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Measuring the stance of monetary policy in zero lower bound environments*

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Abstract

I propose a simple framework that quantifies the stance of monetary policy as a "shadow short rate" when interest rates are near the zero lower bound. The framework is shown to be a close approximation to the Black (1995) approach for modelling the term structure subject to a zero-lower-bound constraint. I demonstrate my framework with a one-factor model applied to Japanese data, including an intuitive economic interpretation of the results, and also discuss the extension to multiple factors.

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1 Introduction

In this article I propose a simple framework for quantifying the stance of monetary policy in terms of a "shadow short rate" when nominal interest rates within the term structure are near the zero lower bound (ZLB).

The ZLB framework I propose is a tractable and close approximation to the Black (1995) framework for modelling the term structure in ZLB environments. The Black framework obtains non-negative short rates as $\underline{\mathrm{r}}(t) = \max\{\mathrm{r}(t), 0\}$, which represents the "real world" option to hold physical currency when the shadow short rate evolves to negative values.[1] Bond prices and yields are then generated from the expected path of $\underline{\mathrm{r}}(t)$. However, as I will discuss in section 4, practical implementations of the Black framework are relatively complex, particularly as the number of factors increase.

Conversely, my ZLB framework is effectively based on non-negative forward rates obtained using bond options to represent the availability of physical currency. I outline the framework in section 2. Section 3 compares a one-factor version of my ZLB framework to the Black framework, and section 4 applies my ZLB framework empirically to Japanese data. I conclude in section 5 and also discuss the important advantages relative to the Black framework for extensions to multiple factors.

2 A non-negative forward rate framework

To establish notation, I introduce a finite-step shadow nominal bond with a price $\mathrm{P}(t+\tau, \delta)$ at time $t+\tau$ that pays 1 at time $t+\tau+\delta$, where $\tau \geq 0$ is any future horizon from time t and $\delta > 0$ represents the time to maturity. I also assume physical currency is always available at time $t+\tau$ with a price of 1 and will pay 1 at time $t+\tau+\delta$.

To maximize their returns, investors will choose the minimum priced investment at time $t+\tau$, i.e. $\min\{1, \mathrm{P}(t+\tau, \delta)\}$.[2] This expression may be re-arranged to $1 - \max\{0, 1 - \mathrm{P}(t+\tau, \delta)\}$, which is a terminal boundary condition in two convenient components. Respectively, the boundary condition of 1 implies a shadow bond price at time t of $\mathrm{P}(t, \tau)$, and $\max\{0, 1 - \mathrm{P}(t+\tau, \delta)\}$ implies a put option price at time t of $\mathrm{Q}(t, \tau, \tau+\delta)$, with a strike price of 1 and expiry at time $t+\tau$. The combined solution $\mathrm{P}(t,\tau) - \mathrm{Q}(t, \tau, \tau+\delta)$ may then be expressed as $\mathrm{P}(t, \tau+\delta) - \mathrm{C}(t, \tau, \tau+\delta)$, where $\mathrm{C}(t, \tau, \tau+\delta)$ is a call option with a strike price of 1 and expiry at time $t+\tau$.[3]

The expression $\mathrm{P}(t, \tau+\delta) - \mathrm{C}(t, \tau, \tau+\delta)$ may be used to obtain forward rates $\underline{\mathrm{f}}(t, \tau)$ that are guaranteed to be non-negative for all maturities. Specifically, the most

[1] A prevalent literature has evolved over several decades with various specifications of short-rate dynamics designed to avoid negative short rates. Examples are Cox, Ingersoll, and Ross (1985)/square-root models, appropriately constrained quadratic-Gaussian models, and log-interest-rate models; James and Webber (2000) pp. 226-33 provides further discussion. However, such models lack the potential information provided by the shadow short rate in the Black framework and in the present article. Note also that the shadow rate, as originally named in Black (1995), is not a shadow price in the usual economic sense; i.e. it is not the marginal change of an objective function with respect to a constraint.

[2] Investors' choices will not be distorted by inflation considerations, because any such effects on the real returns from nominal bond and physical currency will be identical.

[3] The re-expression uses standard put-call parity, i.e. $\mathrm{F} = \mathrm{C} - \mathrm{Q}$, with strike prices of 1. Hence, setting the forward bond price $\mathrm{F} = \mathrm{P}(t, \tau+\delta) \,/\, \mathrm{P}(t,\tau) - 1 = 0$ gives $\mathrm{P}(t, \tau+\delta) - \mathrm{P}(t,\tau) = \mathrm{C} - \mathrm{Q}$, and so $\mathrm{P}(t, \tau+\delta) - \mathrm{C} = \mathrm{P}(t,\tau) - \mathrm{Q}$.

transparent way to obtain what I will refer to as currency-adjusted-bond (CAB) forward rates is the following numerical approximation:[4]

$$\underline{\mathrm{f}}(t,\tau) = -\frac{1}{\delta}\left(\log\left[\frac{\mathrm{P}(t,\tau+\delta) - \mathrm{C}(t,\tau,\tau+\delta)}{\mathrm{P}(t,\tau)}\right]\right) \tag{1}$$

Note that I use an underscore to denote quantities that are constrained by the ZLB, such as $\underline{\mathrm{f}}(t,\tau)$, and omit the underscore to denote shadow quantities that have no ZLB constraint, such as $\mathrm{P}(t,\tau+\delta)$.

CAB interest rates corresponding to $\underline{\mathrm{f}}(t,\tau)$ may be obtained using the standard term structure relationship $\underline{\mathrm{R}}(t,\tau) = \frac{1}{\tau}\int_0^\tau \underline{\mathrm{f}}(t,\upsilon)\mathrm{d}\upsilon$ where υ is a dummy integration variable from zero to the time to maturity. Note that the numerical approximation to $\underline{\mathrm{R}}(t,\tau)$ is conveniently the arithmetic mean of $\underline{\mathrm{f}}(t,\tau)$ when the latter is calculated at uniformly spaced maturities $\Delta\tau$.[5]

3 Comparing the CAB and Black frameworks

In this section I compare results from the Black and CAB frameworks using the risk-neutral Vasicek (1977) model to represent the shadow short rate process and term structure. Specifically, the diffusion process is $\mathrm{d\,r}(t) = \kappa\left[\theta - \mathrm{r}(t)\right]\mathrm{d}\,t + \sigma\mathrm{d}W(t)$, where $\mathrm{r}(t)$ is the shadow short rate (the single state variable), κ, θ, and σ are respectively the mean reversion, steady state level, and volatility (annualized standard deviation) parameters, and $\mathrm{d}W(t)$ are Gaussian unit normal $\mathrm{N}(0,1)$ innovations.

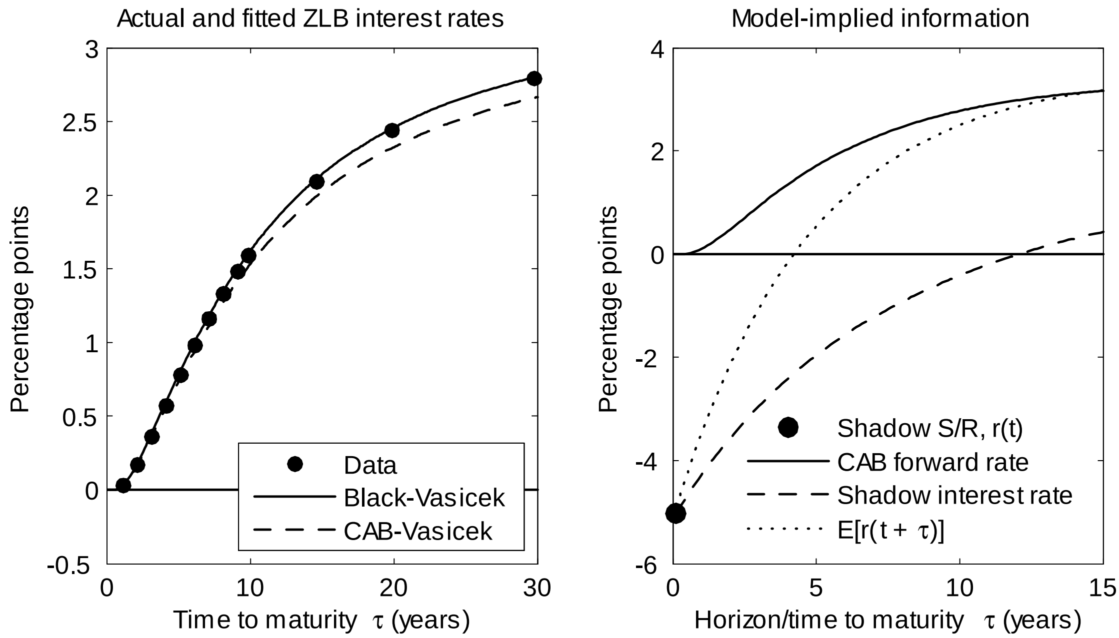

Figure 1: Actual, CAB-Vasicek, and Black-Vasicek term structures for Japan in February 2004, and associated model-implied information.

[4]The expression arises from the standard term structure relationship and intermediate steps as follows: $\underline{\mathrm{f}}(t,\tau) = -\frac{\mathrm{d}}{\mathrm{d}\tau}\log\left[\underline{\mathrm{P}}(t,\tau)\right] \simeq -\frac{1}{\delta}\left(\log\left[\frac{\mathrm{P}(t,\tau+\delta)-\mathrm{C}(t,\tau,\tau+\delta)}{\mathrm{P}(t,\tau)-\mathrm{C}(t,\tau,\tau)}\right]\right)$, and $\mathrm{C}(t,\tau,\tau) = 0$. For cross-checking the results in sections 3 and 4, I have also derived a lengthier analytic expression for $\underline{\mathrm{f}}(t,\tau)$ in the limit as $\delta \to 0$; see appendix A.

[5]That is, $\underline{\mathrm{R}}(t,\tau) = \frac{1}{\tau}\int_0^\tau \underline{\mathrm{f}}(t,\upsilon)\mathrm{d}\upsilon \simeq \frac{1}{\tau}\left[\Delta\tau\sum_{i=1}^{I}\underline{\mathrm{f}}(t,i\Delta\tau)\right] = \frac{1}{I}\sum_{i=1}^{I}\underline{\mathrm{f}}(t,i\Delta\tau)$, where $\Delta\tau = \tau/I$.

Panel 1 of figure 1 summarizes the zero-coupon government bond yield data from Gorovoi and Linetsky (2004) table 7.1, p. 71 and the estimated Black-Vasicek interest rates (i.e. based on the risk-neutral Vasicek model within the Black framework) from the same source. I obtain comparable CAB-Vasicek interest rates $\underline{\mathrm{R}}(t,\tau)$ from values of $\underline{\mathrm{f}}(t,\tau)$ obtained via equation 1. Specifically, I use the closed-form analytic bond price and bond option price formulas for the Vasicek model (as available from standard textbooks; see, for example, Hull (2000) pp. 567-8) and the risk-neutral Vasicek state variable/parameter set from Gorovoi and Linetsky (2004), i.e. $\{\mathrm{r}(t), \kappa, \theta, \sigma\} = \{-0.0512, 0.212, 0.0354, 0.0283\}$ to evaluate $\underline{\mathrm{f}}(t,\tau)$ and then $\underline{\mathrm{R}}(t,\tau)$.

The immediate point to note for the purpose of the present article is that the CAB-Vasicek and Black-Vasicek term structures are not identical despite sharing an identical shadow short rate specification. That difference is fundamental rather than due to numerical approximation,[6] and arises because the Black (1995) framework restricts current and future short rates to be non-negative while the CAB framework restricts all current forward rates to be non-negative.[7] That said, the differences between the two frameworks are very small in this example, i.e. a maximum of 14 basis points (bps) at the 30-year maturity. Parameter sensitivity tests show that long-maturity divergences increase mainly with larger values of σ and smaller values of κ (see appendix B). However, the divergences remain small for typical parameters values, including those estimated in the following section.

Panel 2 of figure 1 illustrates model-implied information associated with panel 1. First, the shadow short rate is the value of the shadow interest rate $\mathrm{R}(t,\tau) = -\log[\mathrm{P}(t,\tau)]/\tau$ in the limit of a zero time to maturity, i.e. $\mathrm{r}(t) = \mathrm{R}(t,0)$. Second, CAB-Vasicek forward rates are non-negative for all times to maturity. Third, I plot the model-implied expected path of the short rate conditional on the prevailing value of the shadow short rate $\mathrm{r}(t)$; i.e. $\mathrm{E}[\mathrm{r}(t+\tau)\,|\,\mathrm{r}(t)]$ which I abbreviate to $\mathrm{E}[\mathrm{r}(t+\tau)]$. That expectation is given by the standard Vasicek expression $\mathrm{E}[\mathrm{r}(t+\tau)] = \theta + \exp(-\kappa\tau)[\mathrm{r}(t) - \theta]$, and so negative values of $\mathrm{r}(t)$ can readily be translated into a horizon τ_0 at which $\mathrm{E}[\mathrm{r}(t+\tau)]$ crosses zero, i.e.:

$$\tau_0 = -\frac{1}{\kappa}\log\left[-\frac{\theta}{r(t)-\theta}\right] \tag{2}$$

The value of the zero horizon τ_0 can be interpreted as the market expectation of a return to a conventional monetary policy environment; i.e. when the ZLB will no longer impose a constraint between the shadow short rate and the actual short rate.[8] Figure 1 has a value of $\tau_0 = 4.2$ years.

[6] I use $\delta = 10^{-6}$ years to obtain $\underline{\mathrm{f}}(t,\tau)$ but the numerical results can be made more precise with smaller values of δ. The analytic expression for $\underline{\mathrm{f}}(t,\tau)$ in appendix A gave practically identical results. Similarly, I use $\Delta\tau = 0.001$ to numerically evaluate $\underline{\mathrm{R}}(t,\tau)$ but the results are insensitive to finer spacing and/or alternative methods of numerical integration.

[7] Therefore, the CAB-Vasicek model offers arbitrage opportunities relative to the Black-Vasicek model, obtainable in principle by selling bonds priced via the CAB-Vasicek framework and investing the proceeds in a rolling investment of $\max\{\mathrm{r}(t), 0\}$.

[8] Interest rates along the term structure will still have ZLB effects to various degrees, given that the prices and yields of securities are based on the expected value of the $\mathrm{r}(t)$ diffusion process which will be constrained by the ZLB.

4 Applying the CAB-Vasicek model to Japan

In this section I provide a simple illustration of applying the CAB-Vasicek model empirically to Japanese data. The data are the end-of-month zero-coupon government bond yields for the 3-month to 7-year maturities shown in each sub-plot of figure 2, plus the 10-, 15-, 20-, and 30-year data for each date, all sourced from Bloomberg. The CAB-Vasicek model applied is as specified in section 3, but I have also allowed for risk premiums by adopting the original Vasicek (1977) model with a constant market price of risk to represent the shadow term structure.[9] The closed-form analytic bond and option price formulas are available from Chaplin (1987) (or from Chaplin (1987) and Chen (1995) by imposing a single factor).

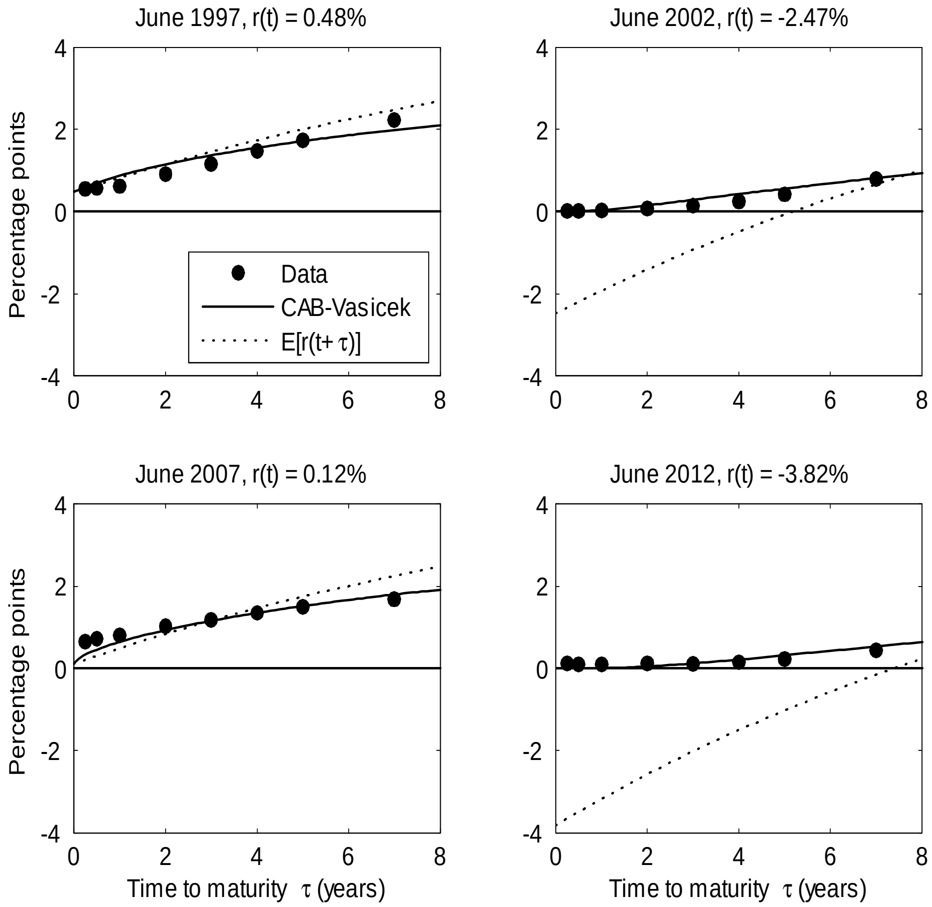

Figure 2: Japanese interest rate data, estimated CAB-Vasicek interest rates $\underline{\mathrm{R}}(t, \tau)$, and model-implied expected paths of the shadow short rate $\mathrm{E}[\mathrm{r}\,(t+\tau)]$.

I use non-linear least squares to jointly estimate the CAB-Vasicek state variables $\mathrm{r}(t)$ for each date and the three parameters across all dates (the latter ensures that

[9]Bond risk premiums will therefore be a time-invariant function of time to maturity. Time-varying risk premiums could readily be allowed for using the essentially affine market price of risk specification from Duffee (2002), but the essentially affine component of Black-Vasicek model is found by Ichiue and Ueno (2006) to be statistically insignificant. Similarly, I found little difference between results obtained with affine and essentially affine Vasicek specifications within the CAB framework, so I have chosen the more parsimonious specification for this article.

the model is intertemporally self-consistent). Regarding divergences with the Black-Vasicek model, I obtain the latter results using Monte Carlo simulations with the estimated CAB-Vasicek state variables $\mathrm{r}(t)$ noted in figure 2 and the shadow Vasicek model parameters estimates $\{\kappa, \theta, \sigma, \gamma\} = \{0.0704, 0.0561, 0.0179, -0.0168\}$.[10] The results are indistinguishable from the CAB-Vasicek results (i.e. a maximum of 4 bps for the 7-year maturity shown, rising to 7 and 27 bps basis points for the 10- and 30-year maturities respectively) so I have omitted them for clarity.

Each sub-figure contains the model-implied expected path of the short rate $\mathrm{E}[\mathrm{r}\,(t+\tau)]$ associated with $\mathrm{r}(t)$. The respective zero horizons for the two negative shadow short rate values as at June 2002 and June 2012 are $\tau_0 = 5.2$ years and $\tau_0 = 7.4$ years.

From an economic perspective, the levels and changes of the shadow short rate $\mathrm{r}(t)$ reflect the stances and changes of monetary policy monetary policy around each date. Specifically: (1) $\mathrm{r}(t)$ is initially positive, and at a level close to the prevailing 0.5 percent official discount rate; (2) $\mathrm{r}(t)$ becomes materially negative following the zero interest rate policy (ZIRP) instigated by the Bank of Japan in February 1999 and subsequent unconventional monetary policy measures (i.e. easings via quantitative money targets)

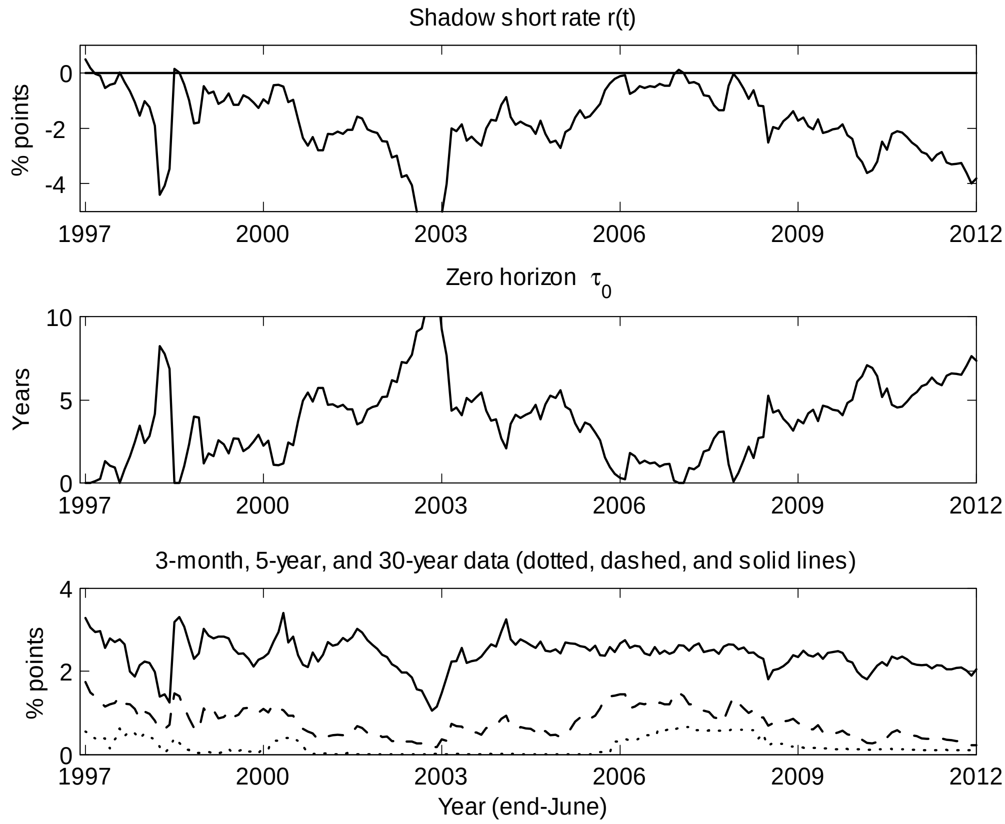

Figure 3: Estimated CAB-Vasicek shadow short rates $\mathrm{r}(t)$ and zero horizon times τ_0 with parameters $\{\kappa, \theta, \sigma, \gamma\} = \{0.0704, 0.0561, 0.0179, -0.0168\}$. The lower panel plots the Japanese data for selected times to maturity.

[10] I use the Euler discretization of the Vasicek diffusion for $\mathrm{r}(t)$ with antithetic draws. All implementations are undertaken to ensure that the standard deviation of the estimated interest rate for each maturity is less than 0.5 basis points.

announced from December 2001; (3) r(t) becomes slightly positive again following the exit from the ZIRP (July 1995);[11] and (4) r(t) becomes very negative following the re-instigation of the ZIRP and quantitative easing measures (October 2010) and subsequent measures in the wake of the Global Financial Crisis.

Figure 3 provides the estimated monthly time series for the shadow short rate r(t) (and the associated zero horizon τ_0) based on the estimated parameters $\{\kappa, \theta, \sigma, \gamma\}$ noted earlier. The local minimum for the most recent estimates is May 2012 with r(t) $= -3.99$ percent ($\tau_0 = 7.6$ years), which is the lowest value since the onset of the Global Financial Crisis during 2007/2008. The local minimum of August 2010, i.e. r(t) $= -3.63$ percent ($\tau_0 = 7.1$ years), corresponds with the U.S. Federal Reserve presaging a second round of unconventional monetary policy measures (i.e. easing via large scale asset purchases) at the Jackson Hole conference, and the likely anticipation of the Bank of Japan's re-instigation of the ZIRP in October 2010.

There are two historical periods when shadow short rates temporarily dipped lower than their most recent values, but those episodes are likely dominated by flow-driven movements rather than representing genuine monetary policy expectations. For example, the global minimum for the sample is May 2003, with r(t) $= -8.27$ percent and $\tau_0 = 12.9$ years (both off scale). That period corresponds to the U.S. deflation scare and new record lows in U.S. bond yields at the time; in sympathy, all Japanese yields with maturities three years or greater reached their global low in April or May 2003 (e.g. the 30-year rate reached 1.05 percent).

The other local minimum is September 1998, with r(t) $= -4.40$ percent and $\tau_0 = 8.2$ years. That period corresponds to the Asian/Russian/Long Term Capital Management crisis, which was accompanied by sharp declines in U.S. bond yields associated with "flight to quality" buying and U.S. monetary policy easing.

The profile of the Black-Vasicek results for r(t) and τ_0 from Ichiue and Ueno (2006) over the comparable dates are similar to my CAB-Vasicek results,[12] although the magnitudes differ. The differences are likely partly due to the different sample period, but mainly because I use 3-month to 30-year interest rate data which results in a smaller estimate of $\kappa = 0.0704$ associated with a larger estimate of $\theta = 0.0561$ (i.e. a steady-state shadow short rate level of 5.61 percent). Ichiue and Ueno (2006) use 6-month to 10-year data over the period 1995 to 2006 and obtain $\kappa = 0.215$ with $\theta = 1.45$ percent.

Appendix C shows that I get results more similar to Ichiue and Ueno (2006) when repeating my estimation over the 1997-2012 period using 3-month to 10-years data. At the same time, the difference in the magnitudes of those results relative to the 3-month to 30-year results illustrates the sensitivities to different data/parameters, hence indicating that it is important to quote shadow short rates and zero horizon times in conjunction with their associated model specification, parameters, and data.

[11] Although, with reference to figure 3, the positive value is only for a single month and it is surrounded by moderately negative values. In other words, the term structure around that time is generally shaped as if the ZIRP and some unconventional monetary policy remained in place or the market expected a return to such an environment.

[12] As noted by Kim and Singleton (2011) p. 25, Ueno, Baba, and Sakurai (2006) obtains implausibly low shadow short rates (a low of around 18 percent). Those results may be due to using a risk-neutral Vasicek model with the non-intemporally consistent approach of separately estimating the state variable and parameters for each term structure observation.

5 Conclusion and extensions

The results in this article suggest that the CAB-Vasicek framework offers a simple, tractable, and close approximation to the Black-Vasicek model for summarizing the stance of monetary policy in a ZLB environment. The estimated shadow short rates from the CAB-Vasicek model are consistent with the evolution of Japanese monetary policy from the late 1990s.

Two obvious examples of the many potential extensions to this article are applying the model to other countries, and improving the model estimation (likely with non-linear filtering and potentially incorporating option price data). The third and most important extension is to multiple factors; first because it is generally accepted that single factor models are not realistic representations of the term structure; and second because Black-Gaussian models increase substantially in "numerical intensity" (i.e. the number of analytic calculations required for implementation) as factors are added.[13] Conversely, the numerical intensity of CAB-Gaussian models does not change with the number of factors because closed-form analytic solutions for bond and option prices are available (see Chen (1995), for example). Finally, if precise Black implementations are required, the CAB framework should facilitate more efficient estimation for one or more factors via Monte Carlo simulations.

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[13] Bomfim (2003), Ueno, Baba, and Sakurai (2006), and Ichiue and Ueno (2007) have respectively used finite-difference grids, Monte Carlo simulations, and interest rate lattices for two-factor Gaussian Black implementations. The numerical intensity of these methods increases to the order of the power of the number of factors. The Gorovoi and Linetsky (2004) approach is semi-analytic, but does not appear to generalize to multiple factors; see Kim and Singleton (2011) p. 11.

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A The analytic expression for CAB-Vasicek forward rates

Appendices A and B in Krippner (2012) contain the expression for CAB forward rates when the generic Gaussian affine term structure model from Chen (1995) is used to represent the shadow-GATSM term structure. To summarize the specification, the shadow short rate is:

$$\mathrm{r}(t) = \sum_{n=1}^{N} s_n(t) \tag{3}$$

where $s_n(t)$ are the N state variables that evolve as a correlated Ornstein-Uhlenbeck process under the physical or $\mathbb{P}$ measure, i.e.:

$$\mathrm{d}s_n(t) = \kappa_n[\theta_n - s_n(t)]\,\mathrm{d}t + \sigma_n \mathrm{d}W_n(t) \tag{4}$$

where θ_n are constants representing the long-run levels of $s_n(t)$, κ_n are positive constants representing the mean reversion rates of $s_n(t)$ to θ_n, σ_n are positive constants representing the volatilities (annualized standard deviations) of $s_n(t)$, $W_n(t)$ are Wiener processes with $\mathrm{d}W_n(t) \sim N(0,1)\mathrm{d}t$, and $\mathbb{E}[\mathrm{d}W_m(t), \mathrm{d}W_m(t)] = \rho_{mn}\mathrm{d}t$, where ρ_{mn} are correlations $-1 \leq \rho_{mn} \leq 1$. The market prices of risk for each factor are constants γ_n.[14]

Krippner (2012) derives the associated CAB forward rate expression as:

$$\underline{\mathrm{f}}(t,\tau) = \mathrm{f}(t,\tau)\cdot \mathrm{N}\left[\frac{\mathrm{f}(t,\tau)}{\omega(\tau)}\right] + \omega(\tau)\cdot\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left[\frac{\mathrm{f}(t,\tau)}{\omega(\tau)}\right]^2\right) \tag{5}$$

[14]The specification could readily be extended to the essentially affine market prices of risk from Duffee (2002); i.e. $\Gamma(t) = \gamma_0 + \gamma_1 s(t)$ in obvious matrix notation, although such an extension is irrelevant for the risk-neutral specification I derive here.

where $\mathrm{f}(t,\tau)$ is the instantaneous shadow forward rate:

$$\begin{aligned} \mathrm{f}\left(t,\tau\right) &= \sum_{n=1}^{N}\theta_n + \left[s_n\left(t\right)-\theta_n\right]\cdot\exp\left(-\kappa_n\tau\right) \\ &\quad +\sum_{n=1}^{N}\sigma_n\gamma_n\cdot G\left(\kappa_n,\tau\right) \\ &\quad -\frac{1}{2}\mathrm{Tr}\left[\Theta\left(\tau\right)\Psi\right] \end{aligned} \tag{6}$$

with $G\left(\kappa_n,\tau\right)=\frac{1}{\kappa_n}\left[1-\exp\left(-\kappa_n\tau\right)\right]$, $\Theta_{ij}\left(\tau\right)=\rho_{ij}\sigma_i\sigma_j\cdot\kappa_i\kappa_j G\left(\kappa_i,\tau\right)G\left(\kappa_j,\tau\right)$, $\Psi_{ij}=\frac{1}{\kappa_i\kappa_j}$, and $\mathrm{Tr}[\cdot]$ the matrix trace operator; and $\omega\left(\tau\right)$ is the instantaneous annualized volatility:

$$\omega\left(\tau\right)=\sqrt{\sum_{n=1}^{N}\sigma_n^2\cdot G\left(2\kappa_n,\tau\right)+2\sum_{m=1}^{N}\sum_{n=m+1}^{N}\rho_{mn}\sigma_m\sigma_n\cdot G\left(\kappa_m+\kappa_n,\tau\right)} \tag{7}$$

The Vasicek (1977) model is a member of the generic GATSM class with $N=1$, $s_1\left(t\right)=\mathrm{r}(t)$, $\kappa_1=\kappa$, $\theta_1=\theta$, $\sigma_1=\sigma$, and $\gamma_1=\gamma$. Making the relevant substitutions for $\mathrm{f}(t,\tau)$ in the first line of equation 6 gives $\theta+\left[\mathrm{r}\left(t\right)-\theta\right]\cdot\exp\left(-\kappa\tau\right)$, the second line gives $\sigma\gamma\cdot G\left(\kappa,\tau\right)$, and the third line gives $\sigma^2\cdot\frac{1}{2}G\left(\kappa,\tau\right)^2$ (given $\Theta_{11}\left(\tau\right)=\sigma^2\cdot\kappa^2\left[G\left(\kappa,\tau\right)\right]^2$, and $\Psi=1/\kappa^2$). The substitutions for $\omega\left(\tau\right)$ give $\omega\left(\tau\right)=\sqrt{\sigma^2\cdot G\left(2\kappa_n,\tau\right)}$.

Therefore, the resulting analytic expression for CAB-Vasicek forward rates is:

$$\underline{\mathrm{f}}\left(t,\tau\right) = \mathrm{f}\left(t,\tau\right)\cdot\mathrm{N}\left[\frac{\mathrm{f}\left(t,\tau\right)}{\omega\left(\tau\right)}\right]+\omega\left(\tau\right)\cdot\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left[\frac{\mathrm{f}\left(t,\tau\right)}{\omega\left(\tau\right)}\right]^2\right) \tag{8a}$$

$$\mathrm{f}\left(t,\tau\right) = \theta+\left[\mathrm{r}\left(t\right)-\theta\right]\cdot\exp\left(-\kappa\tau\right)-\sigma\gamma\cdot G\left(\kappa,\tau\right)-\sigma^2\cdot\frac{1}{2}G\left(\kappa,\tau\right)^2 \tag{8b}$$

$$\omega\left(\tau\right) = \sigma\sqrt{G\left(2\kappa,\tau\right)} \tag{8c}$$

Note that $\int_0^\tau\underline{\mathrm{f}}(t,\upsilon)\mathrm{d}\upsilon$ does not admit a closed-form analytic solution (because the integral of the cumulative normal density function is non-analytic), so $\underline{\mathrm{R}}(t,\tau)$ must be obtained by numerical integration whether $\underline{\mathrm{f}}(t,\tau)$ is obtained with its analytic expression or its numerical approximation. (The integral of the normal density function is also non-analytic, but it is well-tabulated or readily approximated analytically via the error function $\mathrm{erf}\left(x\right)$. Similarly, tabulating the integral of the cumulative normal density function or using an analytic approximation may prove more time-efficient than direct numerical integration.)

B The sensitivity of CAB-Vasicek and Black-Vasicek divergences

Figure 4 illustrates the sensitivity of divergences between the Black-Vasicek and CAB-Vasicek frameworks to changes in the parameters of the shadow short rate specification. The first sub-figure repeats panel 1 from figure 2, i.e. the ZLB models with the state variable parameter set $\{\mathrm{r}(t), \kappa, \theta, \sigma\} = \{-0.0512, 0.212, 0.0354, 0.0283\}$ from Gorovoi and Linetsky (2004), and the second sub-figure plots the divergence between the two frameworks. The remaining sub-figures plot the divergences (not changes in divergences) between the two frameworks when the given parameter changes are made while holding the other parameters at their Gorovoi and Linetsky (2004) values. Note that the divergence increases mainly with larger values of volatility σ and smaller values of mean-reversion κ. The sensitivity of divergences to changes in the steady state level θ and the shadow short rate $\mathrm{r}(t)$ are immaterial.

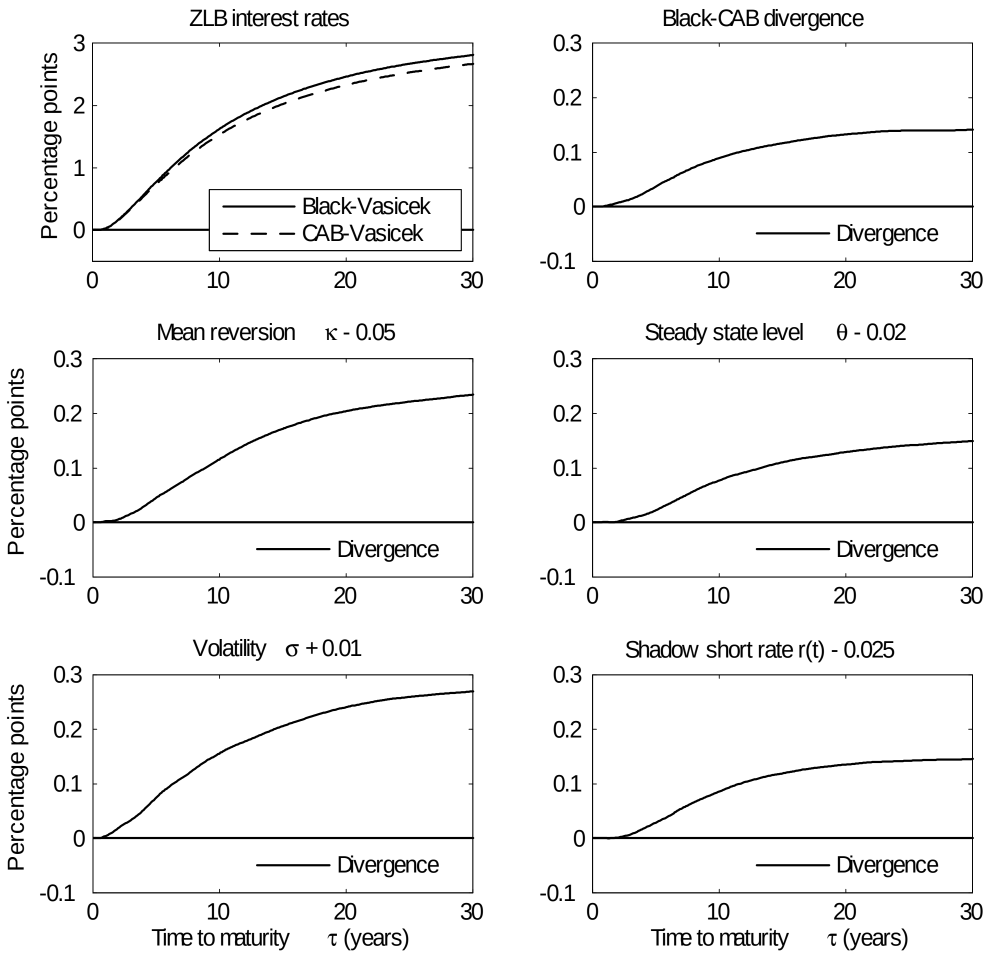

Figure 4: Divergences between the Black-Vasicek and CAB-Vasicek frameworks with the base shadow short rate specification $\{\mathrm{r}(t), \kappa, \theta, \sigma\}$ $= \{-0.0512, 0.212, 0.0354, 0.0283\}$ and with the given parameters changes labelled in subsequent sub-figures.

C Alternative estimated results for the CAB-Vasicek model

Figure 5 illustrates the CAB-Vasicek results estimated as described in the main text, but using 3-month to 10-year time-to-maturity data. The estimated parameters are

$\{\kappa, \theta, \sigma, \gamma\} = \{0.199, 0.0294, 0.0234, -0.00916\}$, which are similar to those in Ichiue and Ueno (2006), as are the associated shadow short rates and zero horizons.

At the same time, while the profiles of $\mathrm{r}(t)$ and τ_0 remain consistent with the results in figure 3, the magnitudes are quite different. That differences indicate that $\mathrm{r}(t)$ and τ_0 are materially sensitive to the parameter sets for the shadow short rate model, which in turn highlights the importance of quoting the results for $\mathrm{r}(t)$ in association with the model specification and estimated parameters.

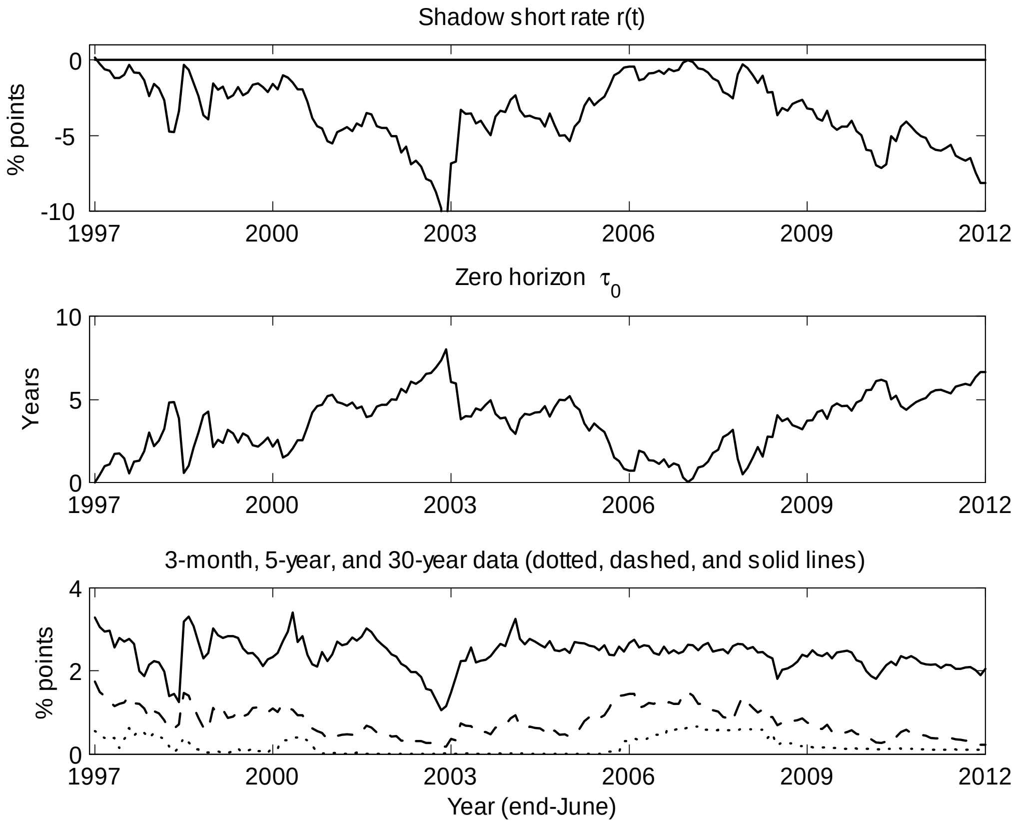

Figure 5: Estimated CAB-Vasicek shadow short rates $\mathrm{r}(t)$ and zero horizon times τ_0 with parameters $\{\kappa, \theta, \sigma, \gamma\} = \{0.199, 0.0294, 0.0234, -0.00916\}$. The lower panel plots data for selected times to maturity.