Subjective Learning

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Motivation - Option Pricing

- Consider a call option with some illiquidity in the underlying asset market - the asset can not be resold immediately.
- At the strike time, the buyer asks himself what the value of the asset will be at the time he can sell.
- At the time he buys the option, he asks himself what he will learn about the resale value by the strike time.
- One reason for the prevalence of options (compared to contracts on fundamentals) might be that this learning is partially subjective.
- We want to talk about the price of options in terms of the buyer's subjective learning (more generally: a subjective model of Option value).
- What happens if the strike time is randomly distributed, with a known distribution (e.g. tied to some random variable exceeding a particular threshold)?

Motivation - More general

- Information is usually taken as an explicit part of economic models.
- If information cannot be observed, or if it varies across economic agents, it should be derived. How can we identify unobserved information?
- We provide an axiomatic characterization of information from choice behavior (preferences over menus of acts).
- The instrumental value of information depends on the available options. The value of a choice set depends on the information structure. Can we identify the link between the (subjective) information structure and preference for flexibility?

Outline

- The relevant domain
- A general model of subjective learning
  - A representation of second-order beliefs
  - A representation of subjective learning
  - Behavioral comparison in terms of learning
- Refinement - Subjective temporal resolution of uncertainty
  - Motivation: Option pricing with random strike times
  - A representation of subjective filtration (exclusive tree)
  - Behavioral comparison in terms of *early* learning
  - Reevaluation of the domain in this context
  - A reinterpretation: Flow utilities
- I will mention the most relevant literature, as I go along.

The relevant domain

- Subjective learning requires some uncertainty.
- Savage (1954):
  - Objective state space S captures all relevant aspects of the world. Space of possible outcomes X
  - DM chooses between acts $f \in \mathcal{F}$, where $f : S \to X$
  - Preferences are described by the DM's beliefs μ over S (and a utility on X)
- If information was objective: DM and modeler observe realization of some random variable to get to new information set $S' \subset S$. Beliefs updated according to Bayes' law
- We could look at choice over two-stage acts, $h : 2^S \to \mathcal{F}$, and expect a representation

$$V(h) = \sum_{S' \in 2^S} \left[\sum_{s \in S'} h(S')(s) \frac{\mu(s)}{\mu(S')} \right] \rho(S')$$

such that $\sum_{S' \in 2^S | s \in S'} \frac{\rho(S')}{\mu(S')} = 1$.

The relevant domain

- Compound acts are not available, if information is subjective. Instead, we have to consider menus of acts, which collect alternatives for choice at a later point in time.
- $S = \{s_1, ..., s_k\}$ is a finite state space.
- An act is a mapping $f : S \to [0, 1]$. $\mathcal{F}$ denotes the set of all acts.
- We interpret payoffs in $[0, 1]$ to be in "utils". We assume that the utility function over outcomes is known and payoffs are stated in its units.
- $\mathcal{K}(\mathcal{F})$ is the set of all non-empty compact subsets of $\mathcal{F}$. A typical menu is $F = \{f, g, h, ...\} \in \mathcal{K}(\mathcal{F})$
- $\succeq$ is a preference relation over $\mathcal{K}(\mathcal{F})$.

Preview -A general representation

Theorem: The relation $\succeq$ satisfies our axioms if and only if there is a probability measure μ on S and a probability measure ρ on 2^S, such that $\sum_{S' \in 2^S | s \in S'} \frac{\rho(S')}{\mu(S')} = 1$ for all s, and

$$V(F) = \sum_{S' \in 2^S} \max_{f \in F} \left[\sum_{s \in S'} f(s) \frac{\mu(s)}{\mu(S')} \right] \rho(S')$$

represents $\succeq$.

Furthermore, the pair (μ, ρ) is unique.

Comparison to DLRS - Domain

"Representing preferences with a unique subjective state space" DLR (2001) and DLRS (2007)

- DLRS: Choice over menus of lotteries, $A = \{p, q, ...\}$, where the space of outcomes X is finite. Lotteries are finite vectors with a probability for every prize. The entries must sum to 1.

- Our model: Finite objective state space, S. Acts are finite vectors with a utility for every state. Entries do not have to sum to 1.

- As a first step we will get a representation that is similar to the one in DLRS by capitalizing on the geometric similarity between the domains.

Axioms

- **Axiom 1 (Ranking)** $\succeq$ is a weak order.
- **Axiom 2 (vNM Continuity)** If $F \succ G \succ H$, then there are $\alpha, \beta \in (0,1)$, such that $\alpha F + (1-\alpha) H \succ G \succ \beta F + (1-\beta) H$.
- **Axiom 3 (Nontriviality)** There are F and G such that $F \succ G$.
- **Axiom 4 (Independence)** For all F, G, H, and $\alpha \in [0,1]$, $F \succeq G \Leftrightarrow \alpha F + (1-\alpha) H \succeq \alpha G + (1-\alpha) H$.
- **Axiom 5 (Set Monotonicity)** $F \subset G$ implies $G \succeq F$.
- **Axiom 6 (Domination)** $f \geq g$ and $f \in F$ imply $F \sim F \cup \{g\}$.

A representation of second-order beliefs

Theorem

The relation $\succeq$ satisfies axioms 1-6 if and only if it can be represented by

$$V(F) = \int_J \max_{f \in F} \left(\sum_{s=1}^{S} f(s)\, \pi_j(s) \right) dp(j)$$

where $\pi_j(\cdot)$ is a unique (up to a set of $p-$measure zero) probability measure on S and $p(\cdot)$ is a unique probability measure on the space of all those measures.

- A similar result appears in Takeoka (2004).

Comparison to DLRS - Representation

- Our representation

$$V(F) = \int_J \max_{f \in F} \left(\sum_{s \in S} f(s) \pi_j(s) \right) dp(j)$$

- DLRS: Choice over menus of lotteries, $A = \{p, q, ...\}$, where space of outcomes, X, is finite.

$$U(A) = \int_S \max_{p \in A} \left(\sum_{x \in X} p(x) u_s(x) \right) d\eta(s)$$

- The following objects are analogous. Objective: $S \leftrightarrow X$, $f(s) \leftrightarrow p(x)$ Subjective: $J \leftrightarrow S$, $p(j) \leftrightarrow \eta(s)$, $\pi_j(s) \leftrightarrow u_s(x)$
- In DLRS the pair (u, η) describes behavior, but the two parameters are only jointly identified.
- In contrast, in our representation the pair (π, p) is unique, because $\sum_S \pi_j(s) = 1$ must hold for all j.

Subjective learning

- Suppose that S fully describes the individual's uncertainty. The modeler does have access to the state space that is relevant for the DM's decisions.
  - Savage idea: " *[S]* represents a description of the world, leaving no relevant aspect undescribed"
- The DM's uncertainty about his beliefs can be understood as uncertainty about the information set he will be in at the time of choosing from the menu.

Definitions

- $J^* \subset J$ is the support of the measure $p(\cdot)$

- Given $f \in \mathcal{F}$, define

$$f^0_{-s}(s') = \begin{cases} f(s') \text{ if } s' \neq s \\ 0 \text{ if } s' = s \end{cases}$$

- Given $f \in \mathcal{F}$ with $f(s) < 1 - \varepsilon$, define

$$f^\varepsilon_s(s') = \begin{cases} f(s') \text{ if } s' \neq s \\ f(s') + \varepsilon \text{ if } s' = s \end{cases}$$

- The support of an act f is $\sigma(f) = \{s \in S : f(s) > 0\}$.

Maximin ("Maximal Fat Free") menus

We provide a general definition of *maximin* menus. Our axioms (including the next one) allow us to establish that all maximin menus are finite. Taking into account that maximin sets are finite, our definition boils down to the following:

- A menu of acts F is *minimal* if for all $f \in F$ and for all $s \in \sigma(f)$, $F \succ (F \backslash f) \cup f_{-s}^{0}$.
  - all acts on a minimal set are "fat-free': reducing an outcome in any state in the support results in an inferior set.
- A menu F is *maximin* if it is minimal and
  1. for all $f \in F$ and for all $s \notin \sigma(f)$, there exists $\bar{\varepsilon} > 0$ such that $F \sim F \cup f_{s}^{\varepsilon}$ for all $\varepsilon < \bar{\varepsilon}$, and
  2. there exists no menu $G \nsubseteq F$ such that $F \cup G$ is minimal.
  - (2) You cannot add items to a Maximin set without making it "fatty"

Identification of acts and beliefs

- **Claim:** A maximin set F always exists
- **Claim:** If F is maximin and finite, then F is isomorphic to the set of first order beliefs, J^*. **Intuition:**
  - Suppose $f \in F$ does best for two (or more) first-order beliefs, i and j
  - Construct g that does better in only one of them $\Rightarrow$ F was not maximin.
  - If $f \in F$ does not do best for any $i \in I$, then F is not minimal, and therefore not maximin.
- **Claim:** Suppose F is maximin and $f \in F$. Let $i(f) \in J$ be the belief such that $f = \arg\max \langle f, \pi(\cdot | i(f)) \rangle$. Then $\sigma(f) = \{s \in S : \pi(s | i(f)) > 0\}$
  - The isomorphism is such that $s \in \sigma(f)$ if and only if s is also in the support of the corresponding first-order belief.

A representation of subjective learning

- **Axiom 7 (Constant MRS)** If F is maximin, $f, g \in F$, and $s, s' \in \sigma(f) \cap \sigma(g)$, then there is $\bar{\varepsilon} > 0$ such that for all $\varepsilon \leq \bar{\varepsilon}$.

$$F \cup f_s^\varepsilon \sim F \cup f_{s'}^\delta \Rightarrow F \cup g_s^\varepsilon \sim F \cup g_{s'}^\delta$$

  - Idea: the relative weight of two states should not change between two information sets that contain both states.

Theorem

$\succeq$ satisfies Axioms 1-7 if and only if there is a probability measure μ on S and a probability measure ρ on 2^S, such that $\sum_{S' \in 2^S | s \in S'} \frac{\rho(S')}{\mu(S')} = 1$ for all s, and

$$V(F) = \sum_{S' \in 2^S} \max_{f \in F} \left[\sum_{s \in S'} f(s) \frac{\mu(s)}{\mu(S')} \right] \rho(S')$$

represents $\succeq$.

Furthermore, the pair (μ, ρ) is unique.

Intuition

- If two beliefs i and j in J^* disagree on the relative weight of two states in their support, then the MRS of utility in one state versus the other must be different, which contradicts Axiom 7.
- In particular, no two beliefs can have the same support $\Rightarrow$ the measure ρ on 2^S is identified by p in Theorem 1
- The measure μ must satisfy

$$\mu(s) = \sum_{J^*} \pi_j(s)\, p(j)$$

- It immediately follows that

$$\mu(s) = \sum_{S' \in 2^S | s \in S'} \frac{\mu(s)}{\mu(S')} \rho(S')$$

or

$$\sum_{S' \in 2^S | s \in S'} \frac{\rho(S')}{\mu(S')} = 1$$

Behavioral comparison in terms of learning

- **Definition:** DM1 has *more preferences for flexibility* than DM2 if for all $f \in \mathcal{F}$ and for all $G \in \mathcal{K}(\mathcal{F})$

$$\{f\} \succeq_1 G \;\Rightarrow\; \{f\} \succeq_2 G$$

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- **Definition:** DM1 has the same beliefs as DM2 if $\mu^1 = \mu^2$

- **Definition:** DM1 *learns more* than DM2, if for all $E \in 2^S$

$$\sum_{E' \subseteq E} \rho_1(E') \geq \sum_{E' \subseteq E} \rho_2(E')$$

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Theorem

DM1 has more preference for flexibility than DM2, if and only if DM1 learns more than DM2 and they have the same beliefs.

Intuition

- **If:** DM1 can always imitate DM2's choice and hence must be weakly better off with a menu rather than a singleton, whenever DM2 is.

Intuition

- **Only if:** Taking $G = \{g\}$ implies that they have same preferences on singletons, hence same beliefs.
- Suppose that there is $E \in 2^S$ with $\sum_{E' \subseteq E} \rho_2(E') > \sum_{E' \subseteq E} \rho_1(E')$. Obviously E is a strict subset of the support of μ.
- Define the act $f := \begin{cases} \delta > 0 \text{ if } s \in E \\ 0 \text{ if } s \notin E \end{cases}$
- Let c denote the constant act that gives $c > 0$ in every state, such that $\delta > c > \frac{\mu(E)}{\mu(S')}\delta$ for all S' that are a strict superset of E. Then $V_i(\{f, c\}) = c + (\delta - c) \sum_{E' \subseteq E} \rho_i(E')$
- Finally, pick c' such that

$$(\delta - c) \sum_{E' \subseteq E} \rho_2(E') > c' - c > (\delta - c) \sum_{E' \subseteq E} \rho_1(E')$$

to find $\{f, c\} \succ_2 \{c'\}$ but $\{c'\} \succ_1 \{f, c\}$, and hence DM1 can not have more preference for flexibility than DM2.

Motivation - options with random strike time

- As before, consider a call option with some illiquidity in the underlying asset market - the asset can not be resold immediately.
- But now assume that the strike time is randomly distributed on time interval $[0, 1]$ with a known distribution (e.g. tied to some random variable exceeding a particular threshold)
- At the time the buyer contemplates the option, he asks himself what he will learn about the resale value by the strike time.
- The question then becomes when exactly the buyer expects to receive information, as his information set at any point in time $t \in [0, 1]$ might end up being the relevant one.
- We want to model a DM who anticipates the (subjective) gradual resolution of uncertainty over time: He faces a filtration $\{\mathcal{P}_t\}$ on S, which is indexed by $t \in [0, 1]$

Preview - a subjective, exclusive tree representation

- **Definition:** The pair $(\mu, \{\mathcal{P}_t\})$ is an exclusive tree, if μ is a probability measure on S and $\{\mathcal{P}_t\}$ is a filtration indexed by $t \in [0, 1]$.
- Suppose for simplicity, that stopping time is distributed uniformly on $[0, 1]$
- **Theorem** $\succ$ satisfies our axioms if and only if there is an exclusive tree, $(\mu, \{\mathcal{P}_t\})$, such that

$$V(F) = \int_{[0,1]} \left(\sum_{P \in \mathcal{P}_t} \max_{f \in F} \left[\sum_{s \in P} f(s) \mu(s) \right] \right) dt$$

represents $\succeq$.

Furthermore, $(\mu, \{\mathcal{P}_t\})$ is unique.

Definitions

- **Definition:** An act f *contains* act g if $\sigma(g) \subset \sigma(f)$.
- **Definition:** Acts f and g *do not intersect* if $\sigma(g) \cap \sigma(f) = \varnothing$.
- **Definition:** $F^{\varepsilon}_{f,s} := F \cup \{g^{\varepsilon}_{s} : g \in F,\ f \text{ contains } g \text{ and } s \in \sigma(g)\}$

Axioms

- **Axiom 8 (Exclusivity):** If F is maximin and $f, g \in F$, then either f and g do not intersect or one contains the other.
  - The collection of supports of the first-order beliefs can be arranged to form a filtration on S. The objective state space S is large enough to capture the subjective resolution of uncertainty, that is, the filtration on the underlying subjective state space is measurable in S.
- **Axiom 9 (Strong Constant MRS):** If F is maximin, $f \in F$, and $s, s' \in \sigma(f)$, then there is $\bar{\varepsilon} > 0$ such that $F \cup f_s^{\varepsilon} \sim F \cup f_{s'}^{\delta}$ implies $F_{f,s}^{\varepsilon} \sim F_{f,s'}^{\delta}$ for all $\varepsilon \leq \bar{\varepsilon}$.
  - If in information set S' the DM assigns probabilities p and p' to two states s and s', respectively, then p and p' are also his expectations of the probabilities of these two states for any future point in time.
- **Axiom 10 (No Immediate Learning):** If F is maximin, then there exist $f \in F$ such that f contains g for all $g \in F$ with $g \neq f$.
  - The (subjective) flow of information can not start before time 0 (at which point his beliefs commence to be relevant for choice from the menu).

A subjective, exclusive tree representation

Theorem

Under the assumptions of Theorem 1, $\succeq$ satisfies Axioms 8 and 9 if and only if there is an exclusive tree, $(\mu, \{\mathcal{P}_t\})$, such that

$$V(F) = \int_{[0,1]} \left(\sum_{P \in \mathcal{P}_t} \max_{f \in F} \left[\sum_{s \in P} f(s)\, \mu(s) \right] \right) dt$$

represents $\succeq$.

If $\succeq$ also satisfies Axiom 10, then there is $\widehat{S} \subseteq S$, such that $\mathcal{P}_0 = \left\{ \widehat{S} \right\}$, that is, the tree $(\mu, \{\mathcal{P}_t\})$ has a unique initial node.

In either case, $(\mu, \{\mathcal{P}_t\})$ is unique.

Intuition

- Axiom 9 implies axiom 7 $\longrightarrow$ Theorem 2.
- Axiom 8 implies that information sets in Theorem 2 can be ordered so as to form a filtration.
- As in Theorem 2, the relative weights assigned to any two states by two different information sets that support them must be the same.
- But the relative weights assigned to two states that are contained in distinct information sets may change.
- The weight reflects both the probabilities and the duration for which the DM expects to be in the relevant information set $\longrightarrow$ change in weights must be absorbed by time, such that the relative weight for any two states is constant over time, from the ex-ante perspective (as dictated by Bayes' law)
- The strengthening from Axiom 7 to Axiom 9 implies that the aggregated time adjustments along every branch of the tree amount to the same.
- There is a unique way to adjust time.

Behavioral comparison in terms of early learning

- **Definition:** DM1 learns earlier than DM2 if $\left\{\mathcal{P}_t^1\right\}$ is weakly finer than $\left\{\mathcal{P}_t^2\right\}$ (DM1's partition is finer than that of DM2 for all $t \in [0,1]$)

Corollary

DM1 has more preference for flexibility than DM2, if and only if DM1 learns earlier than DM2 and they have the same beliefs.

- **Remark**: the property that DM1 learns earlier than DM2 does not imply that DM1's willingness to pay to add options to a given menu is always greater than that of DM2.
- **Remark**: Consider the special case where neither decision maker expects to learn over time. In that case, their respective preferences can be described by a degenerate filtration with $\mathcal{P}_t = \mathcal{P}$ for all $t \in [0,1]$. Then more preferences for flexibility corresponds to having finer partition. This implication of our result is independently observed by Lleras (2011).

Reevaluating the domain

- As argued before, compound acts are not a feasible domain, when information is subjective.
- When considering temporal resolution of subjective uncertainty, it might seem natural to consider compound menus (menus over menus of acts, etc.) At every stage, such a compound menu specifies the set of feasible actions. Takeoka (2007) derives a compound lottery with subjective probabilities based on this approach.
- This domain has two drawbacks:
  1. To specify the domain, the timing of the stages must be objective.
  2. The domain is complicated and very large (menus of infinite dimensional objects)
- We suggest, instead, to reinterpret the familiar domain of menus of acts: The menu specifies the collection of actions that are available to the DM at any point in time. We show that it is rich enough for a full identification, even of the timing of information arrival in continuous time.

A reinterpretation - flow utilities

- At any point $t \in [0, 1]$ the DM holds one act from the menu.
- At time 1, the true state of the world becomes objectively known, and the DM is paid the convex combination of the payoffs specified by all acts on the menu, where the weight assigned to each act is simply the amount of time the DM held it.
- That is, the DM derives a flow utility from holding a particular act, where the state dependent flow is determined ex post, at the point where payments are made.
- The information set at any point in time $t \in [0, 1]$ is relevant for choice from the menu.