RESEARCH PAPER NO. 2058

Existence of Equilibria in All-Pay Auctions

Srihari Govindan

Robert Wilson

July 2010

This work was supported in part by a grant from the U.S. National Science Foundation.

ABSTRACT

For an all-pay sealed-bid auction of an item for which each bidder's realized value can depend on every bidder's privately observed signal, existence of equilibria in behavioral strategies is established using only the assumption that bidders' value functions and the density function of signals are positive and continuous on a product of intervals. Such equilibria have atomless distributions of bids and thus are not affected by how tied bids are resolved.

EXISTENCE OF EQUILIBRIA IN ALL-PAY AUCTIONS

SRIHARI GOVINDAN AND ROBERT WILSON

Abstract. For an all-pay sealed-bid auction of an item for which each bidder's realized value can depend on every bidder's privately observed signal, existence of equilibria in behavioral strategies is established using only the assumption that bidders' value functions and the density function of signals are positive and continuous on a product of intervals. Such equilibria have atomless distributions of bids and thus are not affected by how tied bids are resolved.

1. Introduction

The theory of auctions developed initially without general theorems on existence of equilibria. Recently, Jackson and Swinkels [5] establish existence for a wide variety of auctions and double auctions with private values, i.e. those in which each bidder observes directly his values of traded items. For auctions with interdependent values, the prevalent tactic follows Milgrom and Weber [7] in assuming that each bidder's private information is represented by a real-valued signal, his realized value of the item is a continuous monotone function of all signals, and the joint density of signals satisfies the affiliation inequality. Because these assumptions yield monotone pure strategies, and thus exclude atoms in the distribution of optimal bids, they circumvent payoff discontinuities due to tie-breaking rules. For a first-price sealed-bid auction of a single item, the most general existence theorem using a similar formulation is by Reny and Zamir [8]. However, de Castro [1] emphasizes that affiliation is a strong assumption.[1] The most general analysis with weaker assumptions is by Jackson, Simon, Swinkels, and Zame [4], who establish existence of an equilibrium when ties are resolved endogenously based on bidders' reports of their observed signals, and truthful reporting is incentive compatible.

Date: 25 July 2010. This paper can be downloaded from
`gsbapps.stanford.edu/researchpapers/library/RP2058.pdf`.

Key words and phrases. auction, all-pay, bidding, signals, interdependent values, common value, equilibrium strategies, existence.

JEL subject classifications: C72, D44.

This work was funded in part by a grant from the National Science Foundation of the United States.

[1]The affiliation inequality requires essentially that signals are non-negatively correlated on every product of intervals. De Castro [1, Theorem 3.1] proves that the subset of continuous density functions that violate the affiliation inequality is open and dense. He examines several weaker properties and establishes for some that there need not exist equilibria in monotone pure strategies.

This paper considers an all-pay auction of a single item with an exogenous tie-breaking rule. Rather than assuming symmetry and affiliation as in Krishna and Morgan [6, §4], we assume only that bidders' value functions and the density function of signals are positive and continuous. The main result establishes existence of equilibria in which for every bid there is zero probability of a tie, and thus the tie-breaking rule has no effect. As in [2, 3] for a first-price auction, these equilibria of an all-pay auction are obtained as essential equilibria of a modified auction obtained by slightly altering how bidders respond to possibilities of tied bids.

2. The Auction Game

We consider an N-player game G that represents an all-pay sealed-bid auction of a single item. In the extensive form of the game, first Nature specifies a profile $s = (s_1, \ldots, s_N)$ of signals, one for each bidder $n = 1, \ldots, N$, according to a distribution F. Then each bidder n observes his own signal s_n and chooses a bid $b_n(s_n)$. He wins the item if his bid is strictly higher than others' bids; or when tied with others' bids, if he is selected by an exogenous tie-breaking rule. Finally, bidder n's payoff is $v_n(s) - b_n(s_n)$ if he wins the item and $-b_n(s_n)$ otherwise.[2]

We impose the following assumption on the joint distribution of signals.

Assumption: [Distribution of Signals]

(1) The set of possible signal profiles is a product set $S = \prod_n S_n$. Each bidder's set of possible signals is the same closed real interval, say $S_n = [0, 1]$.
(2) The distribution F of signal profiles has a density $f : S \to \mathbb{R}$ that is continuous and positive.[3]

For each n let $\mathcal{S}_n$ be the collection of Borel measurable subsets of S_n and let λ_n be the Lebesgue measure on S_n. Similarly, $\mathcal{S}_{-n}$ is the Borel measurable subsets of $S_{-n} = \prod_{m \neq n} S_m$ and λ_{-n} is the Lebesgue measure on S_{-n}.

For each bidder n and his signal s_n, let $F_n(\cdot|s_n)$ be the conditional distribution of the signals s_{-n} of n's opponents, and let $f_n(\cdot|s_n)$ be its density function. Assumption (2) ensures that bidder n's conditional densities $f_n(\cdot|s_n) : S_{-n} \to \mathbb{R}$ indexed by his signal $s_n \in S_n$ are an equicontinuous family.

We impose the following assumption on bidders' value functions.

Assumption: [Bidders' Value Functions]

[2]If Nature also chooses unobserved components $\tilde{\omega}$ of a bidder's realized value $\tilde{v}_n(s, \tilde{\omega})$ then his value is $v_n(s) = E[\tilde{v}_n(s, \tilde{\omega})|s]$.

[3]This assumption can be weakened: the distribution F needs only to be absolutely continuous w.r.t. the product of its marginal distributions.

(3) The set of possible profiles of realized values is the product set $V = \prod_n V_n$. Each bidder's set of possible values is the same closed real interval, say $V_n = [v_*, v^*]$, where $0 \leqslant v_* < v^*$.
(4) The joint valuation function $v : S \to V$ is continuous.
(5) For each n and signal $s_n \in S_n$, bidder n's value $v_n(s_n, \cdot)$ is positive outside of a set with λ_{-n} measure zero in S_{-n}.[4]

In view of Assumption (3) it suffices to assume that for each bidder the feasible set of bids is the interval $B = [0, v^*]$. Denote the collection of Borel measurable subsets of B by $\mathbf{B}$.

Because the game has perfect recall, we specify a bidder's strategy in behavioral form as mixtures over bids conditional on his signals. A behavioral strategy for bidder n is a transition-probability function $\sigma_n(\cdot|\cdot) : \mathbf{B} \times S_n \to [0,1]$ such that for each signal s_n, $\sigma_n(\cdot|s_n)$ is a probability measure on B; and for each event $A \in \mathbf{B}$, $\sigma_n(A|\cdot)$ is a measurable function on S_n.[5] Let Σ_n be n's set of behavioral strategies, and define $\Sigma = \prod_n \Sigma_n$.

For each n let $\Sigma_{-n} = \prod_{m \neq n} \Sigma_m$ and $\mathbf{B}_{-n} = \prod_{m \neq n} \mathbf{B}$. For each n and $\sigma_{-n} \in \Sigma_{-n}$, define a transition-probability function $\tilde{\sigma}_{-n} : \mathbf{B}_{-n} \times S_{-n} \to [0,1]$ by $\tilde{\sigma}_{-n}(B_{-n}|s_{-n}) = \prod_{m \neq n} \sigma_m(B_m|s_m)$ for every product $B_{-n} = \prod_{m \neq n} B_m$ of measurable sets. This is a homeomorphism of Σ_{-n} onto its image, so to reduce notation we use σ_{-n} for the function $\tilde{\sigma}_{-n}$ as well. And, we write $\sigma_{-n}(C)$ when $B_m = C$ for every $m \neq n$.

We endow behavioral strategies with the topology of weak convergence; i.e. a sequence σ_n^k of strategies in Σ_n converges to σ_n iff for every continuous function $\eta : B \to \mathbb{R}$ and each event $W_n \in \mathcal{S}_n$,

$$\int_{W_n} \int_B \eta(b)\, d\sigma_n^k(b|s_n)\, ds_n \to \int_{W_n} \int_B \eta(b)\, d\sigma_n(b|s_n)\, ds_n\,.$$

With this topology, Σ_n is a compact (metrizable) space.[6] [7]

Say that a bid b is a *point of continuity* of n's strategy σ_n if $\sigma_n(\{\, b\,\}|\cdot)$ is zero λ_n-a.e. on S_n. That is, for almost no signal s_n does n bid b with positive probability. Similarly, say

[4]This assumption is imposed for expositional simplicity. Without it, there might be an equilibrium with positive probability of ties at the bid zero, but in this event each bidder's value is zero, so any tie-breaking rule would suffice.

[5]Strictly speaking, a behavioral strategy is an equivalence class of transition-probability functions, where σ_n is equivalent to σ_n' if $\sigma_n(\cdot|s_n) = \sigma_n'(\cdot|s_n)$ λ_n-a.e. on S_n.

[6]This definition is weaker than requiring (λ_n-a.e. on S_n) pointwise weak convergence of the sequence of distributions $\sigma_n^k(\cdot|s_n)$. An equivalent definition requires that the displayed integral be u.s.c. (resp. l.s.c.) for each function η that is u.s.c. (resp. l.s.c.).

[7]Instead of the Lebesgue measure λ_{-n} on S_{-n}, if one uses the measure $F_n(\cdot|s_n)$ for some signal $s_n \in S_n$ then it represents bidder n's interim belief after receiving his signal s_n. By Assumption (2) on F, for every signal s_n the induced topology on the profiles σ_{-n} is the same.

that b is a point of continuity of σ_{-n} if $\sigma_{-n}(\{\, b \,\}|\cdot) = 0$ λ_{-n}-a.e. on S_{-n}. In particular, the probability is zero that b is the highest bid among n's opponents.

For the auction game G the expected payoff to a bidder n when his signal is s_n, he bids b, and each $m \neq n$ uses strategy $\sigma_m \in \Sigma_m$ is

$$\pi_n(s_n, b, \sigma_{-n}) = \int_{S_{-n}} v_n(s_n, s_{-n}) P_n(s_{-n}|b, \sigma_{-n}) f_n(s_{-n}|s_n)\, ds_{-n} - b\,,$$

where $P_n(s_{-n}|b, \sigma_{-n})$ is his probability of winning. If ties are broken with equal probabilities for all high bidders then this probability is

$$P_n(s_{-n}|b, \sigma_{-n}) = \sum_{M \subset N \setminus n} (1/[|M| + 1]) \Bigg(\prod_{m \in M} \sigma_m(\{\, b \,\}|s_m) \Bigg) \Bigg(\prod_{m \notin M} \sigma_m([0, b)|s_m) \Bigg)\,,$$

with the convention that if the subset M of opponents who bid b is empty then the first product above is 1, or if $M = N \setminus n$ then the second product is 1. If opponents' strategies do not generate atoms in the distribution of the maximum of their bids then this expected payoff simplifies to

$$\pi_n(s_n, b, \sigma_{-n}) = \int_{S_{-n}} v_n(s_n, s_{-n}) \Bigg(\prod_{m \neq n} \sigma_m([0, b)|s_m) \Bigg) f_n(s_{-n}|s_n)\, ds_{-n} - b\,.$$

3. The Auxiliary Game

This section specifies an auxiliary game G^* that is exactly the same as the auction game G specified in Section 2 except for a change in the way a bidder responds to the possibility of tied bids.

For each bidder n, bid b, and profiles s_{-n} and σ_{-n} of opponents' signals and strategies, define the following two probabilities:

$$\sigma_{-n}(b^+|s_{-n}) = \prod_{m \neq n} \sigma_m([0, b])|s_m) \qquad \text{and} \qquad \sigma_{-n}(b^-|s_{-n}) = \prod_{m \neq n} \sigma_m([0, b)|s_m)\,,$$

which differ only in opponents' probabilities of the bid b. We use these to define the following alternative payoff functions. For each $(s_n, b, \sigma_{-n}) \in S_n \times B \times \Sigma_{-n}$,

$$\pi_n^+(s_n, b, \sigma_{-n}) = \int_{S_{-n}} v_n(s_n, s_{-n}) \sigma_{-n}(b^+|s_{-n}) f_n(s_{-n}|s_n)\, ds_{-n} - b\,,$$

$$\pi_n^-(s_n, b, \sigma_{-n}) = \int_{S_{-n}} v_n(s_n, s_{-n}) \sigma_{-n}(b^-|s_{-n}) f_n(s_{-n}|s_n)\, ds_{-n} - b\,.$$

Note that $\pi_n^+(s_n, b, \sigma_{-n})$ envisions winning if n's bid b exceeds or matches his opponents' bids, whereas $\pi_n^-(s_n, b, \sigma_{-n})$ envisions winning only if n's bid b exceeds his opponents' bids. Since the valuation function is non-negative, π_n^+ and π_n^- represent best- and worst-case payoffs from ties.

If σ_{-n} has no points of discontinuity then both π_n^+ and π_n^- agree with n's payoff function π_n in the auction game G. More generally, if $b^k \downarrow b$ then $\pi_n^+(s_n, b^k, \sigma_{-n})$ and $\pi_n^-(s_n, b^k, s_{-n})$ converge to $\pi_n^+(s_n, b, \sigma_{-n})$, and if $b^k \uparrow b$ then they converge to $\pi_n^-(s_n, b, \sigma_{-n})$.

Now define the auxiliary game G^* as follows. Given $\sigma_{-n} \in \Sigma_{-n}$, say that n's bid b is an optimal reply when his signal is s_n if $\pi_n^+(s_n, b, \sigma_{-n}) \geqslant \pi_n^-(s_n, c, \sigma_{-n})$ for all bids $c \in B$. That is, the best-case payoff from b must be as large as the worst-case payoff from any other bid.

The following lemma establishes the continuity properties of the best-case and worst-case payoffs. Its proof is an immediate consequence of the choice of topology.

Lemma 3.1. *The payoff functions π_n^+ and π_n^- : $S_n \times B \times \Sigma_{-n} \to \mathbb{R}$ are upper and lower semi-continuous respectively.*

For each $(s_n, \sigma_{-n}) \in S_n \times \Sigma_{-n}$ let $\phi_n(s_n, \sigma_{-n}) \subset B$ be the set of bidder n's optimal replies to σ_{-n} when his signal is s_n.

Lemma 3.2. *The correspondence $\phi_n : S_n \times \Sigma_{-n} \twoheadrightarrow B$ is upper semi-continuous and has nonempty and compact images.*

Proof. For each s_n and σ_{-n} the function π_n^+ is upper semi-continuous in b and hence attains a maximum over B. Any maximizer of π_n^+ is trivially an optimal reply and hence ϕ_n has nonempty images. The other two properties follow from the fact that π_n^+ is u.s.c. while π_n^- is l.s.c. $\square$

Let $\Phi_n : \Sigma_{-n} \twoheadrightarrow \Sigma_n$ be the correspondence that assigns to each strategy σ_{-n} of n's opponents the set of n's strategies σ_n such for each signal $s_n \in S_n$ the support of $\sigma_n(\cdot|s_n)$ is a nonempty subset of $\phi_n(s_n, \sigma_{-n})$. Then Φ_n is an u.s.c. correspondence with nonempty, compact, and convex images. And so too is the optimal-reply correspondence $\Phi : \Sigma \twoheadrightarrow \Sigma$ obtained as the product $\Phi(\sigma) = \prod_n \Phi_n(\sigma_{-n})$. By the Fan-Glicksberg fixed-point theorem, therefore, Φ has a fixed point that is an equilibrium of the auxiliary game G^*.

An equilibrium of G^* is not necessarily an equilibrium of the auction game G. However, those equilibria of G^* with nonatomic distributions of bids are equilibria of G.

4. Existence of Equilibria for the Auction Game

This section establishes that an equilibrium exists for the auction game G, regardless of the tie-breaking rule. This is done by showing that the auxiliary game G^* has an equilibrium with nonatomic bid distributions that is then an equilibrium of the auction game G.[8]

Theorem 4.1. *The auction game G has an equilibrium in behavioral strategies. In particular, it has an equilibrium in which ties among the highest bids occur with zero probability.*

The remainder of this section is devoted to proving this existence theorem. One begins with the observation that the optimal-reply correspondence Φ defined above has essential sets of fixed points. An essential set has the property that every sufficiently small perturbation of Φ has a fixed point arbitrarily close to the set. To exploit this property we construct a sequence of perturbed auxiliary games G^k that induce a sequence of perturbed correspondences Φ^k. We then show that the limit points of equilibria of the perturbed games, obtained as fixed points of the perturbed correspondences, are equilibria of both G^* and the auction game G. The perturbed games G^k are obtained simply by supposing that a bidder's bid is distorted by noise before it is received by the auctioneer.

The sequence of perturbed games G^k is constructed as follows. For each positive integer k, the strategy sets in the game G^k are the same as in the auction game G and in the auxiliary game G^*. However, when bidder n bids b the auctioneer perceives n's bid as the sum of b and a random variable uniformly distributed on the interval $[-1/k, 1/k]$. Thus, the payoff functions are defined as follows. First, for each bid $b \in B$ let μ_b^k be the uniform distribution over the interval $[b - 1/k, b + 1/k]$. Next, for each strategy σ_n^k of each bidder n, define a transition-probability distribution $\tilde{\sigma}_n^k$ from S_n to $B^k \equiv [-1/k, v^* + 1/k]$ via its distribution function

$$\tilde{\sigma}_n^k([-1/k, b]|s_n) = \int_B \int_{-1/k}^{b} d\mu_c^k(b')\, d\sigma_n^k(c|s_n)\,.$$

Then $\tilde{\sigma}_n^k([-1/k, b]|s_n)$ is the probability that the bid from n received by the auctioneer is no more than b, given that bids are subject to noise. For each n and each strategy profile σ_{-n}, the corresponding product profile induces a product transition-probability function over B_{-n}^k. Finally, the payoff to bidder n when his signal is s_n, he bids $b_n \in B$, and his opponents play the strategy-profile σ_{-n} is

$$\pi_n^k(s_n, b, \sigma_{-n}) = \int_{B^k} \pi_n^+(s_n, c, \tilde{\sigma}_{-n}^k)\, d\mu_b^k(c)\,,$$

[8] We actually prove the following weaker property: for every bid b, there is at least one bidder n for whom b is a point of continuity of σ_{-n}. This property is equivalent to the statement that the probability is zero of b being the highest bid among n's opponents. Thus, at every bid there is zero probability of a tie.

where the domain of $\pi_n^+(s_n, c, \sigma_{-n})$ is extended now to include bids in $B^k \setminus B$ in the obvious way.[9]

The payoff function π_n^k is continuous in n's bid b and therefore, as in the previous section, the correspondence ϕ_n^k that assigns to each (s_n, σ_{-n}) the set of bids that are optimal for n when his signal is s_n in reply to opponents' strategies σ_{-n} in game G^k is an u.s.c. correspondence with nonempty and compact images. The induced optimal-reply correspondence $\Phi^k : \Sigma \twoheadrightarrow \Sigma$ satisfies the conditions for existence of a fixed point, which is then an equilibrium of G^k.

Hereafter, select a sequence of k's for which a subsequence σ^k of equilibria of G^k converges to some profile $\sigma^* \in \Sigma$.

Lemma 4.2. *Let $(s_n^k, b^k) \in S_n \times B$ be a subsequence converging to (s_n, b). Then:*

$$\pi_n^+(s_n, b, \sigma_{-n}^*) \geqslant \limsup_k \pi_n^k(s_n^k, b^k, \sigma_{-n}^k) \geqslant \liminf_k \pi_n^k(s_n^k, b^k, \sigma_{-n}^k) \geqslant \pi_n^-(s_n, b, \sigma_{-n}^*).$$

Proof. Fix $\varepsilon > 0$. Since π_n^+ is upper semi-continuous and the sequence $(s_n^k, b^k, \sigma_{-n}^k)$ converges to (s_n, b, σ_{-n}^*), there exists K large enough such that for all $k \geqslant K$, $\pi_n^+(s_n^k, b', \sigma_{-n}^k) < \pi_n^+(s_n, b, \sigma_{-n}^*) + \varepsilon$ for all $b' \in [b - 1/K, b + 1/K]$. The payoff $\pi_n^k(s_n^k, b^k, \sigma_{-n}^k)$ is obtained as the average of values of $\pi_n^+(s_n^k, b', \sigma_{-n}^k)$ for b' in a smaller interval, so when k is very large this implies that $\pi_n^+(s_n, b, \sigma_{-n}^*) + \varepsilon \geqslant \limsup_k \pi_n^k(s_n^k, b^k, \sigma_{-n}^k)$. Since ε was arbitrary, this establishes the first of the claimed inequalities. The proof for the inequality involving π_n^- is similar. □

For each k and each bidder n let $\theta_n^k : S_n \to \mathbb{R}$ be the function that assigns to each signal s_n the corresponding equilibrium payoff for n from the equilibrium σ^k of game G^k.

Lemma 4.3. *The family of functions θ_n^k is bounded and equicontinuous, and has a subsequence that converges to a continuous function θ_n^*.*

Proof. For each signal s_n, bidder n's equilibrium payoff $\theta_n^k(s_n)$ is in the interval $[-1/k, v^*]$. Therefore, the θ_n^k's are uniformly bounded. Equicontinuity follows from the continuity of v_n and the equicontinuity property of the conditional densities $f_n(\cdot|s_n)$. Since the θ_n^k's are both bounded and equicontinuous, they are totally bounded; hence there is a Cauchy subsequence that is convergent. □

We assume hereafter that for each bidder n the sequence θ_n^k is itself the Cauchy subsequence identified in the above lemma.

[9]In this definition, one could equivalently replace π_n^+ with π_n^- or, indeed, any function that agrees with π_n^+ at bids that are points of continuity of σ_m.

Lemma 4.4. *For λ_n-a.e. signal s_n of bidder n, every bid b in the support of $\sigma_n^*(\cdot|s_n)$ is an optimal reply to σ_{-n}^* in the auxiliary game G^* and $\pi_n^+(s_n, b, \sigma^*) = \theta_n^*(s_n)$.*

Proof. We first show that $\pi_n^+(s_n, b, \sigma^*) \leqslant \theta_n^*(s_n)$ for all s_n and b. Indeed, if this is not true for some (s_n, b), then if necessary by replacing b with a slightly higher bid, we can assume that b is a point of continuity and $\pi_n^+(s_n, b, \sigma^*) = \pi_n^-(s_n, b, \sigma^*) > \theta_n^*(s_n)$. There is now a closed neighborhood T_n of s_n such that the corresponding inequality is strict for all $t_n \in T_n$. Therefore, for all large k, $\pi_n^k(t_n, b, \sigma^k) > \theta_n^k(t_n)$ for all t_n: indeed, otherwise there is a sequence t_n^k converging to some $t_n \in T_n$ such that $\lim_k \pi_n^k(t_n, b, \sigma^k) = \pi_n^+(t_n, b, \sigma^*) \leqslant \theta_n^*(t_n)$, which is impossible by construction of T_n. Since $\theta_n^k(t_n)$ is the equilibrium payoff of t_n in the game G^k, the inequality $\pi_n^k(t_n, b, \sigma^k) > \theta_n^k(t_n)$ is itself impossible, which proves that $\pi_n^+(s_n, b, s^*) \leqslant \theta_n^*(s_n)$ for all s_n and b.

Let X_n be the set of (s_n, b) such that $\pi_n^+(s_n, b, \sigma_{-n}^*) = \theta_n^*(s_n)$. By the upper semi-continuity of π_n^+ and the continuity of θ_n^*, it follows from the previous paragraph that X_n is a closed set and that if b is a best reply for s_n against σ^* then $(s_n, b) \in X_n$. We claim that for each $(s_n, b) \notin X_n$ there exists an open neighborhood $W \times C$ such that $\int_W \sigma_n^*(C|s_n)\, ds_n = 0$, which would complete the proof of the Lemma. To prove this claim, choose any open neighborhood $W \times C$ of (s_n, b) such that its closure is contained in $(S_n \times B) \setminus X$. We assert that for all large k, $\pi_n^k(s_n', c, \sigma_{-n}^k) < \theta_n^k(s_n')$ for all $(s_n', c) \in W \times C$. Indeed, otherwise there exists a sequence (s_n^k, c^k) converging to some (s_n', c) in the closure of $W \times C$ and such that $\pi_n^k(s_n^k, c^k, \sigma_{-n}^k) = \theta_n^k(s_n^k)$ for all k, and then $\lim_k \pi_n^k(s_n^k, c^k, \sigma_{-n}^k) = \theta_n^*(s_n')$. But that is impossible since, by Lemma 4.2, $\lim_k \pi_n^k(s_n^k, c^k, \sigma_{-n}^k) \leqslant \pi_n^+(s_n', c, \sigma_{-n}^*)$, while by construction, $\pi_n^+(s_n', c, \sigma_{-n}^*) < \theta_n^*(s_n')$. Therefore, for all large k, $\pi_n^k(s_n', c, \sigma_{-n}^k) < \theta_n^k(s_n')$ for all $(s_n', c) \in W \times C$ as asserted. Since the bids in C are suboptimal against σ_{-n}^k for signals in W for all large k, it follows that $\int_W \sigma_n^k(C|s_n)\, ds_n = 0$ for all such k. By the lower semi-continuity of the indicator function on the open set C, $\int_W \sigma_n^*(C|s_n)\, ds_n = 0$, which proves the claim. □

Thus σ^* is an equilibrium of the auxiliary game G^*. The final lemma shows that σ^* is also an equilibrium of the auction game G.

Lemma 4.5. *For the equilibrium σ^*, at every bid there is zero probability of a tie.*

Proof. Suppose to the contrary that a tie occurs with positive probability at some bid $b \in B$. For each bidder n let $T_n(b)$ be the subset of n's signals s_n such that $\sigma_n^*(\{\, b\,\}|s_n) > 0$. Let $N(b)$ be the set of bidders n such that $T_n(b)$ has positive measure. By supposition, $|N(b)| \geqslant 2$, and since b might be a highest bid, $\int_{S_m} \sigma_m^*([0, b)|s_m)\, ds_m > 0$ for all $m \notin N(b)$. Let $T(b) = \prod_{n \in N(b)} T_n(b) \prod_{n \notin N(b)} S_n$.

Choose $\varepsilon > 0$ such that the points $b \pm \varepsilon$ are points of continuity of σ^* and such that for each $n \in N(b)$ and $s_n \in T_n(b)$,

$$2\varepsilon < \int_{T_{-n}(b)} v_n(s_n, s_{-n}) \left(\prod_{n \neq m \in N(b)} \sigma^*_m([b \pm \varepsilon]|s_m) \right) \left(\prod_{m \notin N(b)} \sigma^*_m([0, b+\varepsilon]|s_m) \right) q_n(s_{-n}) f_n(s_{-n}|s_n)\, ds_{-n}\,,$$

where $[b \pm \varepsilon]$ is the interval $[b-\varepsilon, b+\varepsilon]$, for every measurable function $q_n : T_{-n}(b) \to [0,1]$ such that its conditional expectation satisfies

$$\frac{\int_{T_{-n}(b)} q_n(s_{-n}) \left(\prod_{n \neq m \in N(b)} \sigma^*_m([b \pm \varepsilon]|s_m) \right) \left(\prod_{m \notin N(b)} \sigma^*_m([0, b+\varepsilon]|s_m) \right) ds_{-n}}{\int_{T_{-n}(b)} \left(\prod_{n \neq m \in N(b)} \sigma^*_m([b \pm \varepsilon]|s_m) \prod_{m \notin N(b)} \sigma^*_m([0, b+\varepsilon]|s_m) \right) ds_{-n}} \geqslant \frac{N(b)-1}{N(b)}\,.$$

Assumption (5) assures that such an ε exists since if not then there exists a sequence ε^l converging to zero, for some $n \in N(b)$ a sequence s^l_n in $T_n(b)$ converging to some s_n, and a sequence q^l_n converging to some q_n satisfying the conditional expectation inequality and such that

$$\int_{T_{-n}(b)} v_n(s, s_{-n}) \left(\prod_{n \neq m \in N(b)} \sigma^*_m(b|s_m) \right) \left(\prod_{m \notin N(b)} \sigma^*_m([0, b]|s_m) \right) q_n(s_{-n}) f_n(s_{-n}|s_n)\, ds_{-n} = 0,$$

which implies that $v_n(s_n, \cdot)$ is zero on a subset of $T_{-n}(b)$ with positive Lebesgue measure, contradicting Assumption (5).

For each k, $n \in N(b)$, and $s_{-n} \in T_{-n}(b)$, let

$$\hat{\sigma}^k_{-n}(s_{-n}) = \left(\prod_{n \neq m \in N(b)} \sigma^k_m([0, b+\varepsilon+1/k]|s_m) - \prod_{n \neq m \in N(b)} \sigma^k_m([b \pm \varepsilon]|s_m) \right) \prod_{m \notin N(b)} \sigma^k_m([0, b{+}\varepsilon{+}1/k]|s_m)\,.$$

For each $n \in N(b)$, let τ^k_n be a transition function from $T_n(b)$ to $[b \pm \varepsilon]$ given by $\tau^k_n(\cdot|s_n) = \sigma^k_n(\cdot|s_n)$ and let $\bar{\tau}^k_n$ be its normalized transition-probability function given by $\bar{\tau}^k_n(\cdot|s_n) = (\tau^k_n([b \pm \varepsilon]|s_n))^{-1} \tau^k_n(\cdot|s_n)$. Because $b \pm \varepsilon$ are points of continuity of σ^*_n and $\sigma^*_n(\{\, b\,\}|s_n) > 0$ for all $s_n \in T_n(b)$, the limit $\bar{\tau}^*_n$ is well-defined and given by $\bar{\tau}^*_n(\cdot|s_n) = (\sigma^*_n([b \pm \varepsilon]|s_n))^{-1} \sigma^*_n(\cdot|s_n)$. Also, for each $b' \in [b \pm (\varepsilon + 1/k)]$ and $n \in N(b)$, define $\tilde{\tau}^k_n(b'|s_n)$ by

$$\tilde{\tau}^k_n(b'|s_n) = \int_{c \in [b \pm \varepsilon]} \int_{b-\varepsilon-1/k}^{b'} d\mu^k_c(b'')\, d\tau^k_n(c|s_n)\,.$$

Finally, observe that for each $s \in T(b)$ and $n \in N(b)$,

$$p^k_n(s) = \prod_{m \notin N(b)} \sigma^k_m([0, b{+}\varepsilon{+}1/k]|s_m) \int_{c \in [b \pm \varepsilon]} \int_{b' \in [b \pm (\varepsilon+1/k)]} \left(\prod_{n \neq m \in N(b)} \tilde{\tau}^k_m(b'|s_m) \right) d\mu^k_c(b')\, d\tau^k_n(c|s_n)$$

is the probability that when the signal profile is $s \in T(b)$, all bidders in $N(b)$ choose a bid in $[b \pm \varepsilon]$, all bidders not in $N(b)$ bid at most $b+\varepsilon+1/k$ and n has the highest bid among the bidders in $N(b)$. Note that $\sum_{n \in N(b)} p_n^k(s) = \prod_{n \in N(b)} \tau_n^k([b \pm \varepsilon]|s_n) \prod_{n \notin N(b)} \sigma_n^k([0, b+\varepsilon+1/k]|s_n)$. Let $\bar{p}_n^k(s)$ be the normalized probability obtained as $\bar{p}_n^k(s) = \left(\sum_{m \in N(b)} p_m^k(s)\right)^{-1} p_n^k(s)$. Extend the definition of $p_n^k(\cdot)$ and $\bar{p}_n^k(\cdot)$ to product subsets $\tilde{T}(b)$ of $T(b)$ as follows:

$$p_n^k(\tilde{T}(b)) = \int_{\tilde{T}(b)} p_n^k(s)\, ds,$$

and let $\bar{p}_n^k(\tilde{T}(b))$ be the normalization of $p_n^k(\tilde{T}(b))$. By going to a subsequence, we obtain well-defined limit functions p_n^* and $\bar{p}_n^*$.

Now observe that if $n \in N(b)$, $s_n \in T_n(b)$, bidder n plays the strategy $\bar{\tau}_n^k(\cdot|s_n)$ in the game G^k, and $1/k < \varepsilon$, then the payoff $\pi_n^k(s_n, \bar{\tau}_n^k, \sigma_{-n}^k)$—which equals s_n's equilibrium-payoff $\theta_n^k(s_n)$ in G^k—is no more than the payoff bidder n would get if the following events occur: (0) all his opponents "intend" to bid no more than $b + \varepsilon + 1/k$ (i.e. their bids before perturbation are no more than $b + \varepsilon$); (1) the bid n pays is at least $b - \varepsilon$; and (2) he wins in one of the following three mutually exclusive events:

(a) $s_{-n} \notin T_{-n}(b)$; or
(b) $s_{-n} \in T_{-n}(b)$ but at least some bidder $m \in N(b)$ intends with positive probability to bid outside $[b \pm \varepsilon]$; or
(c) $s_{-n} \in T_{-n}(b)$ and each opponent $m \in N(b)$ intends to bid in $[b \pm \varepsilon]$ and the outcome of the perturbed game G^k is that n's bid is the highest among the bidders in $N(b)$.

The gross payoff (before subtracting the cost of the bid that is at least $b - \varepsilon$) from event (2a) is:

$$\int_{s_{-n} \notin T_{-n}(b)} v_n(s) \sigma_{-n}^k([0, b + \varepsilon + 1/k]|s_{-n}) f(s_{-n}|s_n)\, ds_{-n}\,;$$

from event (2b), it is:

$$\int_{s_{-n} \in T_{-n}(b)} v_n(s) \hat{\sigma}_{-n}^k(s_{-n}) f(s_{-n}|s_n)\, ds_{-n}\,;$$

and from event (2c) it is:

$$\int_{s_{-n} \in T_{-n}(b)} v_n(s) \left(\prod_{n \neq m \in N(b)} \sigma_m^k([b \pm \varepsilon]|s_m) \right) \left(\prod_{m \notin N(b)} \sigma_m^k([0, b + \varepsilon + 1/k]|s_m) \right) \bar{p}_n^k(s) f(s_{-n}|s_n)\, ds_{-n}\,.$$

Going to the limit, we have that for each $s_n \in T_n(b)$ the gross payoff from bidding according to τ_n^* equals $\lim_k \pi_n^k(s_n, \bar{\tau}_n^*, \sigma^*) + (b - \varepsilon) = \theta_n^*(s_n) + (b - \varepsilon)$ and is no more than the limit of the sum of the above three terms. Observe that this limit would be exactly the gross payoff from bidding $b+\varepsilon$, i.e. would equal $\pi_n^+(s_n, b+\varepsilon, \sigma^*) + b + \varepsilon$ if $\bar{p}_n^*(s_n, s_{-n}) = 1$ for all $s_{-n} \in T_{-n}(b)$. However, $\bar{p}_n^*(T_n(b)) \leqslant 1/N(b)$ for some $n \in N(b)$, since necessarily these probabilities sum to 1. Therefore, for this n and signals s_n in a subset $T_n^\circ(b)$ of $T_n(b)$ with non-zero measure, $\bar{p}_n^*(s_n, T_{-n}(b)) \leqslant 1/N(b)$. Given any $s_n \in T_n^\circ(b)$, for each $s_{-n} \in T_{-n}(b)$, define $q_n(s_{-n}) = 1 - \bar{p}_n^*(s_n, s_{-n})$. Then

$$\frac{\int_{T_{-n}(b)} q_n(s_{-n}) \left(\prod_{n \neq m \in N(b)} \sigma_m^*([b \pm \varepsilon]|s_m)\right) \left(\prod_{m \notin N(b)} \sigma_m([0, b+\varepsilon]|s_m)\right) ds_{-n}}{\int_{T_{-n}(b)} \left(\prod_{n \neq m \in N(b)} \sigma_m^*([b \pm \varepsilon|s_m) \prod_{m \notin N(b)} \sigma_m([0, b+\varepsilon]|s_m)\right) ds_{-n}} \geqslant \frac{N(b)-1}{N(b)} .$$

Let $\underline{\pi}_n^*(s_n)$ be the limit of event (2c) and let $\overline{\pi}_n^*(s_n)$ be the limit with $\overline{p}_n^*(s)$ replaced uniformly by 1 (as were n surely to win). Then $\overline{\pi}_n^*(s_n) - \underline{\pi}_n^*(s_n)$ equals

$$\int_{s_{-n} \in T_{-n}(b)} v_n(s) \left(\prod_{n \neq m \in N(b)} \sigma_m^*([b \pm \varepsilon]|s_m)\right) \left(\prod_{m \notin N(b)} \sigma_m^*([0, b+\varepsilon]|s_m)\right) q_n(s_{-n}) f(s_{-n}|s_n)\, ds_{-n} ,$$

which is strictly greater than 2ε due to the choice of $\varepsilon > 0$. Therefore, $\overline{\pi}_n^*(s_n) > \underline{\pi}_n^*(s_n) + \varepsilon$. But this implies that for each signal $s_n \in T_n^\circ(b)$ the bid $b + \varepsilon$ is a better reply against σ^* than b, which contradicts Lemma 4.4. Hence the probability of a tie at the bid b must be zero. □

Lemma 4.4 establishes that σ^* is an equilibrium of the auxiliary game G^*, and Lemma 4.5 verifies that it is also an equilibrium of the auction game G because ties have zero probability. This completes the proof of Theorem 4.1.

5. Concluding Remark

This paper fills a gap in the literature on auctions by proving existence of equilibria for an all-pay sealed-bid auction of a single item for which bidders have interdependent values, using only the assumption that payoffs and the signal density are positive and continuous on a product of intervals. Since these equilibria have atomless bid distributions, the probability of tied bids is zero, and therefore they are immune to tie-breaking rules. Because the method of proof circumvents discontinuities due to tied bids, it may be useful in other specifications of auctions' payment rules. The gist is that a slight modification of how bidders respond to possibilities of tied bids yields a well-defined fixed-point problem for which fixed points in an essential set have atomless bid distributions, and thus those in essential sets are also equilibria of the auction regardless of the tie-breaking rule.

References

[1] de Castro, L. (2009), Affiliation and Dependence in Economic Models, Economics Department, University of Illinois.

[2] Govindan, S., and R. Wilson (2010), Existence of Equilibria in Auctions with Private Values, Stanford Business School Research Paper 2056. `gsbapps.stanford.edu/researchpapers/library/RP2056.pdf`

[3] Govindan, S., and R. Wilson (2010), Existence of Equilibria in Auctions with Interdependent Values: Two Symmetric Bidders, Stanford Business School Research Paper 2057. `gsbapps.stanford.edu/researchpapers/library/RP2057.pdf`

[4] Jackson, M., L. Simon, J. Swinkels, and W. Zame (2002), Communication and Equilibrium in Discontinuous Games of Incomplete Information, Econometrica, 70: 1711-1740.

[5] Jackson, M., and J. Swinkels (2005), Existence of Equilibrium in Single and Double Private Value Auctions. Econometrica, 73: 93139.

[6] Krishna, V., and J. Morgan (1997), An Analysis of the War of Attrition and the All-Pay Auction, Journal of Economic Theory, 72: 343-362.

[7] Milgrom, P., and R. Weber (1982), A Theory of Auctions and Competitive Bidding, Econometrica, 50: 1089-1122.

[8] Reny, P., and S. Zamir (2003), On the Existence of Pure Strategy Monotone Equilibria in Asymmetric First-Price Auctions, Econometrica, 72: 11051126.

Department of Economics, University of Iowa, Iowa City IA 52242, USA.
E-mail address: `srihari-govindan@uiowa.edu`

Stanford Business School, Stanford, CA 94305-5015, USA.
E-mail address: `rwilson@stanford.edu`

RESEARCH PAPER NO. 2057

Existence of Equilibria in Auctions with Interdependent Values: Two Symmetric Bidders

Srihari Govindan

Robert Wilson

February 2010

This work was supported in part by a grant from the U.S. National Science Foundation.

ABSTRACT

For two symmetric bidders, weak monotonicity conditions are shown to imply existence of an equilibrium in mixed behavioral strategies for a sealed-bid first-price auction of an item for which each bidder's value depends on every bidder's observed signal. Such an equilibrium has atomless distributions of bids and thus is unaffected by how tied bids are resolved.

EXISTENCE OF EQUILIBRIA IN AUCTIONS WITH INTERDEPENDENT VALUES: TWO SYMMETRIC BIDDERS

SRIHARI GOVINDAN AND ROBERT WILSON

Abstract. For two symmetric bidders, weak monotonicity conditions are shown to imply existence of an equilibrium in mixed behavioral strategies for a sealed-bid first-price auction of an item for which each bidder's value depends on every bidder's observed signal. Such an equilibrium has atomless distributions of bids and thus is unaffected by how tied bids are resolved.

1. Introduction

This paper studies a sealed-bid first-price auction of a single item. It introduces a new method to prove existence of an equilibrium in mixed behavioral strategies. An indirect construction yields an equilibrium with nonatomic distributions of bids, thereby circumventing discontinuities due to tie-breaking rules.

First one establishes existence of equilibria for a modified formulation with a novel specification of how bidders choose their bids. For this modified formulation there is a well-defined fixed-point problem for which equilibria are the solutions. Moreover, there exist essential sets of fixed points for which every small perturbation of the problem has a nearby fixed point. The second step considers such perturbations derived by perturbing bidders' payoffs. One then establishes that limit points of equilibria for these perturbed versions have no atoms in the distributions of bids. The probability of tied bids is therefore zero, which implies that these limit equilibria are equilibria of the original auction regardless of the tie-breaking rule. For the proof here it suffices to consider perturbations in which each bidder anticipates that his submitted bid will be slightly distorted by noise before it is received by the auctioneer.

Date: February 2010. This paper can be downloaded from
`gsbapps.stanford.edu/researchpapers/library/RP2057.pdf`.

Key words and phrases. auction, bidding, signals, interdependent values, common value, equilibrium strategies, existence.

JEL subject classifications: C72, D44.

We are grateful to Philip Reny for comments. Wilson's work was funded partly by a grant from the U.S. National Science Foundation.

This method is applied to establish existence of equilibria in behavioral strategies when bidders' values are interdependent.[1] That is, each bidder n observes a private signal s_n, submits a bid $b_n(s_n)$, and then obtains a payoff that is nonzero only if his bid wins, in which case his payoff is $v_n(s_1, s_2, \ldots) - b_n(s_n)$, where his realized value v_n depends on the signals $s_1, s_2, \ldots$ observed by all bidders. Interdependent values occur, for example, when bidders' signals are informative about an unobserved common-value component. The existence theorem, Theorem 4.1 below, assumes that (a) the joint distribution of signals has a positive and continuous density on a hypercube, (b) bidders' value functions are continuous and nondecreasing over the same domain and range, and (c) a family of conditional expectations of a bidder's value are strictly increasing in his signal (see Assumption 6 below).

For simplicity here we consider only the case of two bidders who are symmetric *ex ante* and establish existence of a symmetric equilibrium. The extension to auctions with N asymmetric bidders is provided in a separate paper.

2. The Auction Game

We consider a two-player game G that represents a first-price sealed-bid auction for a single item. In the extensive form of the game, first Nature specifies a profile $s = (s_1, s_2)$ of signals, one for each bidder $n = 1, 2$, according to a distribution F. Then each bidder n observes his own signal s_n and chooses a bid $b_n(s_n)$. Finally, bidder n's payoff is $v_n(s) - b_n(s_n)$ if he wins the item and zero otherwise. He wins if his bid is strictly higher than others' bids, or when tied with others' bids, if he is selected by a tie-breaking rule.

We impose the following assumptions on the distribution of signals.

Assumption: [Distribution of Signals]

(1) The set of possible signal profiles is a product set $S = \prod_n S_n$. Each bidder's set of possible signals is the same real closed interval, say $S_n = [0, 1]$.
(2) The distribution F of signal profiles has a density f that is positive and continuous on S.[2]

For each n, let $\mathcal{S}_n$ be the Borel measurable subsets of S_n and let λ_n be the Lebesgue measure on S_n.

When considering bidder n, his opponent is denoted m. For each bidder n and his signal s_n, let $F_n(\cdot|s_n)$ be the conditional distribution of the signals s_m of n's opponent, and let

[1] For the special case of private values, [3] provides a simpler proof.

[2] This assumption can be weakened considerably: the distribution F need only be absolutely continuous with respect to the product of its marginal distributions.

$f_n(\cdot|s_n)$ be its density function. Assumption (2) ensures that bidder n's conditional densities $f_n(\cdot|s_n) : S_m \to \mathbb{R}$ indexed by his signal $s_n \in S_n$ are an equicontinuous family.

We impose the following assumptions on bidders' value functions.

Assumption: [Distributions of Values]

(3) The set of possible profiles of realized values is a product set $V = \prod_n V_n$. Each bidder's set of possible values is the same real closed interval, say $V_n = [v_*, v^*]$, where $v_* < v^*$.

(4) The joint valuation function $v : S \to V$ is continuous.

(5) For each bidder n his value function $v_n : S \to V_n$ is nondecreasing in his opponent's signal s_m.

(6) For each bidder n and each non-null, measurable set T_m of S_m, the conditional expectation of n's value given T_m and n's signal s_n, namely[3]

$$\frac{\int_{T_m} v_n(s_n, s_m) f_n(s_m|s_n)\, ds_m}{\int_{T_m} f_n(s_m|s_n)\, ds_m},$$

is strictly increasing in n's signal s_n.

Assumption (6) is stronger than requiring that bidder n's value v_n is strictly increasing in his signal s_n. It requires that a higher signal implies a higher expected value even when bidder n conditions on an informative event about his opponent's signal. The following lemma shows that Assumption (6) is equivalent to a seemingly stronger property, which is used in our proofs.

Lemma 2.1. *Assumption (6) holds iff for each non-null measurable function $\beta_m : S_m \to [0, 1]$,*

$$\frac{\int_{S_m} v_n(s_n, s_m)\beta_m(s_m) f_n(s_m|s_n)\, ds_m}{\int_{S_m} \beta_m(s_m) f_n(s_m|s_n)\, ds_m},$$

is strictly increasing in n's signal s_n.

Proof. Sufficiency is obvious. As for the necessity of the condition, suppose Assumption (6) holds. Fix a non-null measurable function $\beta_m : S_m \to [0, 1]$ and two signals $s < s'$. The functions $v_n(\tilde{s}_n, \cdot) f_n(\cdot|\tilde{s}_n)$ and $f_n(\cdot|\tilde{s}_n)$ for $\tilde{s}_n = s_n, s'_n$ induce non-atomic measures on S_m (via the Lebesgue measure on S_m). By a standard purification theorem (see for instance, [2, Theorem 4]) there exists a non-null, measurable function $\tilde{\beta}_m : S_m \to [0, 1]$ that is an

[3]Here and later the integral is computed using the Lebesgue measure on S_m.

indicator function and such that for each $\tilde{s}_n = s_n, s'_n$,

$$\frac{\int_{S_m} v_n(\tilde{s}_n, s_m)\beta_m(s_m) f_n(s_m|\tilde{s}_n)\, ds_m}{\int_{S_m} \beta_m(s_m) f_n(s_m|\tilde{s}_n)\, ds_m} = \frac{\int_{S_m} v_n(\tilde{s}_n, s_m)\tilde{\beta}_m(s_m) f_n(s_m|\tilde{s}_n)\, ds_m}{\int_{S_m} \tilde{\beta}_m(s_m) f_n(s_m|\tilde{s}_n)\, ds_m},$$

which proves the result. □

Lastly, the present exposition requires that bidders are symmetric *ex ante*.

Assumption: [Symmetry of Players]

(7) The distribution function F and the joint valuation function v are symmetric with respect to bidders.

In view of Assumption (3) it suffices to assume that for each bidder the feasible set of bids is the interval $B = [v_*, v^*]$, the same as the interval of the bidder's possible values. Denote the Borel measurable subsets of B by $\mathbf{B}$.

Because the game has perfect recall, we specify a bidder's strategy in behavioral form as mixtures over bids conditional on his signals. A behavioral strategy for bidder n is a transition probability function $\sigma_n(\cdot|\cdot) : \mathbf{B} \times S_n \to [0, 1]$ such that for each signal s_n, $\sigma_n(\cdot|s_n)$ is a probability measure on B; and for each event $A \in \mathbf{B}$, $\sigma_n(A|\cdot)$ is a measurable function on S_n.[4] Let Σ_n be the set of behavioral strategies of bidder n.

Endow behavioral strategies with the topology of weak convergence; i.e. a sequence σ_n^k of strategies in Σ_n converges to σ_n iff for every continuous function $\eta : B \to \mathbb{R}$ and each event $O_n \in \mathcal{S}_n$,

$$\int_{O_n} \int_B \eta(b)\, d\sigma_n^k(b|s_n)\, ds_n \to \int_{O_n} \int_B \eta(b)\, d\sigma_n(b|s_n)\, ds_n\,.$$

With this topology, Σ_n is a compact (metrizable) space.[5] Instead of the Lebesgue measure λ_n on S_n, if one uses the measure $F_m(\cdot|s_m)$ for some signal $s_m \in S_m$ of bidder $m \neq n$ then it represents bidder m's interim belief after receiving his signal s_m. By Assumption (2) on F, for all m and s_m the induced topology on Σ_n is the same.

Remark: It is convenient here to work with behavioral strategies rather than distributional strategies. Note however that given a behavioral strategy σ_n of bidder n and a signal s_m of bidder $m \neq n$ there is for bidder m a well-defined conditional belief that is the distributional

[4]Strictly speaking a behavioral strategy is an equivalence class of transition probability functions, where σ_n is equivalent to σ'_n if $\sigma_n(\cdot|s_n) = \sigma'_n(\cdot|s_n)$ λ_n-a.e. on S_n.

[5]This definition is weaker than requiring (a.e. on S_n) pointwise weak convergence of the sequence of distributions $\sigma_n^k(\cdot|s_n)$. An equivalent definition requires that the displayed integral be u.s.c. (resp. l.s.c.) for each function η that is u.s.c. (resp. l.s.c.).

strategy $\varsigma_n(\sigma_n, s_m)$ defined on $\mathcal{S}_n \times \mathbf{B}$ by

$$\varsigma_n(\sigma_n, s_m)(O_n \times C) = \int_{O_n} \sigma_n(C|s_n) f_m(s_n|s_m)\, ds_n \,,$$

where $O_n \times C$ is the event that n's signal $s_n \in O_n \in \mathcal{S}_n$ and n's bid is in $C \in \mathbf{B}$. The family of distribution functions ς_n is continuous on $\Sigma_n \times S_m$; in particular, the expectation w.r.t. $\varsigma_n(\sigma_n, s_m)$ of any real-valued continuous (or u.s.c. or l.s.c) function on $S_n \times B$ is continuous (or u.s.c. or l.s.c., respectively) w.r.t. (σ_n, s_m). •

Say that a bid $b \in B$ is a point of continuity of n's strategy σ_n if $\sigma_n(\{\, b \,\}|s_n)$ is zero λ_n-a.e. on S_n, i.e. for almost no signal does n bid b with positive probability. The set of bids that are not points of continuity of σ_n is countable. Moreover, if σ_n^k is a sequence of strategies converging to σ_n and b is a point of continuity of σ_n then, since the indicator function for b is u.s.c., $\int_{S_n} \sigma_n^k(\{b\}|s_n)\, ds_n$ converges to zero. Therefore there exists a subsequence such that $\sigma_n^k(\{\, b \,\}|\cdot)$ converges λ_n-a.e. on S_n to zero.

For the auction game G the expected payoff to a bidder n when his signal is s_n, he bids b_n, and m uses strategy $\sigma_m \in \Sigma_m$ is

$$\pi_n(s_n, b_n, \sigma_m) = \int_{S_m} [v_n(s_n, s_m) - b][\sigma_m([v_*, b_n)|s_m) + \frac{1}{2}\sigma_m(\{\, b_n \,\}|s_m)] f_n(s_m|s_n)\, ds_m \,.$$

Consistent with the assumed symmetry of the bidders, the coefficient $\frac{1}{2}$ assumes that the tie-breaking rule gives the bidders equal chances of winning in the event of a tie at the bid b_n, but actually the tie-breaking rule has no role in the sequel.

3. The Auxiliary Game

This section specifies an auxiliary game G^* that is exactly the same as the auction game G specified in Section 2 except for a change in the way a bidder responds to the possibility of tied bids.

Before specifying the game G^*, first define for each bidder n the following alternative payoff functions. For each $(s_n, b_n, \sigma_m) \in S_n \times B \times \Sigma_m$,

$$\hat{\pi}_n(s_n, b_n, \sigma_m) = \int_{S_m} [v_n(s_n, s_m) - b_n]\sigma_m([v_*, b_n)|s_m) f_n(s_m|s_n)\, ds_m \,,$$

$$\bar{\pi}_n^+(s_n, b_n, \sigma_m) = \int_{S_m} [v_n(s_n, s_m) - b_n]^+ \sigma_m(\{\, b_n \,\}|s_m) f_n(s_m|s_n)\, ds_m \,,$$

$$\bar{\pi}_n^-(s_n, b_n, \sigma_m) = \int_{S_m} [v_n(s_n, s_m) - b_n]^- \sigma_m(\{\, b_n \,\}|s_m) f_n(s_m|s_n)\, ds_m \,.$$

Note that $\hat{\pi}_n(s_n, b_n, \sigma_m)$ envisions winning if n's bid b_n strictly exceeds m's bid, whereas $\bar{\pi}_n^+(s_n, b_n, \sigma_m)$ and $\bar{\pi}_n^-(s_n, b_n, \sigma_m)$ envision only the event that n's bid b_n is tied with m's bid. The latter two differ according to whether n receives the positive or negative part of the payoff $v_n(s_n, s_m) - b_n$, corresponding to the best and worst scenarios.

Now define $\pi_n^+ = \hat{\pi}_n + \bar{\pi}_n^+$ and $\pi_n^- = \hat{\pi}_n + \bar{\pi}_n^-$. Then π_n^+ and π_n^- represent the best and worst payoffs from ties. Note that if m's strategy σ_m does not generate any atoms in the distribution of m's bids then $\bar{\pi}_n^+ = \bar{\pi}_n^- = 0$ and therefore both π_n^+ and π_n^- agree with n's payoff function π_n in the auction game G. The following lemma establishes the continuity properties of the best and worst payoff functions π_n^+ and π_n^-.

Lemma 3.1. *The payoff functions π_n^+ and $\pi_n^- : S_n \times B \times \Sigma_m \to \mathbb{R}$ are upper and lower semi-continuous, respectively.*

Proof. Let $(s_n^k, b_n^k, \sigma_m^k)$ be a sequence converging to (s_n, b_n, σ_m). Let S_m^0 be the set of signals s_m^0 of m such that $v_n(s_n, s_m^0) = b_n$. S_m^0 is a closed interval $[\underline{s}_m^0, \overline{s}_m^0]$ since n's value v_n is continuous and nondecreasing in s_m. Fix $\varepsilon > 0$. Take the ε interval around S_m^0 and choose $0 < \delta \leqslant \varepsilon$ such that for k large enough, b_n^k belongs to the interval $b_n \pm \delta$ and $v_n(s_n^k, s_m) - b_n + \delta$ is negative if $s_m \leqslant \underline{s}_m^0 - \varepsilon$ and positive if $s_m \geqslant \overline{s}_m^0 + \varepsilon$. Then

$$\begin{aligned} \pi_n^+(s_n^k, b_n^k, \sigma_m^k) \leqslant & \int_{s_m < \underline{s}_m^0 - \varepsilon} [v_n(s_n^k, s_m) - b_n + \delta] \sigma_m^k([v_*, b_n - \delta) | s_m) f_n(s_m | s_n^k)\, ds_m \\ + & \int_{s_m \geqslant \overline{s}_m^0 + \varepsilon} [v_n(s_n^k, s_m) - b_n + \delta] \sigma_m^k([v_*, b_n + \delta] | s_m) f_n(s_m | s_n^k)\, ds_m \\ + & \int_{s_m^0 \pm \varepsilon} [v_n(s_n^k, s_m) - b_n + \delta)]^+ \sigma_m^k([v_*, b_n + \delta] | s_m) f_n(s_m | s_n^k)\, ds_m\,. \end{aligned}$$

Going to the limit, one sees that $\limsup_k \pi_n^+(s_n^k, b_n^k, \sigma_m^k)$ on the left side is no more than the right side of the above inequality with $(s_n^k, b_n^k, \sigma_m^k)$ replaced by its limit. Now choose a sequence of ε's and thus δ's decreasing to zero. Then one obtains $\pi_n^+(s_n, b_n, \sigma_m)$ as the limit of the right-hand side, which proves that π_n^+ is upper semi-continuous. The proof of the lower semi-continuity of π_n^- is similar. □

Now define the auxiliary game G^* as follows. Given m's strategy σ_m, say that n's bid b_n is an optimal reply when his signal is s_n if $\pi_n^+(s_n, b_n, \sigma_m) \geqslant \pi_n^-(s_n, c, \sigma_m)$ for every bid $c \in B$. That is, the best payoff from b_n must be as good as the worst payoff from any other bid.

For each (s_n, σ_m) let $\phi_n(s_n, \sigma_m)$ be the set of bidder n's optimal replies to σ_m when n's signal is s_n.

Lemma 3.2. *The correspondence $\phi_n : V_n \times \Sigma_m \twoheadrightarrow B$ is upper semi-continuous and has nonempty and compact images.*

Proof. For each signal s_n of bidder n and strategy σ_m of m, the function π_n^+ is upper semi-continuous in n's bid b and hence attains a maximum over B. Any maximizer of π_n^+ is trivially an optimal reply and hence ϕ_n has nonempty images. The other two properties follow from the fact that π_n^+ is u.s.c. while π_n^- is l.s.c. □

Let $\Phi_n : \Sigma_m \twoheadrightarrow \Sigma_n$ be the correspondence that assigns to each strategy σ_m of m the set of n's strategies σ_n such for each signal $s_n \in S_n$ the support of $\sigma_n(\cdot|s_n)$ is a nonempty subset of $\phi_n(s_n, \sigma_m)$. Then Φ_n is an upper semi-continuous correspondence with nonempty compact and convex images. And so too is the optimal-reply correspondence $\Phi : \Sigma \twoheadrightarrow \Sigma$ obtained as the product of Φ_n and Φ_m. The Fan-Glicksberg fixed-point theorem therefore implies that Φ has a fixed point. Hence the auxiliary game G^* has an equilibrium. The symmetry of the bidders implies further that there is a symmetric fixed point and thus a symmetric equilibrium of G^*.

An equilibrium of G^* is not necessarily an equilibrium of the auction game G. However, those equilibria of G^* with nonatomic distributions of bids are equilibria of G.

4. Existence of Equilibria for the Auction Game

This section establishes that a symmetric equilibrium exists for the auction game G, regardless of the tie-breaking rule. This is done by showing that the auxiliary game G^* has a symmetric equilibrium with nonatomic bid distributions that is then a symmetric equilibrium of the auction game G.

Theorem 4.1. *The auction game G has a symmetric equilibrium. In particular, it has a symmetric equilibrium with no atoms in the distribution of any bidder's bids.*

The remainder of this section is devoted to the proof of this existence theorem. Throughout, by a fixed point or an equilibrium we mean a symmetric one. One begins with the observation that the optimal-reply correspondence Φ defined above has essential sets of fixed points. Each one of these sets has the property that every sufficiently small perturbation of Φ has a fixed point arbitrarily close to the set. To exploit this property we construct a sequence of perturbed auxiliary games G^k converging to G^* that induce a sequence of perturbed correspondences Φ^k converging to Φ. We then show that limit points of equilibria of the perturbed games, obtained as fixed points of the perturbed correspondences, have nonatomic bid distributions. Hence, these limits points are equilibria of both G^* and the auction game G. The perturbed games G^k are obtained simply by supposing that a bidder's bid is distorted by noise before it is received by the auctioneer.

The sequence of perturbed games G^k is constructed as follows. For each positive integer k, the strategy sets in the game G^k are the same as in the auction game G and in the auxiliary game G^*. However, when bidder n bids b the auctioneer perceives n's bid as the sum of b

and a random variable uniformly distributed on the interval $[-1/k, 1/k]$. Thus, the payoff functions are defined as follows. First, for each bid $b \in B$ let μ_b^k be the uniform distribution over the interval $[b-1/k, b+1/k]$. Next, for each strategy σ_m^k of bidder m, define a transition probability distribution $\tilde{\sigma}_m^k$ from S_m to $B^k \equiv [v_* - 1/k, v^* + 1/k]$ via its distribution function

$$\tilde{\sigma}_m^k([v_* - 1/k, b]|s_m) = \int_B \int_{v_*-1/k}^{b} d\mu_c^k(b')\, d\sigma_m^k(c|s_m)\,.$$

Then $\tilde{\sigma}_m^k([v_* - 1/k, b]|s_m)$ is the probability that the bid from m received by the auctioneer is no more than b, given that bids are subject to noise. Finally, the payoff to bidder n when his signal is s_n, he bids $b_n \in B$, and his opponent plays the strategy σ_m is

$$\pi_n^k(s_n, b, \sigma_m) = \int_{B^k} \pi_n^+(s_n, c, \tilde{\sigma}_m^k)\, d\mu_b^k(c)\,,$$

where the domain of $\pi_n^+(s_n, c, \sigma_m)$ is extended now to include bids in $B^k \setminus B$ in the obvious way.[6]

The payoff function π_n^k is continuous. As in the previous section, therefore, the correspondence ϕ_n^k that assigns to each (s_n, σ_m) the set of bids that are optimal for n when his signal is s_n in reply to m's strategy σ_m in game G^k is an u.s.c. correspondence with nonempty and compact images. The induced optimal-reply correspondence $\Phi^k : \Sigma \twoheadrightarrow \Sigma$ satisfies the conditions for existence of a fixed point, which is then an equilibrium of G^k.

Select a subsequence of k's diverging to infinity for which a sequence σ^k of equilibria of the perturbed games G^k converges to some strategy profile $\sigma^* \in \Sigma$. Lemmas 4.4 and 4.5 below establish first that σ^* is an equilibrium of the auxiliary game G^*, and then that it is also an equilibrium of the auction game G because the probability of tied bids is zero. Preceding these are two preliminary lemmas that establish bounds on the equilibrium payoffs in G^k and G^*, and continuity of bidders' equilibrium payoff functions as functions of their signals. In the course of these lemmas, convergent subsequences are selected to ensure regularity properties.

Lemma 4.2. *Let (s_n^k, b_n^k) be a subsequence converging to (s_n, b_n). Then:*

$$\pi_n^+(s_n, b_n, \sigma_m^*) \geqslant \limsup_k \pi_n^k(s_n^k, b_n^k, \sigma_m^k) \geqslant \liminf_k \pi_n^k(s_n^k, b_n^k, \sigma_m^k) \geqslant \pi_n^-(s_n, b_n, \sigma_m^*)\,.$$

Proof. Fix any small $\varepsilon > 0$. Since π_n^+ is upper semi-continuous and $(s_n^k, b_n^k, \sigma_m^k)$ converges to (s_n, b_n, σ_m^*), there exists K large enough such that for all $k \geqslant K$, $\pi_n^+(s_n^k, b_n', \sigma_m^k) < \pi_n^+(s_n, b_n, \sigma_m^*) + \varepsilon$ for all $b_n' \in [b_n - 1/K, b_n + 1/K]$. The payoff $\pi_n^k(s_n^k, b_n^k, \sigma_m^k)$ is obtained as

[6] In this definition, one could equivalently replace π_n^+ with π_n^- or, indeed, any function that agrees with π_n^+ at bids that are points of continuity of σ_m.

the average of values of $\pi_n^+(s_n^k, b_n', \sigma_m^k)$ for b_n' in a smaller interval, so when k is very large this implies that $\pi_n^+(s_n, b_n, \sigma_m^*) + \varepsilon \geqslant \limsup_k \pi_n^k(s_n^k, b_n^k, \sigma_m^k)$. Since ε was arbitrary, this establishes the first of the claimed inequalities. The proof for the inequality involving π_n^- is similar. □

For each k and each bidder n let $\theta_n^k : S_n \to \mathbb{R}$ be the function that assigns to each signal s_n the corresponding equilibrium payoff of bidder n from the equilibrium σ^k of game G^k.

Lemma 4.3. *The family of functions θ_n^k is bounded and equicontinuous, and has a subsequence that converges to a continuous function $\theta_n^* : S_n \to \mathbb{R}$.*

Proof. For each signal s_n, bidder n's equilibrium payoff is clearly in the interval $[-1/k, v^* - v_* + 1/k]$. Therefore, the θ_n^k's are uniformly bounded. Equicontinuity follows from the continuity of v_n and the equicontinuity of the conditional densities $f_n(\cdot|s_n)$. Since the θ_n^k's are both bounded and equicontinuous, they are totally bounded; hence there is a Cauchy subsequence that is convergent. □

Assume hereafter that for each n the sequence θ_n^k is itself the Cauchy subsequence identified in the above lemma. The next lemma establishes that σ^* is an equilibrium of the auxiliary game G^*.

Lemma 4.4. *For a.e. signal s_n of bidder n, every bid b in the support of $\sigma_n^*(\cdot|s_n)$ is an optimal reply to σ_m^* in the auxiliary game G^*.*

Proof. Let X_n be the set of $(s_n, b) \in S_n \times B$ such that $\pi_n^+(s_n, b, \sigma_m^*) \geqslant \theta_n^*(s_n)$. By the upper semi-continuity of π_n^+ and the continuity of θ_n^*, X_n is a closed set. We claim that for each $(s_n, b) \notin X_n$, there exists an open neighborhood $W_n \times C$ such that $\int_{W_n} \sigma_n^*(C|s_n)\, ds_n = 0$. This suffices to prove the lemma since $\pi_n^-(s_n, c, \sigma_m^*) \leqslant \theta_n^*(s_n)$ by Lemma 4.2. To prove this claim, choose any open neighborhood $W_n \times C$ of (s_n, b) such that its closure is contained in $(S_n \times B)\backslash X_n$. We assert that for all large k, $\pi_n^k(s_n', c, \sigma_m^k) < \theta_n^k(s_n')$ for all $(s_n', c) \in W_n \times C$. Indeed, otherwise there exists a sequence (s_n^k, c_n^k) in $W_n \times C$ converging to some (s_n', c_n) in the closure of $W_n \times C$ and such that $\pi_n^k(s_n^k, c_n^k, \sigma_m^k) = \theta_n^k(s_n')$ for all k, and then $\lim_k \pi_n^k(s_n^k, c_n^k, \sigma_m^k) = \theta_n^*(s_n')$. But that is impossible since, by Lemma 4.2, $\lim_k \pi_n^k(s_n^k, c_n^k, \sigma_m^k) \leqslant \pi_n^+(s_n', c_n, \sigma_m^*)$, while by construction, $\pi_n^+(s_n', c_n, \sigma_m^*) < \theta_n^*(s_n')$. Therefore, for all large k, $\pi_n^k(s_n', c, \sigma_m^k) < \theta_n^k(s_n')$ for all $(s_n', c) \in W_n \times C$ as asserted. Since the bids in C are suboptimal against σ_m^k for signals in W_n for all large k, it follows that $\int_{W_n} \sigma_n^k(C|s_n)\, ds_n = 0$ for all such k. By the lower semi-continuity of the indicator function on the open set C, $\int_{W_n} \sigma_n^*(C|s_n)\, ds_n = 0$, which proves the claim. □

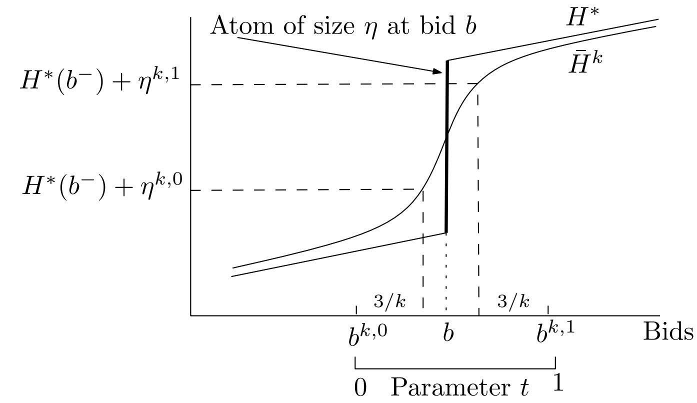

FIGURE 1

We now prove the key lemma, which asserts that in the equilibrium σ^* of G^* each bidder's strategy induces a nonatomic distribution of his bids.

Lemma 4.5. *Every bid b is a point of continuity of σ^*_m, i.e. $\int_{S_m} \sigma^*_m(\{ b \}|s_m)\, ds_m = 0$.*

Proof. Fix a bid $b \in B$. For each bidder m let $W_m \subset S_m$ be the set of signals s_m such that $\sigma^*_m(\{ b \}|s_m) > 0$. Suppose to the contrary that $\int_{W_m} \sigma^*_m(\{ b \}|s_m)\, ds_m \equiv \eta > 0$ for some bidder m. Since we are considering a symmetric equilibrium, the same holds for his opponent n as well. For each bid c, and for each $j = k$ or $*$ define the bid distribution functions

$$H^j(c) = \int_{S_m} \sigma^j_m([v_*, c]|s_m)\, ds_m \qquad \text{and} \qquad H^j(c^-) = \int_{S_m} \sigma^j_m([v_*, c)|s_m)\, ds_m\,.$$

In particular, $H^*(b) = H^*(b^-) + \eta$. Also for each k define the expectation

$$\bar{H}^k(c) = \int_{B^k} H^k(c')\, d\mu^k_c(c')\,.$$

Based on the Lebesgue measure of signals, H^j is the distribution function of bids by bidder m when he plays the strategy σ^j_m, and $\bar{H}^k$ is the expectation of H^k when m's received bids are perturbed via the k-th uniform distribution. Clearly $\bar{H}^k$ is a continuous function on B^k.

We now study the supposed atom of σ^*_m of size η at b by zooming in on small intervals around it that shrink as $k \to \infty$. On these intervals we study the equilibria σ^k of the perturbed games G^k to establish that their limit, the equilibrium σ^* of G^*, has no atoms in the distribution of bids. The gist of the proof is to show that if k is large then bidder n's optimal replies avoid bidding b, instead preferring to bid more or less than b. For the following construction it may be helpful to refer to the illustrative example in Figure 1.

Consider a sequence η^k in the interior of the unit square. Represent each η^k as the pair $(\eta^{k,0}, \eta^{k,1})$. Let $\tilde{b}^{k,0}$ be the highest bid c such that $\bar{H}^k(c) = H^*(b^-) + \eta^{k,0}$, and similarly $\tilde{b}^{k,1}$ is the lowest bid c such that $\bar{H}^k(c) = H^*(b^-) + \eta^{k,1}$. Hereafter suppose that the sequence is such that η^k converges to $(0, \eta)$ and both $\tilde{b}^{k,0}$ and $\tilde{b}^{k,1}$ converge to b. Define $b^{k,0} = \tilde{b}^{k,0} - 3/k$ and $b^{k,1} = \tilde{b}^{k,1} + 3/k$, which also converge to b.

Define $\delta^k = b^{k,1} - b^{k,0}$ and $\varepsilon^k = 1/(k\delta^k)$. Then $\delta^k > 6/k$ and $0 < \varepsilon^k < 1/6$. Now select a subsequence such that ε^k converges to some limit point ε^* in $[0, 1/6]$. Observe that $\varepsilon^* = 0$ iff $k[b^{k,1} - b^{k,0}]$ diverges to infinity. Define the intervals $T^k = [2\varepsilon^k, 1-2\varepsilon^k]$ and $T^* = [2\varepsilon^*, 1-2\varepsilon^*]$.

Next rescale bids in the interval $[b^{k,0}, b^{k,1}]$, using instead the parameter $t \in T \equiv [0, 1]$. For each k, define the linear function $\zeta^k : T \to [b^{k,0}, b^{k,1}]$ via $\zeta^k(t) = b^{k,0} + \delta^k t$. In the following a bid $\hat{b}$ in the interval $[b^{k,0}, b^{k,1}]$ is represented by the parameter $\hat{t}$ for which $\hat{b} = \zeta^k(\hat{t})$ and we refer to $\hat{b}$ and $\hat{t}$ interchangeably.

For a bid parameter $t \in [\varepsilon^k, 1 - \varepsilon^k]$, the bid $\zeta^k(t)$ is in the interval $[b^{k,0} + 1/k, b^{k,1} - 1/k]$. Recall that if the bid $\zeta^k(t)$ is chosen then the bid received by the auctioneer is $\zeta^k(t)$ plus a noise term that is uniformly distributed on the interval $[-1/k, 1/k]$. Hence the uniform distribution of received bids in the interval $[\zeta^k(t) - 1/k, \zeta^k(t) + 1/k]$ induces via ζ^k a distribution ν_t^k over corresponding parameters t' in T, where the distribution ν_t^k is uniform over the interval $[t - \varepsilon^k, t + \varepsilon^k]$. If $t \in [0, \varepsilon^k]$, then define ν_t^k to be the distribution that places probability $(2\varepsilon^k)^{-1}(\varepsilon^k - t)$ on zero and assigns the rest of the probability uniformly on $(0, t + \varepsilon^k]$. The distribution ν_t^k for $t \in [1 - \varepsilon^k, 1]$ is similarly defined, by having a mass point at 1. Hereafter, ν_t^k plays the role of the distribution of the received bid t' conditional on the chosen bid t. For each $t \in T$, as k goes to infinity one obtains a well-defined limit distribution ν_t^* that is uniform over the interval $t \pm \varepsilon^*$ if $t \in I^*$.

The following definitions employ this rescaling.

For each k, from m's strategy σ_m^k construct the induced transition function $\tau_m^k : T \times S_m \to [0, 1]$ via its conditional distribution function

$$\tau_m^k([0, t]|s_m) = \sigma_m^k([\zeta^k(0), \zeta^k(t)]|s_m)\,,$$

which is the conditional probability given his signal s_m that m's bid parameter is in the interval $[0, t]$. Also let $\bar{\tau}_m^k(\cdot|s_m) = (\tau_m^k(T|s_m))^{-1}\tau_m^k(\cdot|s_m)$ be the normalized probability when $\bar{\tau}_m^k(T|s_m) > 0$, and otherwise let it be zero. Then $\bar{\tau}_m^k$ is a probability transition function on S_m, with the measure on S_m that assigns to each S'_m, the probability $\tau_m^k(T|S'_m)/\tau_m^k(T|S_m)$. Select a further subsequence of the k's such that m's induced bid distribution τ_m^k and the normalized version $\bar{\tau}_m^k$ converge to limits τ_m^* and $\bar{\tau}_m^*$ respectively.

Observe that $\int_{S_m} \tau_m^k([2\varepsilon^k, 1-3\varepsilon^k]|s_m)\, ds_m \geqslant \eta^{k,1} - \eta^{k,0}$, and since $\eta^{k,1} - \eta^{k,0}$ converges to η, also $\int_{S_m} \tau_m^*([2\varepsilon^*, 1-3\varepsilon^*]|s_m)\, ds_m \geqslant \eta$. This follows from the fact that for each k, $H^k(\tilde{b}^{k,0} - 1/k) \leqslant H^*(b^-) + \eta^{k,0}$ (since otherwise $\bar{H}^k(\tilde{b}^{k,0}) > H^*(b^-) + \eta^{k,0}$) and $H^k(\tilde{b}^{k,1}) \geqslant H^*(b^-) + \eta^{k,1}$.

For each $j = k$ or $*$, each parameter $\tilde{t} \in [\varepsilon^j, 1-\varepsilon^j]$, and each signal $s_m \in S_m$, define the conditional probability that m's received bid parameter is in the interval $[0, \tilde{t}]$ by its conditional distribution function:

$$\tilde{\tau}_m^j([0,\tilde{t}]|s_m) = \int_{[0,1]} \int_0^{\tilde{t}} d\nu_t(t')\, d\tau_m^j(t|s_m)\,.$$

Also define for each bid parameter $t \in T^j = [2\varepsilon^j, 1-2\varepsilon^j]$ and m's signal $s_m \in S_m$ the conditional expectation of $\tilde{\tau}_m^j$ with respect to the induced noise distribution ν_t^j of bidder n:

$$\hat{\tau}_m^j(t|s_m) = \int_{[0,1]} \tilde{\tau}_m^j([0,t']|s_m)\, d\nu_t^j(t')\,.$$

Note that $\tilde{\tau}_m^j([0,t]|s_m)$ and $\hat{\tau}_m^j(t|s_m)$ are weakly increasing in t for each signal s_m, since $\tau_m^j([0,t]|s_m)$ has that property and ν_t^j is a family of distributions for which higher t implies first-order stochastic dominance.

Observe that from bidder n's perspective, in the perturbed game G^k, if he chooses t and thus the bid $\zeta^k(t)$ then his probability of winning when m's signal is s_m and chooses a bid parameter in T is $\hat{\tau}_m^k(t|s_m)$. (Because of the noise, the probability is zero that both bidders' received bid parameters are any $t' \in T$.)

Finally, for each k and each pair (s_n, s_m) of the bidders' signals, denote by $p_n^k(s_n, s_m)$ the probability that n wins when he chooses a bid in $T^k = [2\varepsilon^k, 1-2\varepsilon^k]$ given his signal s_n and m chooses a bid parameter in T given his signal s_m. That is,

$$p_n^k(s_n, s_m) = \int_{T^k} \hat{\tau}_m^k(t|s_m)\, d\tau_n^k(t|s_n)\,.$$

Then

$$\prod_n \tau_n^k([2\varepsilon^k, 1-2\varepsilon^k]|s_n) \leqslant \sum_n p_n^k(s_n, s_m) \leqslant \prod_n \tau_n^k(T|s_n)\,.$$

Define $\bar{p}_n^k(s_n, s_m) \equiv \left(p_n^k(s_n, s_m) + p_m^k(s_m, s_n)\right)^{-1} p_n^k(s_n, s_m)$.

Analogously, for each pair of measurable subsets $(\tilde{S}_n, \tilde{S}_m)$ of the bidders' signals, define the probability

$$p_n^k(\tilde{S}_n, \tilde{S}_m) = \int_{\tilde{S}_n \times \tilde{S}_m} p_n^k(s_n, s_m)\, ds_n\, ds_m\,,$$

and $\bar{p}^k_n(\tilde{S}_n, \tilde{S}_m) \equiv \big(p^k_n(\tilde{S}_n, \tilde{S}_m) + p^k_m(\tilde{S}_m, \tilde{S}_n)\big)^{-1} p^k_n(\tilde{S}_n, \tilde{S}_m)$.

The remainder of the proof is broken into five steps. Step 1 shows that the limit of the parameterized version accurately represents a bidder's limit strategy σ^*_n.

Step 1. We claim that when viewed as sequences in $L_\infty(S_n, \lambda_n)$, the space of bounded λ_n-measurable functions on S_n, in the weak*-topology: (a) $\tau^k_n(T|\cdot)$ converges to $\sigma^*_n(\{\, b\,\}|\cdot)$, and (b) $\sigma^k_n([v_*, b^{k,0})|\cdot)$ converges to $\sigma^*_n([v_*, b)|\cdot)$. To prove this, given a measurable subset $\tilde{S}_n$ of S_n, let $\eta(\tilde{S}_n)$ be $\int_{\tilde{S}_n} \sigma^*_n(\{\, b\,\}|s_n)\, ds_n$ and let $\eta^-(\tilde{S}_n) = \int_{\tilde{S}_n} \sigma^*_n([v_*, b)|s_n)\, ds_n$. For each $\varepsilon > 0$, choose bids $\underline{b}, \bar{b}$ that are points of continuity of σ^*_n such that $\underline{b} < b < \bar{b}$ and such that

$$\int_{\tilde{S}_n} \sigma^*_n([v_*, \underline{b}]|s_n)\, ds_n \geqslant \eta^-(\tilde{S}_n) - \varepsilon \quad \text{and} \quad \int_{\tilde{S}_n} \sigma^*_n([v_* \bar{b}]|s_n)\, ds_n \leqslant \eta^-(\tilde{S}_n) + \eta(\tilde{S}_n) + \varepsilon\,.$$

For all large k, $\underline{b} < b^{k,0}$ and $b^{k,1} < \bar{b}$. Therefore,

$$\limsup_k \int_{\tilde{S}_n} \tau^k_n(T|s_n)\, ds_n \leqslant \eta(\tilde{S}_n) + 2\varepsilon \quad \text{and} \quad \liminf_k \sigma^k_n([v_*, b^{k,0})|s_n)\, ds_n \geqslant \eta^-(\tilde{S}_n) - \varepsilon\,.$$

Since ε was arbitrary,

$$\limsup_k \int_{\tilde{S}_n} \tau^k_n(T|s_n)\, ds_n \leqslant \eta(\tilde{S}_n) \quad \text{and} \quad \liminf_k \sigma^k_n([v_*, b^{k,0})|s_n)\, ds_n \geqslant \eta^-(\tilde{S}_n)\,.$$

For an arbitrary $\varepsilon' > 0$ and all large k,

$$\int_{S_n} \tau^k_n([2\varepsilon^k, 1 - 3\varepsilon^k]|s_n)\, ds_n \geqslant \eta - \varepsilon'/2\,,$$

while

$$\int_{S_n \setminus \tilde{S}_n} \tau^k_n(T|s_n)\, ds_n \leqslant \int_{S_n \setminus \tilde{S}_n} \tau^*_n(T|s_n)\, ds_n + \varepsilon'/2\,.$$

Therefore, $\int_{\tilde{S}_n} \tau^k_n([\varepsilon^k, 1 - \varepsilon^k]|s_n)\, ds_n \geqslant \eta(\tilde{S}_n) - \varepsilon'$. Thus,

$$\lim_k \int_{\tilde{S}_m} \tau^k_n([\varepsilon^k, 1 - \varepsilon^k]|s_n)\, ds_n = \int_{\tilde{S}_m} \sigma^*_n(\{\, b\,\}|s_n)\, ds_n\,.$$

Finally, for the ε chosen at the beginning of this step, for large k

$$\int_{\tilde{S}_n} [\sigma^k_n([v_*, b^{k,0})|s_n) + \tau^k_n(T|s_n)]\, ds_n \leqslant \eta^-(\tilde{S}_n) + \eta(\tilde{S}_n) + \varepsilon\,.$$

Hence, using the fact that τ^k_n converges to τ^*_n,

$$\limsup_k \int \sigma^k_n([v_*, b^{k,0})|s_n)\, ds_n \leqslant \eta^-(\tilde{S}_n) + \varepsilon\,.$$

Therefore, since ε was arbitrary,

$$\limsup_k \int \sigma_n^k([v_*, b^{k,0})|s_n)\, ds_n \leqslant \eta^-(\tilde{S}_n)\,.$$

As seen above, $\liminf_k \sigma_n^k([v_*, b^{k,0})|s_n)\, ds_n \geqslant \eta^-(\tilde{S}_n)$, so this completes Step 1. •

Recall that $W_n \subset S_n$ is the set of n's signals s_n such that $\sigma_n^*(\{\, b\,\}|s_n) > 0$ and by hypothesis W_n has positive measure η. From Step 1 it follows that $\tau_n^*(T|s_n) = 0$ for a.e. $s_n \in S_n \setminus W_n$, while $\tau_n^*([2\varepsilon^*, 1-3\varepsilon^*]|s_n) = \sigma_n^*(\{\, b\,\}|s_n)$ for a.e. $s_n \in W_n$. Also, by selecting a convergent subsequence, p_n^k converges to a function p_n^* and $\bar{p}_n^k$ converges to $\bar{p}_n^*$, with the property that for each pair $(\tilde{S}_n, \tilde{S}_m)$,

$$\prod_n \int_{\tilde{S}_n} \tau_n^*([2\varepsilon^*, 1-3\varepsilon^*]|s_n)\, ds_n = \sum_n p_n^*(\tilde{S}_n, \tilde{S}_m).$$

To prepare for the next steps, for each k, signal s_n, and a probability distribution μ over T^k, define

$$\rho_n^k(s_n, \mu) = \int_{[0,1]} \int_{S_m} [v_n(s_n, s_m) - b]\hat{\tau}_m^k(t|s_m) f_n(s_m|s_n)\, ds_m\, d\mu(t)\,,$$

which is approximately the portion of n's expected payoff from bidding close to b when his signal is s_n and his bid is matched against bids of his opponent that are also close to b. When n chooses the strategy $\bar{\tau}_n^k(\cdot|s_n)$, then this payoff is:

$$\rho_n^k(s_n, \bar{\tau}_n^k(\cdot|s_n)) = \left(\tau_n^k(T|s_n)\right)^{-1} \int_{S_m} [v_n(s_n, s_m) - b] p_n^k(s_n, s_m) f_n(s_m|s_n)\, ds_m.$$

Similarly, for a distribution μ over T, define

$$\rho_n^*(s_n, \mu) = \sup_{(s_n^k, \mu^k)\to(s,\mu)} \limsup_k \rho_n^k(s_n^k, \mu^k).$$

Because of the equicontinuity of f and the continuity of v, we could take the sequence s_n^k in the above definition to be the constant sequence of s_n.

Finally, let

$$\xi_n^*(s_n) = \int_{S_m} [v(s_n, s_m) - b]\sigma_m^*([v_*, b)|s_m) f_n(s_m|s_n)\, ds_m,$$

which is n's expected payoff based on m's limit strategy σ_m^* from bidding b and winning only if m bids less.

Define $\vartheta_n^*(s_n) = \theta_n^*(s_n) - \xi_n^*(s_n)$. Thus $\vartheta_n^*(s_n)$ is the amount by which the equilibrium payoff $\theta_n^*(s_n)$ exceeds the payoff from bidding b and winning only if m's bid is strictly less. Hence $\vartheta_n^*(s_n)$ is n's incremental payoff from winning with the bid b when matched with the same bid b by m. Steps 2 and 3 show that the incremental payoff $\vartheta_n^*(s_n)$ bounds the limit $\rho_n^*(s_n,\mu)$ of his payoff from bids close to b when his signal is s_n and his bid is matched against m's bids close to b, and show that in fact these two payoffs are equal and positive if t is in the support of his limit strategy $\tau_n^*(\cdot|s_n)$ for those signals $s_n \in W_n$ for which n bids b with positive probability in the equilibrium σ^*.

Step 2. We claim that $\vartheta_n^*(s_n) \geqslant \rho_n^*(s_n,\mu)$ for each signal s_n and μ. More specifically, if (s_n^k,μ^k) is a sequence converging to (s_n,μ^k), then $\limsup \rho_n^k(s_n^k,\mu^k) \leqslant \vartheta_n^*(s_n)$; and this inequality holds as an equality if, letting $\zeta^k(\mu^k)$ be the distribution over bids induced by μ^k via ζ^k, $\pi_n^k(s_n^k,\zeta^k(\mu^k)) = \theta_n^k(s_n)$—which is true, for e.g., if s_n^k is generic and the support of μ^k is contained in the support of $\tau_n^k(\cdot|s_n^k)$. To prove this claim, consider a sequence (s_n^k,μ_n^k) converging to (s_n,μ_n). Analogous to the definition of ξ_n^* above, define

$$\xi_n^k(s_n^k) = \int_{S_m} [v_n(s_n^k,s_m) - b]\sigma_m^k([v_*,b^{k,0})|s_m) f_n(s_m|s_n^k)\, ds_m\,.$$

Since $\sigma_m^k([v_*,b^{k,0})|s_m)$ converges to $\sigma_m^*([v_*,b)|s_m)$, $\xi_n^k(s_n^k)$ converges to $\xi_n^*(s_n)$.

The difference $\pi_n^k(s_n^k,\zeta^k(\mu^k),\sigma_m^k) - \xi_n^k(s_n^k) - \rho_n^k(s_n^k,\mu^k)$ is

$$\int_{T^k}\int_{S_m}\int_{[0,1]} [b-\zeta^k(t')][\sigma_m^k([v_*,b^{k,0})|s_m) + \tilde{\tau}_m^k([0,t')|s_m)] f_n(s_m|s_n^k)\, d\nu_t(t')\, ds_m\, d\mu(t).$$

As k goes to infinity, $\zeta^k(t')$ converges to b for all $t' \in [0,1]$ and hence this difference converges to zero. Therefore, for each s_n and t, since $\pi_n^k(s_n^k,\zeta^k(\mu^k),b) \leqslant \theta_n^k(s_n^k)$, it follows that $\xi_n^*(s_n) + \limsup_k \rho_n^k(s_n^k,\mu^k) \leqslant \theta_n^*(s_n) = \xi_n^*(s_n) + \vartheta_n^*(s_n)$ and this inequality holds with equality under the given conditions as well. •

Step 3. For a.e. $s_n \in W_n$:

$$0 < \vartheta_n^*(s_n) = \big(\tau_n^*(T|s_n)\big)^{-1}\left(\int_{S_m} [v_n(s_n,s_m) - b]p_n^*(s_n,s_m) f(s_m|s_n)\, ds_m\right).$$

As in the previous step, it is easy to see that for each W_n',

$$\lim_k \int_{W_n'} \tau_n^k(T|s_n)[\pi_n^k(s_n,\zeta^k(\bar{\tau}_n^k(T|s_n)),\sigma_m^k) - \xi_n^k(s_n) - \rho_n^k(s_n,\bar{\tau}_n^k(\cdot|s_n))]\, ds_n = 0$$

where $\zeta^k(\bar{\tau}_n^k(T|s_n))$ is the distribution over bids induced by $\bar{\tau}_n^k$. Since the strategies in $\zeta^k(\bar{\tau}_n^k(T|s_n))$ are optimal for each s_n, $\pi_n^k(s_n,\zeta^k(\bar{\tau}_n^k([0,1]|s_n),\sigma_m^k)) = \theta_n^k(s_n)$. Also, the equicontinuity of θ_n^k and ξ_n^k imply that the left-hand side of the above equation equals:

$$\int_{W_n'} \left(\tau_n^*(T|s_n)[\theta_n^*(s_n) - \xi_n^*(s_n)] - \int_{S_m} [v_n(s_n,s_m) - b]p_n^*(s_n,s_m)\right) f_n(s_m|s_n)\, ds_n.$$

As $\theta^*_n(s_n) = \xi^*_n(s_n) + \vartheta^*_n(s_n)$, and W'_n is an arbitrary subset of W_n, the claimed equality follows.

To prove the positivity of ϑ^*_n a.e. on W_n, we first show that it is non-negative. Suppose, to the contrary, that $\vartheta^*_n(s_n)$ is negative for some s_n. Then $\xi^*_n(s_n) > \theta^*_n(s_n)$. Obviously now $b > v_*$, since otherwise $\xi^*_n(s_n) = 0$ and thus $\theta^*_n(s_n)$ would be negative, which is impossible. Observe then that $\xi^*_n(s_n)$ is the limit of $\pi^*_n(s_n, b^l, \sigma^*_m)$ for a sequence b^l of bids approaching b from the left that are points of continuity of σ^*_m. Therefore, one can choose $b' < b$ such that b' is a point of continuity of σ^*_m and $\pi^*_n(s_n, b', \sigma^*_m) > \theta^*_n(s_n)$, which is impossible for generic s_n. Thus, $\vartheta^*_n(s_n)$ is non-negative a.e.

Now fix a subset W'_n of W_n with positive measure. Then $p^*_n(W'_n, S_m) > 0$ since by symmetry $p^*_n(W'_n, W'_n) = 1/2$. Also for each $s_n \in W'_n$, $\vartheta^*_n(s_n) \geqslant 0$. Therefore,

$$\frac{\int_{W'_n}\int_{S_m} v_n(s_n, s_m) p^*_n(s_n, s_m) f_n(s_m|s_n)\, ds_m\, ds_n}{\int_{W'_n}\int_{S_m} p^*_n(s_n, s_m) f_n(s_m|s_n)\, ds_m\, ds_n} \geqslant b\,.$$

By Assumption (6) therefore, for each s_n that is greater than the essential supremum of W'_n:

$$\frac{\int_{W'_n}\int_{S_m} v_n(s_n, s_m) p^*_n(s'_n, s_m) f_n(s_m|s_n)\, ds_m\, ds'_n}{\int_{W'_n}\int_{S_m} p^*_n(s'_n, s_m) f_n(s_m|s_n)\, ds_m\, ds'_n} > b\,.$$

Therefore, $\lim_k \rho^k_n(s_n, \bar{\tau}^k_n(\cdot|W'_n)) > 0$, i.e. by mimicking the strategy of W'_n, in the limit s_n obtains a positive payoff. By Step 2, this implies that $\vartheta^*_n(s_n) > 0$ such an s_n. Since W'_n was an arbitrary set of W_n with positive measure, we have that for all s_n greater than the essential infimum of W_n, $\vartheta^*_n(s_n) > 0$. •

Step 4. We claim that for a.e. $s_n \in W_n$, $\bar{p}^*_n(s_n, \cdot)$ is weakly decreasing almost everywhere on W_m. To prove this claim, for a subset $\tilde{W}_n$ of W_n with positive measure and $x \in [0, 1]$, let $\bar{s}_m(x, \tilde{W}_n)$ be the essential supremum of the set of m's signals s_m such that $\bar{p}^*_n(\tilde{W}_n, s_m) \geqslant x$. If the claim is not true then there exists a set $\tilde{W}_n$ of positive measure and $x \in (0, 1]$ such that there is a positive measure of signals s_m in $[0, \bar{s}_m(x, \tilde{W}_n)] \cap W_m$ for which $\bar{p}^*_n(\tilde{W}_n, s_m) < x$. Choose two closed and disjoint subsets W^0_m and W^1_m of $[0, \bar{s}_m(x, \tilde{W}_n)] \cap W_m$ such that both sets have positive measure and: (i) for each $s^0_m \in W^0_m$, $s^1_m \in W^1_m$, one has $s^0_m < s^1_m$; (ii) $\bar{p}^*_n(\tilde{W}_n, W^0_m) < x$ and $\bar{p}^*_n(\tilde{W}_n, W^1_m) \geqslant x$. Therefore, $\bar{p}^*_m(W^0_m, \tilde{W}_n) - \bar{p}^*_m(W^1_m, \tilde{W}_n) > 0$.

Since for each $i = 0, 1$, $\bar{p}^k_m(W^i_m, \tilde{W}_n)$ converges to $\bar{p}^*_m(W^i_m, \tilde{W}_n)$, we have that for a subsequence there exists for all k a pair $(s^{k,i}_m, t^{k,i})$ converging to some $(s^{\infty,i}_m, t^{\infty,0})$ such that: (i) $s^{k,i}_m \in W^i_m$ for all k; (ii) $t^{k,i}$ is in the support of $\tau^k_m(\cdot|s^{k,i}_m)$ for all k; (iii) letting $\hat{\tau}^{*,i}_n(t^{\infty,i}|\cdot)$ be the limit of $\hat{\tau}^{k,i}_n(t|\cdot)$, $\int_{\tilde{W}_n} \hat{\tau}^{\infty,0}_n(t^{\infty,0}|s_n) - \hat{\tau}^{\infty,1}_n(t^{\infty,1}|s_n)\, ds_n > 0$. Obviously $t^{k,0} > t^{k,1}$ for all k and hence $\int_{S_n} \hat{\tau}^{\infty,0}_n(t^{\infty,0}|s_n) - \hat{\tau}^{\infty,1}_n(t^{\infty,1}|s_n)\, ds_n > 0$ as well.

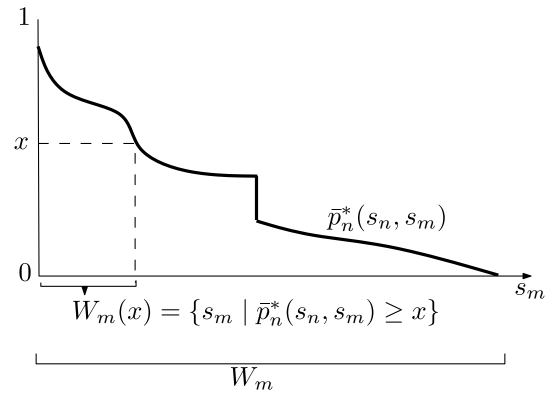

Figure 2

Letting $s_m^{\infty,i}$ be the limit of $s_m^{k,i}$, we have from the previous step that $\rho^*_m(s_m^{\infty,i}, t_m^{\infty,i}) = \int_{S_n} [v_m(s_m^{\infty,i}, s_n) - b]\hat{\tau}^\infty_n(t_m^{\infty,i}|s_n) f_m(s_n|s_m^{\infty,i})\, ds_n = \vartheta_m(s_m^{\infty,i})$. This implies, for $i = 1$, that

$$\frac{\int_{S_n} v_m(s_m^{\infty,1}, s_n)\beta(s_n) f_m(s_n|s_m^{\infty,1})\, ds_n}{\int_{S_n} \beta(s_n) f_m(s_n|s_m^{\infty,1})\, ds_n} \leqslant b$$

where $\beta(s_n) = \hat{\tau}^\infty_n(t^{\infty,0}|s_n) - \hat{\tau}^\infty_n(t^{\infty,1}|s_n)$. By Assumption (6), the same expression using $s_m^{\infty,0}$ in place of $s_m^{\infty,1}$ would be a strict inequality since $s_m^{\infty,0} < s_m^{\infty,1}$. Thus, $\lim_k \rho^k_m(s_m^{k,0}, t^{k,1})$ is strictly greater than $\lim_k \rho^k_m(s_m^{k,0}, t^{k,0})$, which is the desired contradiction. •

It is easy to show using the previous step that the support of τ^*_n is a singleton for a.e. $s_n \in W_n$ and that this function is then weakly increasing, a fact not used below.

We come now to the final step. For those signals $s_n \in W_n$ for which n bids b in the equilibrium σ^* of G^*, Step 5 shows that the amount $\vartheta^*_n(s_n)$ by which the equilibrium payoff $\theta^*_n(s_n)$ exceeds the payoff $\xi^*_n(s_n)$ from bidding b and winning only if m's bid is strictly less, cannot be so large as to imply always winning when m also bids b, and that therefore bidder n prefers to bid slightly more than b. Thus Step 5 obtains a contradiction to the conclusion of Lemma 4.4 that σ^* is an equilibrium of G^*, and thereby a contradiction to the initial hypothesis that m's strategy σ^*_m induces an atom at the bid b.

Step 5. Fix a generic signal $s_n \in W_n$. By Steps 2 and 3, $\vartheta^*_n(s_n)$ and $p^*_n(s_n, W_m)$ are both positive. The incremental payoff is

$$\vartheta^*_n(s_n) = \int_{W_m} [v_n(s_n, s_m) - b]\tau^*_m(T|s_m)\bar{p}^*_n(s_n, s_m) f_n(s_m|s_n)\, ds_m\,.$$

For each $x \in [0, 1]$, define the set $W_m(x) \equiv \{\, s_m \in W_m \mid \bar{p}^*_n(s_n, s_m) \geqslant x \,\}$. Figure 2 illustrates a possible form of $\bar{p}^*_n(s_n, s_m)$ and $W_m(x)$ for the case that W_m is an interval.

By Step 3, $W_m(x)$ is monotonically nonincreasing in x. One can therefore express $\vartheta_n^*(s_n)$ as the double integral

$$\vartheta_n^*(s_n) = \int_{W_m} \int_{[0,1]} [v_n(s_n, s_m) - b] 1_{W_m(x)}(s_m) \tau_m^*(T|s_m) f_n(s_m|s_n)\, dx\, ds_m,$$

where $1_{W_m(x)}(\cdot)$ is the indicator function for $W_m(x)$. For each x and signal s_n, define

$$u_n(x, s_n) = \int_{W_m(x)} [v_n(s_n, s_m) - b] \tau_m^*(T|s_m) f_n(s_m|s_n)\, ds_m\, .$$

Then reversing the order of integration above yields the alternative formula

$$\vartheta_n^*(s_n) = \int_{[0,1]} u_n(x, s_n)\, dx\, .$$

If $u_n(x, s_n) > 0$ for some x then for all $x_1 \leqslant x \leqslant x_2$, $u_n(x_1, s_n) \geqslant u_n(x_2, s_n)$, with the inequality being strict if $W_m(x_1) \setminus W_m(x_2)$ has positive measure. To prove this, let $\overline{s}_m(x)$ be the essential supremum of $W_m(x)$. Since $u_n(x, s_n) > 0$, $W_m(x)$ has positive measure and $\overline{s}_m(x)$ is well-defined. Also, $v_n(s_n, \overline{s}_m(x)) > b$, since otherwise by Assumption (5), $v_n(s_n, s_m) \leqslant b$ for all $s_m \in W_m(x)$ and then $u_n(x, s_n) \leqslant 0$. Fix $x_1 < x \leqslant x_2$ and for each $i = 1, 2$, let $\overline{s}_m(x_i)$ be the essential supremum of $W_m(x_i)$. By the monotonicity of $W_m(\cdot)$, $\overline{s}_m(x_1) \geqslant \overline{s}_m(x_2)$, where the inequality is strict if $W_m(x_1) \setminus W_m(x_2)$ has positive measure. Observe that for each $\overline{s}_m(x) \leqslant s_m \leqslant \overline{s}_m(x_1)$, $v_n(s_n, s_m) > b$, and thus $u_n(x_1, s_n)$ is strictly positive. If $u_n(x_2, s_n)$ is non-positive then one is done. Otherwise, if $u_n(x_2, s_n) > 0$, then $v_n(s_n, \overline{s}_m(x_2)) > b$ and thus, again by Assumption (5), for all $\overline{s}_m(x_2) \leqslant s_m \leqslant \overline{s}_m(x_1)$, $v_n(s_n, s_m) > b$. Thus, $u_n(x_1, s_n) > u_n(x_2, s_n)$.

Since $\vartheta_n^*(s_n)$ is greater than zero, $u_n(x, s_n) > 0$ for some x. Therefore, the preceding paragraph implies that $u_n(x, s_n) \leqslant u_n(0, s_n)$ for all x, and hence

$$\vartheta_n^*(s_n) \leqslant \int_{W_m} [v_n(s_n, s_m) - b] \tau_m^*(T|s_m) f_n(s_m|s_n)\, ds_m\, .$$

Observe that this inequality must be strict for a.e. $s_n \in W_n$. Otherwise, when n's signal is s_n he wins with probability one against every signal $s_m \in W_m$ of bidder m, which in an equilibrium is not possible outside a set of measure zero in W_n.

To finish the proof, we show that when the above inequality is strict bidder n with signal s_n prefers some bid $b' > b$ for all large k. Indeed, first observe that $b < v^*$, since otherwise, the payoff $\theta_n^*(s_n) \leqslant 0$ (the maximum value is v^* and if it equals the bid b, then bidder n cannot make a profit in any event). Then, observe that the total payoff $\xi_n^*(s_n) + \int_{W_m} [v_n(s_n, s_m) - b] \tau_m^*([0, 1]|s_m) f_n(s_m|s_n)\, ds_m$ is the right-hand limit of $\pi_n^+(s_n, b, \sigma_m^*)$. So, if the inequality is strict then one can choose $b' > b$ that is a point of continuity and such that $\pi_n^+(s_n, b', \sigma_m^*) > \xi_n^*(s_n) + \vartheta_n^*(s_n)$. But that is impossible since by definition $\xi_n^*(s_n) + \vartheta_n^*(s_n) = \theta_n^*(s_n)$ and therefore $\pi_n^+(s_n, b', \sigma_m^*) > \theta_n^*(s_n)$, contradicting the initial supposition that $\theta_n^*(s_n)$ is n's equilibrium payoff when his signal is s_n. •

The contradiction obtained in Step 5 implies the falsity of the initial hypothesis that bidder m's strategy induces an atom at the bid b, and thereby concludes the proof of Lemma 4.5. □

In sum, Lemma 4.4 establishes that a limit point σ^* of the equilibria σ^k of the auxiliary games G^k is an equilibrium of G^*, and Lemma 4.5 establishes that σ^* does not induce atoms in the distribution of any bidder's bids. Hence the probability of tied bids is zero and therefore σ^* is also an equilibrium of the auction game G, regardless of the tie-breaking rule. This verifies Theorem 4.1.

Remark. Assumptions (5) and (6) are invoked only in Steps 2, 3, and 4 of Lemma 4.5. Without these assumptions, the proof shows how to generate an endogenous tie-breaking rule, as in [4]. For each bid b at which there is an atom, of which there is a countable number, the limiting probabilities $p_n^*(s_n, s_m)$ and $p_m^*(s_m, s_n)$ give the relative odds of s_n or s_m winning if both bid b and then report their signals truthfully. In the event of a tie at such a bid b, the incentive to truthfully report one's signal is obtained from the fact that the limit payoff $\vartheta_n^*(s_n)$ maximizes $\rho_n^*(s_n, t)$ and that misreporting one's signal, say $\tilde{s}_n \in W_n$ instead of s_n, obtains $\lim_k \rho_n^k(s_n, \tilde{t}^k)$ as the payoff, where $\tilde{t}^k$ is a sequence converging to a $\tilde{t}$ in the support of $\tau_n^*(\cdot|\tilde{s}_n)$, and for which $\tilde{t}^k$ is in the support of $\tau_n^k(\cdot|\tilde{s}_n^k)$ for a sequence of $\tilde{s}_n^k$ converging to $\tilde{s}_n$.

5. Concluding Remarks

In auction theory, most existence theorems focus on equilibria in pure strategies that are strictly increasing in bidders' signals. Like Theorem 4.1 here, monotone pure-strategy equilibria obviate discontinuities due to tie-breaking rules because they induce nonatomic distributions of bids. However, these theorems rely on unrealistically strong assumptions about the joint distribution of signals, such as affiliation.[7] In contrast, the theorem here establishes existence of an equilibrium in mixed strategies using weak assumptions about

[7] De Castro [1, Theorem 3.1] proves that the subset of continuous density functions that are not affiliated is open and dense. He examines various plausible weaker properties and establishes for some that there need not exist equilibria in increasing pure strategies. He also shows that an auction with private values jointly distributed according to a density that is piecewise constant has pure-strategy equilibria, and an example for which no pure strategy equilibrium is increasing.

the distribution of bidders' signals and their value functions.[8] In general we see advantages to distinguishing between those assumptions sufficient for existence of equilibria, and those that ensure pure strategies and monotonicity.

The proof of Theorem 4.1 brings additional advantages. One is that the auxiliary game induces a well-defined fixed-point problem in the space of behavioral strategies, a feature absent from previous work. This problem necessarily has essential sets of fixed points for which every perturbation of the problem has a nearby fixed point. As shown by the proof in Section 4, limit points of these nearby fixed points are equilibria of both the auxiliary game and the auction game. Because essential fixed points are those for which the Leray-Schauder index is nonzero, there is the further possibility of distinguishing between those with positive and negative indices, since it is known that in finite games those with positive and negative indices are dynamically stable and unstable, respectively, under monotone adjustment processes.

References

[1] De Castro, L. (2009), Affiliation and Dependence in Economic Models, Economics Department, University of Illinois.

[2] Dvoretzky, A., A. Wald, and J. Wolfowitz (1951), Relations Among Certain Ranges of Vector Measures, *Pacific J. Math.*, 1: 59-74.

[3] Govindan, S., and R. Wilson (2010), Existence of Equilibria in Auctions with Private Values, Stanford Business School Research Paper 2056. `gsbapps.stanford.edu/researchpapers/library/RP2056.pdf`

[4] Jackson, M., L. Simon, J. Swinkels, and W. Zame (2002), Communication and Equilibrium in Discontinuous Games of Incomplete Information, Econometrica, 70: 1711-1740.

Department of Economics, University of Iowa, Iowa City IA 52242, USA.
E-mail address: `srihari-govindan@uiowa.edu`

Stanford Business School, Stanford, CA 94305-5015, USA.
E-mail address: `rwilson@stanford.edu`

[8]We hope in later work to establish existence of an equilibrium in pure strategies.