the case where $t = 0$ on some interval $[0, a]$ and $t > 0$ elsewhere, for the case where $t > 0$ on $[0, 1]$ is analogous.

Suppose that $t = 0$ on some interval $[0, a]$ and $t > 0$ elsewhere. It suffices to show that there exists $\tau \in \widetilde{\mathcal{T}}(H)\cap\mathcal{K}$ such that $\tau = 0$ on $[0, a-\varepsilon]$. Observe that we may choose a rational number q in $(a-\varepsilon, a)$ such that $\mathcal{T}_q(H)\cap\mathcal{K} \neq \emptyset$. Because $\mathcal{T}_q(H)\cap\mathcal{K} \neq \emptyset$ and $\mathcal{T}_q^o(H)$ is dense in $\mathcal{T}_q(H)$, we may pick f from $\mathcal{T}_q^o(H)\cap\mathcal{K}$. Because $f \in \mathcal{T}^*(H) = \left\{t_1(H), t_2(H), t_3(H), ...\right\}$, there is some k with $t_k(H) = f$. Let η be the radius of $\mathcal{K}$. Then, for n sufficiently high,

$$0 < \varepsilon_n < \min\left\{\eta, q-(a-\varepsilon)\right\},$$

and $\tau_k^{\varepsilon_n}(H)$ is an element of $N_{\varepsilon_n}(t_k(H))$ with $\tau_k^{\varepsilon_n}(H) = 0$ on $[0, q-\varepsilon_n]$. It follows that $t_k(H) \in \mathcal{K}$ and $t_k(H) = 0$ on $[0, a-\varepsilon]$. Since $t_k(H) \in \widetilde{\mathcal{T}}(H)$, the proof is complete. ‖

Lemma 7.9. *Suppose that $F \in \mathcal{F}$. Then any $H \in \mathcal{S}^*(F)$ sufficiently close to F satisfies, for each i, $u_i(\tau, t) > 0$ for some $\tau \in \mathcal{T}(H)$ and every $t \in \mathcal{T}(H)$ with $t = 0$ on $\operatorname{supp}\{p_H\} \cap [0, x_F)$.*

Proof. Suppose that $F \in \mathcal{F}$. Because the set of probability measures on $[0, 1]$ with finite support is dense in $\mathcal{H}$ (Billingsley, 1968), any $H \in \mathcal{S}^*(F)$

sufficiently close to F has $\max\left\{\operatorname{supp}\{p_H\} \cap [0, x_F)\right\}$ sufficiently close to x_F. For H with $\max\left\{\operatorname{supp}\{p_H\} \cap [0, x_F)\right\}$ close to x_F, it is easily seen that $u_i(\tau, t) > 0$ for some $\tau \in \mathcal{T}(H)$ and every $t \in \mathcal{T}(H)$ with $t = 0$ on $\operatorname{supp}\{p_H\} \cap [0, x_F)$. $\|$

Lemma 7.10. *Suppose that $H \in \bigcup_{F \in \mathcal{F}} \mathcal{S}(F)$, $p \in \mathbf{P}(\mathcal{T}(H))$, and $\varepsilon > 0$. Then, for any $\tau \in \mathcal{T}(H)$, there exists $\tau^* \in \widetilde{\mathcal{T}}(H)$ such that*

$$\liminf \int_{\mathcal{T}(H)} (w(\tau^*, t) - w(t, \tau^*)) p_n(dt) \geq \int_{\mathcal{T}(H)} (w(\tau, t) - w(t, \tau)) p(dt) - \varepsilon$$

for every sequence (p_n) converging weakly to p.

Proof. The argument from the proof of Lemma 7.2 can be used to prove this lemma. One needs only to observe that the analogue of x^* must lie in $[x_F, 1]$ and that the analogue of t_{ϵ^*} may actually be chosen from $\widetilde{\mathcal{T}}(H)$ by Lemma 7.8. $\|$

Definition 7.11 (Reny, 1996)**.** A subset $\mathcal{A}$ of $\mathcal{T}(H)$ ***ensures local payoff security of*** U_i ***on*** $\mathbf{P}(\mathcal{T}(H))^2$ if, for each $\mu = (\mu_1, \mu_2) \in \mathbf{P}(\mathcal{T}(H))^2$ and every $\varepsilon > 0$, there exists $\nu_i \in \mathbf{P}(\mathcal{T}(H))$ with $\nu_i(\mathcal{A}) = 1$ such that $U_i(\nu_i, \tilde{\mu}_{-i}) \geq U_i(\mu) - \varepsilon$ for all $\tilde{\mu}_{-i}$ in some open neighborhood of μ_{-i}.

Lemma 7.12. *Suppose that $H \in \bigcup_{F \in \mathcal{F}} \mathcal{S}(F)$. Then the set $\widetilde{\mathcal{T}}(H)$ ensures local payoff security of U_i on $\mathbf{P}(\mathcal{T}(H))^2$ for each i.*

Proof. Suppose that $H \in \bigcup_{F \in \mathcal{F}} \mathcal{S}(F)$. Fix $i \in \{1, 2\}$, $\mu = (\mu_1, \mu_2) \in \mathbf{P}(\mathcal{T}(H))^2$, and $\varepsilon > 0$. We have to show that there exists $\nu_i \in \mathbf{P}(\mathcal{T}(H))$ with $\nu_i(\widetilde{\mathcal{T}}(H)) = 1$ such that $U_i(\nu_i, \tilde{\mu}_{-i}) \geq U_i(\mu) - \varepsilon$ for all $\tilde{\mu}_{-i}$ in some open neighborhood of μ_{-i}.

It is clear that there exists $\tau \in \mathcal{T}(H)$ such that

$$U_i(\tau, \mu_{-i}) > U_i(\mu) - \frac{\varepsilon}{4}. \tag{8}$$

By Lemma 7.10, there exists $\tau^* \in \widetilde{\mathcal{T}}(H)$ such that every sequence $\{\mu_n\}$ converging weakly to μ_{-i} satisfies

$$\liminf \int_{\mathcal{T}(H)} (w(\tau^*, t) - w(t, \tau^*)) \mu_n(dt) \geq \int_{\mathcal{T}(H)} (w(\tau, t) - w(t, \tau)) \mu_{-i}(dt) - \frac{\varepsilon}{2}.$$

Hence, $U_i(\tau^*, \tilde{\mu}_{-i}) \geq U_i(\tau, \mu_{-i}) - \frac{3\varepsilon}{4}$ for every $\tilde{\mu}_{-i}$ in some open neighborhood of μ_{-i}. From these inequalities and (8), we see that $U_i(\tau^*, \tilde{\mu}_{-i}) \geq U_i(\mu) - \varepsilon$ for every $\tilde{\mu}_{-i}$ in some open neighborhood of μ_{-i}. $\|$

Definition 7.13 (Reny, 1996)**.** A ***strategic approximation*** to the game $G(\mathcal{T}(H))$ is a countable set of pure strategies $\mathcal{Q} \subseteq \mathcal{T}(H)$ such that

(i) for every $i \in \{1, 2\}$, $\sup\{U_i(t, \mu_{-i}) : t \in \mathcal{Q}\} = \sup\{U_i(t, \mu_{-i}) : t \in \mathcal{T}(H)\}$ for all $\mu_{-i} \in \mathbf{P}(\mathcal{T}(H))$, and

(ii) whenever $(\mathcal{A}_n)$ is an increasing sequence of finite sets whose union is $\mathcal{Q}$, any limit of equilibria of the sequence of finite games $(G(\mathcal{A}_n))$ is an equilibrium of $G(\mathcal{T}(H))$.[21]

Lemma 7.14. *Suppose that $H \in \bigcup_{F \in \mathcal{F}} \mathcal{S}(F)$. Then, $\mathcal{Q}$ is a strategic approximation to $G(\mathcal{T}(H))$ if $\mathcal{Q}$ is a countable subset of $\mathcal{T}(H)$ that ensures local payoff security of U_i on $\mathbf{P}(\mathcal{T}(H))^2$ for each i.*

Proof. Suppose that $H \in \mathcal{S}(F)$ for some $F \in \mathcal{F}$. Suppose that $\mathcal{Q}$ is a countable subset of $\mathcal{T}(H)$ that ensures local payoff security of U_i on $\mathbf{P}(\mathcal{T}(H))^2$ for each i. The set $\mathcal{T}(H)$ is a compact metric space (Lemma 7.1). Further, the sum of the players' payoff functions is upper semicontinuous on $\mathbf{P}(\mathcal{T}(H))^2$ (in fact, constant on $\mathbf{P}(\mathcal{T}(H))^2$). It follows from Theorem 4 of Reny (1996) that $\mathcal{Q}$ is a strategic approximation to $G(\mathcal{T}(H))$.[22] $\|$

Lemma 7.15. *Suppose that $F \in \mathcal{F}$ satisfies $x_F \leq m_F$, and let $H \in \mathcal{S}^*(F)$ be sufficiently close to F in the sense of Lemma 7.9. If $\mu = (\mu_1, \mu_2)$ is a Nash equilibrium of $G(\mathcal{T}(H))$ with $\mu^n = (\mu_1^n, \mu_2^n) \to \mu$ for some sequence (μ^n), where each μ^n is a Nash equilibrium of $G(\mathcal{A}_n)$ and $(\mathcal{A}_n)$ is an increasing sequence of finite sets whose union is $\widetilde{\mathcal{T}}(H)$, then $\mu_i(\mathcal{T}_{conv}(H)) < 1$ for each i.*

Proof. Fix $F \in \mathcal{F}$ with $x_F \leq m_F$ and choose $H \in \mathcal{S}^*(F)$ sufficiently close to F in the sense of Lemma 7.9. Let $\mu = (\mu_1, \mu_2)$ be a Nash equilibrium of

[21] Finite sets are included among those which are considered countable, so $\mathcal{Q}$ may be finite.

[22] Theorem 4 of Reny (1996) requires that the vector payoff function of the game $G(\mathcal{T}(H))$ satisfy a condition termed *reciprocal upper semicontinuity.* As pointed out by Reny, the upper semicontinuity of the sum of the players' payoffs on $\mathbf{P}(\mathcal{T}(H))^2$ is a sufficient condition for the vector payoff function of the game $G(\mathcal{T}(H))$ to be reciprocally upper semicontinuous.

$G(\mathcal{T}(H))$ with $\mu^n = (\mu_1^n, \mu_2^n) \to \mu$ for some sequence (μ^n), where each μ^n is a Nash equilibrium of $G(\mathcal{A}_n)$ and $(\mathcal{A}_n)$ is an increasing sequence of finite sets whose union is $\widetilde{\mathcal{T}}(H)$. Suppose that $\mu_i(\mathcal{T}_{conv}(H)) = 1$ for some i, say for $i = 2$. We shall obtain a contradiction.

Set
$$\bar{t} := \sup \Big\{ t : t \in \operatorname{supp}\{\mu_2\} \Big\}.$$
A contradiction is easily obtained if $\bar{t} = 0$ on $\operatorname{supp}\{p_H\} \cap [0, x_F)$, for then Lemma 7.9 gives $\tau \in \mathcal{T}(H)$ such that $u_1(\tau, t) > 0$ for every $t \in \operatorname{supp}\{\mu_2\}$, and this cannot hold true if μ is a Nash equilibrium of $G(\mathcal{T}(H))$.

Throughout the sequel, we assume that $\bar{t}(x) > 0$ for some $x \in \operatorname{supp}\{p_H\} \cap [0, x_F)$. Put
$$y := \min \Big\{ x : x \in \operatorname{supp}\{p_H\} \text{ and } \bar{t}(x) > 0 \Big\}.$$
We proceed in a number of steps.

Step 0. To each $\delta > 0$ and every $\epsilon > 0$ there corresponds an N such that, for every $n \geq N$,
$$\mu_2^n \Big(\operatorname{cl} N_\delta \big(\operatorname{supp}\{\mu_2\} \big) \Big) > 1 - \epsilon.$$ [23]

Proof. Fix δ and $\epsilon > 0$. Because $\mu_2^n \to \mu_2$ and $N_\delta\big(\operatorname{supp}\{\mu_2\}\big)$ is open, we have
$$\liminf \mu_2^n \Big(N_\delta \big(\operatorname{supp}\{\mu_2\} \big) \Big) \geq \mu_2 \Big(N_\delta \big(\operatorname{supp}\{\mu_2\} \big) \Big) = 1.$$
It follows that there exists N such that
$$\mu_2^n \Big(\operatorname{cl} N_\delta \big(\operatorname{supp}\{\mu_2\} \big) \Big) \geq \mu_2^n \Big(N_\delta \big(\operatorname{supp}\{\mu_2\} \big) \Big) > 1 - \epsilon$$
for every $n \geq N$. $\|$

Step 1. For some $\alpha > 0$ and any (f_n) with
$$f_n \in \arg\max \left\{ t(y) : t \in \operatorname{supp}\{\mu_2^n\} \cap \operatorname{cl} N_\alpha \Big(\arg\max \big\{ t(y) : t \in \operatorname{supp}\{\mu_2\} \big\} \Big) \right\},$$ [24]

[23] The set $N_\delta\big(\operatorname{supp}\{\mu_2\}\big)$ designates $\bigcup_{t \in \operatorname{supp}\{\mu_2\}} N_\delta(t)$.

[24] Since $\mu_2^n \to \mu_2$, $\operatorname{supp}\{\mu_2^n\} \cap \operatorname{cl} N_\alpha \Big(\arg\max \big\{ t(y) : t \in \operatorname{supp}\{\mu_2\} \big\} \Big)$ is non-empty for n sufficiently high.

$\varepsilon, \gamma > 0$, a subsequence (n_k) of (n), and J may be found such that, for every $k \geq J$, $f^*_{n_k} \in \mathcal{T}(H)$ and $\mathcal{B}_{n_k} \subseteq \mathcal{T}(H)$ with $\mu_2^{n_k}(\mathcal{B}_{n_k}) > \gamma$ exist such that $f^*_{n_k} < f_{n_k} - \varepsilon$ on $(y, 1] \cap \operatorname{supp}\{p_H\}$, $f^*_{n_k} = 0$ on $[0, y) \cap \operatorname{supp}\{p_H\}$, and

$$u_1(\tilde{f}_{n_k}, t) > u_1(f_{n_k}, t) + \varepsilon$$

for each $\tilde{f}_{n_k} \in \Big\{\tilde{t} \in N_\beta(f^*_{n_k}) : \tilde{t} = 0 \text{ on } [0, y) \cap \operatorname{supp}\{p_H\}\Big\}$, every $t \in \mathcal{B}_{n_k}$, and some $\beta > 0$.

Proof. Let

$$\overline{\tau} := \sup \arg\max \Big\{t(y) : t \in \operatorname{supp}\{\mu_2\}\Big\}$$

and

$$\underline{\tau} := \inf \arg\max \Big\{t(y) : t \in \operatorname{supp}\{\mu_2\}\Big\}.$$

Set $\overline{y} := \min\Big\{x : x \in \operatorname{supp}\{p_H\} \cap (y, 1]\Big\}$ and $\underline{y} := \max\Big\{x : x \in \operatorname{supp}\{p_H\} \cap [0, y)\Big\}$. It is easily seen that $\mu_i(\mathcal{T}_{conv}(H)) = 1$ and $y < x_F$ imply $\overline{\tau}(y) < y - \underline{y}$ and

$$\overline{\tau}(y) < \underline{\tau}(\overline{y}) \leq \overline{\tau}(\overline{y}) < \overline{\tau}(y) + \overline{y} - y.$$

Fix m and α with $m > \max\Big\{2, \frac{1 + p_H\{[0,y)\}}{p_H\{y\}}\Big\}$ and

$$0 < \alpha < \frac{1}{m+2} \min\Big\{y - \underline{y} - \overline{\tau}(y), \underline{\tau}(\overline{y}) - \overline{\tau}(y), \overline{\tau}(y) + \overline{y} - y - \overline{\tau}(\overline{y})\Big\}.$$

For each n, let

$$f_n \in \arg\max\Big\{t(y) : t \in \operatorname{supp}\{\mu_2^n\} \cap \operatorname{cl} N_\alpha\Big(\arg\max\Big\{t(y) : t \in \operatorname{supp}\{\mu_2\}\Big\}\Big)\Big\}.$$

Pick $\eta \in \Big(0, \min\Big\{\alpha, \frac{1}{2}(1 - x_F)\Big\}\Big)$. Let

$$\epsilon_{\eta/4} := \min\Big\{H\Big(a + \frac{\eta}{4}\Big) - H(a) : a \in \Big[x_F, 1 - \frac{\eta}{4}\Big]\Big\}.$$

Since H is strictly increasing on $[x_F, 1]$, $\epsilon_{\eta/4} > 0$. Take ε with

$$0 < \varepsilon < \frac{1}{2} \min\Big\{\eta, \epsilon_{\eta/4}\Big\}.$$

Define, for each n,

$$f_n^*(x) := \begin{cases} 0 & \text{if } 0 \le x < \underline{y}, \\ \frac{\overline{\tau}(y)+m\alpha}{y-\underline{y}}(x-\underline{y}) & \text{if } \underline{y} \le x < y, \\ \overline{\tau}(y)+m\alpha+\frac{f_n(\overline{y})-\eta-\overline{\tau}(y)-m\alpha}{\overline{y}-y}(x-y) & \text{if } y \le x < \overline{y}, \\ f_n(x)-\eta & \text{if } \overline{y} \le x \le 1, \end{cases}$$

$$\mathcal{S}_n := \Big\{t \in \operatorname{supp}\{\mu_2^n\} : t \text{ intersects with } \tfrac{1}{2}f_n^* + \tfrac{1}{2}f_n \text{ in } [x_F, 1]\Big\},$$

$$\overline{\mathcal{S}}_n := \Big\{t \in \operatorname{supp}\{\mu_2^n\} : t > \tfrac{1}{2}f_n^* + \tfrac{1}{2}f_n \text{ on } [x_F, 1]\Big\},$$

$$\underline{\mathcal{S}}_n := \Big\{t \in \operatorname{supp}\{\mu_2^n\} : t < \tfrac{1}{2}f_n^* + \tfrac{1}{2}f_n \text{ on } [x_F, 1]\Big\},$$

$$\mathcal{O}_n := \Big\{t \in \operatorname{supp}\{\mu_2^n\} : t(y) > f_n(y)\Big\}.$$

Being a closed subset of the compact space $\mathcal{T}(H)$ (Lemma 7.1),

$$\operatorname{cl} N_\alpha\Big(\arg\max\Big\{t(y) : t \in \operatorname{supp}\{\mu_2\}\Big\}\Big)$$

is itself compact. Because (f_n) lies in $\operatorname{cl} N_\alpha\Big(\arg\max\Big\{t(y) : t \in \operatorname{supp}\{\mu_2\}\Big\}\Big)$, this set contains the limit point f of some subsequence of (f_n). Without loss of generality, let $f_n \to f$. Similarly, we may write (passing to a subsequence if necessary) $f_n^* \to f^*$ for some $f^* \in \mathcal{T}(H)$.

It is easily seen that $x_F \le m_F$ implies the existence of $\theta > 0$ with $u_1(f_n, t) < -\theta$ for each $t \in \underline{\mathcal{S}}_n$ and every n and $u_1(\tau, t) > \theta$ for each

$$(\tau, t) \in \left(\Big\{\tilde{t} \in N_\theta(f_n^*) : \tilde{t} = 0 \text{ on } [0, y) \cap \operatorname{supp}\{p_H\}\Big\} \cup \underline{\mathcal{S}}_n\right) \times \overline{\mathcal{S}}_n$$

and every n. Since $f^* \to f^*$, $f_n^* \in N_{\theta/2}(f^*)$ for every $n \ge L$ and some L. Therefore, $u_1(\tau, t) > \theta$ for each

$$(\tau, t) \in \Big\{\tilde{t} \in N_{\theta/2}(f^*) : \tilde{t} = 0 \text{ on } [0, y) \cap \operatorname{supp}\{p_H\}\Big\} \times \overline{\mathcal{S}}_n$$

and every $n \ge L$. Now Lemma 7.8 gives $f^\circ \in \widetilde{\mathcal{T}}(H)$ with $u_1(f^\circ, t) > \theta$ for each $t \in \overline{\mathcal{S}}_n$ and every $n \ge L$.

If $\mu_2^n(\mathcal{S}_n) \to 0$ and $\mu_2^n(\overline{\mathcal{S}}_n) \to 0$, then $\mu_2^n(\underline{\mathcal{S}}_n) \to 1$, and so

$$U_1(f_n, \mu_2^n) = \sum_{t \in \underline{\mathcal{S}}_n} \mu_2^n(t) u_1(f_n, t) + \sum_{t \in \mathcal{S}_n \cup \overline{\mathcal{S}}_n} \mu_2^n(t) u_1(f_n, t)$$

$$
\begin{aligned}
&\leq \mu_2^n(\underline{\mathcal{S}}_n)\Big(\max_{t\in\underline{\mathcal{S}}_n} u_1(f_n,t)\Big) + \mu_2^n\Big(\mathcal{S}_n \cup \overline{\mathcal{S}}_n\Big)\Big(\max_{t\in\mathcal{S}_n\cup\overline{\mathcal{S}}_n} u_1(f_n,t)\Big)\\
&< -\theta\mu_2^n(\underline{\mathcal{S}}_n) + \mu_2^n\Big(\mathcal{S}_n \cup \overline{\mathcal{S}}_n\Big)\Big(\max_{t\in\mathcal{S}_n\cup\overline{\mathcal{S}}_n} u_1(f_n,t)\Big)\\
&\to -\theta < 0,
\end{aligned}
$$

thereby contradicting that (μ_2^n, μ_2^n) is a Nash equilibrium of $G(\mathcal{A}_n)$.

If $\mu_2^n(\mathcal{S}_n) \to 0$ and $\mu_2^n(\overline{\mathcal{S}}_n) \not\to 0$, there must be a subsequence $\Big(\mu_2^{n_k}(\overline{\mathcal{S}}_{n_k})\Big)$ with $\mu_2^{n_k}(\overline{\mathcal{S}}_{n_k}) \to \rho$ for some $\rho > 0$. Without loss of generality, let $\Big(\mu_2^n(\overline{\mathcal{S}}_n)\Big)$ be one such subsequence. Either $\rho = 1$ or $0 < \rho < 1$.

If $\rho = 1$, then

$$
\begin{aligned}
U_1(f^\circ, \mu_2^n) &= \sum_{t\in\overline{\mathcal{S}}_n} \mu_2^n(t)u_1(f^\circ, t) + \sum_{t\in\mathcal{A}_n\setminus\overline{\mathcal{S}}_n} \mu_2^n(t)u_1(f^\circ, t)\\
&\to \theta > 0.
\end{aligned}
$$

But since $(\mathcal{A}_n)$ is an increasing sequence of sets whose union is $\widetilde{\mathcal{T}}(H)$ and $f^\circ \in \widetilde{\mathcal{T}}(H)$, there exists l with $f^\circ \in \mathcal{A}_l$ and $U_1(f^\circ, \mu_2^l) > 0$, contradicting that (μ_2^l, μ_2^l) is a Nash equilibrium of $G(\mathcal{A}_l)$.

If, on the other hand, $0 < \rho < 1$, then we may write (passing to a subsequence if necessary) $\mu_2^n(\underline{\mathcal{S}}_n) \to \varrho$ for some $\varrho > 0$. Define, for each n, the probability measure $\tilde{\mu}_2^n$ on $\mathcal{A}_n$ as follows:

$$
\tilde{\mu}_2^n(\mathcal{B}) := \frac{\mu_2^n(\mathcal{B} \cap \underline{\mathcal{S}}_n)}{\mu_2^n(\underline{\mathcal{S}}_n)}.
$$

Then,

$$
\begin{aligned}
&U_1(\tilde{\mu}_2^n, \mu_2^n)\\
&\quad= \sum_{\tau\in\mathcal{A}_n} \tilde{\mu}_2^n(\tau)\Big(\sum_{t\in\mathcal{S}_n} \mu_2^n(t)u_1(\tau,t) + \sum_{t\in\overline{\mathcal{S}}_n} \mu_2^n(t)u_1(\tau,t) + \sum_{t\in\underline{\mathcal{S}}_n} \mu_2^n(t)u_1(\tau,t)\Big)\\
&\quad= \sum_{\tau\in\underline{\mathcal{S}}_n} \tilde{\mu}_2^n(\tau)\Big(\sum_{t\in\mathcal{S}_n} \mu_2^n(t)u_1(\tau,t) + \sum_{t\in\overline{\mathcal{S}}_n\cap\mathcal{O}_n} \mu_2^n(t)u_1(\tau,t)\\
&\qquad + \sum_{t\in\overline{\mathcal{S}}_n\setminus\mathcal{O}_n} \mu_2^n(t)u_1(\tau,t) + \sum_{t\in\underline{\mathcal{S}}_n\cap\mathcal{O}_n} \mu_2^n(t)u_1(\tau,t) + \sum_{t\in\underline{\mathcal{S}}_n\setminus\mathcal{O}_n} \mu_2^n(t)u_1(\tau,t)\Big)\\
&\quad= \frac{1}{\mu_2^n(\underline{\mathcal{S}}_n)} \sum_{\tau\in\underline{\mathcal{S}}_n\cap\mathcal{O}_n} \sum_{t\in\mathcal{A}_n} \mu_2^n(\tau)\mu_2^n(t)u_1(\tau,t)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{\mu_2^n(\underline{\mathcal{S}}_n)} \sum_{\tau \in \underline{\mathcal{S}}_n \setminus \mathcal{O}_n} \mu_2^n(\tau) \Bigg(\sum_{t \in \mathcal{S}_n} \mu_2^n(t) u_1(\tau, t) + \sum_{t \in \overline{\mathcal{S}}_n \cap \mathcal{O}_n} \mu_2^n(t) u_1(\tau, t) \\
&+ \sum_{t \in \overline{\mathcal{S}}_n \setminus \mathcal{O}_n} \mu_2^n(t) u_1(\tau, t) + \sum_{t \in \underline{\mathcal{S}}_n \cap \mathcal{O}_n} \mu_2^n(t) u_1(\tau, t) + \sum_{t \in \underline{\mathcal{S}}_n \setminus \mathcal{O}_n} \mu_2^n(t) u_1(\tau, t) \Bigg) \\
> &\frac{1}{\mu_2^n(\underline{\mathcal{S}}_n)} \sum_{\tau \in \underline{\mathcal{S}}_n \cap \mathcal{O}_n} \sum_{t \in \mathcal{A}_n} \mu_2^n(\tau) \mu_2^n(t) u_1(\tau, t) \\
&+ \frac{\theta}{\mu_2^n(\underline{\mathcal{S}}_n)} \mu_2^n\Big(\underline{\mathcal{S}}_n \setminus \mathcal{O}_n\Big) \mu_2^n\Big(\overline{\mathcal{S}}_n \setminus \mathcal{O}_n\Big) + \frac{1}{\mu_2^n(\underline{\mathcal{S}}_n)} \sum_{\tau \in \underline{\mathcal{S}}_n \setminus \mathcal{O}_n} \mu_2^n(\tau) \Bigg(\sum_{t \in \mathcal{S}_n} \mu_2^n(t) u_1(\tau, t) \\
&+ \sum_{t \in \overline{\mathcal{S}}_n \cap \mathcal{O}_n} \mu_2^n(t) u_1(\tau, t) + \sum_{t \in \underline{\mathcal{S}}_n \cap \mathcal{O}_n} \mu_2^n(t) u_1(\tau, t) \Bigg) \\
\to &\frac{\theta \varrho \rho}{\varrho} > 0,
\end{aligned}
$$

for $\mu_2^n(\mathcal{S}_n) \to 0$ by assumption, $\mu_2^n(\mathcal{O}_n) \to 0$, $\mu_2^n(\underline{\mathcal{S}}_n \setminus \mathcal{O}_n) \to \varrho$, and $\mu_2^n(\overline{\mathcal{S}}_n \setminus \mathcal{O}_n) \to \rho$. Thus, a contradiction is obtained in this case as well.

We conclude that $\mu_2^n(\mathcal{S}_n) \not\to 0$, and so a subsequence $\Big(\mu_2^{n_k}(\mathcal{S}_{n_k})\Big)$ may be obtained that satisfies $\mu_2^{n_k}(\mathcal{S}_{n_k}) \to \lambda$ for some $\lambda > 0$. Without loss of generality, let $\Big(\mu_2^n(\mathcal{S}_n)\Big)$ be one such subsequence. Because $\mu_2^n(\mathcal{S}_n) \to \lambda$ and $\mu_2^n(\mathcal{O}_n) \to 0$, $\mu_2^n(\mathcal{S}_n \setminus \mathcal{O}_n) > \gamma$ for all $n \geq K$, some K, and some $\gamma > 0$. Define $\mathcal{B}_n := \mathcal{S}_n \setminus \mathcal{O}_n$ for each n. Choose $\beta \in (0, \eta/8)$.

Fix any $n \geq K$. One can show that $f_n^* \in \mathcal{T}(H)$.[25] Further, we have $\mu_2^n(\mathcal{B}_n) > \gamma$ because $n \geq K$. On the other hand, the definitions of ε and f_n^* entail $f_n^* = 0$ on $[0, y) \cap \operatorname{supp}\{p_H\}$ and $f_n^* < f_n - \varepsilon$ on $(y, 1] \cap \operatorname{supp}\{p_H\}$. Finally, given an arbitrary

$$\tilde{f}_n \in \Big\{\tilde{t} \in N_\beta(f_n^*) : \tilde{t} = 0 \text{ on } [0, y) \cap \operatorname{supp}\{p_H\}\Big\}$$

and $t \in \mathcal{B}_n$, t intersects with $\frac{1}{2}f_n^* + \frac{1}{2}f_n$ in $[x_F, 1]$, say at z. We consider the case where $\Big[z, z + \frac{\eta}{4}\Big] \subseteq [x_F, 1]$ (if this were not the case, we would have $\Big[z - \frac{\eta}{4}, z\Big] \subseteq [x_F, 1]$ and a similar argument would work). Since the right-derivative of t does not exceed 1 everywhere on this map's domain, we have

$$f_n^* = f_n - \eta < t < f_n \text{ on } I := \Big[z, z + \tfrac{\eta}{4}\Big].$$

Therefore, letting I^c be the complement of I in $[0, 1]$,

$$\begin{aligned}
u_1(\tilde{f}_n, t) &= p_H\{\tilde{f}_n < t\} - p_H\{t < \tilde{f}_n\} \\
&= p_H\Big\{\{\tilde{f}_n < t\} \cap I\Big\} + p_H\Big\{\{\tilde{f}_n < t\} \cap I^c\Big\} - p_H\{t < \tilde{f}_n\} \\
&\geq H\Big(z + \frac{\eta}{4}\Big) - H(z) + p_H\Big\{\{f_n < t\} \cap I^c\Big\} - p_H\{t < f_n\} \\
&\geq \epsilon_{\eta/4} + p_H\Big\{\{f_n < t\} \cap I^c\Big\} - p_H\{t < f_n\} \\
&> \varepsilon + p_H\Big\{\{f_n < t\} \cap I\Big\} + p_H\Big\{\{f_n < t\} \cap I^c\Big\} - p_H\{t < f_n\} \\
&> \varepsilon + u_1(f_n, t),
\end{aligned}$$

[25] Verifying that f_n^* is a tax function is immediate after the observation of the following facts. First, the definition of α entails $\frac{\overline{\tau}(y) + m\alpha}{y - \underline{y}} \leq 1$. Second,

$$f_n(\overline{y}) - \eta - \overline{\tau}(y) - m\alpha \geq \underline{\tau}(\overline{y}) - \alpha - \eta - \overline{\tau}(y) - m\alpha \geq 0$$

if $\underline{\tau}(\overline{y}) - \overline{\tau}(y) \geq (2+m)\alpha$, which holds true by the definition of α. Third, $\frac{f_n(\overline{y}) - \eta - \overline{\tau}(y) - m\alpha}{\overline{y} - y} \leq 1$ if

$$(1 - m)\alpha - \eta \leq \overline{y} - y + \overline{\tau}(y) - \overline{\tau}(\overline{y}),$$

which is true by the definitions of α and η. To see that f_n^* is an admissible tax function, observe that

$$\int_0^1 f_n^* dH \geq \int_0^1 f_n dH \geq R_H$$

if

$$\alpha\Big(p_H\{y\}(m - 1) - p_H\{[0, y)\}\Big) \geq \eta\Big(1 - p_H\{y\}\Big),$$

which holds true by the definitions of m, α, and η.

as we sought. ‖

Step 2. There exist $f^\bullet \in \widetilde{\mathcal{T}}(H)$ *and a subsequence* (ν_2^n) *of* (μ_2^n) *such that* $U_1(f^\bullet, \nu_2^n) > 0$ *for every* n.

Proof. For each n, choose

$$f_n \in \arg\max\Big\{t(y) : t \in \operatorname{supp}\{\mu_2^n\} \cap \operatorname{cl} N_\alpha\Big(\arg\max\big\{t(y) : t \in \operatorname{supp}\{\mu_2\}\big\}\Big)\Big\}.$$

Step 1 gives, for each n, $\varepsilon, \gamma > 0$, a subsequence (n_k) of (n), and J such that, for every $k \geq J$, $f_{n_k}^* \in \mathcal{T}(H)$ and $\mathcal{B}_{n_k} \subseteq \mathcal{T}(H)$ with $\mu_2^{n_k}\big(\mathcal{B}_{n_k}\big) > \gamma$ exist with $f_{n_k}^* < f_{n_k} - \varepsilon$ on $(y, 1] \cap \operatorname{supp}\{p_H\}$, $f_{n_k}^* = 0$ on $[0, y) \cap \operatorname{supp}\{p_H\}$, and

$$u_1(\tilde{f}_{n_k}, t) > u_1(f_{n_k}, t) + \varepsilon$$

for each $\tilde{f}_{n_k} \in \Big\{t \in N_\beta(f_{n_k}^*) : t = 0 \text{ on } [0, y) \cap \operatorname{supp}\{p_H\}\Big\}$, every $t \in \mathcal{B}_{n_k}$, and some $\beta > 0$. To ease notation, assume $(n_k) = (n)$. Because (f_n^*) lies in the compact space $\mathcal{T}(H)$ (Lemma 7.1), it contains a subsequence that converges in $\mathcal{T}(H)$. Without loss of generality, let (f_n^*) be one such subsequence, and denote its limit point by f^*.

Since $f_n^* \to f^*$, there exists M such that $f_n^* \in N_{\min\{\beta/2, \varepsilon/2\}}(f^*)$ for every $n \geq M$. Further, $f_n^* \to f^*$ and $f_n^* = 0$ on $[0, y) \cap \operatorname{supp}\{p_H\}$ for every n imply $f^* = 0$ on $[0, y) \cap \operatorname{supp}\{p_H\}$. It follows that, for every $n \geq \max\{J, M\}$, $f^* < f_n - \frac{\varepsilon}{2}$ on $(y, 1] \cap \operatorname{supp}\{p_H\}$, $f^* = 0$ on $[0, y) \cap \operatorname{supp}\{p_H\}$, and

$$u_1(\tilde{f}, t) > u_1(f_n, t) + \varepsilon$$

for each $\tilde{f} \in \Big\{t \in N_{\beta/2}(f^*) : t = 0 \text{ on } [0, y) \cap \operatorname{supp}\{p_H\}\Big\}$ and every $t \in \mathcal{B}_n$. This and Lemma 7.8 yield $f^\bullet \in \widetilde{\mathcal{T}}(H)$ such that, for each $n \geq \max\{J, M\}$,

$$f^\bullet < f_n \text{ on } (y, 1] \cap \operatorname{supp}\{p_H\}, \tag{9}$$

$$f^\bullet = 0 \text{ on } [0, y) \cap \operatorname{supp}\{p_H\}, \tag{10}$$

and

$$u_1(f^\bullet, t) > u_1(f_n, t) + \varepsilon \text{ for every } t \in \mathcal{B}_n. \tag{11}$$

The definition of each f_n, along with (9), (10), $f^\bullet, f_n \in \widetilde{\mathcal{T}}(H)$ for every n (so that $h(y) \neq f_n(y)$ for each $h \in \widetilde{\mathcal{T}}(H)$ and every n), and $H \in \mathcal{S}^*(H)$ (so that $p_H\{y\} < p_H\{(y, 1]\}$), implies

$$\begin{aligned} &u_1(f^\bullet, t) \geq u_1(f_n, t) \\ &\qquad \text{for each } t \in \operatorname{supp}\{\mu_2^n\} \setminus \mathcal{T}_n \text{ and every } n \geq \max\{J, M\}, \end{aligned} \tag{12}$$

where each $\mathcal{T}_n$ is defined as

$$\mathcal{T}_n := \operatorname{supp}\{\mu_2^n\} \setminus \operatorname{cl} N_\alpha\Big(\operatorname{supp}\{\mu_2\}\Big).$$

By Step 0, an N exists such that

$$\mu_2^n(\mathcal{T}_n) < \tfrac{\gamma\varepsilon}{4} \text{ for every } n \geq N.$$

From this, (11), (12), and $\mu_2^n(\mathcal{B}_n) > \gamma$ for every $n \geq J$, we obtain, for $n \geq \max\{J, M, N\}$,

$$\begin{aligned}
U_1(f^\bullet, \mu_2^n) &= \sum_{t \in \mathcal{B}_n} \mu_2^n(t) u_1(f^\bullet, t) + \sum_{t \in \mathcal{A}_n \setminus \mathcal{B}_n} \mu_2^n(t) u_1(f^\bullet, t) \\
&\geq \sum_{t \in \mathcal{B}_n} \mu_2^n(t)\Big(u_1(f_n, t) + \varepsilon\Big) + \sum_{t \in \mathcal{A}_n \setminus \mathcal{B}_n} \mu_2^n(t) u_1(f^\bullet, t) \\
&> \gamma\varepsilon + \sum_{t \in \mathcal{B}_n} \mu_2^n(t) u_1(f_n, t) + \sum_{t \in \mathcal{A}_n \setminus \mathcal{B}_n} \mu_2^n(t) u_1(f^\bullet, t) \\
&= \gamma\varepsilon + \sum_{t \in \mathcal{B}_n} \mu_2^n(t) u_1(f_n, t) \\
&\qquad + \sum_{t \in (\mathcal{A}_n \setminus \mathcal{B}_n) \setminus \mathcal{T}_n} \mu_2^n(t) u_1(f^\bullet, t) + \sum_{t \in (\mathcal{A}_n \setminus \mathcal{B}_n) \cap \mathcal{T}_n} \mu_2^n(t) u_1(f^\bullet, t) \\
&> \frac{\gamma\varepsilon}{2} + \sum_{t \in \mathcal{B}_n} \mu_2^n(t) u_1(f_n, t) \\
&\qquad + \sum_{t \in (\mathcal{A}_n \setminus \mathcal{B}_n) \setminus \mathcal{T}_n} \mu_2^n(t) u_1(f_n, t) + \sum_{t \in (\mathcal{A}_n \setminus \mathcal{B}_n) \cap \mathcal{T}_n} \mu_2^n(t) u_1(f_n, t) \\
&= \frac{\gamma\varepsilon}{2} + \sum_{t \in \mathcal{A}_n} \mu_2^n(t) u_1(f_n, t) \\
&= \frac{\gamma\varepsilon}{2} \\
&> 0,
\end{aligned}$$

where the last equality holds true because $f_n \in \operatorname{supp}\{\mu_2^n\}$ and (μ_2^n, μ_2^n) is a Nash equilibrium of $G(\mathcal{A}_n)$. $\|$

Let $(\mathcal{D}_n)$ be the subsequence of $(\mathcal{A}_n)$ corresponding to (ν_2^n) from Step 2. Since $f^\bullet \in \widetilde{\mathcal{T}}(H)$ and $(\mathcal{A}_n)$ is an increasing sequence whose union is $\widetilde{\mathcal{T}}(H)$, $f^\bullet \in \mathcal{D}_n$ for some n. This and Step 2 yield $f^\bullet \in \mathcal{D}_n$ and $U_1(f^\bullet, \nu_2^n) > 0$ for some n, thereby contradicting that (ν_2^n, ν_2^n) is a Nash equilibrium of $G(\mathcal{D}_n)$. $\|$

Proof of Theorem 5.4. Suppose that $F \in \mathcal{F}$ satisfies $x_F \leq m_F$. Take any $H \in \mathcal{S}^*(F)$ sufficiently close to F in the sense of Lemma 7.9. Let $(\mathcal{A}_n)$ be an increasing sequence of finite sets whose union is $\widetilde{\mathcal{T}}(H)$. To each n there corresponds a Nash equilibrium (μ_1^n, μ_2^n) of $G(\mathcal{A}_n)$.

The set $\widetilde{\mathcal{T}}(H)$ is a countable subset of $\mathcal{T}(H)$ that ensures, by virtue of Lemma 7.12, local payoff security of each U_i on $\mathbf{P}(\mathcal{T}(H))^2$. It follows from Lemma 7.14 that $\widetilde{\mathcal{T}}(H)$ is a strategic approximation to $G(\mathcal{T}(H))$, and so any limit of equilibria of the sequence of finite games $\Big(G(\mathcal{A}_n)\Big)$ is an equilibrium of $G(\mathcal{T}(H))$. Since $\mathbf{P}(\mathcal{T}(H))^2$ is sequentially compact (in fact, compact and metric), a subsequence of the sequence (μ_1^n, μ_2^n) exists that converges in $\mathbf{P}(\mathcal{T}(H))^2$. The limit (μ_1, μ_2) of one such subsequence is therefore a Nash equilibrium of $G(\mathcal{T}(H))$. Now Lemma 7.15 gives $\mu_i(\mathcal{T}_{conv}(H)) < 1$ for each i. $\|$

References

Aliprantis, C. D., and K. C. Border (1999). *Infinite dimensional analysis.* Berlin: Springer-Verlag.

Banks, J. S., Duggan, J., and M. Le Breton (2002), "Bounds for mixed strategy equilibria and the spatial model of elections," *Journal of Economic Theory*, 103, 88-105.

Baye, M. R., G. Tian, and J. Zhou (1993), "Characterizations of the existence of equilibria in games with discontinuous and non-quasiconcave payoffs," *Review of Economic Studies*, 60, 935-948.

Bénabou, R. (1996), "Inequality and growth," *NBER Macroeconimcs Annual 1996*, B. Bernanke and J. Rotemberg (eds.), MIT Press, Cambridge, pp. 11-74.

Billingsley, P. (1968). *Convergence of probability measures.* New York: John Wiley and Sons.

Dutta, B., and J.-F. Laslier (1999), "Comparison functions and choice correspondences," *Social Choice and Welfare*, 16, 513-532.

Carbonell-Nicolau, O., and E. F. Klor (2003), "Representative democracy and marginal rate progressive income taxation," *Journal of Public Economics*, 87, 2339-2366.

Cukierman, A., and A. H. Meltzer (1991), "A political theory of progressive taxation," in *Political Economy*, A. Meltzer, A. Cukierman, and S. F. Richard (eds.), Oxford University Press, Oxford, pp. 76-108.

Glicksberg, I. L. (1952), "A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points," *Proceedings of the American Mathematical Society*, 3, 170-174.

Gouveia, M., and D. Oliver (1996), "Voting over flat taxes in an endowment economy," *Economics Letters*, 50, 251-258.

Jackson, M. O., and J. M. Swinkels (2003), "Existence of equilibrium in single and double private value auctions," forthcoming in *Econometrica.*

Klor, E. F. (2003), "On the popular support for progressive taxation," *Journal of Public Economic Theory*, 5, 593-604.

Kramer, G. H. (1978), "Existence of electoral equilibrium," in *Game Theory and Political Science*, P. Ordeshook (ed.), New York University Press, New York, pp. 223-228.

Kreps, D. M., and J. A. Sheinkman (1983), "Quantity precommitment and Bertrand competition yield Cournot outcomes," *Bell Journal of Economics*, 14, 326-337.

Laslier, J.-F. (2000), "Interpretation of electoral mixed strategies," *Social Choice and Welfare*, 17, 283-292.

Laslier, J.-F., and N. Picard (2002), "Distributive politics and electoral competition," *Journal of Economic Theory*, 103, 106-130.

Lizzeri, A., and N. Persico (2001), "The provision of public goods under alternative electoral systems," *American Economic Review*, 91, 225-239.

Lizzeri, A., and N. Persico (2000), "The drawbacks of electoral competition," University of Pennsylvania Center for Analytic Research in Economics and Social Science (CARESS) Working Paper 00/02.

Marhuenda, F., and I. Ortuño-Ortín (1995), "Popular support for progressive income taxation," *Economics Letters*, 48, 319-324.

Marhuenda, F., and I. Ortuño-Ortín (1998), "Income taxation, uncertainty and stability," *Journal of Public Economics*, 67, 285-300.

Maskin, E. (1986), "The existence of equilibrium with price-setting firms," *American Economic Review*, 76, 382-386.

McKelvey, R. D. (1986), "Covering, dominance, and institution-free properties of social choice," *American Journal of Political Science*, 30, 283-314.

Meltzer, A. H., and S. F. Richard (1981), "A rational theory of the size of the government," *Journal of Political Economy*, 89, 914-927.

Mitra, T., and E. A. Ok (1996), "Personal income taxation and the principle of equal sacrifice revisited," *International Economic Review*, 37, 925-948.

Myerson, R. B. (1993), "Incentives to cultivate favored minorities under alternative electoral systems," *American Political Science Review* , 87, 856-869.

Osborne, M. J., and C. Pitchik (1986), "Price competition in a capacity-constrained duopoly," *Journal of Economic Theory*, 38, 238-260.

Reny, P. J. (1996), "Local payoff security and the existence of pure and mixed strategy Nash equilibria in discontinuous games," mimeo, Department of Economics, University of Pittsburgh.

Reny, P. J. (1999), "On the existence of pure and mixed strategy Nash equilibria in discontinuous games," *Econometrica*, 67, 1029-1056.

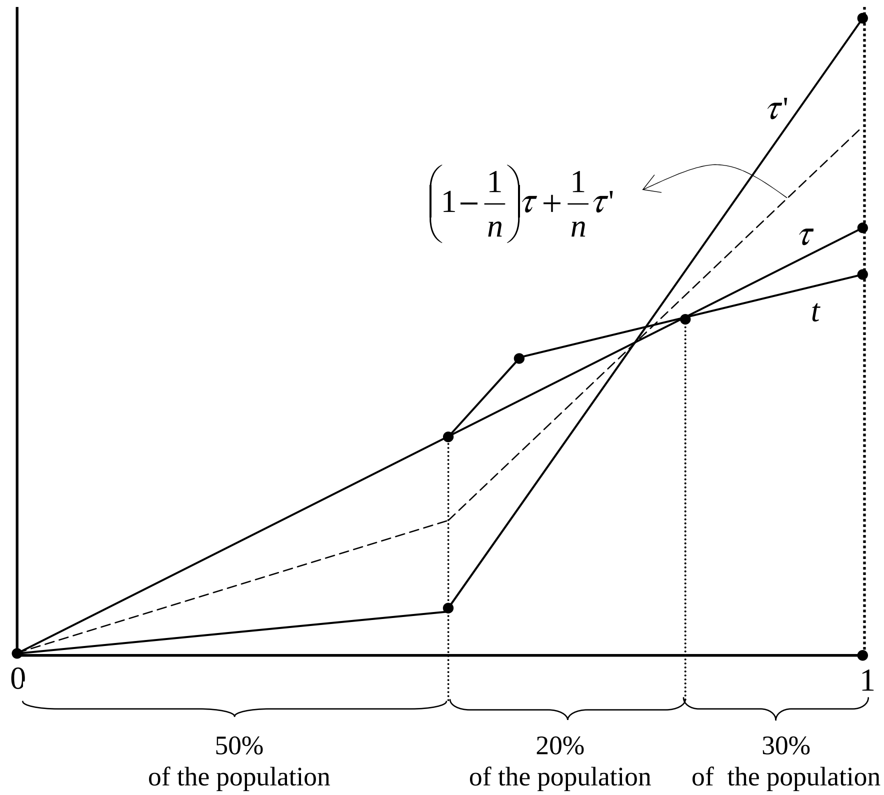
Roemer, J. E. (1999), "The democratic political economy of progressive income taxation," *Econometrica*, 67, 1-19.

Roberts, K. W. S. (1977), "Voting over income tax schedules," *Journal of Public Economics*, 8, 329-340.

Romer, T. (1975), "Individual welfare, majority voting, and the properties of a linear income tax," *Journal of Public Economics*, 4, 163-185.

Simon, L. K., and W. R. Zame (1990), "Discontinuous games and endogenous sharing rules," *Econometrica*, 58, 861-872.

Tan, K.-K., Y. Jian, and X.-Z. Yuan (1995), "Existence theorems of Nash equilibria for non-cooperative n-person games," *International Journal of Game Theory*, 24, 217-222.



τ'
$\left(1-\frac{1}{n}\right)\tau+\frac{1}{n}\tau'$
τ
t
0
1
50%
of the population
20%
of the population
30%
of the population

Figure 1

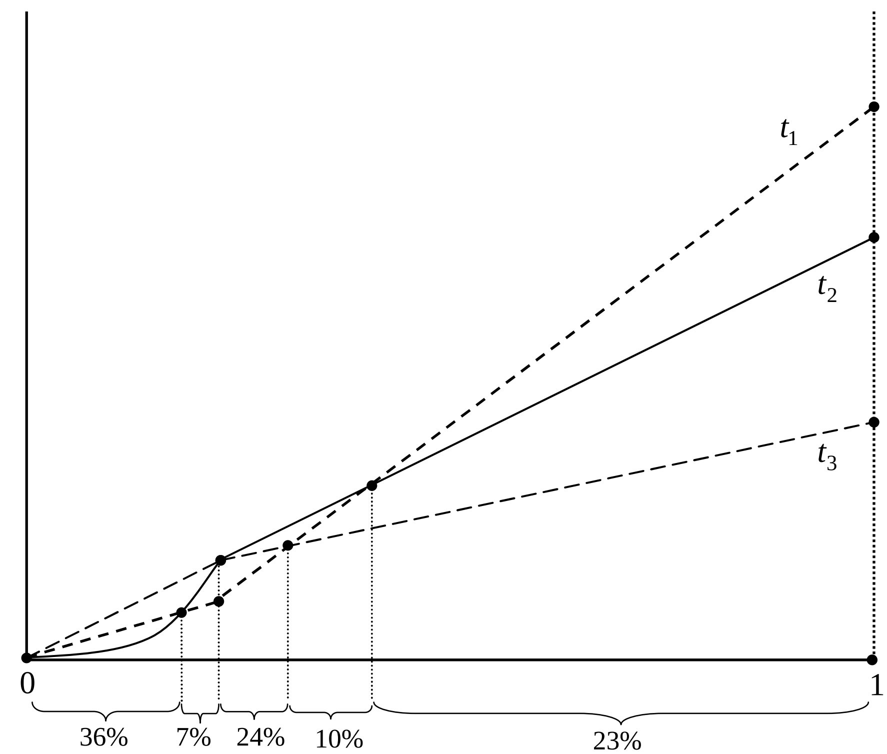


Figure 2