Optimal Auctions with Financially Constrained Bidders *

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Abstract

We consider an environment where ex-ante homogenous potential buyers of an indivisible good have liquidity constraints: they cannot pay more than their 'budget' regardless of their valuation. A buyer's valuation for the good as well as her budget are her private information. We derive the symmetric constrained-efficient and revenue maximizing auctions for this setting. We show how to implement these via a standard auction (all pay) with a modified winning rule: the highest bidder need not win the good outright. Stated differently, the optimal allocation rule has 'pooling' in the middle, despite the usual regularity conditions. Subsidizing bidders with low budgets is undesirable in terms of both revenue and social welfare. From a technical standpoint, the paper contributes to auction design with multidimensional private information by working directly with lower dimensional reduced form allocation rules, abstracting from the higher dimensional ex-post allocation rule.

Keywords: optimal auction, budget constraints, reduced form, multidimensional mechanism design

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1 Introduction

Modeling buyers as budget or liquidity constrained allows one to relax a standard assumption in the auction literature that conflates a buyer's willingness to pay with her ability to pay. The assumption is untenable in a variety of situations. For instance, in government auctions (privatization, license sales etc.), the sale price may well exceed a buyers' liquid assets, and she may need to rely on an imperfect (i.e. costly) capital market to raise funds. These frictions limit her ability to pay, but not her valuation (how much she would pay if she had the money). Indeed, these financial constraints are more palpable to bidders than valuations, which are relatively amorphous.

Here we assume that in addition to private valuations, buyers have privately known 'hard' budget constraints: no buyer can pay more than her budget regardless of her valuation.[1] We do not model the source of these constraints. The reader is referred to the discussion in Che and Gale [6] for possible explanations.

This paper considers the sale of a single indivisible good to ex-ante homogenous buyers whose valuations and budgets are their private information. We provide an analysis of both the expected revenue maximizing and constrained efficient symmetric auctions. Our analysis is for the case where the solution concept is Bayes-Nash. Individual rationality is imposed in an interim sense. In other words, we only require the auction to offer each buyer a non-negative expected surplus. We then show how our analysis can be applied when the auction is required to be dominant strategy implementable or ex-post individually rational.

1.1 An Incorrect First Intuition

It is natural to ask why budgets cannot be accommodated in a straightforward way i.e. consider each agent's type as being the minimum of their valuation v and budget b, and run the appropriate optimal auction. We claim such a mechanism will be strictly suboptimal in terms of revenue. This is because it will (i) pool 'too many' types and, (ii) discourage competition. It is easiest to see this via an example.

Suppose two buyers who both have a commonly known budget of 1. Both have valuations that are i.i.d. draws from a uniform distribution over $[0, 2]$. The resulting distribution over the 'modified types' $\min(v, b)$ is $F(v) = v/2$ for v less than 1, $F(1) = 1$. The optimal auction when types are thusly distributed does not allot to buyers with value less than 1, and the resulting expected revenue is 0.75. The implied allocation rule can be implemented as an all-pay auction- buyers with 'modified type' 1 all make their expected payment of 0.75, lower types pay nothing. In this candidate 'optimal' auction, no type ever pays their budget. This suggests that the proposed auction may be sub-optimal, i.e. there is money left on the table.

[1] The literature also considers the case of 'soft' budget constraints, where bidders may be able to get additional funds from the market at some cost.

What went wrong? Consider the types with values 1 and, say, 1.01. Our intuition from standard mechanism design suggests that the incentive rents that will accrue to the types could result in the expected payment of both being less than the budget constraint of 1. Therefore it is possible that they could be separated. However, by treating buyers with value v as having value $\min(v, b)$, the mechanism forces these two types to be pooled. In general, this is 'too much' pooling: the optimal mechanism will pool a subset of the types that have value larger than their budget.

1.2 Discussion of Results

In this section we describe the main qualitative features of the revenue maximizing auction subject to budget constraints.[2] We describe both the optimal allocation rule and an implementation. We contrast it with the optimal auction when buyers have no budget constraints (Myerson [13]), and when all buyers have the same commonly known budget (Laffont and Robert [10]). Our discussion is for the case where the distribution of buyer values satisfies the monotone hazard rate condition (in particular, uniform). When this condition is not satisfied, additional pooling of types may be required, as originally shown by Myerson.

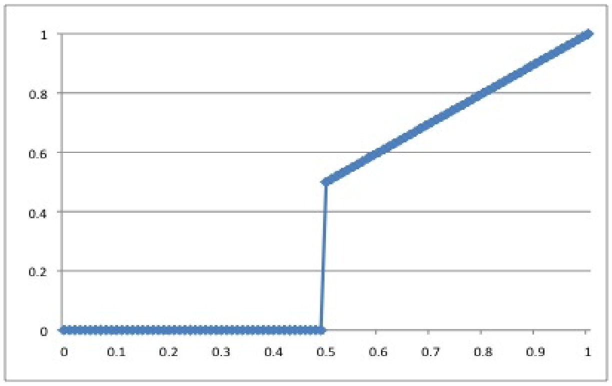
No Budget Constraints When buyers are not budget constrained, the type of an agent is just her valuation, and Myerson [13] applies. In this case we know that at each realized profile of types, the optimal allocation rule allots to the highest valuation above an appropriately chosen reserve $\underline{v}$. The reserve is the lowest valuation with a non-negative 'virtual valuation.' Assuming 2 bidders and valuations to be uniform in $[0, 1]$, the resulting interim allocation probabilities are as graphed in Figure 1(a).

The optimal auction can be implemented as any standard auction (first price, second price, or all pay) with the appropriate reserve.

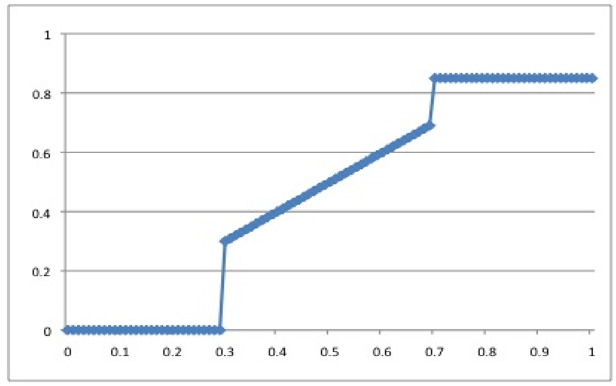
Common Knowledge Common Budgets Suppose all buyers have the same (common knowledge) budget constraint b. The type of an agent is still just her valuation. Laffont and Robert showed that the revenue maximizing auction will 'pool' some types at the top. All types above some $\bar{v}$ will be treated as if they had valuation exactly $\bar{v}$: the budget constraint binds for precisely these types. They argued that the optimal allocation rule would allot the good to the highest valuation subject to this 'pooling,' and subject to it being higher than an appropriately chosen reserve $\underline{v}$. This reserve would be lower than the optimal reserve when there were no budget constraints. The resulting interim allocation probabilities are as graphed in Figure 1(b).

The optimal auction is implemented as an all pay auction with appropriately chosen reserve. In equilibrium, buyers with value $\bar{v}$ and above bid exactly b and therefore get

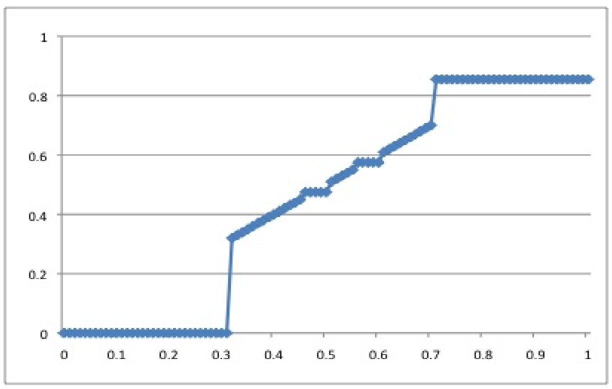
[2] The constrained efficient auction is similar but has no reserve price.



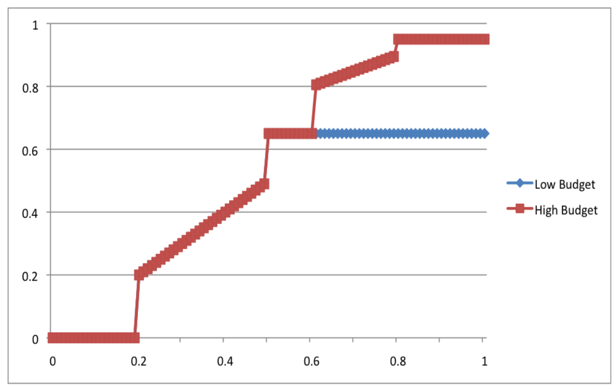
(a) No Budget Constraints



(b) Common Budget Constraint, Decreasing Density



(c) Common Budget Constraint



(d) Private Budgets

FIGURE 1: Optimal Interim Allocation Probability $a(v)$ plotted against v.

pooled.

Under just the standard monotone hazard rate assumption, however, this mechanism need not be optimal. An additional condition that is sufficient for optimality is that the density function of the valuations is (weakly) decreasing.[3] If this condition is violated, our analysis shows that there may be additional pooling in the middle as displayed in Figure 1(c).[4] In this case, the optimal auction cannot be implemented as a 'standard' mechanism.

PRIVATE BUDGETS Finally, suppose bidders have one of 2 budgets $H > L$, with equal probability and independent of their value. The type of a bidder is 2 dimensional- valuation and budget. There will be two cutoffs for pooling at the top: $\bar{v}_H > \bar{v}_L$. All high budget bidders with valuation at least $\bar{v}_H$ will be pooled (and pay H) and all low budget bidders

[3]This was overlooked in the original papers by Laffont and Robert, and Maskin.

[4]Formally, we require that $v - \frac{1-F(v)}{f(v)} + \frac{\eta}{f(v)}$ be increasing in v. η is endogenously determined. As a result we can only provide sufficient conditions in terms of the primitives of the model.

with valuation at least $\bar{v}_L$ will be pooled (and pay L). A bidder with valuation $v < \bar{v}_L$ will get the same allocation whether he is of a high budget or low budget type.

Our key finding is the additional distortion the optimal allocation rule produces for high budget bidders with valuation only slightly larger than $\bar{v}_L$. High budget bidders whose valuations are in the range $[\bar{v}_L, \bar{v}_L + \Delta]$ for an appropriately chosen Δ will be pooled with lower budget bidders with valuation in $[\bar{v}_L, 1]$. The resulting interim allocation probabilities are graphed in Figure 1(d). The case of $k > 2$ possible budgets will involve $k-1$ such intervals in which the outcome is distorted, and k cutoffs where the k budget constraints bind.

The auction can be implemented as an all pay auction with a modified winning rule: the highest bid need not win the good outright. As before there will be (an appropriately chosen) reserve price. Further the auction will treat all bids between $[L, L+\delta)$ as L: i.e. if there are multiple bids in this range, the winner is selected randomly. In the resulting equilibrium, all buyers with a high budget and a valuation in $[\bar{v}_L, \bar{v}_L + \Delta]$ will bid L and be pooled low budget bidders with valuation larger than $\bar{v}_L$. As in the common knowledge budget case, buyers with low budget and valuation larger than $\bar{v}_L$ bid L and get pooled. Similarly, buyers with a high budget and valuation larger than $\bar{v}_H$ bid H and get pooled. If either of the maintained assumptions of monotone hazard rate and decreasing density on the distribution of valuations is violated, additional pooling may be required.

1.2.1 Subsidies

Budget constraints depress revenues because low budget bidders cannot put competitive pressure on high budget bidders. Therefore, it is natural to inquire into instruments to foster competition. In other settings where some bidders are disadvantaged relative to others, subsidies have been suggested.[5]

A subsidy is not the only instrument for encouraging competition. For this reason an analysis of the optimal auction is useful: it may suggest other instruments that are more effective. We find that the optimal mechanism rules out subsidies that are type dependent. In particular, we show that loosening the budget constraints with a lump-sum transfer does not improve revenue. Rather, as we described above, the optimal mechanism favors bidders with small budgets with a higher probability of winning, by distorting the winning rule away from the standard 'highest bidder wins'.

[5] In the FCC spectrum auctions, Ayres and Cramton [1] argued that subsidizing women and minority bidders actually increased revenues since it induced other bidders to bid more aggressively. Their argument was based on the assumption that minority bidders would typically assign lower valuations to the asset than large bidders. In a procurement context, Rothkopf et al [15] find that subsidizing inefficient competitors can be desirable. Zheng [16] studies a stylized setting where bidders are subject to a 'soft' budget constraint and shows that if the auctioneer in this setting has access to cheaper funds, he may wish to subsidize some bidders.

This no-subsidy result applies also to the case when buyers have heterogenous, commonly known budgets. Even in this case the optimal mechanism will not involve (and cannot be interpreted as) a lump-sum transfer to the constrained buyer. The method of analysis thus yields another insight regarding the design of auctions in such settings. Where prior work suggested there may be gains to subsidizing low budget bidders, our analysis shows that the auctioneer would decline to subsidize bidders in the optimal auction. Thus, arguments in favor of subsidies depend on the analysis of specific (i.e. sub-optimal) auction mechanisms.

1.2.2 Technical Contribution

In terms of techniques, this paper is a contribution to the literature on mechanism design when agents' types are multidimensional. Some of the difficulties associated with multidimensional types do not arise here because the extra dimension of private information does not influence valuations. As a result, the expected revenue is still a linear function of the allocation rule.[6]

A difficulty here is that budget constraints render the associated incentive compatibility constraints non-differentiable. Therefore, standard first-order techniques have no bite in this setting, and characterizing incentive compatible allocation rules is hard. We skirt this difficulty by considering a model of discrete types, i.e there are only a finite (if large) number of possible valuations and budgets.[7] This makes the problem of optimal design amenable to the use of tools from linear programming. We characterize the optimal mechanism in the continuum of types case by considering successively finer discrete type spaces.

Another contribution is in working directly with the reduced form, i.e. agents' interim allocation probabilities- their probability of getting the good conditional on their own type and taking expectations over everyone else's type. Compared to the standard profile-by-profile allocation rule, this allows us to reduce dimensionality, thus making the problem more tractable. Border [3] provides necessary and sufficient conditions for an interim allocation rule to be feasible.[8] The conditions are a system of linear inequalities that must be satisfied, and these turn out to be 'easy' to work with. Further, one can recover the optimal ex-post allocation rule from the optimal interim allocation rule. Intuitively, this is because the optimal allocation must be a corner point of the feasible region in a linear program. The ex-post allocation rules implementing these corner points are uniquely identified.

[6] By contrast, for example in Rochet and Choné [14], which IC constraints bind is not known a priori, and therefore payments of different types as a function of the allocation rule is hard to analytically determine.

[7] Readers with long memories will recall that the 'original' optimal auction paper by Harris and Raviv [9] also assumed discrete types.

[8] I.e. for there to exist a feasible profile-by-profile allocation rule resulting in exactly those interim allocation probabilities.

1.3 Related Literature

Revenue Ranking 'Standard' Auctions Theoretical investigations of auctions with budget constraints have mainly been confined to analyzing 'standard' auction formats when bidders are financially constrained. Che and Gale [6], for example, consider the revenue ranking of first price, second price and all pay auctions under financial constraints. Benoit and Krishna [2] look into the effects of budget constraints in multi-good auctions, and they compare sequential to simultaneous auctions. Brusco and Lopomo [5] study strategic demand reduction in simultaneous ascending auctions and show that inefficiencies can emerge even if the probability of bidders having budget constraints is arbitrarily small. This summary is by no means complete and for illustrative purposes only.

Optimal Design Research focused on optimal design is more limited. Laffont and Robert [10] as well as Maskin[12] offer an incomplete analysis of the case when valuations are private information but budgets are common knowledge and identical. Malakhov and Vohra [11] consider the case when one bidder has a known budget constraint and the other does not. Che and Gale [7] compute the revenue maximizing pricing scheme when there is a single buyer whose budget constraint and valuation are both his private information.[9] Borgs et al [4] study a multi-unit auction and design an auction that maximizes worst case revenue when the number of bidders is large. Nisan et al [8] show in a closely related setting that no dominant strategy incentive compatible auction can be Pareto-efficient when bidders are budget constrained.

1.4 Organization of this paper

In Section 2 we describe the model. In Section 3 we examine the special case when all bidders have the same common knowledge budget constraint. This helps build intuition for the more involved private information case. In Section 4 we examine the case when bidders' budgets are private information. In Section 4.6 we discuss the (im)-possibility of profitably subsidizing bidders.

2 A Discrete Formulation

2.1 The Environment

There are N risk neutral buyers interested in a single indivisible good.

[9] Their definition of a financial constraint is akin to the soft constraints described earlier.

Space of Types Each buyer has a private valuation for the good v in $V = \{\epsilon, 2\epsilon, \ldots, m\epsilon\}$.[10] For notational convenience we take $\epsilon = 1$. Further, each buyer has a privately known budget constraint b in $B = \{b_1, b_2, \ldots, b_k\}$, wlog $b_1 < b_2 < \ldots < b_k$. The type of a buyer is a 2-tuple consisting of his valuation and his budget $t = (v, b)$; and the space of types is $T = V \times B$.

Buyers' types are i.i.d. draws from a commonly known distribution π over T. We assume that the valuation and budget components of a bidder's type are independent, and that all budgets are equally likely. The valuation component is distributed with CDF $F(\cdot)$ and density $f(\cdot)$. Therefore:

$$\mathbb{P}\big(t = (v, b)\big) = \pi(t) = \frac{1}{k} f(v). \tag{1}$$

It will be clear from the analysis that the assumption of equally likely budgets is purely for notational convenience. The assumption of independence is harder to relax- we provide further discussion after the proof of our main result.

Buyer Preferences An agent of type $t = (v, b)$ who is given the good with probability a and asked to make a payment p derives utility:

$$u(a, p|(v, b)) = \begin{cases} va - p & \text{if } p \leq b, \\ -\infty & \text{if } p > b. \end{cases}$$

In other words an agent has a standard quasi-linear utility up to his budget constraint, but cannot pay more than his budget constraint under any circumstances.

2.2 Seller's Problem

By the Revelation Principle, we confine ourselves without loss of generality to direct revelation mechanisms. The seller must specify an allocation rule and a payment rule. The former determines how the good is to be allocated as a function of the profile of reported types and the latter the payments each agent must make as a function of the reported types. Denote the implied interim expected allocation and payment for a bidder of type t as $a(t)$ and $p(t)$ respectively.

To ensure participation of all agents we require interim individual rationality:

$$\forall\, t \in T,\, t = (v, b): \qquad va(t) - p(t) \geq 0. \tag{2}$$

[10] The assumption that valuations are equally spaced is for economy of notation only.

The budget constraint and individual rationality require that no type's ex-post payments exceed their budget. For now we impose only an expected budget constraint. We discuss why this is without loss of generality in the next paragraph.

$$\forall\, t \in T,\, t = (v, b) : \qquad p(t) \leq b. \tag{3}$$

We require that Bayesian incentive compatibility hold. Due to the budget constraint, the incentive constraints will only require that a type $t = (v, b)$ has no incentive to misreport as types t' such that $p(t') \leq b$. We can write this as:

$$\forall\, t, t' \in T,\, t = (v, b) : \qquad va(t) - p(t) \geq \chi_{\{p(t') \leq b\}} \left(va(t') - p(t') \right), \tag{4}$$

where χ is the characteristic function. Note that the presence of this characteristic function renders the incentive compatibility constraints non-differentiable, and thus standard first order conditions do not apply.

All our constraints are written in terms of expected payments. Three comments are due:

1. Since our solution concept is Bayes-Nash, we impose only interim individual rationality and incentive compatibility constraints. We discuss strengthening these in Section 5.1.

2. One may worry that by writing the budget constraint (3) only on interim rather than the more natural ex-post payments, we are relaxing the problem. However, if expected payments do not exceed the budget, then there exists a profile-by-profile payment rule such that these payments never exceed the budget *ex-post*: the 'all-pay' payment rule where type t pays $p(t)$ regardless of other agents' reports.

3. By contrast, in writing the IC constraints (4) in terms of expected rather than ex-post payments, the problem may be over-constrained. If we were to work with ex-post payments, the IC constraints where a buyer over-reports his budget can be relaxed. Why? Because if a buyer reported his budget as b, he could be required to 'prove' it by making a payment of b with some vanishing probability ϵ. A buyer with true budget $b' < b$ would therefore be unwilling to risk over-reporting. Nevertheless, in our solution, these constraints do not bind.

As a result, working directly with expected payments is justified.

Feasible Interim Allocation Rules A key prior result we use in this paper is from Border [3]. Border provides a set of linear inequalities which characterize the space of feasible interim allocation probabilities given the distribution over types and the number of buyers. In other words, they characterize which interim allocation probabilities can be achieved by some feasible profile-by-profile allocation rule. These inequalities simplify our problem significantly, since we now search over the (lower dimensional) space of interim allocation probabilities, rather than concerning ourselves with the allocation rule profile by profile. The Border inequalities state that a set of interim allocation probabilities $\{a(t)\}_{t\in T}$ is feasible if and only if the $a(t)$'s satisfies:

$$\forall\, t \in T: \qquad a(t) \geq 0, \tag{5}$$

$$\forall\, T' \subseteq T: \qquad N \sum_{t \in T'} \pi(t) a(t) \leq 1 - \left(\sum_{t \notin T'} \pi(t) \right)^N . \tag{6}$$

The left hand side of (6) is the expected probability the good is allocated to an agent with a type in T', which must be less than the probability that at least one agent has a type in T'.

Equivalent Optimization Program Therefore, the problem of finding the revenue maximizing auction can be written as:

$$\max_{a,p} \sum_t \pi(t) p(t) \tag{RevOpt}$$
$$\text{s.t.:} \qquad (2-6).$$

Similarly, the problem of finding the constrained efficient auction can be written as:

$$\max_{a,p} \sum_t \pi(t) v a(t) \tag{ConsEff}$$
$$\text{s.t.:} \qquad (2-6).$$

2.3 Overview of Linear Programming Approach

To orient the reader, we give an overview of the approach taken. By working with a discrete type space, we will reformulate the problem as a linear program of the following form:

$$\begin{aligned} Z = \max_x \; & mx \\ \text{s.t.} \; & Cx \leq d \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

The first set of constraints, $Cx \leq d$, corresponding to (2 - 4), are 'complicated.' The second set, $Ax \leq b$, correspond to the feasibility constraints (6). This set is 'easy' in the sense that A is an upper triangular matrix. An upper triangular matrix in a linear program is easy because the corresponding dual constraints will also have a triangular component. A triangular system of equations is easy to solve by Gaussian elimination. As a result the solution of this linear program is easy to characterize by complementary slackness. Stated alternately,

$$\begin{aligned} Z(\lambda) = \max_{x} \ & mx + \lambda(d - Cx) \\ \text{s.t. } \ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

For each $\lambda \geq 0$, $Z(\lambda)$ is easy to compute because A is upper triangular. By the duality theorem of linear programming,

$$Z = \min_{\lambda \geq 0} Z(\lambda).$$

Thus our task reduces to identifying the non-negative λ that minimizes $Z(\lambda)$. Now, $Z(\lambda)$ is a piecewise linear function of λ with a finite number of breakpoints. We find an indirect way to enumerate the breakpoints without explicitly listing them. In this way we compute the value Z.

In the auction context, the coefficients of x in the objective function, i.e. $(m - \lambda C)$, have an interpretation as 'virtual values.'

3 The Common Knowledge Budget Case

In this section, we analyze the case where all bidders have the same, commonly known budget. This helps us build intuition and familiarity with the proof methods used subsequently to analyze the general case. We examine the case of revenue maximization.

Since all bidders have the same budget constraint b, a bidder's type is just her valuation. Further, we can drop the characteristic function in the IC constraints since, by individual rationality, all types must have a payment of at most b. Given these simplifications, the problem of maximizing revenue, (RevOpt), becomes:

$$\max_{a,p} \sum f(v)p(v) \tag{RevOptCK}$$

$$\begin{aligned}
\text{s.t. } \forall\, v: &\quad p(v) \leq b \\
\forall\, v, v': &\quad va(v) - p(v) \geq va(v') - p(v') \\
\forall\, v: &\quad va(v) - p(v) \geq 0 \\
\forall\, V' \subseteq V: &\quad N \sum_{v \in V'} f(v)a(v) \leq 1 - \left(\sum_{v \notin V'} f(v) \right)^N \\
\forall\, v: &\quad a(v) \geq 0
\end{aligned}$$

First, add a 'dummy' type 0 to the space of types, and define $a(0) = p(0) = 0$. We can subsume the IR constraint, by requiring IC over the extended type space $V' = V \bigcup \{0\}$. Standard arguments imply that an allocation rule $a(\cdot)$ can be part of an incentive compatible mechanism if and only if $a(v)$ is non-decreasing in v. Further, the payment rule that maximizes revenue associated with this allocation rule is:

$$p(v) = va(v) - \sum_{1}^{v-1} a(v'). \tag{7}$$

Note the absence of a constant. We have set this to zero, ruling out subsidies in the form of lump-sum transfers. In effect we compute the optimal subsidy-free mechanism.[11] In Section 4.6 we show that any lump-sum transfer will necessarily reduce expected revenue.

Substituting (7) back into (RevOptCK), we can rewrite it as:

$$\max_{a} \sum f(v)\varphi(v)a(v) \tag{REVOPTCK2}$$

$$\forall\, v: \quad va(v) - \sum_{v'=1}^{v-1} a(v') \leq b \tag{8a}$$

$$\forall\, V' \subseteq V: \quad N \sum_{v \in V'} f(v)a(v) \leq 1 - \left(\sum_{v \notin V'} f(v) \right)^N \tag{8b}$$

$$\begin{aligned}
\forall\, v: &\quad a(v) - a(v+1) \leq 0 \\
\forall\, v: &\quad a(v) \geq 0
\end{aligned}$$

Here $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ is type v's 'virtual valuation,' as in Myerson [13].

Monotonicity of the allocation rule makes many of the constraints in (8b) redundant:

[11]Note that incentive compatibility rules out type dependent subsidies.

PROPOSITION 1 (Border). *Let $a : T \to [0, 1]$ be the interim probability of allocation for a type space T. For each $\alpha \in [0, 1]$, set*

$$E_\alpha = \{t : a(t) \geq \alpha\}.$$

Then a is feasible if and only if for each E_α:

$$N \sum_{t \in E_\alpha} a(t) f(t) \leq 1 - \left(\sum_{T - E_\alpha} f(t) \right)^N . \tag{9}$$

This makes the problem easier by decreasing the number of constraints we need consider. The original Border constraints would require constraints corresponding to all subsets of types. By Proposition 1, the number of constraints is at most the number types. Since IC constraints normally imply some sort of monotonicity of the allocation rule, the sets E_α are easy to characterize, and therefore these constraints are easy to use. For instance, in this setting:

COROLLARY 1. *If $a(\cdot)$ is increasing in v, it is feasible if and only if :*

$$\forall\, v \in V : \qquad N \sum_{v}^{m} f(v') a(v') \leq 1 - F^N(v - 1). \tag{10}$$

For notational convenience define $c_v = \frac{1 - F^N(v-1)}{N}$ for each v.
From (7), it should be easy to see:

$$\begin{aligned} a(v+1) > a(v) &\implies p(v+1) > p(v), \\ a(v+1) = a(v) &\implies p(v+1) = p(v). \end{aligned}$$

Therefore, if the budget constraint (8a) binds for some valuation $\bar{v}$, it must bind for all valuations $v \geq \bar{v}$. If the budget constraint does not bind in the optimal solution, the solution must be the same as Myerson's. Thus for simplicity, we assume the budget constraint binds in the optimal solution. We summarize this in the following observation.

OBSERVATION 1. If a^* is an optimal solution to (RevOptCK), the budget constraint must bind for some types $\{\bar{v}, \bar{v} + 1, \ldots, m\}$. Further, all these types are pooled, i.e. for all $v \geq \bar{v}$, $a^*(v) = a^*(\bar{v})$.

Therefore the optimal solution must pool some types at the top. Looking ahead, we will find that that the 'adjusted' virtual valuation of a type v is $\varphi(v) + \frac{\eta}{f(v)}$, where $\varphi(v)$ is the Myersonian virtual valuation, and η is the dual variable corresponding to the budget constraint. The latter term, $\frac{\eta}{f(v)}$, adjusts for the budget constraint. Increasing allocations to lower types reduces the payment of the high types, and hence 'relaxes' the budget constraint. As in Myerson, we require that the adjusted virtual valuation $\varphi(v) + \frac{\eta}{f(v)}$ be increasing in v. A sufficient condition for this is that $f(v)$ is weakly decreasing and satisfies the monotone hazard rate condition. By analogy with Myerson, the lowest type that will be allotted is the lowest type ($\underline{v}$) whose adjusted virtual valuation is non-negative. Finally, the optimal allocation rule will be efficient between types $\underline{v}$ and $\bar{v}-1$. We summarize this in the following proposition, a formal proof follows.

Proposition 2. *Suppose $f(v)$ is weakly decreasing, and the virtual valuation $\varphi(v)$ is increasing in v. Then the solution of (RevOptCK) can be described as follows: there will exist two cutoffs $\bar{v}$ and $\underline{v}$. No valuation less than $\underline{v}$ will be allotted. All types $\bar{v}$ and above will be pooled, and the budget constraint will bind for exactly those types. The allocation rule will be efficient between types $\underline{v}$ and $\bar{v} - 1$. Finally, $\underline{v}$ is the lowest type such that*

$$\varphi(\underline{v}) + \frac{\eta}{f(\underline{v})} \geq 0,$$
$$\eta = \frac{(1 - F(\bar{v} - 1))(1 - F(\bar{v} - 2))}{\bar{v}f(\bar{v} - 1) + (1 - F(\bar{v} - 1))}.$$

If $f(v)$ is not weakly decreasing in v or does not satisfy increasing virtual values, the optimal solution may require pooling in the middle.

We can describe the analogous optimal mechanism if types are drawn from a continuum by considering successively finer discretizations (see Appendix B).

Proposition* (Continuous Types, Laffont and Robert). *Suppose $f(v)$ is weakly decreasing and the virtual valuation $\varphi(v)$ is increasing in v. The revenue maximizing mechanism can be described as follows: there will be two cutoffs $\bar{v}$ and $\underline{v}$. Valuations less than $\underline{v}$ will not be allotted. Types $\bar{v}$ and above will be pooled, and the budget constraint will bind for exactly those types. The allocation rule will be efficient in the interval $[\underline{v}, \bar{v}]$. Finally, $\underline{v}$ is the lowest type such that:*

$$\varphi(\underline{v}) + \frac{\eta}{f(\underline{v})} \geq 0$$
$$\eta = \frac{(1 - F(\bar{v}))^2}{\bar{v}f(\bar{v}) + (1 - F(\bar{v}))}.$$

If $f(v)$ is not weakly decreasing in v or does not satisfy increasing virtual values, the optimal solution may require pooling in the middle.

Implementation First consider what one can call the regular case- i.e. f satisfies both the monotone hazard rate and weakly decreasing density conditions. In this case, the implementation will be as described by Laffont and Robert: the auction will be implemented as an all pay auction, with an appropriately chosen reserve price. The reserve price p will be the payment of type $\underline{v}$ in the auction described above. If f is not regular- i.e. it violates either the monotone hazard rate or decreasing density conditions, then additional pooling may be required. In particular the all-pay auction described above may not be optimal. The pooling identified by the optimal auction must then be implemented by further modifying the winning rule in the all-pay auction.

Proof of Proposition 2

The proof is broken into four main steps for expositional reasons. First, we use Observation 1 to rewrite the seller's problem. Next, we flip to the dual, and work with the constraints of the dual. We then construct a dual feasible solution that complements the proposed primal optimal, proving optimality.

Step 1: Rewriting the Primal

Suppose the lowest type for which the budget constraint binds in the optimal solution a^* is $\bar{v}$. Substituting into (REVOPTCK2); and dropping the redundant Border constraints by Lemma 1, we conclude that a^* must be a solution to the program below. The symbols in parentheses on the right of the constraints refer to the corresponding dual variables.

$$\max_a \left(\sum_1^{\bar{v}-1} f(v)\varphi(v)a(v) \right) + (1 - F(\bar{v}-1))\bar{v}a(\bar{v})$$

$$\text{s.t.} \qquad -\sum_1^{\bar{v}-1} a(v') + \bar{v}a(\bar{v}) = b \qquad (\eta)$$

$$\forall\, v \leq \bar{v}: \qquad \sum_v^{\bar{v}-1} f(v')a(v') + (1 - F(\bar{v}-1))a(\bar{v}) \leq c_v \qquad (\beta_v)$$

$$\forall\, v: \qquad a(v) - a(v+1) \leq 0 \qquad (\mu_v)$$

$$\forall\, v: \qquad a(v) \geq 0$$

Step 2: Taking the Dual

The dual is shown below, the primal variable associated with each dual constraint displayed in parentheses next to the constraint.

$$\min_{\eta,\beta,\mu} b\eta + \sum_{1}^{\bar{v}} c_v \beta_v$$

$$\text{s.t.} \quad \bar{v}\eta + (1 - F(\bar{v}-1)) \sum_{1}^{\bar{v}} \beta_v - \mu_{\bar{v}-1} \geq (1 - F(\bar{v}-1))\bar{v} \qquad (a(\bar{v}))$$

$$\forall\, v \leq (\bar{v}-1): \quad -\eta + f(v) \sum_{1}^{v} \beta_{v'} + \mu_v - \mu_{v-1} \geq f(v)\varphi(v) \qquad (a(v))$$

$$\forall\, v: \quad \beta_v, \mu_v \geq 0$$

Let $\underline{v}$ be the lowest valuation for which $a^*(\underline{v}) > 0$. By complementary slackness we have:

$$\bar{v}\eta + (1 - F(\bar{v}-1)) \sum_{1}^{\bar{v}} \beta_v - \mu_{\bar{v}-1} = (1 - F(\bar{v}-1))\bar{v},$$

$$\underline{v} \leq v \leq (\bar{v}-1): \quad -\eta + f(v) \sum_{1}^{v} \beta_{v'} + \mu_v - \mu_{v-1} = f(v)\varphi(v).$$

Rewriting, we get:

$$\sum_{1}^{\bar{v}} \beta_v - \frac{\mu_{\bar{v}-1}}{1 - F(\bar{v}-1)} = \bar{v} - \frac{\bar{v}\eta}{1 - F(\bar{v}-1)},$$

$$\underline{v} \leq v \leq (\bar{v}-1): \quad \sum_{1}^{v} \beta_{v'} + \frac{\mu_v}{f(v)} - \frac{\mu_{v-1}}{f(v)} = \varphi(v) + \frac{\eta}{f(v)}.$$

Step 3: Constructing a Dual Feasible Solution

The proof proceeds by constructing dual variables that complement the primal solution described in the statement of the proposition.

Since $a^*(v) = 0$ for $v < \underline{v}$ and $f(v) > 0$ for all v, the corresponding Border constraints (8b) do not bind at optimality. Therefore $\beta_v = 0$ for all $v < \underline{v}$. Further $0 = a^*(\underline{v}-1) < a^*(\underline{v})$ by definition of $\underline{v}$, and so, by complementary slackness, $\mu_{\underline{v}-1} = 0$. Similarly, since $\bar{v}$ is the lowest type for which the budget constraint binds, $a^*(\bar{v}) > a^*(\bar{v}-1)$, implying that $\mu_{\bar{v}-1} = 0$.

Subtracting the dual constraints corresponding to types $\bar{v}$ and $\bar{v}-1$ and using the fact that $\mu_{\underline{v}-1} = 0$, we have:[12]

$$\beta_{\bar{v}} + \frac{\mu_{\bar{v}-2}}{f(\bar{v}-1)} = \bar{v} - \bar{v}\frac{\eta}{1-F(\bar{v}-1)} - \varphi(\bar{v}-1) - \frac{\eta}{f(\bar{v}-1)}. \tag{11a}$$

Subtracting the dual constraints corresponding to v and $v-1$, where $\underline{v}+1 \leq v \leq \bar{v}-1$, we have:

$$\beta_v + \frac{\mu_v}{f(v)} - \frac{\mu_{v-1}}{f(v)} - \frac{\mu_{v-1}}{f(v-1)} + \frac{\mu_{v-2}}{f(v-1)} = \varphi(v) - \varphi(v-1) + \eta\left(\frac{1}{f(v)} - \frac{1}{f(v-1)}\right). \tag{11b}$$

Finally, the dual constraint corresponding to type $\underline{v}$ reduces to:

$$\beta_{\underline{v}} + \frac{\mu_{\underline{v}}}{f(\underline{v})} = \varphi(\underline{v}) + \frac{\eta}{f(\underline{v})}. \tag{11c}$$

It suffices to identify a non-negative solution to the system (11a-11c) such that $\beta_v = 0$ for all $v < \underline{v}$ and $\mu_{\underline{v}-1} = 0$. Consider the following solution, with all other dual variables set to 0:

$$\begin{aligned} \underline{v}+1 \leq v \leq \bar{v}-1: \qquad & \beta_v = \varphi(v) - \varphi(v-1) + \eta\left(\frac{1}{f(v)} - \frac{1}{f(v-1)}\right), \\ & \eta = \frac{(1-F(\bar{v}-1))(1-F(\bar{v}-2))}{\bar{v}f(\bar{v}-1) + (1-F(\bar{v}-1))}. \end{aligned}$$

Step 4: Verification

Direct computation verifies that the given solution satisfies (11a-11c). In fact it is the unique solution to (11a-11c) with all μ's equal to zero. All variables are non-negative. In particular, β_v for $\underline{v}+1 \leq v \leq \bar{v}-1$ is positive. Since $f(\cdot)$ satisfies the increasing virtual value and decreasing density conditions, for any v,

$$\varphi(v) - \varphi(v-1) + \eta\left(\frac{1}{f(v)} - \frac{1}{f(v-1)}\right) > 0.$$

The dual constraint for $a(\underline{v}-1)$ reduces to:

$$0 \geq \varphi(\underline{v}-1) + \frac{\eta}{f(\underline{v}-1)},$$

verifying that types below $\underline{v}$ have a non-positive adjusted virtual value $\varphi(v) + \frac{\eta}{f(v)}$.

Furthermore, it complements the primal solution described in the statement of the proposition. This concludes the case where our regularity condition on the distribution of types

[12] This step is where the upper triangular constraint matrix is helpful.

(monotone hazard rate, decreasing density) are met.

Now suppose our sufficient condition is violated, i.e. $\varphi(v)-\varphi(v-1)+\eta\big(\frac{1}{f(v)}-\frac{1}{f(v-1)}\big)<0$ for some v. The dual solution proposed above will be infeasible since $\beta_v<0$. More generally, there can be no dual solution that satisfies (11a-11c) with all $\mu_v=0$. Hence, there must be at least one v between $\underline{v}$ and $\bar{v}-1$ such that $\mu_v>0$. This implies, by complementary slackness, that the corresponding primal constraint, $a(v)-a(v+1)\leq 0$ binds at optimality, implying pooling. Further the corresponding μ's are therefore non-zero, and are the 'ironing' multipliers a la Myerson. □

If such further pooling or ironing is required, one can further restrict the set of optimal dual solutions. A proof follows from observation of the 'hierarchical allocation rules' of Border [3].

Lemma 1 (Border). *In any solution to the primal problem (REVOPTCK2), at most one of the Border constraint (8b) corresponding to type v, and the monotonicity constraint corresponding to type $v-1$ can bind. Further, by complementary slackness:*

$$\forall\, v: \qquad \beta_v\mu_{v-1}=0 \tag{12}$$

The solution to the system of equations (11a-11c), (12) constitutes the optimal dual solution. It is easily seen that this solution is unique- therefore even in the case where ironing is required, there is a unique solution.

We are also in a position to describe the constrained efficient auction for this setting. The proof is very similar to that of Proposition 2 (objective function has v instead of $\varphi(v)$), and is therefore omitted.

Proposition 3 (Constrained Efficient Auction). *Suppose $f(v)$ is weakly decreasing in v. Then the constrained efficient auction in this setting can be described as follows: there will exist a cutoff $\bar{v}$. All types $\bar{v}$ and above will receive the same interim allocation probability, and the budget constraint will bind for exactly those types. The allocation rule will be efficient for types below $\bar{v}-1$. If the sufficient conditions are not met, the optimal solution may require pooling in the middle.*

In other words, if the distribution of types has decreasing density, the constrained efficient mechanism is an all-pay auction (with no reserve).

4 The General Case

Recall the original program (RevOpt):

$$
\begin{aligned}
& \max_{a,p} \sum_{j=1}^{k} \sum_{v=1}^{m} \frac{1}{k} f(v) p(v, b_j) \\
\forall\, (v,b) \in T: \quad & va(v,b) - p(v,b) \geq 0 \\
\forall\, (v,b) \in T: \quad & p(v,b) \leq b \\
\forall\, (v,b),(v',b') \in T: \quad & va(v,b) - p(v,b) \geq \chi_{\{p(v',b') \leq b\}} \left(va(v',b') - p(v',b') \right) \\
\forall\, T' \subseteq T: \quad & N \sum_{t \in T'} \pi(t) a(t) \leq 1 - \left(\sum_{t \notin T'} \pi(t) \right)^{N} \\
\forall\, t \in T: \quad & a(t) \geq 0
\end{aligned}
$$

The incentive compatibility constraints can be separated into 3 categories:

1. Misreport value: $va(v,b) - p(v,b) \geq va(v',b) - p(v',b)$. (13a)
2. Misreport budget: $va(v,b) - p(v,b) \geq \chi_{\{p(v,b') \leq b\}} \left(va(v,b') - p(v,b') \right)$. (13b)
3. Misreport both: $va(v,b) - p(v,b) \geq \chi_{\{p(v',b') \leq b\}} \left((va(v',b') - p(v',b') \right)$. (13c)

Standard arguments imply that the IC constraints corresponding to a misreport of value, (13a), can be satisfied by some pricing rule if and only if, for any b, $v \geq v'$ implies that $a(v,b) \geq a(v',b)$. Incentive compatibility and individual rationality imply

$$p(v,b) \leq va(v,b) - \sum_{1}^{v-1} a(v',b).$$

The difficulty stems from the IC constraints relating to misreport of budget, (13b) and (13c). In particular, we need (further) constraints on the allocation rule such that there exists an incentive compatible pricing rule.

4.1 Identifying Incentive Compatible Allocation Rules

We now identify conditions on interim allocation rules such that each type's payment is the maximum possible, i.e.

$$p(v,b) = va(v,b) - \sum_{1}^{v-1} a(v',b). \tag{14}$$

The following observation is analogous to Observation 1 in the common knowledge budget case. It simply says that fixing a budget b, if the allocation rule is such that the budget constraint binds for a type v, then it binds for all types v' larger than v, and all those types are pooled.

Observation 2. For any budget b, an allocation rule a is incentive compatible only if:

$$\forall\, v' \geq v: \qquad p(v,b) = b \implies a(v',b) = a(v,b).$$

Recall that fixing budget, a buyer's expected payment is increasing in his value. The next lemma compares types with the same value v, but different budgets $b' > b$. It shows that as long as buyer (v, b') has an expected payment less than b, it receives the same allocation as (v, b).

Lemma 2. *Fix an allocation rule a such that a is incentive compatible and individually rational with pricing rule (14). Fix two budgets $b' > b$. Let $\underline{v}_b$ be the largest v such that $p(\underline{v}_b, b') \leq b$. Then,*

$$\forall\, v \leq \underline{v}_b: \qquad a(v,b') = a(v,b).$$

Further, $a(\underline{v}_b + 1, b') > a(m,b)$.

A word on our choice of notation is in order here:

- $\underline{v}_i$ is the highest valuation with budget greater than b_i that pays at most b_i.
- $\overline{v}_i$ is the lowest valuation with budget b_i for whom the budget constraint binds.

We do not require that $\underline{v}_i \leq \overline{v}_i$ (nor will this generally be true). The reason for this slightly unintuitive notation is that $\overline{v}_i$ corresponds to pooling cutoffs, hence the upper-bar, while, $\underline{v}_i$ can be thought of as a 'reserve.'

We summarize this in the following definition:

Definition 1. *Given an allocation rule a that is incentive compatible and individually rational with pricing rule (14), define cutoffs:*

$$\forall\, i \leq k: \qquad \bar{v}_i = \arg\min\{v : p(v, b_i) = b_i\},$$
$$\forall\, i \leq k-1: \qquad \underline{v}_i = \arg\max\{v : p(v, b_{i+1}) \leq b_i\}.$$

Note that $\underline{v}_i < \bar{v}_{i+1}$. Further, define :

$$\overline{V} = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\},$$
$$\underline{V} = \{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_{k-1}\}.$$

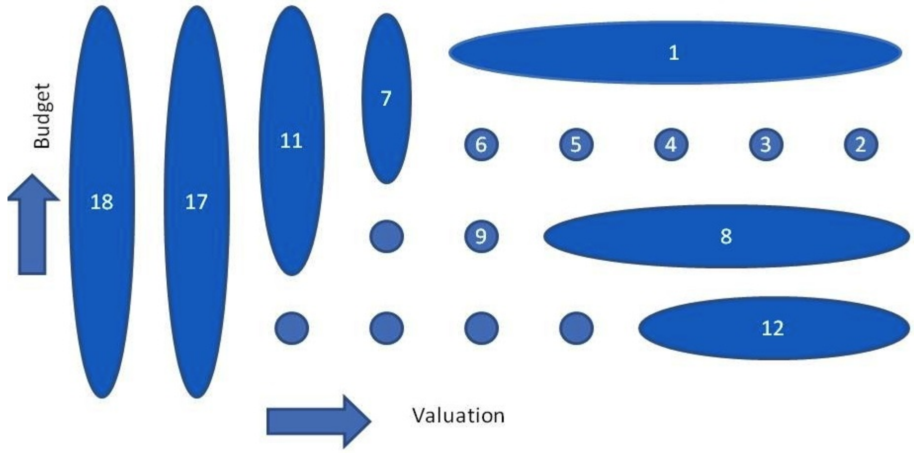


FIGURE 2: AN IC ALLOCATION RULE

Observation 2 and Lemma 2 imply:

OBSERVATION 3. An allocation rule $a : T \to [0,1]$ is consistent with cutoffs $\bar{V} = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$ and $\underline{V} = \{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_{k-1}\}$, where $\underline{v}_i \leq \bar{v}_{i+1}$ for all i, and pricing rule (14), incentive compatible and individually rational if and only if:

$$\forall\, v, b: \quad a(v,b) \leq a(v+1,b) \tag{15a}$$
$$\forall\, i: \quad a(\bar{v}_i - 1, b_i) < a(\bar{v}_i, b_i) \tag{15b}$$
$$\forall\, i: \quad p(\bar{v}_i, b_i) = b_i \tag{15c}$$
$$\forall\, i, v \geq \bar{v}_i: \quad a(v, b_i) = a(\bar{v}_i, b_i) \tag{15d}$$
$$\forall\, i, v \leq \underline{v}_i: \quad a(v, b_i) = a(v, b_{i+1}) \tag{15e}$$
$$\forall\, i: \quad p(\underline{v}_i + 1, b_i + 1) > b_i \tag{15f}$$
$$\forall\, i: \quad a(\underline{v}_i + 1, b_i + 1) > a(m, b_i) \tag{15g}$$

Figure 2 depicts an incentive compatible allocation rule for a type space with 10 possible valuations and 4 possible budgets. The blobs represent types that are pooled. The numbering reflects decreasing allocations: i.e. the blob numbered 1 has the highest interim probability of getting the good, 2 has the second highest and so on.

Given a collection of cut-offs we describe how to find an allocation rule consistent with those cutoffs that maximizes revenue. By Observation 3 we can drop the individual rationality, budget, and incentive compatibility constraints in (RevOpt) and substitute instead

(15a-15g). Therefore, we have:

$$\max_{a} \quad \sum_{i=1}^{k}\sum_{v=1}^{m}\frac{f(v)}{k}\varphi(v)a(v,b_i) \qquad \text{(REVOPT)}$$
$$\text{s.t.} \quad (15a-15g), (5), (6).$$

To ensure a well defined program the strict inequalities in (15b) and (15g) have to be relaxed to a weak inequality. If for a given set of cutoffs, the optimal solution to (REVOPT) binds at inequality (15b) or (15g), we know that the set of cutoffs being considered cannot be feasible. Hence we can restrict attention to cut-offs where (the weak version of) the inequalities do not bind at optimality.

Therefore, having fixed the cutoffs $\underline{V}, \overline{V}$, by (15a) and (15g), most of the Border constraints are rendered redundant by Proposition 1. In particular consider type (v, b_i); $v \leq \bar{v}_i$. Then, by Observation 3, $E_{(v,b_i)}$, the set of all types t such that $a(t) \geq a(v, b_i)$ is:

$$E_{(v,b_i)} = \bigcup_{j=i+1}^{k} \{(v', b_j) : v' \geq \min(v, \underline{v}_i + 1, \ldots, \underline{v}_{j-1} + 1)\} \bigcup \{(v', b_i) : v' \geq v\}.$$

It follows from Proposition 1 that the relevant Border constraints to be considered are:

$$\forall\, i, v \leq \bar{v}_i : \qquad N \sum_{t \in E_{(v,b_i)}} a(t)f(t) \leq 1 - \left(\sum_{T-E_{(v,b_i)}} f(t)\right)^N . \tag{16}$$

4.2 Potentially Optimal Allocation Rules

The next lemma further restricts the configurations of cutoffs in a revenue maximizing rule. Fix a budget b. Recall that $\underline{v}_b$ is the maximum value v such that type (v, b'), for any budget b' larger than b, pays at most b. The lemma says that $\underline{v}_b + 1$ is at least as large as the the lowest value v such that type (v, b)'s budget constraint binds.

Lemma 3. *Let a^* solve (REVOPT). Then the cutoffs $\overline{V}, \underline{V}$ as defined in Definition 1 must satisfy:*

$$\forall\, i \leq k-1 : \qquad \underline{v}_i \geq \bar{v}_i - 1. \tag{17}$$

Lemma 3 is a statement that the cutoffs are ordered in the intuitive sense. The budget constraint corresponding to budget b_i can bind in an optimal solution only if the budget constraints corresponding to each $b_j < b_i$ bind. Therefore, in any optimal solution, there must be a largest budget b_i such that the budget constraints corresponding to $b \leq b_i$ bind,

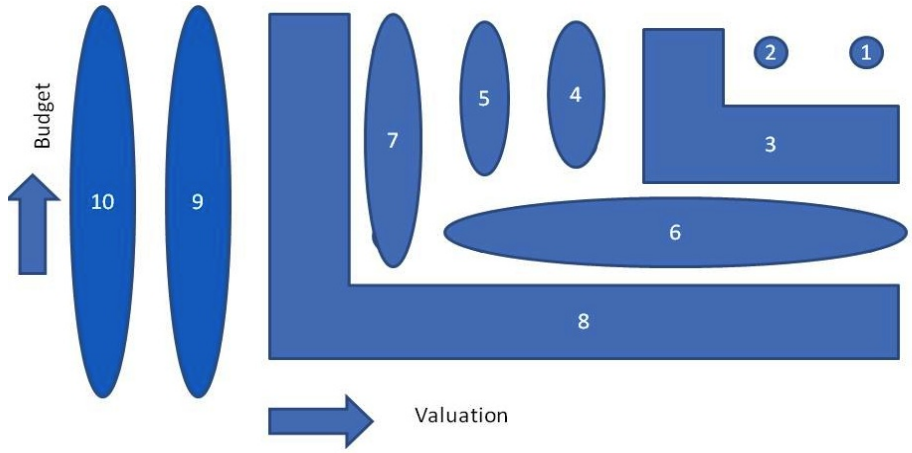


FIGURE 3: A POTENTIALLY OPTIMAL IC RULE

and the budget constraints corresponding to $b > b_i$ are slack. For notational simplicity we assume that in the optimal solution, all budget constraints bind.

With this added restriction on cutoffs; the set of incentive compatible and individually rational rules are summarized in Observation 4. Since $\underline{v}_i \geq \bar{v}_i - 1$, (15f) and (15g) are satisfied automatically, and

$$\underline{v}_i \equiv \arg\min\{v : a(v+1, b_{i+1}) > a(\bar{v}_i, b_i)\}.$$

OBSERVATION 4. An allocation rule $a : T \to [0,1]$, consistent with the cutoffs $\bar{V} = \{\bar{v}_1 \leq \bar{v}_2 \leq \ldots \leq \bar{v}_k\}$ and pricing rule (14) is incentive compatible and individually rational if and only if there exist, $x : V \to [0,1]$ and $y : \bar{V} \to [0,1]$ such that:

$$\forall\, i \leq k, v \geq \bar{v}_i : \qquad a(v, b_i) = y(\bar{v}_i), \tag{18a}$$

$$\forall\, i \leq k, \bar{v}_{i-1} + 1 \leq v \leq \bar{v}_i - 1, j \geq i : \qquad a(v, b_j) = x(v), \tag{18b}$$

$$\forall\, i \leq k : \qquad \bar{v}_i y(\bar{v}_i) - \sum_{1}^{\bar{v}_i - 1} x(v) = b_i, \tag{18c}$$

$$\forall\, v : \qquad x(v) \leq x(v+1), \tag{18d}$$

$$\forall\, i \leq k : \qquad x(\bar{v}_i - 1) < y(\bar{v}_i), \tag{18e}$$

$$\forall\, i < k : \qquad y(\bar{v}_i) \leq x(\bar{v}_i). \tag{18f}$$

Figure 3 displays an allocation rule whose cutoffs satisfy Lemma 3. As before, the blobs represent types that are pooled. The numbering reflects decreasing allocations: i.e. the blob numbered 1 has the highest interim probability of getting the good, 2 has the second highest and so on.

4.3 Characterizing the Optimal

We are now in a position to describe the expected revenue maximizing mechanism.

Proposition 4. *Suppose $f(v)$ is weakly decreasing, and the virtual value $\varphi(v)$ is increasing in v. Then, there is an optimal solution $a^*(v,b)$ to (RevOpt) that can be described as follows: there will exist cutoffs $\bar{v}_1 \leq \underline{v}_1 \leq \bar{v}_2 \leq \dots \underline{v}_{k-1} \leq \bar{v}_k$ and $\underline{v}_0$. No valuation less than $\underline{v}_0$ will be allotted. The allocation rule will satisfy (18a-18f). The allocation will be efficient between each $\underline{v}_i$ and $\bar{v}_{i+1}$. Types (v,b) with $b > b_i$ and $\bar{v}_i \leq v \leq \underline{v}_i$ will be pooled with types (v,b_i) whose budget constraints bind, i.e. $v \geq \bar{v}_i$.*

If the sufficient conditions are not met, the optimal solution may require additional pooling in the middle.

Proposition* (Continuous Valuations). *Suppose $f(v)$ is weakly decreasing in v, and $\frac{1-F(v)}{f(v)}$ is increasing in v. Then, there is an optimal solution $a^*(v,b)$ to (RevOpt) that can be described as follows: there will exist cutoffs $\bar{v}_1 \leq \underline{v}_1 \leq \bar{v}_2 \leq \dots \underline{v}_{k-1} \leq \bar{v}_k$ and $\underline{v}_0$. No valuation less than $\underline{v}_0$ will be allotted. There will exist functions $x : V \to [0,1]$ and $y : V \to [0,1]$ such that the allocation rule satisfies:*

$$\begin{aligned}
\forall\, i \leq k, v \geq \bar{v}_i : \qquad & a(v,b_i) = y(\bar{v}_i), \\
\forall\, i \leq k, \bar{v}_{i-1} \leq v < \bar{v}_i, j \geq i : \qquad & a(v,b_j) = x(v), \\
\forall\, i \leq k : \qquad & \bar{v}_i y(\bar{v}_i) - \int_0^{\bar{v}_i} x(v) = b_i, \\
\forall\, v : \qquad & x(v) \uparrow \text{ in } v, \\
\forall\, i \leq k : \qquad & \lim_{v \uparrow \bar{v}_i} x(\bar{v}_i) < y(\bar{v}_i), \\
\forall\, i < k : \qquad & y(\bar{v}_i) \leq \lim_{v \downarrow \bar{v}_i} x(v).
\end{aligned}$$

The allocation will be efficient between each $\underline{v}_i$ and $\bar{v}_{i+1}$. Further, for all $b > b_i$ and $\bar{v}_i \leq v \leq \underline{v}_i$, $a^*(v,b) = y(\bar{v}_i)$. If the sufficient conditions are not met, the optimal solution may require additional pooling in the middle.

4.4 Implementation

The above propositions describe the optimal mechanism directly in terms of reduced form. To help understand it, we describe the implementation of the optimal auction. Consider the 'regular' case, i.e. f satisfies the monotone hazard rate and decreasing density conditions. The implementation is akin to an all-pay auction.

Continuous Types Case For simplicity let us first consider the continuous case. There will be a reserve price r. Further for each budget $b_i < b_k$,there will be a $\triangle_i > 0$.[13] If there are multiple bids in the interval $[b_i, b_i + \triangle_i)$, and no bid exceeding $b_i + \triangle_i$, the auctioneer will award the good randomly to one of these bidders(rather than to the highest bid). In all other cases, the good will be awarded to the highest bidder.In equilibrium there will be no bids in the interval $(b_i, b_i + \triangle_i)$.

Due to the extra pooling being implemented, there will be 2 sorts of bidders who bid b_i in equilibrium:

1. Types with budget b_i and valuation larger than a cutoff valuation $\bar{v}_i$- these are the type for whom the budget constraint binds.

2. Types with a budget larger than b_i, and valuation in the interval $[\bar{v}_i, \underline{v}_i)$: these types would prefer to bid b_i and risk being pooled, rather than bid $b_i + \triangle_i$ and win the good outright if at all.

Discrete Types Case If types are discrete, the implementation is slightly convoluted. There will be a reserve price r, and a (finite) set of valid bids larger than r. As in a standard all-pay auction, the highest bid will win, ties are broken randomly, and all bidders pay their bid. The set of valid bids is exactly the set of $p(v, b)$'s, $(v, b) \in T$, identified in the statement of Proposition 4.

To see how this corresponds to the description of the proposition, it is instructive to consider the bids submitted by various types in equilibrium. The reserve price r is such that no-type with a valuation less than $\underline{v}_0$ would find it profitable to submit a bid other than 0 in equilibrium. The type $(\bar{v}_i, b_i)$ is the lowest valuation with budget b_i to bid her budget. Critically, the lowest valid bid larger than b_i is such that types with valuations $\bar{v}_i$ through $\underline{v}_i - 1$ and budgets larger than b_i prefer to bid b_i (and potentially be pooled with other bidders), rather than bid higher and win the good outright. All other valid bids, i.e. bids that are not equal to any of the budgets b_1 through b_k, are separating. In other words, for any other valid bid, there is exactly one valuation that makes that bid in equilibrium.

In the event that f violates either the monotone hazard rate or decreasing density condition, the optimal auction may require further pooling (over and above the pooling already required by the regular case). These extra pooling intervals can be identified by explicitly constructing feasible solutions to the program (REVOPT) and its dual which complement each other, and also satisfy the constraints identified in Lemma 1.

[13]These can be computed by solving the continuous versions of the appropriate equations in the proof of Proposition 4.

4.5 Proof of Proposition 4

The proof of Proposition 4 might appear convoluted, but indeed it follows the same basic strategy as the (hopefully, more palatable) proof of Proposition 2. First, the set of feasible IC, IR allocation rules (as identified in Observation 4) is used to further 'massage' the sellers problem into something more amenable for analysis. Once that is done, the dual is computed. The allocation rule identified is shown to be optimal by constructing a feasible dual solution that complements it. The proof is broken into major steps on these lines to orient the reader.

Step 1: Reformulating the Seller's Problem

Substituting (18a) and (18b) into (16), the Border constraints to be considered are:

$$\forall\, i \le k,\ \bar{v}_{i-1} \le v \le \bar{v}_i - 1:$$
$$\sum_{v'=v}^{\bar{v}_i-1} \tfrac{k-i+1}{k} f(v')x(v') + \sum_{j=i+1}^{k}\sum_{\bar{v}_{j-1}}^{\bar{v}_j-1} \tfrac{k-j+1}{k} f(v')x(v') + \sum_{j=i}^{k} \frac{(1-F(\bar{v}_j-1))}{k} y(\bar{v}_j) \le c_v, \tag{19a}$$
$$\forall\, i \le k:$$
$$\sum_{j=i+1}^{k} \frac{(1-F(\bar{v}_j-1))}{k} y(\bar{v}_j) + \sum_{j=i+1}^{k}\sum_{\bar{v}_{j-1}}^{\bar{v}_j-1} \tfrac{k-j+1}{k} f(v)x(v) \le \frac{1-(1-\frac{k-i}{k}(1-F(\bar{v}_i-1)))^N}{N}. \tag{19b}$$

where the c's are the right hand side of the appropriate Border inequality (16), i.e.

$$\bar{v}_{i-1} \le v \le \bar{v}_i - 1: \qquad N c_v = 1 - \left(1 - \tfrac{k-i}{k}(1-F(v-1))\right)^N.$$

Making the appropriate substitutions, (REVOPT) becomes:

$$\max_{x,y\ge 0} \quad \sum_{j=1}^{k}\sum_{\bar{v}_{j-1}}^{\bar{v}_j-1} \frac{k-j+1}{k} f(v)\varphi(v)x(v) + \frac{\sum_{i=1}^{k} \bar{v}_i(1-F(\bar{v}_i-1))y(\bar{v}_i)}{k} \tag{REVOPT2}$$
$$\text{s.t.} \quad (18c - 18f, 19a, 19b).$$

Step 2: Constructing the Dual

Let η_i be the dual variable associated with the budget constraint (18c). Since we assume constraint (18e) does not bind at optimality, the corresponding dual variable will be 0, and therefore is dropped. Let μ_v be the dual variable associated with the monotonicity constraint (18d), and $\bar{\mu}_{\bar{v}_i}$ the dual variable associated with the constraint (18f). Denote by β_v the dual variable associated with (19a), and $\bar{\beta}_{\bar{v}_i}$, the dual variable associated with (19b). The dual program is:

$$\min_{\eta,\mu,\beta} \quad \sum_{i=1}^{k} b_i\eta_i + \sum_{v=1}^{\bar{v}_k} c_v\beta_v + \sum_{i=1}^{k} \bar{c}_{\bar{v}_i}\bar{\beta}_{\bar{v}_i}$$

$\forall\, i \leq k, (\bar{v}_{i-1}+1) \leq v \leq (\bar{v}_i - 1):$

$$-\sum_{j=i}^{k}\eta_j + \tfrac{k-i+1}{k} f(v)\left(\sum_{1}^{v}\beta_v + \sum_{j=1}^{i-1}\bar{\beta}_{\bar{v}_j}\right) + \mu_v - \mu_{v-1} \geq \tfrac{k-i+1}{k} f(v)\varphi(v), \tag{20a}$$

$$\forall\, i \leq k: \quad -\sum_{j=i+1}^{k}\eta_j + \tfrac{k-i}{k} f(v)\left(\sum_{1}^{\bar{v}_i}\beta_v + \sum_{j=1}^{i}\bar{\beta}_{\bar{v}_j}\right) + \mu_{\bar{v}_i} - \bar{\mu}_{\bar{v}_i} \geq \tfrac{k-i}{k} f(\bar{v}_i)\varphi(\bar{v}_i), \tag{20b}$$

$$\forall\, i \leq k: \quad \bar{v}_i\eta_i + \tfrac{(1-F(\bar{v}_i-1))}{k}\left(\sum_{1}^{\bar{v}_i-1}\beta_v + \sum_{j=1}^{i}\bar{\beta}_{\bar{v}_j}\right) + \bar{\mu}_{\bar{v}_i} \geq \tfrac{(1-F(\bar{v}_i-1))}{k}\bar{v}_i, \tag{20c}$$

$$\eta, \beta, \mu \geq 0.$$

Here, (20a) is the dual inequality corresponding to primal variable $x(v)$ where $(\bar{v}_{i-1}+1) \leq v \leq (\bar{v}_i - 1)$, (20b) the dual inequality corresponding to $x(\bar{v}_i)$ and (20c) the dual inequality corresponding to $y(\bar{v}_i)$.

Step 3: Analyzing the Dual

Fix an optimal primal solution(x^*, y^*) and let $\underline{v}$ be the lowest valuation which gets allotted in that solution. Therefore any type (v, b) where $v \geq \underline{v}$ gets allotted. It is easy to see that $\underline{v} \leq \bar{v}_1$. Complementary slackness implies that the inequalities (20a) bind for all $v \geq \underline{v}$, as do (20b, 20c) for all i. Rewriting (20a-20c) as in the common knowledge case:

$$\left(\sum_{1}^{v}\beta_v + \sum_{j=1}^{i-1}\bar{\beta}_{\bar{v}_j}\right) + \frac{(\mu_v - \mu_{v-1})}{\frac{(k-i+1)}{k} f(v)} = \varphi(v) + \frac{\sum_{j=i}^{k}\eta_j}{\frac{(k-i+1)}{k} f(v)}, \tag{21a}$$

$$\left(\sum_{1}^{\bar{v}_i}\beta_v + \sum_{j=1}^{i}\bar{\beta}_{\bar{v}_j}\right) + \frac{(\mu_{\bar{v}_i} - \bar{\mu}_{\bar{v}_i})}{\frac{(k-i)}{k} f(\bar{v}_i)} = \varphi(\bar{v}_i) + \frac{\sum_{j=i+1}^{k}\eta_j}{\frac{(k-i)}{k} f(\bar{v}_i)}, \tag{21b}$$

$$\left(\sum_{1}^{\bar{v}_i-1}\beta_v + \sum_{j=1}^{i}\bar{\beta}_{\bar{v}_j}\right) + \frac{\bar{\mu}_{\bar{v}_i}}{\frac{(1-F(\bar{v}_i-1))}{k}} = \bar{v}_i - \frac{\bar{v}_i\eta_i}{\frac{(1-F(\bar{v}_i-1))}{k}}. \tag{21c}$$

Subtracting the equation (21a) corresponding to $v-1$ from the equation corresponding to v for $\bar{v}_{i-1}+2 \le v \le \bar{v}_i - 1$, we have:

$$\beta_v + \frac{(\mu_v - \mu_{v-1})}{\frac{(k-i+1)}{k} f(v)} - \frac{(\mu_{v-1} - \mu_{v-2})}{\frac{(k-i+1)}{k} f(v-1)} = \varphi(v) - \varphi(v-1) + \frac{\sum_{j=i}^{k} \eta_j}{\frac{k-i+1}{k}} \left(\frac{1}{f(v)} - \frac{1}{f(v-1)}\right). \tag{22a}$$

Subtracting the equation (21b) corresponding to $\bar{v}_i$ from equation (21a) corresponding to $\bar{v}_i + 1$, we have:

$$\beta_{\bar{v}_i+1} + \frac{(\mu_{\bar{v}_i+1} - \mu_{\bar{v}_i})}{\frac{(k-i+1)}{k} f(\bar{v}_i+1)} - \frac{(\mu_{\bar{v}_i} - \bar{\mu}_{\bar{v}_i})}{\frac{(k-i)}{k} f(\bar{v}_i)} = \varphi(\bar{v}_i+1) - \varphi(\bar{v}_i) + \frac{\sum_{j=i}^{k} \eta_j}{\frac{k-i}{k}} \left(\frac{1}{f(\bar{v}_i+1)} - \frac{1}{f(\bar{v}_i)}\right).$$

Similarly, subtracting the equation (21c) corresponding to $\bar{v}_i$ from (21b) corresponding to $\bar{v}_i$ we have:

$$\beta_{\bar{v}_i} + \frac{(\mu_{\bar{v}_i} - \bar{\mu}_{\bar{v}_i})}{\frac{(k-i)}{k} f(\bar{v}_i)} - \frac{\bar{\mu}_{\bar{v}_i}}{\frac{(1-F(\bar{v}_i-1))}{k}} = \varphi(\bar{v}_i) - \bar{v}_i + \frac{\sum_{j=i+1}^{k} \eta_j}{\frac{(k-i)}{k} f(\bar{v}_i)} + \frac{\bar{v}_i \eta_i}{\frac{(1-F(\bar{v}_i-1))}{k}} \tag{22b}$$

Finally, subtracting (21a) corresponding to $\bar{v}_i - 1$ from (21c) corresponding to $\bar{v}_i$, we have:

$$\bar{\beta}_{\bar{v}_i} + \frac{\bar{\mu}_{\bar{v}_i}}{\frac{(1-F(\bar{v}_i-1))}{k}} - \frac{\mu_{\bar{v}_i-2}}{\frac{(k-i)}{k} f(\bar{v}_i-1)} = \bar{v}_i - \varphi(\bar{v}_i-1) - \frac{\bar{v}_i \eta_i}{\frac{(1-F(\bar{v}_i-1))}{k}} - \frac{\sum_{j=i}^{k} \eta_j}{\frac{(k-i+1)}{k} f(\bar{v}_i-1)} \tag{22c}$$

If the optimal solution a^* is strictly monotone, the inequalities (18d-18f) do not bind. Complementary slackness implies all the μ's are 0. As in the common knowledge budget case we set $\bar{\beta}_{\bar{v}_i} = 0$ for all i since this will satisfy complementary slackness. Therefore, from (22c), we have that:

$$\eta_k = \frac{1}{k} \frac{(1-F(\bar{v}_k-1))(1-F(\bar{v}_k-2))}{\bar{v}_k f(\bar{v}_k-1) + (1-F(\bar{v}_k-1))},$$

$$\eta_i = \frac{1}{k} \frac{(1-F(\bar{v}_i-1))(1-F(\bar{v}_i-2))}{(k-i+1)\bar{v}_i f(\bar{v}_i-1) + (1-F(\bar{v}_i-1))} - \frac{(1-F(\bar{v}_i-1)) \sum_{i+1}^{k} \eta_j}{(k-i+1)\bar{v}_i f(\bar{v}_i-1) + (1-F(\bar{v}_i-1))}$$

It is easily verified that the η's as specified are non-negative and therefore dual feasible. Further, one can show that $i \le j \implies \eta_i \ge \eta_j$, in other words, as one would suspect, smaller budgets have larger shadow prices. Substituting into (22a) we have, $\forall\, v : \bar{v}_{i-1} < v < \bar{v}_i$,

$$\beta_v = \varphi(v) - \varphi(v-1) + \frac{\sum_{j=i}^{k} \eta_j}{\frac{k-i+1}{k}} \left(\frac{1}{f(v)} - \frac{1}{f(v-1)}\right).$$

Note that β_v for all v such that $\bar{v}_{i-1} < v < \bar{v}_i$ will be positive if f is weakly decreasing. Finally, substituting the η's into (22b), we have:

$$\beta_{\bar{v}_i} = \varphi(\bar{v}_i) - \varphi(\bar{v}_i - 1) + \frac{\sum_{i+1}^k \eta_j}{\frac{(k-i)}{k} f(\bar{v}_i)} - \frac{\sum_i^k \eta_j}{\frac{(k-i+1)}{k} f(\bar{v}_i - 1)}.$$

Step 4: Non Monotonicity in Adjusted Virtual Values

The adjusted virtual value of a type with valuation $\bar{v}_i$ and budget strictly larger than b_i is

$$\varphi(\bar{v}_i) + \frac{\sum_{i+1}^k \eta_j}{\frac{(k-i)}{k} f(\bar{v}_i)}.$$

This may be smaller than the adjusted virtual valuation of a type with value $\bar{v}_i - 1$ and budget b_i or larger,

$$\varphi(\bar{v}_i - 1) + \frac{\sum_i^k \eta_j}{\frac{(k-i+1)}{k} f(\bar{v}_i - 1)},$$

even if f is weakly decreasing and satisfies the monotone hazard rate.

Intuitively, this happens because allocating to types with valuation $\bar{v}_i - 1$ and budget $\geq b_i$ also 'relaxes' the budget constraints corresponding to budget b_i and larger. Allocating to types with value $\bar{v}_i$ and budget $> b_i$ only relaxes the budget constraints for budgets strictly larger b_i. Therefore, the adjusted virtual valuation may be non monotone at this point. As a result, the $\beta_{\bar{v}_i}$ shown above can be negative, and thus infeasible. Additional pooling will be required.

As described in the proposition, for each budget b_i there will be an additional cutoff $\underline{v}_i$. Types (v, b) where $\bar{v}_i \leq v \leq \underline{v}_i$ and $b > b_i$ will be pooled with the types (v, b_i), $v \geq \bar{v}_i$ (i.e. the types for whom the budget constraint binds). The allocation rule will be efficient between $\underline{v}_i$ and $\bar{v}_{i+1}$.

Step 5: Constructing the Complementary Dual Solution

Since $a^*(v, b) = 0$ for all $v < \underline{v}_0$, the corresponding Border constraints must be slack, and therefore $\beta_v = 0$ for all $v < \underline{v}_0$. Since $x^*(\underline{v}_i + 1) > y^*(\bar{v}_i)$, $\mu_{\underline{v}} = 0$.

The β_v for $\underline{v}_i + 2 \leq v \leq \bar{v}_{i+1} - 1$ is as specified in (22a), with the corresponding μ's set to 0. By Lemma 1, β_v for $\bar{v}_i \leq v \leq \underline{v}_i$ is 0 since, by the statement of the proposition, $a^*(v, b) = y(\bar{v}_i)$ for all $b > b_i$. The relevant μ's can be calculated from the relevant equations.

Instead of computing these μ's, we can instead suppose that the types which have been ironed, $\{(v, b_i) \text{ for } v \geq \bar{v}_i\} \bigcup \{(v, b_j) \text{ for } j > i, \bar{v}_i \leq v \leq \underline{v}_i\}$, all correspond to one 'artificial' type, t_i. The probability of t_i is

$$\pi(t_i) = \frac{(k-i)}{k}(F(\underline{v}_i) - F(\bar{v}_i - 1)) + \frac{1}{k}(1 - F(\bar{v}_i - 1)).$$

Further, its adjusted virtual valuation is:

$$\varphi(t_i) = \bar{v}_i - \frac{(k-i)(\underline{v}_i - \bar{v}_i + 1)(1 - F(\underline{v}_i))}{k\pi(t_i)} + \frac{(\underline{v}_i - \bar{v}_i + 1)\sum_{j=i+1}^k \eta_j}{\pi(t_i)} - \frac{\bar{v}_i \eta_i}{\pi(t_i)}$$

Since the budget constraint for budget b_i binds at $\bar{v}_i$, analogous to the proof of Proposition 2, $\bar{\beta}_{\bar{v}_i}$ is 0, and therefore we can solve for η_i from:

$$\varphi(t_i) - \varphi(\bar{v}_i - 1) - \frac{k\sum_{j=i}^k \eta_j}{(k-i+1)f(\bar{v}_i - 1)} = 0. \tag{23}$$

Note that the adjusted virtual valuation of $\bar{v}_i - 1$ can be written as:

$$\varphi(\bar{v}_i - 1) + \frac{k\sum_{j=i+1}^k \eta_j}{(k-i+1)f(\bar{v}_i - 1)} + \frac{k\eta_i}{(k-i+1)f(\bar{v}_i - 1)}.$$

To see that the η_i that solves (23) is positive, we need to show that:

$$\varphi(\bar{v}_i - 1) + \frac{k\sum_{j=i+1}^k \eta_j}{(k-i+1)f(\bar{v}_i - 1)} < \bar{v}_i - \frac{(k-i)(1 - F(\underline{v}_i))}{k\pi(t_i)} + \frac{(\underline{v}_i - \bar{v}_i + 1)\sum_{j=i+1}^k \eta_j}{\pi(t_i)} \tag{24}$$

The right hand side of this inequality equals

$$\frac{1}{\pi(t_i)}\left(\bar{v}_i\frac{(1 - F(\bar{v}_i - 1))}{k} + \sum_{v=\bar{v}_i}^{\underline{v}_i}\left(\tfrac{(k-i)}{k}f(v)\varphi(v) + \sum_{j=i+1}^{k}\lambda_j\right)\right).$$

However,

$$\varphi(\bar{v}_i - 1) + \frac{k\sum_{j=i+1}^k \eta_j}{(k-i+1)f(\bar{v}_i - 1)} < \bar{v}_i,$$

since $\sum_{j=i+1}^k \eta_j$ is appropriately small (see Proposition 6). Further, for $\bar{v}_i \le v \le \underline{v}_i$,

$$\varphi(\bar{v}_i - 1) + \frac{k\sum_{j=i+1}^k \eta_j}{(k-i+1)f(\bar{v}_i - 1)} < \varphi(v) + \frac{k\sum_{j=i+1}^k \eta_j}{(k-i)f(v)},$$

follows from the monotone hazard rate and decreasing density conditions. (24) follows since its right hand side is a weighted average of the right hand side of the latter two inequalities.

Further, we have that:

$$\beta_{\underline{v}_i+1} = \varphi(\underline{v}_i + 1) + \frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i)f(\underline{v}_i + 1)} - \varphi(t_i). \tag{25}$$

To ensure that $\beta_{\underline{v}_i+1} \geq 0$ it suffices by inequality (23) that cutoffs $\bar{v}_i$ and $\underline{v}_i$ satisfy:

$$\varphi(\underline{v}_i + 1) + \frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i)f(\underline{v}_i + 1)} \geq \varphi(\bar{v}_i - 1) + \frac{k\sum_{i}^{k}\eta_j}{(k-i+1)f(\bar{v}_i - 1)}.$$

Finally, note that $\underline{v}_0$ will be the lowest valuation such that $\varphi(v) + \frac{\sum_1^k \eta_j}{f(v)} \geq 0$; and

$$\beta_{\underline{v}_0} = \varphi(v) + \frac{\sum_1^k \eta_j}{f(v)}.$$

The partial solution identified above, with all other dual variables set to 0, is an optimal dual solution. Since $\beta_v > 0$ for all $\underline{v}_i + 1 \leq v \leq \bar{v}_{i+1} - 1$, by complementary slackness, the corresponding Border constraints (8b) bind. This concludes the case where our regularity condition on the distribution of types (monotone hazard rate, decreasing density) are met. If the monotone hazard rate or decreasing density assumptions are not satisfied then the dual solution identified may be infeasible, and therefore additional pooling will be required due to Lemma 1. □

We can also describe the constrained efficient auction for this setting. The proof is similar, and omitted. Intuitively, the mechanism is similar in structure to the case of revenue maximization. The key difference is that there is no reserve price.

Proposition 5 (Constrained Efficient Auction). *Suppose $f(v)$ is decreasing in v. Then the solution of (ConsEff) can be described as follows: there will exist cutoffs $\bar{v}_1 \leq \underline{v}_1 \leq \bar{v}_2 \leq \dots \underline{v}_{k-1} \leq \bar{v}_k$ and $\underline{v}_0 = 0$. The allocation rule will satisfy (18a-18f).The allocation will be efficient between each $\underline{v}_i$ and $\bar{v}_{i+1}$. Further, for all $b > b_i$ and $\bar{v}_i \leq v \leq \underline{v}_i$, $a(v, b) = y(\bar{v}_i)$. If the sufficient conditions are not met, the optimal solution may require additional pooling in the middle.*

4.6 Subsidies

Since budget constrained bidders are unable to effectively compete in the auction, this will depress auction revenues. To get around this problem, prior work has examined various kinds of subsidies (lump sum transfer, discounts) and their effect in a particular auction setting.

In our setting, there is only one possible (incentive compatible) means of subsidy- a lump sum transfer from the auctioneer to the agents. This is because, given an allocation rule, incentive compatibility determines prices up to a constant:

$$p(v,b) = va(v,b) - \sum_{1}^{v-1} a(v',b) + c.$$

Let us consider a subsidy via lump sum payment of some amount ϵ. This costs the auctioneer ϵ per agent. The effect of this subsidy is to relax the budget constraints by ϵ. Therefore the gain in revenue is (at most) $\epsilon \sum_i \eta_i$.[14] We show below that $\sum_i \eta_i \leq 1$, and thus $\epsilon \sum_i \eta_i \leq \epsilon$. As a result, if the auctioneer were running the optimal auction, he should not offer subsidies. This result remains true even when bidders' budgets are common knowledge.

Proposition 6. *For all i,*

$$\sum_{i}^{k} \eta_j \leq \tfrac{(k-i+1)}{k}(1 - F(\bar{v}_i - 1)). \tag{26}$$

How then does the optimal auction encourage competition? Recall that for each i, types $\{(v, b_i) \text{ for } v \geq \bar{v}_i\} \bigcup \{(v, b_j) \text{ for } j > i, \bar{v}_i \leq v \leq \underline{v}_i\}$ are pooled. The pooling serves to allot the good to disadvantaged bidder types (v, b_i), $v \geq \bar{v}_i$ even in profiles where there are bidders with higher valuations and budgets present. Intuitively, favoring bidders in this way is better than lump-sum transfers because there are more degrees of freedom: a lump-sum transfer must be given to a bidder regardless of his type in order to maintain incentive compatibility.

5 Discussion

5.1 Strengthening Interim IC and IR

The optimal auction we identify is implemented as an all-pay auction with a modified winning rule. All-pay auctions are generally considered 'unappealing.' In our opinion, this is a critique of the interim individual rationality constraint. One can impose ex-post IR, rather than interim IR. The optimal auction with this more stringent constraint can be characterized using similar techniques to ours. The resulting implementation will be a first price auction with a modified winning rule.This is because any ex-post IR and interim IC mechanism can be supported by an ex-post payment rule that requires the bidder to pay some fixed price p (contingent on his report) if he wins, and 0 if he loses. The resulting expected revenue from imposing ex-post individual rationality will be strictly less than if we required interim individual rationality.

In a similar vein one can ask what the optimal auction is if one desires dominant strategy implementability rather than Bayes-Nash (interim IC). An analogous argument shows that

[14] Recall that η_i is the shadow price of the budget constraint.

the optimal auction can be implemented as a second price auction, with a similarly modified winning rule. The resulting expected revenue will be lower still.

5.2 Relaxing Independence

The assumption that a buyer's valuation is independent of his budget may worry some readers. It can be shown that the full power of independence is not needed. All we need is that

$$b > b' \implies \varphi(v, b) \leq \varphi(v, b). \tag{27}$$

Here $\varphi(v, b)$ is the virtual valuation of value v when the distribution over values is the distribution conditional on value being b. Assume that the distribution of values conditional on any budget satisfies the monotone hazard rate condition. Then (27) implies that the distribution of values conditional on a higher budget b first order stochastically dominates the distribution of values conditional on a lower budget b', which we would claim is the economically interesting case.

If the correlation between values and budgets goes in the opposite direction, then the proof of Lemma 3 fails. The optimal allocation rule may look like the sort identified in Figure 2. High budget constraints may bind, while low budget constraints are slack. A clean characterization appears to be out of reach for that case.

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A Counterexamples

A.1 Laffont and Robert's Solution

In this section, we examine the classical formulation, with a continuum of types. We show by counterexample that the Laffont and Robert solution is not optimal for all distributions that meet the monotone hazard rate condition– if the decreasing density assumption fails, additional pooling is required. We can now use the intuition gleaned from Section 3 to identify a flaw in the L-R solution in the event that densities are not decreasing. Pick a $v' \in (\underline{v}, \overline{v})$; and 'iron' some small interval of types $[v', v' + \epsilon]$. The new allotment rule is therefore:

$$a'(v) = \begin{cases} \frac{1-F^N(\overline{v})}{N(1-F(\overline{v}))} & v \geq \overline{v} \\ \frac{F^N(v'+\epsilon)-F^N(v')}{N(F(v'+\epsilon)-F(v'))} & v \in [v', v' + \epsilon] \\ F^{N-1}(v) & v \in [\underline{v}, v') \bigcup (v' + \epsilon, \overline{v}] \\ 0 & o.w. \end{cases} \tag{28}$$

By Lemma 4 (see Appendix A.3), if $f(v)$ is increasing in the interval; then

$$g_\epsilon \equiv \epsilon \frac{F^N(v'+\epsilon) - F^N(v')}{N(F(v'+\epsilon) - F(v'))} - \int_{v'}^{v'+\epsilon} F^{N-1}(v)dv > 0.$$

Let us assume that $f(v)$ is increasing in the range $[0, 1]$. As a result, the budget constraint is now slack. We can now potentially improve on the revenue by 'un-pooling' v_2 to $v_2 + \delta$.

First, δ solves the implicit equation:

$$\delta \frac{F^N(\overline{v}+\epsilon) - F^N(\overline{v})}{N(F(\overline{v}+\epsilon) - F(\overline{v}))} - \int_{\overline{v}}^{\overline{v}+\delta} F^{N-1}(v)dv = g_\epsilon. \tag{29}$$

The change in revenue from ironing types $[v', v'+\epsilon]$ is:

$$\Delta(v_3, \epsilon) \equiv \int_{v'}^{v'+\epsilon} \varphi(v)f(v)\left[\frac{F^N(v'+\epsilon) - F^N(v')}{N(F(v'+\epsilon) - F(v'))} - F^{N-1}(v)\right]dv.$$

Similarly, the change in revenue from 'un-pooling' $[\overline{v}, \overline{v}+\delta]$ is:

$$\Delta(\overline{v}, \epsilon) \equiv -\int_{\overline{v}}^{\overline{v}+\delta} \varphi(v)f(v)\left[\frac{F^N(\overline{v}+\delta) - F^N(\overline{v})}{N(F(\overline{v}+\delta) - F(\overline{v}))} - F^{N-1}(v)\right]dv.$$

Therefore the total change in revenue is:

$$\Delta = \Delta(v', \epsilon) + \Delta(\overline{v}, \epsilon).$$

Since $\varphi(\cdot)$ and $f(\cdot)$ are both increasing; $\Delta(\overline{v}, \epsilon) \geq 0 \geq \Delta(v', \epsilon)$. Potentially, $\Delta \geq 0$ for some suitable parameter choices. In other words, our perturbation of the L-R solution can increase expected revenue, therefore the L-R solution is not optimal. We flesh out a numerical example below.

A.1.1 An Example

Suppose 2 bidders, i.e. $N = 2$. Both have valuations in the interval $[0, 1]$ which are drawn i.i.d. with density $f(v) = 2v$; $F(v) = v^2$. Both have a common budget constraint $b = 0.5$. The 'virtual value' of a bidder of valuation v, $\varphi(v) = \frac{3v^2-1}{2v}$, which is increasing on the interval $[0, 1]$. If there was no budget constraint, the optimal auction would be a second price auction with reserve price $v_0 = \frac{1}{\sqrt{3}}$, i.e. $\varphi(v_0) = 0$.

Recall that the L-R solution would require us to compute $\underline{v}$ and $\overline{v}$ to solve:

$$\frac{\overline{v}}{2} + \frac{\overline{v}^3}{6} + \frac{\underline{v}^3}{3} = 0.5,$$
$$3\underline{v}^2 + \frac{(1-\overline{v}^2)^2}{1+\overline{v}^2} = 1.$$

Solving, we get $\underline{v} = 0.5415$ and $\overline{v} = .7523$. Therefore; $\underline{v} < v_0 < \overline{v}$. For the perturbation we outlined above, select $v' = \frac{1}{\sqrt{3}}(= v_0)$; and $\epsilon = 10^{-4}$. Our functional forms lend themselves to easy analytic calculation– some calculation yields:

$$\begin{aligned}
g_\epsilon &= \frac{\epsilon^3}{6} \\
\delta &= \epsilon \\
\Delta(v', \epsilon) &= -\frac{(3v'^2+1)\epsilon^3}{6} - \frac{v'\epsilon^4}{2} + \epsilon^5 \\
\Delta(\overline{v}, \epsilon) &= +\frac{(3\overline{v}^2+1)\epsilon^3}{6} + \frac{\overline{v}\epsilon^4}{2} - \epsilon^5
\end{aligned}$$

Substituting we see that net change in revenue

$$\Delta \approx \frac{(\overline{v}^2 - v'^2)\epsilon^3}{2} > 0.$$

A.2 Maskin

The counterexample also works for the case of constrained efficiency. Once again, suppose 2 bidders, i.e. $N = 2$. Both have valuations in the interval $[0, 1]$ which are drawn i.i.d. with density $f(v) = 2v$; $F(v) = v^2$. Both have a common budget constraint $b = 0.5$. The Maskin solution would require us to pick a cutoff $\bar{v}$ to solve:

$$\frac{\bar{v}}{2} + \frac{\bar{v}^3}{6} = 0.5,$$

i.e. $\bar{v} = 0.8177$. Let us now pick $\epsilon \ll 1$, and iron $[0, \epsilon]$. It can be shown that the budget constraint for type $\bar{v}$ is relaxed by $\frac{\epsilon^3}{6}$. Therefore we can now have the efficient allocation for types $[\bar{v}, \bar{v} + \epsilon)$ and still satisfy the budget constraint. Further, one can show the expected loss of efficiency from ironing the interval $[0, \epsilon]$ is $O(\epsilon^5)$, while the gain in efficiency from unpooling the types $[\bar{v}, \bar{v} + \epsilon)$ is roughly $\frac{1}{3}\overline{v}^2\epsilon^3$.

A.3 Ironing

Let $f(.)$ be the density function for some distribution on $\Re$, and let $F(.)$ be the associated cumulative distibution function.

Lemma 4. *If $f(.)$ is (strictly) increasing on some interval $[v_1, v_2]$, then for any $N > 1$, we have:*

$$(v_2 - v_1)\frac{F^N(v_2) - F^N(v_1)}{N(F(v_2) - F(v_1))} > \int_{v_1}^{v_2} F^{N-1}(v)dv$$

Proof. Rewriting, we have to show that

$$\frac{\int_{v_1}^{v_2} f(v)F^{n-1}(v)dv}{\int_{v_1}^{v_2} f(v)dv} \quad > \quad \frac{\int_{v_1}^{v_2} F^{n-1}(v)dv}{\int_{v_1}^{v_2} dv}$$

This is true if and only if

$$\int_{v_1}^{v_2} dv \int_{v_1}^{v_2} f(v)F^{n-1}(v)dv > \int_{v_1}^{v_2} F^{n-1}(v)dv \int_{v_1}^{v_2} f(v)dv$$

Note that both sides are equal (to zero) at $v_2 = v_1$. Therefore we have the desired inequality if the derivative w.r.t v_2 of the left hand side is greater than the right hand side. Differentiating both sides w.r.t. v_2 and rearranging we have that this is true if and only if:

$$F^{N-1}(v_2)\left(f(v_2)(v_2 - v_1) - \int_{v_1}^{v_2} f(v)dv\right) + \int_{v_1}^{v_2}(f(v) - f(v_2))F^{N-1}(v)dv > 0$$

The inequality now follows by observing that $f(v)$ is increasing in v, therefore

$$\int_{v_1}^{v_2}(f(v) - f(v_2))F^{N-1}(v)dv > F^{N-1}(v_2)\int_{v_1}^{v_2}(f(v) - f(v_2))dv \qquad \square$$

B Proofs from Section 3

B.1 Optimal Auction with Continuous Types

Suppose bidders all have a common knowledge budget b. Their valuations belong to the interval $[0, V]$, and are drawn i.i.d. from a distribution with density f and cumulative distribution F. Further, assume that f satisfies the decreasing density and increasing hazard rate conditions.

Suppose we now discretize this space- we assume that all bidder valuations belong to the set $\{\epsilon, 2\epsilon, \dots, m\epsilon\}$, where $m\epsilon = V$. The 'density' of any type, $k\epsilon$ is $f^\epsilon(k\epsilon) = \int_{(k-1)\epsilon}^{k\epsilon} f(v)dv$. The cumulative distribution F^ϵ is defined analogously. Note that f^ϵ will satisfy both the decreasing density and monotone hazard rate conditions.

Our solution in this discrete type space can be summarized thus: there will exist two cutoffs $\underline{v}$ and $\bar{v}$. No valuation less than $\underline{v}$ will be allotted. Types $\bar{v}$ and above will be pooled, and the budget constraint will bind for those types. The allocation rule will be efficient between $\underline{v}$ and $\bar{v}-1$. Therefore the optimal allocation rule $a(v)$ satisfies: [15]

$$\bar{v}a(v) - \epsilon \sum_{j=1}^{\bar{k}-1} a(j\epsilon) = b, \;\; \bar{k}\epsilon = \bar{v},$$

and $\underline{v}$ is the lowest type such that:

$$\underline{v} - \epsilon \frac{1 - F^{\epsilon}(\underline{v})}{f^{\epsilon}(\underline{v})} + \frac{\epsilon \eta}{f^{\epsilon}} \geq 0,$$

where

$$\eta = \frac{(1 - F(\bar{v} - \epsilon))(1 - F(\bar{v} - 2\epsilon))}{\bar{v}\frac{f(\bar{v}-\epsilon)}{\epsilon} + (1 - F(\bar{v} - \epsilon))}.$$

Note that as $\epsilon \to 0$:

$$\begin{aligned} p(v) &\to va(v) - \int_0^v a(v')dv' \\ \eta &\to \frac{(1 - F(\bar{v}))^2}{\bar{v}f(\bar{v}) - (1 - F(\bar{v}))}. \end{aligned}$$

As a result as $\epsilon \downarrow 0$, the solution identified in Proposition 2 converges to the solution identified in Laffont and Robert.

C Proofs from Section 4

C.1 Proof of Lemma 2

By assumption, $p(v, b') \leq b$ for any $v \leq \underline{v}_b$. By individual rationality, $p(v, b) \leq b$ for any v. Therefore the incentive compatibility constraints (13b) corresponding to type (v, b) misreporting as (v, b') and type (v, b') misreporting as (v, b) for any $v \leq v_b$ imply that:

$$\begin{aligned} \forall v \leq \underline{v}_b : &\qquad va(v, b) - p(v, b) = va(v, b') - p(v, b') \\ \implies \forall v \leq \underline{v}_b : &\qquad \sum_{1}^{v-1} a(v', b) = \sum_{1}^{v-1} a(v', b') \\ \implies \forall v \leq (\underline{v}_b - 1) : &\qquad a(v, b) = a(v, b'). \end{aligned}$$

[15] To see this, note that the payment rule will be $p(v) = va(v) - \epsilon \sum_{v'<v} a(v')$, and virtual valuations etc. must be updated accordingly.

To see that $a(\underline{v}_b, b) = a(\underline{v}_b, b')$, first consider the IC constraint corresponding to type $(\underline{v}_b + 1, b)$ misreporting as type $(\underline{v}_b, b')$. By assumption $p(\underline{v}_b, b') \leq b$, therefore we can drop the characteristic function and write the IC constraint as:

$$
\begin{aligned}
& (\underline{v}_b + 1)a(\underline{v}_b + 1, b) - p(\underline{v}_b + 1, b) \geq (\underline{v}_b + 1)a(\underline{v}_b, b') - p(\underline{v}_b, b') \\
\implies \quad & \sum_1^{\underline{v}_b} a(v, b) \geq \sum_1^{\underline{v}_b} a(v, b') \\
\implies \quad & a(\underline{v}_b, b) \geq a(\underline{v}_b, b').
\end{aligned}
$$

The last inequality follows since $\sum_1^{\underline{v}_b - 1} a(v, b) = \sum_1^{\underline{v}_b - 1} a(v, b')$. Similarly one can show that $a(\underline{v}_b, b) \leq a(\underline{v}_b, b')$.

Finally, we need to show that $a(\underline{v}_b + 1, b') > a(m, b)$. By assumption,

$$
\begin{aligned}
& p(\underline{v}_b + 1, b') > b \geq p(m, b) \\
\implies \quad & (\underline{v}_b + 1)a(\underline{v}_b + 1, b') - \sum_1^{\underline{v}_b} a(v, b') > ma(m, b') - \sum_1^{m-1} a(v, b) \\
\implies \quad & (\underline{v}_b + 1)a(\underline{v}_b + 1, b') > ma(m, b') - \sum_{\underline{v}_b + 1}^{m-1} a(v, b) \\
\implies \quad & \qquad > (\underline{v}_b + 1)a(m, b).
\end{aligned}
$$

The last inequality follows since for any v, $a(v + 1, b) \geq a(v, b)$. □

C.2 Proof of Lemma 3

First an expositional note. This proof borrows notation and methodology from the proof of Proposition 4. We recommend the reader first read that proof before verifying this one. If one likes, one may think of Proposition 4 as characterizing the optimal auction assuming that cutoffs are ordered as (17), and this Lemma following to show that the assumption is without loss.

Consider a candidate optimal allocation rule a not satisfying (17). We will show how to construct another feasible allocation rule a' which achieves a strictly higher revenue via perturbation. It will be clear that feasibility of this perturbation depends on (17) not being satisfied.

To this end, let $j \equiv \max\{i : \underline{v}_i < \overline{v}_i - 1\}$. Therefore $\underline{v}_i \geq \overline{v}_i - 1$ for all $i > j$. It follows that $a(v, b)$, for $v > \underline{v}_j$ and $b > b_j$, satisfies the constraints on properly ordered cutoffs summarized in Observation 4, i.e. the constraint system (18).

By Observation 3, (15g), we know that $a(\underline{v}_j + 1, b) > a(\overline{v}_j, b_j)$ for $b > b_j$. Our perturbation proceeds as follows:

1. Reduce the allocation of $a(\underline{v}_j + 1, b_{\geq j+1})$ by ϵ.

2. Increase the allocation $a(\geq \overline{v}_j, b_j)$ by δ, where δ is such that the allocation rule is still feasible, $\delta = \frac{(k-j)f(\underline{v}_j+1)}{1-F(\overline{v}_j-1)}\epsilon$.

Step 1: Confirming the Perturbation is Revenue Increasing

The 'adjusted' virtual value of the types $(\underline{v}_j + 1, b_{\geq j+1})$ is

$$\xi(\underline{v}_j + 1) = \varphi(\underline{v}_j + 1) + \frac{\sum_{i=j+1}^{k} \eta_i}{\frac{k-j}{k} f(\underline{v}_j + 1)}.$$

The adjusted virtual value of the types $(\geq \overline{v}_j, b_j)$ is

$$\xi(\overline{v}_j) = \xi(\overline{v}_j - 1) = \varphi(\overline{v}_j - 1) + \frac{\eta_j}{\frac{1}{k} f(\overline{v}_j - 1)}.$$

If we can show that $\xi(\underline{v}_j + 1) < \xi(\overline{v}_j)$, the perturbation is indeed revenue increasing.

Step 2: Restoring Feasibility in a Revenue Neutral Way

Our perturbation will violate the budget constraints of all types $(\geq \overline{v}_i, b_i)$ for $i \geq j$. However, restoring budget feasibility in a revenue neutral way is easy. Recall that for each budget $b_i > b_j$, the adjusted virtual value of $(\geq \overline{v}_i, b_i)$ is equal to the adjusted virtual value of $(\overline{v}_i - 1, \geq b_i)$. Further, the adjusted virtual value of $(\geq \overline{v}_j, b_j)$ equals the adjusted virtual value of $(\overline{v}_j - 1, b_j)$. Budget feasibility can be restored by appropriately reducing the allocation of $(\geq \overline{v}_i, b_i)$ and increasing the allocation of $(\overline{v}_i - 1, \geq b_i)$ for each $i > j$, and similarly reducing the allocation of $(\geq \overline{v}_j, b_j)$ and increasing the allocation of $(\overline{v}_j - 1, b_j)$.

C.3 Private Information Budgets

Suppose valuations are from some interval $[0, V]$, while budgets are one of a finite number $b_1, \ldots b_k$. Valuations and budgets are independently determined, and each bidder's valuation is an i.i.d. draw from some distribution F with density f, and the budget is drawn i.i.d. according to a uniform distribution. Further suppose f satisfies the decreasing density and increasing hazard rate conditions.

The problem of maximizing the revenue in the continuous case can be written as:

$$\begin{aligned} \max_{a,p} \sum_{i=1}^{k} \frac{1}{k} \int_0^V p(v,b_i) f(v) dv \qquad \text{(Cont Rev Opt)} \\ \text{s.t. } & \text{Incentive Compatibility} \\ & \text{Individual Rationality} \\ & p(v,b_i) \leq b_i \\ & a \text{ feasible} \end{aligned}$$

Standard arguments imply that Incentive compatibility and individual rationality pin down the pricing rule to the standard rent extraction formula, i.e. :

$$p(v,b) = va(v,b) - \int_0^v a(x,b) dx \qquad (30)$$

Therefore, substituting (30) into Cont Rev Opt, we can eliminate the variables p, leading to:

$$\max_{a \in C} \sum_{i=1}^{k} \frac{1}{k} \int_0^V \varphi(v) a(v,b_i) f(v) dv. \qquad \text{(Cont Rev Opt2)}$$

Once again, we can discretize this space- we assume that all bidder valuations belong to the set $\{\epsilon, 2\epsilon, \ldots, m\epsilon\}$, where $m\epsilon = V$. The 'density' of any type, $k\epsilon$ is $f^{\epsilon}(j\epsilon) = \int_{(j-1)\epsilon}^{j\epsilon} f(v)dv$. The cumulative distribution F^{ϵ} is defined analogously. Note that f^{ϵ} will satisfy both the decreasing density and monotone hazard rate conditions if f does.

This discretization can be thought of as an extra constraint, i.e. that each type $v \in [(k-1)\epsilon, k\epsilon)$ must be pooled for all $k = 1, 2, \ldots m$. Let us denote the feasible region with this extra constraint as C_{ϵ}. Consider the revenue maximizing mechanism for this discretization(as identified in Proposition 4), let us denote it $(a^{\epsilon}, p^{\epsilon})$. Note that a^{ϵ} solves:

$$\max_{a \in C_{\epsilon}} \sum_{i=1}^{k} \frac{1}{k} \int_0^V \varphi(v) a(v,b_i) f(v) dv,$$

and p^{ϵ} is then defined by (30).

Note that $C_{\epsilon} \subseteq C$, and that both are compact subsets of the L_1 space defined by the measure f. Further, the operator $T(a) = \sum_{i=1}^{k} \frac{1}{k} \int_0^V \varphi(v) a(v,b_i) f(v) dv$ is a bounded linear operator from the L_1-space of allocation rules to $\Re$. Therefore T achieves its maximum on each set C_{ϵ} and C.

Since $C_\epsilon \uparrow C$ pointwise, it must be that:

$$\lim_{\epsilon \to 0} \max_{a \in C_\epsilon} \sum_{i=1}^{k} \frac{1}{k} \int_0^V \varphi(v) a(v, b_i) f(v) dv$$
$$= \max_{a \in C} \sum_{i=1}^{k} \frac{1}{k} \int_0^V \varphi(v) a(v, b_i) f(v) dv$$

Finally, recalling that $T(a)$ is a bounded linear operator: [16]

$$lim_{\epsilon \to 0} a^\epsilon \in \{\arg\max_{a \in C} \sum_{i=1}^{k} \frac{1}{k} \int_0^V \varphi(v) a(v, b_i) f(v) dv\}.$$

The result follows by noting that the the sequence cutoffs identified in Proposition 4 for each discretized ϵ will converge. □

C.4 Subsidies

First, we provide a technical result needed in the proof of Proposition 6

Lemma 5. *The function*

$$\phi(\pi) = \frac{(1 - F(v-2))\pi + vf(v-1)(\frac{k-i+1}{k}(1 - F(v-1)) - \pi)}{\frac{k-i+1}{k} vf(v-1) + \pi}$$

is decreasing in π.

Proof. We are done if we can show that $\phi'(\pi) \leq 0$. Writing $\phi(\pi) = \frac{N(\pi)}{D(\pi)}$ with $N(\cdot), D(\cdot)$ appropriately defined,

$$\phi'(\pi) = \frac{N'(\pi)D(\pi) - D'(\pi)N(\pi)}{D^2(\pi)}.$$

Therefore we are done if we can show that $N'(\pi)D(\pi) - D'(\pi)N(\pi) < 0$. Note that

$$D'(\pi) = 1,$$
$$N'(\pi) = (1 - F(v-2)) - vf(v-1).$$

[16] Technically we should also formally show that the sequence $\{a^\epsilon\}$ has a pointwise limit, but this follows by inspection of the a^ϵ's characterized by Proposition 4.

Therefore

$$
\begin{aligned}
&N'(\pi)D(\pi) - D'(\pi)N(\pi) \\
=&((1-F(v-2)) - vf(v-1))\left(\tfrac{k-i+1}{k}vf(v-1)+\pi\right) \\
&-\left((1-F(v-2))\pi + vf(v-1)\left(\tfrac{k-i+1}{k}(1-F(v-1))-\pi\right)\right) \\
=&((1-F(v-2)) - vf(v-1))\left(\tfrac{k-i+1}{k}vf(v-1)\right) - \tfrac{k-i+1}{k}vf(v-1)(1-F(v-1)) \\
=&(-(v-1)f(v-1))\left(\tfrac{k-i+1}{k}vf(v-1)\right) \\
\leq&0.
\end{aligned}
$$

□

Proof of Proposition 6. We prove by induction on i. For $i = k$, we know that

$$
\begin{aligned}
\eta_k &= \frac{1}{k}\frac{(1-F(\bar{v}_k-1))(1-F(\bar{v}_k-2))}{\bar{v}_k f(\bar{v}_k-1)+(1-F(\bar{v}_k-1))} \\
&= \frac{(1-F(\bar{v}_k-1))}{k}\frac{(1-F(\bar{v}_k-2))}{(\bar{v}_k-1)f(\bar{v}_k-1)+(1-F(\bar{v}_k-2))} \\
&\leq \frac{(1-F(\bar{v}_k-1))}{k}.
\end{aligned}
$$

For the induction hypothesis, assume that

$$\sum_{i+1}^{k}\eta_j \leq \tfrac{(k-i)}{k}(1-F(\bar{v}_{i+1}-1)).$$

Therefore we are left to show (26).

Recall from the proof of Proposition 4 that at optimality, η_i, the dual variable corresponding to the budget constraint corresponding to b_i, solves

$$\varphi(t_i) - \varphi(\bar{v}_i - 1) - \frac{k\sum_{j=i}^{k}\eta_j}{(k-i+1)f(\bar{v}_i-1)} = 0,$$

where

$$\varphi(t_i) = \bar{v}_i - \frac{(\underline{v}_i - \bar{v}_i + 1)}{\pi(t_i)}\left(\frac{k-i}{k}(1-F(\underline{v}_i)) - \sum_{i+1}^{k}\eta_j\right) - \frac{\bar{v}_i}{\pi(t_i)}\eta_i,$$

and,

$$\pi(t_i) = \frac{1}{k}(1-F(\bar{v}_i-1)) + \frac{k-1}{k}(F(\underline{v}_i) - F(\bar{v}_i-1)).$$

By the induction hypothesis,

$$\varphi(t_i) \leq \bar{v}_i - \frac{\bar{v}_i}{\pi(t_i)}\eta_i,$$

and therefore

$$\begin{aligned}
\bar{v}_i - \varphi(\bar{v}_i - 1) &\geq \frac{\bar{v}_i}{\pi(t_i)}\eta_i + \frac{k\sum_{j=i}^{k}\eta_j}{(k-i+1)f(\bar{v}_i-1)} \\
\implies \frac{1-F(\bar{v}_i-2)}{f(\bar{v}_i-1)} &\geq \frac{\bar{v}_i}{\pi(t_i)}\eta_i + \frac{k\sum_{j=i}^{k}\eta_j}{(k-i+1)f(\bar{v}_i-1)}.
\end{aligned}$$

Rearranging terms, we have

$$\frac{k-i+1}{k}\,\frac{(1-F(\bar{v}_i-2))\pi(t_i) + \bar{v}_i f(\bar{v}_i-1)\sum_{i+1}^{k}\eta_j}{\frac{k-i+1}{k}\bar{v}_i f(\bar{v}_i-1)+\pi(t_i)} \geq \sum_{i}^{k}\eta_j. \tag{31}$$

Once again, by the induction hypothesis,

$$\sum_{i+1}^{k}\eta_j \leq \tfrac{k-i}{k}(1-F(\underline{v}_i)) = \tfrac{k-i+1}{k}\big(1-F(\bar{v}_i-1)\big) - \pi(t_i). \tag{32}$$

Substituting (32) into (31),

$$\begin{aligned}
\sum_{i}^{k}\eta_j &\leq \left(\frac{k-i+1}{k}\right)\frac{(1-F(\bar{v}_i-2))\pi(t_i) + \bar{v}_i f(\bar{v}_i-1)\Big(\frac{k-i+1}{k}\big(1-F(\bar{v}_i-1)\big)-\pi(t_i)\Big)}{\frac{k-i+1}{k}\bar{v}_i f(\bar{v}_i-1)+\pi(t_i)} \\
&\equiv \phi(\bar{v}_i, \pi(t_i)).
\end{aligned}$$

Lemma 5 shows that $\phi(\cdot)$ is decreasing in its second argument. Given $\bar{v}_i$, the lowest possible value for $\pi(t_i)$ is $\frac{1}{k}(1-F(\bar{v}_i-1))$, at which the left hand side of the bound will be maximized. Therefore, substituting $\pi(t_i) = \frac{1}{k}(1-F(\bar{v}_i-1))$,

$$\begin{aligned}
\sum_{i}^{k}\eta_j &\leq \left(\frac{k-i+1}{k}\right)\frac{(1-F(\bar{v}_i-2))(1-F(\bar{v}_i-1)) + (k-i)\bar{v}_i f(\bar{v}_i-1)(1-F(\bar{v}_i-1))}{(k-i+1)\bar{v}_i f(\bar{v}_i-1)+(1-F(\bar{v}_i-1))} \\
&\leq \left(\frac{k-i+1}{k}\right)(1-F(\bar{v}_i-1))
\end{aligned}$$

□