Weak Selection versus Strong Selection of Rationalizability via Perturbations of Higher-order Beliefs[*,†]

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March 8, 2011

Abstract

We distinguish two different selections of rationalizable outcomes: the strong selection (in Rubinstein (1989) and Carlsson and Van Damme (1993)) and the weak selection (in Weinstein and Yildiz (2007)). In contrast to Weinstein and Yildiz's result that every rationalizable action can be weakly selected, we show strong selection is generically impossible for types with multiple rationalizable actions. Furthermore, we fully characterize the actions which can be strongly selected for all finite types.

[*]Preliminary and incomplete.

[†]We thank Eddie Dekel, Marcin Pęski, Johnathan Weistein, Muhamet Yildiz and seminar participants at University of Western Ontario for very helpful comments. All remaining errors are our own.

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1 Introduction

One major challenge faced by game theory is the prevalence of multiple equilibria which substantially limits prediction power. A large literature pursues the idea of refining predictions by perturbing players' higher-order beliefs. This literature starts with two seminal papers. First, Rubinstein (1989) studies a 2×2 coordination game with two strict Nash equilibria. Rubinstein observes that only the Pareto-dominated equilibrium remains rationalizable when the game is mutually known up to any arbitrarily high but finite level. In a similar vein, Carlsson and Van Damme (1993) consider another 2×2 coordination game with two strict Nash equilibria. They show that by introducing a vanishingly small noisy signal (which amounts to a perturbation of higher-order beliefs), only the risk-dominant equilibrium survives.

In a recent paper, Weinstein and Yildiz (2007) (hereafter, WY) prove the following striking result. In a fixed finite game with no common knowledge restriction on payoffs, for any rationalizable action a of any (Harsanyi) type t, we can slightly perturb the higher-order beliefs of t to make a new type t' for which a is the unique rationalizable action. This result suggests that the idea of selecting an equilibrium using perturbations of higher-order beliefs is problematic; as WY say:

> In this paper, we generalize both uniqueness and noise dependence results in a strong way: (i) we can make any game dominance-solvable by introducing a suitable form of small incomplete information, but (ii) by varying the form of incomplete information, we can select any rationalizable strategy in the original game, weakening the selection argument (Weinstein and Yildiz, 2007, p.389).

In this paper, we revisit this issue by pointing out a subtle difference between the notion of selection highlighted in Carlsson and Van Damme (1993) and Rubinstein (1989) and that studied in WY. To see the difference, consider a rationalizable action a of a type t which has multiple rationalizable actions, and a sequence of types $\{t_n\}$ whose beliefs (weak*-)approximate those of t up to any finite order, i.e., t_n converges to t in the product topology (see Section 4 for a formal definition).[1] When n is large, t_n can be viewed as a small

[1] Product topology natually generalize the approximations in Rubinstein (1989) and Carlsson and

perturbation of t. We distinguish two kinds of selections : an action a is *strong selected* for t if for some $\varepsilon > 0$, a is the *only* $\varepsilon-$rationalizable action for any t_n, whereas a is *weakly selected* for t if a the *only* ($0-$)rationalizable action for any t_n.[2] Consequently, a strong selection must be a weak selection, while a weak selection may not be a strong selection. To see this, consider the following single-agent decision problem.

Example 1. There are two actions $\{a, b\}$, two payoff parameters $\{\theta_0, \theta_a\}$ and one agent. The agent's payoff depend on both the payoff parameter and the action chosen, which is illustrated as follows.

action	payoff
a	0
b	0

$\theta = \theta_0$

action	payoff
a	0
b	-1

$\theta = \theta_a$

In this single-agent decision problem, a type is identified with the agent's belief over the payoff parameters $\theta \in \{\theta_0, \theta_a\}$, and hence convergence of beliefs in product topology is reduced to convergence in distribution. Consider the following types.

$$t_n [\theta] \equiv \begin{cases} 1 - \frac{1}{n}, & \text{if } \theta = \theta_0, \\ \frac{1}{n}, & \text{if } \theta = \theta_a, \end{cases} \quad \forall n = 1, 2, ..., \infty.$$

Clearly, $\{t_n\}$ converges to t_∞, and a is the unique best reply and hence uniquely rationalizable for every t_n. That is, a is weakly selected for t. However, a cannot be strongly selected for t: for any $\varepsilon > 0$, b is also an $\varepsilon-$best reply and hence $\varepsilon-$rationalizable for t_n with sufficiently large n.

Rubinstein's e-mail game and the global game are classical examples to illustrate impacts of higher-order beliefs on strategic behaviors. To summarize strategic behaviors encoded in types, Dekel, Fudenberg, and Morris (2006) define the strategic topology for types: types are close in strategic topology if and only if they have similar rationalizable actions. Intuitively, $\{t_n\}$ converges to t_∞ in the strategic topology means

Van Damme (1993). See Weinstein and Yildiz (2007) for more discussion.

[2]See Section 3 for formal definitions of $\varepsilon-$rationalizable actions and $0-$rationalizable actions. These definitions are adopted from Dekel, Fudenberg, and Morris (2006, 2007) and are the correleated incomplete-information counterparts of the Berheim-Pearce definition for complete-information games.

$\forall \varepsilon > 0$, every rationalizable action a of t_∞ is $\varepsilon -$ rationalizable for all but finite t_n.

Dekel, Fudenberg, and Morris (2006) show that strategic convergence implies product convergence, but the converse is not true. Namely, we may have t_n converges to t_∞ in product topology, but t_n does not converge to t_∞ in the strategic topology — Ely and Peski (2010) and Chen and Xiong (2011) call such a phenomenon *strategic discontinuity.* Therefore, Rubinstein's e-mail game and the global game are classical examples of strategic discontinuity.

The major difference between the two selection notions described above is that strong selection captures the idea of strategic discontinuity, but weak selection does not. To see this, consider a type t with two rationalizable actions a and b. If a can be strongly selected along a sequence of types $\{t_n\}$, then b is rationalizable for t but b is not $\varepsilon-$rationalizable for some $\varepsilon > 0$ and any t_n, i.e., strong selection implies strategic discontinuity. However, weak selection does not imply strategic discontinuity. For instance, in Example 1, though a is weakly selected for t and a is uniquely rationalizable for every t_n, both actions a and b are $\varepsilon-$rationalizable for all but finite t_n for any $\varepsilon > 0$, i.e., no strategic discontinuity occurs.

Taking exactly the same setup as Weinstein and Yildiz (2007), this paper adopts strong selection instead of weak selection. We try to pin down the exact effect due to different selection notions adopted. To achieved this goal, we restrict our attention to the space of types with multiple rationalizable actions, because these are the types which need equilibrium (or rationalizability) selection. WY show that every type in this space admits a weak selection. In sharp contrast to WY's result, our first main result (Theorem 2) shows that strong selection is generically impossible.

We prove Theorem 2 by considering the following set of types.

$$B_\infty \equiv \left\{ \begin{array}{c} \text{type } t \text{ has multiple} \\ \text{rationalizable actions} \end{array} : \begin{array}{c} t \text{ believes all of her opponents} \\ \text{have unique rationalizable actions} \end{array} \right\} \tag{1}$$

The set B_∞ is generic in the sense that within the space of types with multiple rationalizable actions, B_∞ is a dense set and it can be written as a countable intersection of open sets in the product topology. Facing types with unique rationalizable actions, every type in B_∞ has a unique conjecture about their opponents' rationalizable actions. Hence, solving rationalizable actions for these types can be recasted as solving optimal actions in a single-person decision (as in Example 1). Consequently, these types do not admit any strong selection.

Our second main result (Theorem 4) fully characterizes actions which can be strongly selected. To see the idea, consider a complete-information type. Proposition 1 in WY shows that an action can be weakly selected if and only if it is rationalizable. It well known that rationalizability can be formulated using the best-reply sets. Formally, let R_i be a set of actions of player i and we say $\times_{j\in N} R_j$ is a best-reply set iff

$$\text{every } a_j \in R_j \text{ is a best reply to some belief } \sigma_{-j} \in \triangle\left(R_{-j}\right).$$

Then, an action a_i is rationalizable for player i if and only if $a_i \in R_i$ for some best reply sets $\times_{j\in N} R_j$.

First, using similar argument as WY, we find a sufficient condition for strong selection. let R_i° be a set of actions of player i and we say $\times_{j\in N} R_j^\circ$ is a strict best-reply set iff

$$\text{every } a_i \in R_i^\circ \text{ is a strict best reply to some belief } \sigma_{-i} \in \triangle\left(R_{-i}^\circ\right).$$

We say an action a_i is strictly rationalizable if $a_i \in R_i^\circ$ for some strict best-reply set $\times_{j\in N} R_j^\circ$. Then, we show a_i can be strongly selected *if* it is strictly rationalizable (Proposition 1).

However, strict rationalizability is not necessary for strong selection. — This is shown by Exmple 3 in Section 2. To get a full characterization, we propose a notion for *collections of subsets* of actions, called the *strict best–reply collection* (Definition 6). Analogous to WY, we will prove that an action a_i is strongly selected for player i if and only if $\{a_i\} \in \mathcal{R}_i$ for some strict best-reply collection $\times_{j\in N} \mathcal{R}_j$ (Theorem 4). This full characterization of strong selection can be extended to a larger class of types, including all finite types, which is studied in Appendix 4.

The rest of the paper proceeds as follows. Section 2 provides two examples to illustrate our main ideas. Section 3 provides formal notations and definitions. Section 4 presents our two main results. Section 5 concludes. All technical proofs are relegated to the Appendix.

2 Examples

In this section, we provide two examples (Example 2 and 3) to illustrate our main idea. Example 2 is a variant of Example 3 in WY and it demonstrates the denseness of B_∞ (defined in (1)). Example 3 shows that strict rationalizability is sufficient but not necessary for strong selection. In Section 4.2.2, we will revisit Example 3 and show that the notion of strict best-reply collection fully characterizes strong selection.

Example 2. There are two players. Each player can choose one of the two actions, "Attack" and "No Attack." Their payoffs depend on an unknown parameter $\theta \in \left\{-\frac{2}{5}, \frac{2}{5}, \frac{6}{5}\right\}$ as shown in the following matrix:

	Attack	No Attack
Attack	θ, θ	$\theta - 1, 0$
No Attack	$0, \theta - 1$	$0, 0$

Let $t^{CK}\left(\frac{2}{5}\right)$ denote the type which have common knowledge of $\theta = \frac{2}{5}$, i.e.,

$$t^{CK}\left(\frac{2}{5}\right)\left[\left(\theta = \frac{2}{5}, t^{CK}\left(\frac{2}{5}\right)\right)\right] = 1.$$

Observe that both actions are rationalizable for $t^{CK}\left(\frac{2}{5}\right)$. Now, we construct a sequence of types in B_∞ which converge to $t^{CK}\left(\frac{2}{5}\right)$ in product topology. That is, for any positive integer k, we will find a type $t'\ (\in B_\infty)$ which is arbitrarily close to $t^{CK}\left(\frac{2}{5}\right)$ in k-th order beliefs.

First, consider the belief σ_2 of player 1 defined as follows.

$$\sigma_2[a_2] \equiv \begin{cases} \frac{2}{5}, & \text{if } a_2 = \text{No Attack}; \\ \frac{3}{5}, & \text{if } a_2 = \text{Attack}. \end{cases}$$

Given the belief σ_2, player 1's type $t^{CK}\left(\frac{2}{5}\right)$ is indifferent between taking "No Attack" and "Attack". Second, by Theorem 1 in WY, there are types t_{k-1}^{NA} and t_{k-1}^{A} which are arbitrarily close to $t^{CK}(2/5)$ in $(k-1)$-th order beliefs and have "No Attack" and "Attack" as their unique rationalizable actions respectively. Third, consider the following type t' for player 1.

$$t'[(\theta, t_2)] = \begin{cases} \frac{2}{5}, & \text{if } (\theta, t_2) = \left(\frac{2}{5}, t_{k-1}^{NA}\right); \\ \frac{3}{5}, & \text{if } (\theta, t_2) = \left(\frac{3}{5}, t_{k-1}^{A}\right). \end{cases}$$

Clearly, t' is arbitrarily close to $t^{CK}\left(\frac{2}{5}\right)$ in k-th order beliefs. Furthermore, both "No Attack" and "Attack" are rationalizable for t', while he believes all of her opponents have unique rationalizable actions. That is, $t' \in B_\infty$.

Example 3. Consider a two-player game.

	c	d	e
a	$1,1$	$0,0$	$0,0$
b	$0,0$	$1,1$	$1,1$

$\theta = \theta_0$

	c	d	e
a	$0,1$	$0,0$	$0,0$
b	$0,1$	$0,0$	$0,0$

$\theta = \theta_c$

	c	d	e
a	$0,0$	$0,1$	$0,0$
b	$0,0$	$0,1$	$0,0$

$\theta = \theta_d$

For $\theta = \theta_0, \theta_c, \theta_d$, let $t^{CK}(\theta)$ denote the type which has common knowledge of θ. Then, both action a and action b are rationalizable for player 1's type $t^{CK}(\theta_0)$.

First, action a is strictly rationalizable in the complete-information game $\theta = \theta_0$ and a can be strongly selected for $t^{CK}(\theta_0)$ as follows.

Define the following sequence of types $\{t_n\}$,

$$t_0 \equiv t^{CK}(\theta_c), t_1[(\theta_0, t_0)] = 1, t_2[(\theta_0, t_1)] = 1, ..., t_n[(\theta_0, t_{n-1})] = 1, ...$$

where t_{2n-1} is player 1's type and t_{2n} is player 2's type. Then, actions a and c are the unique $\frac{1}{2}-$rationalizable actions for type t_{2n-1} and t_{2n} respectively. Furthermore, $\{t_{2n-1}\}$ converges to $t^{CK}(\theta_0)$ as $n \to \infty$. Therefore, a is strongly selected for $t^{CK}(\theta_0)$.

Second, it is easy to check action b is not strictly rationalizable for player 1 in the complete-information game $\theta = \theta_0$. However, we now show b can be strongly selected for player 1's type $t^{CK}(\theta_0)$. Define a sequence of types $\{s_n\}$ as follows,

$$s_0 \equiv t^{CK}(\theta_d), s_1[(\theta_0, s_0)] = 1, s_2[(\theta_0, s_1)] = 1..., s_n[(\theta_0, s_{n-1})] = 1, ...$$

where s_{2n-1} is player 1's type and s_{2n} is player 2's type. Then, for any $n > 0$, type s_{2n-1} has a unique $\frac{1}{2}-$rationalizable action b and type s_{2n} has two $\frac{1}{2}-$rationalizable actions d and e. Furthermore, s_{2n-1} converges to $t^{CK}(\theta_0)$ as $n \to \infty$. Therefore, b is strongly selected for $t^{CK}(\theta_0)$.

3 Definitions

Fix a finite set of players $N = \{1, 2, ..., n\}$ and a finite set of payoff-relevant parameters Θ. Each player i has a finite action space A_i and utility function $u_i : \Theta \times A \to \mathbb{R}$ (where $A \equiv \Pi_{j \in N} A_j$). Following WY, we impose the following richness assumption.

Assumption 1 *The game satisfies the Richness assumption that for each i and each a_i, there exists $\theta^{a_i} \in \Theta$ such that $u_i(\theta^{a_i}, a_i, a_{-i}) > u_i(\theta^{a_i}, a_i', a_{-i})$, $\forall a_i' \neq a_i$, $\forall a_{-i}$.*

Throughout the note, for any metric space Y with metric d_Y, let $\Delta(Y)$ denote the space of all probability measures on the Borel σ-algebra of Y endowed with the weak*-topology. Every product space is endowed with the product topology and every subspace is endowed with the relative topology. Every finite or countable set is endowed with the discrete topology and $|E|$ denotes the cardinality of a finite set E. For any $\mu \in \Delta(X \times Y)$, we let $\text{marg}_X \mu$ denote the marginal distribution of μ on X. For any $\mu \in \Delta(Y)$, let $\text{supp}\mu$ denote the support of μ, i.e., the minimal closed set with $\mu-$measure 1.

By a model, we mean a pair (T, κ), where $T = T_1 \times T_2 \times \cdots \times T_n$ is a compact metric space. Each $t_i \in T_i$ is called a type of player i and is associated with a belief $\kappa_{t_i} \in \Delta(\Theta \times T_{-i})$. Assume that $t_i \mapsto \kappa_{t_i}$ is a continuous mapping. A finite model is a model such that $|T| < \infty$. A type is finite if it comes from a finite model.

Given any type t_i in a model $(\Theta \times T, \kappa)$, we can compute the first-order belief of t_i (i.e., his belief on Θ) by setting $t_i^1 = \text{marg}_\Theta \kappa_{t_i}$. We then compute the second-order belief of t_i (i.e., his belief about (θ, t_{-i}^1)) by setting

$$t_i^2(F) = \kappa_{t_i}\left(\left\{(\theta, t_{-i}) : \left(\theta, t_{-i}^1\right) \in F\right\}\right)$$

for each measurable $F \subseteq \Theta \times \Delta(\Theta)^{n-1}$. We can compute the entire hierarchy of beliefs $\left(t_i^1, t_i^2, ..., t_i^k, ...\right)$ by proceeding in this way and write $h_i(t_i) = \left(t_i^1, t_i^2, ..., t_i^k, ...\right)$ for the resulting hierarchy.

We endow Θ with the discrete metric. Let $Y^0 = \Theta$ and $Y^{k+1} = Y^k \times \Delta\left(Y^k\right)^{n-1}$ for every $k \geq 0$. We will work with the universal type space T_i^* constructed in Mertens and

Zamir (1985) which is a subset of $\times_{k=0}^{\infty}\Delta\left(Y^k\right)$. Mertens and Zamir show that for any type t_i in any model, there is some $s_i \in T_i^*$ such that s_i and t_i have the same hierarchy of beliefs (i.e., $h_i(t_i) = s_i$), and moreover, T_i^* (endowed with the product topology) is a compact metric space homeomorphic to $\Delta\left(\Theta \times T_{-i}^*\right)$ (endowed with the weak* topology). Denote the homeomorphism by κ_i^*. Then, (T^*, κ^*) is itself a model where $\kappa_{t_i}^* \equiv \kappa_i^*(t_i)$ for every $t_i \in T_i^*$. For a sequence of types $\{t_{i,m}\}$ and a type t_i, we write $t_{i,m} \to t_i$ when $\{t_{i,m}\}$ converges to t_i in the product topology. That is, $t_{i,m} \to t_i$ iff for every $k \geq 1$, $t_{i,m}^k \to t_i^k$ in the weak*-topology.

Following WY, we adopt the solution concept of interim correlated rationalizability (ICR) proposed in Dekel, Fudenberg, and Morris (2006, 2007) and restrict attention to the universal type space.[3] For any $\pi \in \Delta(\Theta \times T_{-i} \times A_{-i})$, we will abuse the notation by writing $\pi(\theta, a_{-i})$ for $\text{marg}_{\Theta\times A_{-i}}\pi$ and use $BR_i(\pi,\varepsilon)$ to denote the set of $\varepsilon-$best replies to $\text{marg}_{\Theta\times A_{-i}}\pi$. That is,

$$BR_i(\pi,\varepsilon) = \left\{ a_i \in A_i : \sum_{\theta,a_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i(\theta, a_i', a_{-i})\right]\pi(\theta, a_{-i}) \geq -\varepsilon,\ \forall a_i' \in A_i \right\} \quad (2)$$

Given a model (T,κ) and $\varepsilon \geq 0$, the $\varepsilon-$ICR of a type t_i, denoted by $S_i^\infty[t_i,\varepsilon]$, is defined as follows.

$$S_i^0[t_i,\varepsilon] = A_i \text{ and for } k \in \mathbb{N},$$

$$S_i^k[t_i,\varepsilon] = \left\{ a_i \in A_i : \begin{array}{c} a_i \in BR_i(\pi,\varepsilon) \text{ for some } \pi \in \Delta(\Theta \times T_{-i} \times A_{-i}) \text{ s.t.} \\ \text{marg}_{\Theta\times T_{-i}}\pi = \kappa_{t_i} \text{ and} \\ \pi\left(\left\{(\theta, t_{-i}, a_{-i}) \in \Theta \times T_{-i} \times A_{-i} : a_{-i} \in S_{-i}^{k-1}[t_{-i},\varepsilon]\right\}\right) = 1. \end{array} \right\}$$

$$S_i^\infty[t_i,\varepsilon] = \bigcap_{k=0}^{\infty} S_i^k[t_i,\varepsilon].$$

We write $S^\infty[t,\varepsilon] = \Pi_{j\in N} S_j^\infty[t_j,\varepsilon]$ and $S_{-i}^\infty[t_{-i},\varepsilon] = \Pi_{j\neq i} S_j^\infty[t_j,\varepsilon]$. We say that $\pi \in \Delta(\Theta \times T_{-i} \times A_{-i})$ is *valid* for type t_i if $\text{marg}_{\Theta\times T_{-i}}\pi = \kappa_{t_i}$ and $\pi\left(a_{-i} \in S_{-i}^\infty[t_{-i},0]\right) = 1$. Dekel, Fudenberg, and Morris (2006, 2007) show that an action $a_i \in S_i^\infty[t_i,0]$ iff $a_i \in BR_i(\pi,0)$ for some π valid for t_i. For notational simplicity, we will write $S_i^\infty[t_i]$ for $S_i^\infty[t_i,0]$, and similarly, all notations without a reference to ε should be understood as having an implicit reference to 0.

[3]This is without loss of generality because Dekel, Fudenberg, and Morris (2006, 2007) show that the $(\varepsilon-)$ICR actions of a type are fully determined by its belief hierarhy and also because the universal type space contains all belief hierarchies.

WY partition the universal type space into two parts: the set of types with multiple rationalizable actions, denoted by M_i, and the set of types with unique rationalizable actions, denoted by $T_i^* \backslash M_i$. That is,

$$M_i = \{t_i \in T_i^* : |S_i^\infty [t_i]| > 1\} \text{ and } T_i^* \backslash M_i = \{t_i \in T_i^* : |S_i^\infty [t_i]| = 1\}.$$

We focus on M_i because we need to refine our prediction only when a type has multiple rationalizable actions.

The following are our key definitions:

Definition 1 *For $t_i \in M_i$ and $a_i \in S_i^\infty [t_i]$, we say a_i can be weakly selected for t_i, if there exists some $t_{i,m} \to t_i$ such that $\{a_i\} = S_i^\infty [t_{i,m}]$ for all m. If some a_i can be weakly selected for t_i, we say t_i admits a weak selection.*

Definition 2 *For $t_i \in M_i$ and $a_i \in S_i^\infty [t_i]$, we say a_i can be strongly selected for t_i, if there exists some $\varepsilon > 0$ and some $t_{i,m} \to t_i$ such that $\{a_i\} = S_i^\infty [t_{i,m}, \varepsilon]$ for all m. If some a_i can be strongly selected for t_i, we say t_i admits a strong selection.*

Finally, we consider the following topological notion of genericity.

Definition 3 *In a topological space X, a set $F\,(\subset X)$ is a residual set if it contains a countable intersection of open and dense set. A set $E\,(\subset X)$ is a meager set if it is the complement of a residual set.*

We view a residual set as a generic set and view a meager set as a non-generic set.

4 Main Results

4.1 Weak Selection *versus* Strong Selection

In this subsection, we explore the difference between weak selection and strong selection. We will restrict our attention to M_i, because we need to refine our prediction only when a type has multiple rationalizable actions.

To ease comparison, we state WY's result as follows. Let M_i^w denote the set of types which have multiple rationalizable actions and admit a weak selection.

Theorem 1 (Weinstein and Yildiz (2007)) $M_i^w = M_i$.

Let M_i^s denote the set of types which have multiple rationalizable actions and admit a strong selection. Our first main result is the following.

Theorem 2 *M_i^s is a meager set in M_i.*

To prove Theorem 2, we show $M_i \backslash M_i^s$ is a dense set which can be written as a countable intersection of open sets. We need the following notations:

$$\begin{aligned} B_{i,\infty} &\equiv \left\{t_i \in M_i : \kappa^*_{t_i}\left[\Theta \times \left(T^*_{-i} \backslash M_{-i}\right)\right] = 1\right\}; \\ B_{i,n} &\equiv \left\{t_i \in M_i : \kappa^*_{t_i}\left[\Theta \times \left(T^*_{-i} \backslash M_{-i}\right)\right] > 1 - \frac{1}{n}\right\}, \forall n \in \mathbb{N}. \end{aligned}$$

Theorem 2 is then a direct consequence of the following three lemmas.

Lemma 1 *$B_{i,n}$ is open in M_i.*

Lemma 2 *$B_{i,\infty}$ is dense in M_i.*

Lemma 3 *$B_{i,\infty} \subset M_i \backslash M_i^s$.*

Proof of Theorem 2. Clearly, $B_{i,\infty} \subset B_{i,n}$ and $\cap_{n=1}^{\infty} B_{i,n} = B_{i,\infty}$. Hence, by Lemma 1 and 2, $B_{i,n}$ is open and dense in M_i. Consequently, $B_{i,\infty}$ is a residual set in M_i. Therefore, M_i^s is a meager set in M_i by Lemma 3.■

4.2 A Full Characterization of Strong Selection

In this section, we will characterize strong selection. For expositional ease, we will restrict our attention to complete-information types. First, following WY's argument, we find a sufficient condition for strong selection. Then, in Theorem 4, we fully characterize strong selection. In Appendix, we extend our fully characterization to a large class of types which includes all finite types.[4]

4.2.1 A sufficient condition

First, we restate WY's characterization as follows.

Definition 4 *In a complete information game $(A_i, u_i)_{i\in N}$, a set $\times_{i\in N} R_i$ with $R_i \subset A_i$ for all $i \in N$ is a best-reply set if for any $a_i \in R_i$, there exists $\sigma_{-i} \in \Delta(R_{-i})$ such that*

$$\sum_{a_{-i}\in R_{-i}} \sigma_{-i}(a_{-i}) \left[\, u_i(a_i, a_{-i}) - u_i(a_i', a_{-i})\right] \geq 0 \text{ for any } a_i' \in A_i.$$

Theorem 3 (WY Proposition 1) *An action a can be weakly selected for player i's type $t^{CK}(\theta_0)$ iff there exists a best-reply set $\times_{j\in N} R_j$ in game θ_0 such that $a \in R_i$.*

Similar to a best-reply set, we define the strict best-reply set as follows.

Definition 5 *In a complete information game $(A_i, u_i)_{i\in N}$, a nonempty set $\times_{i\in N} R_i$ with $R_i \subset A_i$ for all $i \in N$ is a strict best-reply set if there exists $\gamma > 0$ such that for any $a_i \in R_i$, there exists $\sigma_{-i} \in \Delta(R_{-i})$ and*

$$\sum_{a_{-i}\in R_{-i}} \sigma_{-i}(a_{-i}) \left[\, u_i(a_i, a_{-i}) - u_i(a_i', a_{-i})\right] \geq \gamma \text{ for any } a_i' \neq a_i.$$

The following proposition provides a sufficient condition for strong selection.

Proposition 1 *An action a can be strongly selected for player i's type $t^{CK}(\theta_0)$ if there exists a strict best-reply set $\times_{i\in N} R_i$ with $R_i \subset A_i$ for all $i \in N$ in game θ_0 such that $a \in R_i$.*

The proof of Proposition 1 is almost the same as the proof of Proposition 1 in WY. Also, the intuition is illustrated in Example 3. Hence, we relegate the proof to Appendix 1.

[4] The full characterization for any type remains an open question to us.

4.2.2 A necessary and sufficient condition

As illustrated in Example 3, the strict best-reply property is not enough to fully characterize strong selection. The notion proposed below suffices.

Definition 6 *In a complete information game* $(A_i, u_i)_{i\in N}$*,* $\times_{i\in N}\left\{R_i^1, R_i^2, ..., R_i^{L_i}\right\}$ *with non-empty* $R_i^l \subset A_i$ *for each* $i \in N$ *and* $l \in \{1, 2, ..., L_i\}$ *is called a strict best-reply collection if there exists* $\gamma > 0$ *such that for any* $i \in N$*, any* $l \in \{1, 2, ..., L_i\}$ *and any* $a_i \in R_i^l$*, there exists* $\sigma_{-i} \in \Delta\left(R_{-i}^1, R_{-i}^2, ..., R_{-i}^{L_{-i}}\right)$ *such that*

$$\begin{aligned} & a_i \in R_i^l \iff \\ & \text{there exists } \zeta_{-i}^{l'} \in \Delta\left(R_{-i}^{l'}\right) \text{ for each } l' \in \{1, 2, ..., L_{-i}\} \text{ such that} \\ & \sum_{l'=1}^{L_i} \sigma_{-i}\left(R_{-i}^{l'}\right)\left[\sum_{a_{-i}\in R_{-i}^{l'}} \zeta_{-i}^{l'}(a_{-i})\left[\, u_i(a_i, a_{-i}) - u_i(a_i', a_{-i})\right]\right] \geq -\gamma, \forall a_i' \in A_i. \end{aligned} \tag{3}$$

The following is our second main result.

Theorem 4 *An action* a *can be strongly selected for player* i*'s type* $t^{CK}(\theta_0)$ *iff there exists a strict best-reply collection* $\times_{i\in N}\left\{R_i^1, R_i^2, ..., R_i^{L_i}\right\}$ *in game* θ_0 *such that* $\{a\} = R_i^{l^*}$ *for some* l^**.*

The proof of Theorem 4 is quite notationally and technically involved, and it is relegated to Appendices 3. We provide the intuition here.

Consider a game θ_0, in which $\times_{i\in N}\left\{R_i^1, R_i^2, ..., R_i^{L_i}\right\}$ is a best-reply collection such that $\{a\} = R_i^{l^*}$. We will show action a can be strongly selected for type $t^{CK}(\theta_0)$. Specifically, for $i \in N$ and $l \in \{1, 2, ..., L_i\}$, each action set R_i^l represents a type sufficiently close to $t^{CK}(\theta_0)$ in the $(k-1)$-th order beliefs. Denote this type by $t_{k-1}^{R_i^l}$. Since $\times_{i\in N}\left\{R_i^1, R_i^2, ..., R_i^{L_i}\right\}$ is a best-reply collection, each R_i^l corresponds to a belief $\sigma_{-i}^{R_i^l} \in \Delta\left(R_{-i}^1, R_{-i}^2, ..., R_{-i}^{L_{-i}}\right)$ which can be used to define types sufficiently close to $t^{CK}(\theta_0)$ in the k-th order beliefs as follows.

$$t_k^{R_i^l} : t_k^{R_i^l}\left[\left(\theta_0, t_{k-1}^{R_{-i}^{l'}}\right)\right] = \sigma_{-i}^{R_i^l}\left(t_{k-1}^{R_{-i}^{l'}}\right) \text{ for } l' = 1, 2, ..., L_{-i}.$$

Then, type $t_k^{R_i^l}$ is sufficiently close to $t^{CK}(\theta_0)$ in the k-th order beliefs. Further, (3) requires $S_i^{\infty}\left[t_k^{R_i^l}, \gamma\right] = R_i^l$. Finally, since $\{a\} \in \left\{R_i^1, R_i^2, ..., R_i^{L_i}\right\}$, we can follow steps above to construct a sequence of types $\left\{t_k^{\{a\}}\right\}$ which converges to $t^{CK}(\theta_0)$ in product topology and $S_i^{\infty}\left[t_k^{\{a\}}, \gamma\right] = \{a\}$ for every k, i.e., a is strongly selected for type $t^{CK}(\theta_0)$.

Example 3 (continued). Recall that

	c	d	e
a	$1,1$	$0,0$	$0,0$
b	$0,0$	$1,1$	$1,1$

$\theta = \theta_0$

We verify that both $\{\{a\}\} \times \{\{c\}\}$ and $\{\{b\}\} \times \{\{d,e\}\}$ are strict best-reply collections in the complete-information game $\theta = \theta_0$. Consider $\{\{b\}\} \times \{\{d,e\}\}$ and the case with $\{\{a\}\} \times \{\{c\}\}$ is similar. To see this, we choose

$$\sigma_1[\{\{b\}\}] = \sigma_2[\{\{d,e\}\}] = \zeta_{1,\{b\}}[\{b\}] = \zeta_{2,\{d,e\}}[\{d\}] = 1.$$

Then, (3) is satisfied.

5 Concluding Remarks

Ely and Peski (2010) define regular types as types around which convergence in the product topology guarantees convergence of rationalizable behavior in *any* finite game. Formally, a type t is regular iff for any finite game, any sequence of types $\{t_n\}$ with $t_n \to t$, and any $\varepsilon > 0$, every rationalizable action of t is $\varepsilon-$rationalizable for any t_n with sufficiently large n. Therefore, in a fixed finite game (such as the one WY and we consider here), regular types do not admit any strong selection. Ely and Peski (2010) show that regular types are generic in the universal type space in the sense they contain a residual set in the product topology. This result does *not* imply our genericity result because in a fixed game, it is possible that regular types are all in the open and dense set of types with unique rationalizable actions.

A Appendix

A.1 Lemma 1 and 3

The proofs of Lemma 1 and 3 need the following result.

Lemma 4 *For any i, if $t_{i,m} \to t_i$, then*

$$\liminf_{m\to\infty} \kappa^*_{t_{i,m}}(G) \geq \kappa^*_{t_i}(G) \text{ for any open set } G \subset \Theta \times T^*_{-i}; \tag{4}$$

$$\limsup_{m\to\infty} \kappa^*_{t_{i,m}}(F) \leq \kappa^*_{t_i}(F) \text{ for any closed set } F \subset \Theta \times T^*_{-i}. \tag{5}$$

Proof. Recall that κ^*_i be the homeomorphism between T^*_i and $\Delta\left(\Theta \times T^*_{-i}\right)$, where T^*_i is endowed with the product topology and $\Delta\left(\Theta \times T^*_{-i}\right)$ is endowed with the weak*-topology. Since $t_{i,m} \to t_i$, we have $\kappa^*_{t_{i,m}} \to \kappa^*_{t_i}$ in weak* topology. Then, (4) and (5) follow from the definition of weak*-topology (see (Dudley, 2002, 11.1.1. Theorem)).■

A.1.1 Proof of Lemma 1

Lemma 1. $B_{i,n}$ is open in M_i.

Proof. Recall

$$B_{i,n} \equiv \left\{t_i \in M_i : \kappa^*_{t_i}\left[\Theta \times \left(T^*_{-i} \backslash M_{-i}\right)\right] > 1 - \frac{1}{n}\right\}.$$

By Proposition 2 in WY, $T^*_{-i} \backslash M_{-i}$ is open. Suppose $B_{i,n}$ is not open in M_i. That is, there is some $t_i \in B_{i,n}$ and some sequence $\{t_{i,m}\}$ with $t_{i,m} \to t_i$ such that $t_{i,m} \in M_i$ and $\kappa^*_{t_{i,m}}\left[\Theta \times \left(T^*_{-i} \backslash M_{-i}\right)\right] \leq 1 - \frac{1}{n}$ for all m. Hence,

$$\liminf_{m\to\infty} \kappa^*_{t_{i,m}}\left[\Theta \times \left(T^*_{-i} \backslash M_{-i}\right)\right] \leq 1 - \frac{1}{n} < \kappa^*_{t_i}\left[\Theta \times \left(T^*_{-i} \backslash M_{-i}\right)\right]. \tag{6}$$

where the last inequality follows because $t_i \in B_{i,n}$. Then, (6) contradicts to (4) in Lemma 4.■

A.1.2 Proof of Lemma 2

Lemma 2. $B_{i,\infty}$ is dense in M_i.

Proof. Recall that T_i^* is a compact metric space. In this proof, we denote by d_i the metric on T_i^*. We now divide the proof into three steps.

Step 1 *For any finite type $t_i \in M_i$, there is some $\pi \in \Delta\left(\Theta \times T_{-i}^* \times A_{-i}\right)$ valid for t_i such that $BR_i\left(\pi\right)$ has more than one actions.*

Since t_i is a finite type, $T_{-i} = \left\{t_{-i} \in T_{-i}^* : \kappa_{t_i}^*\left[t_{-i}\right] > 0\right\}$ is a finite set. Since $t_i \in M_i$, there are at least two distinct actions a_i' and a_i'' in $S_i^\infty\left[t_i\right]$. Thus, there are $\pi', \pi'' \in \Delta\left(\Theta \times T_{-i} \times A_{-i}\right)$ which are valid for t_i such that $a_i' \in BR_i\left(\pi'\right)$ and $a_i'' \in BR_i\left(\pi''\right)$. Step 1 holds if either $BR_i\left(\pi'\right)$ or $BR_i\left(\pi''\right)$ has more than one actions. Now suppose that $BR_i\left(\pi'\right)$ and $BR_i\left(\pi''\right)$ both have only one actions. Define

$$\begin{aligned} \Sigma &\equiv \left\{\pi \in \Delta\left(\Theta \times T_{-i} \times A_{-i}\right) : \pi \text{ is valid for } t_i\right\}; \\ P_{a_i} &\equiv \left\{\pi \in \mathbf{\Sigma} : \left\{a_i\right\} = BR_i\left(\pi\right)\right\}. \end{aligned}$$

Observe that Σ is convex, and hence also a connected set in $\mathbb{R}^{\left|\Theta \times T_{-i} \times A_{-i}\right|}$ endowed with the Euclidean topology. Then, for every $a_i \in A_i$, P_{a_i} is an (Euclidean-)open set in Σ and $P_{a_i} \cap P_{b_i} = \varnothing$ if $a_i \neq b_i$. Moreover, $P_{a_i'} \neq \varnothing$ and $P_{a_i''} \neq \varnothing$ because $\pi' \in P_{a_i'}$ and $\pi'' \in P_{a_i''}$. Since $\mathbf{\Sigma}$ is connected, we have $\cup_{a_i \in A_i} P_{a_i} \subsetneqq \mathbf{\Sigma}$. Thus, there is some $\pi^* \in \mathbf{\Sigma}$ such that $\pi^* \notin P_{a_i}$ for all $a_i \in A_i$. Since $BR_i\left(\pi^*\right) \neq \varnothing$, $BR_i\left(\pi^*\right)$ has more than one actions.■

Step 2 *For any finite type $t_i \in M_i$, any $\varepsilon > 0$ and any $k \geq 1$, there is a sequence of finite types type $\left\{t_{i,m}\right\}$ such that $t_{i,m} \in B_{i,\infty}$ for all m and $t_{i,m} \rightarrow t_i$.*

Fix $\varepsilon > 0$ and $k \geq 1$. Since t_i is a finite type, $T_{-i} = \left\{t_{-i} \in T_{-i}^* : \kappa_{t_i}^*\left[t_{-i}\right] > 0\right\}$ is a finite set. By step 1, there is some $\pi \in \Delta\left(\Theta \times T_{-i} \times A_{-i}\right)$ valid for t_i such that $BR_i\left(\pi\right)$ has at least two actions. Define

$$\eta = \min\left\{d_{-i}\left(t_{-i}, s_{-i}\right) : t_{-i}, s_{-i} \in T_{-i} \text{ and } t_{-i} \neq s_{-i}\right\}$$

where $\eta > 0$ because T_{-i} is a finite set.

By Proposition 1 in Weinstein and Yildiz (2007), for each $t_{-i} \in T_{-i}$ and $a_{-i} \in S_{-i}^{\infty}[t_{-i}]$, there is some type $\widetilde{t}_{-i}[t_{-i}, a_{-i}] \in T_{-i}^{*}$ such that

$$S_{-i}^{\infty}\left[\widetilde{t}_{-i}[a_{-i}, t_{-i}]\right] = \{a_{-i}\}; \tag{7}$$

$$d_{-i}\left(t_{-i}, \widetilde{t}_{-i}[t_{-i}, a_{-i}]\right) < \min\{\eta, 1/m\}. \tag{8}$$

Define $\kappa_{t_{i,m}}^{*}$ as follows.

$$\kappa_{t_{i,m}}^{*}(\theta, t_{-i}) = \begin{cases} \pi(\theta, a_{-i}), & \text{if } t_{-i} = \widetilde{t}_{-i}[t_{-i}, a_{-i}] \text{ for some } t_{-i} \in T_{-i} \text{ and } a_{-i} \in S_{-i}^{\infty}[t_{-i}]; \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\kappa_{t_{i,m}}^{*} \in \Delta(\Theta \times T_i^{*})$ uniquely determines a type $t_{i,m}$ in T_i^{*} because T_i^{*} is homeomorphic to $\Delta\left(\Theta \times T_{-i}^{*}\right)$. Clearly, $\kappa_{t_{i,m}}^{*}\left(T_{-i}^{*} \backslash M_{-i}\right) = 1$ by (7). Then, because (8) holds for all $t_{-i} \in T_{-i}$, $\kappa_{t_{i,m}}^{*} \to \kappa_{t_i}^{*}$ and hence $t_{i,m} \to t_i$ as $m \to 0$.

We now show that $t_{i,m}$ has multiple rationalizable actions. By our construction, any $\pi^{m} \in \Delta(\Theta \times T_{-i} \times A_{-i})$ valid for $t_{i,m}$ satisfies the following.

$$\pi^{m}(\theta, a_{-i}) = \pi(\theta, a_{-i}).$$

Hence, by (2), $BR_i(\pi^{m}) = BR_i(\pi)$. Since $|BR_i(\pi)| > 1$, we have $|BR_i(\pi^{m})| > 1$. Therefore, $t_{i,m}$ has multiple rationalizable actions, i.e., $t_{i,m} \in B_{i,\infty}$.

Step 3 *$B_{i,\infty}$ is dense in M_i.*

Take any $t_i \in M_i$ and any $\varepsilon > 0$. First, by (Dekel, Fudenberg, and Morris, 2006, Lemmas 13 and 14 and Theorem 1), there is some finite type $t_i' \in T_i^{*}$ such that $S_i^{\infty}[t_i] = S_i^{\infty}[t_i']$ and $d_i(t_i, t_i') < \varepsilon/2$. Then, for any $\varepsilon > 0$, by step 2, there is some $t_i'' \in B_{i,\infty}$ such that $d_i(t_i', t_i'') < \varepsilon/2$. Hence, $d_i(t_i, t_i'') < \varepsilon$. Therefore, $B_{i,\infty}$ is dense in M_i.■

A.2 Proof of Lemma 3

Lemma 3. $B_{i,\infty} \subset M_i \backslash M_i^{s}$.

Proof. Pick any $t_i \in B_{i,\infty}$ and we will show $t_i \in M_i \backslash M_i^s$. Since $B_{i,\infty} \subset M_i$ by the definition of $B_{i,\infty}$, it remains to prove $t_i \notin M_i^s$. To see this, fix a sequence of types $\{t_{i,m}\}$ with $t_{i,m} \to t_i$, $\varepsilon > 0$, and $a_i \in S_i^\infty [t_i]$. We will prove that $a_i \in S_i^\infty [t_{i,m}, \varepsilon]$ for all sufficiently large m. Then, since this is true for any $a_i \in S_i^\infty [t_i]$, any $\varepsilon > 0$, and any sequence of types $\{t_{i,m}\}$ with $t_{i,m} \to t_i$, we have $t_i \notin M_i^s$.

For $a_{-i} \in A_{-i}$, define

$$U_{-i}^{a_{-i}} := \left\{ t_{-i} \in T_{-i}^* : S_{-i}^\infty [t_{-i}] = \{a_{-i}\} \right\}.$$

Moreover, for every $\theta \in \Theta$, $\{\theta\} \times U_{-i}^{a_{-i}}$ is open by Proposition 2 of WY. Observe that $\{\Theta \times M_{-i}\} \cup \left\{ \{\theta\} \times U_{-i}^{a_{-i}} : \theta \in \Theta, a_{-i} \in A_{-i} \right\}$ is a partition of $\Theta \times T_{-i}^*$.

Claim 1

$$\lim_{m\to\infty} \kappa_{t_{i,m}}^* [\Theta \times M_i] = \kappa_{t_i}^* [\Theta \times M_i] = 0. \tag{9}$$

$$\lim_{m\to\infty} \kappa_{t_{i,m}}^* \left[\{\theta\} \times U_{-i}^{a_{-i}} \right] = \kappa_{t_i}^* \left[\{\theta\} \times U_{-i}^{a_{-i}} \right]. \tag{10}$$

By Proposition 4 of Dekel, Fudenberg, and Morris (2007), $a_i \in S_i^\infty [t_i]$ implies that there is some $\sigma_{-i} : \Theta \times T_{-i}^* \to \Delta (A_{-i})$ such that[5]

$$\text{supp}\sigma_{-i} (\theta, t_{-i}) \subset S_{-i}^\infty [t_{-i}], \forall (\theta, t_{-i}); \tag{11}$$

$$\int_{\Theta \times T_{-i}^*} \sum_{a_{-i}\in A_{-i}} [u_i (\theta, a_i, a_{-i}) - u_i (\theta, a_i', a_{-i})] \sigma_{-i} (\theta, t_{-i}) [a_{-i}] \kappa_{t_i}^* [(\theta, t_{-i})] \geq 0. \tag{12}$$

Since $t_i \in B_{i,\infty}$, $\kappa_{t_i}^* [\Theta \times M_i] = 0$. Moreover, since $\{\Theta \times M_{-i}\} \cup \left\{ \{\theta\} \times U_{-i}^{a_{-i}} : \theta \in \Theta, a_{-i} \in A_{-i} \right\}$ is a partition of $\Theta \times T_{-i}^*$, (11) implies that

$$\begin{aligned} &\int_{\Theta \times T_{-i}^*} \sum_{a_{-i}\in A_{-i}} [u_i (\theta, a_i, a_{-i}) - u_i (\theta, a_i', a_{-i})] \sigma_{-i} (\theta, t_{-i}) [a_{-i}] \kappa_{t_i}^* [(\theta, t_{-i})] \\ = &\sum_{(\theta, a_{-i})\in \Theta\times A_{-i}} [u_i (\theta, a_i, a_{-i}) - u_i (\theta, a_i', a_{-i})] \kappa_{t_i}^* \left[\{\theta\} \times U_{-i}^{a_{-i}} \right] \end{aligned} \tag{13}$$

[5] See, for example, footnote 16 in Chen, Di Tillio, Faingold, and Xiong (2010) for an explanation why we can make $\text{supp}\sigma_{-i} (\theta, t_{-i}) \subset S_{-i}^\infty [t_{-i}]$ for *all* (θ, t_{-i}) instead of for $\kappa_{t_i}^*$−almost all (θ, t_{-i}).

Let $K = \max_{i,a,\theta} |u_i(\theta, a)|$. Then,

$$\begin{aligned}
&\int_{\Theta \times T^*_{-i}} \sum_{a_{-i} \in A_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i(\theta, a'_i, a_{-i})\right] \sigma_{-i}(\theta, t_{-i})[a_{-i}] \, \kappa^*_{t_{i,m}}[(\theta, t_{-i})] \\
\geq &\sum_{(\theta, a_{-i}) \in \Theta \times A_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i(\theta, a'_i, a_{-i})\right] \kappa^*_{t_{i,m}}\left[\{\theta\} \times U^{a_{-i}}_{-i}\right] - 2K\kappa^*_{t_{i,m}}[\Theta \times M_i]
\end{aligned}$$

Since $\kappa^*_{t_{i,m}}[\Theta \times M_i] \to 0$ as $m \to \infty$ by (9) in Claim 1, it follows that

$$\begin{aligned}
&\lim_{m\to\infty} \int_{\Theta \times T^*_{-i}} \sum_{a_{-i} \in A_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i(\theta, a'_i, a_{-i})\right] \sigma_{-i}(\theta, t_{-i})[a_{-i}] \, \kappa^*_{t_{i,m}}[(\theta, t_{-i})] \\
\geq &\lim_{m\to\infty} \sum_{(\theta, a_{-i}) \in \Theta \times A_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i(\theta, a'_i, a_{-i})\right] \kappa^*_{t_{i,m}}\left[\{\theta\} \times U^{a_{-i}}_{-i}\right] \\
= &\sum_{(\theta, a_{-i}) \in \Theta \times A_{-i}} \left[u_i(\theta, a_i, a_{-i}) - u_i(\theta, a'_i, a_{-i})\right] \kappa^*_{t_i}\left[\{\theta\} \times U^{a_{-i}}_{-i}\right] \geq 0 \qquad (14)
\end{aligned}$$

where the equality follows from (10) in Claim 1 and the last inequality follows from (12) and (13). By (14), for sufficiently large m, $a_i \in BR_i(\pi_m, \varepsilon)$ where $\pi_m \in \Delta\left(\Theta \times T^*_{-i} \times A_{-i}\right)$ is defined for any $(\theta, a_{-i}) \in \Theta \times A_{-i}$ and any measurable set $E \subset T^*_{-i}$ as

$$\pi_m[(\theta, a_{-i}) \times E] = \int_E \sigma_{-i}(\theta, t_{-i})[a_{-i}] \, d\kappa^*_{t_{i,m}}(\theta, t_{-i}).$$

Note that π_m is valid for $t_{i,m}$ by (11). Hence, $a_i \in S^\infty_i[t_{i,m}, \varepsilon]$ for sufficiently large m.■

A.2.1 Proof of Claim 1

Claim 1.

$$\lim_{m\to\infty} \kappa^*_{t_{i,m}}[\Theta \times M_i] = \kappa^*_{t_i}[\Theta \times M_i] = 0. \qquad (15)$$

$$\lim_{m\to\infty} \kappa^*_{t_{i,m}}\left[\{\theta\} \times U^{a_{-i}}_{-i}\right] = \kappa^*_{t_i}\left[\{\theta\} \times U^{a_{-i}}_{-i}\right]. \qquad (16)$$

Proof. First, we prove (15). By the definition of $B_{i,\infty}$, $t_i \in B_{i,\infty}$ implies

$$\kappa^*_{t_i}[\Theta \times M_i] = 0. \qquad (17)$$

By Proposition 2 in WY, $\Theta \times M_i$ is closed in $\Theta \times T^*_{-i}$. Hence,

$$0 \leq \lim\inf_{m\to\infty} \kappa^*_{t_{i,m}}[\Theta \times M_i] \leq \lim\sup_{m\to\infty} \kappa^*_{t_{i,m}}[\Theta \times M_i] \leq \kappa^*_{t_i}[\Theta \times M_i] = 0.$$

where the last inequality follows from Lemma 4. Hence, (15) holds.

Second, we prove (16). Since $\{\theta\} \times U_{-i}^{a_{-i}}$ is open in $\Theta \times T_{-i}^{*}$, by Lemma 4,

$$\liminf_{m\to\infty} \kappa_{t_{i,m}}^{*}\left[\{\theta\} \times U_{-i}^{a_{-i}}\right] \geq \kappa_{t_i}^{*}\left[\{\theta\} \times U_{-i}^{a_{-i}}\right]. \tag{18}$$

Moreover, $(\Theta \times M_i) \cup \left(\{\theta\} \times U_{-i}^{a_{-i}}\right)$ is closed, because $\{\theta'\} \times U_{-i}^{a'_{-i}}$ is open for any $\theta' \in \Theta$ and $a'_{-i} \in A_{-i}$ and

$$(\Theta \times M_i) \cup \left(\{\theta\} \times U_{-i}^{a_{-i}}\right) = \left(\Theta \times T_{-i}^{*}\right) \setminus \left[\bigcup_{\theta'\neq\theta, a'_{-i}\neq a_{-i}} \left(\{\theta'\} \times U_{-i}^{a'_{-i}}\right)\right].$$

Then,

$$\begin{aligned}
\limsup_{m\to\infty} \kappa_{t_{i,m}}^{*}\left[\{\theta\} \times U_{-i}^{a_{-i}}\right] &\leq \limsup_{m\to\infty} \kappa_{t_{i,m}}^{*}\left[(\Theta \times M_i) \cup \left(\{\theta\} \times U_{-i}^{a_{-i}}\right)\right] \\
&\leq \kappa_{t_i}^{*}\left[(\Theta \times M_i) \cup \left(\{\theta\} \times U_{-i}^{a_{-i}}\right)\right] \\
&= \kappa_{t_i}^{*}\left[\Theta \times M_i\right] + \kappa_{t_i}^{*}\left[\{\theta\} \times U_{-i}^{a_{-i}}\right] \\
&= \kappa_{t_i}^{*}\left[\{\theta\} \times U_{-i}^{a_{-i}}\right],
\end{aligned} \tag{19}$$

where the second inequality follows from (5) in Lemma 4 and the fact that $(\Theta \times M_i) \cup \left(\{\theta\} \times U_{-i}^{a_{-i}}\right)$ is closed; the last equality follows from (17). Finally, (18) and (19) imply (16).■

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