Efficient Firm Dynamics in a Frictional Labor Market *

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Abstract

We develop and analyze a labor search model in which heterogeneous firms operate under decreasing returns and compete for labor by posting long-term contracts. Firms achieve faster growth by offering higher lifetime wages, which allows them to fill vacancies with higher probability, consistent with recent empirical findings. The model also captures several other regularities about firm size, job flows and pay, and generates sluggish aggregate dynamics of labor market variables. In contrast to existing bargaining models, efficiency obtains on all margins of job creation and destruction, and allows a tractable characterization over the business cycle.

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1 Introduction

Search models of the labor market following the Diamond-Mortensen-Pissarides framework have traditionally treated the labor-demand side rather simplistically: Firms operate under constant returns to scale, they post vacancies to attract job seekers, and they fill these vacancies at a rate which depends on aggregate market conditions but is independent of firm characteristics (see, e.g., Rogerson et al. (2005)). Yet, recent evidence shows that empirical job-filling rates vary considerably between employers: firms expand faster not only by posting more vacancies, but especially by filling these vacancies at higher rates (see Davis et al. (2013)). Moreover, by relying on constant-returns specifications, standard search and matching models are silent about all aspects that relate to firm size, even though firm size and firm dynamics are empirically important for wages, job flows and aggregate employment.[1]

This work proposes and analyzes a new model framework to think about firm dynamics in a frictional labor market. The notion of a firm is operationalized through decreasing returns in production. The idea that different firms can achieve different hiring probabilities is captured through wage competition: Firms can publicly post long-term wage contracts to attract unemployed workers. Therefore, matching rates are not an aggregate object but are firm-specific, and growing firms can attract more applicants to their vacancies through better offers. They are willing to do so if it is increasingly costly to post additional vacancies, which arises, for example, when recruitment takes up time of the existing workers (Shimer (2010)), so that firms expand their workforce slowly over time. We argue that this feature not only generates varying job-filling rates at

[1]For example, larger firms are more productive, pay more, and have lower job flow rates (e.g., Davis et al. (1996))); younger firms have higher exit rates and pay higher wages (e.g., Haltiwanger et al. (2013), Brown and Medoff (2003)); and Moscarini and Postel-Vinay (2012) find that small and large firms contribute to the business cycle in different ways.

the micro level, but also gives rise to sensible aggregate dynamics. Particularly, key labor market variables, such as the job-finding rate, react with delay to aggregate shocks. While such sluggish adjustment is consistent with the evidence from vector autoregressions (e.g. Fujita and Ramey (2007)), it is hard to reconcile with the textbook search and matching model (Shimer (2005)). We show that a plausibly calibrated version of our model can in fact not only replicate a substantial part of the cross-firm variation of job-filling rates, together with other cross-sectional features, but that it also generates slow adjustment of aggregate job-finding rates and other desirable business-cycle properties.

To address these applied issues, we first outline our labor market theory, which is built on the theory of competitive search (Moen (1997)) extended to heterogeneous firms with rich idiosyncratic and aggregate dynamics, while maintaining full tractability. Thereby we formulate an alternative to the current workhorse model for heterogeneous firms in frictional labor markets which is based on random search with intra-firm bargaining, pioneered by Stole and Zwiebel (1996) and Smith (1999) and applied by a large body of recent work.[2] While this approach is successful in many dimensions, random matching together with bilateral intra-firm bargaining (and no commitment over future wages) generate some undesirable implications. Due to random search, the rate at which a firm fills its vacancies is independent of the firm's growth rate. Moreover, due to the absence of commitment, a worker in a growing firm usually sees his wages decline with tenure. This happens because additional employees reduce the marginal product, and renegotiations lead to lower wages for existing workers, yielding a within-firm externality. Finally, the normative implications are very different

[2]A subset of this literature considers, for example, unemployment and efficiency (Bertola and Caballero (1994), Acemoglu and Hawkins (2013), Mortensen (2009)), labor and product market regulation (Koeniger and Prat (2007), Ebell and Haefke (2009)), business cycles (e.g., Elsby and Michaels (2013), Fujita and Nakajima (2013)), and international trade and its labor market implications (Helpman and Itskhoki (2010)).

from standard (constant returns) search and from our model: the within-firm externality leads firms to hire excessively in order to depress wages of its existing workers (see e.g. Smith (1999)).[3]

Our model framework generates different positive and normative implications. Competition endows wages with an allocative element akin to that in classical economies, allowing more productive firms to hire faster by attracting more workers per job. We show analytically in a steady-state version of the model that faster firm growth comes along with higher job-filling rates.[4] We also show analytically that the qualitative properties exhibit several of the empirical connections between firm size, growth and pay.

Regarding normative implications, the decentralized economy creates and destroys jobs efficiently both on the *extensive* margins of firm entry/exit and on the *intensive* margins of firm expansion/contraction. This gives a natural efficiency benchmark, extending the insights of the competitive theory of Hopenhayn and Rogerson (1993) to an environment with frictional unemployment. The competitive element of wage posting with full commitment eliminates the within-firm externality, as it decouples a worker's earnings from the employment of other workers. It also eliminates across-firm externalities, because different firms can offer different contracts and fill vacancies at different rates which correspond to a modified Hosios condition. While this condition is at the heart of many efficiency arguments in the competitive search literature, the subtle nature of search markets does not always render it sufficient to induce constrained efficiency, es-

[3]The inefficiency cannot be corrected by an appropriate level of the bargaining power parameter satisfying the Hosios (1990) condition. Even with wage commitments, the randomness of the search process generates an across-firm externality that impedes efficiency (see Hawkins (2010)).

[4]Though this is a very natural channel, the actual proof is non-trivial. It requires the firms' value functions to be supermodular in firm size and productivity, which does not follow from standard arguments. Clearly, there are other possible explanations for the relation between firm growth and job-filling rates; see further discussion of this issue in Sections 2 and 4 of this paper.

pecially when choices along different margins interact.[5] We are not aware of a formal efficiency result for large firms operating under decreasing returns.[6]

Next to these properties, we also establish that our environment is particularly tractable, even in the presence of aggregate shocks. The main difficulty in equilibrium models with heterogeneous firms is that each firm's optimal choice depends on the distribution of other firms in the market. When workers can decide for which wage contracts to search and new firms enter every period, we find that optimal firm policies only depend on the current aggregate productivity state but not on the firm distribution, which substantially reduces computational complexity and avoids approximation techniques like those of Krusell and Smith (1998) that are usually necessary to study business cycles with heterogeneous large firms (Elsby and Michaels (2013), Fujita and Nakajima (2013)). The reason is that in equilibrium workers are indifferent between applying to new entrants or to existing firms, so that their reservation wages are tied to new firms, the value of which is determined only by the current aggregate productivity state.

The idea that firms' policy functions are jump variables with respect to aggregate shocks goes back to Pissarides (2000) for random search and to Shi (2009) and Menzio and Shi (2010, 2011) in directed search. Since there is free entry at each wage contract, job-finding rates jump together with aggregate shocks and there is no sluggishness. In our framework, most wage contracts are posted by existing firms and are not governed by free entry, which generates empirically meaningful

[5] Posting and full specification in payments upon hiring are present for example in Galenianos and Kircher (2009), but efficiency fails because of an intensive margin (search intensity) on the workers' side which is not internalized. Guerrieri (2008) introduces an intensive margin through moral hazard and finds efficiency in steady state but not out of steady state. These subtleties indicate a lack of an easily applicable general proof on which we could draw to establish efficiency in our context, and renders an efficiency proof necessary both in and out of steady state.

[6] Hawkins (2013) suggests such an outcome on the basis of a static model, but his results are complicated by the stochastic nature of the hiring process and they do not extend to dynamic settings with shocks. Menzio and Moen (2010) do not obtain efficiency because they focus on lack of commitment, and Garibaldi and Moen (2010) abstract from decreasing returns.

persistent effects for workers' job-finding rates. Since the link between firm-level dynamics and aggregate dynamics is important, we explore this feature in more detail in the quantitative section of this paper. Indeed we demonstrate that the calibrated model generates aggregate dynamics that are largely in line with the U.S. business cycle. As this makes our model potentially useful for policy analysis, we conduct an initial exploration of the effects of hiring subsidies on labor market dynamics. Somewhat surprisingly, we find that these subsidies, especially when implemented in a counter-cyclical manner, have a destabilizing effect on the labor market since they foster more labor reallocation during recessions.

Our work describes the recruitment behavior of firms competing for unemployed workers. One could envision additionally competition for employed workers. Burdett and Mortensen (1998), Postel-Vinay and Robin (2002), and Moscarini and Postel-Vinay (2013) explore this in random search environments, but the complexity of these models makes it difficult to study firm dynamics, as firms are usually assumed to face neither idiosyncratic nor aggregate shocks.[7] In the competitive-search literature, job-to-job movements have been considered by Shi (2009), Menzio and Shi (2010, 2011), Garibaldi and Moen (2010) and recently Schaal (2010). Except for the last contribution, firm size in these models is not restricted by the operated technology, circumventing considerations induced by the difference between average and marginal product. Schaal (2010) differs from ours by assuming linear recruitment costs, which have the implication that firms immediately jump to their desired sizes, they are indifferent between all submarkets and hence face identical job-filling rates, and there is no aggregate sluggishness.

[7] Moscarini and Postel-Vinay (2013) do allow for aggregate shocks, but their requirement of rank-preserving hiring prevents the study of firm entry and firm-specific shocks. To our knowledge, the only model that explicitly focuses on firm dynamics is Lentz and Mortensen (2010), which combines decreasing returns with on-the-job search, but again it has no idiosyncratic or aggregate shocks.

The next section builds intuition for our model in a simplified version without productivity shocks. This allows for a teachable representation, it establishes the most important insights for the dynamics of employment, job-filling rates and wage offers analytically, and it demonstrates the efficiency of the decentralized allocation. In Section 3, we lay out the notationally more complex analysis that takes account of aggregate and idiosyncratic shocks, we prove the equivalence between the efficient and the decentralized allocations and show that the framework retains tractability. Beyond the theoretical contribution, we consider a calibrated numerical example in Section 4 to address the quantitative features of our model. Section 5 concludes the paper. All proofs and some extensions are relegated to the Appendix.

2 A Stationary Model of Firm Growth

2.1 The Environment

The model is set in discrete time and we consider a stationary environment. That is, there are neither idiosyncratic nor aggregate shocks in this section. The labor market within a given period operates in three stages. First, new firms enter and draw their productivity. Second, production and search activities take place. Third, vacancies and unemployed workers are matched, and a fraction of workers leave their firms. Afterwards some firms exit, and the next period starts. The following explains each part in turn.

The economy consists of a continuum of workers and firms. The mass of workers is normalized to one. Each worker is infinitely-lived, risk-neutral, and discounts future income with factor $\beta < 1$. A worker supplies one unit of labor per period when employed and receives income $b \geq 0$ when unemployed. Only unemployed workers search for employment, so there are no job-to-job transitions. On the other side of the labor market is an endogenous mass of firms. Firms are large relative to workers, in the sense that each active firm employs a continuum of workers. Firms are also risk neutral and have the same discount factor β.

An entrant firm pays setup cost $K > 0$ to start production. At this point it draws productivity x with probability $\pi_0(x)$ from the finite set $x \in X$. In each period, a firm produces output $xF(L)$ with $L \geq 0$ workers, where F is a twice differentiable, strictly increasing and strictly concave function satisfying $F'(0) = \infty$ and $F'(\infty) = 0$. Firms die with exogenous probability $\delta > 0$, in which case all workers are laid off into unemployment. Furthermore, each employed worker separates from the firm with exogenous probability $s \geq 0$. Thus, in this section, a firm's productivity stays constant throughout its life, and any worker's retention probability is exogenous at $\varphi \equiv (1-\delta)(1-s)$.

Search for new hires is a costly activity. A firm with workforce L and productivity x that posts V vacancies incurs recruitment cost $C(V, L, x)$. Apart from twice differentiability, we assume that a firm's output net of recruitment costs is strictly increasing in (L, x) and strictly concave in (V, L). In particular, this requires that C is strictly convex in V. Popular functional form are

$$C(V, L, x) = xF(L) - xF(L - hV) + k(V) \quad \text{or} \quad C(V, L, x) = \frac{c}{1+\gamma}\left(\frac{V}{L}\right)^{\gamma} V \; . \tag{1}$$

In the first specification, $k(V)$ captures some convex monetary costs (see e.g. Cooper et al. (2007)) and hV captures labor input in recruitment (see e.g. Shimer (2010)). Even in the absence of monetary costs and despite linearity of the labor input, this leads to convex vacancy costs because of decreasing returns in production.[8] The second, constant-returns specification, which is borrowed from

[8] Clearly no more workers can be engaged in hiring than are present at the firm. To get the hiring process started for entrant firms, we need to assume that a new firm is endowed with initial labor input of the entrepreneur L_e so that the actual labor input is $\tilde{L} = L_e + L$. Recruitment activities are then constrained by $hV \leq L + L_e$, and Inada conditions on F ensure that this constraint never binds. A similar adjustment is needed for the second specification

Merz and Yashiv (2007), assumes that the average cost per vacancy increases in the vacancy rate (i.e. vacancies divided by employment) and it also allows larger firms to hire a given number of workers at lower costs. In either setting, firms cannot instantaneously grow large simply by posting enough vacancies at constant marginal cost. For some proofs of cross-sectional relationships derived below (Proposition 1 and subsequent corollaries), we focus on cost functions such as those in (1) which satisfy the following property:

$$\text{(C)} \qquad C_{12} \leq 0 \ , \ C_{13} \geq 0 \ , \quad \text{and} \ \ C_{12}C_{13} + C_{11}[F' - C_{23}] \geq 0 \ .$$

In order to attract workers, a recruiting firm announces a flat flow wage income w to be paid to its new hires for the duration of the employment relation. The assumption that the firm offers the same wage to all its new hires turns out not to entail a restriction; see the discussion following equation (6) below. Further, because of risk neutrality, only the net present value that a firm promises to the worker matters. Flat wages are one way of delivering these promises.[9]

Unemployed workers direct their job search towards the most attractive offers: they can observe all wage offers and choose for which wage to search. At any wage, job seekers and vacancies are matched according to a matching function. In particular, a firm fills its vacancies with probability m only if it offers a wage that attracts $\lambda(m)$ unemployed job seekers per vacancy.[10] Standard assumptions on the matching function guarantee that this function is twice differentiable, strictly

in (1) to avoid division by zero at entrant firms (see Section 4).

[9]This is a theory of the present value of offered wages. Constant wages might be viewed as the limiting case of risk-neutral firms and risk-averse workers, as risk-aversion vanishes. But other payment patterns are conceivable; for further discussion about this issue, see Section 3.4.

[10]Note that we adopt the standard assumption in the literature on large firms in search models that each job has its own matching probability, i.e., applicants from one job cannot be hired at another job in the same firm, which arises, for example, if different jobs require different qualifications. Only few papers explore the idea that workers are literally identical and can be hired for another job than the one they applied for (see e.g. Burdett et al. (2001), Hawkins (2013), Lester (2010)).

increasing and strictly convex in m, with $\lambda(0) = 0$, $\lambda'(0) \geq 1$ and $\lambda'(1) = \infty$.[11] It is increasing since firms achieve a higher matching probability only if more workers are searching for their vacancies. It is convex since it becomes increasingly difficult to improve matching prospects any further when more workers are attracted to the job. The worker's matching probability is $m/\lambda(m)$, which is strictly decreasing.

To understand what wage $w(m)$ a firm has to offer in order to achieve matching probability m, notice that in a stationary environment an unemployed worker who is seeking for a particular wage in one period is willing to search for that wage in every period. Let U denote the discounted present value from such job search. It is given by the following asset value equation:[12]

$$(1-\beta)U = b + \underbrace{\frac{m}{\lambda(m)}\beta(1-\delta)\frac{w(m)-(1-\beta)U}{1-\beta\varphi}}_{\equiv\rho} \quad . \tag{2}$$

It states that the flow value of unemployment equals the current period unemployment income b together with an option value from searching, denoted by ρ. The search value is the probability of finding a job multiplied with the worker's discounted job surplus. Since workers have a choice where to search for a job, their flow value from unemployment must be equal in all markets that attract workers. Therefore, ρ is a global value that is common to all markets, which means that a firm has to offer the following wage to achieve matching rate

[11] Function λ is simply the inverse of the standard reduced-form matching function $\tilde{m} : [0,\infty) \to [0,1)$ that maps the realized unemployed-vacancy ratio $\tilde{\lambda}$ into the hiring probability. Typically, $\tilde{m}$ is assumed to be strictly increasing and strictly concave, and $\tilde{m}(\tilde{\lambda}) \leq \min(1,\tilde{\lambda})$ guarantees that $\tilde{m}'(0) \leq 1$. Therefore, we can define $\lambda(m) = \tilde{m}^{-1}(m)$, and the properties in the text follow.

[12] Bellman equations for employed and unemployed workers are $E = w + \beta[\varphi E + (1-\varphi)U]$ and $U = b+\beta[m\lambda(m)^{-1}(1-\delta)E+(1-m\lambda(m)^{-1}(1-\delta))U]$. Equation (2) follows by substituting the first into the second.

$m > 0$:

$$w(m) = b + \rho + \frac{1-\beta\varphi}{\beta(1-\delta)}\frac{\lambda(m)}{m}\rho \ . \tag{3}$$

This relation says that a firm can only recruit workers when its wage offer matches the workers' unemployment value $(1-\beta)U = b + \rho$ plus a premium which is needed to attract workers to vacancies with job-filling rate m. This premium is increasing in m, which is a crucial insight. The relationship between job-filling rates and wage offers is standard in the competitive search literature.

2.2 The Firms' Recruitment Policies

Consider the problem of a firm that takes the search value of unemployed workers and the associated relationship (3) as given. Later, the search value will be determined as an equilibrium object that depends on the number of firms and their wage offers.

Let $J^x(L, W)$ be the present profit value of a firm that has productivity x, employs L workers and is committed to a wage bill of W. An entrant firm's profit value is then $J^x(0, 0)$. The firm's recruitment choice involves deciding the number of posted vacancies V as well as the job-filling probability m, which requires a wage offer of $w(m)$. Its recursive profit maximization problem is expressed as

$$\begin{aligned} J^x(L,W) \quad &= \max_{(m,V)\in[0,1]\times\mathbb{R}_+} xF(L) - W - C(V,L,x) + \beta(1-\delta)J^x(L_+,W_+) \ , \\ &\text{s.t. } L_+ = L(1-s) + mV \ , \ W_+ = W(1-s) + mw(m)V \ . \end{aligned} \tag{4}$$

The first line reflects the value of output net of wage and hiring costs, plus the discounted value of continuation with an adjusted workforce and its associated wage commitment. The second line captures that employment next period consists of the retained workers and the new hires. For the wages, since separations are random they reduce the wage bill proportionally, and new commitments are added for the new hires.

This problem can be simplified by noting that wages are commitments that have to be fulfilled as long as the worker does not separate, irrespective of future recruitment decisions. This has the implication that $J^x(L, W) = J^x(L, 0) - W/(1-\beta\varphi)$, which eliminates the wage bill as a state variable, so that (4) readily yields

$$\begin{aligned} J^x(L,0) \quad &= \quad \max_{(m,V)\in[0,1]\times\mathbb{R}_+} xF(L) - C(V,L,x) - D(m)V + \beta(1-\delta)J^x(L_+,0) \ , \\ &\quad\ \ \text{s.t. } L_+ = L(1-s) + mV \ , \end{aligned} \tag{5}$$

where $D(m) \equiv mw(m)\beta(1-\delta)/(1-\beta\varphi)$ captures the cost of increasing the matching probability by raising wage costs. Note that D is increasing and strictly convex since the matching function $\lambda(m)$ has these properties. Problem (5) makes it readily apparent that a firm has two channels to hire workers in a given period. It can increase the number of vacancies and associated costs C, or it can increase the filling rate of each job and associated costs D. The optimality conditions for the control variables in (5) are derived rigorously in the Appendix, but we provide some intuition here for the main trade-offs. The optimal choices for the number of vacancies and their matching probability are governed by one intratemporal and one intertemporal optimality condition.

Regarding the intratemporal optimality condition, consider a firm that aims to hire H workers in this period. It faces the problem of choosing the number of vacancies and the job-filling probability to minimize costs $C(V, .) + D(m)V$ subject to $H = mV$. The first-order condition for this problem is

$$C_1(V,L,x) = D'(m)m - D(m) = \rho[m\lambda'(m) - \lambda(m)] \ . \tag{6}$$

This links the marginal recruitment costs to the marginal increase in wage costs associated with increasing the job-filling probability.

Relationship (6) offers a number of insights. It defines the optimal policy for vacancy postings $V = V^x(m, L)$ as a function of the job-filling rate (and firm size). Because of convex recruitment costs, this policy function is increasing in m; thus, vacancy postings and job-filling rates are complementary tools in the firm's recruitment strategy. This captures the basic stylized fact highlighted by Davis et al. (2013) that firms use both more vacancies as well as higher job-filling rates to achieve faster growth.[13] In contrast, under constant marginal recruitment costs ($C_1(V, L, x) = c$), as assumed in much of the literature, the job-filling rate would be constant and independent of firm characteristics, while all employment adjustment is instantaneous and is achieved through the number of vacancies. Finally, note that equation (6) balances the wage costs for new hires against recruitment costs at a unique point, which shows why a firm would not want to offer different wages at a given point in time even if this were permissable.

The firm also decides how to structure hiring over time. This is governed by an intertemporal optimality condition which reads

$$xF'(L_+) - C_2(V_+, L_+, x) - b - \rho = \frac{\rho}{\beta(1-\delta)} \Big[\lambda'(m) - \beta\varphi\lambda'(m_+)\Big] \ . \qquad (7)$$

Here L_+, V_+, and m_+ are employment, vacancy postings and the job-filling rate in the next period. The left-hand side of (7) gives the marginal benefit of a higher workforce in the next period. If this is high, then the firm rather hires more now than to wait and hire next period, as expressed by the right-hand side which is increasing in the current job-filling rate m and decreasing in m_+. In particular, a more productive firm wants to achieve fast growth by offering a more attractive contract now rather than later, thus raising the current job-filling rate. Equation

[13]The first equation in (6) suggests that this argument holds in a broader class of models in which firms can influence job-filling rates. In our model, job-filling rates are increased via higher wage offers which reflects the allocative role of wages in the labor market.

(7) implicitly defines the optimal job-filling policy $m^x(L)$. Starting from $L = 0$, this determines the firm's growth path through $L_+ = L(1-s) + m^x(L)V$, where $V = V^x(m^x(L), L)$ comes from the static optimality condition (6).

An illustration how a firm grows over time is provided in Figure 1 which shows the phase diagram in (L, m) space for the firm's problem with recruitment costs $C(V, L, x) = xF(L) - xF(L - hV) + cV$ for which the optimality conditions become especially tractable.[14] Initially the firm is small and the optimal job-filling rate exceeds the optimal long-run rate m^*. This rate is the firm's policy after it converges to its long-run optimal size $L^* > 0$ where it only conducts replacement hiring. The downward-sloping saddle path depicts the firm's policy function $m^x(L)$ and describes the adjustment process to the long-run optimal size, along which the firm spreads recruitment costs over time. This is in contrast to a model with linear recruitment costs ($C(V, L, x) = cV$) in which firms would jump directly to (L^*, m^*). In terms of comparative statics, this example also shows that the stationary firm size and the job-filling rates along the transition depend positively on x: a more productive firm grows larger and offers higher lifetime wages on its transition to the long-run employment level. The following proposition and its corollaries provide broader comparative statics results. The job-filling rate is linked via (3) to the wage offer, so that the findings carry over to the net present value of wages to new hires.[15]

Proposition 1: *Consider recruitment cost functions satisfying property (C). The firm's value function $J^x(L, W)$ is strictly increasing and strictly concave*

[14] In Lemma 3 of the Appendix we show that equations (6) and (7) simplify to only one equation linking m_t and m_{t+1} which is independent of L_t. This equation has a unique long-run job-filling probability $m^* > 0$ if h is low enough, and m_t converges to m^* from any initial value $m_0 > 0$. Employment adjusts according to $L_{t+1} = L_t(1-s) + m_t V^x(m_t, L_t)$. Using (6), it is easy to see that the curve $L_{t+1} = L_t$ is downward-sloping in (L, m) space, so that the saddle path lies above this curve when $L_t < L^*$.

[15] These characterization results depend crucially on the supermodularity of the value function, which renders this proof non-trivial. While standard techniques (Amir (1996)) can be applied when the cost function is independent of firm size and productivity, this is not true in general, as we discuss in the Appendix.

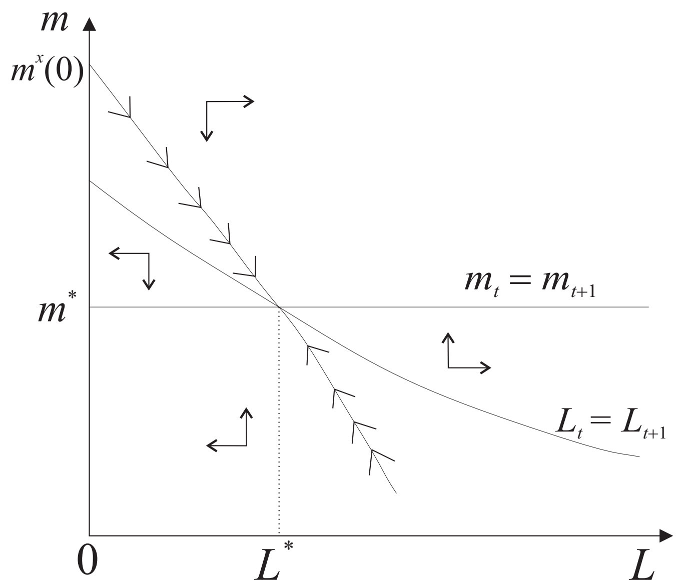

Figure 1: The firm's optimal recruitment policy follows the declining saddle path.

in its workforce L, strictly increasing in productivity x, strictly supermodular in (x, L) and decreasing in the worker's search value ρ. The job-filling rate $m^x(L)$ is strictly increasing in productivity x and strictly decreasing in the workforce L. Posted vacancies $V^x(m, L)$ are increasing in L and strictly increasing in the desired job-filling rate m.

Since these results hold for any search value ρ, they also apply when this value is determined in general equilibrium. These results imply relationships between size, productivity, pay, and hiring:

Corollary 1: *Consider recruitment cost functions satisfying property (C). Conditional on size, more productive firms pay higher lifetime wages and have a higher job-filling rate. Conditional on productivity, younger/smaller firms pay higher lifetime wages and have a higher job-filling rate.*

In the Appendix we also prove the following connection to firm growth rates.

Corollary 2: *If recruitment costs are given by either specification in (1) with parameter h sufficiently small, more productive firms have a higher growth rate,* *conditional on size; and larger/older firms have a lower growth rate, conditional on productivity.*

While it already follows from (6) that vacancy postings and job-filling rates are positively related, the two corollaries link these policies to the firm's growth rate. They point out that job-filling rates and firm growth rates are positively correlated, depending positively on x and negatively on L. This cross-sectional relationship has been highlighted recently by Davis et al. (2013), and we further explore in Section 4 how well our model captures this quantitatively. Furthermore, since higher job-filling rates are directly associated with higher earnings for new hires, the two corollaries also imply that faster-growing firms offer higher lifetime wages. Belzil (2000) documents such patterns after controlling for size and worker characteristics; he shows that wages, particularly those of new hires, are positively related to a firm's job creation. Our findings that younger firms grow faster (conditional on survival) and pay higher wages (to workers with the same characteristics) are consistent with the evidence (see Haltiwanger et al. (2013) and Brown and Medoff (2003)). Moreover, a positive wage-size relation emerges in our model if the dispersion in productivity is large enough.[16]

2.3 Firm Entry, General Equilibrium, and Efficiency

Free entry of firms implies that no entrant makes a positive profit, that is,

$$\sum_{x \in X} \pi(x) J^x(0,0) \leq K \ , \tag{8}$$

with equality if entry is positive. This condition implicitly pins down the worker's job surplus ρ and therefore the relationship between wages and job-filling rates. In a stationary equilibrium, a constant mass of N_0 firms enters the market in

[16]We note that enough productivity dispersion is also required in models with intra-firm bargaining, and even more so because wages of *all* workers decline in a growing firm.

every period, so that there are $N_a = N_0(1-\delta)^a$ firms of age a in any period. Let $(L_a^x, m_a^x, V_a^x, w_a^x)_{a\geq 0}$ be the employment/recruitment path for a firm with productivity x. Then, a firm of age a has L_a^x employed workers, and $\lambda(m_a^x)V_a^x$ unemployed workers are searching for jobs with offered wage w_a^x. Therefore, the mass of entrant firms N_0 is uniquely pinned down from aggregate resource feasibility:

$$1 = \sum_{a\geq 0} N_0(1-\delta)^a \sum_{x\in X} \pi(x)[L_a^x + \lambda(m_a^x)V_a^x] \ . \tag{9}$$

This equation says that the unit mass of workers is either employed or unemployed. We now define a stationary equilibrium.

Definition: *A stationary competitive search equilibrium is a list* $\left(\rho, N_0, (L_a^x, m_a^x, V_a^x, w_a^x)_{x\in X, a\geq 0}\right)$ *with the following properties. Unemployed workers' job search strategies maximize utility: (3) holds for all* (w_a^x, m_a^x)*. Firms' recruitment policies are optimal:* $(L_a^x, m_a^x, V_a^x)_{a\geq 0}$ *solve (5) for all* $x \in X$*. There is free entry of firms: (8) and* $N_0 \geq 0$ *hold with complementary slackness. Aggregate resource feasibility (9) holds.*

Since firms' behavior has already been characterized, it remains to explore equilibrium existence and uniqueness.

Proposition 2: *A stationary competitive search equilibrium exists and is unique. There is strictly positive firm entry provided that* K *is sufficiently small.*

The previous section already outlined that this model generates sensible relationships between productivity, size, growth, and job-filling rates. It is relevant to understand whether these patterns are actually socially efficient, especially since existing models with intra-firm bargaining always entail inefficiencies, as discussed in the introduction. To this end, consider a social planner who decides at each point in time about firm entry, vacancy postings and job-filling rates for all firms. The planner takes as given the numbers of firms that entered in some earlier period, as well as the employment stocks of all these firms. Formally, the planner's state vector is $\sigma = (N_a, L_a^x)_{a\geq 1, x\in X}$ where N_a is the mass of firms of age $a \geq 1$, and L_a^x is employment of a firm with productivity x and age a. The planner maximizes the present value of output net of opportunity costs of employment and net of the costs of entry and recruitment, subject to the economy's resource constraint. With $\sigma_+ = (N_{a,+}, L_{a,+}^x)_{a\geq 1, x\in X}$ denoting the state vector in the next period, the recursive formulation of the social planning problem is

$$\begin{aligned} S(\sigma) \;=\; & \max_{N_0,(V_a^x,m_a^x)_{a\geq 0}} \left\{ \sum_{a\geq 0} N_a \sum_{x\in X} \pi(x) \Big[xF(L_a^x) - bL_a^x - C(V_a^x, L_a^x, x) \Big] \right\} \\ & \qquad -KN_0 + \beta S(\sigma_+) \qquad (10) \\ & \text{s.t. } L_0^x = 0,\ L_{a+1,+}^x = (1-s)L_a^x + m_a^x V_a^x,\ a \geq 0,\ x \in X \ , \\ & N_{a+1,+} = (1-\delta) N_a \ ,\ a \geq 0 \ , \\ & \sum_{a\geq 0} N_a \sum_{x\in X} \pi(x) \Big(L_a^x + \lambda(m_a^x) V_a^x \Big) \leq 1 \ . \end{aligned}$$

We say that a solution to problem (10) is *socially optimal*.

Proposition 3: *The stationary competitive search equilibrium is socially optimal.*

The efficiency of equilibrium can be linked to a variant of the well-known Hosios (1990) condition.[17] It says that efficient job creation requires that the firm's surplus share for the marginal vacancy is related to the elasticity of the matching function. Write the worker's search value $\rho = \frac{m}{\lambda(m)} S^w$ as the product between the match probability and the worker's job surplus S^w. Then, equation (6) can be rewritten as

$$C_1(V, L, x) = \frac{1 - \varepsilon_{m,\lambda}}{\varepsilon_{m,\lambda}} m S^w \ ,$$

where $\varepsilon_{m,\lambda} = \frac{\lambda(m)}{\lambda'(m)m} \in [0, 1]$ is the matching-function elasticity.

[17] See also the Hosios condition in a large-firm model with intra-firm bargaining in Hawkins (2010).

3 Productivity Shocks and Firm Dynamics

In this section we show that our framework can be extended to include much richer dynamics, both at the firm level and in the aggregate, while retaining tractability and efficiency. We include both firm-specific and aggregate productivity shocks to explore not only two margins of job creation (firm entry and firm growth), but also the two margins of job destruction (endogeneous firm exit and firm contraction). This extension allows us to study in the next section to which extent the model can quantitatively account not only for the micro-level heterogeneity in the firms' recruitment behavior, but also for sluggish aggregate dynamics.

Assume now that output of a firm with L workers is $xzF(L)$ where $x \in X$ is idiosyncratic productivity and $z \in Z$ is aggregate productivity. Both x and z follow Markov processes on finite state spaces X and Z with respective transition probabilities $\pi(x_+|x)$ and $\psi(z_+|z)$. An entrant firm pays fixed cost $K(z)$, possibly dependent on the aggregate state, and draws an initial productivity level $x_0 \in X$ with probability $\pi_0(x_0)$. For a firm of age $a \geq 0$, let $x^a = (x_0, \ldots, x_a) \in X^{a+1}$ denote the history of idiosyncratic productivity, and let $z^t = (z_0, \ldots, z_t)$ be the history of aggregate states at time t with corresponding probability $\psi(z^t)$.

We assume that an active firm incurs a fixed operating cost $f \geq 0$ per period, which is required to obtain a non-trivial exit margin. In this section we are as agnostic as possible about the recruitment cost function; we only assume that $C(V, L, xz)$ is strictly increasing and convex in posted vacancies. Firms exit with exogenous probability $\delta_0 \geq 0$ which is a lower bound for the actual exit rates $\delta \geq \delta_0$. Similarly, workers quit a job with exogenous rate $s_0 \geq 0$ which provides a lower bound for the actual separation rates $s \geq s_0$.

The timing within each period is analogous to the previous section. *First*, aggregate and idiosyncratic productivities are revealed and new firms enter. *Second*, firms produce and they decide about hires, layoffs, and possibly about exiting at the end of the period. And *third*, workers and firms are matched. We start to describe and characterize the planning problem before we show its equivalence to a competitive search equilibrium.

3.1 The Planning Problem

The planner decides at each point in time about firm entry and exit, layoffs and hires (i.e. vacancy postings and matching probabilities) for all firm types, knowing that matching probability m requires $\lambda(m)$ unemployed workers per vacancy. In a given aggregate history z^t, we denote by $N(x^a, z^t)$ the mass of firms of age a with idiosyncratic history x^a. Similarly, $L(x^a, z^t)$ is the employment stock of any of these firms. At every history node z^t and for every firm type x^a, the planner decides an exit probability $\delta(x^a, z^t) \geq \delta_0$, a separation rate $s(x^a, z^t) \geq s_0$, vacancy postings $V(x^a, z^t) \geq 0$, and a matching probability $m(x^a, z^t)$.[18] The numbers of firm types change between periods t and $t+1$ according to

$$N(x^{a+1}, z^{t+1}) = [1 - \delta(x^a, z^t)]\pi(x_{a+1}|x_a)\psi(z_{t+1}|z_t)N(x^a, z^t) \ , \tag{11}$$

and the workforce at any of these firms adjusts to

$$L(x^{a+1}, z^{t+1}) = [1 - s(x^a, z^t)]L(x^a, z^t) + m(x^a, z^t)V(x^a, z^t) \ . \tag{12}$$

At time $t = 0$, the planner takes as given the numbers of firms that entered the economy in some earlier period, as well as the employment stock of each of

[18] To save on notation, we do not allow the planner to discriminate between workers with different firm tenure. Given that there is no learning-on-the-job, there is clearly no reason for the planner to do so. Nonetheless, the competitive search equilibrium considered in Section 3.4 allows firms to treat workers in different cohorts differently, which is necessary because firms offer contracts sequentially and are committed to these contracts. See the proof of Proposition 5 for further elaboration of this issue.

these firms. Hence, the state vector at date 0 is summarized by the initial firm distribution $(N(x^a, z^0), L(x^a, z^0))_{a \geq 1, x^a \in X^{a+1}}$. In a given history z^t, the planner also decides the mass of new entrants $N_0(z^t) \geq 0$, so that

$$N(x_0, z^t) = \pi_0(x_0) N_0(z^t) \text{ and } L(x_0, z^t) = 0 \ . \tag{13}$$

The sequential planning problem is

$$\max_{\delta, s, V, m, N_0} \sum_{t \geq 0, z^t} \beta^t \psi(z^t) \Bigg\{ -K(z_t) N_0(z^t) + \sum_{a \geq 0, x^a} N(x^a, z^t) \Big[x_a z_t F(L(x^a, z^t)) \\ -bL(x^a, z^t) - f - C(V(x^a, z^t), L(x^a, z^t), x_a z_t) \Big] \Bigg\} \ , \tag{14}$$

subject to the dynamic equations for N and L, namely (11), (12) and (13), and subject to the resource constraints, for all $z^t \in Z^{t+1}$,

$$\sum_{a \geq 0, x^a} N(x^a, z^t) \Big[L(x^a, z^t) + \lambda(m(x^a, z^t)) V(x^a, z^t) \Big] \leq 1 \ . \tag{15}$$

This constraint says that the labor force (employment plus unemployment) cannot exceed the given unit mass of workers. We summarize a solution to the planning problem by a vector $(\mathbf{N}, \mathbf{L}, \mathbf{V}, \mathbf{m}, \mathbf{s}, \boldsymbol{\delta})$, with $\mathbf{N} = (N(x^a, z^t))_{a, t \geq 0}$ and similar notation for the other variables.

3.2 Characterization of the Planning Solutions

We show that there is a convenient characterization of a planning solution which says that hiring, layoff and exit decisions follow a recursive equation at the level of the individual firm. Specifically, for any existing firm, the planner maximizes the social value of the firm, taking into account the social value of each worker tied to the firm. This social worker value is given by the multiplier on the resource constraint (15) which we denote by $\mu(z^t)$ and which generally depends on the initial firm distribution and on the full state history z^t.

A particularly powerful characterization can be obtained under the provision that firm entry is positive in all states of the planning solution. When this is the case, the social value of a worker (and thus firm-level value and policy functions) depend only on the current aggregate state but are independent of the state history and of the firm distribution. This is our aggregate-arbitrage property, which relates to the concept of block recursivity introduced by Shi (2009) and Menzio and Shi (2010, 2011).

To gain intuition for the independence from the distribution of existing firms, envision any period in which the planner can assign unemployed workers either to existing firms or to new firms. If there are many existing firms, there are fewer workers left to be assigned to new firms. Nevertheless, the social value of any worker that is assigned to a new firm does not change: Each new firm has an optimal size, and if less workers are assigned to new firms, then proportionally less new firms will be created, leaving the marginal value of each worker unchanged. Therefore, efficient hiring by existing firms requires their marginal social benefit of hiring to be equal to the social benefit at the new firms which depends on the aggregate state alone.

To see the independence of value functions from the firm distribution formally, suppose there are n aggregate states $z \in Z = \{z_1, \ldots, z_n\}$, and let μ_i be the social value of a worker in state z_i. Write $M = (\mu_1, \ldots, \mu_n)$ for the vector of social values. Let $G(L, x, i; M)$ be the social value of a firm with employment stock L, idiosyncratic productivity x and aggregate productivity z_i, satisfying the Bellman equations

$$\begin{aligned} G(L,x,i;M) = \max_{\delta,s,V,m} \quad & xz_iF(L) - bL - f - \mu_i[L + \lambda(m)V] - C(V, L, xz_i) \\ & + \beta(1-\delta)E_{x,i}G(L_+, x_+, i_+; M) \ , \end{aligned} \tag{16}$$

where maximization is subject to $L_+ = (1-s)L+mV$, $\delta \in [\delta_0, 1]$, $s \in [s_0, 1]$, $m \in [0, 1]$ and $V \geq 0$. The interpretation of this equation is rather straightforward. A firm's social value encompasses flow output net of the opportunity cost of employment, net of fixed costs and recruitment costs, and net of the social cost of workers tied to the firm in this period; these workers include the current workforce L and also λV unemployed workers who are assigned to this firm.

Positive entry in all aggregate states requires that the expected social value of a new firm is equal to the entry cost,

$$\sum_{x \in X} \pi_0(x) G(0, x, i; M) = K(z_i) \ . \tag{17}$$

This characterization of planning solutions by (G, M) is particularly helpful for numerical applications. Despite considerable firm heterogeneity, the model can be solved by a recursive problem on a low-dimensional state space (16) and the (simultaneous) solution of a finite-dimensional fixed-point problem (17). Importantly, the distribution of firms is irrelevant for this computation. After the corresponding policy functions have been calculated, firm entry follows as a residual of the economy's resource constraint and *does depend* on the distribution of existing firms: in every period with aggregate state i each existing firm with productivity x and size L attracts $V(L, x, i)\lambda(m(L, x, i))$ job seekers according to the policy functions, while a number $N_0(z^t)$ of new firms enter to absorb the remaining job seekers. Since job-finding prospects differ between firms, the aggregate job-finding rate therefore also depends on the firm-size distribution, as does the evolution of aggregate employment, output and job flows. As we see in the next section, these aggregate variables in fact adjust with delay to aggregate shocks. Because of the dependence of N_0 on the distribution of employment among existing firms, it cannot generally be guaranteed that the planning solution has positive entry in all state histories. Therefore, this property can only be checked ex-post in simulations of the model. Analytically, we prove that any solution of (16)–(17) which gives rise to positive entry in all state histories describes indeed a solution to the planner's problem. We also find that a unique solution of these equations exists for small aggregate shocks.

Proposition 4:

(a) *Suppose that a solution of (16) and (17) exists with associated allocation* $\mathbf{A} = (\mathbf{N}, \mathbf{L}, \mathbf{V}, \mathbf{m}, \mathbf{s}, \boldsymbol{\delta})$ *satisfying* $N(z^t) > 0$ *for all* z^t*. Then* $\mathbf{A}$ *is a solution of the sequential planning problem (14).*

(b) *If* $K(z)$*,* f*, and* b *are sufficiently small and if* $z_1 = \ldots = z_n = \overline{z}$*, equations (16) and (17) have a unique solution* (G, M)*. Moreover, if the transition matrix* $\psi(z_j|z_i)$ *is strictly diagonally dominant and if* $|z_i - \overline{z}|$ *is sufficiently small for all* i*, equations (16) and (17) have a unique solution.*

3.3 Recruitment and Layoff Strategies

The reduction of the planning problem to (16) permits a straightforward characterization of the optimal layoff and hiring strategies. For a growing firm, it follows from the first-order conditions for m and V, similar to equation (6), that

$$C_1(V, L, xz_i) = \mu_i[m\lambda'(m) - \lambda(m)] \ . \tag{18}$$

As in the previous section, this equation implies an increasing relation between matching probabilities and the number of posted vacancies at the firm. With higher m, the planner is willing to post more vacancies at higher marginal recruiting cost. Denote the solution to equation (18) by $V = V(m, L, x, i)$, which is positive for $m > \underline{m}(L, x, i)$. The planner's optimal choice of m for firm (L, x)

in aggregate state i satisfies[19]

$$\beta(1-\delta_0)E_{x,i}\frac{dG}{dL}(L_+,x_+,i_+;M)=\mu_i\lambda'(m) \ ,$$

with $L_+ = L(1-s_0)+mV(m,L,x,i)$. Therefore, the firm hires workers, if and only if

$$\beta(1-\delta_0)E_{x,i}\frac{dG}{dL}(L(1-s_0),x_+,i_+;M)>\mu_i\lambda'(\underline{m}(L,x,i)) \ . \tag{19}$$

Conversely, the planner wants the firm to lay off workers if

$$E_{x,i}\frac{dG}{dL}(L(1-s_0),x_+,i_+;M)<0 \ . \tag{20}$$

The two conditions (19) and (20) show how the firm's strategy depends on its characteristics. Small and productive firms recruit workers and grow, whereas large and unproductive firms dismiss workers and shrink. There is also an open set of characteristics where firms do not adjust their workforce (cf. Bentolila and Bertola (1990) and Elsby and Michaels (2013)).

3.4 Decentralization

As in Section 2, a competitive search equilibrium gives rise to the same allocation as the planning solution. Consider firms that offer workers a sequence of state-contingent wages, to be paid for the duration of the match. They also commit to cohort-specific and state-contingent retention probabilities. Contracts are contingent on the idiosyncratic productivity history of the firm at age k, x^k, and on the aggregate state history z^t at time t. Formally, a contract offered by

[19]Note that $\delta=\delta_0$ and $s=s_0$ if the firm hires workers.

a firm of age a at time T takes the form

$$\mathcal{C}_a = \Big(w_a(x^k, z^t), \varphi_a(x^k, z^t)\Big)_{k>a, t=T+k-a},$$

where $w_a(x^k, z^t)$ is the wage paid to the worker in history (x^k, z^t), conditional on the worker being still employed by the firm in that instant. $\varphi_a(x^k, z^t)$, for $k > a$, is the probability of retaining the worker at the end of the period, so $1 - \varphi_a(x^k, z^t)$ is the separation probability.

In Appendix B, we describe the workers' and the firms' search problems and we define a competitive search equilibrium, analogously to the stationary model. We also prove the following welfare theorem, extending Proposition 3.

Proposition 5: *A competitive search equilibrium is socially optimal.*

It is not hard to see that a wage commitment is sufficient for a firm to implement its desired policy, even if it cannot commit to retention rates. Given risk neutrality, the firm can set the wages following any future history exactly equal to the reservation wage (i.e. the flow value of unemployment) which is the sum of unemployment income and the worker's shadow value, $b + \mu(z^t)$. It can achieve any initial transfer to attract workers through a hiring bonus. In this decentralization, the costs of an existing worker are always equal to his social value in the alternative: unemployment and search for another job. Since the flow surplus for any retained worker equals his shadow value, the firm's problem in this case coincides with the planner's problem, so that firing and exiting will be exactly up to the socially optimal level even though the firm does not commit to retention rates. Workers do not have any incentive to quit the job unilaterally, either, because they are exactly compensated for their social shadow value from searching. If the workers also cannot commit to stay, this is the unique wage policy that overcomes the commitment problem on both sides of the market and implements the socially efficient outcome. Alternatively, even a slight degree of risk aversion on the workers' side would give rise to flat wage profiles to offer insurance. This clarifies that the current model determines surplus sharing only, whereas the time path of payments depends on additional details, like the ability to commit to specific actions (see Schaal (2010) for a related point).

4 A Calibrated Example

The previous sections outlined that this model can capture important features at the micro level (e.g. varying job-filling rates) and it is tractable for studying business cycle dynamics with potentially sluggish adjustment of aggregate variables. In this section we calibrate our model to the U.S. labor market in order to investigate how well the model is able to capture the main features in the data quantitatively. We first explore the model's cross-sectional properties, showing among other results how it generates differential job-filling rates as in Davis et al. (2013). We then show that the same parameterizations give rise to aggregate sluggishness. We conclude with a short exploration of the effects of hiring credits for business-cycle stabilization.

We briefly sketch the model calibration, referring to Appendix C for details and for the parameter values. We calibrate the model at weekly frequency and choose the firm-specific productivities to match firm and employment shares in the three size classes $1-49$, $50-499$, and ≥ 500.[20] For the recruitment technology, we choose the employment-scaled form[21] $c(V) = \frac{c}{1+\gamma}(\frac{V}{L})^{1+\gamma}L$. In our benchmark calibration we take a cubic function ($\gamma = 2$). While this is in line with Merz

[20] We calibrate the model to match the size distribution of firms (rather than establishments) in the Business Employment Dynamics program of the Bureau of Labor Statistics (BLS). We note that those results relating to establishment-level statistics (e.g. Figure 2 below) are robust when we restrict the model sample to the first two size classes (less than 500 employees) which largely represent one-establishment businesses.

[21] To avoid division by zero at entrant firms, we assume that actual labor input $\tilde{L} = 1 + L$ is the sum of the labor inputs of the (single) owner and of the employed workers.

and Yashiv (2007) who estimate a similar cubic hiring technology,[22] we take an agnostic view about this parameter value. Therefore, we compare the benchmark results with those obtained with a nearly linear recruitment technology ($\gamma = 0.1$) and with a much higher elasticity ($\gamma = 8$). In all versions, the scale parameter c is recalibrated to match our target for the weekly job-filling rate. The unemployment income b is set at the same value (relative to earnings) as in Hagedorn and Manovskii (2008) to ensure that reasonably small aggregate productivity shocks have quantitatively significant labor market responses. Robustness regarding this parameter, as well as regarding the returns-to-scale parameter, is explored in Appendix D.

We first simulate the model for a stationary cross-section of firms. Besides matching the calibration targets, our model generates negative relationships between firm size and quarterly job creation and job destruction rates in different size classes (see Table 1). This is despite the fact that we do not calibrate idiosyncratic productivity processes separately for each size class. A similar negative relationship between firm size and job flows obtains at entrant and exiting firms. Our model also performs reasonably well in matching the cross-sectional dispersion of quarterly employment growth rates across firms (see Table 2).

One dimension of particular interest is the relationship between employment growth, the vacancy rate and the vacancy yield, which are positively related for growing firms in the Job Openings and Labor Turnover Survey (Davis et al. (2013)). This indicates that the matching rate varies across firms, a feature that is not present in most standard models. To see whether our model can trace this relationship quantitatively, we calculate monthly model statistics for hires, vacancies, layoffs and employment growth rates.[23] Figure 2 shows the

[22]Merz and Yashiv (2007) consider hires instead of vacancies, and they use aggregate quarterly data, so that their estimation results are not applicable to our model.

[23]When L_{t-1} and V_{t-1} denotes employment and vacancies at the end of month $t-1$ and H_t are hires during month t, the hires rate is $h_t = H_t/L_{t-1}$, the vacancy rate is $v^r_{t-1} = V_{t-1}/L_{t-1}$

Table 1: Firm size, employment shares and quarterly job flows

	Data			Model ($\gamma = 2$)		
Size class	1–49	50–499	$\geq$ 500	1–49	50–499	$\geq$ 500
Firm shares	94.9	4.6	0.4	95.1	4.5	0.4
Employment shares	29.9	25.9	44.1	31.5	26.0	42.5
Job creation	10.6	5.7	3.1	9.2	6.1	5.4
Job destruction	10.4	5.4	2.9	9.3	6.1	5.4
Job creation (openings)	3.0	0.27	0.02	1.9	0.15	0.04
Job destruction (closings)	2.9	0.32	0.04	2.9	0.31	0.1

Notes: The first two rows report firm and employment shares in the three size classes 1-49, 50-499, and ≥ 500 (calibrated). The last four rows are quarterly job creation and destruction rates in the three size classes, expressed as shares of employment. Data statistics are from the Business Employment Dynamics (1992-2011) of the BLS.

cross-sectional relationships from the data and for the three parameterizations of our model.[24] In the data, firms grow larger both by posting more vacancies and by filling vacancies faster, with the vacancy yield accounting for most of the variation. The benchmark calibration with a cubic hiring cost function can account for around two thirds of the observed variation in vacancy yields (see the blue curve in the upper right graph). Employers that expand more rapidly offer more attractive contracts and fill these vacancies faster. There can be many different reasons why vacancy yields are higher in faster-growing firms. For example, strongly expanding firms may search more intensively or they may use alternative recruitment channels. Time aggregation can also account for part of this variation; see Davis et al. (2013) for a discussion. Our results suggest that competitive search can be one important, but perhaps not the only mechanism responsible for the observed heterogeneity in vacancy yields and vacancy rates.

and the vacancy yield is $v_t^y = H_t/V_{t-1}$, so that $h_t = v_{t-1}^r v_t^y$. We use this definition, which is slightly different from Davis et al. (2013), for the model and data statistics. We are grateful to Jason Faberman for providing these data.

[24]To smooth the relationships, all figures in the graphs are calculated as five-bin centered moving averages, as in Davis et al. (2013).

Table 2: Distribution of employment growth

Growth rate interval	Data	Model ($\gamma = 2$)
-2 (exit)	0.7	0.5
$(-2, -0.2]$	7.5	11.7
$(-0.2, -0.05]$	16.5	12.7
$(-0.05, -0.02]$	9.6	9.1
$(-0.02, 0.02)$	30.9	32.9
$[0.02, 0.05)$	9.9	6.3
$[0.05, 0.2)$	16.7	14.6
$[0.2, 2)$	7.5	11.9
2 (entry)	0.7	0.4

Notes: The table reports employment shares for intervals of quarterly employment growth rates. The empirical distribution is taken from Table 2 of Davis et al. (2010). Model statistics are calculated for the benchmark calibration from a cross-section of $4.7 \cdot 10^6$ firms.

Figure 2 further shows the model results for the nearly linear recruitment technology ($\gamma = 0.1$) and for the one with high curvature ($\gamma = 8$). With linear vacancy costs, *weekly* vacancy yields m are constant and hence do not vary with employment growth. Variations in the *monthly* vacancy yield are solely explained by time aggregation. The green curve in the upper right graph of Figure 2 shows that the vacancy yield is indeed nearly flat for employment growth below 10 percent. Time aggregation (i.e., firms post and fill unrecorded vacancies during the month) accounts for the variation in vacancy yields beyond that point. On the other hand, as indicated by the red curves in the figure, our model can principally account for the full variation in vacancy yields and vacancy rates if the curvature of the recruitment technology is sufficiently large. On a related note, Davis et al. (2013) show that vacancy yields (and vacancy rates) vary substantially by industry and employer size groups. While we have not introduced industry-specific parameters into our model, we can study the effect of size and find that smaller employers indeed have higher vacancy yields, albeit the variation is smaller than in the data. Specifically, in our benchmark calibration the vacancy yield at firms

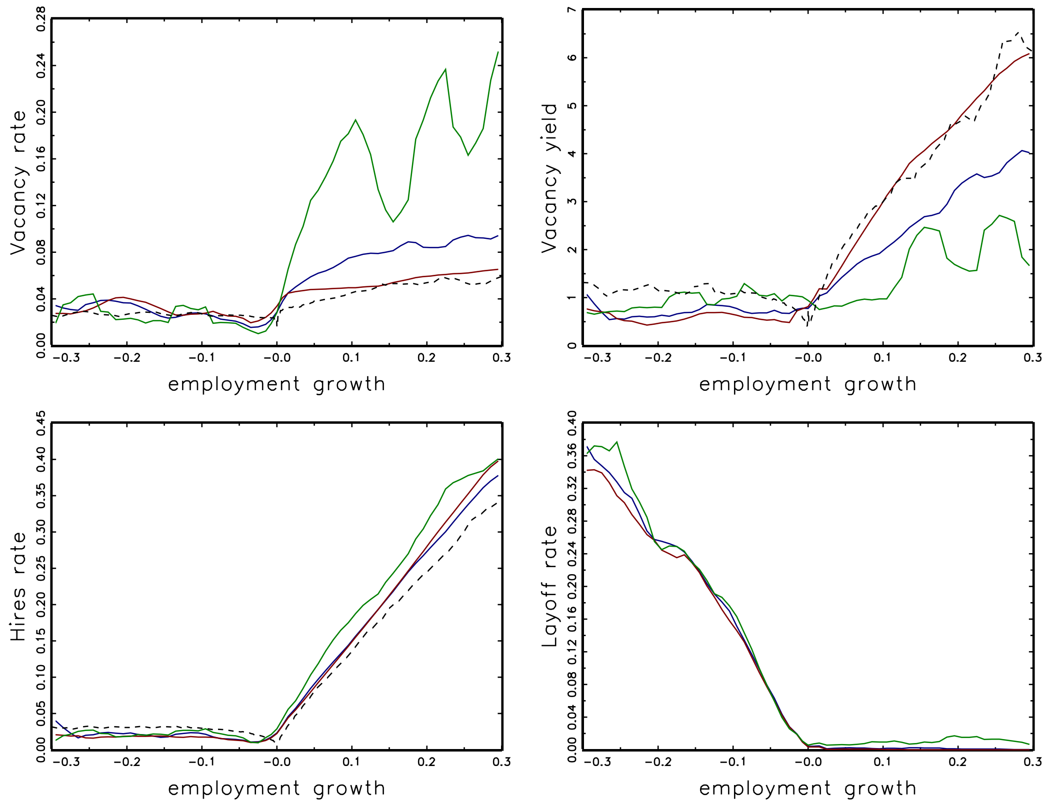

Figure 2: Cross-sectional relationships between monthly employment growth and the vacancy rate, the vacancy yield, the hires rate and the layoff rate. The dashed curves are from the data used in Davis et al. (2013), the solid blue curves are for the model with cubic hiring costs ($\gamma = 2$), the solid green curves are for $\gamma = 0.1$ and the solid red curves are for $\gamma = 8$. Model statistics are calculated from a cross-section of $4.7 \cdot 10^6$ firms.

with less than 50 workers exceeds the one at firms with more than 500 workers by 10 percent, while in the data the difference is almost a factor of two.[25]

The bottom graphs in Figure 2 show that our model largely accounts for the relationships between employment growth, hires rates and layoff rates, both for growing and for shrinking firms, and regardless of the curvature parameter in the

[25]We expect that more flexible forms of the recruitment technology should give larger variation by employer size: for instance, if C had decreasing returns in (V, L), vacancy postings in larger firms would be less costly so that these firms prefer to recruit less intensively, reducing job-filling rates further.

hiring cost technology.[26]

To explore the impact of aggregate shocks, we first consider the model response to a permanent increase in the aggregate productivity parameter by one percent. In response to the shock, we let entry costs increase by the same factor.[27] The new steady-state equilibrium features more firms and higher aggregate output. Since firms are on average smaller, labor productivity increases by 1.1 percent, output increases by 1.8 percent and unemployment falls by 8 percent. In Figure 3, we compare impulse responses for the three calibrations with different curvature parameters. Relative to the model with nearly linear recruitment costs, convex costs generate a pronounced labor market propagation, featuring sluggish adjustment of the job–finding rate and of the vacancy–unemployment ratio. In reduced–form vector autoregressions, Fujita and Ramey (2007) and Fujita (2011) find that productivity shocks induce sluggish responses of the vacancy–unemployment ratio and of the job–finding rate. Fujita and Ramey (2007) and Shimer (2005) argue that standard search and matching models cannot generate such patterns because market tightness and the job-finding rate are jump variables which correlate perfectly with aggregate productivity. The bottom graphs in Figure 3 show that this is also true in our model when vacancy costs are linear,[28] but not when they are convex in which case both variables lag behind aggregate productivity by 2-3 quarters.[29]

We emphasize that the sluggish model dynamics comes about for the same pa-

[26] For the empirical relationship between employment growth and layoffs, see Davis et al. (2010) who find that layoffs dominate quits for large employment contractions. In our model, the quit rate is exogenous at s_0 so that variations in layoffs necessarily capture all variations in separations.

[27] Without the proportional increase in entry costs, firm entry would exhibit an implausible spike at the time of the shock. There are many reasons why entry costs vary with the business cycle, e.g. procyclical rental rates, capital prices, or outside opportunities of entrepreneurs.

[28] Equation (18) implies that m is a function of the aggregate state μ_i alone if marginal vacancy costs are constant.

[29] Fujita and Ramey (2007) find a peak response of market tightness after four quarters. As they consider temporary shocks, the results are not directly comparable.

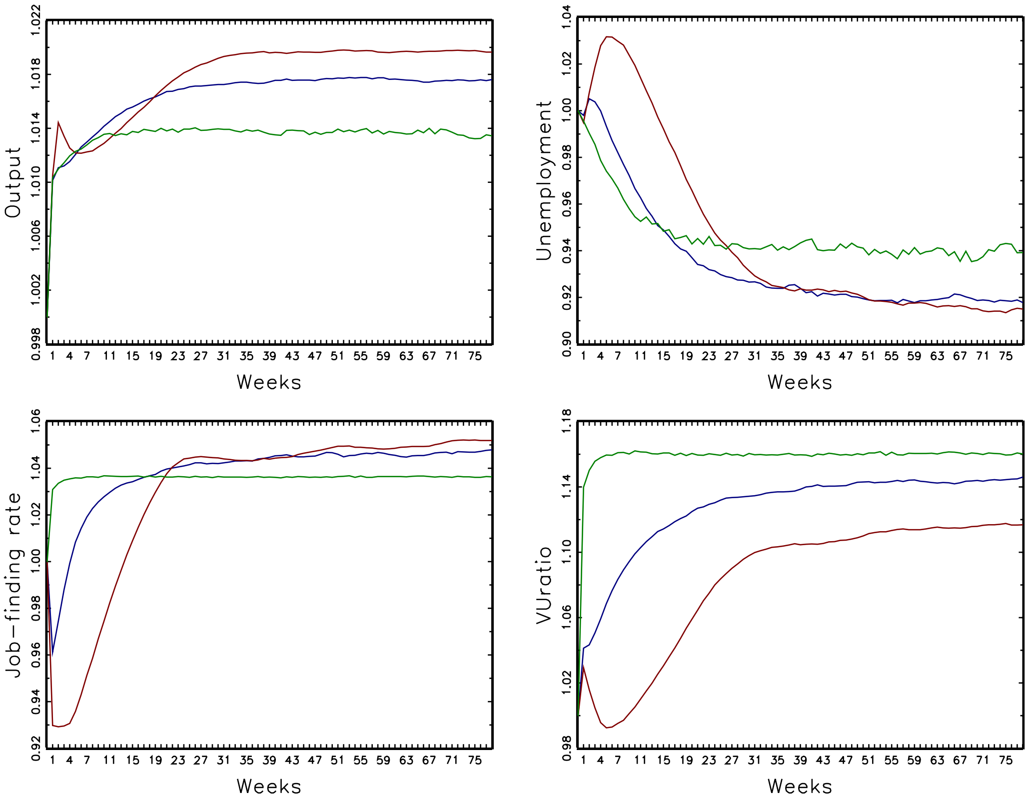

Figure 3: Impulse response to a permanent 1% increase in aggregate productivity. The blue curves are for the model with cubic hiring costs ($\gamma = 2$), the green curves are for $\gamma = 0.1$ and the red curves are for $\gamma = 8$.

rameterizations of the recruitment technology which also give rise to plausible variations of vacancy yields across firms. Micro-level features are thus directly linked to the dynamics at the aggregate level. Lagged responses to productivity shocks are neither picked up by most random search models, nor by existing models with directed search, such as Shi (2009), Menzio and Shi (2010, 2011), and Schaal (2010). In our model, convexity of recruitment technologies in combination with the entry of new firms contribute to the delayed response of the labor market: the positive shock triggers a surge of entrant firms who create only few jobs when they are small but more as they grow larger. With linear recruitment costs, all firms (young and old) would directly jump to their optimal sizes.

To study business cycle properties, we solve the model as outlined in Section 3.2. The aggregate productivity parameter attains five equally distant values in the interval $[z_{min}, 2-z_{min}]$, and the Markov process for z is a mean–reverting process with transition probability ψ, as described in Appendix C of Shimer (2005). The two parameters $(z_{min}, \psi) = (0.95, 0.015)$ are set to target a quarterly standard deviation and autocorrelation of labor productivity around trend of 0.013 and 0.76. As before, we allow the entry cost K to vary with the aggregate state, so as to reduce the volatility of job creation at opening firms.[30] Table 3 shows the outcome of this experiment for volatility and comovement with aggregate output. The key labor market variables are amplified almost as much as in the data, which is not too surprising given that we calibrate the opportunity cost of work rather close to marginal labor productivity. Relative to a model with homogeneous firms (Hagedorn and Manovskii (2008)), firm heterogeneity and decreasing returns add no more amplification.[31] Besides amplification, our model generates correlation patterns with aggregate output which are consistent with the data. Particularly, it captures a negative comovement of unemployment and vacancies, despite the feature that separation rates are counter-cyclical. We also note that the correlation between labor productivity and the job-finding rate is positive though not perfect, in contrast to Shimer's (2005) calibration of the search and matching model with homogeneous firms.

We conclude this section by providing some first exploration of the positive implications of policy interventions in our environment. Given that key aggregate variables react with delay to aggregate shocks - in contrast to most existing models - we think that our model is useful to study how policy affects employment

[30]Specifically, we let K vary between 199.75 in the lowest productivity state and 209.67 in the highest state.

[31]This is consistent with Krause and Lubik (2007), Faccini and Ortigueira (2010) and Hawkins (2011) who obtain little amplification of technology shocks in labor market models with intra-firm bargaining.

Table 3: Business cycle statistics

	Data		Model	
	Relative volatility	Correlation with output	Relative volatility	Correlation with output
Productivity	0.797	0.645	0.454	0.881
Unemployment	8.610	-0.861	6.93	-0.929
Vacancies	8.343	0.428	5.32	0.578
Job–finding rate	4.940	0.830	4.121	0.928
Separation rate	3.577	-0.628	3.743	-0.768

Notes: All variables are logged and HP filtered with parameter 1600. Relative volatility measures the standard deviation of a variable divided by the standard deviation of output. Data are for the U.S. labor market (1948Q1–2007Q1, Vacancies: 1951Q1–2006Q3); the job–finding rate and separation rate series were constructed by Robert Shimer (see Shimer (2012) and his webpage http://sites.google.com/site/robertshimer/research/flows). The model statistics (benchmark calibration with $\gamma = 2$) are obtained from a simulation of $2 \cdot 10^5$ firms over a period of 26000 weeks. Weekly series are converted into quarterly series by time averaging.

volatility over the cycle.[32]

For this initial exploration, we focus on hiring subsidies (hiring credits), as these have been extensively deployed to stimulate job growth in past recessions and have received renewed attention during the Great Recession.[33] Indeed, it is conceivable that they succeed in stabilizing business cycle fluctuations, especially when they are used in a counter-cyclical way. However, we find that the contrary is the case. We compare time-invariant and counter-cyclical subsidies, financed by lump-sum taxes. We solve the model as the solution to a quasi-planner's problem who maximizes social welfare subject to given government policy (cf. Veracierto (2008)). We set the subsidy per hire to 0.03 which corre-

[32] We do not dwell too much on the normative implications, as our baseline environment is constrained efficient. Normative implications become more interesting in extensions to risk-sharing that we outline in the conclusions.

[33] The Hiring Incentives to Restore Employment Act (HIRE) of 2010 includes tax exemptions from employer social security contributions and business income tax breaks for workers hired from unemployment; hiring credits were also an element of the American Jobs Act proposed by the Obama administration in 2011. See Neumark (2013) for an overview of various hiring credits in the United States.

sponds to 8% of a monthly wage so that government expenditures on hiring subsidies are 0.3 percent of GDP. With a counter-cyclical policy, hiring firms receive the subsidy only when the aggregate productivity state is below its mean. Table 4 shows the outcome of this exercise. While both policies succeed in stabilizing the job-finding rate to some extent, they dramatically increase the volatility of separations and unemployment.[34] Perhaps surprisingly, these destabilizing forces are stronger for the counter-cyclical policy where low-productivity firms lay off even more workers during recessions and fewer workers during booms. These findings suggest that hiring subsidies are not particularly useful to stabilize the cycle, at least when they are not accompanied by additional policies aiming to dampen separations during recessions. More work on these issues will obviously be needed to explore the impact of such policies in broader environments.

Table 4: Business cycle effects of hiring subsidies

	Laissez faire	Constant policy	Cyclical policy
Unemployment	10.3	17.7	20.8
Vacancies	7.9	4.0	12.6
Output	1.5	2.9	2.7
Job–finding rate	6.2	4.8	4.8
Separation rate	5.6	17.5	22.7

Notes: The table reports the standard deviations of logged and HP filtered (parameter 1600) quarterly variables, where model statistics are obtained as in Table 3.

[34]In steady state, both the separation and the hiring rate increase by 50 percent in response to the (time-invariant) policy. This results in much more volatile firm dynamics with substantially more worker reallocation between employment and unemployment, which ultimately also increases the steady-state unemployment rate. We also find that the two hiring margins do not react equally to the policy: firms hire more workers by using more vacancy postings, while the aggregate vacancy yield increases only slightly.

5 Conclusions

This paper investigates job reallocation in a model where firms actively compete for workers in a frictional labor market. Meaningful dynamics arise when firms cannot instantly post vacancies at constant marginal costs - for example because existing workers are required for recruitment. Firms that want to expand quickly are willing to pay higher salaries to attract more workers and hence fill vacancies faster. Matching rates are therefore not an aggregate object, as in most of the search literature, but are firm-specific as recently documented by Davis et al. (2013). Calibrated versions of the model show that it can account for this variation in vacancy yields, alongside other cross-sectional features. The same reasons that let firms vary their vacancy yields also induce delayed aggregate responses of key labor market variables to productivity shocks. As a first application of this model, we explore the effects of (counter-cyclical) hiring credits on labor markets over the business cycle.

Apart from this contribution, this paper lays out a competitive search model with heterogeneous firms facing convex recruitment costs. This model provides an alternative to the current workhorse models for large firms in search markets which are based on random search and intra-firm bargaining. We establish substantial differences between these environments: Competition for workers induces different vacancy yields at different firms, while they are identical in standard (random search) models. Multi-worker firms in that environment always engage in inefficient hiring, whereas we show that our environment retains the main efficiency properties known from competition in economies with single-worker firms. Finally, we show that our model remains tractable both in and out of steady state, which makes it particularly useful for applied purposes.

We conclude by noting that this framework is flexible for extensions. It is straightforward to allow for variable capital investment or for worker heterogeneity, as long as the firm-level production functions retain decreasing returns in variable inputs. A further extension is to introduce risk aversion. In constant-returns environments with exogenous separation rates, Acemoglu and Shimer (1999) and Rudanko (2011) introduce long-term contracting and analyze the implications for risk sharing, unemployment insurance and labor market dynamics. Our model with exogenous separations could also be augmented along these lines. If workers are risk averse and have no access to capital markets, risk-neutral firms offer flat wage contracts. Similar to our exposition in Section 2, firms solve a recursive problem subject to a workers' participation constraint which takes into account $u(w)$ instead of w. But different from our results, equilibrium ceases to be socially efficient, provided that the planner is allowed to redistribute income to the unemployed. Lack of unemployment insurance induces workers to search too much for low-paying but easy-to-get jobs (as in Acemoglu and Shimer (1999)), and should lead to excess employment in low-productivity firms and therefore to a misallocation of labor between heterogeneous firms.

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Appendix A: Proofs

Proof of Proposition 1:
Rewrite problem (5) to express the dependence of the value function on x and on the worker's search value ρ as the solution to the dynamic programming problem

$$\begin{aligned} G(L,x;\rho) &= \max_{(m,V)\geq 0} xF(L) - C(V,L,x) - D(m)V + \beta(1-\delta)G(L_+,x;\rho) \\ &\quad \text{s.t. } L_+ = L(1-s) + mV \ , \ V,m \geq 0 \ , \end{aligned} \tag{21}$$

where function $D(m)$ is defined in the text. It is increasing, strictly convex in m and increasing in ρ. This problem is equivalently defined on a compact state space $L \in [0,\overline{L}]$ where $\overline{L}$ is so large that it never binds. This is possible because of the Inada condition $\lim_{L\to\infty} F'(L) = 0$. The RHS in problem (21) defines an operator T which maps a continuous function $G_0(L,x;\rho)$, defined on $\mathcal{S} = [0,\overline{L}] \times [0,\overline{x}] \times [0,\overline{\rho}]$ into a continuous function $G_1(L,x;\rho) = T(G_0)(L,x;\rho)$ defined on the same domain. Here $\overline{x}$ and $\overline{\rho}$ are arbitrary upper bounds on x and ρ. Operator T is a contraction, therefore there exists a unique fixed point G^* which is a continuous function and which is the limit of any sequence G_n defined by $G_n = T(G_{n-1})$.
Starting from a continuous G_0 that is differentiable and weakly increasing in L and x and weakly decreasing in ρ, successive application of T yields a sequence G_n where each element shares these properties. Since the subset of continuous functions on $\mathcal{S}$ that are weakly increasing in L and x and weakly decreasing in ρ is closed under the sup norm, the limit G^* of sequence G_n is in this set. Because $xF(L) - C(V,L,x)$ is strictly increasing in (L,x) and since $D(m)$ is strictly decreasing in ρ, the limit G is strictly increasing in x and L and strictly decreasing in ρ.
We show in subsequent Lemmata 1 and 2 that T maps functions that are differentiable and concave in L and supermodular in L and x into functions with the same properties. Since the subset of concave and supermodular functions is closed, the same arguments as above imply that the unique fixed point G^* is concave in L and supermodular in (L,x). Since function $xF(L) - C(V,L,x)$ is strictly concave in L, G^* is also strictly concave in L. Concavity in L and differentiability of $xF(L) - C(V,L,x)$ together with the theorem of Benveniste and Scheinkman establishes differentiability of G^* in L.
Before we establish the remaining results, rewrite (21) in terms of hirings $H = mV$. Dropping argument ρ from G, we can equivalently write (21) as

$$G(L,x) = \max_H xF(L) - \mathcal{C}(H,L,x) + \beta(1-\delta)G(L(1-s)+H,x) \tag{22}$$

where

$$\mathcal{C}(H,L,x) \equiv \min_m C\Big(\frac{H}{m},L,x\Big) + D(m)\frac{H}{m} \ . \tag{23}$$

The right hand side of (22) is an equivalent expression of the fixed-point operator T. As will become clear, the per period return $xF(L) - \mathcal{C}(H,L,x)$ is supermodular in (L,H) but when $C_{13} > 0$ (which arises in first specification in (1) for $h > 0$) the per period return is strictly submodular in (H,x) and in (L_+,x) when one writes

$H = L_+ - (1-s)L$, which renders standard tools to prove supermodularity (e.g., Amir (1996)) inapplicable. To proceed, the optimality condition for problem (23) is

$$C_1\Big(\frac{H}{m}, L, x\Big) = D'(m)m - D(m) \ . \tag{24}$$

Differentiate this equation to obtain

$$\begin{aligned} \frac{dm}{dH} &= \frac{C_{11}}{C_{11}\dfrac{H}{m} + D''(m)m^2} > 0 \ , & (25)\\ \frac{dm}{dL} &= \frac{C_{12}m}{C_{11}\dfrac{H}{m} + D''(m)m^2} = \frac{C_{12}m}{C_{11}}\frac{dm}{dH} \leq 0 \ , & (26)\\ \frac{dm}{dx} &= \frac{C_{13}m}{C_{11}\dfrac{H}{m} + D''(m)m^2} = \frac{C_{13}m}{C_{11}}\frac{dm}{dH} \geq 0 \ . & (27) \end{aligned}$$

Therefore, we can express the derivatives of cost function $\mathcal{C}$ as

$$\begin{aligned} \mathcal{C}_1 &= D'(m) > 0 \ , & \\ \mathcal{C}_2 &= C_2 \ , & \\ \mathcal{C}_{11} &= D''(m)\frac{dm}{dH} > 0 \ , & (28)\\ \mathcal{C}_{12} &= D''(m)\frac{dm}{dL} \leq 0 \ , & (29)\\ \mathcal{C}_{22} &= C_{22} - C_{12}\frac{H}{m^2}\frac{dm}{dL} \ , & (30)\\ \mathcal{C}_{13} &= D''(m)\frac{dm}{dx} \geq 0 \ , & (31)\\ \mathcal{C}_{23} &= C_{23} - C_{12}\frac{H}{m^2}\frac{dm}{dx} \ . & (32) \end{aligned}$$

Lemma 1: Suppose that G is twice differentiable and concave in L. Then $T(G)$ is twice differentiable and

(a) concave in L if the following condition holds:

$$\mathcal{C}_{12}^2 + \mathcal{C}_{11}[xF'' - \mathcal{C}_{22}] \leq 0 \ . \tag{33}$$

(b) concave in L and supermodular in (L, x) if G is supermodular in (L, x) and if (33) and the following condition hold:

$$\mathcal{C}_{12}\mathcal{C}_{13} + \mathcal{C}_{11}[F' - \mathcal{C}_{23}] \geq 0 \ . \tag{34}$$

Lemma 2:

(a) Condition (33) holds under the following condition on the original cost function

C:

$$C_{12}^2 + C_{11}[xF'' - C_{22}] \leq 0 \ . \tag{35}$$

(b) Condition (34) holds under the following condition on the original cost function C:

$$C_{12}C_{13} + C_{11}[F' - C_{23}] \geq 0 \ . \tag{36}$$

Proof of Lemma 1: Consider $T(G)$ defined by the RHS of (22).
Part (a). Since G is a concave and twice differentiable function of L, $T(G)$ is also twice differentiable, and a policy function exists and is differentiable. Differentiate $T(G)$ twice with respect to L to obtain

$$\frac{d^2(TG)}{dL^2} = xF'' - \mathcal{C}_{22} + \beta\varphi(1-s)G_{11} + \Big[-\mathcal{C}_{12} + \beta\varphi G_{11} \Big] \frac{dH}{dL} \ . \tag{37}$$

Differentiate the FOC $\mathcal{C}_1 = \beta(1-\delta)G_1$ with respect to L to obtain

$$\frac{dH}{dL} = \frac{\beta\varphi G_{11} - \mathcal{C}_{12}}{\mathcal{C}_{11} - \beta(1-\delta)G_{11}} \ . \tag{38}$$

Substitute this into (37) to obtain

$$\frac{d^2(TG)}{dL^2} = xF'' - \mathcal{C}_{22} + \frac{\beta\varphi(1-s)G_{11}\mathcal{C}_{11} + \mathcal{C}_{12}^2 - 2\beta\varphi G_{11}\mathcal{C}_{12}}{\mathcal{C}_{11} - \beta(1-\delta)G_{11}} \ .$$

In the last term, the denominator is positive and larger than $\mathcal{C}_{11}$. In the numerator, all terms involving G_{11} are negative (due to (28) and (29)); hence the numerator is smaller than $\mathcal{C}_{12}^2$. Therefore,

$$\frac{d^2(TG)}{dL^2} \leq xF'' - \mathcal{C}_{22} + \frac{\mathcal{C}_{12}^2}{\mathcal{C}_{11}} \ ,$$

which is non-positive under (33). Hence, T maps a concave and twice differentiable function into a function with the same properties.
Part (b). Since G is a concave, supermodular and twice differentiable function of (L, x), $T(G)$ is twice differentiable and a differentiable policy function exists. Differentiate $T(G)$ twice with respect to L and x to obtain

$$\frac{d^2(TG)}{dLdx} = F' - \mathcal{C}_{23} + \beta\varphi G_{12} + \Big[-\mathcal{C}_{12} + \beta\varphi G_{11} \Big] \frac{dH}{dx} \ . \tag{39}$$

Differentiate the FOC $\mathcal{C}_1 = \beta(1-\delta)G_1$ with respect to x to obtain

$$\frac{dH}{dx} = \frac{\beta(1-\delta)G_{12} - \mathcal{C}_{13}}{\mathcal{C}_{11} - \beta(1-\delta)G_{11}} \ . \tag{40}$$

Substitute this into (39) to obtain

$$\frac{d^2(TG)}{dLdx} = F' - \mathcal{C}_{23} + \frac{\beta\varphi G_{12}\mathcal{C}_{11} + \mathcal{C}_{12}\mathcal{C}_{13} - \beta(1-\delta)G_{12}\mathcal{C}_{12} - \beta\varphi G_{11}\mathcal{C}_{13}}{\mathcal{C}_{11} - \beta(1-\delta)G_{11}} \ .$$

In the last term, the denominator is positive and larger than $\mathcal{C}_{11}$. In the numerator, all terms involving G_{11} and G_{12} are non-negative (due to (28), (29) and (31)); hence the numerator is greater than $\mathcal{C}_{12}\mathcal{C}_{13} \leq 0$. Therefore,

$$\frac{d^2(TG)}{dLdx} \geq F' - \mathcal{C}_{23} + \frac{\mathcal{C}_{12}\mathcal{C}_{13}}{\mathcal{C}_{11}} \ ,$$

which is non-negative under (34). Hence, $T(G)$ is supermodular. □

Proof of Lemma 2: From (26), (27), (28), (29) and (31) follows that

$$\mathcal{C}_{12} = \frac{\mathcal{C}_{11}C_{12}m}{C_{11}} \ , \tag{41}$$

$$\mathcal{C}_{13} = \frac{\mathcal{C}_{11}C_{13}m}{C_{11}} \ . \tag{42}$$

Furthermore, substituting (29) into (26), and substituting (31) into (27) to eliminate $D''(m)$ imply that

$$\mathcal{C}_{22} = C_{22} - \frac{C_{12}^2}{C_{11}} + \frac{mC_{12}}{C_{11}}\mathcal{C}_{12} \ , \tag{43}$$

$$\mathcal{C}_{23} = C_{23} - \frac{C_{12}}{C_{11}}\Big[C_{13} - m\mathcal{C}_{13}\Big] \ . \tag{44}$$

Part (a): Rewrite (33) using (41) and (43) to obtain the equivalent condition

$$xF'' - C_{22} + \frac{C_{12}^2}{C_{11}} \leq 0 \ .$$

Because of $C_{11} > 0$, this condition is equivalent to (35).
Part (b): Rewrite (34) using (41), (42) and (44) to obtain the equivalent condition

$$F' - C_{23} + \frac{C_{12}C_{13}}{C_{11}} \geq 0 \ .$$

Because of $C_{11} > 0$, this condition is equivalent to (36). □

It follows from Lemma 1 and 2 that the value function $G(L,x)$ is concave in L and supermodular in (L,x) because property (C) guarantees both (35) and (36).
Because of strict concavity of problem (21), policy functions $m^x(L)$ and $V^x(m^x(L),L)$ exist. To derive first-order conditions (6) and (7) is straightforward: The first condition directly follows from (24); the second follows from the intertemporal optimality condition $\mathcal{C}_1(H,L,x) = \beta(1-\delta)G_1(L(1-s)+H,x)$ and from using the envelope theorem and (6).
The properties of V^x stated in Proposition 1 were already established in the main text.

To see how $m^x(L)$ depends on L, use (26) and (38) to get

$$\frac{dm^x(L)}{dL} = \frac{dm(H,L,x)}{dL} + \frac{dm(H,L,x)}{dH}\frac{dH}{dL} = \frac{dm}{dH}\left[\frac{C_{12}m}{C_{11}} + \frac{\beta\varphi G_{11} - \mathcal{C}_{12}}{\mathcal{C}_{11} - \beta(1-\delta)G_{11}}\right] .$$

Because of

$$\frac{C_{12}m}{C_{11}} = \frac{\mathcal{C}_{12}}{\mathcal{C}_{11}} \leq \frac{\mathcal{C}_{12}}{\mathcal{C}_{11} - \beta(1-\delta)G_{11}} ,$$

the term in [.] is negative, and so is $dm^x/(dL)$.
To verify that m is increasing in x, use (27) and (40) to get

$$\frac{dm^x(L)}{dx} = \frac{dm(H,L,x)}{dx} + \frac{dm(H,L,x)}{dH}\frac{dH}{dx} = \frac{dm}{dH}\left[\frac{C_{13}m}{C_{11}} + \frac{\beta(1-\delta)G_{12} - \mathcal{C}_{13}}{\mathcal{C}_{11} - \beta(1-\delta)G_{11}}\right] .$$

Because of

$$\frac{C_{13}m}{C_{11}} = \frac{\mathcal{C}_{13}}{\mathcal{C}_{11}} \geq \frac{\mathcal{C}_{13}}{\mathcal{C}_{11} - \beta(1-\delta)G_{11}} ,$$

the term in [.] is positive, and so is $dm^x/(dx)$. □

Proof of Corollary 2: Because of exogenous separations, the growth rate of a firm, $[mV - sL]/L$ is perfectly correlated with the job-creation rate,

$$\text{JCR}(x,L) = m^x(L)\frac{V^x(m^x(L),L)}{L} .$$

Differentiation of the job-creation rate with respect to x implies

$$\frac{d\text{JCR}}{dx} = \frac{dm^x}{dx}\frac{V^x}{L} + \frac{m^x}{L}\frac{dV^x}{dx} + \frac{m^x}{L}\frac{dV^x}{dm}\frac{dm^x}{dx} .$$

In this expression, the first and the third term are strictly positive. Under the second cost function in (1), the second term is zero. Under the first cost function in (1), the second term is zero when $h = 0$, and negative but small if h is small. Thus, $d\text{JCR}/(dx)$ is positive if h is sufficiently small.
Differentiation of the job-creation rate with respect to L implies

$$\frac{d\text{JCR}}{dL} = \frac{dm^x}{dL}\frac{V^x}{L} + \frac{m^x}{L}\frac{dV^x}{dL} + \frac{m^x}{L}\frac{dV^x}{dm}\frac{dm^x}{dL} - m\frac{V^x}{L^2} .$$

In this expression, the first, the third and the fourth term are strictly negative. Under the second cost function in (1), $\frac{dV^x}{dL} = \frac{V^x}{L}$, and the second and forth terms cancel out. Under the first cost function in (1), the second term is zero when $h = 0$, and positive but small if h is small. Thus, $d\text{JCR}/(dL)$ is negative if h is sufficiently small. □

Lemma 3: In the model of Section 2 with recruitment cost $C(V,L,x) = xF(L) - xF(L-hV) + cV$, job-filling rates in the optimal firm's problem follow the dynamic equation

$$\rho\Big[m_{t+1}\lambda'(m_{t+1}) - \lambda(m_{t+1})\Big] - (b+\rho)h - c = \frac{\rho h}{\beta(1-\delta)}\Big[\lambda'(m_t) - \beta\varphi\lambda'(m_{t+1})\Big]. \quad (45)$$

It has a unique steady state solution $m^* > 0$ if, and only if,

$$h < \frac{\beta(1-\delta)\overline{m}}{1-\beta\varphi} \ , \tag{46}$$

with $\overline{m} \equiv \lim_{m\to 1} m - \dfrac{\lambda(m)}{\lambda'(m)} > 0$. Under this condition, any sequence $m_t > 0$ satisfying this dynamic equation converges to m^*.

Proof of Lemma 3: It is straightforward to derive (45) by substitution of (6) into (7). A steady state m^* must satisfy the condition

$$\rho\Big[m - \frac{\lambda(m)}{\lambda'(m)}\Big] = \frac{\rho h(1-\beta\varphi)}{\beta(1-\delta)} + \frac{(b+\rho)h + c}{\lambda'(m)} \ . \tag{47}$$

The LHS is strictly increasing and goes from 0 to $\rho\overline{m}$ as m goes from 0 to 1. The RHS is decreasing in m with limit $\rho h(1-\beta\varphi)/[\beta(1-\delta)]$ for $m \to 1$. Hence, a unique steady state m^* exists iff (46) holds.[35] Furthermore, differentiation of (45) at m^* implies that

$$\frac{dm_{t+1}}{dm_t}\Big|_{m^*} = \frac{h}{\beta(1-\delta)m^* + h\beta\varphi} \ ,$$

which is positive and smaller than one iff

$$h < \frac{\beta(1-\delta)m^*}{1-\beta\varphi} \ .$$

But this inequality must be true because (47) implies

$$h = \frac{\rho[m^*\lambda'(m^*) - \lambda(m^*)] - c}{\frac{\rho[1-\beta\varphi]}{\beta(1-\delta)}\lambda'(m^*) + b + \rho} < \frac{\beta(1-\delta)m^*}{1-\beta\varphi} \ .$$

Therefore, the steady state m^* is locally stable. Moreover, the dynamic equation defines a continuous, increasing relation between m_{t+1} and m_t which has only one intersection with the 45-degree line. Hence, $m_{t+1} > m_t$ for any $m_t < m^*$ and $m_{t+1} < m_t$ for any $m_t > m^*$, which implies that m_t converges to m^* from any initial value $m_0 > 0$. $\square$

Proof of Proposition 2:
It remains to prove existence and uniqueness. From Proposition 1 follows that the entrant's value function $J^x(0,0)$ is decreasing and continuous in ρ. Hence the expected profit prior to entry,

$$\Pi^*(\rho) \equiv \sum_{x\in X} \pi(x) J^x(0,0)$$

is a decreasing and continuous function of ρ. Moreover, the function is strictly decreasing in ρ whenever it is positive. This also follows from the proof of Proposition 1 which shows that $G(0,x;\rho)$ is *strictly* decreasing in ρ when the new firm x recruits

[35] If this condition fails, firms cannot profitably recruit workers.

workers ($V^x(m^x(0),0) > 0$). If no new firm recruits workers, expected profit of an entrant cannot be positive. Hence, equation (8) can have at most one solution for any $K > 0$. This implies uniqueness, with entry of firms if (8) can be fulfilled or without entry of firms otherwise. A solution to (8) exists provided that K is sufficiently small. To see this, $\Pi^*(0)$ is strictly positive because of $F'(0) = \infty$: some entrants will recruit workers since the marginal product $G_1(mV, x; \rho)$ is sufficiently large relative to the cost of recruitment and relative to the wage cost which are, for $\rho = 0$, equal to mVb (see equation (21)). But when $\Pi^*(0) > 0$, a sufficiently small value of K guarantees that (8) has a solution since $\lim_{\rho\to\infty} \Pi^*(\rho) = 0$. □

Proof of Proposition 3:
We will show that the first-order conditions that uniquely characterize the decentralized allocation are also first order conditions to the planner's problem. The same auxiliary problem that we employ in the proof of Lemma 4 part (b) (see the proof of Proposition 4) then establishes that the planner cannot improve upon this allocation. We denote by $S_{N,a}$ the derivative of S with respect to N_a and by $S_{L,a,x}$ the derivative of S with respect to L_a^x. The multiplier on the resource constraint is $\mu \geq 0$. First-order conditions with respect to N_0, V_a^x, and m_a^x, $a \geq 0$, are

$$\sum_{x\in X} \pi(x)\Big[xF(0) - C(V_0^x, 0, x)\Big] - K + \beta(1-\delta)S_{N,1} - \mu \sum_{x\in X} \pi(x)\lambda(m_0^x)V_0^x = 0 \ , \quad (48)$$

$$-N_a\pi(x)\Big[C_1(V_a^x, L_a^x, x) + \mu\lambda(m_a^x)\Big] + \beta S_{L,a+1,x} m_a^x \leq 0 \ , \ V_a^x \geq 0 \ , \quad (49)$$

$$\beta S_{L,a+1,x} - \mu N_a \pi(x)\lambda'(m_a^x) = 0 \ . \quad (50)$$

Here condition (49) holds with complementary slackness. The envelope conditions are, for $a \geq 1$ and $x \in X$,

$$S_{L,a,x} = N_a\pi(x)\Big[xF'(L_a^x) - C_2'(V_a^x, L_a^x, x) - b - \mu\Big] + \beta(1-s)S_{L,a+1,x} \ , \quad (51)$$

$$S_{N,a} = \sum_{x\in X} \pi(x)\Big[xF(L_a^x) - C(V_a^x, L_a^x, x) - bL_a^x\Big] - \mu \sum_{x\in X} \pi(x)\Big(L_a^x + \lambda(m_a^x)V_a^x\Big) + \beta(1-\delta)S_{N,a+1} \ . \quad (52)$$

Use (50) to substitute $S_{L,a,x}$ into (51) to obtain

$$xF'(L_{a+1}^x) - C_2(V_{a+1}^x, L_{a+1}^x, x) - b - \mu = \frac{\mu}{\beta(1-\delta)}[\lambda'(m_a^x) - \beta\varphi\lambda'(m_{a+1}^x)] \ .$$

This equation is the planner's intertemporal optimality condition; it coincides with equation (7) for $\mu = \rho$. This is intuitive: when the social value of an unemployed worker μ coincides with the surplus value that an unemployed worker obtains in search equilibrium, the firm's recruitment policy is efficient. Next substitute (50) into (49) to obtain, for $a \geq 0$ and $x \in X$,

$$C_1(V_a^x, L_a^x, x) \geq \mu[m_a^x\lambda'(m_a^x) - \lambda(m_a^x)] \ , \ V_a^x \geq 0 \ . \quad (53)$$

Again for $\mu = \rho$, this condition coincides with the firm's intratemporal optimality condition in competitive search equilibrium, equation (6). Lastly, it remains to verify that entry is socially efficient when the value of a jobless worker is $\mu = \rho$. The planner's choice of firm entry, condition (48), together with the recursive equation for the marginal firm surplus $S_{N,a}$, equation (52), shows that

$$K = \sum_{a\geq 0}[\beta(1-\delta)]^a \sum_{x\in X} \pi(x)\Big[xF(L_a^x) - bL_a^x - C(V_a^x, L_a^x, x) - \mu(L_a^x + \lambda(m_a^x)V_a^x)\Big] . \quad (54)$$

On the other hand, the expected profit value of a new firm is

$$\sum_{x\in X} \pi(x)J^x(0,0) = \sum_{a\geq 0}[\beta(1-\delta)]^a \sum_{x\in X} \pi(x)\Big[xF(L_a^x) - W_a^x - C(V_a^x, L_a^x, x)\Big] .$$

Hence, the free-entry condition in search equilibrium, equation (8), coincides with condition (54) for $\mu = \rho$ if, for all $x \in X$,

$$\sum_{a\geq 0}[\beta(1-\delta)]^a \Big[(b+\mu)L_a^x + \mu\lambda(m_a^x)V_a^x - W_a^x\Big] = 0 . \quad (55)$$

Now after substitution of

$$\begin{aligned} L_a^x &= \sum_{k=0}^{a-1}(1-s)^{a-1-k} m_k^x V_k^x \ , \text{ and} \\ W_a^x &= \sum_{k=0}^{a-1}(1-s)^{a-1-k} V_k^x \Big[\frac{\rho\lambda(m_k^x)(1-\beta\varphi)}{\beta(1-\delta)} + m_k^x(b+\rho)\Big] \end{aligned}$$

into (55), it is straightforward to see that the equation is satisfied for $\mu = \rho$. □

Proof of Proposition 4:
Part (a):
Let $\beta^t\psi(z^t)\mu(z^t) \geq 0$ be the multiplier on the resource constraint (15) in history node z^t. That is, $\mu(z^t)$ is the social value of a worker in history z^t. Write $\boldsymbol{\mu} = (\mu(z^t))$ for the vector of multipliers. Let $G_t(L, x, z^t)$ denote the social value of an existing firm with employment stock L, idiosyncratic productivity x and aggregate productivity history z^t. The sequence G_t obeys the recursive equations

$$\begin{aligned} G_t(L, x, z^t) = \max_{\delta,s,V,m}\ & xz_tF(L) - bL - \mu(z^t)[L + \lambda(m)V] - C(V, L, xz_t) - f \quad (56) \\ & +\beta(1-\delta)E_{x,z^t}G_{t+1}(L_+, x_+, z^{t+1}) \\ \text{s.t.}\quad & L_+ = (1-s)L + mV \ , \\ & \delta \in [\delta_0, 1],\ s \in [s_0, 1],\ m \in [0, 1],\ V \geq 0 . \end{aligned}$$

We first prove the equivalence between problem (56) and the planner's problem (14) (Lemma 4). Then we show that the reduced problem (16) solves (56) if entry is positive in all states.

Lemma 4:

(a) *For given multipliers* $\mu(z^t)$*, there exist value functions* $G_t : \mathbb{R}_+ \times X \times Z^{t+1} \to \mathbb{R}$*,* $t \geq 0$*, satisfying the system of recursive equations (56).*

(b) *If* $\mathbf{X} = (\mathbf{N}, \mathbf{L}, \mathbf{V}, \mathbf{m}, \mathbf{s}, \boldsymbol{\delta})$ *is a solution of the planning problem (14) with multipliers* $\boldsymbol{\mu} = (\mu(z^t))$*, then the corresponding firm policies also solve problem (56) and the complementary-slackness condition*

$$\sum_{x \in X} \pi_0(x) G_t(0, x, z^t) \leq K(z_t) \ , \ N_0(z^t) \geq 0 \ , \tag{57}$$

is satisfied for all z^t*. Conversely, if* $\mathbf{X}$ *solves for every firm problem (56) with multipliers* $\boldsymbol{\mu}$*, and if condition (57) and the resource constraint (15) hold for all* z^t*, then* $\mathbf{X}$ *is a solution of the planning problem (14).*

Proof of Lemma 4:
Part (a): The RHS in the system of equations in (56) defines an operator T which maps a sequence of bounded functions $G = (G_t)_{t \geq 0}$, with $G_t : [0, \overline{L}] \times X \times Z^t \to \mathbb{R}$ such that $\|G\| \equiv \sup_t \|G_t\| < \infty$, into another sequence of bounded functions $\tilde{G} = (\tilde{G}_t)_{t \geq 0}$ with $\|\tilde{G}\| = \sup_t \|\tilde{G}_t\| < \infty$. Here $\overline{L}$ is sufficiently large such that the bound $L_+ \leq \overline{L}$ does not bind for any $L \in [0, \overline{L}]$. The existence of $\overline{L}$ follows from the Inada condition for F: the marginal product of an additional worker $xzF'(L_+) - b$ must be negative for any $x \in X$, $z \in Z$, for all $L_+ \geq \overline{L}$ with sufficiently large $\overline{L}$; hence no hiring will occur beyond $\overline{L}$. Because the operator satisfies Blackwell's sufficient conditions, it is a contraction in the space of bounded function sequences G. Hence, the operator T has a unique fixed point which is a sequence of bounded functions.
Part (b): Take first a solution $\mathbf{X}$ of the planning problem, and write $\beta^t \psi(z^t) \mu(z^t) \geq 0$ for the multipliers on constraints (15). Then $\mathbf{X}$ maximizes the Lagrange function

$$\begin{aligned} \mathcal{L} \ &= \ \max \sum_{t \geq 0, z^t} \beta^t \psi(z^t) \Big\{ - K(z_t) N_0(z^t) + \sum_{a \geq 0, x^a} N(x^a, z^t) \Big[x_a z_t F(L(x^a, z^t)) - bL(x^a, z^t) \\ & \quad -f - C(V(x^a, z^t), L(x^a, z^t), x_a z_t) - \mu(z^t) \Big[L(x^a, z^t) + \lambda(m(x^a, z^t)) V(x^a, z^t) \Big] \Big] \Big\} \end{aligned}$$

For each individual firm, this problem is the sequential formulation of the recursive problem (56) with multipliers $\mu(z^t)$. Hence, firm policies also solve the recursive problem; furthermore, the maximum of the Lagrange function is the same as the sum of the social values of entrant firms plus the social values of firms which already exist at

$t = 0$, namely,

$$\mathcal{L} = \max_{N_0(.)} \sum_{t,z^t} \beta^t \psi(z^t) N_0(z^t) \Big[- K(z_t) + \sum_x \pi_0(x) G_t(0, x, z^t) \Big] + \sum_{z \in Z} \psi(z^0) \sum_{a \geq 1, x^a} N(x^a, z^0) G_0(L(x^a, z^0), x_a, z^0) \ .$$

This also proves that the complementary-slackness condition (57) describes optimal entry.

To prove the converse, suppose that $\mathbf{X}$ solves for every firm the recursive problem (56) with given multipliers $\mu(z^t)$, and that (57) and the resource constraints (15) are satisfied. Define an auxiliary problem (AP) as an extension of the original planning problem (14) which allows the planner to rent additional workers (or to rent out existing workers) at rental rate $\mu(z^t)$ in period t. Formally, the (AP) differs from the original problem in that the resource constraint (15) is replaced by

$$\sum_{a \geq 0, x^a} N(x^a, z^t) \Big[L(x^a, z^t) + \lambda(m(x^a, z^t)) V(x^a, z^t) \Big] \leq \mathcal{M}(z^t) \ , \tag{58}$$

with $\mathcal{M}(z^t) - 1 > 0$ workers hired or $\mathcal{M}(z^t) - 1 < 0$ workers hired out. Further, the rental cost (rental income) term $-\mu(z^t)[\mathcal{M}(z^t) - 1]$ is added into the braces in the objective function (14). Then it follows immediately that the multiplier on constraint (58) is equal to $\mu(z^t)$. We further claim that allocation $\mathbf{X}$ solves problem (AP), and hence also solves the original planning problem. To see this, suppose that there is an allocation $(\mathbf{X}', \mathcal{M})$ which is feasible for problem (AP) and which strictly dominates $\mathbf{X}$. Write

$$O(x^a, z^t) \equiv x_a z_t F(L(x^a, z^t)) - bL(x^a, z^t) - f - C(V(x^a, z^t), L(x^a, z^t), x_a z_t)$$

for the net output created by firm (x^a, z^t) in allocation $\mathbf{X}$ and write $O'(x^a, z^t)$ for the same object in allocation $\mathbf{X}'$. Further, write S for the total surplus value in allocation $(\mathbf{X}, \mathbf{1})$ and write $S' > S$ for the surplus value in allocation $(\mathbf{X}', \mathcal{M})$. Then

$$\begin{aligned} S' &= \sum_{t \geq 0, z^t} \beta^t \psi(z^t) \Bigg\{ - K(z_t) N_0'(z^t) + \sum_{a \geq 0, x^a} N'(x^a, z^t) O'(x^a, z^t) - \mu(z^t)[\mathcal{M}(z^t) - 1] \Bigg\} \\ &\leq \sum_{t \geq 0, z^t} \beta^t \psi(z^t) \Bigg\{ - K(z_t) N_0'(z^t) + \mu(z^t) \\ &\quad + \sum_{a \geq 0, x^a} N'(x^a, z^t) \Big[O'(x^a, z^t) - \mu(z^t) \Big(L'(x^a, z^t) + \lambda(m'(x^a, z^t)) V'(x^a, z^t) \Big) \Big] \Bigg\} \\ &\leq \sum_{t \geq 0, z^t} \beta^t \psi(z^t) N_0'(z^t) \Big[- K(z_t) + \sum_x \pi_0(x) G_t(0, x, z^t) \Big] \end{aligned}$$

$$
\begin{aligned}
&+\sum_{z\in Z}\psi(z^0)\sum_{a\geq 1,x^a}N(x^a,z^0)G_0(L(x^a,z^0),x_a,z^0)+\sum_{t,z^t}\beta^t\psi(z^t)\mu(z^t)\\
\leq\; &\sum_{t\geq 0,z^t}\beta^t\psi(z^t)N_0(z^t)\Big[-K(z_t)+\sum_x\pi_0(x)G_t(0,x,z^t)\Big]\\
&+\sum_{z\in Z}\psi(z^0)\sum_{a\geq 1,x^a}N(x^a,z^0)G_0(L(x^a,z^0),x_a,z^0)+\sum_{t,z^t}\beta^t\psi(z^t)\mu(z^t)=S\;.
\end{aligned}
$$

Here the first inequality follows from resource constraint (58). The second inequality follows since the discounted sum of surplus values for an individual firm which is of age a at time t, namely

$$
\sum_{\tau\geq t}\beta^{\tau-t}\sum_{x^{a+\tau-t}z^\tau}\psi(z^\tau|z^t)\pi(x^{a+\tau-t}|x^a)\prod_{k=t}^{\tau-1}[1-\delta(x^{a+k-t},z^k)]
$$

$$
\Big[O'(x^{a+\tau-t},z^\tau)-\mu(z^\tau)[L'(x^{a+\tau-t},z^\tau)+\lambda(m'(x^{a+\tau-t},z^\tau))V'(x^{a+\tau-t},z^\tau)]\Big]\;,
$$

is bounded above $G_t(0,x_0,z_t)$ (for new firms, $a=0$) or by $G_0(L(x^a,z^0),x_a,z^0)$ (for firms of age $a>0$ existing at $t=0$) by definition of G_t. The third inequality follows from the complementary-slackness condition (57): either the term $-K(z_t)+\sum_x\pi_0(x)G_t(0,x,z^t)$ is zero in which case the first summand is zero on both sides of the inequality; or it is strictly negative in which case $N_0(z^t)=0$ and $N_0'(z^t)\geq 0$. The last equality follows from the definition of surplus value S and the assumption that allocation $\mathbf{X}$ solves problem (56) at the level of each individual firm. This proves $S'\leq S$ and hence contradicts the hypothesis $S'>S$. This completes the proof of Lemma 4. □

To complete the proof of Prop. 4, part (a), let μ_i be the multiplier in aggregate state z_i, defined by (16) and (17), and write $M=(\mu_1,\ldots,\mu_n)$. With $\mu(z^t)\equiv\mu_i$ for $z_t=z_i$, the unique solution of (56) coincides with the one of (16), i.e. $G_t(L,x,z^t)=G(L,x,i;M)$ for $z_t=z_i$, and also the firm-level policies coincide. If they give rise to an allocation $\mathbf{X}$ with positive entry in all aggregate states z^t, (17) implies that (57) holds for all z^t. Hence Lemma 4(b) implies that $\mathbf{X}$ is a solution of the planning problem.
Part (b): Solving (16) in the stationary case $z=\overline{z}$ involves to find a single value function $G(L,x;M)$. Application of the contraction mapping theorem implies that such a solution exists, is unique, and is continuous and non-increasing in $\mu\in\mathbb{R}$ and strictly decreasing in μ when $G(.)>0$.
Therefore, the function $\Gamma(\mu)\equiv\sum_x\pi_0(x)G(0,x;\mu)\geq 0$ is continuous, strictly decreasing when positive, and zero for large enough μ. Furthermore, when f and b are sufficiently small, $\Gamma(0)>0$; hence when $K>0$ is sufficiently small, there exists a unique $\overline{\mu}\geq 0$ satisfying equation (17).
In the stochastic case $z\in\{z_1,\ldots,z_n\}$ and for any given vector $M=(\mu_1,\ldots,\mu_n)\in R_+^n$, the system of recursive equations (16) has a unique solution $G(.;M)$. Again this follows from the application of the contraction-mapping theorem. Furthermore, G is differen-

tiable in M, and all elements of the Jacobian $(dG(L, x, i; M)/(d\mu_j))_{i,j}$ are non-positive. The RHS of (16) defines an operator mapping a function $G(L, x, i; M)$ with a strictly diagonally dominant Jacobian matrix $(dG(L, x, i; M)/(d\mu_j))_{i,j}$ into another function $\tilde{G}$ whose Jacobian matrix $(d\tilde{G}(L, x, i; M)/(d\mu_j))_{i,j}$ is diagonally dominant. This follows since the transition matrix $\psi(z_j|z_i)$ is strictly diagonally dominant and since all elements of $(d\tilde{G}(L, x, i; M)/(d\mu_j))$ have the same (non-positive) sign. Therefore, the unique fixed point has a strictly diagonally dominant Jacobian. Now suppose that $(z_1, \ldots, z_n)$ is close to $(\overline{z}, \ldots, \overline{z})$ and consider the solution $\mu_1 = \ldots = \mu_n = \overline{\mu}$ of the stationary problem. Since the Jacobian matrix $(dG(0, x, i; M)/(d\mu_j))_{i,j}$ is strictly diagonally dominant, it is invertible. By the implicit function theorem, a unique solution M to equation (17) exits. $\square$

Appendix B: Decentralization

The Workers' Search Problem

Let $U(z^t)$ be the utility value of an unemployed worker in history z^t, and let $W(\mathcal{C}_a, x^k, z^t)$ be the utility value of a worker hired by a firm of age a in contract $\mathcal{C}_a$ who is currently employed at that firm in history x^k, with $k > a$. The latter satisfies the recursive equation

$$W(\mathcal{C}_a, x^k, z^t) = w_a(x^k, z^t) + \beta\Big\{(1 - \varphi_a(x^k, z^t))E_{z^t}U(z^{t+1}) \qquad (59)$$
$$+\varphi_a(x^k, z^t)E_{x^k,z^t}W(\mathcal{C}_a, x^{k+1}, z^{t+1})\Big\} \ .$$

An unemployed worker searches for contracts which promise the highest expected utility, considering that more attractive contracts are less likely to sign. The worker observes all contracts $\mathcal{C}_a$ and he knows that the probability to sign a contract is $m/\lambda(m)$ when m is the firm's matching probability at the offered contract. That is, potential contracts are parameterized by the tuple $(m, \mathcal{C}_a)$. Unemployed workers apply for those contracts where expected surplus is maximized:

$$\rho(z^t) = \max_{(m,\mathcal{C}_a)} \frac{m}{\lambda(m)}(1 - \delta(x^a, z^t))\beta E_{x^a,z^t}\Big[W(\mathcal{C}_a, x^{a+1}, z^{t+1}) - U(z^{t+1})\Big] \ . \qquad (60)$$

The Bellman equation for an unemployed worker reads as

$$U(z^t) = b + \rho(z^t) + \beta E_{z^t}U(z^{t+1}) \ . \qquad (61)$$

The Firms' Problem

A firm of age a in history (x^a, z^t) takes as given the employment stocks of workers hired in some earlier period, $(L_\tau)_{\tau=0}^{a-1}$, as well as the contracts signed with these workers, $(\mathcal{C}_\tau)_{\tau=0}^{a-1}$. For the contracts to be consistent with the firm's constraints on exit and separations, the retention probabilities must satisfy $\varphi_\tau(x^a, z^t) \le (1 - s_0)(1 - \delta_0)$. The firm chooses an actual exit probability $\delta \ge \delta_0$ and cohort-specific layoff probabilities s_τ. For these probabilities to be consistent with separation probabilities specified in existing contracts, it must hold that $\delta \le 1 - \varphi_\tau(x^a, z^t)$ for all $\tau \le a - 1$, and $s_\tau = 1 - \varphi_\tau(x^a, z^t)/(1 - \delta)$ when $\delta < 1$, with arbitrary choice of s_τ when $\delta = 1$. The firm also decides new contracts $\mathcal{C}_a$ to be posted in V vacancies with desired matching probability m. It is no restriction to presuppose that the firm offers only one type of contract. When J_a is the value function of a firm of age a, the firm's problem is written as

$$J_a\Big[(\mathcal{C}_\tau)_{\tau=0}^{a-1}, (L_\tau)_{\tau=0}^{a-1}, x^a, z^t\Big] = \max_{(\delta,m,V,\mathcal{C}_a)} x_a z_t F(L) - W - C(V, L, x_a z_t) \qquad (62)$$
$$-f + \beta(1 - \delta)E_{x^a,z^t}J_{a+1}\Big[(\mathcal{C}_\tau)_{\tau=0}^{a}, (L_{\tau+})_{\tau=0}^{a}, x^{a+1}, z^{t+1}\Big]$$

$$\text{s.t.} \quad L_{a+} = mV, \ m \in [0, 1], \ V \ge 0, \ L_{\tau+} = L_\tau \frac{\varphi_\tau(x^a, z^t)}{1 - \delta}, \ \tau \le a - 1 \ , \qquad (63)$$

$$\delta \in [\delta_0, \min_{0\le\tau\le a-1} 1-\varphi_\tau(x^a,z^t)],\ s_0 \le 1-\varphi_\tau(x^a,z^t)/(1-\delta)\ , \tag{64}$$

$$W = \sum_{\tau=0}^{a-1} w_\tau(x^a,z^t)L_\tau\ ,\ L = \sum_{\tau=0}^{a-1} L_\tau, \tag{65}$$

$$\rho(z^t) = \frac{m}{\lambda(m)}(1-\delta)\beta E_{x^a,z^t}\Big[W(\mathcal{C}_a, x^{a+1}, z^{t+1}) - U(z^{t+1})\Big] \text{ if } m>0\ . \tag{66}$$

The last condition is the workers' participation constraint; it specifies the minimum expected utility that contract $\mathcal{C}_a$ must promise in order to attract a worker queue of length $\lambda(m)$ per vacancy.

Definition: A competitive search equilibrium is a list

$$\Big[U(z^t), \rho(z^t), \mathcal{C}_a(x^a,z^t), m(x^a,z^t), V(x^a,z^t), \delta(x^a,z^t), J_a(.), L_\tau(x^a,z^t), N(x^a,z^t), N_0(z^t)\Big],$$

for all $t \ge 0$, $a \ge 0$, $x^a \in X^{a+1}$, $z^t \in Z^{t+1}$, $0 \le \tau \le a$, and for a given initial firm distribution, such that

(a) Firms' exit, hiring and layoff strategies are optimal. That is, J_a is the value function and $\mathcal{C}_a(.)$, $\delta(.)$, $m(.)$, and $V(.)$ are the policy functions for problem (62)-(66).

(b) Employment evolves according to

$$\begin{aligned} L_\tau(x^a,z^t) &= L_\tau(x^{a-1},z^{t-1})\frac{\varphi_\tau(x^a,z^t)}{1-\delta(x^a,z^t)}\ ,\ 0 \le \tau \le a-1\ , \\ L_a(x^a,z^t) &= m(x^a,z^t)V(x^a,z^t)\ ,\ a \ge 0\ . \end{aligned}$$

(c) Firm entry is optimal. That is, the complementary slackness condition

$$\sum_x \pi_0(x)J_0(x,z^t) \le K(z_t)\ ,\ N_0(z^t) \ge 0\ , \tag{67}$$

holds for all z^t, and the number of firms evolves according to (11) and (13).

(d) Workers' search strategies are optimal, i.e. (ρ, U) satisfy equations (60) and (61).

(e) Aggregate resource feasibility; for all z^t,

$$\sum_{a\ge0,x^a} N(x^a,z^t)\Big[\lambda(m(x^a,z^t))V(x^a,z^t) + \sum_{\tau=0}^{a-1} L_\tau(x^a,z^t)\Big] = 1\ . \tag{68}$$

Proposition 5: *A competitive search equilibrium is socially optimal.*
Proof: The proof proceeds in two steps. First, substitute the participation constraint (66) into the firm's problem and make use of the contracts' recursive equations (59)

to show that the firms' recursive profit maximization problem is identical to the maximization of the social surplus of a firm. Second, show that a competitive search equilibrium is socially optimal.
First, define the social surplus of a firm with history (x^a, z^t) and with predetermined contracts and employment levels as follows:

$$G_a\Big[(\mathcal{C}_\tau)_{\tau=0}^{a-1},(L_\tau)_{\tau=0}^{a-1},x^a,z^t\Big] \equiv J_a\Big[(\mathcal{C}_\tau)_{\tau=0}^{a-1},(L_\tau)_{\tau=0}^{a-1},x^a,z^t\Big]+\sum_{\tau=0}^{a-1}L_\tau\Big[W(\mathcal{C}_\tau,x^a,z^t)-U(z^t)\Big]. \tag{69}$$

Using (59) and (61), the worker surplus satisfies

$$W(\mathcal{C}_\tau,x^a,z^t)-U(z^t) = w_\tau(x^a,z^t)-b-\rho(z^t)+\beta\varphi_\tau(x^a,z^t)E_{x^a,z^t}\Big[W(\mathcal{C}_\tau,x^{a+1},z^{t+1})-U(z^{t+1})\Big].$$

Now substitute this equation and (62) into (69), and write

$$\sigma \equiv \Big[(\mathcal{C}_\tau)_{\tau=0}^{a-1},(L_\tau)_{\tau=0}^{a-1},x^a,z^t\Big] \text{ and } \sigma_+ \equiv \Big[(\mathcal{C}_\tau)_{\tau=0}^{a},(L_{\tau+})_{\tau=0}^{a},x^{a+1},z^{t+1}\Big],$$

with $L_{\tau+}$ as defined in (63) and $L=\sum_{\tau=0}^{a-1}L_\tau$, to obtain

$$\begin{aligned}
G_a(\sigma) &= \max_{\delta,m,V,\mathcal{C}_a}\Bigg\{x_a z_t F(L)-C(V,L,x_a z_t)-f-\sum_{\tau=0}^{a-1}L_\tau w_\tau(x^a,z^t) \\
&\quad +\beta(1-\delta)E_{x^a,z^t}J_{a+1}(\sigma_+)\Bigg\}+\sum_{\tau=0}^{a-1}L_\tau\Bigg[w_\tau(x^a,z^t)-b-\rho(z^t) \\
&\quad +\beta\varphi_\tau(x^a,z^t)E_{x^a,z^t}\Big[W(\mathcal{C}_\tau,x^{a+1},z^{t+1})-U(z^{t+1})\Big]\Bigg] \\
&= \max_{\delta,m,V,\mathcal{C}_a}\Bigg\{x_a z_t F(L)-[b+\rho(z^t)]L-f-C(V,L,x_a z_t)+\beta(1-\delta)E_{x^a,z^t}J_{a+1}(\sigma_+) \\
&\quad +\beta\sum_{\tau=0}^{a-1}L_\tau\varphi_\tau(x^a,z^t)E_{x^a,z^t}\Big[W(\mathcal{C}_\tau,x^{a+1},z^{t+1})-U(z^{t+1})\Big]\Bigg\} \\
&= \max_{\delta,m,V,\mathcal{C}_a}\Bigg\{x_a z_t F(L)-bL-\rho(z^t)[L+\lambda(m)V]-f-C(V,L,x_a z_t) \\
&\quad +\beta(1-\delta)E_{x^a,z^t}J_{a+1}(\sigma_+) \\
&\quad +\beta(1-\delta)\sum_{\tau=0}^{a}L_{\tau_+}E_{x^a,z^t}\Big[W(\mathcal{C}_\tau,x^{a+1},z^{t+1})-U(z^{t+1})\Big]\Bigg\} \\
&= \max_{\delta,m,V,\mathcal{C}_a}\Bigg\{x_a z_t F(L)-bL-\rho(z^t)[L+\lambda(m)V]-f
\end{aligned} \tag{70}$$

$$-C(V,L,x_a z_t)+\beta(1-\delta)E_{x^a,z^t}G_{a+1}(\sigma_+)\Bigg\}\ .$$

Here maximization is always subject to (63) and (64), the third equation makes use of

$$(1-\delta)L_{\tau+}=\varphi_\tau(x^a,z^t)L_\tau\ ,$$

for $\tau\leq a-1$, and

$$\rho(z^t)\lambda(m)V=\beta(1-\delta)L_{a+}E_{x^a,z^t}\Big[W(\mathcal{C}_a,x^{a+1},z^{t+1})-U(z^{t+1})\Big]\ ,$$

and the last equation makes use of (69) for G_{a+1}. This shows that the firm solves a surplus maximization problem which is identical to the one of the planner specified in (56) provided that $\rho(z^t)=\mu(z^t)$ holds for all z^t, where μ is the social value of an unemployed worker as defined in the proof of Proposition 4. The only difference between the two problems is that the firm commits to cohort-specific separation probabilities, whereas the planner chooses in every period an identical separation probability for all workers (and he clearly has no reason to do otherwise). Nonetheless, both problems have the same solution: they are dynamic optimization problems of a single decision maker in which payoff functions are the same and the decision sets are the same. Further, time inconsistency is not an issue since there is no strategic interaction and since discounting is exponential. Hence solutions to the two problems, with respect to firm exit, layoffs and hiring strategies, are identical. In both problems the decision maker could discriminate between different cohorts in principal. Because such differential treatment does not raise social firm value, there is also no reason for competitive search to produce such an outcome. Nonetheless, there can be equilibrium allocations where different cohorts have different separation probabilities, but these equilibria must also be socially optimal because they maximize social firm value.

It remains to verify that competitive search gives indeed rise to socially efficient firm entry. When $\mu(z^t)=\rho(z^t)$, $G_0(x,z^t)=J_0(x,z^t)$ as defined in (69) coincides with $G_0(0,x,z^t)$, as defined in (56). Hence, the free-entry condition (67) coincides with the condition for socially optimal firm entry (57). Because of aggregate resource feasibility (68), the planner's resource constraint (15) is also satisfied. Since the allocation of a competitive search equilibrium satisfies all the requirements of Lemma 4(b), it is socially optimal. $\square$

Appendix C: Calibration details

We choose the period length to be one week and set $\beta = 0.999$ so that the annual interest rate is about 5 percent. We assume a CES matching function $m(\lambda) = (1+k\lambda^{-r})^{-1/r}$ and set the two parameters k and r to target a weekly job-finding rate of 0.129 and an elasticity of the job-finding rate with respect to the vacancy-unemployment ratio of 0.28 (Shimer (2005)).[36] Below we also target the (average) weekly job-filling rate at 0.3, which corresponds to a monthly vacancy yield of 1.3 (Davis et al. (2013)). Thus we calculate the parameters k and r to attain the two targets at $\lambda = 0.3/0.129 = 2.326$.
The production technology is Cobb-Douglas with xL^α where the firm's idiosyncratic productivity $x = x_0 x_1$ contains a time-invariant component x_0 and a transitory component x_1 (cf. Elsby and Michaels (2013)). The time-invariant component is drawn upon firm entry from one of three values x_0^i, $i = 1, 2, 3$, with entry shares σ^i where (x_0^i, σ^i) are chosen to match the firm and employment shares within the three size classes 1-49, 50-499 and ≥ 500. The transitory component x_1 is drawn from one of five equidistant values in the range $[1 - \overline{x}, 1 + \overline{x}]$ and is redrawn every period with probability π. Parameters π and $\overline{x}$ are chosen to match a monthly separation rate of 4.2 percent and the observation that about two thirds of employment is at firms with monthly employment growth rates in the range $[-0.02, 0.02]$ (see Davis et al. (2010)). Firm exit is exogenous; that is, we set the operating cost to $f = 0$ and choose exit probabilities specific for the three firm types δ^i, $i = 1, 2, 3$, to match job losses at closing firms for the three size classes. Parameter α is set to 0.7 which gives rise to a labor share of roughly 2/3.[37]
We choose unemployment income b at 97.7 percent of mean wage earnings. This value corresponds to the parameter value chosen by Hagedorn and Manovskii (2008) which ensures that reasonably small aggregate productivity shocks have quantitatively significant labor market responses. This is only 68 percent of labor productivity, but 96.8 percent of the average (employment-weighted) marginal product. We explore robustness to a much lower value of b in Appendix D. The exogenous quit rate is set at $s_0 = 0.0048$ to match a monthly quit rate of 2 percent. The entry cost parameter K can be normalized arbitrarily since all firm value functions (and thus the free-entry condition) are linearly homogeneous in the vector (x, b, c, K).
As mentioned in the main text, the recruitment technology has the form $c(V) = \frac{c}{1+\gamma}(\frac{V}{L})^{1+\gamma} L$, where we take a cubic function ($\gamma = 2$) for the benchmark calibration. When we compare the benchmark results with those for $\gamma = 0.1$ and for $\gamma = 8$, we recalibrate parameters c and b (or K) to target the average unemployment-vacancy ratio $\lambda = 2.32$ which gives rise to an average weekly job-filling rate of 0.3 and the same b/w ratio as in the benchmark.[38] We note that recruitment costs per hire are

[36]Note that there is no third parameter in the CES matching function since we require that $\lim_{\lambda\to\infty} m(\lambda) = 1$.

[37]Given that all capital is fixed at the level of a firm, this calculation of factor shares ignores capital income accruing from variable capital investment which would suggest a higher value of α. For a robustness analysis, see Appendix D.

[38]Deviating from Table 5, we set $c = 0.00295$, $K = 208.64$ for $\gamma = 0.1$ and $c = 1.84 \cdot 10^7$,

reasonably low for all three parameterizations (below 1% of quarterly earnings). Table 5 summarizes the parameter choices for the benchmark calibration.

Table 5: Parameter choices in the benchmark calibration ($\gamma = 2$).

Parameter	Value	Description	Explanation
β	0.999	Discount factor	Annual interest rate 5%
k	6.276	Matching fct. scale	weekly job-finding rate 0.129
r	1.057	Matching fct. elasticity	0.28 (Shimer (2005))
α	0.7	Prod. fct. elasticity	Labor share
c	0.409	Vacancy cost parameter	weekly job-filling rate 0.3
(x_0^i)	$(.274, .621, 1.488)$	permanent productivity	employment shares (3 size classes)
(σ^i)	$(99.2, .765, .035)\%$	share at entry	firm shares (3 size classes)
(δ^i)	$(2.24, .25, .03)‰$	exit rates	job losses at exiting firms
$\overline{x}$	0.11	Productivity range	monthly separation rate 4.2%
π	0.06	Adjustment prob.	2/3 of employment in firms with employment growth in [-0.02,0.02]
b	0.1	unemployment income	97.7% of wage income
K	205.0	Entry cost	Arbitrary normalization
s_0	0.48%	Quit rate	Monthly quit rate 2%

$K = 186.92$ for $\gamma = 8$ (fixing $b = 0.1$ throughout).

Appendix D: Robustness

We explore the robustness of the calibration exercise regarding different parameter choices for unemployment income b and for the returns-to-scale parameter α. Departing from the benchmark calibration with cubic vacancy costs we consider two variations. First, we consider the alternative of setting unemployment income to 70 percent of average wages (46% of labor productivity), instead of 97.7 percent as in the benchmark. Second, relative to the benchmark with $\alpha = 0.7$ which gives rise to a plausible labor share (with fixed capital at any individual firm) we consider the alternative of $\alpha = 0.95$ which is more in line with a model where capital can be adjusted at the firm level. In both variations, parameters c, $\overline{x}$ and (x_0^i) are readjusted so that the model hits the same calibration targets as in the benchmark calibration.

Figure 4 shows that the cross-sectional behavior of vacancy rates, vacancy yields, hires rates and layoff rates is almost unchanged relative to the benchmark calibration. That is, irrespective of the parameter values for b and α, the model with cubic vacancy costs explains more than half of the cross-sectional variation in vacancy yields for firm growth rates below 20 percent, although the curves flatten out at firm growth above 20 percent relative to the benchmark calibration (blue curve).

Figure 5 shows the impulse responses to a one-percent increase in aggregate productivity. Here the two variations exhibit markedly different patterns, but this is little surprising. First, the model with a lower value of unemployment income clearly generates less amplification (red curves), which is in line with the well-known finding of Shimer (2005) that search and matching models with high match surplus generate too little labor market volatility. The propagation of the shock is similar to the benchmark, however. In the model with a higher returns-to-scale parameter (green curves), the productivity increase generates larger (and hump-shaped) responses of the job-finding rate and of the vacancy-unemployment ratio, but this does not imply that the model exhibits more labor market amplification, since the output response is stronger as well.

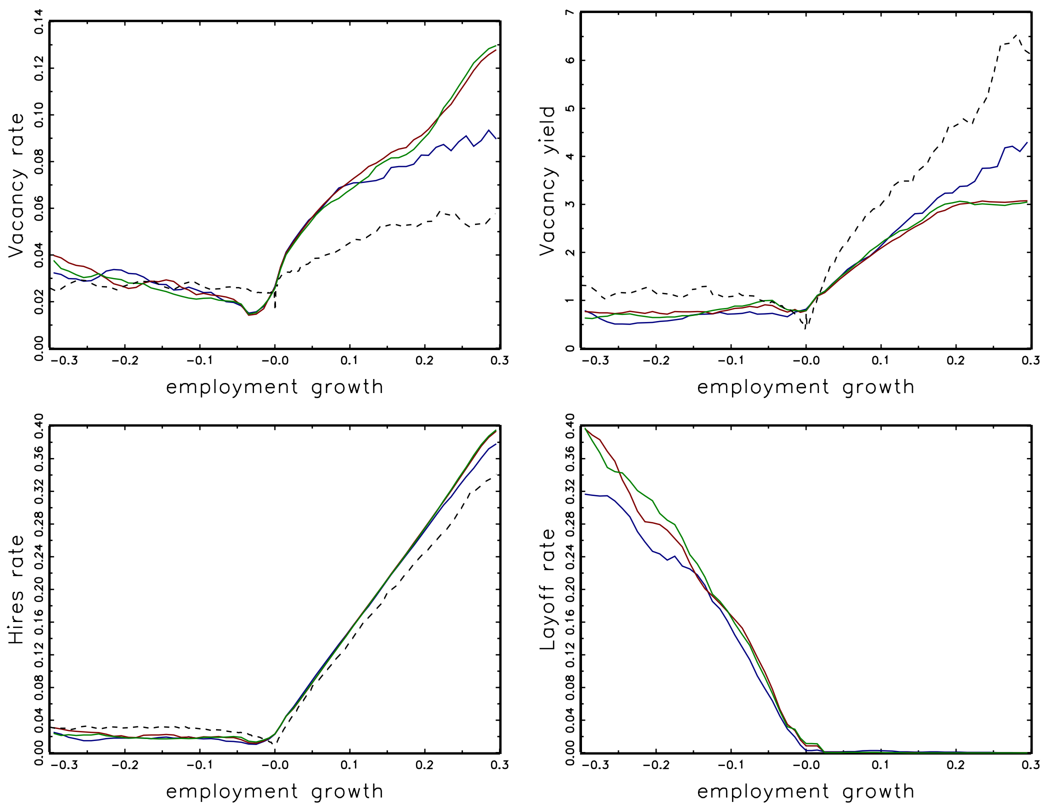

Figure 4: Cross-sectional relationships between monthly employment growth and the vacancy rate, the vacancy yield, the hires rate and the layoff rate. The dashed curves are from the data used in Davis et al. (2013), the solid blue curves are for the benchmark parameterization ($b/w \approx 0.977$, $\alpha = 0.7$), the solid red curves are for the calibration with $b/w \approx 0.7$ and the solid green curves are for $\alpha = 0.95$.

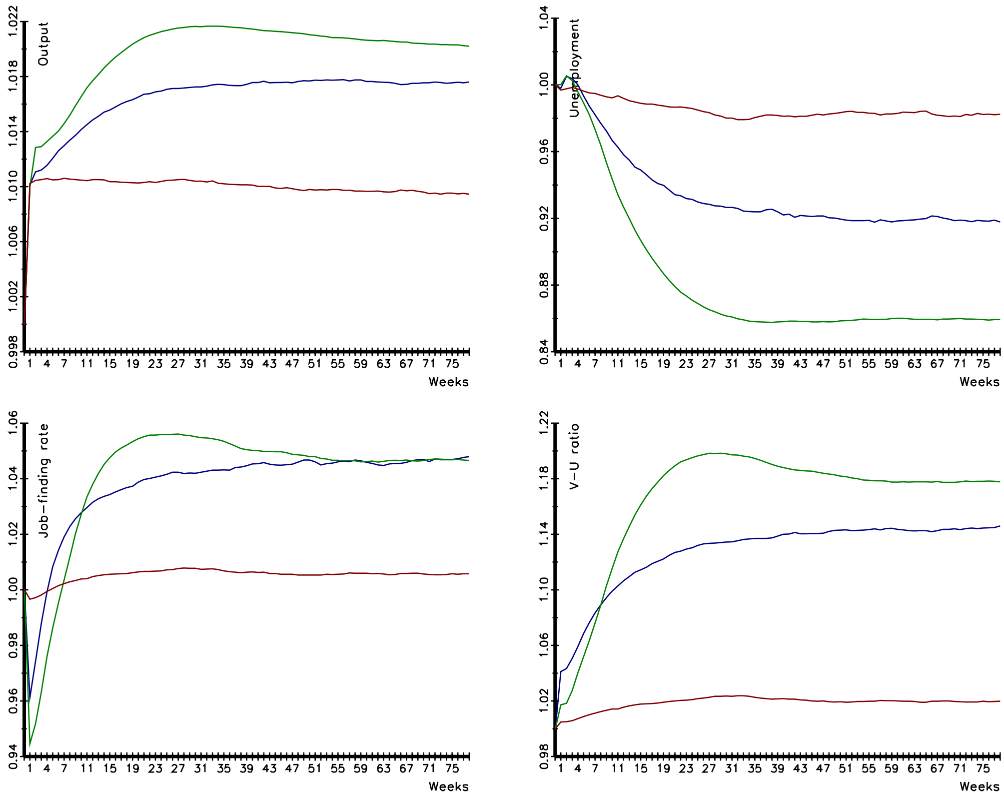

Figure 5: Impulse response to a permanent 1% increase in aggregate productivity. The blue curves are for the benchmark parameterization, the red curves are for $b/w = 0.7$ and the green curves are for $\alpha = 0.95$.