Testing Models of Low-Frequency Variability*

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Abstract

We develop a framework to assess how successfully standard time series models explain low frequency variability of a data series. The low-frequency information is extracted by computing a finite number of weighted averages of the original data, where the weights are low frequency trigonometric series. The properties of these weighted averages are then compared to the asymptotic implications of a number of common time series models. We apply the framework to twenty-one U.S. macroeconomic and financial time series using frequencies lower than the business cycle.

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1 Introduction

Persistence and low-frequency variability has been an important and ongoing empirical issue in macroeconomics and finance. Nelson and Plosser (1982) sparked the debate in macroeconomics by arguing that many macroeconomic aggregates follow unit-root autoregressions. Beveridge and Nelson (1981) used the logic of the unit-root model to extract stochastic trends from macro series, and showed that variations in these stochastic trends were a large, sometimes dominant, source of variability in the series. Meese and Rogoff's (1983) finding that the random walk model produces forecasts of exchange rates more accurate than other models of the time focused attention on the unit root model in international finance. And in finance, interest in the random walk model arose naturally because of its relation to the efficient markets hypothesis (Fama (1970)).

This empirical interest led to the development of econometric methods for testing the unit root hypothesis, and for estimation and inference in systems that contain integrated series. More recently, the focus has shifted towards more general models of persistence, such as the fractional (or long-memory) model and the local-to-unity autoregression, which nest the unit root model as a special case. While these models are designed to explain low-frequency behavior of time series, they also have implications for higher frequency variation. In fully parametric models, efficient statistical procedures thus exploit both low and high frequency variations for inference. This raises the natural concern about the robustness of such inference to alternative sources of higher frequency variability. These concerns have been addressed by, for example, constructing unit-root tests using AR models that are augmented with additional lags as in Said and Dickey (1984), by grafting stationary ARMA models onto fractional models as in Granger and Joyeux (1980), or by using various nonparametric estimators for long-run covariance matrices and (as in Geweke and Porter-Hudak (1983) (GPH)) for the fractional parameter.

As useful as these approaches are, there still remains a question of how successful these various methods are in controlling for unknown or misspecified high frequency variability. These concerns are evident from Monte Carlo experiments for unit root tests (e.g. Schwert (1989)) and long-run variance estimators (e.g. den Haan and Levin (1997)). In the typical asymptotic modelling of these issues, the difficulty is represented as problem of bandwidth choice: large bandwidths generate estimates with low variance, but with potentially large biases. Small bandwidths lead to more robust but potentially inefficient inference. In addition, for small bandwidths, the asymptotic approximation based on the bandwidth diverging to infinity becomes poor, so that alternative asymptotic nestings are required – see, for instance, Ng and Perron (2001) for the question of lag-length choice of unit root tests, and Kiefer and Vogelsang (2005) for long-run variance estimation.

In this paper, we address the issue of robustness directly. The approach investigated here *begins* by specifying the low-frequency band of interest. For example, in our empirical analysis, we focus on frequencies lower than the business cycle, that is periods greater than eight years. We extract the low frequency component of the series of interest by computing weighted averages of the data, where the weights are low frequency trigonometric series. Inference about the low-frequency variability of the series is exclusively based on the properties of these weighted averages, disregarding other aspects of the original data. Letting q denote the number of weighted averages that capture below business cycle frequency variability, we find that q is small in typical applications. For example, $q = 13$ for the post-war quarterly macroeconomic data studied here. This suggests basing inference on asymptotic approximations in which q is fixed as the sample sizes tends to infinity. Such asymptotics yield a q-dimensional multivariate Gaussian limiting distribution for the weighted averages, with a covariance matrix that depends on the specific model of low-frequency variability. Inference about alternative models or model parameters can thus draw on the well-developed statistical theory concerning multivariate normal distributions.

Several papers have addressed other empirical and theoretical questions in similar frameworks. Bierens (1997) derives estimation and inference procedures for cointegration relationships based on a finite number of weighted averages of the original data, with a joint Gaussian limiting distribution. Phillips (2006) pursues a similar approach with an infinite number of weighted averages. Phillips (1998) provides a theoretical analysis of 'spurious regressions' of various persistent time series on a finite (and also infinite) number of deterministic regressors. Müller (2004) finds that long-run variance estimators based on a finite number of outer products of trigonometrically weighted averages is optimal in a certain sense. All these approaches exploit the known asymptotic properties of weighted averages for a given stochastic model of low-frequency variability. In contrast, the focus of this paper is to test alternative models of low frequency variability (and their parameters) in a robust way.

The plan of the paper is as follows. In the next section we introduce the three classes of models that we will consider: fractional models, local-to-unity autoregressions, and the local-level model, parameterized as an unobserved components model with a large I(0) component and a small unit root component. We discuss the asymptotic properties of weighted averages for these models, our choice of weights and the tests we perform. Section 3 contains the empirical analysis for 21 U.S. series. Proofs are collected in an appendix.

2 Methodology

2.1 Models and Asymptotic Approximations

Let y_t, $t = 1, \cdots, T$ denote the observed time series, and consider the decomposition of y_t into unobserved deterministic and stochastic components

$$y_t = d_t + u_t.$$

Our attention will focus on the low-frequency variability of the stochastic component u_t. In this sense, the deterministic component is a nuisance and is modelled as a constant $d_t = \mu$, or as a constant plus linear trend $d_t = \mu + \beta t$, with unknown parameters μ and β.

Several different models have been proposed to model the low frequency variability of u_t for macroeconomic and financial time series, and we consider three leading models. The first is a fractional ("long-memory") model; stationary versions of the model yield a spectral density $S(\lambda) \propto |\lambda|^{-2d}$ as $\lambda \to 0$, where d is the fractional parameter. The second model is the AR model with largest root close to unity; using standard notation we will write the dominant AR coefficient as $\rho_T = (1 - c/T)$, so that the process is characterized by the local-to-unity parameter c. For this model, normalized versions of u_t behave like random variables generated by an Ornstein-Uhlenbeck process with diffusion parameter $-c$. The third model that we consider decomposes u_t into an $I(0)$ and $I(1)$ component, $u_t = w_t + (g/T) \sum_{s=1}^{t} \eta_t$, where $(w_t, \eta_t)'$ are $I(0)$ with long-run covariance matrix $\omega^2 I_2$, and g is a parameter that governs the relative importance of the $I(1)$ component. In this "Local Level" model (c.f. Harvey (1989)) both components are important for the low-frequency variability of u_t.

As we show below in Theorem 1, the low frequency variability implied by each of these models can be characterized by the stochastic properties of the partial sum process for u_t, so for our purposes it suffices to define each model in terms of the behavior of these partial sums. Letting W denote a Wiener process and ω a generic non-zero scalar constant, the specific assumptions for each model, and their integrated counterparts are given as

1(a) Stationary Fractional Model (*FR*): u_t follows a stationary fractional model with parameter $-1/2 < d < 1/2$, where $T^{-1/2-d}\sum_{t=1}^{[\cdot T]} u_t \Rightarrow \omega W^d(\cdot)$, where W^d is a "type I" fractional Brownian motion defined as $W^d(s) = A(d)\int_{-\infty}^{0}\left[(s-l)^d-(-l)^d\right]dW(l) + A(d)\int_0^s (s-l)^d dW(l)$, where $A(d) = \left(\frac{1}{2d+1}+\int_0^\infty\left[(1+l)^d-l^d\right]dl\right)^{-1/2}$.

1(b) Integrated fractional model: u_t follows a fractional model with parameter $1/2 < d < 3/2$, when the first differences $u_t - u_{t-1}$ (with $u_0 = 0$) follow a stationary fractional model with parameter $d-1$.

2 Local-to-unity model (*OU*): u_t follows $u_t = \rho_T u_{t-1} + \eta_t$, $\rho_T = 1 - c/T$ and $T^{-1/2}\sum_{t=1}^{[\cdot T]}\eta_t \Rightarrow \omega W(\cdot)$. Assumptions about the initial condition, u_0, depend on whether the model is stable ($c > 0$) or not ($c \leq 0$). In the stable model, u_0 is drawn from the stationary limiting distribution and $T^{-1/2}u_{[\cdot T]} \Rightarrow \omega J_c(s)$; the stationary Ornstein-Uhlenbeck (*OU*) process $J_c(s) = Ze^{-sc}/\sqrt{2c} + \int_0^s e^{-c(s-l)}dW(l)$, with $Z \sim \mathcal{N}(0,1)$ independent of W. In the unstable model ($c \leq 0$), $u_0 = 0$, and $T^{-1/2}u_{[sT]} \Rightarrow \omega\int_0^s e^{-c(s-l)}dW(l)$.

3 Integrated local-to-unity model (*I-OU*): u_t follows an integrated local-to-unity model with parameter c if the first differences $u_t - u_{t-1}$ (with $u_0 = 0$) follow a local-to-unity model, where for simplicity we restrict the analysis to the stable model with $c > 0$.

4 Local-Level model (*LL*): u_t follows a local level model with parameter $g \geq 0$, when $u_t = w_t + \frac{g}{T}\sum_{s=1}^{t}\eta_s$, and $(T^{-1/2}\sum_{t=1}^{[\cdot T]} w_t, T^{-1/2}\sum_{t=1}^{[\cdot T]}\eta_t)' \Rightarrow \omega(W_1(\cdot), W_2(\cdot))'$, where W_1 and W_2 are independent Wiener processes.

5 Integrated local-level model (*I-LL*): u_t follows an integrated local-level model with parameter $g \geq 0$ if the first differences $u_t - u_{t-1}$ (with $u_0 = 0$) follow a local level

model with parameter g.

Strictly speaking, the specifications (2)-(5) require u_t and y_t to be modelled as double arrays, but we omit any dependence on T to ease notation.

Table 1 summarizes the assumptions about convergence of the partial sum process for each model and provides and a description of the covariance kernel of the limiting process. A large number of primitive conditions have been used to justify these convergences. Specifically, for the stationary fractional model (1a), weak convergence to the fractional Wiener process W^d has been established under various primitive conditions for u_t by Taqqu (1975) and Chan and Terrin (1995)—see Marinucci and Robinson (1999) for additional references and discussion. Mandelbrot and Ness (1968) showed that W^d so defined has almost surely continuous sample paths. Model (1b) uses Velasco's (1999) definition of a fractional process for $1/2 < d < 3/2$. The local-to-unity model (2) and local level model (4) rely on a Functional Central Limit Theorem applied to $(w_t\ \eta_t)'$; various primitive conditions are given, for example, in McLeish (1974), Woolridge and White (1988), Phillips and Solo (1992), and Davidson (2002); see Stock (1994) for general discussion.

The unit root and $I(0)$ models are nested in several of the models in Table 1. The unit root model corresponds to the integrated fractional model (1b) with $d = 1$, the local-to-unity model (2) with $c = 0$, and the integrated local-level model (5) with $g = 0$. Similarly, the $I(0)$ model corresponds to the stationary fractional model (1a) with $d = 0$ and the local-level model (4) with $g = 0$.

The objective of this paper is to assess how well these specifications do in modelling the low-frequency variability of u_t. Since the deterministic component d_t is unknown, we restrict attention to statistics that are functions of the least-square residuals of a regression of y_t on a constant (denoted u_t^μ) or on a constant and time trend (denoted u_t^τ). Because $\{u_t^i\}_{t=1}^T$, $i = \mu, \tau$ are maximal invariants to the groups of transformations $\{y_t\}_{t=1}^T \to \{y_t + m\}_{t=1}^T$ and $\{y_t\}_{t=1}^T \to \{y_t + m + bt\}_{t=1}^T$, respectively, there is no loss of generality in basing inference on functions of $\{u_t^i\}_{t=1}^T$ for tests that are invariant to these transformations.

We extract the information about the low frequency variability of u_t by considering a finite number of weighted averages of u_t^i, $i = \mu, \tau$, where the weights are known and deterministic low-frequency trigonometric series. We discuss specific choices for these functions below, but first provide a general result about the asymptotic properties of these weighted averages. Here and below, the limits of integrals are understood to be zero and one, if not indicated otherwise.

Theorem 1 *Suppose there exists* α *and* $\omega > 0$ *such that* $T^{-\alpha}\sum_{t=1}^{[\cdot T]} u_t \Rightarrow \omega G(\cdot)$, *where* G *is a mean-zero Gaussian process with almost surely continuous sample paths and* $k(r,s) = E[G(r)G(s)]$. *Define*

$$\begin{aligned} k^{\mu}(r,s) &= k(r,s) - rk(1,s) - sk(r,1) + rsk(1,1) \qquad (1) \\ k^{\tau}(r,s) &= k^{\mu}(r,s) - 6s(1-s)\int k^{\mu}(r,l)dl \\ &\quad -6r(1-r)\int k^{\mu}(l,s)dl + 36rs(1-s)(1-r)\int\int k^{\mu}(l,\lambda)dld\lambda, \end{aligned}$$

and let $\Psi(\cdot) = (\Psi_1(\cdot), \cdots, \Psi_q(\cdot))'$, *where* $\Psi_l : [0,1] \mapsto \mathbb{R}$, $l = 1, \cdots, q$, *are functions with continuous derivative* ψ_l. *Then*

$$X_T \equiv T^{-\alpha}\sum_{t=1}^{T} \Psi(t/T)u_t^i \Rightarrow \mathcal{N}(0, \omega^2\Sigma)$$

where the j,l*th element of* Σ *is given by*

$$\int\int \psi_j(r)\psi_l(s)k^i(r,s)drds.$$

The joint distribution of the q weighted averages of u_t^i, $i = \mu$, τ is thus asymptotically normal with a covariance matrix that is—up to scale—determined by the covariance kernel of G. As Theorem 1 demonstrates, for an asymptotically justified analysis of the weighted averages X_T, the only relevant property of the models (1)-(5) is the limiting behavior of appropriately scaled partial sums of u_t.

If X_T captures the information in y_t about the low-frequency variability of u_t, then the question of model fit for a specific stochastic model simply becomes the question whether X_T is approximately distributed $\mathcal{N}(0, \omega^2\Sigma)$. For the models introduced above, $\Sigma = \Sigma(\theta)$ is a known function of the model parameter $\theta \in \{d, c, g\}$ for the fractional, local-to-unity, and long memory model respectively, and ω^2 is an unknown constant governing the low-frequency scale of the process. (For example, ω^2 is the long-run variance of η_t in the local-to-unity model.) Because q is finite, that is our asymptotics keep q fixed as $T \to \infty$, it is not possible to estimate ω^2 consistently using the q elements in X_T. This suggests restricting attention to scale invariant tests of X_T. Imposing scale invariance has the additional advantage that the value of α in $X_T = T^{-\alpha} \sum_{t=1}^{T} \Psi(t/T) u_t^i$ does not need to be known.

Thus, consider the following maximal invariant to the group of transformation $X_T \to cX_T$, $c \neq 0$,

$$v_T = X_T / \sqrt{X_T' X_T}.$$

If $X \sim \mathcal{N}(0, \omega^2 \Sigma)$ then the density of $v = (v_1, \cdots, v_q)' = X/\sqrt{X'X}$ with respect to the uniform measure on the surface of a q dimensional unit sphere is proportional to (see, for instance, Kariya (1980) or King (1980))

$$f_v(\Sigma) = |\Sigma|^{-1/2} \left(v' \Sigma^{-1} v\right)^{-q/2}. \tag{2}$$

For a given model for u_t, the asymptotic distribution of v_T depends only on q and the model parameter θ. Our strategy therefore is to base inference about the appropriateness of the asymptotic implication that X_T is approximately distributed $\mathcal{N}(0, \omega^2 \Sigma(\theta))$ for the models in Table 1 using tests based on v_T, $\Sigma(\theta)$, and the density (2).

2.2 Continuity of the Fractional and Local-to-Unity Models

Before discussing the choice of functions Ψ and test statistics, it is useful to take a short digression to discuss the continuity of $\Sigma(\theta)$ for two of the models. In the local-to-unity model, there is a discontinuity at $c = 0$ in our treatment of the initial condition and this leads to different covariance kernels in Table 1; similarly, in the fractional model there is a discontinuity at $d = 1/2$ as we move from the stationary to the integrated version of the model. As it turns out, these discontinuities do not lead to discontinuities of the density of v in (2) as a function of c and d.

This is easily seen in the local-to-unity model (2). Location invariance implies that it suffices to consider the asymptotic distribution of $T^{-1/2}(u_{[\cdot T]} - u_1)$. As noted by Elliott (1999), in the stationary model $T^{-1/2}(u_{[\cdot T]} - u_1) \Rightarrow J^c(\cdot) - J^c(0) = Z(e^{-sc} - 1)/\sqrt{2c} + \int_0^s e^{-c(s-l)} dW(l)$, and $\lim_{c \to 0} (e^{-sc} - 1)/\sqrt{2c} = 0$, so that the asymptotic distribution of $T^{-1/2}(u_{[\cdot T]} - u_1)$ is continuous at $c = 0$.

The calculation for the fractional model somewhat more involved. First, note that the density (2) of v remains unchanged when Σ is replaced by $\Sigma/\operatorname{tr}\Sigma$. Let $k^{\mu}_{FR(d)}(r,s)$ and $k^{\mu}_{I\text{-}FR(d)}(r,s)$ be defined as (1) for the covariance kernels of models (1a) and (1b) from Table 1, respectively. As shown in the appendix, the limits

$$\lim_{d\uparrow 1/2} \frac{k^{\mu}_{FR(d)}(r,s)}{1/2-d} \quad \text{and} \quad 2\lim_{d\downarrow 1/2} \frac{k^{\mu}_{I\text{-}FR(d)}(r,s)}{d-1/2} \tag{3}$$

exist and coincide, so that $\Sigma(d)/\operatorname{tr}\Sigma(d)$ can be continuously extended at $d=1/2$ in the mean and trend case. This result suggests a definition of a *demeaned* fractional process with $d=1/2$ as any process whose partial sums converge to a Gaussian process with covariance kernel (3). The possibility of a continuous extension across all values of d renders Velasco's (1999) definition of fractional processes with $d\in(1/2,3/2)$ as the partial sums of a stationary fractional process with parameter $d-1$ considerably more attractive, as it does not lead to a discontinuity at the boundary $d=1/2$, at least for demeaned data with appropriately chosen scale.

2.3 Choice of Ψ and the resulting $\Sigma(\theta)$

Recall that the functions $\Psi=(\Psi_1,\cdots,\Psi_q)'$ were introduced to extract the low-frequency variations of u_t uncontaminated by higher frequency variations. One natural measure of accuracy for a candidate Ψ is the R^2 of a continuous time regression of a generic periodic series $\sin(\pi rs+a)$ on $\Psi_1(s),\cdots,\Psi_q(s)$, where $r\geq 0$ and $a\in[0,\pi)$. Ideally, this R^2 would equal unity for $r\leq q$ and zero for $r>q$. Consider two choices of Ψ : $\Psi^{\cos}_l(s)=\sqrt{2}\cos(\pi ls)$, the first q functions of the Fourier cosine expansion (excluding the constant, since $\sum_{t=1}^T u^i_t=0$ for $i=\mu,\tau$); and $\Psi^{\text{Fourier}}_l(s)=\sqrt{2}\cos(2\pi\frac{l+1/2}{2}s)$ for $l=1,3,\cdots,q-1$ and $\Psi^{\text{Fourier}}_l(s)=\sqrt{2}\sin(2\pi\frac{l}{2}s)$ for $l=2,4,\cdots,q$, the first $q/2$ elements of the usual Fourier expansion (where we assume q is even for convenience).

Regressing $\sin(\pi r s + a)$ on these choices of Ψ yields the following values of R^2

$$\begin{aligned} R^2_{\cos} &= \frac{8r^3}{\pi(2\pi r + \sin(2a) - \sin(2(a+r\pi))}\sum_{l=1}^{q}\frac{(\cos(a) - (-1)^q\cos(a+r\pi))^2}{(r^2-l^2)^2} \\ R^2_{\text{Fourier}} &= \frac{8r^3(\cos(a) - \cos(a+r\pi))^2}{\pi(2\pi r + \sin(2a) - \sin(2(a+r\pi))}\sum_{l=1}^{q/2}\frac{1}{(r^2-4l^2)^2} \\ &\quad + \frac{8r(\sin(a) - \sin(a+r\pi))^2}{\pi(2\pi r + \sin(2a) - \sin(2(a+r\pi))}\sum_{l=1}^{q/2}\frac{4l^2}{(r^2-4l^2)^2} \end{aligned}$$

(with these expression extended by continuous limits at $r = 1, \cdots, q$). Evidently, $R^2_{\cos}$ and R^2_{Fourier} converge to zero as $r \to \infty$ for all fixed values of q and a, so that these choices for Ψ do not extract any high frequency information. To get a better sense of the properties for medium range values of r, Figure 1 depicts $R^2_{\cos}$ and R^2_{Fourier} as a function of r for two empirically relevant values $q = 14$ and $q = 32$. In the top panel, for each value of r, $R^2_{\cos}$ and R^2_{Fourier} are averaged over all values for the phase shift $a \in [0, \pi)$, in the middle panel, $R^2_{\cos}$ and R^2_{Fourier} are individually maximized over a and in the bottom panel, $R^2_{\cos}$ and R^2_{Fourier} are individually minimized over a. For both values of q, these functions come reasonably close to the ideal of extracting all information about cycles of frequency $r \leq q$ ($R^2 = 1$) and no information about cycles of frequency $r > q$ ($R^2 = 0$).

Based on these figures, there is little reason to choose one of the functions over the other. Of course, different choices yield different expressions for $\Sigma(\theta)$, and, as it turns out, the form of $\Sigma(\theta)$ implied by the cosine expansion is somewhat more convenient than the form yielded by the Fourier expansion. In particular, in the demeaned case with $d_t = \mu$, the cosine expansion produces values of $\Sigma(\theta)$ that are exactly diagonal for the $I(0)$ and unit root model, and nearly diagonal for the other models of empirical relevance. In contrast, the the off-diagonal elements $\Sigma(\theta)$ are large for many models using the Fourier expansion (c.f. Akdi and Dickey (1998) for an analysis of the unit root model).

Figure 2 plots the square roots of the diagonal elements of $\Sigma(\theta)$ for the cosine expansion for the fractional, local-to-unity, and local level models, for various values of $\theta = \{d, c, g\}$ in the demeaned case. Evidently, more persistent models produce larger variances for low-frequency components, a generalization of the familiar 'periodogram' intuition that for stationary u_t, the variance of $\sqrt{2/T}\sum_{t=1}^{T}\cos(\pi lt/T)u_t$ is an approximately unbiased estimator of the spectral density at frequency $l/2T$. For example, for the unit root model ($d = 1$ in the fractional model or $c = 0$ in the local-to-unity model), the standard deviation of X_1 is approximately 15 times larger than the standard deviation of X_{15}. In contrast, when $d = 0.25$ in the fractional model the relative standard deviation of X_1 falls to 2, and when $c = 5$ in the local-to-unity model, the relative standard deviation of X_1 is 7. In the $I(0)$ model ($d = 0$ in the fractional model or $g = 0$ in the local-level model), $\Sigma = I_q$, and all of the standard deviations are unity.

Table 2 shows the simple average of the absolute vales of all pair-wise correlations implied by $\Sigma(\theta)$ for $q = 15$ for the various models considered in Figure 2. The correlation is identically zero for the unit root and I(0) models, and is very nearly zero for the other parameter values.

Figure 2 and Table 2 summarize characteristics of $\Sigma(\theta)$ for the demeaned versions of the data; the covariance matrices for the detrended data share many of these characteristics. For example, the detrended data covariance matrix also exhibits pronounced heteroskedasticity, but with attenuated values for small values of l. As in the demeaned case, the off-diagonal elements of $\Sigma(\theta)$ for the detrended data are small in the empirically relevant models: all of entries in the version of Table 2 for detrended series are less than 0.04.

2.4 Test Statistics

As is evident in Figure 2 and Table 2, the major difference in the models involves the diagonal elements of $\Sigma(\theta)$. Said differently, the models imply different forms of heteroskedasticity for the elements of X_T. Thus, we begin by discussing tests of specific forms of heteroskedasticity against flexible alternatives. To be specific, let Σ_0 denote the value of Σ under a particular null model and parameter θ_0. We nest this value of Σ in the more general model $\Sigma = \Lambda\Sigma_0\Lambda$, where $\Lambda = \text{diag}(\exp(\delta_1), \cdots, \exp(\delta_q))$, and $\delta = (\delta_1, \cdots, \delta_q)'$ is a mean zero mean Gaussian vector with $E[\delta\delta'] = \gamma^2\Omega$. Under the null hypothesis $\gamma = 0$ and $\Sigma = \Sigma_0$; under the alternative $\gamma \neq 0$, and the deviation from the null depends on the realization of δ. The covariance matrix Ω determines which kind of deviations are more likely. Modelling δ as a random vector allows the alternative to flexibly capture a wide range of specific alternatives. Conditional on δ, the alternative covariance matrix $\Lambda\Sigma_0\Lambda$ has the lth diagonal elements multiplied by $\exp(2\delta_l)$ compared to Σ_0, while the correlation structure remains unchanged.

Formally, consider the null and alternative hypotheses

$$\begin{aligned} H_0 &: v \text{ has density } f_v(\Sigma_0) \\ H_1 &: v \text{ has density } E_\delta f_v(\Lambda \Sigma_0 \Lambda) \end{aligned}$$

where E_δ denotes integration over the measure of δ and f_v is defined in (2). Let e be a $q \times 1$ vector of ones, and ι_j the $q \times 1$ vector with a one in the jth row and zeros elsewhere. After calculations that closely mirror those of Nyblom (1989), one obtains that the locally best test at $\gamma = 0$ rejects for large values of

$$\text{LB} = (q/2+1)\frac{d'\Omega d}{(v'\Sigma_0^{-1}v)^2} + 2\frac{e'\Omega d}{v'\Sigma_0^{-1}v} - \frac{\text{tr}\, D\Omega}{v'\Sigma_0^{-1}v}$$

where d and D are a $q \times 1$ vector and a $q \times q$ matrix, respectively, with elements

$$\begin{aligned} d_j &= v_j \iota_j' \Sigma_0^{-1} v \\ D_{jl} &= \begin{cases} v_l v_j \iota_l' \Sigma_0^{-1} \iota_j \text{ for } l \neq j \\ v_j \iota_j' \Sigma_0^{-1} v + v_j^2 \iota_j' \Sigma_0^{-1} \iota_j \text{ for } l = j \end{cases} \end{aligned}$$

In the empirical work presented in the next section we consider three choices for Ω that correspond to the following stochastic specifications for δ: (i) a break at $[q/2+1]$, so that $\tilde{\delta}_l = \mathbf{1}[l > q/2]\varepsilon_1$, (ii) a Gaussian martingale model $\tilde{\delta}_l = \tilde{\delta}_{l-1} + \varepsilon_l$ with $\tilde{\delta}_0 = 0$ and (iii) a linear trend of random slope $\tilde{\delta}_l = \varepsilon_1 l$, where $\varepsilon_l \sim iid\mathcal{N}(0,1)$. In all three cases, $\delta_l = \tilde{\delta}_l - q^{-1}\sum_{j=1}^q \tilde{\delta}_j$, $l = 1, \cdots, l$, where the demeaning centers the alternative model for Σ at the null model. (The demeaning also results in a simplification of the statistic because $e'\Omega = 0$ for a demeaned δ.)

The critical value of the LB statistic can easily be computed by simulation, however, its null distribution depends on Σ_0. An alternative is to base the test instead on $v^* = Qv$ for some matrix Q satisfying $Q\Sigma_0 Q' = I_q$; in this case the null hypothesis about the covariance matrix Σ^* of v^* becomes $H_0 : \Sigma^* = I_q$. The empirical analysis in the next section uses this approach, where Q is chosen as the Cholesky factor of Σ_0. Because $\Sigma(\theta)$ is approximately diagonal for most empirically relevant models, there are only small differences between tests based on v and tests based on v^*.

For $\Sigma_0 = I_q$, $d_l = v_l^2$ and D becomes a diagonal matrix with elements $D_{ll} = 2v_l^2$, resulting in familiar formulae for the LB test statistic. For example, with q even, the LB statistic for a break becomes the usual F ratio that tests for a break in the mean of v_l^2. The test statistic for martingale variation (which we label as LBIM to reflect that fact that is the local best test with $\Sigma_0 = I_q$ for martingale variation) simplifies to

$$\text{LBIM} = (q/2+1)\sum_{l=1}^{q}\sum_{j=1}^{l}(v_j^2 - \overline{v^2})^2 - \frac{1}{3q}\sum_{l=1}^{q}[6l^2 - 6l(1+q) + (1+q)(1+2q)]v_l^2.$$

where $\overline{v^2} = q^{-1}\sum_{l=1}^{q} v_l^2$. The first term, which dominates the statistic for large q, is the usual Nyblom (1989) locally best test statistic for a martingale variation in the mean of v_l^2.

Thus far we have not discussed volatility in the underlying data y_t despite the large empirical literature documenting heteroskedasticity in financial data (e.g., Bollerslev, Engle, and Nelson (1994) and Andersen, Bollerslev, Christoffersen, and Diebold (2006)) and macroeconomic data (e.g., Balke and Gordon (1989), Kim and Nelson (1999) and McConnell and Perez-Quiros (2000)). The reason, of course, is the asymptotic results shown in Table 1 are robust to moderate amounts of heteroskedasticity. However, severe heteroskedasticity leads to changes in the form of $\Sigma(\theta)$, and it is interesting to test for these kind of alternatives. A simple example motivates the test that we will use: suppose that u_t is white noise with time varying variance $E[u_t^2] = \sigma^2(t/T) = 1 + 2a\cos(\pi t/T)$ for $|a| < 1/2$. For this process, the j, lth element of $E[X_T X_T']$ converges to $\int \Psi_l(s)\Psi_j(s)\sigma^2(s)ds$, and with $\Psi_l(s) = \sqrt{2}\cos(\pi ls)$, we obtain

$$\lim_{T\to\infty} E[X_T X_T'] = \begin{pmatrix} 1 & a & 0 & \cdots & 0 & 0 \\ a & 1 & a & \cdots & 0 & 0 \\ 0 & a & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 1 \end{pmatrix}$$

which is recognized as the covariance matrix of a MA(1) process. This suggests checking for serial correlation in v^* (the transformed version of X that eliminates any model specific correlation in $\Sigma(\theta)$). We do this using $\widehat{\rho} = \sum_{l=2}^{q} v_l^* v_{l-1}^* / \sum_{l=2}^{q}(v_l^*)^2$ which forms the basis of the (one-sided) best local test for the MA(1) or AR(1) alternative (Godfrey (1981), Kariya (1988)).

Finally, we test the unit root null hypothesis using a low-frequency point-optimal-invariant test. Specifically, in the context of the local-to-unity model we test the unit root model

$c = c_0 = 0$ versus an alternative model with $c = c_a$ using the likelihood ratio statistic

$$\text{LFUR} = v'\Sigma(c_0)^{-1}v/v'\Sigma(c_a)^{-1}v$$

where the values of c_a are those suggested by Elliott, Rothenberg, and Stock (1996) ($c_a = 7.5$ for demeaned series and $c_a = 13.5$ for detrended series). We label the statistic LFUR as a reminder that it is a low-frequency unit root test statistic.

3 Empirical Results

3.1 Data

In this section we study twenty-one macroeconomic and financial time series using the low-frequency methods discussed in the last section. We analyze post-war quarterly versions of important macroeconomic aggregates (real GDP, aggregate inflation, nominal and real interest rates, productivity, and employment) and longer annual versions of related series (real GNP from 1869-2004, nominal and real bond yields from 1900-2004, and so forth). We also study several cointegrating relations by analyzing differences between series (such as long-short interest rate spreads) or logarithms of ratios (such as consumption-income or dividend-price ratios). A detailed description of the data is given in the Data Appendix. As usual, several of the data series are transformed by taking logarithms, and as discussed above, the deterministic component of each series is modeled as a constant or a linear trend. Table A.1 summarize these transformations for each series.

3.2 Statistics Reported

The empirical results are summarized in Tables 3 and 4. Table 3 concentrates on "traditional" statistics for persistence. Specifically, it presents p-values for the DF-GLS unit root test of Elliott, Rothenberg, and Stock (1996), and estimates of the fractional parameter d using a version of the semi-parametric regression of Geweke and Porter-Hudak (1983) (GPH). Table 3 also reports the p-value for the low-frequency unit root test (LFUR) discussed in the last section. Table 4 presents results from the other tests presented in the last section and a summary of the likelihood based on v_T and density (2). For reasons discussed in introduction, these traditional statistics reported in Table 3 may (or may not) lead to valid inference about the models. We include them as useful benchmarks for comparison with the results in Table 4.

The remainder of the subsection summarizes the computational details used to produce these results, and the next subsection discusses the empirical results.

The DF-GLS tests were constructed using an autoregressions that included 4 lags of first differences of the data. Following the discussion in Robinson (2003), the GPH estimator is constructed as $-1/2$ times the OLS regression slope coefficient from the regression of the first m log-periodogram ordinates (excluding frequency zero) onto log-frequency and a constant, and the standard error is computed as $\pi^2/24m$. The estimator is implemented using $m = [n^\delta]$, where results are shown for $\delta = 0.5$ and 0.65. Because inference for the GPH estimator assumes stationarity ($-0.5 < d < 0.5$), results are shown for both the level and first difference of the series.

As discussed above, the data was transformed to the q observations $\sum_{t=1}^{T} \cos(l\pi t/T)u_t^i$, $l = 1, ..., q$, where $i = \mu, \tau$ (as indicated in Table A.1), and q was chosen to isolate frequencies lower than the business cycle. Using the standard 6-32 quarter definition of business cycle periodicity, this means that attention is restricted to frequencies lower than $2\pi/32$ for quarterly series and $2\pi/8$ for annual series. The post-war quarterly series span the period 1952:1-2004:4, so that $T = 212$, and $q = [2T/32] = 13$. Each annual time series is available for a different sample period (real GNP is available from 1869-2004, while bond rates are available from 1900-2004, for example), so the value of q is series-specific.

Confidence intervals (90%) are used Table 4 to summarize the results of the LBIM and $|\widehat{\rho}|$ tests. These confidence regions were computed numerically by inverting the relevant test using a fine grid of values of the relevant parameter. The values that are reported as the confidence regions are those parameters that are not rejected by the test using a 10% significance level. For the fractional model d was restricted to be in the range $-0.49 \leq d \leq 1.49$; for the local-to-unity models $-3.0 \leq c \leq 30.0$; and for the LLM models $0.0 \leq g \leq 30.0$.

Table 4 also presents a (crude) summary of the likelihood for each of the model based on v_T, normalized to unity for the I(0) model. It reports the maximum value of the log-likelihood for the ranges of parameters given in the last paragraph, the corresponding maximum likelihood estimate, and the set of parameter values whose likelihood value is within 2 log-points of the maximized likelihood.

3.3 Results

The purpose of the empirical analysis is to assess the empirical adequacy of standard models of low-frequency variability for the observed low-frequency variability. To this end, it is useful to focus on four questions:

1. Is the unit root model ($d = 1$ in the fractional model, $c = 0$ in the local-level model, $g = 0$ in the integrated local level model) consistent with data?

2. Is the $I(0)$ model ($d = 0$ in the fractional model, $g = 0$ in the local-level model) consistent with data?

3. Are there entire classes of models that are rejected by the tests?

4. Are the results from the low-frequency components of the data similar to the results obtained using standard methods (unit root tests, GPH estimators, and so forth)?

Real GDP/GNP. The post-war quarterly real GDP data are consistent with a unit-root model, but not the $I(0)$ model. The unit-root model is included in the confidence intervals of Table 4, while the $I(0)$ model is not. From Table 3, the unit-root model is not rejected by the low-frequency POI test (p-value $= 0.13$), nor by the DF-GLS test (p-value $= 0.39$). The GPH regression using the differences of the data with $n^{0.65}$ yields an insignificant t-statistic ($-0.09/0.06$), consistent with the unit root model for the level of the series. The absolute value of the t-statistic for the $n^{0.5}$ GPH estimator exceeds 2.0, but the regression uses only 14 observations, so the asymptotic normal distribution is likely to be a poor approximation to the exact small distribution. (All of the post-war series have a common sample period with $n = 212$, $[n^{0.5}] = 14$, and $[n^{0.65}] = 32$.) The log-likelihood for the unit root model is 8.81; this corresponds to the MLE in the I-LLM model, and is close the maximized value in the fractional and OU models (where the maximized values are 8.95 and 9.02, respectively).

The results are somewhat less clear using the annual observations on real GNP from 1869-2004. The unit-root model is not rejected by the LBIM and $|\widehat{\rho}|$ tests, but Table 3 reports a p-value of 0.05 for the low-frequency unit-root test statistic, and the DF-GLS p-value is 0.02. The log-likelihood for the unit root model (11.01) is more than 2.5 log points from the maximized value of the OU log-likelihood, where the MLE is $\widehat{c}_{MLE} = 24.5$. On the other hand, the $I(0)$ model is rejected. The GPH results are rather nonsensical, perhaps because they are based on few observations ($[n^{0.65}] = 24$).

Inflation. The unit-root model for inflation is not rejected using the post-war quarterly data, while the $I(0)$ model is rejected. Results are shown for inflation based on the GDP deflator, but similar conclusions follow from the PCE deflator and CPI. Stock and Watson (2005) document instability in the "size" of the unit root component (corresponding to the value of g in the local-level model) over the post-war period, but apparently this instability is not severe enough to lead to rejections based on the tests considered here. Both the unit-root and $I(0)$ models are rejected using the annual observations on inflation over the 1869-2004. Indeed the OU, I-OU and I-LL models are rejected for all of the parameter values considered. The series appears consistent with a stationary fractional model with positive value of d. This is perhaps not surprising given the instability in monetary policy regimes over the past two centuries and the connection between regime switching and long-memory models discussed in Diebold and Inoue (2001).

Labor productivity and employee hours. The LBIM test rejects both the unit-root and $I(0)$ model for productivity. Productivity appears to be better characterized by a process that remains persistent even after taking first differences. This additional persistence is consistent with long-swings in trend productivity growth in the post-war period such as the productivity slowdown of the 1970's and 1980's and the productivity rebound of the 1990's. This persistence is missed by the DF-GLS statistic, which does not reject a unit root for the level of productivity, but does reject a unit root for the first differences (not reported in the Tables). It is also missed by the GPH regressions, which are consistent with the unit root model.

The behavior of employee hours has received considerable attention in the recent VAR literature (see Gali (1999), Christiano, Eichenbaum, and Vigfusson (2003), and Francis and

Ramey (2006a)). Results based on the LBIM test, unit root tests and GPH regression are all consistent with unit-root but not $I(0)$ low-frequency behavior, and Francis and Ramey (2006b) discuss demographic trends that are potentially responsible for the persistence in this series. Note however that $|\widehat{\rho}|$ is too large to be consistent with the unit-root model. Examination of the series suggests this rejection is associated with an increase in the low-frequency variability in the second half of the sample period.

Interest rates. Post-war nominal interest rates are generally consistent with a unit-root but not an $I(0)$ process. The only contrary evidence is a p-value of 0.04 for the low-frequency unit root test for the 1-year Treasury bond rate. Similarly, unit root test statistics, GPH regressions, LBIM tests, and likelihood values for annual data from 1900-2004 on rates on long-term high-grade industrial bonds are consistent with the unit-root model. However, the low-frequency transformations of this series are highly negatively correlated ($\widehat{\rho} = -0.64$), which is inconsistent with the unit root model. Examination of the time series plot shows a substantial increase in volatility in this series post-1960, and this appears to be the cause of the rejection. Post-war real interest rates are also consistent with the unit root but not the $I(0)$ models. Finally, the long annual time series on real industrial bond rates is not well described by any of the models. Variability in the low frequency components is consistent with a model with considerable mean reversion (the OU model with large value of c or the fractional model with value of d close to zero), but the components are highly serially correlated ($\widehat{\rho} = 0.60$) which is inconsistent with these models. Again, the high serial correlation is caused by a large change in volatility (in this case a drop in the volatility of postwar values of the series relative to pre-war values).

Real exchange rates. The persistence of real exchange rates is well established, and a large empirical literature has examined the unit root or near unit root behavior of real exchange rates. The data used here—annual observations on the real dollar/pound real exchange rate from 1791-1990—come from one important empirical study (Lothian and Taylor (1996)) in this literature. Low frequency variability in this series is inconsistent with both the unit-root and $I(0)$ models, both of which are rejected by the LBIM statistic; the unit-root is rejected by the unit root test statistics (DF-GLS and its low-frequency counterpart) and the GPH regressions. Local-to-unity models with large values of c are not rejected by the LBIM statistics, but this model fits the data poorly relative to fractional model with d close to 0.5 or a local-level model with a reasonably large random walk component ($g = 20$).

Cointegrating relations. Several of the data series, such as the spread between 10-year and 1-year Treasury bond rates, represent error correction terms from putative cointegrating relationships. Under the hypothesis of cointegration, these series should be $I(0)$. The $I(0)$ model is consistent with the results reported in Table 4, while the unit root model is rejected in Table 4 and by the unit root tests reported in Table 3. Real unit labor costs (the logarithm of the ratio of labor productivity to real wages, $y-n-w$ in familiar notation) display similar behavior: results in Table 4 are consistent with the $I(0)$ model but reject the unit-root model, as do the unit root tests in Table 3. In both cases, the GPH regression using $n^{0.65}$ suggest more persistence, but again, these regressions use 32 periodogram ordinates corresponding to periods 6.6 quarters and higher.

The "balanced growth" cointegrating relation between consumption and income (e.g., King, Plosser, Stock, and Watson (1991)) fares less well, where the unit-root is not rejected and the $I(0)$ model is rejected. The apparent source of this rejection is the large increase in this ratio over the 1985-2004 period, a subject that has attracted much recent attention (for example, see Lettau and Ludvigson (2004) for an explanation based on increases in asset values.) The investment-income relationship also appears to be at odds with the null of cointegration, although this rejection depends in part on the particular series used for investment and its deflator.

Finally, the stability of the logarithm of the dividend-stock price ratio, and its implication for the predictability of stock prices, has been an ongoing subject of controversy (see Campbell and Yogo (2006) for a recent discussion). Using Campbell and Yogo's (2006) annual data for the SP500 from 1880-2002, the unit root model is rejected by the LBIM test; the LBIM confidence intervals suggest less persistence than a unit root (for example the LBIM confidence interval for the fractional model includes $0.49 \leq d \leq 0.86$ in the fractional model). Similar results hold for the earning-price ratio. The shorter (1928-2004) CRSP dividend-yield (also from Campbell and Yogo (2006)), displays more low-frequency persistence and is consistent with the unit-root model.

Volatility of stock returns. Ding, Granger, and Engle (1993) analyzed the absolute value of daily returns form the SP500 and showed that the autocorrelations decayed in a way that was remarkably consistent with a fractional process. Low frequency characteristics of the data summarized in Table 4 are consistent with this finding. Both the unit-root and $I(0)$ models are rejected by the LBIM statistic, but models with somewhat less persistence than the unit root, such as the fraction model with $0.19 < d < 0.71$, are not rejected. The GPH statistics (which are now based on a large number of observations, $n = 20643$ so that $[n^{0.5}] = 143$, and $[n^{0.65}] = 637$) suggest some instability across frequencies: $\widehat{d} = 0.38$ using $n^{0.65}$ and $\widehat{d} = 0.46$ using $n^{0.5}$ with small standard errors in both cases. This suggests an important role for the role of the frequency cut-off for the analysis, a point made by Andersen and Bollerslev (1997) in the context of volatility modeling and by Bollerslev and Mikkelsen (1999) in their study of long-term equity anticipation securities (LEAPS) on the SP500. Also, there is again some evidence for too much autocorrelation in the transformed series; only rather large values of d are not rejected by a test based on $|\hat{\rho}|$. It seems that the absolute returns are subject to substantial low-frequency volatility, which is not captured well by the stationary fractional model.

3.4 Additional Empirical Results

This section will contain additional empirical results investigating the robustness of conclusions reached in the body of the paper. Robustness calculations will include

1. Using the standard Fourier expansion in place of the Fourier Cosine expansion
2. Using alternative versions of Ω (for the LB test)
3. Using alternative tests for volatility of low-frequency volatility in the underlying series
4. Using different choices of q
5. Likelihood ratio tests comparing the different models (for example, local-to-unity vs. fractional

4 Conclusions

Standard specification tests for time series models examine the model's appropriateness over the whole spectrum. In contrast, the methodology developed here isolates a model's low frequency implications by focusing exclusively on the properties of a finite number of weighted averages of the original data. For example, by choosing the weights as trigonometric series with periods larger than eight years, our empirical analysis considers whether any of five standard models of peristence successfully explain the variability of twenty-one macro and financial time series at frequencies lower than the business cycle.

Despite this narrow focus, we find some fairly strong empirical results. Very few of the series are compatible with the $I(0)$ model, including many putative cointegration error correction terms. The unit root model fares better and successfully explains low frequency variability of many post-war macroeconomic time series. However, for a number of series that have traditionally been analyzed in an autoregressive framework, the fractional model seems to provide a better fit at low frequencies.

Somewhat surprisingly, the simple one-parameter models considered here seem to be reasonably successful at explaining the low-frequency variability of typical macro and financial time series. Specific models are often rejected for specific series, but all of the models are not rejected for any series. At the same time, for several series, there seems to be too much low-frequency variability in the second moment to provide good fits for any of the models. From an economic perspective, this underlines the importance of understanding the sources and implications of such low frequency volatility changes. From a statistical perspective, this finding motivates further research into models that allow for substantial changes in second moments.

5 Appendix

Proof of Theorem 1:

Define $S_t = \sum_{s=1}^{t} u_s$ and $S_t^i = \sum_{s=1}^{t} u_s^i$, $i = \mu, \tau$. With $T^{-\alpha} S_{[\cdot T]} \Rightarrow \omega G(\cdot)$, we find by least squares algebra and the CMT

$$\begin{aligned} T^{-\alpha}S^{\mu}_{[\lambda T]} &= T^{-\alpha}S_{[\lambda T]} - \lambda T^{-\alpha}S_T + R^{\mu}_T(\lambda) \\ T^{-\alpha}S^{\tau}_{[\lambda T]} &= T^{-\alpha}S^{\mu}_{[\lambda T]} - 6\lambda(1-\lambda)\int S^{\mu}_{[lT]}dl + R^{\tau}_T(\lambda) \end{aligned}$$

where $\sup_{\lambda\in[0,1]}|R^i_T(\lambda)| \xrightarrow{p} 0$ for $i=\mu,\tau$. Thus, by the CMT

$$\begin{aligned} T^{-\alpha}\omega^{-1}S^{\mu}_{[\lambda T]} &\Rightarrow G(\lambda) - \lambda G(1) \equiv G^{\mu}(\lambda) && (4) \\ T^{-\alpha}\omega^{-1}S^{\tau}_{[\lambda T]} &\Rightarrow G^{\mu}(\lambda) - 6\lambda(1-\lambda)\int G^{\mu}(l)dl \equiv G^{\tau}(\lambda) && (5) \end{aligned}$$

and

$$\begin{aligned} E[G^{\mu}(r)G^{\mu}(s)] &= E[(G(r)-rG(1))(G(s)-sG(1))] \\ &= k^{\mu}(r,s) \end{aligned}$$

and similarly, $k^{\tau}(r,s) = E[G^{\tau}(r)G^{\tau}(s)]$. By summation by parts

$$\begin{aligned} \sum_{t=1}^{T}\Psi_l(t/T)u^i_t &= S^i_T\Psi_l(1) - \sum_{t=1}^{T}S^i_{t-1}(\Psi_l(t/T) - \Psi_l((t-1)/T)) \\ &= -\int S^i_{[\lambda T]}\psi_l(\lambda)d\lambda + \int S^i_{[\lambda T]}(\psi_l(\lambda) - \frac{\Psi_l([\lambda T]/T + T^{-1}) - \Psi_l([\lambda T]/T)}{T^{-1}})d\lambda, \end{aligned}$$

since $S^i_T = 0$ for $i=\mu,\tau$. Application of the mean-value theorem yields

$$\sup_{\lambda\in[0,1]}|\psi_l(\lambda) - \frac{\Psi_l([\lambda T]/T + T^{-1}) - \Psi_l([\lambda T]/T)}{T^{-1}}| \leq \sup_{\lambda\in[0,1]}\sup_{\lambda'\in[0,1],|\lambda'-\lambda|\leq T^{-1}}|\psi_l(\lambda) - \psi_l(\lambda')| \to 0$$

and the uniform convergence follows from continuity (and hence uniform continuity) of $\psi_l(\cdot)$ on the unit interval. Thus

$$\begin{aligned} & T^{-\alpha}|\int S^i_{[\lambda T]}(\psi_l(\lambda) - \frac{\Psi_l([\lambda T]/T + T^{-1}) - \Psi_l([\lambda T]/T)}{T^{-1}})d\lambda| \\ \leq & \sup_{\lambda\in[0,1]}|\psi_l(\lambda) - \frac{\Psi_l([\lambda T]/T + T^{-1}) - \Psi_l([\lambda T]/T)}{T^{-1}}|\int|T^{-\alpha}S^i_{[\lambda T]}|d\lambda \xrightarrow{p} 0 \end{aligned}$$

since $\int|T^{-\alpha}S^i_{[\lambda T]}|d\lambda \Rightarrow \omega\int|G^i(\lambda)|d\lambda$ by the CMT. Using this result row by row and the convergences (4) and (5), we obtain by the CMT

$$\begin{aligned} X_T &= T^{-\alpha}\sum_{t=1}^{T}\Psi(t/T)u^i_t \\ &= -\int S^i_{[\lambda T]}\psi(\lambda)d\lambda + o_p(1) \Rightarrow -\int G^i(\lambda)\psi(\lambda)d\lambda \end{aligned}$$

where $\psi(\cdot) = (\psi_1(\cdot), \cdots, \psi_q(\cdot))'$, and the result follows.

Continuity of fractional process at $d = 1/2$:
By the definition of $k^{\mu}_{FR(d)}(r,s)$ and $k^{\mu}_{I\text{-}FR(d)}(r,s)$, we find for $s \leq r$

$$\begin{aligned} k^{\mu}_{FR(d)}(r,s) = \tfrac{1}{2}[&s^{1+2d} + r^{1+2d} - (r-s)^{1+2d} + 2rs \\ &- s(1-(1-r)^{1+2d} + r^{1+2d}) - r(1-(1-s)^{1+2d} + s^{1+2d})] \end{aligned}$$

and

$$\begin{aligned} k^{\mu}_{I\text{-}FR(d)}(r,s) = \frac{1}{4d(1+2d)}[&-r^{1+2d}(1-s) - s(s^{2d} + (r-s)^{2d} + (1-r)^{2d} - 1) \\ &+ r(s^{1+2d} + 1 - (1-s)^{2d} + (r-s)^{2d}) + sr((1-s)^{2d} + (1-r)^{2d} - 2))] \end{aligned}$$

so that

$$\lim_{d \uparrow 1/2} k^{\mu}_{FR(d)}(r,s) = \lim_{d \downarrow 1/2} k^{\mu}_{I\text{-}FR(d)}(r,s) = 0.$$

Now for $0 < s < r$, using that for any real $a > 0$, $\lim_{x \downarrow 0}(a^x - 1)/x = \ln a$, we find

$$\begin{aligned} \lim_{d \uparrow 1/2} \frac{k^{\mu}_{FR(d)}(r,s)}{1/2 - d} = &-(1-r)^2 s \ln(1-r) - r^2(1-s)\ln r - r(1-s)^2 \ln(1-s) \\ &+ (r-s)^2 \ln(r-s) + (r-1)s^2 \ln(s) \end{aligned}$$

and

$$\lim_{d \uparrow 1/2} \frac{k^{\mu}_{FR(d)}(r,r)}{1/2 - d} = 2(1-r)r(-(1-r)\ln(1-r) - r\ln r).$$

Performing the same computation for $k^{\mu}_{I\text{-}FR(d)}(r,s)$ yields the result.

6 Data Appendix

Table A1 lists the series used in section 3, the sample period, transformation, and data source and notes. In the column labeled Smpl (sample period), PWQ denotes postwar quarterly observations over 1952:1-2004:4, other series are annual over the period shown with the exception on absolute returns, which are daily daily over the period shown. In the column labeled Trans (transformation), the transformation are: demeaned levels (lev μ), detrended levels (lev τ), demeaned logarithms (ln μ), detrended logarithms (ln τ), and ln(rat) denotes the logarithm of the indicated ratio. In the column labeled Source and Notes, DRI denotes the DRI Economics Database (formerly Citibase) and NBER denotes the NBER historical data base.

References

Akdi, Y., and D. Dickey (1998): "Periodograms of Unit Root Time Series: Distributions and Tests," *Communications in Statistics: Theory and Methods*, 27, 69–87.

Andersen, T., and T. Bollerslev (1997): "Heterogeneous Information Arrivals and Return Volatility Dynamics: Uncovering the Long-Run in High Frequency Returns," *Journal of Finance*, 52, 975–1005.

Andersen, T., T. Bollerslev, P. Christoffersen, and F. Diebold (2006): *Volatility: Practical Methods for Financial Applications*. Princeton University Press, Princeton.

Balke, N., and R. Gordon (1989): "The Estimation of Prewar Gross National Product: Methodology and New Evidence," *Journal of Political Economy*, 94, 38–92.

Beveridge, S., and C. Nelson (1981): "A New Approach to Decomposition of Economics Time Series Into Permanent and Transitory Components with Particular Attention to Measurement of the Business Cycle," *Journal of Monetary Economics*, 7, 151–174.

Bierens, H. (1997): "Nonparametric Cointegration Analysis," *Journal of Econometrics*, 77, 379–404.

Bollerslev, T., R. Engle, and D. Nelson (1994): "ARCH Models," in *Handbook of Econometrics Vol. IV*, ed. by R. Engle, and D. McFadden. Elsevier Science, Amsterdam.

Bollerslev, T., and H. Mikkelsen (1999): "Long-Term Equity Anticipation Securities and Stock Market Volatility Dynamics," *Journal of Econometrics*, 92, 75–99.

Campbell, J., and M. Yogo (2006): "Efficient Tests of Stock Return Predictability," *forthcoming in Journal of Financial Economics.*

Chan, N., and N. Terrin (1995): "Inference for Unstable Long-Memory Processes with Applications to Fractional Unit Root Autoregressions," *Annals of Statistics*, 23, 1662–1683.

Christiano, L., M. Eichenbaum, and R. Vigfusson (2003): "What Happens After a Technology Shock," *NBER Working Paper 9819.*

Davidson, J. (2002): "Establishing Conditions for the Functional Central Limit Theorem in Nonlinear and Semiparametric Time Series Processes," *Journal of Econometrics*, 106, 243–269.

den Haan, W., and A. Levin (1997): "A Practitioner's Guide to Robust Covariance Matrix Estimation," in *Handbook of Statistics 15*, ed. by G. Maddala, and C. Rao, pp. 309–327. Elsevier, Amsterdam.

Diebold, F., and A. Inoue (2001): "Long Memory and Regime Switching," *Journal of Econometrics*, 105, 131–159.

Ding, Z., C. Granger, and R. Engle (1993): "A Long Memory Property of Stock Market Returns and a New Model," *Journal of Empirical Finance*, 1, 83–116.

Elliott, G. (1999): "Efficient Tests for a Unit Root When the Initial Observation is Drawn From its Unconditional Distribution," *International Economic Review*, 40, 767–783.

Elliott, G., T. Rothenberg, and J. Stock (1996): "Efficient Tests for an Autoregressive Unit Root," *Econometrica*, 64, 813–836.

Fama, E. (1970): "Efficient Capital Markets: A Review of Theory and Empirical Work," *Journal of Finance*, 25, 383–417.

Francis, N., and V. Ramey (2006a): "Is the Technology-Driven Business Cycle Hypothesis Dead?," *forthcoming in Journal of Monetary Economics.*

——— (2006b): "Measures of Per Capita Hours and their Implications for the Technology-Hours Debate," *mimeo, U.C. San Diego.*

Gali, J. (1999): "Technology, Employment, and the Business Cycle: Do Technology Shocks Explain Aggregate Fluctuations?," *American Economic Review*, 89, 249–271.

Geweke, J., and S. Porter-Hudak (1983): "The Estimation and Application of Long Memory Time Series Models," *Journal of Time Series Analysis*, 4, 221–238.

GODFREY, L. (1981): "On the Invariance of the Lagrange Multiplier Test with Respect to Certain Changes in the Alternative Hypothesis," *Econometrica*, 49, 1443–1455.

GRANGER, C., AND R. JOYEUX (1980): "An Introduction to Long-Memory Time Series Models and Fractional Differencing," *Journal of Time Series Analysis*, 1, 15–29.

HARVEY, A. (1989): *Forecasting, Structural Time Series Models and the Kalman Filter*. Cambridge University Press.

KARIYA, T. (1980): "Locally Robust Test for Serial Correlation in Least Squares Regression," *Annals of Statistics*, 8, 1065–1070.

——— (1988): "The Class of Models for Which the Durbin-Watson Test is Locally Optimal," *International Economic Review*, 29, 167–175.

KIEFER, N., AND T. VOGELSANG (2005): "A New Asymptotic Theory for Heteroskedasticity-Autocorrelation Robust Tests," *Econometric Theory*, 21, 1130–1164.

KIM, C.-J., AND C. NELSON (1999): "Has the Economy Become More Stable? A Bayesian Approach Based on a Markov-Switching Model of the Business Cycle," *The Review of Economics and Statistics*, 81, 608–616.

KING, M. (1980): "Robust Tests for Spherical Symmetry and their Application to Least Squares Regression," *The Annals of Statistics*, 8, 1265–1271.

KING, R., C. PLOSSER, J. STOCK, AND M. WATSON (1991): "Stochastic Trends and Economic Fluctuations," *American Economic Review*, 81, 819–840.

LETTAU, M., AND S. LUDVIGSON (2004): "Understanding Trend and Cycle in Asset Values: Reevaluating the Wealth Effect on Consumption," *American Economic Review*, 94, 276–299.

LOTHIAN, J., AND M. TAYLOR (1996): "Real Exchange Rate Behavior: The Recent Float from the Perspective of the Past Two Centuries," *Journal of Political Economy*, 104, 488–509.

Mandelbrot, B., and J. V. Ness (1968): “Fractional Brownian Motions, Fractional Noise and Applications,” *SIAM Review*, 10, 422–437.

Marinucci, D., and P. Robinson (1999): “Alternative Forms of Fractional Brownian Motion,” *Journal of Statistical Planning and Inference*, 80, 111–122.

McConnell, M., and G. Perez-Quiros (2000): “Output Fluctuations in the United States: What Has Changed Since the Early 1980’s,” *American Economics Review*, 90, 1464–1476.

McLeish, D. (1974): “Dependent Central Limit Theorems and Invariance Principles,” *The Annals of Probability*, 2, 620–628.

Meese, R., and K. Rogoff (1983): “Empirical Exchange Rate Models of the Seventies: Do They Fit Out of Sample?,” *Journal of International Economics*, 14, 3–24.

Müller, U. (2004): “A Theory of Robust Long-Run Variance Estimation,” *mimeo, Princeton University.*

Nelson, C., and C. Plosser (1982): “Trends and Random Walks in Macroeconomic Time Series — Some Evidence and Implications,” *Journal of Monetary Economics*, 10, 139–162.

Ng, S., and P. Perron (2001): “Lag Length Selection and the Construction of Unit Root Tests with Good Size and Power,” *Econometrica*, 69, 1519–1554.

Nyblom, J. (1989): “Testing for the Constancy of Parameters Over Time,” *Journal of the American Statistical Association*, 84, 223–230.

Phillips, P. (1998): “New Tools for Understanding Spurious Regression,” *Econometrica*, 66, 1299–1325.

——— (2006): “Optimal Estimation of Cointegrated Systems with Irrelevant Instruments,” *Cowles Foundation Discussion Paper 1547.*

Phillips, P., and V. Solo (1992): “Asymptotics for Linear Processes,” *Annals of Statistics*, 20, 971–1001.

Robinson, P. (2003): "Long-Memory Time Series," in *Time Series with Long Memory*, ed. by P. Robinson, pp. 4–32. Oxford University Press, Oxford.

Said, S., and D. Dickey (1984): "Testing for Unit Roots in Autoregressive-Moving Average Models of Unknown Order," *Biometrika*, 71, 2599–607.

Schwert, G. (1989): "Tests for Unit Roots: A Monte Carlo Investigation," *Journal of Business & Economic Statistics*, 7, 147–159.

Stock, J. (1994): "Unit Roots, Structural Breaks and Trends," in *Handbook of Econometrics*, ed. by R. Engle, and D. McFadden, vol. 4, pp. 2740–2841. North Holland, New York.

Stock, J., and M. Watson (2005): "Has Inflation Become Harder to Forecast?," *mimeo, Princeton University.*

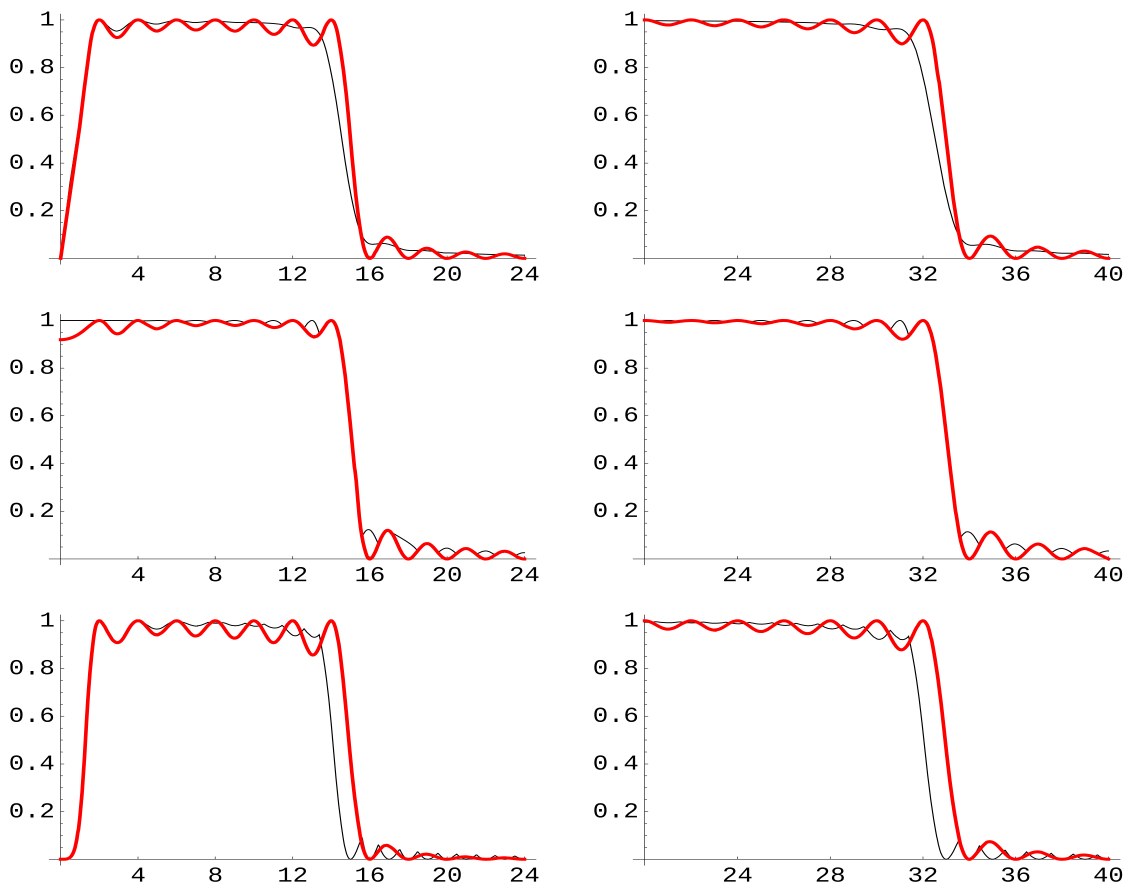
Taqqu, M. (1975): "Convergence of Integrated Processes of Arbitrary Hermite Rank," *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 50, 53–83.

Velasco, C. (1999): "Non-Stationary Log-Periodogram Regression," *Journal of Econometrics*, 91, 325–371.

Woolridge, J., and H. White (1988): "Some Invariance Principles and Central Limit Theorems for Dependent Heterogeneous Processes," *Econometric Theory*, 4, 210–230.

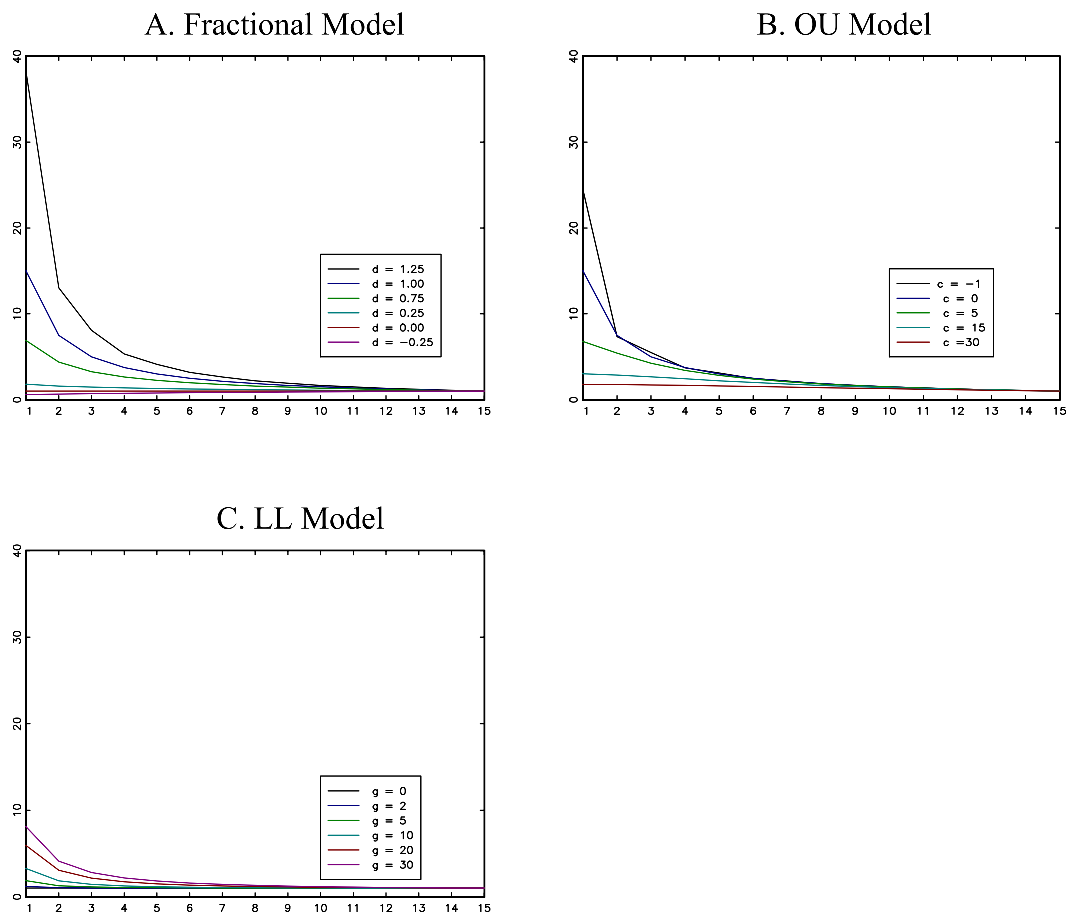
Figure 1

R^2 From Regression of $sin(r\pi s + a)$ onto $\Psi_1(s) \ldots \Psi_q(s)$



Notes: The first column shows the R^2 for $q = 14$ and the second column for $q = 32$. The first row shows average values over values of the phase a, the second column shows the R^2 maximized over a, and the final row shows the R^2 minimized over a. The (thin) black curve shows results for the Fourier cosine expansion and the (thick) red curve for the Fourier expansion.

Figure 2
Standard Deviation of X_l Implied by Different Models



Notes: These figures show the square roots of the diagonal elements of $\Sigma(\theta)$ for different values of the parameter $\theta = (d, c, g)$, where $\Sigma(\theta)$ is computed for the demeaned data.

Table 1: Asymptotic Properties of Partial Sums of Popular Time Series Models

	Process	Parameter	Partial sum convergence	covariance kernel $k(r,s)$, $s \leq r$
1a	FR	$-\frac{1}{2} < d < \frac{1}{2}$	$T^{-1/2-d}\omega^{-1}\sum_{t=1}^{[\cdot T]} u_t \Rightarrow W^d(\cdot)$	$\frac{1}{2}(r^{2d+1} + s^{2d+1} - (r-s)^{2d+1})$
1b	FR	$\frac{1}{2} < d < \frac{3}{2}$	$T^{-1/2-d}\omega^{-1}\sum_{t=1}^{[\cdot T]} u_t \Rightarrow \int_0^{\cdot} W^{d-1}(l)dl$	$\frac{(r-s)^{2d+1}+(1+2d)(rs^{2d}+r^{2d}s)-r^{2d+1}-s^{2d+1}}{4d(1+2d)}$
2a	OU	$c > 0$	$T^{-3/2}\omega^{-1}\sum_{t=1}^{[\cdot T]} u_t \Rightarrow \int_0^{\cdot} J^c(l)dl$	$\frac{2cs-1+e^{-cs}+(e^{-cr}-e^{-c(r-s)})}{2c^3}$
2b	OU	$c \leq 0$	$T^{-3/2}\omega^{-1}\sum_{t=1}^{[\cdot T]} u_t \Rightarrow \int_0^{\cdot}\int_0^{\lambda} e^{-c(\lambda-l)}dW(l)d\lambda$	$\begin{cases} \frac{1+2c^2rs-c(r+s)+e^{-2cs}(c(r-s)-1)}{4c^3} \text{ for } c>0 \\ \frac{1}{6}(3rs^2 - s^3) \text{ for } c=0 \end{cases}$
3	I-OU	$c > 0$	$T^{-5/2}\omega^{-1}\sum_{t=1}^{[\cdot T]} u_t \Rightarrow \int_0^{\cdot}\int_0^{\lambda} J^c(l)dld\lambda$	$\frac{3-sc(3+c^2s^2)+3rc(1-cs+c^2s^2)-3e^{-cs}(1+cr)-3e^{-cr}(1+cs-e^{cs})}{6c^5}$
4	LL	$g \geq 0$	$T^{-1/2}\omega^{-1}\sum_{t=1}^{[\cdot T]} u_t \Rightarrow W_1(\cdot) + g\int_0^{\cdot} W_2(l)dl$	$s + \frac{1}{6}g(3rs^2 - s^3)$
5	I-LL	$g \geq 0$	$T^{-3/2}\omega^{-1}\sum_{t=1}^{[\cdot T]} u_t \Rightarrow \int_0^{\cdot} W_1(l)dl + g\int_0^{\cdot}\int_0^{\lambda} W_2(l)dld\lambda$	$\frac{1}{6}(3rs^2 - s^3) + \frac{1}{120}g(10r^2s^3 - 5rs^4 + s^5)$

Notes: W, W_1 and W_2 are independent standard Wiener processes, W^d is "type I" fractional Brownian Motion defined as $W^d(s) = A(d)\int_{-\infty}^0 \left[(s-l)^d - (-l)^d\right] dW(l) + A(d)\int_0^s (s-l)^d dW(l)$ where $A(d) = \left(\frac{1}{2d+1} + \int_0^\infty \left[(1+l)^d - l^d\right] dl\right)^{-1/2}$ and J^c is the stationary Ornstein-Uhlenbeck process $J^c(s) = Ze^{-sc}/\sqrt{2c} + \int_0^s e^{-c(s-l)}dW(l)$ with $Z \sim \mathcal{N}(0,1)$ independent of W.

Table 2
Average Absolute Correlations for $\Sigma(\theta)$ for Demeaned Series.

Fractional Model	d = −0.25	d = 0.00	d = 0.25	d = 0.75	d = 1.00	d = 1.25
	0.03	0.00	0.01	0.01	0.00	0.03
Local-to-Unity Model	c = 30	c = 20	c = 20	c = 5	c = 0	c = − 1
	0.02	0.02	0.02	0.02	0.00	0.03
Local-Level Model	g = 0	g = 2	G = 5	g = 10	g = 20	g = 30
	0.00	0.00	0.00	0.00	0.00	0.00

Notes: Entries in the table are the average values of the absolute values of the correlations associated with $\Sigma(\theta)$ with q = 15 for the various models associated with demeaned series.

Table 3
Unit Root and GPH Regression Statistics

Series	Sample Period	Unit Root Statistics (p-values)		GPH Regressions			
				Levels		Differences	
		DF-GLS	LFUR	$n^{0.5}$	$n^{0.65}$	$n^{0.5}$	$n^{0.65}$
Real GDP	PWQ	0.39	0.13	1.00 (0.09)	0.98 (0.06)	-0.19 (0.09)	-0.09 (0.06)
Real GNP	1869-2004	0.02	0.05	0.98 (0.10)	0.98 (0.07)	-0.84 (0.10)	-0.46 (0.07)
Inflation	PWQ	0.22	0.07	0.84 (0.09)	0.91 (0.06)	-0.10 (0.09)	-0.02 (0.06)
Inflation	1870-2004	0.00	0.01	0.52 (0.10)	0.30 (0.07)	-0.85 (0.10)	-0.93 (0.07)
Productivity	PWQ	0.96	0.71	0.95 (0.09)	0.97 (0.06)	0.07 (0.09)	-0.03 (0.06)
Hours	PWQ	0.54	0.42	0.75 (0.09)	0.99 (0.06)	-0.11 (0.09)	0.09 (0.06)
10YrTBond	PWQ	0.47	0.13	1.05 (0.09)	1.08 (0.06)	0.13 (0.09)	0.10 (0.06)
1YrTBond	PWQ	0.23	0.04	0.85 (0.09)	0.95 (0.06)	-0.05 (0.09)	-0.03 (0.06)
3mthTbill	PWQ	0.23	0.07	0.74 (0.09)	1.00 (0.06)	-0.21 (0.09)	0.04 (0.06)
Bond Rate	1900-2004	0.32	0.16	1.07 (0.10)	1.22 (0.07)	0.08 (0.10)	0.09 (0.07)
Real Tbill Rate	PWQ	0.06	0.01	0.72 (0.09)	0.79 (0.06)	-0.19 (0.09)	-0.12 (0.06)
Real Bond Rate	1900-2004	<0.01	<0.01	0.46 (0.10)	0.35 (0.07)	-0.55 (0.10)	-0.65 (0.07)
Dollar/Pound Real Ex. Rate	1792-1990	<0.01	0.01	0.58 (0.09)	0.49 (0.06)	-0.48 (0.09)	-0.55 (0.06)
TBond Spread	PWQ	<0.01	.	0.18 (0.09)	0.61 (0.06)	-0.80 (0.09)	-0.41 (0.06)
Unit Labor Cost	PWQ	0.03	0.00	0.57 (0.09)	0.75 (0.06)	-0.55 (0.09)	-0.31 (0.06)
real C-GDP	PWQ	0.84	0.81	0.96 (0.09)	0.96 (0.06)	0.19 (0.09)	-0.10 (0.06)
real I-GDP	PWQ	0.66	0.24	0.62 (0.09)	0.87 (0.06)	-0.74 (0.09)	-0.25 (0.06)
Div/Price (SP500)	1880-2002	0.19	0.17	0.35 (0.10)	0.63 (0.07)	-0.31 (0.10)	-0.34 (0.07)
Earnings/Price (SP500)	1880-2002	0.03	0.01	0.70 (0.10)	0.62 (0.07)	-0.36 (0.10)	-0.35 (0.07)
Div/Price (CRSP)	1926-2002	0.46	0.47	0.72 (0.11)	0.72 (0.08)	-0.28 (0.11)	-0.43 (0.08)
Abs.Returns (SP500)	Daily	<0.01	<0.01	0.46 (0.03)	0.38 (0.01)	-0.52 (0.03)	-0.61 (0.01)

Notes: Smpl is the sample period where PWQ denotes post-war quarterly (1952:1-2004:4), Daily denotes daily data from January 3, 1928 – November 22, 2005, and the other entries denote annual observations over the period listed. Entries under DF-GLS and LFUR are *p*-values for the tests. The values under *GPH* are the estimated values of *d* using the levels and differences of the data using the number of periodogram ordinates listed with estimated standard error in parentheses.

Table 4
Confidence Intervals and Likelihood Summary for Low-Frequency Components

Model	90% CI		LLF MLE (LLR < 2)
	LBIM	$\lvert \hat{\rho} \rvert$	
A. Real GDP (1952:1-2004:4)			
Frac. (*d*)	(0.24 1.49)	(-0.49 1.49)	8.95 0.83 (0.26 1.49)
OU (*c*)	(-3.00 28.00)	(-3.00 30.00)	9.02 6.00 (-3.00 30.00)
LL (*g*)	(7.00 30.00)	(0.00 30.00)	8.45 30.00 (6.50 30.00)
I -OU (*c*)	(17.50 30.00)	(11.50 30.00)	8.37 30.00 (10.50 30.00)
I – LL (*g*)	(0.00 19.00)	(0.00 22.50)	8.81 0.00 (0.00 12.00)
B. Real GNP (1869-2004)			
Frac. (*d*)	(0.31 1.12)	(-0.49 1.49)	12.71 0.64 (0.28 1.03)
OU (*c*)	(-3.00 30.00)	(-3.00 30.00)	13.52 24.50 (4.50 30.00)
LL (*g*)	(25.00 30.00)	(0.00 30.00)	10.36 30.00 (13.50 30.00)
I -OU (*c*)	(. .)	(-3.00 30.00)	3.03 30.00 (22.00 30.00)
I – LL (*g*)	(0.00 16.00)	(0.00 30.00)	11.01 0.00 (0.00 8.50)
C. Inflation, GDP Deflator (1952:1-2004:4)			
Frac. (*d*)	(0.41 1.42)	(-0.49 1.49)	4.33 0.79 (0.28 1.30)
OU (*c*)	(-3.00 19.50)	(-3.00 30.00)	4.61 5.00 (-2.00 23.00)
LL (*g*)	(12.00 30.00)	(0.00 30.00)	4.12 30.00 (9.00 30.00)
I -OU (*c*)	(23.00 30.00)	(-3.00 30.00)	3.10 30.00 (12.00 30.00)
I – LL (*g*)	(0.00 15.00)	(0.00 30.00)	4.01 0.00 (0.00 8.00)
D. Inflation, GNP Deflator (1870-2004)			
Frac. (*d*)	(0.06 0.49)	(-0.49 0.93)	2.70 0.30 (0.05 0.60)
OU (*c*)	(. .)	(18.00 30.00)	0.62 30.00 (17.00 30.00)
LL (*g*)	(3.00 30.00)	(0.00 30.00)	2.38 9.00 (1.50 30.00)
I -OU (*c*)	(. .)	(. .)	-20.26 30.00 (24.00 30.00)
I – LL (*g*)	(. .)	(. .)	-6.33 0.00 (0.00 5.00)
E. Labor Productivity (1952:1-2004:4)			
Frac. (*d*)	(1.03 1.49)	(-0.49 1.49)	17.35 1.49 (1.00 1.49)
OU (*c*)	(. .)	(-3.00 30.00)	15.36 0.00 (-2.50 7.00)
LL (*g*)	(. .)	(0.00 30.00)	13.36 30.00 (17.50 30.00)
I -OU (*c*)	(-3.00 30.00)	(-3.00 30.00)	17.57 10.50 (-1.50 30.00)
I – LL (*g*)	(1.50 30.00)	(0.00 30.00)	16.99 21.50 (0.00 30.00)
F. Employee Hours per Capita (1952:1-2004:4)			
Frac. (*d*)	(0.41 1.49)	(-0.49 -0.22)	9.59 0.90 (0.32 1.49)
OU (*c*)	(-3.00 17.50)	(-3.00 -2.00)	9.57 3.00 (-3.00 23.50)
LL (*g*)	(10.50 30.00)	(. .)	9.28 30.00 (8.00 30.00)
I -OU (*c*)	(17.00 30.00)	(. .)	9.01 30.00 (10.50 30.00)
I – LL (*g*)	(0.00 16.00)	(3.50 11.50)	9.53 0.00 (0.00 15.00)
G. 10Yr Treasury Bond Rate (1952:1-2004:4)			
Frac. (*d*)	(0.61 1.49)	(-0.49 1.49)	7.09 0.99 (0.52 1.41)
OU (*c*)	(-3.00 10.50)	(-3.00 30.00)	7.25 1.50 (-2.00 11.00)
LL (*g*)	(17.50 30.00)	(0.00 30.00)	6.65 30.00 (12.00 30.00)
I -OU (*c*)	(12.50 30.00)	(-3.00 30.00)	6.71 30.00 (10.50 30.00)
I – LL (*g*)	(0.00 23.50)	(0.00 30.00)	7.09 0.00 (0.00 12.50)

Table 4 Continued

Model	90% CI		LLF MLE (LLR < 2)
	LBIM	$\lvert \hat{\rho} \rvert$	
H. 10Yr Treasury Bond Rate (1952:1-2004:4)			
Frac. (*d*)	(0.32 1.08)	(-0.49 1.49)	3.64 0.71 (0.21 1.18)
OU (*c*)	(-3.00 24.00)	(-3.00 30.00)	3.43 5.50 (-1.50 30.00)
LL (*g*)	(9.00 30.00)	(0.00 30.00)	3.81 24.00 (6.50 30.00)
I -OU (*c*)	(. .)	(-3.00 30.00)	1.13 30.00 (15.50 30.00)
I – LL (*g*)	(0.00 3.00)	(0.00 30.00)	2.89 0.00 (0.00 7.50)
I. 3-Month Treasury Bill Rate (1952:1-2004:4)			
Frac. (*d*)	(0.30 1.11)	(-0.49 1.49)	3.68 0.71 (0.21 1.19)
OU (*c*)	(-3.00 25.00)	(-3.00 30.00)	3.48 5.50 (-2.00 30.00)
LL (*g*)	(8.50 30.00)	(0.00 30.00)	3.84 24.00 (6.50 30.00)
I -OU (*c*)	(. .)	(-3.00 30.00)	1.25 30.00 (15.50 30.00)
I – LL (*g*)	(0.00 4.00)	(0.00 30.00)	2.95 0.00 (0.00 7.50)
J. Long-term Industrial Bond Rate (1900-2004)			
Frac. (*d*)	(0.75 1.37)	(. .)	21.03 0.99 (0.69 1.32)
OU (*c*)	(-3.00 10.00)	(. .)	21.50 3.50 (-2.00 10.50)
LL (*g*)	(. .)	(. .)	17.48 30.00 (22.50 30.00)
I -OU (*c*)	(. .)	(. .)	18.43 30.00 (19.00 30.00)
I – LL (*g*)	(0.00 22.00)	(. .)	21.02 0.00 (0.00 13.50)
K. Real 3-Month Treasury Bill Rate (1952:1-2004:4)			
Frac. (*d*)	(-0.20 1.49)	(-0.49 1.49)	0.57 0.39 (-0.28 1.21)
OU (*c*)	(-3.00 30.00)	(-3.00 30.00)	1.14 15.50 (-0.50 30.00)
LL (*g*)	(0.00 30.00)	(0.00 30.00)	0.03 2.00 (0.00 30.00)
I -OU (*c*)	(-3.00 30.00)	(-3.00 30.00)	-0.77 30.00 (9.50 30.00)
I – LL (*g*)	(0.00 30.00)	(0.00 30.00)	-0.49 0.00 (0.00 8.50)
L. Real Long-Term Industrial Bond Rates (1900-2004)			
Frac. (*d*)	(-0.06 0.58)	(-0.49 -0.08)	0.97 0.24 (-0.10 0.62)
OU (*c*)	(28.00 30.00)	(. .)	0.32 30.00 (15.50 30.00)
LL (*g*)	(0.00 30.00)	(. .)	0.09 4.00 (0.00 30.00)
I -OU (*c*)	(. .)	(. .)	-14.55 30.00 (22.50 30.00)
I – LL (*g*)	(. .)	(. .)	-6.03 0.00 (0.00 5.00)
M. Real Dollar/Pound Exchange Rate (1792-1990)			
Frac. (*d*)	(0.27 0.62)	(-0.49 1.49)	16.78 0.46 (0.29 0.69)
OU (*c*)	(23.00 30.00)	(-3.00 30.00)	13.30 27.00 (9.00 30.00)
LL (*g*)	(11.00 30.00)	(0.00 30.00)	17.94 21.50 (9.50 30.00)
I -OU (*c*)	(. .)	(0.50 30.00)	-6.31 30.00 (23.50 30.00)
I – LL (*g*)	(. .)	(0.00 30.00)	8.91 0.00 (0.00 5.50)
N. Interest Rate Term Spread (10Yr-1Yr, 1952:1-2004:4)			
Frac. (*d*)	(-0.49 0.33)	(-0.49 1.49)	0.04 0.07 (-0.39 0.57)
OU (*c*)	(23.50 30.00)	(-3.00 30.00)	-0.76 30.00 (11.50 30.00)
LL (*g*)	(0.00 8.00)	(0.00 30.00)	0.41 4.00 (0.00 17.00)
I -OU (*c*)	(. .)	(-3.00 30.00)	-8.63 30.00 (18.00 30.00)
I – LL (*g*)	(. .)	(0.00 30.00)	-5.71 0.00 (0.00 5.00)
O. Real Unit Labor Cost (*y-n-w*, 1952:1-2004:4)			
Frac. (*d*)	(-0.04 0.53)	(-0.49 1.49)	1.19 0.34 (-0.10 0.83)
OU (*c*)	(12.50 30.00)	(-3.00 30.00)	0.42 26.50 (2.50 30.00)
LL (*g*)	(0.00 13.50)	(0.00 30.00)	2.03 7.00 (0.50 25.50)
I -OU (*c*)	(. .)	(-3.00 30.00)	-3.95 30.00 (15.50 30.00)
I – LL (*g*)	(. .)	(0.00 30.00)	-2.06 0.00 (0.00 5.50)

Table 4 Continued

Model	90% CI		LLF MLE (LLR < 2)
	LBIM	$\lvert\hat{\rho}\rvert$	
P. Consumption-Income Ratio (1952:1-2004:4)			
Frac. (*d*)	(0.58 1.49)	(-0.49 1.49)	10.14 1.10 (0.69 1.46)
OU (*c*)	(-3.00 8.50)	(-2.50 30.00)	10.79 -1.50 (-3.00 2.00)
LL (*g*)	(11.00 30.00)	(0.00 30.00)	9.70 30.00 (9.00 30.00)
I -OU (*c*)	(9.50 30.00)	(-3.00 30.00)	9.86 30.00 (8.50 30.00)
I – LL (*g*)	(0.00 30.00)	(0.00 30.00)	10.24 2.00 (0.00 10.50)
Q. Investment/Income Ratio (1952:1-2004:4)			
Frac. (*d*)	(0.52 1.21)	(-0.49 0.68)	8.08 0.89 (0.51 1.29)
OU (*c*)	(-3.00 10.50)	(3.50 30.00)	8.03 -0.50 (-3.00 7.50)
LL (*g*)	(9.50 30.00)	(0.00 8.00)	8.42 28.50 (8.50 30.00)
I -OU (*c*)	(. .)	(. .)	6.60 30.00 (14.50 30.00)
I – LL (*g*)	(0.00 5.50)	(. .)	7.94 1.00 (0.00 8.00)
R. Dividend-Price Ratio (SP500, 1880-2002)			
Frac. (*d*)	(0.49 0.86)	(0.69 1.49)	12.45 0.72 (0.45 1.00)
OU (*c*)	(6.00 25.00)	(-3.00 14.00)	11.15 7.50 (-3.00 24.50)
LL (*g*)	(26.00 30.00)	(. .)	12.43 30.00 (17.50 30.00)
I -OU (*c*)	(. .)	(-3.00 30.00)	0.78 30.00 (23.00 30.00)
I – LL (*g*)	(. .)	(0.00 30.00)	10.52 0.00 (0.00 6.50)
S. Earnings-Price (SP500, 1880-2002)			
Frac. (*d*)	(0.37 0.83)	(-0.49 -0.29) (0.23 1.49)	7.71 0.59 (0.30 0.90)
OU (*c*)	(10.00 30.00)	(-3.00 30.00)	7.01 18.50 (2.50 30.00)
LL (*g*)	(19.00 30.00)	(6.00 30.00)	7.26 30.00 (13.50 30.00)
I -OU (*c*)	(. .)	(-3.00 30.00)	-5.32 30.00 (23.00 30.00)
I – LL (*g*)	(. .)	(0.00 30.00)	4.42 0.00 (0.00 5.50)
T. Dividend/Price (CRSP 1926-2002)			
Frac. (*d*)	(0.65 1.23)	(-0.49 -0.04) (0.49 1.49)	10.00 0.93 (0.57 1.28)
OU (*c*)	(-3.00 12.00)	(-3.00 24.00)	9.95 1.00 (-3.00 12.00)
LL (*g*)	(29.00 30.00)	(15.00 30.00)	9.20 30.00 (16.50 30.00)
I -OU (*c*)	(. .)	(9.50 30.00)	7.46 30.00 (18.50 30.00)
I – LL (*g*)	(0.00 10.00)	(0.00 30.00)	9.96 1.00 (0.00 10.50)
U. Absolute Returns (SP500)			
Frac. (*d*)	(0.19 0.71)	(0.60 1.49)	2.96 0.48 (0.10 0.88)
OU (*c*)	(9.50 30.00)	(-3.00 13.00)	2.41 18.00 (2.50 30.00)
LL (*g*)	(5.50 30.00)	(20.50 30.00)	3.37 16.50 (5.00 30.00)
I -OU (*c*)	(. .)	(0.50 30.00)	-3.58 30.00 (18.00 30.00)
I – LL (*g*)	(. .)	(0.00 30.00)	-0.15 0.00 (0.00 5.50)

Notes: Entries under the 90% CI columns are 90% confidence intervals for the parameter of the model obtained by inverting 10% level tests based on the LBIM and $\lvert\hat{\rho}\rvert$ statistics, respectively. The four numbers in the fourth column are the maximum of the log-likelihood, the parameter that maximizes the log-likelihood, and the set of values of the parameter that yield likelihoods within 2 log-points of the maximum.

Table A1
Data Description and Sources

Series	SMPL	Trans	Source and Notes
Real GDP	PWQ	ln τ	CB: GDP157
Real GNP	1869-2004	ln τ	1869-1928: Balke and Gordon (1989) 1929-2004: BEA (Series are linked in 1929)
Inflation (GDP Deflator)	PWQ	lev μ	CB: 400×ln(GDP272(t)/GDP272(t−1))
Inflation (GNP Deflator)	1870-2004	lev μ	GNP Deflator (PGNP): 1869-1928: Balke and Gordon (1989) 1929-2004: BEA (Series are linked in 1929) Inflation Series is 100×ln(PGNP(t)/PGNP(t−1))
Productivity	PWQ	ln τ	CB: LBOUT (Output per hour, business sector)
Hours	PWQ	ln μ	CB: LBMN(t)/P16(t) (Employee hours/population)
10YrTBond	PWQ	lev μ	CB: FYGT10
1YrTBond	PWQ	lev μ	CB: FYGT1
3mthTbill	PWQ	lev μ	CB:FYGM3
Bond Rate: Industrial Bonds Highest Rating	1900-2004	lev μ	NBER: M13108 (1900-1946) CB: FYAAAI (1947-2004)
Real Tbill Rate: Ex-post 3-month real rate using PCE inflation	PWQ	lev μ	CB: FYGM3(t)-400×ln(GDP273(t+1)/GDP273(t))
Real Bond Rate: Nominal Rates minus Inflation (GNP Defl.)	1900-2004	lev μ	R(t) – 100×ln(PGNP(t)/PGNP(t−1)) R(t) = Bond Rate (described above) PGNP = GNP deflator (described above)
Dollar/Pound Real Ex. Rate	1792-1990	ln μ	Lothian and Taylor (1996)
TBond Spread	PWQ	lev μ	CB: FYGT10-FYGT1
Unit Labor Cost (Real, Bus. Sector)	PWQ	ln μ	CB: LBLCP(t)/LBGDP(t)
real C-GDP	PWQ	ln(rat) μ	CB: GDP 158/GDP157
real I-GDP	PWQ	ln(rat) μ	CB: GDP 177/ GDP 157
Div/Price (SP500)	1880-2002	ln(rat) μ	Campbell and Yogo (2006)
Earnings/Price (SP500)	1880-2002	ln(rat) μ	Campbell and Yogo (2006)
Div/Price (CRSP)	1926-2002	ln(rat) μ	Campbell and Yogo (2006)
Abs.Returns (SP500)	Daily 1/3/1928 – 11/22/2005	lev μ	SP: SP500(t) is the closing price at date t. Absolute returns are |ln[SP500(t)/SP500(t−1)]|