Multi-Agent Search with Deadline*

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First version: January 11, 2011
This version: December 31, 2011

Abstract

We study a multi-agent search problem with a deadline: for instance, the situation that arises when a husband and a wife need to find an apartment by September 1. We provide an understanding of the factors that determine the positive search duration in reality. Specifically, we show that the expected search duration does not shrink to zero even in the limit as the search friction vanishes. Additionally, we find that the limit duration increases as more agents are involved, for two reasons: the *ascending acceptability effect* and the *preference heterogeneity effect*. The convergence speed is high, suggesting that the mere existence of *some* search friction is the main driving force of the positive duration in reality. Welfare implications and a number of discussions are provided.

*We thank David Ahn, Attila Ambrus, Pol Antràs, Katie Baldiga, Alessandro Bonatti, Georgy Egorov, Drew Fudenberg, Chiaki Hara, Johannes Hörner, Chong Huang, Haruo Imai, Yuhta Ishii, Atsushi Kajii, Fuhito Kojima, David Laibson, Bart Lipman, Mihai Manea, Jordi Massó, Akihiko Matsui, Sho Miyamoto, Akira Okada, Wojciech Olszewski, Daisuke Oyama, Debraj Ray, Al Roth, Yuval Salant, Larry Samuelson, Bruno Strulovici, Tomasz Strzalecki, Takuo Sugaya, Satrou Takahashi, Kentaro Tomoeda, Takahiro Watanabe, Alex Wolitzky, Yuichi Yamamoto, Yosuke Yasuda, and seminar/conference participants at Universitat Autònoma de Barcelona, Brown University, Columbia University, Harvard University, Hitotsubashi University, Kyoto University, University of Tokyo, Yale University, the 22nd Summer Festival on Game Theory (International Conference on Game Theory) at Stony Brook, SWET 2011 at Hokkaido University, and GDRI Workshop Marseille for helpful comments. In addition, Morgan McClellon read through the previous version of this paper and gave us very detailed comments, which significantly improved the presentation of the paper. A portion of this research was conducted while Kamada was visiting Institut d'Anàlisi Econòmia at Universitat Autònoma de Barcelona; he thanks the university and especially Joan Esteban for the hospitality during the stay. Kamada thanks his advisors, Attila Ambrus, Al Roth, Tomasz Strzalecki, and especially Drew Fudenberg for extensive discussions and comments as well as continual guidance and support.

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1 Introduction

This paper studies a search problem with two features that arise in many real-life situations: The decision to stop searching is made by *multiple individuals*, and there is a *predetermined deadline* by which a decision has to be made. Our primary goal is to provide an understanding of the factors that determine the positive search duration in reality.

To fix ideas, imagine a couple who must find an apartment in a new city by September 1, as the contract with their current landlord terminates at the end of August. Since they are not familiar with the city, they ask a broker to identify new apartments as they become available. The availability of new apartments depends on many factors; there is no guarantee that a new apartment will become available every day. Whenever the broker finds an apartment, the husband and wife both express whether they are willing to rent it or not. If they cannot agree, they forfeit the offered apartment—since the market is seller's market, there is no option to "hold" an offer while searching for a better one. Although the couple agree on the need to rent some apartment, their preferences over specific apartments are not necessarily aligned. The search ends once an agreement is made; if the couple cannot agree on an apartment by September 1, they will be homeless.

To analyze these situations, we consider an n-player search problem with a deadline. Time is continuous and "opporunties" arrive according to a Poisson process. Opportunties are i.i.d. realizations of payoffs for each player. After viewing an opportunity, the players respond with "yes" or "no." The search ends if and only if all players say "yes." If the search does not end by the deadline, players obtain an a priori specified fixed payoff. Notice that the arrival rate of Poisson process captures "friction" inherent in the search process: larger arrival rates correspond to smaller friction. Since there is a trivial subgame perfect equilibrium in which all players always reject, we analyze an (appropriately defined) trembling-hand equilibrium of this game.

Our analysis consists of three steps. *In the first step*, we show that for any number of players and under very weak distributional assumptions, the expected duration of search does not shrink to zero even in the limit as the friction of search vanishes. Hence the mere existence of some search friction has a nonvanishing impact on the search duration. This result is intuitive but by no means obvious.[1] The incentives are complicated. Waiting for a future opportunity to arrive offers an incremental gain in payoffs, but an increased possibility of reaching the deadline. Both the rewards and the costs go to zero as the search friction vanishes; the optimal balance is difficult to quantify because agents need to make decision of before observing all future realizations of offers. For this reason, we employ an indirect proof that bounds the acceptance probability at each moment.

In the second step, we show that in the limit, expected duration increases with the

[1] Indeed, we offer examples where our assumptions do not hold and the result fails.

number of agents involved in the search. This happens for two reasons, which we call the "ascending acceptability effect" and the "preference heterogeneity effect." Roughly put, the ascending acceptability effect refers to the fact that a player faces a larger incentive to wait if there are more opponents, as in equilibrium the opponents become increasingly willing to accept offers as time goes on. The preference heterogeneity effect refers to the fact that such these future opportunities include increasingly favorable offers for the player due to heterogeneity of preferences.

In the third step, we show that the speed of convergence for the expected search duration is fast. Moreover, we use numerical examples to show that as the friction disappears, the limit duration of search is actually close to durations with nonnegligible search friction. This provides evidence that our limit analysis contains economically-meaningful content, and the mere existence of some friction is actually the main driving force of the positive duration in reality—so the effects that we identify in the first and second steps are the keys to understand the positive duration in reality.

In Figures 1 and 2, we depict how the duration can be decomposed into the effects mentioned above when there are two players, the offer distribution is uniform over a feasible payoff set that has all nonnegative payoff profiles with the sum of coordinates being no more than 1 (Figure 3), the arrival rate is 10, and the horizon length is 1. This corresponds to the case where there are ten weeks to search an apartment, and the information of a new apartment comes once every week on average—quite a high friction. Even in this case, it is clear in the figure that the finiteness of arrival rates has a very little effect on the duration, while other effects are significant. The limit expected duration is directly computed from a key variable r determined by the details of the model (X, μ). The larger the r is, the longer the duration is. The increase in r from the one-player model to the two-player model is accounted for by the ascending acceptability effect and the preference heterogeneity effect. In this example, the former effect is larger than the latter.[2]

The two key features in our model, *deadline* and *multiple agents*, give rise to new theoretical challenges. In particular, these two things *interact* with each other. First, the existence of a deadline implies that the problem is *nonstationary*: the problems faced by the agents at different moments of time are different. Nonstationarity often results in intractability, but we partially overcome this by taking an indirect approach: we first analyze the limit expected duration (the first and the second steps) which is easier to characterize, and then argue that the limit case approximates the case with finite arrival rates reasonably well (the third step). Second, one may argue that since each player's decision at any given opportunity is essentially conditional on the situation where all other agents say "yes," the problem essentially boils down to a single-player

[2]At the end of the main section (Section 4), we will be explicit about how we conducted this decomposition.

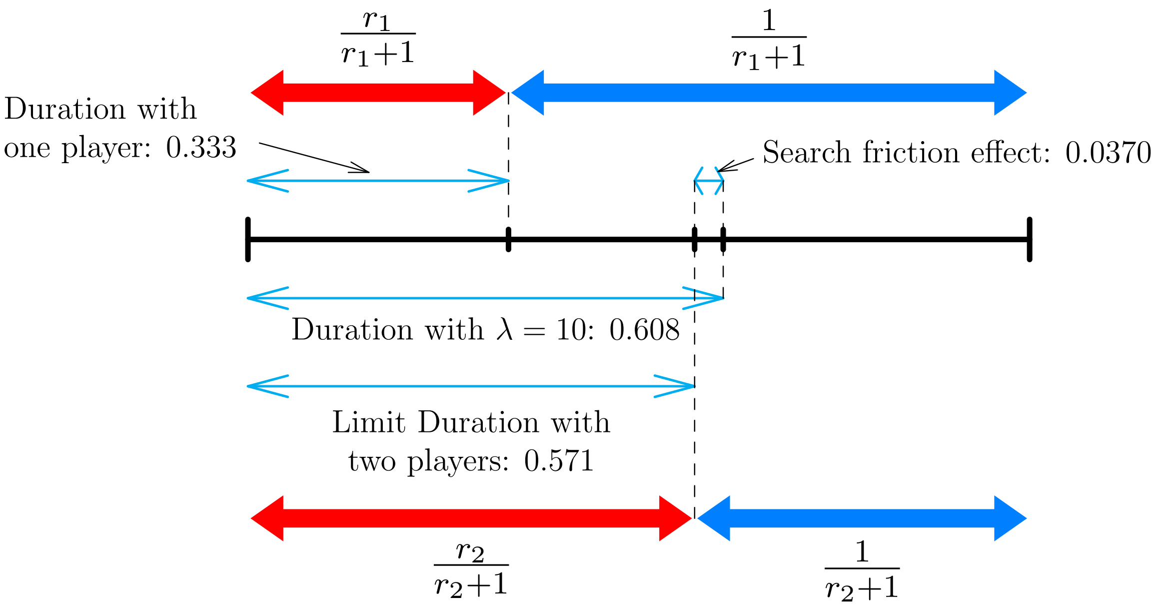

Figure 1: Decomposition of search durations: The case with uniform distribution over the space depicted in Figure 3 and the horizon length of 1. The one player duration is computed by assuming uniform distribution over the unit interval. r_1 and r_2 are illustrated in Figure 2.

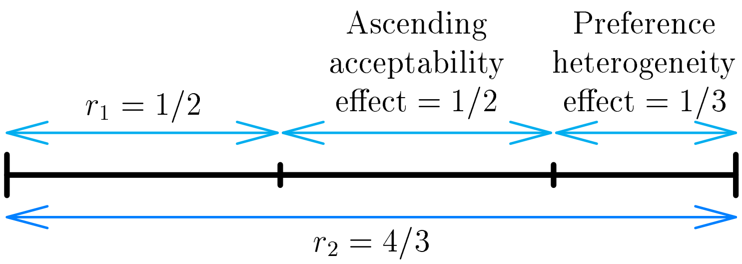

Figure 2: Decomposition of $r_2 - r_1$.

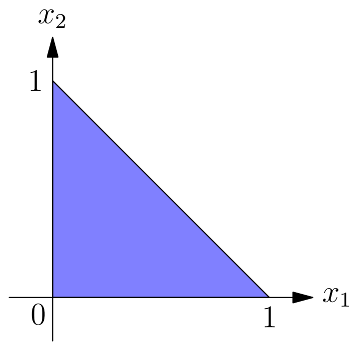

Figure 3: An example of the domain of feasible payoff profiles

search problem. This argument misses an important key property of our model. It is indeed true that at each given opportunity the decisions by the opponents do not affect a player's decision. However, the player's expectation about the opponents' future decisions affects her decision today, and such *future* decisions by opponents are in turn affected by other agents' decisions even *further in the future*. The two "futures" discussed in the previous sentence are different precisely due to the nonstationarity—hence the two features interact. It will become clear in our analysis that it is this interaction that is crucial to our argument in the three steps.

Besides the results on duration, we provide a number of additional results. Most prominently, we study welfare implications our model. In order to isolate the effects of multiple agents and a finite horizon as cleanly as possible, the departure from the standard model is kept minimal. This enables us to modify our model in a wide variety of directions and also to conduct comparative statics. To give some examples, we study the case when payoffs realize upon agreement (corresponding to the situation where the couple can rent an apartment as soon as they sign a contract); the robustness of our results to different arrival processes; the case with the presence of fixed time costs; offer distributions varying over time; changes in bargaining power over time; the optimal choice of horizon length (in a market-design context); the case of majority rule rather than unanimity; the possibility of negotiation. All these and many other things can be and will be discussed in our framework.

1.1 Literature Review

Finite vs. infinite horizon with multiple agents.

First, although there is a large body of literature on search problems with a single agent and an infinite horizon, there are only few papers that diverge from these two assumptions.[3] Some recent papers in game theory discuss infinite-horizon search models in which a group of decision-makers determine when to stop. Wilson (2001), Compte and Jehiel (2010), and Cho and Matsui (2011) consider search models in which a unanimous agreement is required to accept an alternative, and show that the equilibrium outcome is close to the Nash bargaining solution when players are patient. Despite the absence of a deadline, these convergence results to the Nash bargaining solution have a similar flavor to our result in Section 6 where payoffs realize as soon as an agreement is reached. In Section 7.3, we will discuss a common logic behind these convergence results. Compte and Jehiel (2010) also analyze general majority rules to discuss the power of each individual to affect outcomes of search, and the size of the set of limit equilibrium outcomes. Albrecht et al. (2010) consider general majority rules, and show that the decision-makers are less picky than the agent in the corresponding single-person search model, and the expected duration of search is shorter if they are sufficiently patient. Alpern and Gal (2009), and Alpern et al. (2010) analyze a search model in which a realized object is chosen when one of two decision-makers accepts it, unless one of them casts a veto which can be exercised only a finite number of times in the entire search process. Moldovanu and Shi (2010) analyze an infinite-horizon two-agent search problem with interdependent preferences with respect to private signals of the payoffs realized in every period. They also show that the duration becomes longer if the number of decision-makers increases from one to two while retaining the information structure.[4] Bergemann and Välimäki (2011) provide an efficient dynamic mechanism with a presence of monetary transfer in an n-agent model with private signals of agents' private values. Importantly, in all of these papers the

[3]See Rogerson et al. (2005) for a survey.

[4]Moldovanu and Shi (2010) show that agents are pickier when there is a larger conflict in preferences, whereas if the signals are public, they are less picky and the duration is shorter with a larger conflict.

search duration converges to zero as the frequency of offer arrivals tends to infinity. The key distinction is that discounting is assumed and payoffs realize upon agreement in these papers, while in our model payoffs realize at the deadline, the assumption that fits to our motivating example of apartment search.

Multiple vs. single agent search with deadline.
A single-agent search problem with deadline is explored in much detail in the operations research literature on the so-called "secretary problem." There is an important difference between this literature and our model. In secretary problems, there are n potential candidates (secretaries) who arrive each date, and the decision maker makes acceptance decisions. The key difference from our analysis is that in secretary problems the decision maker does not have cardinal preferences but ordinal preferences, and attempts to maximize the probability that the best candidate is chosen. Since the number of candidates is finite, this is technically a search problem with finite horizon. The optimal policy as the number of candidates grows to infinity is to disregard all candidates for some time before choosing, so this model also has a positive limit search duration. The reason for positive duration is, however, different from ours. In secretary problems, the decision maker must gather information about available alternatives to make sure what she chooses is reasonably well-ranked. In our setting with cardinal preferences and known distribution of payoffs, there is no information gathering. Rather, what underlies the positive duration is the tradeoff between the potential gain from waiting for a better allocation in the future and the loss from reaching the deadline. This tradeoff is not an issue in secretary problems as the decision maker benefits only from the best candidate. See Ferguson (1989) for an extensive survey of the literature.

Single-agent search with infinite horizon.
The so-called "search theory" literature has focused mainly on a single-agent search problem with infinite horizon and extended such a model to the context of large population. Seminal papers by McCall (1970) and Mortensen (1970) explore models in which a single agent faces an iid draw of payoffs over an infinite horizon. These models are extended in many directions.[5] A common feature in these papers is that the model has some form of "waiting costs" either as a discounting or as a search cost, irrespective of the length of the horizon (finite or infinite). This assumption would be a reasonable one in their context as their main application was job search, where the overall horizon length (in finite horizon models) is several decades, and one period corresponds to a year or a month. On the other hand, our interest is in the case where the horizon length is rather short, as in the apartment search example we provided in the introduction. This naturally gives rise to the assumption that payoffs realize at the deadline—which would not have made

[5] An extensive survey of the literature can be found in Lippman and McCall (1976).

sense in the job search application. Because of this difference, the limit search duration as the friction goes away in models of this line of the literature is zero, so they could not implement the exercise that we do in this paper. Later work extended the model to a large population model in which the search friction is given endogenously through a "matching function." Again, in a nutshell, these analyses are more or less extensions of the single-agent search model with infinite horizon, and thus there has been no question on the "limit duration" as the friction vanishes.

Multi-agent search with finite horizon.

A few papers consider multi-person decision problems (See Ferguson (2005) and Abdelaziz and Krichen (2007) for surveys), but none has looked at the search duration. Sakaguchi (1973) was the first to study a multi-agent search model with finite horizon. Sakaguchi (1978) proposed a two-agent continuous-time finite-horizon stopping game in which opportunities arrive according to a Poisson process as in our model. He derived the same ordinary differential equations (ODE) as ours and provided several characterizations,[6] and then computed equilibrium strategies in several specific examples.[7] However, no analysis on duration appeared in his papers. Note that obtaining the ODE constitutes only a preliminary part of our contribution; our focus is on the search duration implied by this equation.

Ferguson (2005)'s main interest is in existence and uniqueness of the stationary cutoff subgame perfect equilibrium with discrete time, general voting rules, varying distributions over time, and presence of fixed costs of search.[8,9] The sufficient condition for uniqueness that he obtains is different from ours.[10]

Multi-agent search vs. bargaining.

The multi-agent search problems are similar to bargaining problems in that both predict what outcome in a prespecified domain is chosen as a consequence of strategic interaction between agents. However, as discussed by Compte and Jehiel (2004, 2010), the search models are different from bargaining models in that in the former, players just make an acceptance decision on what is exogenously provided to them, while in the latter,

[6]Specifically, he showed that (a) the cutoffs are nondecreasing and concave in the time variable, and (b) in the independent environment, players are less picky than in the single player case.

[7]Examples he examined are (1) the Bernoulli distribution on a binary domain, (2) $h(x,y) = f(x)g(y)(1+\gamma(1-2F(x))(1-2G(x)))$ for f,g being arbitrary density functions, and γ being a parameter that measures correlation, (3) an exponential distribution, and (4) a direct product of exponential and uniform distributions. Apart from case (1) in which the limit search duration is trivially zero, our results imply that all cases have positive limit durations.

[8]He mentions the idea of trembling-hand equilibrium only verbally, and does not provide a formal definition. Instead, there is an assumption that agreement probability is always positive.

[9]He also analyzes an exponential case and does a comparative statics in terms of individual search costs.

[10]The condition states that the distribution of offers is independent across agents and the distance to the conditional expectation above value v_i is decreasing in v_i for all player i.

players have full control over what to propose. Our model is a search model, and thus in our model players are "passively" assess exogenous opportunities. This assumption captures the feature of situations that we would like to analyze. For example many potential tenants do not design their houses for themselves, but they simply wait for a broker to pass them information regarding new apartments. The distinction between these "passive" and "active" players is important when we consider the difference between our work and the standard bargaining literature.[11]

Another important issue in relation with the bargaining literature is the distinction between positive search duration and so-called "bargaining delay." Bargaining delay is particularly important because it is often associated with inefficiency caused by discounting. In our model payoffs realize at the deadline (so in essence agents do not discount the future), so the positive-duration result does not necessary imply inefficiency. Actually, we prove that generically the expected payoff profile cannot be Pareto inefficient in the limit as the search friction vanishes. We do not view this as necessarily detrimental to our contribution, as our primary aim is to provide a deeper understanding of the positive duration in reality.[12]

Multi-agent search vs. bargaining with finite horizon.
Ambrus and Lu (2010a), Gomes et al. (1999) and Imai and Salonen (2011) consider a bargaining model with finite horizon, in which players obtain an opportunity to propose a share distribution of the surplus at asynchronous timings, having full control over proposals, and analyze the equilibrium payoffs.[13] The important distinction from our search model is that without any further assumptions (such as private information) that can be resolved over time or an "option to wait" as assumed in Ma and Manove (1993), the first player who obtains the opportunity makes an offer that all players would accept in equilibrium. This is in line with the intuition of Rubinstein (1982)'s canonical model of alternating-offer bargaining, and implies that as the timing of proposals becomes frequent the duration until the agreement can become arbitrarily small.[14] In our model, however, there is a trade-off as the search friction decreases between more arrivals today and more arrivals in the future. Our main objective of this paper is to discuss the effects driven (at least in part) by this trade-off, while bargaining models do not have such a trade-off (thus a question on duration is trivial).

[11]Cho and Matsui (2011) present another view: A drawn payoff profile in the search process can be considered as an outcome of a (unique) equilibrium in a bargaining game which is not explicitly described in the model and does not depend on the future equilibrium strategy profile. According to this interpretation, every player is "active" although the "activeness" is embedded in the model.

[12]In our framework, we can also ask a normative question: In Section 7.9, we examine a market designer's problem to tune parameters of the model (the horizon length and the distribution of offers) when the search friction is finite so the Pareto-efficiency result does not have bite.

[13]See Ambrus and Lu (2010b) for an application of their model to legislative processes.

[14]A finite horizon version of Rubinstein (1982)'s model with Poisson opportunities is a special case of Ambrus and Lu (2010a)'s model, so the limit duration is zero in such a model.

A part of results by Gomes et al. (1999) and Imai and Salonen (2011) shows that in some cases the limit equilibrium is the Nash bargaining solution. Although these results about equilibrium payoff profiles is reminiscent of our result in Section 6, the results are different in the conditions that determine the limit profiles.[15]

Revision games.
Broadly, this paper is part of a rapidly growing literature on "revision games," which explores implications of adding a revision phase before a predetermined deadline at which actions are implemented and players receive payoffs. The first papers on revision games by Kamada and Kandori (2009, 2011) show the possibility of cooperation in such a setting,[16] and Calcagno and Lovo (2010), Kamada and Sugaya (2010a), and Ishii and Kamada (2011) examine the effect of asynchronous timings of revisions on the equilibrium outcome in revision games. Kamada and Sugaya (2010b) apply the revision games setting to election campaigns. Romm (2011) analyzes the implication of introducing a "reputational type" in a variant of a revision game introduced by Kamada and Sugaya (2010a). General insights from these works are that when the action space is finite (as in our case) the set of equilibria is typically small and the solution can be obtained by (appropriately implemented) backwards induction, and that a differential equation is useful when characterizing the equilibrium. In our paper we follow and extend these methods to characterize equilibria and apply the framework to the context of search situations that often arise in reality. Some examples we provide in this paper are reminiscent of those provided in Kamada and Sugaya (2010a).

The paper is organized as follows. Section 2 provides a model. In Section 3 we provide some preliminary results. In particular, we show that trembling-hand equilibria take the form of cutoff strategies, by which we mean each player at each moment of time has a "cutoff" of payoffs below which they reject offers and otherwise accept. Section 4 is the main section of the paper. Subsections 4.1, 4.2, and 4.3 correspond to Steps 1, 2, and 3 of our argument, respectively. Section 5 provides a welfare analysis of our main model. Section 6 considers the case in which payoffs realize upon agreement and there is a discounting—the case analogous to analyses in the previous work. In Section 7, we provide a number of discussions. Among others, we show that even if agents can negotiate and transfer utilities after each realization of payoffs, our basic result of positive duration is still valid. Section 8 concludes. Proofs are given in the Appendix unless otherwise noted.

[15] See Remark 2 in the previous version of this paper (Kamada and Muto (2011b)) for a more comprehensive comparison between our work and these papers. There we argue that under different conditions the limit equilibrium payoff profile is the Nash bargaining solution in each model when the discount rate and the frequency of opportunities converge simultaneously.

[16] See Ambrus and Burns (2010) for a related work on an analysis of eBay-like auctions.

2 Model

The Basic Setup

There are n players searching for an indivisible object. Let $N = \{1, \dots, n\}$ be the set of players. A typical player is denoted by i, and the other players are denoted by $-i$. The players search within a finite time interval $[-T, 0]$ with $T > 0$, on which opportunities of agreement arrive according to the Poisson process with arrival rate $\lambda > 0$. At each opportunity, nature draws an indivisible object which is characterized by a payoff profile $x = (x_1, \dots, x_n)$ following an identical and independent probability measure μ defined on the Borel sets of $\mathbb{R}^n$. A payoff profile $x \in \mathbb{R}^n$ is often referred to as an allocation. After allocation x is realized, each player simultaneously responds by either accepting or rejecting x without a lapse of time. Let $B = \{\text{accept}, \text{reject}\}$ be the set of responses in this search process. If all players accept, the search ends, and at time 0 the players receive the corresponding payoff profile x. If at least one of the players rejects the offer, then they continue to search. If players reach no agreement before the deadline at time 0, they obtain the disagreement payoff profile normalized at $x^d = (0, \dots, 0) \in \mathbb{R}^n$.[17]

Support and Pareto Efficiency

Let $X = \{x \in \mathbb{R}^n \mid \mu(Y) > 0 \text{ for all open } Y \ni x\}$ be the support of μ. Note that $X \subseteq \mathbb{R}^n$ is a closed subset on which μ is full support. Without loss of generality, we assume that $X \subseteq \mathbb{R}^n_+$.[18] An allocation $x = (x_1, \dots, x_n) \in X$ is *Pareto efficient* in X if there is no allocation $y = (y_1, \dots, y_n) \in X$ such that $y_i \geq x_i$ for all $i \in N$ and $y_j > x_j$ for some $j \in N$. An allocation $x \in X$ is *weakly Pareto efficient* in X if there is no allocation $y \in X$ such that $y_i > x_i$ for all $i \in N$. The set of all (weakly) Pareto efficient allocations in X is called the (*weak*, resp.) *Pareto frontier* of X. We sometimes consider weak Pareto efficiency also on $\hat{X} = \{v \in \mathbb{R}^n_+ \mid x \geq v \text{ for some } x \in X\}$ which is the nonnegative region of the comprehensive extension of X.

Assumptions

We make the following weak assumptions throughout the paper.

Assumption 1. (a) The expectation $\int_X x_i \, d\mu$ is finite for all $i \in N$.

(b) If $n \geq 2$, for all $i \in N$, i's marginal distribution of μ has a density function that is locally bounded.[19]

[17] This is without loss of generality as long as payoffs realize at the deadline. When the payoffs realize upon agreement as in Section 6, this does change some of the analysis (the initial condition of the differential equation (10) changes), but we restrict ourselves to $x^d = (0, \dots, 0)$ as the change is minor.

[18] This is without loss of generality as long as there is a positive probability in $\mathbb{R}^n_+$ since the strategic environment is identical to the case where the arrival rate is adjusted to $\mu(\mathbb{R}^n_+)\lambda$ because players prefer to reject any negative payoffs.

[19] A function $g(y)$ defined on $\mathbb{R}$ is locally bounded if for all y, there exists $C > 0$ and $\varepsilon > 0$ such that $|g(y')| \leq C$ for all $y' \in (y - \varepsilon, y + \varepsilon)$.

If condition (a) is violated, a player always wants to wait for better payoffs before the deadline, so a best response does not exist. Condition (b) rules out a distribution which has infinitely large density at some point, while it still allows for a distribution under which there is a positive probability that an allocation falls on degenerate subsets such as a line segment which is not horizontal or vertical. In Section 7.5, we will provide an example that demonstrates the need for Condition (b) for our main results to hold.

Histories and Strategies

Let us define strategies in this game. A history at $-t \in [-T, 0]$ where players observed k (≥ 0) offers in $[-T, -t)$ consists of

1. a sequence of times $(t^1, \dots, t^k)$ when there were Poisson arrivals before $-t$, where $-T \leq -t^1 < -t^2 < \cdots < -t^k < -t$,
2. allocations $x^1, \dots, x^k$ drawn at opportunities $t^1, \dots, t^k$, respectively,
3. acceptance/rejection decision profiles $(b^1, \dots, b^k)$, where each decision profile b^l $(l = 1, \dots, k)$ is contained in $B^n \setminus \{(\text{accept}, \dots, \text{accept})\}$,
4. allocation $x \in X \cup \{\varnothing\}$ at $-t$ ($x = \varnothing$ if no Poisson opportunity arrives at $-t$).

We denote a history at time $-t$ by $\big((t^1, x^1, b^1), \dots, (t^k, x^k, b^k), (t, x)\big)$. Let $\tilde{\mathcal{H}}_t^k$ be the set of all such histories at time $-t$, $\tilde{\mathcal{H}}_t = \bigcup_{k=0,1,2,\dots} \tilde{\mathcal{H}}_t^k$ and $\tilde{\mathcal{H}} = \bigcup_{-t \in [-T,0]} \tilde{\mathcal{H}}_t$.[20] Let

$$\mathcal{H}_t^k = \left\{ \big((t^1, x^1, b^1), \dots, (t^k, x^k, b^k), (t, x)\big) \in \tilde{\mathcal{H}}_t^k \,\middle|\, x \neq \varnothing \right\}$$

be the history at time $-t$ when players have an opportunity and there have been k opportunities in the past. Let $\mathcal{H}_t = \bigcup_{k=0,1,2,\dots} \mathcal{H}_t^k$ and $\mathcal{H} = \bigcup_{-t \in [-T,0]} \mathcal{H}_t$. A (behavioral) *strategy* σ_i of player i is a function from $\mathcal{H}$ to the set of probability distributions over the set of responses B. Let Σ_i be the set of all strategies of i, and $\Sigma = \times_{i \in N} \Sigma_i$. For $\sigma \in \Sigma$, let $u_i(\sigma)$ be the expected payoff of player i when players play σ.[21]

Equilibrium Notions

A strategy profile $\sigma \in \Sigma$ is a *Nash equilibrium* if $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$ for all $\sigma_i' \in \Sigma_i$ and all $i \in N$. Let $u_i(\sigma \,|\, h)$ be the expected continuation payoff of player i given that a history $h \in \tilde{\mathcal{H}}$ is realized and strategies taken after h is given by σ. A strategy profile

[20] Precisely speaking, there are histories in which infinitely many opportunities arrive. We ignore these possibilities since such histories happen with probability zero.

[21] The function $u(\sigma)$ is well-defined for the following reason: $\mathcal{H}^k := \bigcup_t \mathcal{H}_t^k$ is seen as a subset of $\mathbb{R}^{(2n+1)k+(n+1)}$, and thus endowed with a Borel sigma-algebra. We assume that $\mathcal{H} = \bigcup_k \mathcal{H}^k$ is endowed with a sigma-algebra induced by these sigma-algebras on $\mathcal{H}^k$, and a strategy must be measurable with respect to this sigma-algebra. The measurability ensures that a strategy profile generates a probability measure on the set of terminal nodes. See Stinchcombe (1992) for a detailed treatment of strategies in general continous-time games.

$\sigma \in \Sigma$ is a *subgame perfect equilibrium* if $u_i(\sigma_i, \sigma_{-i} \,|\, h) \geq u_i(\sigma_i', \sigma_{-i} \,|\, h)$ for all $\sigma_i' \in \Sigma_i$, $h \in \mathcal{H}$, and all $i \in N$. A strategy $\sigma_i \in \Sigma_i$ of player i is a *Markov strategy* if for history $h \in \mathcal{H}_t$ at $-t$, $\sigma_i(h)$ depends only on the time $-t$ and the drawn allocation x. A strategy profile $\sigma \in \Sigma$ is a *Markov perfect equilibrium* if σ is a subgame perfect equilibrium and σ_i is a Markov strategy for all $i \in N$. We will later show that players play a Markov perfect equilibrium (except at histories in a zero-measure set) if they follow a trembling-hand equilibrium defined below. For $\varepsilon \in (0, 1/2)$, let Σ^ε be the set of strategy profiles which prescribe probability at least ε for both responses in $\{\text{accept}, \text{reject}\}$ after all histories in $\mathcal{H}$. A strategy profile $\sigma \in \Sigma$ is a *trembling-hand equilibrium* if there exists a sequence $(\varepsilon^m)_{m=1,2,\ldots}$ and a sequence of strategy profiles $(\sigma^m)_{m=1,2,\ldots}$ such that $\varepsilon^m > 0$ for all m, $\lim_{m\to\infty} \varepsilon^k = 0$, $\sigma^m \in \Sigma^{\varepsilon^m}$, σ^m is a Nash equilibrium in the game with a restricted set of strategies Σ^{ε^m} (ε^m-constrained game) for all m, and $\lim_{m\to\infty} \sigma^m(h) = \sigma(h)$ for all $h \in \mathcal{H}$ according to the pointwise convergence in histories.[22]

3 Preliminary Results

In this section, we present preliminary results which will become useful in the subsequent sections. We will show that there exists an essentially unique trembling-hand equilibrium, in which every player plays a "cutoff strategy." We will derive an ordinary differential equation that characterizes the cutoff profile in the equilibrium. In addition, we will observe a basic invariance: The change in equilibrium continuation payoffs when raising the arrival rate is the same as that when stretching the duration from the deadline with the same ratio. Finally, by examining the differential equation, the limit equilibrium payoff as $\lambda \to \infty$ is shown to be weakly efficient.

The next proposition shows that trembling-hand equilibrium is essentially unique and Markov.

Proposition 1. *Suppose that σ and σ' are two trembling-hand equilibria. Then $u_i(\sigma \,|\, h) = u_i(\sigma' \,|\, h')$ for almost all histories $h, h' \in \tilde{\mathcal{H}}_t \setminus \mathcal{H}_t$ and all $i \in N$.*

That is, regardless of the history, any two trembling-hand equilibria give rise to the same continuation payoff at time $-t$. Three remarks are in order: First, we ruled out histories in $\mathcal{H}_t$, because different realization of payoffs that players accept clearly give rise to different continuation payoffs on the equilibrium path. Second, since agents move simultaneously, there exist subgame perfect equilibria in which all players reject any allocations.[23] We introduced the trembling-hand equilibrium to rule out such trivial

[22] This equilibrium concept is an analog of extensive-form trembling-hand equilibrium, as opposed to its normal-form counterpart. Although our extensive-form game involves uncountably many nodes and hence the standard definitions of trembling-hand equilibria are not directly applicable, it is for this reason that we call this notion a trembling-hand equilibrium.

[23] If players respond sequentially, we can show that any subgame perfect equilibrium consists of cutoff strategies. Therefore our results are essentially independent of the timing of responses of players.

equilibria. In an ε-constrained game, a player will optimally accept a favorable allocation for herself, expecting the others to accept it with a small probability. Third, and relatedly, there exist sequential equilibria in which every player has a strict incentive at almost all histories.[24] Our trembling-hand equilibrium rules out such equilibria.

A Markov strategy σ_i of player $i \in N$ is a *cutoff strategy* with cutoff $v_i(t) \geq 0$ if player i who is to respond at time $-t$ accepts allocation $x \in X$ whenever $x_i \geq v_i(t)$, and rejects it otherwise. For a profile $v = (v_1, \ldots, v_n) \in \mathbb{R}^n_+$, we define a set of allocations by $A(v) = \{x \in X \mid x_i \geq v_i \text{ for all } i \in N\}$. When players play cutoff strategies with cutoff profile $(v_1(t), \ldots, v_n(t))$, we sometimes call $A(v(t))$ an "acceptance set" as they agree with an allocation x at time $-t$ if and only if $x \in A(v(t))$. We often denote this acceptance set by $A(t)$ with a slight abuse of notation when the cutoff profile in consideration is not ambiguous. Suppose that all players play Markov strategies σ, and there is no Poisson arrival at time $-t \in [-T, 0]$. Then player i has an expected payoff $u_i(\sigma \mid h)$ at $-t$ independent of history $h \in \tilde{\mathcal{H}}_t \setminus \mathcal{H}_t$ played before time $-t$. We denote the continuation payoff at time $-t$ by $v_i(t, \sigma) = u_i(\sigma \mid h)$. For simplicity of notations, we hereafter omit to write a cutoff strategy profile σ explicitly, and denote by $v_i(t)$ the continuation payoff of player i at time $-t$.

The following proposition shows that there exists a trembling-hand equilibrium that consists of cutoff strategies, and characterizes the path of cutoffs.

Proposition 2. *There exists a trembling-hand equilibrium that consists of (Markov) cutoff strategies. Moreover, an equilibrium continuation payoff profile $v(t) = (v_1(t), \ldots, v_n(t))$ at time $-t \in [-T, 0]$ is given by a solution of the following ordinary differential equation (ODE)*

$$v'(t) = \lambda \int_{A(t)} \big(x - v(t)\big)\, d\mu \tag{1}$$

with an initial condition $v(0) = (0, \ldots, 0)$.

This proposition is shown by the following argument. An equilibrium continuation payoff $v_i(t)$ supported by a cutoff strategy profile is given by the following recursive expression: For $i \in N$,

$$\begin{aligned} v_i(t) &= \int_0^t \left(\int_{X \setminus A(\tau)} v_i(\tau)\, d\mu + \int_{A(\tau)} x_i \, d\mu \right) \lambda e^{-\lambda(t-\tau)}\, d\tau \\ &= \int_0^t \left(v_i(\tau) + \int_{A(\tau)} \big(x_i - v_i(\tau)\big)\, d\mu \right) \lambda e^{-\lambda(t-\tau)}\, d\tau. \end{aligned} \tag{2}$$

After time $-t$, players receive the first Poisson opportunity at time $-\tau$ with probability

[24] An example similar to the one in Cho and Matsui (2011, Proposition 4.4) can be used to show this result.

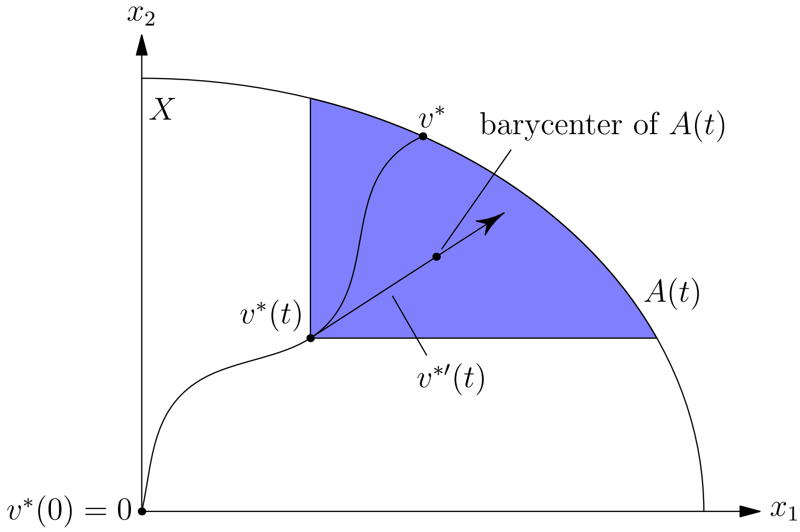

Figure 4: The path and the velocity vector of ODE (1)

density $\lambda e^{-\lambda(t-\tau)}$. If player i finds that the drawn payoff x_i to her is no worse than her continuation payoff $v_i(\tau)$, i optimally accepts this allocation x, otherwise, i rejects it. If all players accept x, i.e., $x \in A(\tau)$, then they reach an agreement with x. If some player rejects x, then search continues with continuation payoff profile $v(\tau)$. This discussion shows that a cutoff strategy profile with cutoffs $v(t)$ characterized by equation (2) is a Markov perfect equilibrium.

Bellman equation (2) implies that $v_i(t)$ is differentiable in t. Multiplying both sides of (2) by $e^{\lambda t}$ and differentiating both sides yield the ordinary differential equation (1) of continuation payoff profile $v(t)$ defined in $\hat{X}$.

Now, a standard argument of ordinary differential equations shows that ODE (1) has a solution whenever Assumption 1 holds.[25] The above argument only shows that the cutoff strategy profile with a cutoff profile given by this solution of ODE (1) is a Markov perfect equilibrium. In the Appendix, we will show that it is in fact a trembling-hand equilibrium.

By Proposition 1, the solution of ODE (1) is unique. Therefore the game has essentially a unique trembling-hand equilibrium for any given X and μ satisfying Assumption 1. Let us denote the unique solution of (1) by $v^*(t;\lambda)$, the continuation payoff profile in the trembling-hand equilibrium. We simply denote this by $v^*(t)$ as long as there is no room for confusion. The probability that all players accept a realized allocation at time $-t$ on the equilibrium path conditional on the event that an opportunity arrives at $-t$ is referred to as the "acceptance probability" at time $-t$.

Let us make a couple of observations about ODE (1). Figure 4 describes an illustration of a typical path and the velocity vector that appear in this ODE for $n = 2$. The shaded

[25] This is because Assumption 1 (b) ensures continuity in v of the right hand side of (1). See Coddington and Levinson (1955, Chapter 1) for a general discussion about ODE.

area shows the acceptance set $A(t)$, whose barycenter with respect to the probability measure μ is $\int_{A(t)} x d\mu / \mu(A(t))$. Therefore the velocity vector $v^{*\prime}(t)$ is parallel to the vector from $v^*(t)$ to the barycenter of $A(t)$, which represents the gain upon agreement relative to $v^*(t)$. The absolute value of $v^{*\prime}(t)$ is proportional to the weight $\mu(A(t))$. Note that ODE (1) immediately implies $v_i^{*\prime}(t) \geq 0$ for all t and $i \in N$, and $v_i^{*\prime}(t) = 0$ if and only if $\mu(A(t)) = 0$. For each $i \in N$, the continuation payoff $v_i^*(t)$ grows as t increases, and eventually either converges to a limit payoff v_i^*, or diverges to infinity.

Since the right hand side of ODE (1) is linear in λ, we have $v^*(t; \alpha\lambda) = v^*(\alpha t; \lambda)$ for all $\alpha > 0$ and all t such that $-t, -\alpha t \in [-T, 0]$. By considering the limit as $\alpha \to \infty$, we have the following proposition:

Proposition 3. *The two limits of* $v^*(T; \lambda)$ *coincide, i.e.,* $\lim_{\lambda\to\infty} v^*(T; \lambda) = \lim_{T\to\infty} v^*(T; \lambda)$, *if one of them exists.*

We henceforth denote this limit by v^*. In the next section, we sometimes deal with these two limits interchangeably. Note that the equality implies $\lim_{\lambda\to\infty} v^*(T; \lambda)$ does not depend on $T > 0$.

Finally, we argue weak Pareto efficiency of the limit allocation. Suppose that $v^* = \lim_{\lambda\to\infty} v^*(T) = \lim_{T\to\infty} v^*(T)$ exists but is not weakly Pareto efficient. Then there exists $x \in X$ that strictly Pareto dominates v^*. Since x belongs to the support of μ, $\mu(Y) > 0$ for any open set $Y \subseteq \mathbb{R}_+^n$ that includes x. For Y sufficiently small, $A(v^*)$ contains Y, and thus $\mu(A(v^*)) > 0$. This implies that the right hand side of ODE (1) is positive, contradicting the fact that $v^* = \lim_{\lambda\to\infty} v^*(t) = \lim_{t\to\infty} v^*(t)$. Hence we obtain the following proposition:

Proposition 4. *Let* $t > 0$ *fixed, and suppose that the solution* $v^*(t; \lambda)$ *of equation* (1) *converges to* $v^* \in \hat{X}$ *as* $\lambda \to \infty$. *Then* v^* *is weakly Pareto efficient.*[26]

We will have further discussions about efficiency in Section 5, in which we will show that the limit allocation v^* is Pareto efficient for almost all distributions μ satisfying mild assumptions, and Pareto efficient for all convex X.

4 Duration of Search

In this section, we will discuss the duration of search in our model. Our argument consists of three steps: In Section 4.1 we will show that even under the quite weak conditions in Assumption 1, search takes a positive time even in the limit as the friction vanishes. In Section 4.2, we argue that the limit duration becomes longer as the number of involved

[26] Note that this does not necessarily imply weak Pareto efficiency in the convex hull when X is non-convex. That is, the convex hull can contain allocations that Pareto-dominate the limit expected payoff profile, while such allocations cannot be achieved under a trembling-hand equilibrium. See footnote 37 for a further discussion on this.

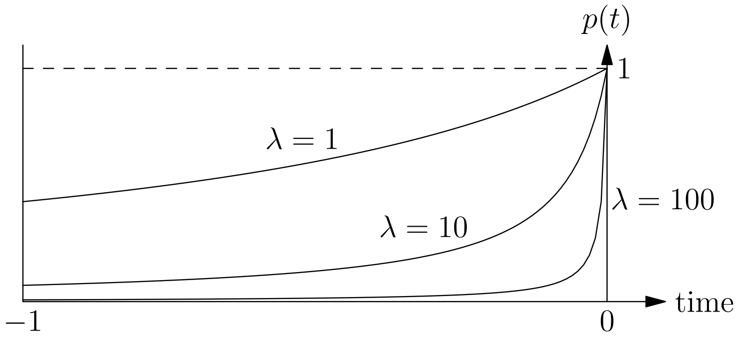

Figure 5: A numerical example of $p(t)$ for the case when $n = 2$, $X = \{x \in \mathbb{R}^2_+ \mid x_1 + x_2 \leq 1\}$, μ is the uniform distribution on X, and $T = 1$.

agents gets larger. This extra duration is caused by two effects called the "ascending acceptability effect" and the "preference heterogeneity effect." We will provide a method to decompose the extra duration by these two effects. In Section 4.3, we demonstrate that the limit duration is "close" to the durations for finite arrival rates. This provides evidence that our limit analysis contains economically-meaningful content, and the mere existence of some friction is actually the main driving force of positive duration in reality—so the effects that we identify in Steps 1 and 2 are the keys to understand the positive duration in reality.

First, let us explain how we compute the expected duration. Given arrival rate $\lambda > 0$, by the differential equation (1) we can compute the equilibrium path of the cutoff profile $v^*(t; \lambda)$. Given $v^*(t; \lambda)$, one can compute the acceptance probability $p(t; \lambda)$ at each time $-t$ as follows:

$$p(t; \lambda) = \mu(A(v^*(t; \lambda))).$$

Let $P(t; \lambda)$ be the probability that there is no agreement until time $-t$:

$$P(t; \lambda) = e^{-\int_t^T \lambda p(s;\lambda)\, ds}. \tag{3}$$

Notice that $\frac{dP(t;\lambda)}{dt} = \lambda p(t; \lambda) P(t; \lambda)$. We often omit λ and simply denote $p(t)$ and $P(t)$ when there is no room for confusion. As an example, Figure 5 graphs $p(t)$ for $\lambda = 1, 10, 100$ when $n = 2$, $X = \{x \in \mathbb{R}^2_+ \mid x_1 + x_2 \leq 1\}$, μ is the uniform distribution on X, and $T = 1$. Figure 6 shows a graph of $P(t)$ in the same environment.

Let $D(\lambda)$ be the expected duration in the equilibrium for given λ when $T = 1$. Since we have $v^*(T; \lambda) = v^*(1; \lambda T)$ as discussed in Section 3, the search duration is proportional to T, and thus the expected duration for general T is written as $D(\lambda)T$. We use these

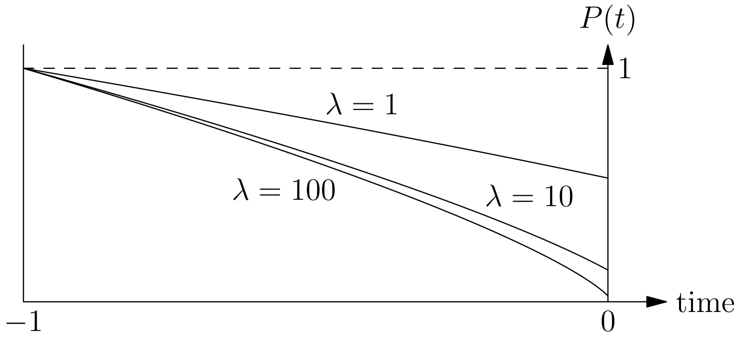

Figure 6: A numerical example of $P(t)$ for the case when $n = 2$, $X = \{x \in \mathbb{R}^2_+ \mid x_1 + x_2 \leq 1\}$, μ is the uniform distribution on X, and $T = 1$.

$p(t; \lambda)$ and $P(t; \lambda)$ to solve for $D(\lambda)$, using integration by parts:

$$
\begin{aligned}
D(\lambda)T = T \cdot & \underbrace{P(0;\lambda)}_{\substack{\text{The probability of no agreement} \\ \text{until time 0}}} \\
& + \int_0^T \underbrace{(T-t)}_{\substack{\text{The duration when the search} \\ \text{ends at time } -t}} \cdot \underbrace{P(t;\lambda)}_{\substack{\text{The probability that} \\ \text{the search does not end until } -t}} \cdot \underbrace{\lambda p(t;\lambda)}_{\substack{\text{The probability density of} \\ \text{agreement at time } -t}} dt \\
&= T \cdot P(0;\lambda) + [(T-t)P(t;\lambda)]_0^T + \int_0^T P(t;\lambda)\, dt \\
&= \int_0^T P(t;\lambda)\, dt. \qquad (4)
\end{aligned}
$$

This final expression has a direct interpretation: For each time $-t$, $P(t)$ is the probability that the duration is greater than $T - t$. Since $P(t) > P(t')$ for $t > t'$, $P(t)$ is integrated for the length of $T - t$ (from T to t). Thus the expression measures the expected duration.

Finally, define

$$D(\infty) = \lim_{\lambda \to \infty} D(\lambda)$$

whenever the limit is well-defined.

In Steps 1 and 2, we will analyze $D(\infty)$. Then in Step 3 we will demonstrate that $D(\lambda)$ converges to $D(\infty)$ reasonably fast.

4.1 Step 1: Positive Duration

The first step of our argument shows the following: *For any number of players n and any probability distribution over feasible allocations μ satisfying fairly weak assumptions, the limit expected search duration as the search friction vanishes is strictly positive.*

We first show the result for the case with $n = 1$ (Theorem 1) and detail the intuition.

Then, using this result, we generalize to an arbitrary number of players (Theorem 2).

4.1.1 Single Agent

Roughly, there are two effects of having a higher arrival rate. On one hand, for any (small) given time interval, there will be an increasing number of opportunities, thus it becomes easier to get a lucky draw. On the other hand, since there will be more and more opportunities in the future as well, the player becomes pickier. Our result shows that these two effects balance each other out. The incentives are complicated. Waiting for a future opportunity to arrive offers an incremental gain in payoffs, but an increased possibility of reaching the deadline. Both the rewards and the costs go to zero as the search friction vanishes; the optimal balance is difficult to quantify because agents need to make decision of before observing all future options.

To explain the detailed intuition for our result, let us specialize to the case of $X = [0, 1]$ and μ being the uniform distribution. We first show that if the acceptance probability at each time $-t$ is $O(\frac{1}{\lambda t})$ then the limit duration is strictly positive.[27] Then we show that the acceptance probability must be indeed $O(\frac{1}{\lambda t})$.

Suppose the acceptance probability at each time $-t$ is $O(\frac{1}{\lambda t})$. Then, the probability that the agreement does not take place by time $-\frac{T}{2}$ is at least

$$e^{-\lambda C \frac{1}{\lambda T/2}} = e^{-2\frac{C}{T}}$$

for some constant $C > 0$, and this is strictly positive. This means that the limit expected duration is at least a strict convex combination of 0 and $\frac{T}{2}$, and therefore is strictly positive.

Now we explain why we expect such a small acceptance probability. Fix time $-t$. Note that the cutoff at $-t$ must be equated with the continuation payoff at $-t$ by optimality at $-t$, and the continuation payoff must be at least as good as the expected payoff by playing some arbitrarily specified strategy from time $-t$ on by optimality in the future. Also, the cutoff at $-t$ uniquely determines the acceptance probability at $-t$. That is, by specifying a future strategy, we can obtain a lower bound of continuation payoff which must be equal to the cutoff, and this gives us an upper bound of the acceptance probability:

$$\begin{aligned}
&\text{The acceptance probability} \\
&= 1 - \text{ the cutoff of the optimal strategy} \\
&= 1 - \text{ the continuation payoff from the optimal future strategy} \\
&\leq 1 - \text{ the continuation payoff from an arbitrarily specified future strategy.}
\end{aligned}$$

[27]For functions $g(y)$ and $h(y)$, we say that $g(y) = O(h(y))$, if there exist $C > 0$ and $\bar{y}$ such that $|g(y)| \leq C \cdot |h(y)|$ for all $y \geq \bar{y}$.

To see what type of future strategy will generate an interesting bound, first consider specifying a constant cutoff from $-t$ on. Suppose that at any time $-s$ after $-t$ the cutoff is $1 - O(\frac{1}{\lambda t})$. Then, a lower bound of the probability that there will be no acceptance in the future can be calculated as

$$e^{-\lambda C \frac{1}{\lambda t}} = e^{-\frac{C}{t}}$$

for some constant $C > 0$, and this is strictly greater than 0 irrespective of λ. This means that even in the limit as $\lambda \to \infty$, the probability of no agreement at time 0 does not shrink to zero. But then, the continuation payoff from this strategy must be at most a strict convex combination of a number at most 1 (the best possible payoff) and 0 irrespective of λ, which means that the acceptance probability is at least a positive number independent of λ. Hence $p(t)$ cannot be $O(\frac{1}{\lambda t})$.

Next, consider a future strategy such that at any time $-s$ after $-t$ the cutoff is such that the player accepts with a higher order than $\frac{1}{\lambda t}$ (thus she accepts with a higher probability; e.g., $\frac{1}{\sqrt{\lambda t}}$). Then the probability of acceptance in the future indeed tends to 1 as $\lambda \to \infty$, but the payoff conditional on acceptance is smaller than the best payoff (i.e., 1) by the amount of the order higher than $\frac{1}{\lambda t}$. Hence the cutoff at $-t$ must be smaller than the best payoff by such an amount, which means that the acceptance probability at $-t$ is of the order higher than $\frac{1}{\lambda t}$.

The analysis of the above two scenarios reveals the tradeoff faced by the player: Setting a high cutoff gives her a high payoff conditional on acceptance, but reduces the acceptance probability. On the other hand, setting a low cutoff results in a low payoff conditional on acceptance but raises the acceptance probability. This suggests thata good strategy must specify a high cutoff for a sufficiently long time to ensure a high payoff conditional on acceptance, and lower cutoffs towards the end to ensure a high enough acceptance probability. Specifically, consider the cutoff $1 - \frac{2}{\lambda s+2}$ for each time $-s$ after time $-t$. This plan has a feature that for any finite time $s > 0$, the acceptance probability is

$$\frac{2}{\lambda s + 2} = \frac{\lambda t + 2}{\lambda s + 2} \cdot \frac{2}{\lambda t + 2} = O\Big(\frac{1}{\lambda t}\Big),$$

thus for any positive future time, the player's payoff conditional on acceptance is smaller than the best payoff by the amount $O(\frac{1}{\lambda t})$. Yet this gives us the limit acceptance probability of 1, as the probability for no acceptance can be calculated as:

$$e^{-\int_0^t \lambda \frac{2}{\lambda s+2} ds} = e^{-[2\ln(\lambda s+2)]_0^t} = \Big(\frac{2}{\lambda t + 2}\Big)^2 \to 0 \text{ as } \lambda \to \infty.$$

A rough intuition for why this can achieve the limit acceptance probability of 1 is that, for each time subinterval $[-\frac{T}{2^{k-1}}, -\frac{T}{2^k}]$ for $k = 1, 2, \ldots$, this strategy makes an acceptance with probability at least

$$1 - e^{-\lambda \cdot \frac{2}{\lambda(T/2^{k-1})+2} \cdot \frac{T}{2^k}} = 1 - e^{-\frac{\lambda T}{\lambda T + 2^k}} \to 1 - \frac{1}{e} > 0 \quad \text{as } \lambda \to \infty.$$

Since $1 - \frac{1}{e}$ is a positive constant independent of k, the acceptance probability increases with an exponential speed as k increases. We can indeed check that the future cutoff scheme $\frac{2}{\lambda s+2}$ gives the player a continuation payoff of $\frac{2}{\lambda t+2} = O(\frac{1}{\lambda t})$ at time $-t$.

Overall, we have shown that when $X = [0, 1]$ and the distribution μ is uniform, the limit expected duration is strictly positive. This argument is generalized to the cases of general distributions satisfying Assumption 1 and the following assumption. Let $F(x)$ be the cumulative distribution function of μ.

Assumption 2. There exists concave function φ such that $1 - \varphi(x)$ is of the same order as $1 - F(x)$ in $\{x \in \mathbb{R} \mid F(x) < 1\}$.[28]

To see what this assumption implies, consider two separate cases—bounded X and unbounded X. If X is bounded, besides pathological cases where F is non-differentiable at infinitely many points, the assumption amounts to say that the slope of the cdf F cannot diverge to infinity at the maximum payoff. If X is unbounded, a simple sufficient condition to guarantee that the assumption holds is to require there exists $\tilde{x}$ such that F is concave on $(\tilde{x}, \infty)$, or equivalently, there exists a nonincreasing density function f on $(\tilde{x}, \infty)$. Concavity of φ lets us invoke the Jensen's inequality to bound the cumulative acceptance probability.

Recall that $D(\lambda)T$ is the expected duration in the equilibrium for given arrival rate λ. Then we obtain the following:

Theorem 1. *Suppose $n = 1$. Under Assumptions 1 and 2,* $\liminf_{\lambda \to \infty} D(\lambda) > 0$.

In the Appendix we also provide a proof that the conclusion of this result holds when X is bounded with another assumption: for $\bar{x} \in X$, $\ln\big(\mu\big(\{x \in X \mid |\bar{x} - x| \leq \varepsilon\}\big)\big)$ is of the same order as $\ln \varepsilon$ when $\varepsilon > 0$ is small. A sufficient condition for this is that there exists $\alpha > 0$ such that $\mu\big(\{x \in X \mid |\bar{x} - x| \leq \varepsilon\}\big)$ is of the same order as ε^α when $\varepsilon > 0$ is small.

4.1.2 Multiple Agents

Now we extend our argument to the case of $n \geq 2$. The basic argument is the same as in the case of $n = 1$: We fix some strategies for players other than i, and consider bounding i's continuation payoff. However, it is not the case that we can implement this proof for any given strategies by the opponents. To see this point, consider the case of 2 players with $X = \{x \in \mathbb{R}^2_+ \mid x_1 = x_2 \leq 1\}$ and the uniform distribution. Suppose that we are given

[28] For functions $g(y)$ and $h(y)$, we say that $g(y)$ is *of the same order as* $h(y)$ in $Y \subseteq \mathbb{R}$ if there exist $c, C > 0$ and $\bar{y} < \sup(Y)$ such that $c|h(y)| \leq |g(y)| \leq C|h(y)|$ for all $y \geq \bar{y}$.

player 2's strategy to set the cutoff $v_2 = 0$ for the time interval $\left[-t, -\left(t - \frac{1}{\sqrt{\lambda t}}\right)\right]$, and then the cutoff $v_2 = 1$ for the rest of the time. Then, player 1's upper bound of acceptance probability cannot be given by $O(\frac{1}{\lambda t})$ because, to ensure the acceptance of a positive payoff, player 1 must accept within the time interval $\left[-t, -\left(t - \frac{1}{\sqrt{\lambda t}}\right)\right]$, and to do so she must set a low enough cutoff.[29]

What is missing in the above strategy of player 2 is the feature that a player's cutoff must be decreasing over time. In the above strategy, the cutoff starts from 0 and then jumps up to 1. We use the decreasingness to show our result.

To see how the decreasingness helps, fix t and consider player $-i$'s equilibrium cutoffs at time $-t$, and suppose for the moment that they will keep using these cutoffs in the future as well. Then, by the result in the case of $n = 1$, we know that the acceptance probability at $-t$ by playing optimally in the future against such strategies is $O(\frac{1}{\lambda t})$ as long as Assumption 2 is met for any cutoff profiles of the other players (sufficient conditions for this to hold are analogous to what we discussed after introducing Assumption 2). Let $p(s)$ for $s < t$ be the acceptance probability given by i's optimal strategy against $-i$'s fixed strategies. Now, consider the actual equilibrium cutoff strategy for $-i$ and consider a new future strategy for player i, which is to accept at each time $-s$ with probability $p(s)$. Notice that, since each opponent's cutoff is decreasing, the expected payoff conditional on acceptance at each time $-s$ must be greater than the case with fixed cutoffs for $-i$, while at each moment the acceptance probability is identical to that case. This means that i's continuation payoff at $-t$ must be higher than in the original case, which implies that the acceptance probability at $-t$ must be $O(\frac{1}{\lambda t})$.

Hence, we obtained the following: Recall that $D(\lambda)T$ is the expected duration in the equilibrium for given arrival rate λ. Let $F_i^{v_{-i}}$ be the marginal cumulative distribution function of player i's payoff conditional on cutoff profiles v_{-i} of the other players with $\mu(A(0, v_{-i})) > 0$.

Assumption 2′. There is $i \in N$ such that for all v^{-i} with $\mu(A(0, v_{-i})) > 0$, there exists a concave function φ such that $1 - \varphi(x_i)$ is of the same order as $1 - F_i^{v_{-i}}(x_i)$ in $\{x_i \in \mathbb{R} \mid F_i^{v_{-i}}(x_i) < 1\}$.[30]

Theorem 2. *Suppose $n \geq 2$. Under Assumptions 1 and 2′,* $\liminf_{\lambda \to \infty} D(\lambda) > 0$.

4.2 Step 2: Effects of the Larger Number of Agents

The second step of our argument concerns the effect of having a larger number of players.

[29] There also exist strategies for player 2 that are independent of λ and still give rise to a low cutoff for player 1, such as $v_2(t) = e^{-(T-t)}$.

[30] This assumption reduces to Assumption 2 when $n = 1$.

In what follows we demonstrate that *there are two reasons that we expect longer durations when there are more players.* The effects that underlie these reasons are called *ascending acceptability effect* and *preference heterogeneity effect.* We explain these effects in turn.

4.2.1 Ascending Acceptability Effect

In Section 4.1.2 we demonstrated that the decreasingness of the opponents' cutoffs can be used to reduce the acceptance probability (through the rise of continuation payoffs). The ascending acceptability effect is also based on the fact that the opponents' cutoffs are decreasing.

To isolate such an effect, let us consider the case when we add players whose preferences are independent of those of the existing players. Specifically, let two problems $(X, \mu), (Y, \gamma)$ satisfy Assumption 1 where $X \subseteq \mathbb{R}^n_+$, $Y \subseteq \mathbb{R}^m_+$, $\mu \in \Delta(X)$, $\gamma \in \Delta(Y)$, and $n, m \geq 1$. Consider three models: (i) n player model (X, μ), (ii) m player model (Y, γ), and (iii) $n + m$ player model $(X \times Y, \mu \times \gamma)$.

Theorem 3. *Suppose that the limit expected durations exist for models (i) and (ii) with $T = 1$, denoted by D_X and D_Y, respectively. Then the limit expected duration D_{XY} also exists in model (iii) with $T = 1$, and satisfies*

$$D_{XY} = 1 - \frac{(1 - D_X)(1 - D_Y)}{1 - D_X D_Y} \tag{5}$$

if $D_X D_Y < 1$, and $D_{XY} = 1$ if $D_X D_Y = 1$.

The reasoning of this proposition will be given in Section 4.2.2. Theorem 3 implies an immediate corollary:

Corollary 5. *Under the assumption in Theorem 3, $D_{XY} > \max\{D_X, D_Y\}$ if $D_X, D_Y \in (0, 1)$.*

In Section 4.2.2, we provide the explicit formula for the probability distribution of the expected duration. The formula in particular implies that the distributions of the durations in models (i) and (ii) are first-order stochastically dominated by that of model (iii), which implies Corollay 5.

There is a simple reasoning behind Corollay 5. Note first that the locus of the path in model (iii) projected on X is identical to the one in model (i) because, by (1), the direction of the vector is determined by the position of the barycenter in the acceptance set. Notice further that if we exogenously specify the strategies of additional m players to be the ones that accept any payoff profiles, then the time path of the cutoffs for the original n players should remain unchanged. In equilibrium, however, these m players' cutoffs are decreasing, so there will be more chances for desirable draws to be accepted ("ascending acceptability"). This is why we expect a longer duration with more players. Another way to put this is that the increase in the acceptance probability caused by additional m players corresponds to an increase in arrival rates over time. This means that a larger fraction of opportunities comes at the late stage of the game, so we expect a longer duration.

To understand the formula (5) in Theorem 3, manipulate it to get:

$$\frac{\text{The expected remaining time in model (iii)}}{\text{The expected remaining time in model (i)}} = \frac{1 - D_{XY}}{1 - D_X} = \frac{1 - D_Y}{1 - D_X D_Y} < 1. \tag{6}$$

Notice that $1 - D_X$ denotes the expected remaining time until the deadline at the time of agreement given X, and the same interpretation is valid for $1 - D_{XY}$. Thus, the left hand side of equation (6) is the ratio of remaining time with $X \times Y$ compared to that of X. This ratio is strictly smaller than 1 whenever $D_X < 1$, by the expression in the right hand side, and increases as D_Y grows. This is intuitive: Higher D_Y implies a slower speed for the continuation payoffs of the additional players to move. Thus players in X have more incentives to wait than in the case with a lower D_Y.

4.2.2 Preference Heterogeneity Effect

Theorem 3 considers the case where preferences of players in model (i) and those of players in model (ii) are independent. In many relevant cases, players' preferences are not independent. Specifically, they are often heterogeneous. We now analyze how heterogeneity in preferences, captured by the change in X and μ, affects the duration. In this subsection, we first provide a general duration formula to understand how preference heterogeneity affects the duration. Then we use this formula in specific examples to analyze the effect of preference heterogeneity.

Let us define values $\underline{r}, \overline{r}$ as follows:

$$\underline{r} = \liminf_{t\to\infty} \sum_{i\in N} \underline{d}_i(v^*(t)) \cdot b_i(v^*(t)), \quad \overline{r} = \limsup_{t\to\infty} \sum_{i\in N} \overline{d}_i(v^*(t)) \cdot b_i(v^*(t)) \tag{7}$$

where

$$\begin{aligned}
&b_i(v) = g_i(A(v)) - v_i, \quad b(v) = (b_1(v), \ldots, b_n(v)),\\
&\underline{d}_i(v) = \frac{1}{\mu(A(v))} \liminf_{\varepsilon\to 0} \frac{\mu(A((v_j + \varepsilon b_j(v))_{j<i}, (v_j)_{j\geq i})) - \mu(A((v_j + \varepsilon b_j(v))_{j\leq i}, (v_j)_{j>i}))}{\varepsilon |b(v)|},\\
&\overline{d}_i(v) = \frac{1}{\mu(A(v))} \limsup_{\varepsilon\to 0} \frac{\mu(A((v_j + \varepsilon b_j(v))_{j<i}, (v_j)_{j\geq i})) - \mu(A((v_j + \varepsilon b_j(v))_{j\leq i}, (v_j)_{j>i}))}{\varepsilon |b(v)|},
\end{aligned}$$

and $g(Y) = (g_1(Y), \ldots, g_n(Y))$ denotes a barycenter of the set $Y \subseteq \mathbb{R}^n$ with respect to μ. Recall that $P(t; \lambda)$ is the probability of no agreement until time $-t$, and $D(\lambda)$ is the limit expected duration when $T = 0$. Now we can show that $P(t;\infty) = \lim_{\lambda\to\infty} P(t;\lambda)$ can be written in the following way:

Theorem 4. *Under Assumption 1, for all $-t \in [-T, 0]$*

$$\left(\frac{t}{T}\right)^{1/\underline{r}} \le \liminf_{\lambda\to\infty} P(t;\lambda) \quad \text{and} \quad \limsup_{\lambda\to\infty} P(t;\lambda) \le \left(\frac{t}{T}\right)^{1/\overline{r}}, \text{ and}$$
$$\frac{1}{1+\underline{r}^{-1}} \le \liminf_{\lambda\to\infty} D(\lambda) \quad \text{and} \quad \limsup_{\lambda\to\infty} D(\lambda) \le \frac{1}{1+\overline{r}^{-1}}.$$

Thus, if $\underline{r} = \overline{r} =: r$, then for all $-t \in [-T, 0]$

$$P(t;\infty) = \left(\frac{t}{T}\right)^{1/r} \quad \text{and} \quad D(\infty) = \frac{1}{1+r^{-1}}.$$

Proof Sketch. Let us provide a proof when $r = \underline{r} = \overline{r}$, and μ has a density function. A formal proof in the general case is given in the Appendix. To show the result, we first prove

$$\lim_{\lambda\to\infty} p(t)\cdot\lambda t = \frac{1}{r}$$

where $p(t) = \mu(A(v^*(t)))$. By the ODE (1), $v^{*\prime}(t) = \lambda(g(A(v^*(t))) - v^*(t))\cdot p(t)$. Differentiating $p(t) = \mu(A(v^*(t)))$,

$$\begin{aligned}
p'(t) &= \sum_{i\in N} \frac{\partial\mu(A(v))}{\partial v_i}\bigg|_{v=v^*(t)} v_i^{*\prime}(t) \\
&= \sum_{i\in N} \frac{\partial\mu(A(v))}{\partial v_i}\bigg|_{v=v^*(t)} \lambda\cdot(g_i(A(v^*(t))) - v_i^*(t))\cdot p(t) \\
&= -\sum_{i\in N} d_i(v^*(t))p(t)\cdot\lambda b_i(v^*(t))\cdot p(t).
\end{aligned}$$

Therefore

$$\frac{p'(t)}{\lambda p(t)^2} = -\sum_{i\in N} d_i(v^*(t))b_i(v^*(t)).$$

This implies that r is the limit of $-p'(t)/\lambda p(t)^2$ as $t\to\infty$. If the limit exists, for any $\varepsilon > 0$ there exists $\bar{t}$ such that $t \ge \bar{t}$ implies

$$r - \varepsilon \le -\frac{p'(t)}{\lambda p(t)^2} \le r + \varepsilon. \tag{8}$$

This means that $p(t)$ is approximated by the solution of ODE $p'(t) = -r\lambda p(t)^2$ with an initial condition at $t = \bar{t}$. Solving this equation, for large t,

$$p(t) \approx \frac{1}{r\lambda(t-\bar{t}) + p(\bar{t})^{-1}}.$$

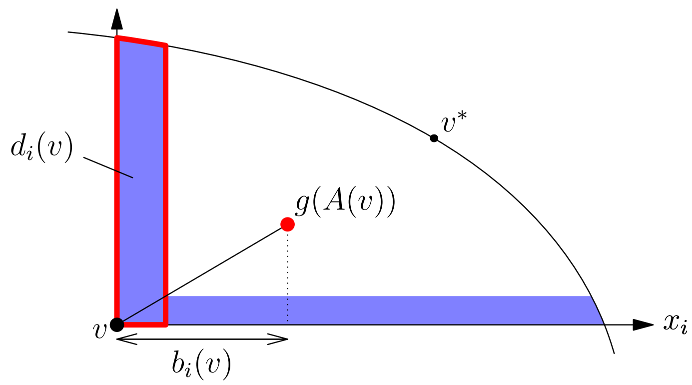

Figure 7: Density term and barycenter term

Hence we get $\lim_{\lambda\to\infty} p(t)\cdot\lambda t = \frac{1}{r}$. We can compute the approximated probability of disagreement P by following formula (3), showing that $P(t;\infty) = \left(\frac{t}{T}\right)^{\frac{1}{r}}$.[31] Applying formula (4), one can easily obtain $D(\infty) = \frac{1}{1+r^{-1}}$. □

Moreover, Theorem 4 immediately implies that if $\underline{r} = \overline{r} = r$, then $P(t;\infty)$ is increasing in t, and (a) for $r < 1$, $P(t;\infty)$ is concave and $\lim_{t\to 0} P'(t;\infty) = \infty$, (b) for $r = 1$, $P(t;\infty)$ is linear and $P'(t;\infty) = \frac{1}{T}$, and (c) for $r > 1$, $P(t;\infty)$ is convex and $\lim_{t\to 0} P'(t;\infty) = 0$.

The graphical intuition for the formula in Theorem 4 is depicted in Figure 7. The first term $d_i(v^*(t))$, which we call the *density term*, is i's marginal density at her continuation payoff conditional on the distribution restricted to the acceptance set. The second term $b_i(v^*(t))$, which we call the *barycenter term*, measures the distance between the barycenter of the acceptance set and the cutoff. Remember that the speed at which the cutoff moves towards the limit point is determined by this distance, by equation (1). Hence the formula for r in equation (7) measures the speed at which the acceptance set shrinks. This is consistent with the fact that the duration formula in equation (7) is increasing in $r > 0$ because if the acceptance probability shrinks quickly, then players reject with high probability for a long time, resulting in a long duration.

Theorem 4 also explains the reasoning behind the formula in Theorem 3. Let r_X, r_Y, and r_{XY} be associated with models (i), (ii), and (iii), respectively. r_X and r_Y are well-defined as D_X and D_Y exist. Then, by definition, r_{XY} must be equal to $r_X + r_Y$. Hence the limit expected duration in model (iii) exists and it is $\frac{1}{1+r_{XY}^{-1}} = \frac{1}{1+(r_X+r_Y)^{-1}}$. Rearranging terms, we get the formula in Theorem 3.

Now we use the formula given in Theorem 4 to analyze specific classes of games to understand the preference heterogeneity effect.

Example 1 (Bounded X, smooth boundary, and continuous and strictly positive density).

[31] The computation is given in Lemma 24 in the Appendix.

Number of agents	1	2	3	5	10	100
Limit expected duration	.333	.571	.692	.806	.901	.990

Table 1: Limit expected duration of search as opportunities arrive more and more frequently.

Here we impose assumptions employed often in the literature on multi-agent search (Wilson (2001), Compte and Jehiel (2010), and Cho and Matsui (2011)).

Assumption 3. (a) X is convex and compact subset of $\mathbb{R}^n_+$, and has a smooth Pareto frontier.

(b) The probability measure μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$, and admits a probability density function f that is continuous and bounded away from zero, i.e., $\inf_{x\in X} f(x) > 0$.

In this case, when the cutoffs for players are high enough, the acceptance set can be approximated with an n-dimensional pyramid by the assumption of smooth boundary, and the distribution over this acceptance can be approximated with a uniform distribution due to the assumption of continuous and strictly positive density. This allows us to explicitly compute the limit expected duration, as follows:

Proposition 6. *Under Assumptions 1 and 3,* $\lim_{\lambda\to\infty} D(\lambda) = \dfrac{n^2}{n^2+n+1}$.

Corollary 7. *Under Assumptions 1 and 3,* $\lim_{\lambda\to\infty} D(\lambda)$ *is increasing in* n.

In the proof in the Appendix, we show this result under much more general assumptions (Assumptions 1, 4, and 7).

The solution of the expected duration provided in Proposition 6 implies that, if only two players are involved in search, the expected duration is $\frac{4}{7}T$, and it monotonically increases to approach T as n gets larger. Table 1 shows the limit expected duration for several values of n when $T = 1$.

If Assumption 3 holds, then the limits in $\overline{d}_i$ and $\underline{d}_i$ coincide. Let us denote $d_i(v) = \overline{d}_i(v) = \underline{d}_i(v)$. When X is bounded, as assumed in Assumption 3, $d_i(v)$ grows to infinity as v comes close to the Pareto frontier, while $b_i(v)$ decreases to zero. For this reason we normalize these terms as follows: For $Y \subseteq X$, let $s(Y) = V(Y)^{\frac{1}{n}}$ be the "size" of Y in X. Let us define the normalized terms as

$$\begin{aligned}
\tilde{d}_i(v) &= d_i(v)s(A(v)) \\
&= \frac{s(A(v))}{\mu(A(v))} \lim_{\varepsilon\to 0} \frac{\mu(A((v_j+\varepsilon b_j(v))_{j<i},(v_j)_{j\geq i})) - \mu(A((v_j+\varepsilon b_j(v))_{j\leq i},(v_j)_{j>i})))}{\varepsilon|b(v)|}, \\
\tilde{b}_i(v) &= \frac{b_i(v)}{s(A(v))} = \frac{g_i(A(v)) - v_i}{s(A(v))}.
\end{aligned}$$

Notice that if $V(A(v)) = 1$, then $\tilde{d}_i(v) = d_i(v)$ and $\tilde{b}_i(v) = b_i(v)$.

When the acceptance set is approximated by an n-dimensional pyramid whose area is normalized to 1, the density term is $\left(n^{\frac{n-1}{n}}\right) \times n$ where "$\times n$" accounts for the fact that there are n agents, and the barycenter term is $n^{\frac{1}{n}}/(n+1)$.

Under Assumption 3, the ascending acceptability effect can be seen in Figure 7 by noting that the area that corresponds to the density term has two segments (n segments in the case of n players), each corresponding to each player. Thus, adding a player results in an extra region of payoffs that will be accepted in the future. The probability density in the extra region increases not only because the number of segments increases, but also because the length of each segment increases. This happens precisely because players' preferences are heterogeneous so the density of the marginal distribution of a player's payoff increases as her payoff decreases. This means that the "extra region" that a player's opponents accept in the future contains relatively more favorable allocations for the player when there are more opponents. Although the barycenter term deceases due to this preference heterogeneity as well, the overall effect is positive. We call this effect the preference heterogeneity effect. Note well that the effect of preference heterogeneity is to (only) magnify the ascending acceptability effect by lengthening the length of each segment.

The argument so far suggests that, under Assumption 3, we can make the following decomposition:

$$\text{The limit duration with } n \text{ agents} = \frac{1}{1 + \left(n \cdot \frac{n}{n+1}\right)^{-1}} = \frac{1}{1 + \left(\underbrace{n \cdot n^{\frac{n-1}{n}}}_{\text{density term}} \cdot \underbrace{\frac{n^{\frac{1}{n}}}{n+1}}_{\text{barycenter term}}\right)^{-1}}. \qquad \square$$

We next use the formula (7) to conduct several comparative statics with respect to preference heterogeneity, holding fixed the number of players.

Example 2 (**Change in the shape of X around the limit payoff profile**).

First, consider the two-player case, under Assumptions 1 and 3 but dropping the assumption that the Pareto frontier is smooth (assumed in Assumption 3 (b)). In this environment, generally X has a kink at the limit expected payoff, so that the acceptance set when t is large can be approximated by a quadrilateral similar to $co\{(0,0)(1,0)(0,1)(q,q)\}$ after rescaling each axis. In this case the limit duration is computed as

$$\text{The limit duration } = \frac{1}{1+\left(2\cdot\frac{2q+1}{6q}\right)^{-1}} = \frac{1}{1+\left(\underbrace{2\cdot\frac{1}{\sqrt{q}}}_{\text{density term}}\cdot\underbrace{\frac{2q+1}{6\sqrt{q}}}_{\text{barycenter term}}\right)^{-1}}. \tag{9}$$

Notice that the term corresponding to the preference heterogeneity effect, $\frac{2q+1}{6q}$, is decreasing in q. This is consistent with how the shape of acceptance set changes with respect to q. As q grows, the kink of the boundary at the limit payoff becomes sharper, so the preferences among the players become less heterogeneous. This means that the "extra region" that a player's opponent accepts in the future does not contain relatively favorable allocations for the player. As a result, the limit duration can be calculated as $D(\infty) = \frac{2q+1}{5q+1}$, and this is decreasing in q. □

Example 3 (**Change in the shape of X under Assumptions 1 and 3**).

Next, consider n-player symmetric X and μ. Consider a transformation of this problem in the following sense: let X^q and $\bar{X}^a$ defined by

$$X^q = \{x \in X \mid \max_{i\in N} x_i - \min_{j\in N} x_j \leq q\} \quad \text{and} \quad \bar{X}^a = \{y^a(x) \,|\, x \in X\}$$

where $y^a(x) = ax + (1-a)x^e$, $a \in (0,1]$, with $x^e = \left(\frac{x_1+\cdots+x_n}{n}, \ldots, \frac{x_1+\cdots+x_n}{n}\right)$. Define μ^q by $\mu^q(C) = \frac{1}{\mu(X^q)} \cdot \mu(C \cap X^q)$ and $\bar{\mu}^a$ by $\bar{\mu}^a\big(\{y^a(x) \,|\, x \in C\}\big) = \mu(C)$ for any $C \subseteq X$.

Both μ^q and $\bar{\mu}^a$ shrink the distribution to the middle: μ^q takes out the offers that give agents "too asymmetric" payoffs, while $\bar{\mu}^a$ moves each point by the amount proportional to the original distance to the equi-payoff line. See Figure 8 for a graphical description in the case of two players. Proposition 6 shows that as long as Assumptions 1 and 3 are met, expected duration is unaffected by the specificality of distribution μ. That is,

Proposition 8. *If (X, μ) satisfies Assumptions 1 and 3, then the limit expected durations with (X^q, μ^q) and $(\bar{X}^a, \bar{\mu}^a)$ are the same as in the case with (X, μ) for any $a \in [0, 1)$.*

The intuition is simple. In both cases, the distribution is still uniform around the limit point and the boundary is smooth even under $\bar{\mu}^a$, so exactly the same calculation as in the case with μ suggests that the limit duration is $\frac{n^2}{n^2+n+1}$. In this case, however, durations with finite arrival rates are affected by the change in preferences. Table 2 shows that the duration becomes shorter as we make the preferences less heterogeneous, in the case $X = \{x \in \mathbb{R}^2_+ \,|\, x_1 + x_2 \leq 1\}$. □

Example 4 (**Change in the distribution over unbounded X**).

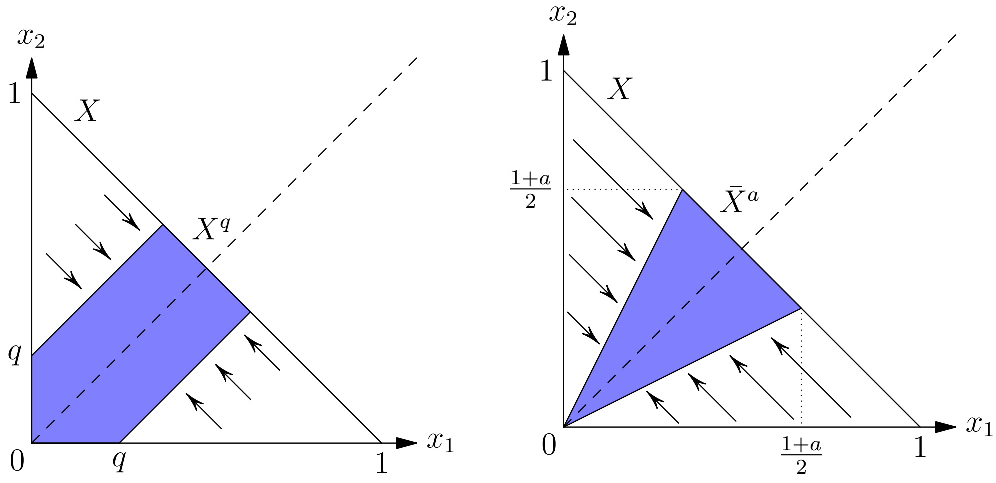

Figure 8: Transfer of allocations in the negotiation for μ^q (left) and $\bar{\mu}^a$ (right).

Consider 2-player symmetric $X = \mathbb{R}^2_+$ and μ which is associated with a density function f_σ parameterized by σ as follows:

$$f_\sigma(x_1, x_2) \propto \begin{cases} e^{-(x_1+x_2)} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_1-x_2)^2}{2\sigma^2}} & \text{if } (x_1, x_2) \in \mathbb{R}^2_+. \\ 0 & \text{otherwise,} \end{cases}$$

That is, we consider an exponential distribution in the direction of 45 degree line, and a normal distribution with variance σ^2 in the direction of 135 degree line. The parameter σ measures the heterogeneity of preferences. Notice that the limit distribution as $\sigma \to 0$ is the (degenerate) exponential distribution over 45 degree line, and the limit distribution as $\sigma \to \infty$ is the product measure in which each player's marginal distribution is an exponential distribution with parameter $\sqrt{2}$. We can solve for the limit duration in these two cases analytically using the duration formula in Theorem 4. In the former case the problem is isomorphic to that of one-player case and the limit expected duration as $\lambda \to \infty$ is $\frac{1}{2}$, and in the latter case it is $\frac{2}{3}$. We use the duration formula to numerically compute the limit duration in the intermediate values of σ, and the result is given in the graph of Figure 9. For any $\sigma > 0$, the density term is a constant $\frac{1}{\sqrt{2}}$. It is the barycenter term that varies with σ. Specifically, the barycenter term rises with σ from $\frac{1}{\sqrt{2}}$ (when $\lambda \to 0$) to $\sqrt{2}$ (when $\lambda \to \infty$). This is because, the more heterogeneous the preferences are, the more realizations of payoffs are scattered outside of the acceptance set.[32] Since it is more difficult for a realization to stay in the acceptance set if the sum of payoffs are smaller, heterogeneity implies that, conditional on acceptance, payoffs are high on average. Thus, if preferences are more heterogeneous (σ is larger), the opponent's gain relative to the continuation payoff conditional on acceptance is higher. This means that

[32]The effect that the total probability on the acceptance set decreases for a given value of v does not matter, as we take the limit as $\lambda \to \infty$.

μ^q	λ				
	10	20	30	100	∞
$q = 1$	0.608	0.591	0.585	0.576	0.571
$q = 0.8$	0.607	0.590	0.584	0.575	0.571
$q = 0.6$	0.600	0.586	0.581	0.575	0.571
$q = 0.4$	0.579	0.574	0.573	0.572	0.571
$q = 0.2$	0.515	0.534	0.544	0.562	0.571
$q = 0$	0.398	0.366	0.355	0.340	0.333

$\bar{\mu}^a$	λ				
	10	20	30	100	∞
$a = 1$	0.608	0.591	0.585	0.576	0.571
$a = 0.8$	0.625	0.601	0.591	0.578	0.571
$a = 0.6$	0.604	0.588	0.583	0.575	0.571
$a = 0.4$	0.567	0.566	0.567	0.570	0.571
$a = 0.2$	0.489	0.512	0.528	0.557	0.571
$a = 0$	0.398	0.366	0.355	0.340	0.333

Table 2: Preference heterogeneity effect under Assumptions 1 and 3. q and a measure heterogeneity of preferences.

the loss from a unit time passing is larger, so the opponent will decrease the cutoff faster. This makes the incentive to wait larger, implying a longer expected duration. □

4.3 Step 3: Finite Arrival Rate

Our results on the expected duration so far suggest that there are reasons to expect a positive duration of search even in the limit as the friction of search vanishes. To evaluate the significance of these reasons, we now consider cases with finite arrival rates. We will show that *the expected duration converges to the limit duration very fast*, provides evidence

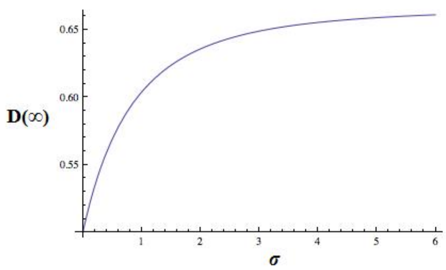

Figure 9: Preference heterogeneity (measured by σ) and the limit search duration.

that our limit analysis contains economically-meaningful content—so the effects that we identify in the previous discussion are the keys to understand the positive duration in reality.

First, we show that the convergence speed of the duration is high. Recall that $D(\lambda)$ and $D(\infty)$ are the expected durations under arrival rate λ and the limit expected duration for $T = 1$, respectively. Theorem 4 ensures the existence of $D(\infty)$ if $\underline{r} = \overline{r}$.

Theorem 5. *Under Assumption 1, if* $\underline{r} = \overline{r}$*, then* $|D(\lambda) - D(\infty)| = O(\frac{1}{\lambda})$.

This is a fast rate of convergence; for example, when payoffs realize upon agreement and there is a positive discount rate (with a finite or infinite horizon), $|D(\lambda) - D(\infty)|$ is of the same order as $\frac{1}{\lambda^{\frac{1}{n+1}}}$ under Assumptions 1 and 3.

We further support our claim numerically through a number of examples. We find that the limit duration of Proposition 6 is not far away from those with finite λ in many cases. The differential equation (1) does not have a closed-form solution in general, and even if it does, $D(\lambda)$ may not have a closed-form solution as it involves further integration. For this reason, we solve the differential equation and integration numerically to obtain the values of $D(\lambda)$ for specific values of λ. We considered the following distributions standard in the literature with $T = 1$. Note that, in the apartment search example, if the couple has ten weeks before the deadline and a broker provides information of an apartment once per week on average (very *in*frequent case), the situation corresponds to $\lambda = 10$.

Case 1: μ is the uniform distribution over $X = \{x \in \mathbb{R}^n_+ \mid \sum_{i\in N} x_i \leq 1\}$ for $n = 1, 2, 3$ and $\lambda = 10, 20, 30, 100$.

Case 2: μ is the uniform distribution over $X = \{x \in \mathbb{R}^n_+ \mid \sum_{i\in N} x_i^2 \leq 1\}$ for $n = 1, 2$ and $\lambda = 10, 20, 30, 100, 1000$.

Case 3: μ is the uniform distribution over $X = \{x \in \mathbb{R}^n_+ \mid \max_{i\in N} x_i \leq 1\}$ for $n = 1, 2, 3$ and $\lambda = 10, 20, 30, 100$.

Case 4: μ is the product measure over $X = \mathbb{R}^n_+$ where each marginal corresponds to an exponential distribution with parameter $a_i > 0$ for $n = 1, 2, 3, 10$ and $\lambda = 10, 20, 30, 100$.

Case 5: μ is the product measure over $X = \mathbb{R}^n_+$ where each marginal corresponds to a log-normal distribution with mean 0 and standard deviation $\sigma = \frac{1}{4}, 1, 4$ for $n = 1$ and $\lambda = 10, 20, 30, 100$.

Figure 10 shows a graph of the cumulative probability of agreement for $\lambda = 10$ (i.e., $1 - P(t; 10)$) and for $\lambda \to \infty$ (i.e., $1 - \lim_{\lambda\to\infty} P(t; \lambda)$) of Case 1 with $n = 2$. Also,

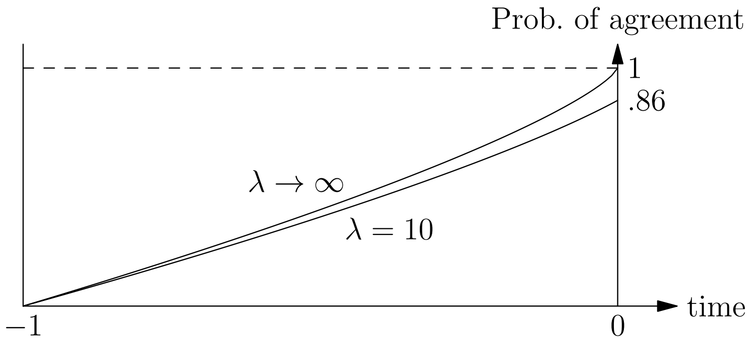

Figure 10: A numerical example of the cumulative probability of agreement.

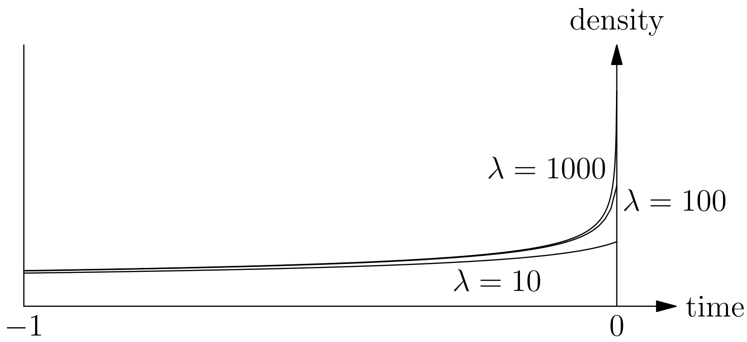

Figure 11: The probability density of the duration of search.

Figure 11 shows the probability density function of the duration of search in such a case (i.e., $P(t;\lambda) \cdot p(t;\lambda)$).

In Table 3, we provide the computed values for selected choices of parameter values and cases. We provide the complete description of all the computed values in the Appendix.[33]

According to our calculation, $D(\lambda)$ is within 10% difference from $D(\infty)$ except for a single case where the difference is 19.4%, which happens in Case 1 with $n = 1$. Generally the percentage falls as the number of agents becomes larger and the arrival rate goes up.[34] For example if we add another player in Case 1, the difference falls down dramatically to 6.5%, and if we increase the arrival rate to 20 (fixing the number of players at $n = 1$), the difference becomes 9.9%. In all other cases the difference is much smaller and often less than 5%.[35] Notice that we predict "over-shooting" of the expected duration in Case 2 than in 1. This is because when the continuation value is far away from the boundary, the shape of the acceptance set is close to a square with which we expect a shorter duration,

[33]Some values are computed analytically: The results for Case 1 for $n = 1, 2$, Case 2 for $n = 1$, Case 4 for $n = 1, 2, 3, 10$ are analytical.

[34]The monotonicity with respect to arrival rates can be analytically proven for Case 1 with $n = 2$. However, the monotonicity fails in general. To see this, consider the case in which $D(\infty) = 1$. By optimality it must be the case that $D(\lambda) < 1$ for any finite λ, so in this case the duration cannot be decreasing in λ. Note also that in Case 2, after the "overshooting" the duration comes back to the limit duration, thus $D(\lambda)$ is nonmonotonic.

[35]We are planning to extend the analysis to more cases beyond the setting provided here.

Case 1

		λ				
		10	20	30	100	∞
$n=1$	Expected duration	0.398	0.366	0.355	0.340	0.333
	Percentage (%)	19.4	9.92	6.64	2.00	0
$n=2$	Expected duration	0.608	0.591	0.585	0.576	0.571
	Percentage (%)	6.48	3.44	2.35	0.731	0

Case 2

		λ					
		10	20	30	100	1000	∞
$n=1$	Expected duration	0.398	0.366	0.355	0.340	0.334	0.333
	Percentage (%)	19.4	9.92	6.64	2.00	0.200	0
$n=2$	Expected duration	0.582	0.568	0.565	0.562	0.567	0.571
	Percentage (%)	1.90	−0.541	−1.21	−1.61	−0.798	0

Case 4

		λ				
		10	20	30	100	∞
$n=1$	Expected duration	0.545	0.524	0.516	0.505	0.5
	Percentage (%)	9.09	4.76	3.23	0.990	0
$n=2$	Expected duration	0.693	0.681	0.676	0.670	0.667
	Percentage (%)	3.91	2.11	1.45	0.465	0

Table 3: Expected durations for finite arrival rates

and gradually the shape approaches a triangle (precisely, the density effect *would* be smaller than the case of a triangle *if* the limit shape of the acceptance set *were* the same as that of X). This suggests that convexity of the set of available allocations, which is often assumed in the literature, facilitates a faster convergence. When X is unbounded, the computed difference was much smaller (Cases 4 and 5).

The discussion so far enables us to perform the decomposition mentioned in the Introduction. The expected duration in the 2-player model with the uniform distribution over $X = \{x \in \mathbb{R}^2_+ \mid x_1 + x_2 \leq 1\}$ and $\lambda = 10$ is 0.608. The limit expected duration as $\lambda \to \infty$ in this case is $\frac{4}{7}$, so the difference is 0.037. This limit duration $\frac{4}{7}$ is calculated from the number r that we denote by $r_2 := \frac{4}{3}$. When there is only one player and the distribution is uniform over $[0, 1]$, the limit duration is $\frac{1}{3}$, and the number r is $r_1 := \frac{1}{3}$. The difference between r_2 and r_1—the difference caused by adding one more player—is determined by two effects, the ascending acceptability effect and the preference heterogeneity effect. To calculate the ascending acceptability effect, we compute r that we would obtain if this additional agent's distribution over feasible payoffs is independent of the original player's, and the distribution corresponds to the uniform distribution over $[0,1]$. The duration and r in this case are $\frac{1}{2}$ and $r_{\text{op}} := 1$, respectively, and the difference in terms of r is given by $r_{\text{op}} - r_1 = 1 - \frac{1}{2} = \frac{1}{2}$. Now the preference heterogeneity effect is the change in r caused by the change in distribution from this product measure to X. This is given by $r_2 - r_{\text{op}} = \frac{4}{3} - 1 = \frac{1}{3}$. In general, fixing an n-player model (X, μ) and an $(n+m)$-player model (Y, γ), we can solve for the ascending acceptability effect by computing the difference between the r in the model (X, μ) and the r in the model $(X \times [0,1]^m, \mu \times (U[0,1])^m)$. Then the preference heterogeneity effect can be computed by solving for the difference in the latter r and the r in the model (Y, γ).[36] This decomposition is well-defined in the sense that the ascending acceptability effect (resp. preference heterogeneity effect) of changing the models from an n-player model (X, μ) to an $(n+m)$-player model (Y, γ) is identical to the sum of ascending acceptability effect (resp. preference heterogeneity effect) of changing the models from an n-player model (X, μ) to an $(n+l)$-player model (Z, δ) and the ascending acceptability effect (resp. preference heterogeneity effect) of changing models from an $(n+l)$-player model (Z, δ) to an $(n+m)$-player model (Y, γ) where $l < m$, since r is additive.

5 Welfare Implications

In Section 3, we showed that the limit expected payoff must be weakly Pareto efficient if the limit exists. In this section we seek further welfare implications. Let us impose the following assumption to rule out uninteresting cases:

Assumption 4. (a) X is a compact subset of $\mathbb{R}^n$.

(b) X coincides with the closure of its interior (with respect to the standard topology of $\mathbb{R}^n$).

(c) The probability measure μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$, and admits a probability density function f.

(d) The probability density f is bounded above and away from zero, i.e., $\sup_{x \in X} f(x) < \infty$ and $\inf_{x \in X} f(x) > 0$.

Condition (a) in Assumption 4 is a standard assumption when we consider welfare implications. Note that we do not assume convexity here. Condition (b) rules out irregularities involving lower dimensional subsets. For example, if X has an isolated point

[36] The uniform distribution over $[0,1]$ can be replaced with any distribution with a positive continuous density over a compact interval without changing the computation.

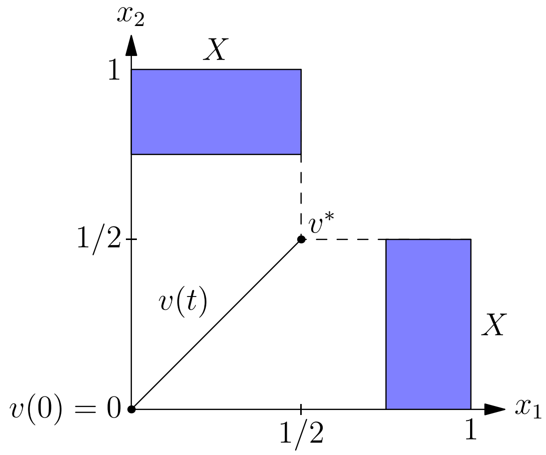

Figure 12: A path that converges to a weakly Pareto efficient allocation.

this condition is violated, because the interior of X does not contain any isolated points. Condition (c) implies that $\mu(Y) = 0$ for any $Y \subseteq X$ that has (n-dimensional) Lebesgue measure zero. Condition (d) is a condition that makes our analysis tractable.

In general, v^* is not necessarily (strictly) Pareto efficient in X even if v^* exists. There is an example of a distribution μ satisfying Assumptions 1 and 4 in which $v^*(t)$ converges to an allocation that is not Pareto efficient.

Example 5. Let $n = 2$, $X = \big([0, 1/2]\times[3/4, 1]\big)\cup\big([3/4, 1]\times[0, 1/2]\big)$, and f be the uniform density function on X, which is shown in Figure 12. By the symmetry with respect to the 45 degree line, we must have $v_1^*(t) = v_2^*(t)$ for all t. Therefore $v^* = (1/2, 1/2)$, which is not Pareto efficient in X.[37] □

Note that v^* is weakly Pareto efficient, and that X is a non-convex set in this example. In fact, we can show that v^* is strictly Pareto efficient if X is convex. Furthermore, even if X is not convex, we can show v^* is "generically" Pareto efficient, that is, v^* is Pareto efficient in X for any generic f that satisfies Assumptions 1 and 4.

Formally, let $\mathcal{F}$ be the set of density functions that satisfy Assumptions 1 and 4. We consider a topology on $\mathcal{F}$ defined by the following distance in $\mathcal{F}$: For $f, \tilde{f} \in \mathcal{F}$,

$$\big|f - \tilde{f}\big| = \sup_{x\in X}\big|f(x) - \tilde{f}(x)\big|.$$

[37] There are (non-trembling-hand) subgame perfect equilibria in which players obtain a more efficient payoff profile than $(1/2, 1/2)$. For example, consider a strategy profile in which players agree with allocations close to $(1, 1/2)$ or $(1/2, 1)$, and if one of the players rejects such allocations, both players reject all allocations after the deviation. This is a subgame perfect equilibrium and gives players expected payoffs close to $(3/4, 3/4)$ in the limit. Similar constructions show that any allocations in the convex hull of general nonconvex X can be an expected payoff profile supported by a subgame perfect equilibrium. However, we rule out such subgame perfect equilibria in a view that rejecting anything after deviation is not a credible threat if a player expects the others to accept with a small probability.

Proposition 9. *Under Assumptions 1 and 4, the set* $\{f \in \mathcal{F} \mid v^* \text{ is Pareto efficient in } X\}$ *is open and dense in* $\mathcal{F}$.

This proposition shows that v^* is efficient only for generic f. However, if X is convex, then v^* is efficient for all f.

Proposition 10. *Suppose that* X *is a convex set. Under Assumptions 1 and 4,* v^* *is Pareto efficient in* X.

Pareto efficiency implies that players reach an agreement almost surely if t is very large. To see this, let $\pi(t)$ be the probability that players reach an agreement in equilibrium before the deadline given that no agreement has been reached until time $-t$. Then the expected continuation payoffs $v^*(t)$ must fall in the set $\{\pi(t)v \mid v \in co(\hat{X})\}$ where $co(\hat{X})$ is the convex hull of $\hat{X}$. This implies $v^*(t)/\pi(t) \in co(\hat{X})$. We have $v_i^*(t) > 0$ for all $t > 0$ and $i \in N$ since $v_i^*(t)$ is nondecreasing and $v_i^{*\prime}(0) > 0$ by equation (1). Since there is a positive probability that no opportunity arrives before the deadline, $\pi(t)$ is smaller than one. Therefore $v^*(t)/\pi(t) \in co(\hat{X})$ Pareto dominates $v^*(t)$. Since $co(\hat{X}) \cap A(v^*(t))$ converges to a singleton as $v^*(t)$ goes to v^* if v^* is Pareto efficient, this implies $\lim_{t\to\infty} \pi(t) = 1$. That is, we have the following proposition:

Proposition 11. *Suppose that* v^* *is Pareto efficient. Then the probability of agreement before the deadline converges to one as* $\lambda \to \infty$.

We note that this proposition fails if v^* is only weakly Pareto efficient. In Example 5, it is evident that players reach no agreement before the deadline with a positive probability, since the limit allocation is $(1/2, 1/2)$ while players should find a good allocations close to $(1, 1/2)$ or $(1/2, 1)$ in the limit as $T \to \infty$.

In Propositions 9 and 10, we showed that $v^*(t)$ almost always converges to the Pareto frontier of X. Now, we consider an inverse problem: For any Pareto efficient allocation w in X which is not at the edge of the Pareto frontier,[38] we show that one can find a density f that satisfies Assumptions 1 and 4 such that the limit of the solution $v^*(t)$ of equation (1) is w.

Proposition 12. *Suppose that* $X \subseteq \mathbb{R}^n_+$ *satisfies Assumption 4 (a), (b). Suppose that* $w \in \mathbb{R}^n_{++}$ *is a Pareto efficient allocation in* X*, and is not located at the edge of the Pareto frontier of* X*. Then there exists a probability measure* μ *with support* X *such that Assumptions 1 and 4 hold, and* $\lim_{\lambda\to\infty} v^*(t) = w$ *for all* $t \in (0, T]$.

In the proof, we construct a probability density function f to have a large weight near $w \in X$, and show that the limit continuation payoff profile is w if there is a sufficiently large weight near w. Note that this claim is not so obvious as it seems. Indeed, we will see in Section 6 that the limit is independent of density f if there is a positive discount rate $\rho > 0$, as long as Assumptions 1 and 4 hold.

[38]We formally define this property in the proof given in the Appendix.

6 The Payoffs Realizing upon Agreement

In this section, we consider the case where the payoffs realize as soon as an agreement is reached, as opposed to the assumption in the previous sections that the payoffs realize only at the deadline. We suppose that if a payoff profile $x = (x_1, \ldots, x_n)$ is accepted by all players at time $-t \in [-T, 0]$ then player i obtains a payoff $x_i e^{-\rho(T-t)}$ where $\rho \geq 0$ is a discount rate. If no agreement has been reached until time 0, each player obtains the payoff 0.[39] First, we note that if $\rho = 0$, exactly the same analyses as in the previous sections apply. This is because with $\rho = 0$, player i's payoff when an agreement occurs at time $-t$ is $x_i e^{-\rho(T-t)} = x_i e^{-0\cdot(T-t)} = x_i$, which is independent of t. In this section, we focus on the case where $\rho > 0$. Under Assumption 1, an easy computation shows that the differential equation (1) is modified in the following way:

$$v'(t) = -\rho v(t) + \lambda \int_{A(t)} \big(x - v(t)\big)\, d\mu \tag{10}$$

with an initial condition $v(0) = (0, \ldots, 0) \in \mathbb{R}^n$.

Suppose Assumptions 1 and 4 hold. Let $v^*(t; \rho, \lambda)$ be the (unique) solution of ODE (10).[40] If λ is large, the right hand side of equation (10) is approximated by the right hand side of equation (1) when the value of the integral is not too small. Therefore, $v^*(t; \rho, \lambda)$ is close to the solution of equation (1) in the case of $\rho = 0$, for λ large relative to ρ. This resemblance of trajectories holds until $v^*(t; \rho, \lambda)$ approaches the boundary of $\hat{X}$. In particular, we can show that the path of $v^*(t; \rho, \lambda)$ approaches $v^*(t; 0, \infty) = \lim_{\lambda\to\infty} v^*(t; \rho, \lambda)$ arbitrarily closely as $\lambda \to \infty$, where $v^*(t; 0, \infty)$ is the limit of the solution of equation (1).

Proposition 13. *For all $\varepsilon > 0$, there exists $\bar{\lambda} > 0$ such that for all $\lambda \geq \bar{\lambda}$,*

$$|v^*(t; 0, \infty) - v^*(t; \rho, \lambda)| \leq \varepsilon \quad \textit{for some } t.$$

Remark 1. Before analyzing $v^* = \lim_{\lambda\to\infty} v^*(t; \rho, \lambda)$, let us consider another limit $v^*(\infty) = \lim_{t\to\infty} v^*(t; \rho, \lambda)$. Since the right hand side of equation (10) is not proportional to λ, these two limits do not coincide for positive $\rho > 0$. If the limit $v^*(\infty)$ exists, this must satisfy

$$\rho v^*(\infty) = \lambda \int_{A(v^*(\infty))} \big(x - v^*(\infty)\big)\, d\mu. \tag{11}$$

For $\rho > 0$, equality (11) shows $\mu(A(v^*(\infty))) > 0$, which implies that $v^*(\infty)$ is Pareto inefficient in X. This will contrast with efficiency of $v^* = \lim_{\lambda\to\infty} v^*(t; \rho, \lambda)$ that we will show in Proposition 14.

[39]This is with loss of generality but setting a nonzero threat point payoff leads only to minor modifications of the statements of our results.

[40]Essential uniqueness of trembling-hand equilibrium is obtained by a proof analogous to that for Proposition 1. The unique solution of equation (10) gives the cutoff profile that characterizes a trembling-hand equilibrium.

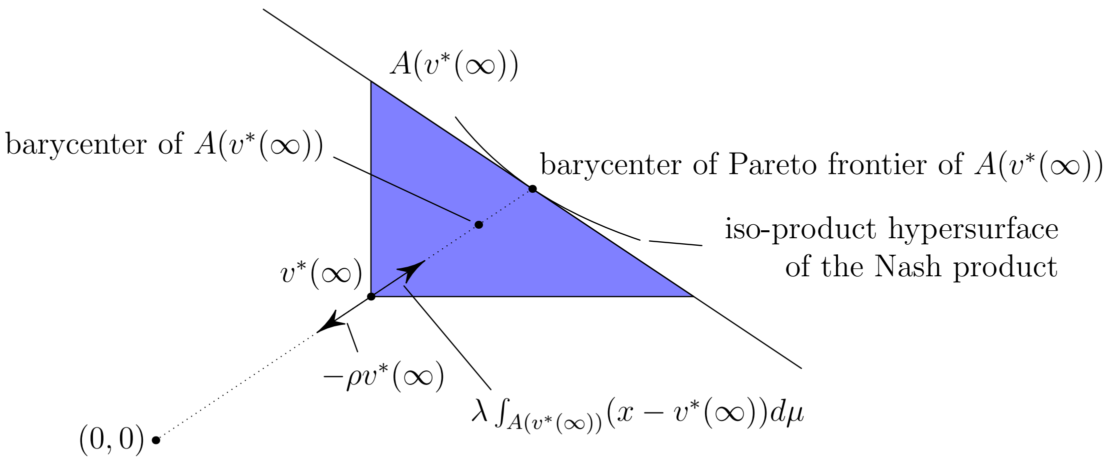

Figure 13: Vectors when $t \to \infty$.

Equality (11) also implies that $v^*(\infty)$ is parallel to the vector from $v^*(\infty)$ to the barycenter of $A(v^*(\infty))$, as shown in Figure 13 in the two-dimensional case. □

To avoid complications, we impose the following assumption in addition to Assumptions 1 and 4:

Assumption 5. (a) The weak Pareto frontier of $\hat{X}$ is smooth.

(b) Every component of the normal vector at any Pareto efficient allocation in X is strictly positive.

(c) There exists $\varepsilon > 0$ such that X contains a set $\{x \in \mathbb{R}^n_+ \mid w \geq x$, and $|w - x| \leq \varepsilon$ for some weakly Pareto efficient $w \in X\}$.

(d) The density function f is continuous.

Now suppose that λ is very large. Then $\mu(A(v^*(\infty)))$ must be very small, which means that $v^*(\infty)$ is very close to the Pareto frontier of X, where $v^*(\infty)$ is defined as in Remark 1. By Assumptions 1 and 4, the density f is approximately uniform in $A(v^*(\infty))$ if $A(v^*(\infty))$ is a set with a very small area. To obtain an intuition, suppose that $A(v^*(\infty))$ is a small n-dimensional pyramid. The vector in the right hand side of equality (11) is parallel to the vector from $v^*(\infty)$ to the barycenter of $A(v^*(\infty))$. We use this property to show that the boundary of $A(v^*(\infty))$ at its barycenter is tangent to the hypersurface defined by $\prod_{i \in N} x_i = a$ for some constant a. We refer to such a Pareto efficient allocation as a *Nash point*, and the set of all Nash points as the *Nash set* of $(\hat{X}, 0)$ (Maschler et al. (1988), Herrero (1989)). The Nash set contains all local maximizers and all local minimizers of the Nash product. If X is convex, there exists a unique Nash point, called the Nash bargaining solution.

The above observation leads to the next proposition.

Proposition 14. *Suppose that Assumptions 1, 4, and 5 hold, and that any Nash point is isolated in X. Then the limit $v^* = \lim_{\lambda\to\infty} v^*(t;\rho,\lambda)$ exists and belongs to the Nash set of the problem $(\hat{X}, 0)$ for all $t > 0$. If X is convex, this limit coincides with the Nash bargaining solution of $(X, 0)$.*

Therefore, the trajectory of $v^*(t)$ for very large λ starts at $v^*(t) = 0$, approaches $v^*(t; 0, \infty)$, and moves along the Pareto frontier until reaching a point close to a Nash point.

Finally we consider the duration of search in the equilibrium. In contrast to Theorems 1 and 2 in the case when payoffs realize at the deadline (or $\rho = 0$), we show that an agreement is reached almost immediately if λ is very large.

Proposition 15. *Suppose that Assumptions 1, 4, and 5 hold. If $\rho > 0$, then $D(\infty) = 0$.*

7 Discussions

7.1 Non-Poisson Arrival Processes

In the main sections we considered Poisson processes to make the presentation of the results easier. The Poisson process assumes that the probability of opportunity arrival is zero at any moment, so in particular the probability of receiving one more opportunity shrinks continuously to zero as the deadline approaches. However in some circumstances it would be more realistic to assume there is a well-defined "final period" that can be reached with a positive probability. In this section we generalize our model to encompass such cases and show that our results are unaffected.

Specifically, consider dividing the time horizon of length T into small subintervals each with length Δt (so there are $\frac{T}{\Delta t}$ periods in total). At the end of each subinterval, players obtain an opportunity with probability $\pi(\Delta t)$. Notice that the Poisson process corresponds to the case when $\pi(\Delta t) = \lambda \Delta t$ for some $\lambda > 0$ and we let $\Delta t \to 0$. Here we allow for general π function, such as $\pi(\Delta t) = a$ or $\pi(\Delta t) = a\sqrt{\Delta t}$ for some constant $a > 0$. Let $v_i(n)$ be the continuation payoff at time $n\Delta t$. Then,

$$\begin{aligned} v_i\Big(\frac{t}{\Delta t}+1\Big) &= \big(1-\pi(\Delta t)\big)v_i\Big(\frac{t}{\Delta t}\Big) + \pi(\Delta t)\left(\int_{X\setminus A(v(\frac{t}{\Delta t}))} v_i\Big(\frac{t}{\Delta t}\Big)\, d\mu + \int_{A(v(\frac{t}{\Delta t}))} x_i\, d\mu\right) \\ &= v_i\Big(\frac{t}{\Delta t}\Big) + \pi(\Delta t)\int_{A(v(\frac{t}{\Delta t}))} \Big(x_i - v_i\Big(\frac{t}{\Delta t}\Big)\Big)\, d\mu. \end{aligned}$$

Hence,

$$v_i\Big(\frac{t}{\Delta t}+1\Big) - v_i\Big(\frac{t}{\Delta t}\Big) = \pi(\Delta t)\int_{A(v(\frac{t}{\Delta t}))} \Big(x_i - v_i\Big(\frac{t}{\Delta t}\Big)\Big)\, d\mu. \tag{12}$$

Notice that if we set $\pi(\Delta t) = \lambda \Delta t$ and take the limit as $\Delta t \to 0$, the left hand side divided by Δt converges to $v_i'(t)$ in the Poisson model and the right hand side divided by Δt converges to $\lambda \int_{A(v(t))} (x_i - v_i(t)) d\mu$, consistent with equation (1).

Proposition 16. *If* $\lim_{\Delta t \to 0} \frac{\pi(\Delta t)}{\Delta t} = \infty$*, under Assumptions 1 and 3, the limit expected duration is* $\frac{n^2}{n^2+n+1} T$.

Note that this result is consistent with Proposition 6 where we consider the Poisson process and take a limit of $\lambda \to \infty$. Thus our limit result is robust to the move structure.

7.2 Relative Importance of Discounting and Search Friction

In the main sections, we have shown that if $\rho = 0$, the limit expected duration as $\lambda \to \infty$ is positive under certain assumptions, and the limit equilibrium payoff profiles are efficient but depend on the distribution μ. In Section 6, in contrast, the limit duration is zero, and the limit payoffs are the Nash bargaining solution if $\rho > 0$ is fixed. In this section, we show that the limit duration and the limit equilibrium payoffs as $\lambda \to \infty$ and $\rho \to 0$ simultaneously depend on the limit of $\lambda \rho^n$.

Proposition 17. *Suppose that Assumptions 1, 3, and 5 hold. The limit expected duration* $D(\infty)$ *and the limit allocation* $v^* = \lim_{\lambda \to \infty, \rho \to 0} v^*(t; \rho, \lambda)$ *satisfy the following claims: (i) If* $\lambda \rho^n \to 0$*, then* $D(\infty) > 0$*, and* $v^* = \lim_{\lambda \to \infty} v^*(t; 0, \lambda)$*, which is the limit analyzed in Sections 4 and 5. (ii) If* $\lambda \rho^n \to \infty$*, then* $D(\infty) = 0$*, and* $v^* = \lim_{\lambda \to \infty} v^*(t; \rho, \lambda)$ *for* $\rho > 0$*, which is the limit shown in Section 6.*

An insight behind this result is as follows: The limit of the expected payoffs depend on whether the first term in ODE (10) is negligible or not when compared to the second term. Let $z(t; \rho, \lambda)$ be the Hausdorff distance from $v^*(t; \rho, \lambda)$ to the Pareto frontier of X. If ρ is very small and λ is not very large, Proposition 13 shows that $v^*(t; \rho, \lambda)$ is close to $\lim_{\lambda \to \infty} v^*(t; 0, \lambda)$ which is on the Pareto frontier. Then we can apply an analogous argument to the one provided in the discussion in the proof sketch of Theorem 4 to show that $z(t; \rho, \lambda)$ is approximately proportional to $\lambda^{-1/n}$. Since $\mu(A(t))$ approximates $z(t; \rho, \lambda)^n$ (times some constant), and the length of the vector from $v^*(t; \rho, \lambda)$ to the barycenter of $A(t)$ is linear in $z(t; \rho, \lambda)$, the second term is of order $\lambda \cdot \lambda^{-1/n} \cdot \lambda^{-1} = \lambda^{-1/n}$. Therefore if $\lambda \rho^n \to 0$ the first term, which approximates ρv^*, is negligible because ρ vanishes more rapidly than $\lambda^{-1/n}$. Thus the limits in this case are the same as in Sections 4 and 5. If $\lambda \rho^n \to \infty$, the first term is significant because ρ does not vanish rapidly compared to $\lambda^{-1/n}$. This corresponds to Section 6. An analogous argument can be made for the limit of durations.

7.3 Infinite-Horizon and Static Games

Although we consider a finite-horizon model, our convergence result in Proposition 14 is suggestive of that in infinite-horizon models such as Wilson (2001), Compte and Jehiel (2010), and Cho and Matsui (2011), all of whom consider the limit of stationary equilibrium outcomes as the discount factor goes to one in discrete-time infinite-horizon models. This is because the threatening power of disagreement at the deadline is quite weak if the horizon is very far away, and thus the infinite-horizon model is similar to a finite-horizon model with $T \to \infty$ if $\rho > 0$. In fact, we can show that the iterated limit as $T \to \infty$ and then $\rho \to 0$ is the Nash bargaining solution in our model if X is convex. By Proposition 17, $\lim_{\rho\to 0} v^*(T; \alpha\rho, \alpha\lambda)$ with $\alpha = \rho^{-a}$ is the Nash bargaining solution for all $a > n/(n+1)$. As $a \to \infty$, we see that the iterated limit $\lim_{\rho\to 0} \lim_{\alpha\to\infty} v^*(T; \alpha\rho, \alpha\lambda)$ is also the Nash bargaining solution. Since enlarging T is equivalent to raising both λ and ρ in the same ratio by the form of ODE (10), the iterated limit as $T \to \infty$ first and then $\rho \to 0$ must be the Nash bargaining solution. For the same reason, the expected duration in the limit as the discount factor goes to one in the infinite-horizon model is zero, being analogous to our Proposition 15 in which we send λ to ∞ while $\rho > 0$ is fixed. Therefore we obtained the following proposition:

Proposition 18. *In the infinite-horizon search model, the expected duration in a stationary equilibrium converges to zero as the discount factor goes to one.*

Propositions 1 and 2 imply that the limit continuation payoff of a player is essentially equal to the cutoff, which is expressed by a single variable. In this sense, there is some connection between our model and a static game considered by Nash (1953) himself, who provided a characterization of the Nash bargaining solution by introducing a static demand game with perturbation described as follows.[41] Suppose that X is convex. The basic demand game is a one-shot strategic-form game in which each player i calls a demand $x_i \in \mathbb{R}_+$. Players obtain $x = (x_1, \dots, x_n)$ if $x \in X$, or 0 otherwise. In the perturbed demand game, players fail to obtain $x \in X$ with a positive probability if x is close to the Pareto frontier. Under certain conditions, he showed that the Nash equilibrium of the perturbed demand game converges to the Nash bargaining solution as the perturbation vanishes.

Let us compare the perturbed Nash demand game with our multi-agent search model with a positive discount rate. Let $p(x) = \mu\big(A(x)\big)$ be the probability that players come across an allocation which Pareto dominates or equals $x \in X$ at an opportunity. If T is very large and t is close to T, players at time $-t$ choose almost the same cutoff profile, say x, contained in the interior of X. The average duration that players wait for an allocation falling into $A(x)$ is almost $1/\lambda p(x)$. During this time interval, payoffs are discounted at

[41]We here follow a slightly modified game considered by Osborne and Rubinstein (1990, Section 4.3). Despite the difference, the model conveys the same insight as the original.

rate ρ. Since x_i must be equal to her continuation payoff in an equilibrium, i would lose nearly $(1 - e^{-\rho/\lambda p(x)})y_i$ on average by insisting on cutoff x_i where y is the expected allocation conditional on $y \in A(x)$. Note that this loss vanishes as $\rho \to 0$ for every x in the interior of X. Let probability $P(y)$ satisfy $P(y) = e^{-\rho/\lambda p(x)}$. Player i loses the same expected payoff when $y \in X$ is demanded in the perturbed demand game where the probability of successful agreement is $P(y)$.

The key tradeoff in this game, the attraction to larger demands or the fear of failure of agreement, is parallel to that in the multi-agent search, to be pickier or to avoid loss from discounting.

7.4 Time Costs

In the model of the main sections, whether or not players discount the future does not affect the outcome of the game, as payoffs are received at the deadline. However, there may still be a time cost associated with search. In this subsection we analyze a model with time costs, and show numerically that the search durations with "reasonable parameter values" are close to the limit duration with zero time cost that we solved for in the main sections.

Consider a model in which each player incurs a flow cost $c > 0$ until the search ends. In this model, it is straightforward to see that the differential equation (1) is modified in the following way:

$$v_i'(t) = -c + \lambda \int_{A(t)} \big(x_i - v_i(t)\big)\, d\mu \tag{13}$$

for each $i \in N$, with an initial condition $v(0) = (0, \ldots, 0) \in \mathbb{R}^n$.

The analysis of this differential equation is similar to the one in Section 6, with an exception that under Assumptions 1, 3, and 5, the limit expected payoff profile as $\lambda \to \infty$ for a fixed cost $c > 0$ is now a point that maximizes the sum of the payoffs, denoted v^S. Let $v^*(t; c, \lambda)$ be the expected payoff at time $-t$ when parameters c and λ are given. A proof similar to the one for Proposition 17 shows the following:

Proposition 19. *Suppose that Assumptions 1, 3, and 5 hold. The limit expected duration $D(\infty)$ and the limit allocation $v^* = \lim_{\lambda\to\infty, c\to 0} v^*(t; c, \lambda)$ satisfy the following claims: (i) If $\lambda c^n \to 0$, then $D(\infty) > 0$, and $v^* = \lim_{\lambda\to\infty} v^*(t; 0, \lambda)$, which is the limit analyzed in Sections 4 and 5. (ii) If $\lambda c^n \to \infty$, then $D(\infty) = 0$, and $v^* = v^S$ for $c > 0$.*

The proposition suggests that for a high arrival rate λ, the expected duration does not change so much when we increase the cost from zero to a small but positive number. Combined with our argument in Step 3, this suggests that whenever the cost is sufficiently small, our limit arguments in Steps 1 and 2 are economically meaningful. Now

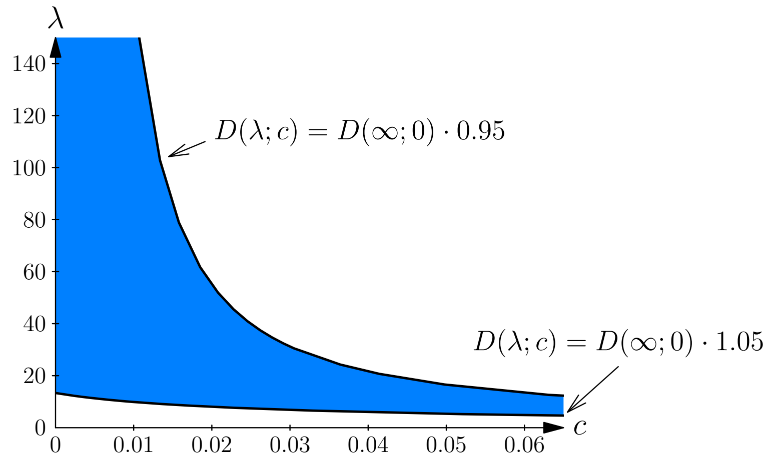

Figure 14: Time costs and arrival rates. The shaded region describes the set of pairs (c, λ) with which the expected search duration is within 5% difference from the limit duration.

we numerically show that the degree to which the cost should be small is not too extreme. Specifically, we consider the case when $n = 2$ and μ is a uniform distribution over $X = \{x \in \mathbb{R}^2_+ \mid x_1 + x_2 \leq 1\}$, and solve for the range of pairs of costs and arrival rates such that the expected search duration is within 5% difference from the limit duration. As shown in Figure 14, such a range contains a wide variety of pairs of parameter values (note that the expected limit payoff is 0.5 in this game, so the cost of 0.05 corresponds to the setting with a fairly high cost). When $n = 2$ and μ is an independent distribution such that each player's marginal is an exponential distribution, whenever the cost c is less than 10% of the expected payoff given with $c = 0$ and $\lambda = 100$, we find that λ for which the expected duration is of 95% of the limit duration is more than 100, and that of 105% is less than 10.[42] These results suggest that the limit argument that we conducted in Steps 1 and 2 of the main sections is economically reasonable.[43]

7.5 Counterexamples of Positive Duration of Search

In Theorems 1 and 2, we showed that the limit expected duration of search is positive if certain assumptions hold. In this section, we present examples of distributions under which assumptions are not satisfied and the expected duration may not be positive as $\lambda \to \infty$.

First, note that it is straightforward to see that if μ assigns a point mass to a point that Pareto-dominates all other points in the support of μ, the limit expected duration is zero. Less obvious is the situation where μ allows for point masses while no point Pareto-dominates all the other points. Even in this case, Kamada and Sugaya (2010a)'s

[42] In this case, we can show that the expected duration is positive even in the limit as $\lambda \to \infty$, exhibiting a stark contrast to Proposition 15. See Kamada and Muto (2011a) for details.

[43] We are planning to extend this argument to more cases beyond the setting provided here.

"three-state example" shows that trembling-hand equilibrium may not be unique, and the duration can be zero in a trembling-hand equilibrium.[44] Our Assumption 1 (b) requires a more stringent condition that the marginal distribution must have a locally bounded density function, and thus does not have a point mass. Here we present an example in which μ does not have a point mass while its marginal has a point mass, and there are multiple equilibria and some of them have zero limit duration.

Example 6. Consider $X = co\{(2,1),(1,1)\} \cup co\{(1,2),(1,1)\}$ and let μ be a uniform distribution over this X. First, consider a strategy profile in which agents accept a payoff strictly above 1 until time $-t^*$ and accept all offers after $-t^*$, where t^* satisfies the indifference condition at $-t^*$:

$$1 = \frac{1 - e^{-\lambda t^*}}{2}(1 + 1.5) + e^{-\lambda t^*} \cdot 0,$$

or $t^* = \frac{1}{\lambda}\ln(5)$. Since given this strategy profile the continuation payoff for both players is 1 if $-t \le -t^*$ and it is strictly less than 1 otherwise, this indeed constitutes a trembling-hand equilibrium.

However, there exist other equilibria. For example, consider a strategy profile which is exactly the same as the above one except that both agents accept the first offer regardless of its realization. Since the continuation payoff at the time of the first arrival is 1 for both players as we have argued, this also constitutes an trembling-hand equilibrium. Thus there are multiple trembling-hand equilibria. Also the limit expected duration under the second equilibrium is trivially zero, suggesting the need for Assumption 1 (b) for Theorem 2 to hold.

The key to multiplicity and zero duration is the fact that payoff profiles at which players are indifferent arrive with positive probability due to the atom on marginals. Assumption 1 rules out such a situation. □

Next, we show that even if μ has no point mass, the limit expected duration may be zero when μ does not satisfy Assumption 2 (nor Assumption 6 in Appendix B.4).

Example 7. For $n = 1$, let F be a cumulative distribution function defined by $F(x) = 1 + \dfrac{1}{\ln(1-x)}$ for $x \in [1 - e^{-1}, 1)$, and $F(1) = 1$. The density is $f(x) = \frac{1}{(1-x)(\ln(1-x))^2}$. Recalling formula (7), the density term is

$$d(v) = \frac{f(v)}{1 - F(v)} = -\frac{1}{(1-v)\ln(1-v)},$$

[44] Consider μ that assigns equal probabilities to $(2,1)$ and $(1,2)$.

and the barycenter term $b_1(v)$ is clearly smaller than $1-v$. Since $\lim_{\lambda\to\infty} v^*(t) = 1$,

$$\begin{aligned} r &= \lim_{v\to 1} d_1(v)b_1(v) \\ &\leq \lim_{v\to 1} \frac{-1}{\ln(1-v)} = 0. \end{aligned}$$

By Theorem 4, the limit duration is zero.

In this example, it is easy to show that for all $\alpha > 0$, there exists $\varepsilon > 0$ such that $1-(1-x)^\alpha \geq F(x)$ for all $x \in [1-\varepsilon, 1]$. Distribution F is very close to a discrete distribution, in that $F(x)$ converges to 1 as $x \to 1$ at a speed slower than any polynomial functions. In such a case, the above computation shows that the limit duration can be zero, which is the same as the case with discrete distributions. □

7.6 The Effect of a Slight Change in the Distribution

The limit result in Proposition 6 depends crucially on the assumption of smooth boundary and continuous positive density. Although this is the assumption that is often invoked in the literature, it is desirable to know how robust this result is. To this end, consider $X = \mathbb{R}^n_+$ and a distribution over X, μ, which may or may not be full-support. Introduce a notion of distance between two distributions, $d(\mu,\gamma) = \sup_{A\subseteq X} |\mu(A) - \gamma(A)|$.

A standard argument on ordinary differential equations shows the following:

Proposition 20. *For any λ, the limit duration is continuous in distribution almost everywhere.*

That is, for any finite arrival rate, the limit duration is not substantially affected by a slight change in distribution. Combined with the result that our limit result approximates the situation with a finite but high arrival rate, this suggests that our limit duration is relevant even for the distributions that are not very different from a distribution that satisfies our assumptions (Assumptions 1 and 4).

7.7 Time Varying Distributions

In the main model we considered the case in which the distribution μ is time-independent. This benchmark analysis is useful in understanding the basic incentive problems that agents face, but in some situations it might be more realistic that the distribution changes over time. In this section, we examine whether the positive duration result in Theorem 1 (the case with a single agent) is robust to this independence assumption. An analogous argument can be made for the multiple-agent case. Let F_t be the cumulative distribution function of the payoff at time $-t$.

First, consider the case in which the distribution becomes better over time in the sense of first order stochastic dominance. In this case, it is easy to see that the expected duration is still positive and it becomes longer at least in certain cases: For each t, consider the cutoff at each time $-s \in (-t, 0]$ that equates the acceptance probability with the one that the agent would get at $-s$ if the distribution in the future were fixed at F_t. This gives a higher continuation payoff at $-t$ as the distribution becomes better over time. Thus the cutoff at $-t$ must be greater than the continuation payoff at $-t$ that the agent would obtain by fixing the distribution at F_t ever after. This means that at any $-t$, the acceptance probability is smaller than the one obtained by fixing the distribution at F_t ever after. Hence the acceptance probability at $-t$ is $O(\frac{1}{\lambda t})$, so we have a positive duration. If $(F_t)_{t \in [0,T]}$ is such that the acceptance probability at time $-s$, $p(s)$, when the payoff is drawn by the fixed F_t independently over time does not depend on t, then the above argument also implies that the duration becomes longer.

Now consider the case when the distribution may become worse off. First, if the support of the distribution becomes worse off, then there is no guarantee of positive duration. For example, if the upper bound of the support decreases exponentially then the analysis of the duration becomes equivalent to that for the case with discounting, in which Proposition 15 has already shown that the limit expected duration is zero.

If the support does not change, then the positive duration result holds quite generally: In the proof of Theorem 1 provided in the Appendix, we did not use the fact that F does not depend on t. The following modification of Assumption 2 guarantees the positive duration.

Assumption 2$''$. There exists a concave function φ such that $1 - \varphi(x)$ is of the same order as $1 - F_t(x)$ in $\{x \in \mathbb{R} \,|\, F_t(x) < 1\}$ for all t.

Notice that we require the existence of $\bar{x}$ and φ that are applicable to all F_t.

Proposition 21. *Suppose* $n = 1$. *Under Assumptions 1 and 2$''$,* $\liminf_{\lambda \to \infty} D(\lambda) > 0$.

7.8 Dynamics of the Bargaining Powers

Consider the case where $X = \{x \in \mathbb{R}^2_+ \,|\, x_1 + x_2 \leq 1\}$ and a density f such that $f(x) > f(x')$ if $x_2 - x_1 > x'_2 - x'_1$. Suppose that the payoff realizes upon agreement as in Section 6, and the discount rate $\rho > 0$ is very small. In this case, the limit of the solution of ODE (1) with $\rho = 0$, denoted $v^*(T; 0, \infty)$, locates at the boundary of X by Proposition 10, and it is to the north-west of $(\frac{1}{2}, \frac{1}{2})$, which is the Nash bargaining solution and is the limit of the solution of ODE (10). Hence, by Proposition 13, the continuation payoff when the players receive payoffs upon the agreement starts at a point close to $(\frac{1}{2}, \frac{1}{2})$, and goes up along the boundary of X and reaches a point close to $v^*(T; 0, \infty)$, and then goes down to $(0, 0)$. On this path of play, player 1's expected payoff is monotonically decreasing over time. On the other hand, player 2's expected payoff changes non-monotonically.

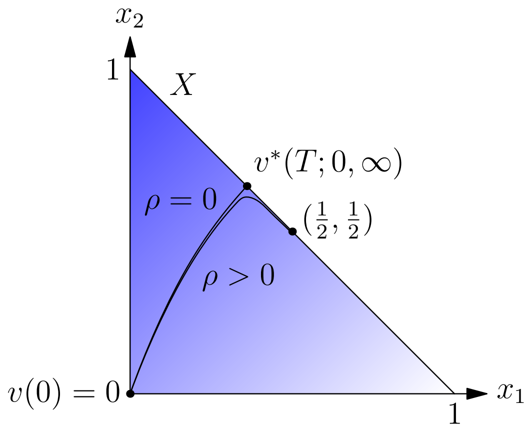

Figure 15: Paths of continuation payoffs. The probability density is low near $(1,0)$, and high near $(0,1)$.

Specifically, it rises up until it reaches close to $v_2^*(T;0,\infty)$, and then decreases over time. Figure 15 illustrates this path.

Underlying this non-monotonicity is the change in the bargaining powers between the players. When the deadline is far away, there will be a lot of opportunities left until the deadline, so it is unlikely that players will accept allocations that are far from the Pareto efficient allocations, so the probability distribution over such allocations matters less. Since X is convex and symmetric, two players expect roughly the same payoffs. However, as the time passes, the deadline comes closer, so players expect more possibility that Pareto-inefficient allocations will be accepted. Since player 2 expects more realizations favorable to her than player 1 does, player 2's expected payoff rises while player 1's goes down. Finally, as the deadline comes even closer, player 2 starts fearing the possibility of reaching no agreement, so she becomes less pickier and the cutoff goes down accordingly.

7.9 Market Designer's Problem

In this section we consider problems faced by a market designer who has a control over some parameters of the model.

First, consider the case when the payoffs realize at the deadline, and the designer can tune the horizon length T. In this case there is no point in making the horizon shorter, as the continuation payoff $v(t)$ is increasing in t.

Second, still in the case with payoffs realizing at the deadline, suppose that the designer can instead affect the probability distribution over potential payoff profiles, by "holding off" some offers. Formally, given μ, let the designer choose a distribution μ' such that $\mu'(C) \leq \mu(C)$ for all $C \subseteq X$.[45] In this case the designer faces a tradeoff: On one hand, tuning the distribution can affect the path of continuation payoffs and the

[45]Note that μ' may not be a probability measure because it might be the case that $\mu'(X) < 1$.

ex ante expected payoff at time $-T$ (an analogous argument to Proposition 12). On the other hand, however, changing the distribution will decrease the expected number of offer arrivals in the finite horizon, so $v(T)$ is lower than the case when the distribution is instead given by μ'' such that $\mu''(C) = \frac{\mu'(C)}{\mu(X')}$ for all $C \subseteq X$. The explicit form of an optimal design would depend on the specificalities of the problem at hand and the objective function of the designer, but basically if the horizon length T is high then reducing probabilities would not lead to too much loss.

Next, consider the case with payoffs realizing upon agreement. In this case there can be a benefit from reducing T. As in the case with payoffs realizing at the deadline, lower T means that the expected payoff at time $-T$ is less close to the Pareto boundary. However, if the solution when payoffs realizing at the deadline, $v^*(T; 0, \infty)$, is socially desirable than the Nash bargaining solution, then by reducing T appropriately the expected payoff profile will come closer to $v^*(T; 0, \infty)$ (provided that the expected payoffs are in between these two payoffs before shortening T; remember that by Proposition 13 the expected payoffs for intermediate values of time t is close to $v^*(T; 0, \infty)$).

On the other hand, tuning the distribution has a smaller effect than the case with payoffs realizing at the deadline, as we know that the payoffs eventually converge to the Nash bargaining solution. However, since $v^*(T; 0, \infty)$ depends on the distribution, Proposition 13 implies that the direction from which the payoff converges varies as the designer varies the distribution.

7.10 Majority Rule

In the main sections we considered the case when players use *unanimous rule* for their decision making. This is a reasonable assumption in many applications such as the apartment search, but there are certain other applications in which *majority rule* fits the reality better. This section is devoted to the analysis of such a case.

Precisely speaking, by majority rule we mean the decision rule such that the object of search is accepted if and only if $k < n$ players say "accept" upon its arrival.

First of all, it is straightforward to check that Propositions 1 and 2 (the trembling-hand equilibrium is essentially unique and players use cutoff strategies) carry over to this case. If X is convex, and satisfies Assumption 4, the limit expected payoff cannot be weakly Pareto efficient. To see this, suppose that the limit payoff is weakly Pareto efficient. Then, as shown in Figure 16, there is a region with a positive measure such that the acceptance takes place. However the barycenter of these regions is in the interior of X by convexity, and hence the limit payoff profile must be an interior point as well. This contradicts the assumption that the limit payoff profile is weakly Pareto efficient. Now, let the true limit point be $\tilde{v}$. Since any payoffs that strictly Pareto dominate $\tilde{v}$ must be accepted by all players, and the measure of this region is strictly positive, the limit

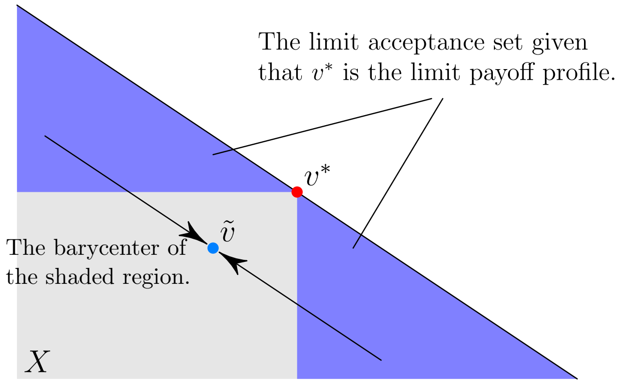

Figure 16: Equilibrium continuation payoffs under a majority rule: If the point v^* were the limit payoff profile, then all payoffs in the shaded region are in the acceptance set, and the barycenter of the shade region, $\tilde{v}$, is in the interior of X as X is convex.

duration must be zero.[46] We summarize our finding as follows:

Proposition 22. *Under the majority rule with $k < n$, if X is convex and satisfies Assumption 4, then the limit expected payoff profile is not weakly efficient, and the limit expected duration of search is zero.*

We note that the conclusion of this proposition still holds even if the smoothness assumption in Assumption 4 is replaced by the following assumption, which essentially says that the Pareto frontier is downward-sloping with respect to other players' payoffs: For all $i \in N$, $\overline{x}_i(x_{-i}) = \sup\{x_i \,|\, (x_i, x_{-i}) \in X\}$ is decreasing in x_j for all $j \neq i$.

7.11 Negotiation

Our model assumes that players cannot transfer utility after agreeing on an allocation. We believe our model keeps the deviation from the standard single-agent infinite-horizon search model minimal so that the analysis purifies the effect of modifying the number of agents and the length of the horizon. Also, our primary interest is in the case where such negotiation is impossible or the case where the stake of the object is very high so even if players could negotiate, the impact on the outcome is negligible. However, in some cases negotiation may not be negligible. Here we discuss such cases. We will show that the duration continuously changes with respect to the degree of impact of negotiation, hence our results are robust with respect to the introduction of negotiation. Our extension also lets us obtain intuitive comparative statics results.

Suppose that players can negotiate after they observe a payoff profile $x \in X$ at each opportunity at time $-t$. Players can shift their payoff profile by making a transfer, and may agree with the resulting allocation. We assume that the allocation they agree with

[46]This discussion is parallel to Compte and Jehiel (2010, Proposition 7) who consider majority rules in a discrete-time infinite-horizon search model.

is the Nash bargaining solution where a disagreement point is the continuation payoff profile at the time $-t$ in the equilibrium defined for this modified game.[47] When making a transfer, we suppose that a linear cost is incurred: If player i gives player j a transfer z, j obtains only az for $a \in [0, 1)$. This cost may be interpreted as a misspecification of resource allocation among agents, or a proportional tax assessed on the monetary transfer. Note that a measures the degree of impact of negotiation. Our model in the main sections corresponds to the case of $a = 0$.

To simplify our argument we restrict attention to a specific model with two players.[48] Specifically, we consider the case with costly transferable utility: Suppose that $X = \{x \in \mathbb{R}^2_+ \mid x_1 + x_2 \leq 1\}$ with the uniform distribution μ on X. For each arrival of payoff profile x, players can negotiate among the set of feasible allocations defined by

$$S(x) = \left\{ x' \in X \;\middle|\; \begin{array}{ll} a(x_1 - x_1') \geq x_2' - x_2 & \text{if } x_1 \geq x_1', \\ x_1' - x_1 \geq a(x_2 - x_2') & \text{if } x_1 < x_1' \end{array} \right\}.$$

We suppose that each player says either "accept" or "reject" to the Nash bargaining solution obtained from the feasible payoff set $S(x)$ and the disagreement point given by the continuation payoff profile $v(t)$.

By looking at geometric properties of the Nash bargaining solution, we can compute the limit expected duration in this environment.

Proposition 23. *Under Assumptions 1, 4, and 7, the limit expected duration in the game with negotiation is* $D(\infty) = \dfrac{4 + 4a^2}{7 + 6a + 7a^2}$ *for* $a \in [0, 1)$.

Since $D(\infty)$ is decreasing in a, the limit expected duration becomes shorter in the presence of negotiation. This is intuitive, as negotiation essentially precludes extreme heterogeneity in the offer realization, thus the agreement can be reached soon. Notice also that the proposition claims that the duration must be strictly positive even with negotiation, and $D(\infty)$ converges to 4/7 as $a \to 0$, which is the same duration as we claimed in Proposition 6. That is, our main result is robust to the introduction of negotiation.

Note that the proposition does not apply to the case in which utilities are perfectly transferable, i.e., $a = 1$. However, this is due to the fact that the analysis above is in a knife-edge case because the Pareto frontier consists of a straight line: If X is strictly convex, then even if $a = 1$, the acceptance set shrinks with a faster speed than the case that we analyzed above, and the resulting duration is longer in such case.[49]

[47] This use of Nash bargaining solution is not critical to our result. Similar implications are obtained from other bargaining solutions such as the one given by take-it-or-leave-it offers by a randomly selected player.

[48] We expect that nothing substantial would change even if we extended the argument to the cases of three or more players.

[49] This suggests that with negotiation, preference heterogeneity may help shortening the search dura-

8 Conclusion

This paper analyzed a modification of the standard search problem by introducing multiplicity of players and a finite horizon. Together, these extensions significantly complicate the usual analysis. Our main results identified the reasons behind the widely-observed phenomenon that such searches often take a long (or at least some) time. We first showed that the search duration in the limit as the search friction vanishes is still positive, hence the mere existence of some search friction has a nonvanishing impact on the search duration. This limit duration is increasing in the number of players as a result of two effects: the ascending acceptability effect and the preference heterogeneity effect. In short, the ascending acceptability effect states that a player has an extra incentive to wait as the opponents accept more offers in the future, and the preference heterogeneity effect states that such "extra offers" include increasingly favorable offers for a player due to heterogeneity of preferences. Then we showed that the convergence speed of the duration as the friction vanishes is high, and numerically demonstrated that durations with positive frictions are reasonably close to the limit duration in our examples. This provides evidence that our limit analysis contains economically-meaningful content, and the mere existence of some friction is actually the main driving force of the positive duration in reality—so the effects that we identify in Steps 1 and 2 are the keys to understand the positive duration in reality.

We also conducted a welfare analysis, and showed that the limit expected payoff is generically Pareto efficient, and depends on the distribution of offers. Lastly, we provided a wealth of discussions to examine the robustness of our main conclusions and to analyze a variety of alternative specifications of the model.

Our paper raises many interesting questions for future research. First, it would be interesting to consider the case where agents can search for another offer even after they agree on an offer (i.e., search with recall). In this case the search duration in equilibrium must be always T, but the duration until the first agreement is not obvious, because players' preferences are heterogeneous: Player 1 may not want to agree on the offer that gives player 2 a high payoff, expecting 2's future reluctance to accept further offers. In our continuation work Kamada and Muto (2011c) we analyze this case and find that under some assumptions, the duration until the first acceptance is positive even in the limit as the friction vanishes. In that paper we also find that players may no longer use cutoff strategies, and as a result the shape of the acceptance set is quite complicated.

Second, it would be interesting to consider a large market model where at each period a fixed number of agents from a large population are matched and some payoff profile is realized. If all agents agree on the profile, they leave the market. There are at least two possible specifications for such a model. First, we can consider the situation where

tion. We plan to explore this issue in our continuation work.

an overlapping generation model with agents facing different deadlines, and there is a constant inflow of agents. In our ongoing research, we solve for a steady state equilibrium strategy and characterize the expected search duration of each agent in the population under certain regularity assumptions. On the other hand, if all agents share the same deadline, the arrival rate must decrease or the set of feasible payoffs must shrink over time to reflect the change in the measure of agents who remain in the market, and it is not obvious whether the positive duration results carry over. Our result on time-varying distributions in Section 7.7 may be useful in such an analysis.

Finally, in order to isolate the effects of multiple agents and a finite horizon as cleanly as possible, we attempted to minimize the departure from the standard model. Inevitably, this entailed ruling out some properties that would be relevant in particular applications. For example, in some cases there may be uncertainty (perhaps resolving over time) about the distribution over outcomes or the opponents' preferences. We conjecture such uncertainty would increase the duration. Another example would be the possibility of agents using effort to increase the arrival rate or perhaps sacrificing a monetary cost to postpone the deadline. Again this would increase the search duration, as players could make these decisions conditional on the time left to the deadline. These extensions of our model are left for future work.

Appendix A: Numerical Results for Finite Arrival Rates

A.1 Uniform Distribution over Multi-Dimensional Triangle (Case 1)

Consider the distribution given by the uniform distribution over $\{x \in \mathbb{R}^n_+ \mid \sum_{i \in N} x_i \leq 1\}$.

		λ				
		10	20	30	100	∞
$n = 1$	Expected duration	0.398	0.366	0.355	0.340	0.333
	Percentage (%)	19.4	9.92	6.64	2.00	0
$n = 2$	Expected duration	0.608	0.591	0.585	0.576	0.57143
	Percentage (%)	6.48	3.44	2.35	0.731	0
$n = 3$	Expected duration	0.734	0.716	0.709	0.698	0.692
	Percentage (%)	5.97	3.35	2.35	0.780	0

A.2 Uniform Distribution over a Sphere (Case 2)

Consider the uniform distribution over $\{x \in \mathbb{R}^n_+ \mid \sum_{i \in N} x_i^2 \leq 1\}$. We get the following. Note that the limit duration for $n = 1$ is the same as in the case of uniform distribution over $\{x \in \mathbb{R}^n_+ \mid \sum_{i \in N} x_i \leq 1\}$.

		λ					
		10	20	30	100	1000	∞
$n = 1$	Expected duration	0.398	0.366	0.355	0.340	0.334	0.333
	Percentage (%)	19.4	9.92	6.64	2.00	0.200	0
$n = 2$	Expected duration	0.582	0.568	0.565	0.562	0.567	0.571
	Percentage (%)	1.90	−0.541	−1.21	−1.61	−0.798	0

A.3 Uniform Distribution over a Cube (Case 3)

Consider the distribution given by the uniform distribution over $\{x \in \mathbb{R}^n_+ \mid \max_{i \in N} x_i \leq 1\}$.

		λ				
		10	20	30	100	∞
$n = 1$	Expected duration	0.398	0.366	0.355	0.340	0.333
	Percentage (%)	19.4	9.92	6.64	2.00	0
$n = 2$	Expected duration	0.545	0.524	0.516	0.505	0.5
	Percentage (%)	9.09	4.76	3.23	0.990	0
$n = 3$	Expected duration	0.634	0.618	0.612	0.604	0.6
	Percentage (%)	5.62	3.00	2.05	0.643	0

A.4 Exponential Distribution (Case 4)

Consider the exponential distribution with parameter a_i for each player i.

		λ				
		10	20	30	100	∞
$n = 1$	Expected duration	0.545	0.524	0.516	0.505	0.5
	Percentage (%)	9.09	4.76	3.23	0.990	0
$n = 2$	Expected duration	0.693	0.681	0.676	0.670	0.667
	Percentage (%)	3.91	2.11	1.45	0.465	0
$n = 3$	Expected duration	0.767	0.759	0.756	0.752	0.75
	Percentage (%)	2.27	1.24	0.864	0.284	0
$n = 10$	Expected duration	0.912	0.911	0.910	0.910	0.909
	Percentage (%)	0.370	0.206	0.145	0.0499	0

A.5 Log-Normal Distribution (Case 5)

Consider the log-normal distribution with the following pdf:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}.$$

Assume $\mu = 0$. The expected durations can be calculated as follows:

		λ			
		10	20	30	100
	$\sigma = \frac{1}{4}$	0.449	0.462	0.469	0.484
$n = 1$	$\sigma = 1$	0.612	0.595	0.588	0.575
	$\sigma = 4$	0.961	0.952	0.946	0.926

Appendix B: Proofs of the Results

B.1 Computation of the Limit Durations

We here prove a lemma that computes the limit cumulative disagreement probability and the limit expected duration when the agreement probability $p(t)$ at time $-t$ is of the same order as $\frac{1}{\lambda t}$.

Lemma 24. *The following three statements hold:*

(i) If for all $\varepsilon > 0$, there exist $C > 0$ and $\bar{\lambda}$ such that $p(t) \leq \frac{C}{\lambda t}$ for all $t \geq \varepsilon$ and all $\lambda \geq \bar{\lambda}$, then $\liminf_{\lambda\to\infty} P(t;\lambda) \geq \left(\frac{t}{T}\right)^C$ for all $t \geq 0$, and $\liminf_{\lambda\to\infty} D(\lambda) \geq \frac{1}{1+C}$.

(ii) If for all $\varepsilon > 0$, there exist $c > 0$ and $\bar{\lambda}$ such that $p(t) \geq \frac{c}{\lambda t}$ for all $t \geq \varepsilon$ and all $\lambda \geq \bar{\lambda}$, then $\limsup_{\lambda\to\infty} P(t;\lambda) \leq \left(\frac{t}{T}\right)^c$ for all $t \geq 0$, and $\limsup_{\lambda\to\infty} D(\lambda) \leq \frac{1}{1+c}$.

(iii) If $\lim_{\lambda\to\infty} p(t)\lambda t = a > 0$ for all $t > 0$, then $P(t;\infty) = \left(\frac{t}{T}\right)^a$ for all $t \geq 0$, and $D(\infty) = \frac{1}{1+a}$.

Proof. First we prove (i). Let us fix $0 < \varepsilon < T$. By formula (3), for all $\lambda \geq \bar{\lambda}$ and all $t \geq \varepsilon$,

$$e^{-\int_t^T (C/s)ds} \leq P(t;\lambda)$$
$$\left(\frac{t}{T}\right)^C \leq P(t;\lambda).$$

Since the above inequality is satisfied for all $\varepsilon > 0$ and sufficiently large λ, we have $\liminf_{\lambda\to\infty} P(t;\lambda) \geq \left(\frac{t}{T}\right)^C$ for all $t \geq 0$. By formula (4), $D(\lambda)T = \int_0^T P(t)dt$ is bounded as follows:

$$\int_{\varepsilon}^{T} \left(\frac{t}{T}\right)^{C} dt \leq D(\lambda)T$$
$$\frac{T^{1+C} - \varepsilon^{1+C}}{(1+C)T^{C}} \leq D(\lambda)T.$$

Since the above inequality is satisfied for all $\varepsilon > 0$ and sufficiently large λ, we have $\liminf_{\lambda\to\infty} D(\lambda) \geq \frac{T^{1+C}}{(1+C)T^{C}\cdot T} = \frac{1}{1+C}$.

Next, a parallel argument shows (ii). Finally, (i) and (ii) together imply (iii). □

B.2 Proof of Proposition 1

Suppose that there exists at least one trembling-hand equilibrium. We show that the continuation payoff of player i at time $-t$ is unique for almost all histories in any trembling-hand equilibrium.

By Assumption 1 (a), the set of player i's expected payoffs given by any play of the game within $[-T, 0]$ is bounded by a value $\overline{x}_i$ for each $i \in N$. By Assumption 1 (b), we can find a Lipschitz constant L_i for $i \in N$ such that $\mu(\{x \in X \mid x_i \in [x'_i, x''_i]\}) \leq L_i |x'_i - x''_i|$ for all x'_i, x''_i in the above domain of payoffs. Let $L = \max_i L_i$.

Let $S_i(\sigma, t) \subseteq \mathbb{R}$ be the support of the continuation payoffs $u_i(\sigma \mid h)$ of player i after histories $h \in \tilde{\mathcal{H}}_t \setminus \mathcal{H}_t$ realized at time $-t$ given a strategy profile σ. For $\varepsilon \in (0, \frac{1}{2})$, let $\overline{v}_i^{\varepsilon}(t)$ and $\underline{v}_i^{\varepsilon}(t)$ be the supremum and the infimum of

$$\bigcup_{\sigma:\ \text{Nash equilibrium in } \Sigma^{\varepsilon}} S_i(\sigma, t).$$

(Note that Assumption 1 (a) ensures boundedness of the support for finite t.) Let $w_i^{\varepsilon}(t) = \overline{v}_i^{\varepsilon}(t) - \underline{v}_i^{\varepsilon}(t)$, and $\bar{w}^{\varepsilon}(t) = \max_{i\in N} w_i^{\varepsilon}(t)$. We will show that $\bar{w}^{\varepsilon}(t) = 0$ for all $\varepsilon > 0$ for any time $-t \in [-T, 0]$. Note that $\bar{w}^{\varepsilon}(0) = 0$ for all ε.

Let us consider the ε-constrained game. If player i accepts an allocation $x \in X$ at time $-t$, she will obtain x_i with probability at least ε^{n-1}. Accepting x is a dominant action of player i if the following inequality holds:

$$\varepsilon^{n-1} x_i + (1 - \varepsilon^{n-1}) \underline{v}_i^{\varepsilon}(t) > \overline{v}_i^{\varepsilon}(t).$$

Rearranging this, we have

$$x_i > \overline{v}_i^{\varepsilon}(t) + \frac{1 - \varepsilon^{n-1}}{\varepsilon^{n-1}} w_i^{\varepsilon}(t).$$

Let $\tilde{v}_i^{\varepsilon}(t) = \overline{v}_i^{\varepsilon}(t) + \frac{1-\varepsilon^{n-1}}{\varepsilon^{n-1}} w_i^{\varepsilon}(t)$, the right hand side of the above inequality. Then $\tilde{v}_i^{\varepsilon}(t) -$

$\underline{v}_i^\varepsilon(t) = \frac{1}{\varepsilon^{n-1}} w_i^\varepsilon(t)$.

Let $X_i^1(t) = \{x \in X \mid x_i > \tilde{v}_i^\varepsilon(t)\}$, $X_i^m(t) = \{x \in X \mid \underline{v}_i^\varepsilon(t) \leq x_i \leq \tilde{v}_i^\varepsilon(t)\}$, and $X_i^0(t) = \{x \in X \mid x_i < \underline{v}_i^\varepsilon(t)\}$. Then $\mu(X_i^m) \leq \frac{L}{\varepsilon^{n-1}} w_i^\varepsilon(t)$. Any player i accepts $x \in X_i^1(t)$ and rejects $x \in X_i^0(t)$ with probability $1-\varepsilon$ after almost all histories at time $-t$. Note that $X = \left(\bigcup_{j\in N} X_j^m(t)\right) \cup \left(\bigcup_{(s_1,\ldots,s_n)\in\{0,1\}^n} \bigcap_{j\in N} X_j^{s_j}(t)\right)$ (where $X_j^m(t)$'s have a nonempty intersection). Then

$$\begin{aligned}\overline{v}_i^\varepsilon(t) \leq \int_0^t \Big(& \sum_{j\in N} \int_{X_j^m(\tau)} \overline{x}_i d\mu \\ & + \sum_{(s_1,\ldots,s_n)\in\{0,1\}^n} \int_{\bigcap_{j\in N} X_j^{s_j}(\tau)} \big((1-\varepsilon)^{\sum_{j\in N} s_j} \varepsilon^{\sum_{j\in N}(1-s_j)} x_i \\ & + (1-(1-\varepsilon)^{\sum_{j\in N} s_j} \varepsilon^{\sum_{j\in N}(1-s_j)}) \overline{v}_i^\varepsilon(\tau)\big) d\mu \Big) \lambda e^{-\lambda(t-\tau)} d\tau,\end{aligned}$$

and

$$\begin{aligned}\underline{v}_i^\varepsilon(t) \geq \int_0^t \Big(& \sum_{(s_1,\ldots,s_n)\in\{0,1\}^n} \int_{\bigcap_{j\in N} X_j^{s_j}(\tau)} \big((1-\varepsilon)^{\sum_{j\in N} s_j} \varepsilon^{\sum_{j\in N}(1-s_j)} x_i \\ & + (1-(1-\varepsilon)^{\sum_{j\in N} s_j} \varepsilon^{\sum_{j\in N}(1-s_j)}) \underline{v}_i^\varepsilon(\tau)\big) d\mu \Big) \lambda e^{-\lambda(t-\tau)} d\tau.\end{aligned}$$

Therefore $w_i^\varepsilon(t) = \overline{v}_i(t) - \underline{v}_i(t)$ is bounded as follows:

$$\begin{aligned}w_i^\varepsilon(t) & \leq \int_0^t \Big(\sum_{j\in N} \int_{X_j^m(\tau)} \overline{x}_i d\mu \\ & \quad + \sum_{(s_1,\ldots,s_n)\in\{0,1\}^n} \int_{\bigcap_{j\in N} X_j^{s_j}(\tau)} \big(1-(1-\varepsilon)^{\sum_{j\in N} s_j} \varepsilon^{\sum_{j\in N}(1-s_j)}\big) w_i^\varepsilon(\tau) d\mu \Big) \lambda e^{-\lambda(t-\tau)} d\tau \\ & \leq \int_0^t \Big(\sum_{j\in N} \overline{x}_i \frac{L}{\varepsilon^{n-1}} w_j^\varepsilon(\tau) \\ & \quad + \sum_{(s_1,\ldots,s_n)\in\{0,1\}^n} \int_X \big(1-(1-\varepsilon)^{\sum_{j\in N} s_j} \varepsilon^{\sum_{j\in N}(1-s_j)}\big) w_i^\varepsilon(\tau) d\mu \Big) \lambda e^{-\lambda(t-\tau)} d\tau \\ & \leq \int_0^t \Big(\sum_{j\in N} \max_{k\in N}\{\overline{x}_k\} \frac{L}{\varepsilon^{n-1}} \\ & \quad + \sum_{(s_1,\ldots,s_n)\in\{0,1\}^n} \big(1-(1-\varepsilon)^{\sum_{j\in N} s_j} \varepsilon^{\sum_{j\in N}(1-s_j)}\big) \Big) \overline{w}^\varepsilon(\tau) \lambda e^{-\lambda(t-\tau)} d\tau.\end{aligned}$$

Since the above inequality holds for all $i \in N$, there exists a constant $M > 0$ such that the following inequality holds:

$$\overline{w}^\varepsilon(t) \leq \int_0^t M \overline{w}^\varepsilon(\tau) e^{-\lambda(t-\tau)} d\tau.$$

Let $W^\varepsilon(t) = \int_0^t \overline{w}^\varepsilon(\tau)e^{\lambda\tau}d\tau$. Then

$$\begin{aligned} W^{\varepsilon\prime}(t) &= \overline{w}^\varepsilon(t)e^{\lambda t} \\ &\leq MW^\varepsilon(t). \end{aligned}$$

Therefore we have $\frac{d}{dt}\big(W^\varepsilon(t)e^{-Mt}\big) = \big(W^{\varepsilon\prime}(t) - MW^\varepsilon(t)\big)e^{-Mt} \leq 0$. Since $W^\varepsilon(0) = 0$ by the definition of $W^\varepsilon(t)$, $W^\varepsilon(t)e^{-Mt} \leq 0$ for all $t \geq 0$. This implies that $\overline{w}^\varepsilon(t) \leq MW^\varepsilon(t)e^{-\lambda t} \leq 0$ for all $t \geq 0$. Hence, $\overline{w}^\varepsilon(t) = 0$ for all $t \geq 0$ and all $\varepsilon \in (0, \frac{1}{2})$. Any trembling-hand equilibria yield the same continuation payoffs after almost all histories at time $-t \in [-T, 0]$.

B.3 Proof of Proposition 2

We show that a solution $v^*(t)$ of ODE (1) characterizes a trembling-hand equilibrium. For $s_i \in \{+, -\}$, and $v_i \in [0, \infty)$ let

$$I_i^{s_i}(v_i) = \begin{cases} [0, v_i] & \text{if } s_i = +, \\ [v_i, \infty) & \text{if } s_i = -, \end{cases}$$

and $p^+ = 1 - \varepsilon$, $p^- = \varepsilon$. For $\varepsilon > 0$, let us write down a Bellman equation similar to (2) with respect to a continuation payoff profile $v^\varepsilon(t)$ in the ε-constrained game:

$$\begin{aligned} v_i^\varepsilon(t) = \int_0^t \Big(\sum_{s \in \{+,-\}^n} \int_{(I_1^{s_1}(v_i^\varepsilon(\tau)) \times \cdots \times I_n^{s_n}(v_i^\varepsilon(\tau))) \cap X} \big(p^{s_1} \dots p^{s_n} \cdot x_i + (1 - p^{s_1} \dots p^{s_n})v_i^\varepsilon(\tau)\big) d\mu \Big) \\ \cdot \lambda e^{-(\lambda+\rho)(t-\tau)} d\tau \end{aligned}$$

This implies that

$$\begin{aligned} v_i^{\varepsilon\prime}(t) &= -(\lambda + \rho)v_i^\varepsilon(t) \\ &\quad + \lambda \sum_{s \in \{+,-\}^n} \int_{(I_1^{s_1}(v_i^\varepsilon(t)) \times \cdots \times I_n^{s_n}(v_i^\varepsilon(t))) \cap X} \big(p^{s_1} \dots p^{s_n} \cdot x_i + (1 - p^{s_1} \dots p^{s_n})v_i^\varepsilon(t)\big) d\mu. \end{aligned}$$

This ODE has a unique solution because the right hand side is Lipschitz continuous in v_i^ε. Let $v^\varepsilon(t)$ be this solution, which is a cutoff profile of a Nash equilibrium in the ε-constrained game by construction. Since $A(t) = (I_1^+(v_i^\varepsilon(\tau)) \times \cdots \times I_n^+(v_i^\varepsilon(\tau))) \cap X$, and $p^+ \to 1$, $p^- \to 0$ as $\varepsilon \to 0$, ODE (1) is obtained by letting $\varepsilon \to 0$. Therefore $v^\varepsilon(t)$ converges to $v^*(t)$ as $\varepsilon \to 0$ because the above ODE is continuous in ε.[50] Hence the cutoff strategy profile with cutoffs $v^*(t)$ is a trembling-hand equilibrium.

[50] See, e.g., Coddington and Levinson (1955, Theorem 7.4 in Chapter 1).

B.4 Proof of Theorem 1

By Assumption 2, there exists a nondecreasing and concave function φ and $\kappa \geq 1$ such that

$$1 - \varphi(x) \leq 1 - F(x) \leq \kappa\big(1 - \varphi(x)\big)$$

for all $x \geq 0$. Let us consider a cutoff strategy with the following cutoff $w(t)$:

$$w(t) = F^{-1}\Big(1 - \frac{2}{\lambda t + 2}\Big),$$

namely, the strategy with acceptance probability $\frac{2}{\lambda t+2}$ at time $-t$. By Assumption 2, we have $w(t) \geq \varphi^{-1}\big(1 - \frac{2}{\lambda t+2}\big)$. Let $P(t)$ be the probability that the search stops before time $-t$ when $w(t)$ is played. Then

$$\begin{aligned} P(t) &= 1 - \exp\Big(-\int_t^T \frac{2}{\lambda\tau + 2} \cdot \lambda d\tau\Big) \\ &= 1 - \Big(\frac{\lambda t + 2}{\lambda T + 2}\Big)^2. \end{aligned}$$

The expected continuation payoff obtained from this strategy is larger than

$$\int_t^0 w(\tau) dP(\tau) \geq \int_0^t \varphi^{-1}\Big(1 - \frac{2}{\lambda\tau + 2}\Big) d\big(1 - P(\tau)\big).$$

Let $W(t)$ be the payoff on the right hand side, and $Q(t)$ be the probability that the search stops before time $-t$ when the player plays a cutoff strategy with cutoff $W(t)$.

$$\begin{aligned} Q(t) &= 1 - \exp\left(-\int_t^T \big(1 - F(W(\tau))\big)\lambda d\tau\right) \\ &\leq 1 - \exp\left(-\int_t^T \kappa\big(1 - \varphi(W(\tau))\big)\lambda d\tau\right). \end{aligned}$$

By concavity of φ, $\varphi(W(\tau))$ is bounded as follows:

$$\begin{aligned} \varphi(W(t)) &= \varphi\left(\int_0^t \varphi^{-1}\Big(1 - \frac{2}{\lambda\tau + 2}\Big) d\big(1 - P(\tau)\big)\right) \\ &\geq \int_0^t \varphi\left(\varphi^{-1}\Big(1 - \frac{2}{\lambda\tau + 2}\Big)\right) d\big(1 - P(\tau)\big) \\ &= \int_0^t \Big(1 - \frac{2}{\lambda\tau + 2}\Big) d\Big(\Big(\frac{\lambda\tau + 2}{\lambda T + 2}\Big)^2\Big) \\ &= 1 - \frac{4}{\lambda T + 2} + \frac{4}{(\lambda T + 2)^2} \\ &\geq 1 - \frac{4}{\lambda T + 2}. \end{aligned}$$

Therefore,

$$\begin{aligned} Q(t) &\leq 1 - \exp\left(-\int_t^T \kappa\Big(\frac{4}{\lambda T+2}\Big)\lambda d\tau\right) \\ &= 1 - \exp\left(-4\kappa \ln\Big(\frac{\lambda T+2}{\lambda t+2}\Big)\right) \\ &= 1 - \Big(\frac{\lambda t+2}{\lambda T+2}\Big)^{4\kappa}, \end{aligned}$$

which is strictly lower than 1 for all $\lambda > 0$ and all $-t \in (-T, 0]$. Since $W(t)$ is the continuation payoff calculated from a strategy that is not necessarily optimal, an optimal strategy gives the player continuation payoffs larger than or equal to $W(t)$. Therefore an optimal strategy must possess a cutoff higher than or equal to $W(t)$. Hence, for all $-t \in (-T, 0]$, the search stops with probability strictly lower than 1 before time $-t$. This proves Theorem 1.

Next, we show a proposition under an independent assumption of Assumption 2 when the support is bounded.

Assumption 6. If X is bounded, then for $\bar{x} = \sup X$, there exists $\alpha \geq \beta > 0$ such that $\varepsilon^\alpha \leq \mu\big([\bar{x}-\varepsilon, \bar{x}]\big) \leq \varepsilon^\beta$ for all $\varepsilon \in (0,1)$.

Proposition 25. *Suppose that X is bounded, and Assumptions 1 and 6 hold. Then* $\liminf_{\lambda\to\infty} D(\lambda) > 0$.

Proof. By ODE (1), the equilibrium continuation payoff $v^*(t)$ is the solution of

$$v'(t) = \lambda \int_{v(t)}^{\bar{x}} \big(x - v(t)\big) d\mu(x).$$

Let $z(t) = \bar{x} - v^*(t)$. Since $z(t) \to 0$ as $t \to \infty$, there exists $\bar{t}$ such that $z(t) < 1$ for all $t \geq \bar{t}$. For $t \geq \bar{t}$, $z(t)$ satisfies

$$\begin{aligned} z'(t) &= -\lambda \int_{\bar{x}-z(t)}^{\bar{x}} \big(\bar{x} - x + z(t)\big) d\mu(x) \\ &\leq -\lambda \int_{\bar{x}-\frac{z(t)}{2}}^{\bar{x}} \frac{z(t)}{2} d\mu(x) \\ &\leq -\lambda \cdot \frac{z(t)}{2} \cdot \Big(\frac{z(t)}{2}\Big)^\alpha. \end{aligned}$$

Solving this, $z(t) \leq \big(2^{-(1+\alpha)}\lambda(t-\bar{t}) + z(\bar{t})^{-\alpha}\big)^{-\frac{1}{\alpha}}$. Therefore,

$$\begin{aligned} p(t) &= \mu\big([\bar{x} - z(t), \bar{x}]\big) \\ &\leq z(t)^\beta \\ &\leq \big(2^{-(1+\alpha)}\lambda(t-\bar{t}) + z(\bar{t})^{-\alpha}\big)^{-\frac{\beta}{\alpha}}. \end{aligned}$$

By formulas (3) and (4), if $p(t)$ is of the order of $\frac{1}{\lambda t}$ or less, then by Lemma 24 we have that $\liminf_{\lambda\to\infty} D(\lambda) > 0$. This is the desired result. $\square$

B.5 Proof of Theorem 4

First, we show that Assumption 1 (b) implies that $\mu(A(v))$ is continuous in $v \in \mathbb{R}^n$.

By ODE (1), $v_i'(t) = \lambda b_i(v(t)) \cdot p(t)$ for each $i \in N$. Since $\mu(A(v))$ is continuous in v,

$$\begin{aligned}\liminf_{\Delta t\to 0} \frac{p(t) - p(t+\Delta t)}{\Delta t} &\leq \sum_{i\in N} \underline{d}_i(v(t))p(t) \cdot v_i'(t) \\ &= \sum_{i\in N} \underline{d}_i(v(t))p(t) \cdot \lambda b_i(v(t))p(t).\end{aligned}$$

By the definition of $\underline{r}$, for all $\varepsilon > 0$, there exists $\bar{t}$ such that for all $t \geq \bar{t}$,

$$\frac{\liminf_{\Delta t\to 0} \frac{p(t)-p(t+\Delta t)}{\Delta t}}{\lambda p(t)^2} \geq \underline{r} - \varepsilon. \tag{B.1}$$

Integrating the both sides and letting $\lambda \to \infty$, we have

$$\limsup_{\lambda\to\infty} p(t) \cdot \lambda t \leq \underline{r}^{-1}. \tag{B.2}$$

An analogous argument shows that

$$\liminf_{\lambda\to\infty} p(t) \cdot \lambda t \geq \overline{r}^{-1}. \tag{B.3}$$

By Lemma 24, we obtain

$$\begin{aligned}\left(\frac{t}{T}\right)^{1/\underline{r}} \leq \liminf_{\lambda\to\infty} P(t;\lambda) \quad &\text{and} \quad \limsup_{\lambda\to\infty} P(t;\lambda) \leq \left(\frac{t}{T}\right)^{1/\overline{r}}, \text{ and} \\ \frac{1}{1+\underline{r}^{-1}} \leq \liminf_{\lambda\to\infty} D(\lambda) \quad &\text{and} \quad \limsup_{\lambda\to\infty} D(\lambda) \leq \frac{1}{1+\overline{r}^{-1}}.\end{aligned}$$

B.6 Proof of Theorem 5

Let $r = \overline{r} = \underline{r}$. Then we can follow the discussion in the proof sketch of Theorem 4, and obtain inequality (8). Since this inequality holds for any λ, for all $\varepsilon > 0$, there is a large $\bar{t}$ such that for all $t \geq \bar{t}$

$$-(r+\varepsilon)p(t;1)^2 \leq p'(t;1) \leq -(r-\varepsilon)p(t;1)^2.$$

For any small $\eta > 0$, let $\bar{\lambda} = \bar{t}/\eta$. Since $p(t;\lambda) = p(t/\lambda;1)$ for all t and λ, we have

$$-(r+\varepsilon)p(t;\lambda)^2 \leq p'(t;\lambda) \leq -(r-\varepsilon)p(t;\lambda)^2$$

for all $t \geq \eta$, and all $\lambda \geq \bar{\lambda}$. Solving this with an initial condition at η, for $t \geq \eta$ and $\lambda \geq \bar{\lambda}$,

$$\frac{1}{(r-\varepsilon)\lambda(t-\eta)+p(\eta)^{-1}} \leq p(t) \leq \frac{1}{(r+\varepsilon)\lambda(t-\eta)+p(\eta)^{-1}}.$$

By formula (3), for $t \geq \eta$ and $\lambda \geq \bar{\lambda}$, we have

$$e^{-\int_t^T \frac{1}{(r+\varepsilon)(s-\eta)+p(\eta)^{-1}/\lambda} ds} \leq P(t) \leq e^{-\int_t^T \frac{1}{(r-\varepsilon)(s-\eta)+p(\eta)^{-1}/\lambda} ds}$$

$$\Big(\frac{r\lambda(t-\eta)+p(\eta)^{-1}}{r\lambda(T-\eta)+p(\eta)^{-1}}\Big)^{(r-\varepsilon)^{-1}} \leq P(t) \leq \Big(\frac{r\lambda(t-\eta)+p(\eta)^{-1}}{r\lambda(T-\eta)+p(\eta)^{-1}}\Big)^{(r+\varepsilon)^{-1}}.$$

By formula (4), we have

$$\int_\eta^T \Big(\frac{r\lambda(t-\eta)+p(\eta)^{-1}}{r\lambda(T-\eta)+p(\eta)^{-1}}\Big)^{(r-\varepsilon)^{-1}} dt$$

$$\leq D(\lambda)T \leq \eta + \int_\eta^T \Big(\frac{r\lambda(t-\eta)+p(\eta)^{-1}}{r\lambda(T-\eta)+p(\eta)^{-1}}\Big)^{(r+\varepsilon)^{-1}} dt$$

$$\frac{1}{1+(r-\varepsilon)^{-1}}\Big(T-\eta+\frac{1}{r\lambda p(\eta)}-\frac{1}{r\lambda p(\eta)(T-\eta)+1}\Big)$$

$$\leq D(\lambda)T \leq \frac{1}{1+(r+\varepsilon)^{-1}}\Big(T-\eta+\frac{1}{r\lambda p(\eta)}-\frac{1}{r\lambda p(\eta)(T-\eta)+1}\Big).$$

Since the above inequalities are satisfied for all $\varepsilon > 0$ and $\eta > 0$ in the limit as $\lambda \to \infty$, and $D(\infty) = \frac{1}{1+r^{-1}}$, $|D(\lambda) - D(\infty)| = O(\frac{1}{\lambda})$.

B.7 Proof of Proposition 6

We prove Proposition 6 in an environment more general than Assumption 3.

Assumption 7. (a) The limit $v^* = \lim_{\lambda\to\infty} v^*(t)$ is Pareto efficient in X.

(b) The Pareto frontier of X is smooth in a neighborhood of v^*.

(c) For the unit normal vector $\alpha \in \mathbb{R}^n_+$ at v^*, $\alpha_i > 0$ for all $i \in N$.[51]

(d) For all $\eta > 0$, there exists $\varepsilon > 0$ such that $\{x \in \mathbb{R}^n_+ \mid |v^* - x| \leq \varepsilon, \alpha \cdot (x - v^*) \leq -\eta\}$ is contained in X, where "$\cdot$" denotes the inner product in $\mathbb{R}^n$.

(e) μ has a continuous density function.

Proposition 6′. Under Assumptions 1, 4, and 7, $\lim_{\lambda\to\infty} D(\lambda) = \dfrac{n^2}{n^2+n+1}$.

[51] We can show basically the same results without this assumption. We avoid complications derived from the indeterminacy of a normal vector on the boundary of X.

Proof. Let $f_H(t) = \max_{x \in A(t)} f(x)$, and $f_L(t) = \min_{x \in A(t)} f(x)$. Since f is continuous, both $f_H(t)$ and $f_L(t)$ are continuous and converge to $f(v^*)$ as $t \to \infty$. For $\varepsilon > 0$, there is $\bar{t}$ such that $|v^* - v^*(t)| \leq \varepsilon$ for all $t \geq \bar{t}$. For $\eta > 0$, let

$$\underline{A}(t) = \{x \in \mathbb{R}^n_+ \mid x \geq v^*(t), \alpha \cdot (x - v^*) \leq -\eta\},$$
$$\overline{A}(t) = \{x \in \mathbb{R}^n_+ \mid x \geq v^*(t), \alpha \cdot (x - v^*) \leq \eta\}.$$

The volume of $\underline{A}(t)$ (with respect to the Lebesgue measure on $\mathbb{R}^n$) is

$$V(\underline{A}(t)) = \frac{1}{n} \prod_{j \in N} \Big(\frac{\alpha \cdot (v^* - v^*(t)) - \eta}{\alpha_j} \Big), \tag{B.4}$$

and the volume of $\overline{A}(t)$ is

$$V(\overline{A}(t)) = \frac{1}{n} \prod_{j \in N} \Big(\frac{\alpha \cdot (v^* - v^*(t)) + \eta}{\alpha_j} \Big). \tag{B.5}$$

Suppose that $\varepsilon > 0$ is small and $\bar{t}$ is large. Then by Assumption 7, $\underline{A}(t) \subset A(t) \subset \overline{A}(t)$ holds for all $\eta > 0$ and all $t \geq \bar{t}$. The rest of the proof consists of two steps.

Step 1: We show that for any two players $i, j \in N$, $\lim_{t \to \infty} v_j^{*\prime}(t)/v_i^{*\prime}(t) = \alpha_i/\alpha_j$. The ith coordinate of the right hand side of equation (1) is bounded as

$$\begin{aligned} &f_L(\bar{t}) \int_{\underline{A}(t)} \big(x_i - v_i^*(t)\big) dx \\ &\quad \leq \int_{A(t)} \big(x_i - v_i^*(t)\big) f(x) dx \leq f_H(\bar{t}) \int_{\overline{A}(t)} \big(x_i - v_i^*(t)\big) dx. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{\lambda f_L(\bar{t}) V(\underline{A}(t))}{n+1} \Big(\frac{\alpha \cdot (v^* - v^*(t)) - \eta}{\alpha_i} \Big) \\ &\quad \leq v_i^{*\prime}(t) \leq \frac{\lambda f_H(\bar{t}) V(\overline{A}(t))}{n+1} \Big(\frac{\alpha \cdot (v^* - v^*(t)) + \eta}{\alpha_i} \Big) \end{aligned}$$

for all $t \geq \bar{t}$ and $i \in N$. By substituting (B.4) and (B.5),

$$\begin{aligned} &\frac{\lambda f_L(\bar{t})}{n(n+1)} \Big(\frac{\alpha \cdot (v^* - v^*(t)) - \eta}{\alpha_i} \Big) \prod_{j \in N} \Big(\frac{\alpha \cdot (v^* - v^*(t)) - \eta}{\alpha_j} \Big) \\ &\quad \leq v_i^{*\prime}(t) \leq \frac{\lambda f_H(\bar{t})}{n(n+1)} \Big(\frac{\alpha \cdot (v^* - v^*(t)) + \eta}{\alpha_i} \Big) \prod_{j \in N} \Big(\frac{\alpha \cdot (v^* - v^*(t)) + \eta}{\alpha_j} \Big) \end{aligned} \tag{B.6}$$

for all $t \geq \bar{t}$ and $i \in N$. By letting $\eta \to 0$, $\varepsilon \to 0$, and $t \to \infty$, we have $\lim_{t \to \infty} v_j^{*\prime}(t)/v_i^{*\prime}(t) = \alpha_i/\alpha_j$.

Step 2: By Step 1, for i and small $\delta > 0$, there exists $\tilde{t} \geq \bar{t}$ such that

$$(1-\delta)\frac{\alpha_i}{\alpha_j} \leq \frac{v_j^* - v_j^*(t)}{v_i^* - v_i^*(t)} \leq (1+\delta)\frac{\alpha_i}{\alpha_j}$$

for all $t \geq \tilde{t}$ and $j \in N$. Therefore,

$$n(1-\delta)(v_i^* - v_i^*(t)) \leq \frac{\alpha \cdot (v^* - v^*(t))}{\alpha_i} \leq n(1+\delta)(v_i^* - v_i^*(t)).$$

By inequality (B.6), we have

$$\begin{aligned}
&\frac{\lambda f_L(\bar{t})}{n(n+1)}\Big(n(1-\delta)(v_i^* - v_i^*(t)) - \frac{\eta}{\alpha_i}\Big)\prod_{j\in N}\Big(n(1-\delta)\frac{\alpha_i}{\alpha_j}(v_i^* - v_i^*(t)) - \frac{\eta}{\alpha_j}\Big)\\
&\quad \leq v_i^{*\prime}(t) \leq \frac{\lambda f_H(\bar{t})}{n(n+1)}\Big(n(1+\delta)(v_i^* - v_i^*(t)) + \frac{\eta}{\alpha_i}\Big)\prod_{j\in N}\Big(n(1+\delta)\frac{\alpha_i}{\alpha_j}(v_i^* - v_i^*(t)) + \frac{\eta}{\alpha_j}\Big)
\end{aligned}$$

for all $t \geq \tilde{t}$ and $j \in N$. Therefore,

$$\begin{aligned}
&\Big(n(1-\delta)(v_i^* - v_i^*(t)) - \frac{\eta}{\alpha_i}\Big)' \leq -\frac{\lambda f_L(\bar{t})(1-\delta)}{n+1}\Big(\prod_{j\neq i}\frac{\alpha_i}{\alpha_j}\Big)\Big(n(1-\delta)(v_i^* - v_i^*(t)) - \frac{\eta}{\alpha_i}\Big)^{n+1}, \text{ and}\\
&\Big(n(1+\delta)(v_i^* - v_i^*(t)) + \frac{\eta}{\alpha_i}\Big)' \geq -\frac{\lambda f_H(\bar{t})(1+\delta)}{n+1}\Big(\prod_{j\neq i}\frac{\alpha_i}{\alpha_j}\Big)\Big(n(1+\delta)(v_i^* - v_i^*(t)) + \frac{\eta}{\alpha_i}\Big)^{n+1}
\end{aligned}$$

for all $t \geq \tilde{t}$ and $j \in N$. By solving differential equations given by the above inequalities with equality, we have

$$\begin{aligned}
n(1-\delta)(v_i^* - v_i^*(t)) - \frac{\eta}{\alpha_i} &\leq \Big(C_L + \frac{\lambda f_L(\bar{t})(1-\delta)nt}{n+1}\prod_{j\neq i}\frac{\alpha_i}{\alpha_j}\Big)^{-\frac{1}{n}}\\
n(1+\delta)(v_i^* - v_i^*(t)) + \frac{\eta}{\alpha_i} &\geq \Big(C_H + \frac{\lambda f_H(\bar{t})(1+\delta)nt}{n+1}\prod_{j\neq i}\frac{\alpha_i}{\alpha_j}\Big)^{-\frac{1}{n}}
\end{aligned}$$

where C_L, C_H are constants determined by the initial condition at $t = \tilde{t}$. Deforming the above inequalities,

$$\begin{aligned}
&\frac{1}{n(1+\delta)}\Big(\frac{C_H}{\lambda t} + \frac{f_H(\bar{t})(1+\delta)n}{n+1}\prod_{j\neq i}\frac{\alpha_i}{\alpha_j}\Big)^{-\frac{1}{n}} - \frac{(\lambda t)^{\frac{1}{n}}\eta}{n(1+\delta)\alpha_i}\\
&\quad \leq (v_i^* - v_i^*(t))(\lambda t)^{\frac{1}{n}} \leq \frac{1}{n(1-\delta)}\Big(\frac{C_L}{\lambda t} + \frac{f_L(\bar{t})(1-\delta)n}{n+1}\prod_{j\neq i}\frac{\alpha_i}{\alpha_j}\Big)^{-\frac{1}{n}} + \frac{(\lambda t)^{\frac{1}{n}}\eta}{n(1-\delta)\alpha_i}.
\end{aligned}$$

As $\eta \to 0$, $\bar{t} \to \infty$, $\delta \to 0$, and $t \to \infty$, we have

$$\lim_{t\to\infty}(v_i^* - v_i^*(t))(\lambda t)^{\frac{1}{n}} = \frac{1}{n}\Big(\frac{f(v^*)n}{n+1}\prod_{j\neq i}\frac{\alpha_i}{\alpha_j}\Big)^{-\frac{1}{n}}$$
$$\lim_{t\to\infty}\alpha_i(v_i^* - v_i^*(t))(\lambda t)^{\frac{1}{n}} = \Big(\frac{n+1}{f(v^*)n^{n+1}}\prod_{j\in N}\alpha_j\Big)^{\frac{1}{n}},$$

which is a positive constant.

Step 3: By the definition of $\underline{A}(t), \overline{A}(t)$,

$$f_L(t)V(\underline{A}(t)) \leq p(t) \leq f_H(t)V(\overline{A}(t)).$$

Inequalities (B.4), (B.5) implies that by letting $\eta \to 0$ we have

$$\lim_{t\to\infty} p(t)\cdot\lambda t = \lim_{t\to\infty}\frac{f(v^*)}{n}\prod_{j\in N}\Big(\frac{\sum_{i\in N}\alpha_i(v_i^* - v_i^*(t))(\lambda t)^{\frac{1}{n}}}{\alpha_j}\Big).$$

By the result of Step 2, this limit exists and computed as follows:

$$\lim_{t\to\infty} p(t)\cdot\lambda t = \frac{f(v^*)}{n}\prod_{j\in N}\left(\frac{n}{\alpha_j}\Big(\frac{n+1}{f(v^*)n^{n+1}}\prod_{k\in N}\alpha_k\Big)^{\frac{1}{n}}\right)$$
$$= \frac{n+1}{n^2}.$$

By Lemma 24, the limit expected duration is

$$D(\infty) = \frac{1}{1+\frac{n+1}{n^2}} = \frac{n^2}{n^2+n+1}.$$

□

B.8 Proof of Proposition 9

Let $v^*(t; f)$ be the solution of ODE (1) for density $f \in \mathcal{F}$, and $v^*(f) = \lim_{\lambda\to\infty} v^*(t; f) = \lim_{t\to\infty} v^*(t; f)$.

First we show that the set is open, i.e., for all $f \in \mathcal{F}$ with $v^*(f)$ Pareto efficient, $\varepsilon > 0$, and a sequence $f_k \in \mathcal{F}$ $(k = 1, 2, \dots)$ with $|f_k - f| \to 0$ $(k \to \infty)$, there exist $\delta > 0$ and $\bar{k}$ such that

$$|v^*(f_k) - v^*(f)| \leq \varepsilon$$

for all $k \geq \bar{k}$.

Since $\lim_{t\to\infty} v^*(t; f) = v^*(f)$, for all $\delta > 0$ there exists $\bar{t} > 0$ such that $|v^*(f) -$

$v^*(t;f)| \leq \delta$ for all $t \geq \bar{t}$. By Pareto efficiency of $v^*(f)$, let $\delta > 0$ be sufficiently small so that $A\big(v^*(\bar{t};f) - (\delta,\delta,\ldots,\delta)\big)$ is contained in the ε-ball centered at $v^*(f)$. Since the right hand side of ODE (1) is continuous in v by Assumption 4, the unique solution of (1) is continuous with respect to parameters in (1). Therefore, for a finite time interval $[0,T]$ including $\bar{t}$, there exists $\bar{k}$ such that $|v^*(t;f_k) - v^*(t;f)| \leq \delta$ for all $t \in [0,T]$ and all $k \geq \bar{k}$. This implies that $v^*(t;f_k) \in A\big(v^*(\bar{t};f) - (\delta,\delta,\ldots,\delta)\big)$, thereby $v^*(f_k) \in A\big(v^*(\bar{t};f) - (\delta,\delta,\ldots,\delta)\big)$. Hence we have $|v^*(f_k) - v^*(f)| \leq \varepsilon$.

Second we show that the set is dense, i.e., for all $f \in \mathcal{F}$ with $v^*(f)$ not strictly Pareto efficient in X and all $\varepsilon > 0$, there exists $\tilde{f} \in \mathcal{F}$ such that $|f - \tilde{f}| \leq \varepsilon$ and $v^*(\tilde{f})$ is Pareto efficient. Since $v^*(f)$ is only weakly Pareto efficient in $\hat{X}$, there exists Pareto efficient $y \in X$ which Pareto dominates $v^*(f)$. Let $I = \{i \in N \,|\, y_i = v_i^*(f)\}$ and $J = N \setminus I$. Since y is Pareto efficient, there is $\delta > 0$ such that if $x \in X$ is weakly Pareto efficient, satisfies $|y - x| \leq \delta$, and $y_i = x_i$ for some $i \in N$, then there is no $\tilde{x} \in X$ such that $\tilde{x}_i > y_i$ and $|y - \tilde{x}| \leq \delta$.

By Assumption 4, for any small $\delta/2 > \eta > 0$, there is a small ball contained in X centered at $\tilde{y}$ with $|y - \tilde{y}| \leq \eta$. Let g be a continuous density function whose support is the above small ball, takes zero on the boundary of the ball, and the expectation of g is exactly $\tilde{y}$. Let $\tilde{f} = (1 - \frac{\varepsilon}{|f|+|g|})f + \frac{\varepsilon}{|f|+|g|}g \in \mathcal{F}$. Since f and g are bounded from above, $|f - \tilde{f}| \leq \varepsilon$.

Since $v^*(f)$ is weakly Pareto efficient, if $v^*(f) \in A(v)$, then $A(v) \subseteq \bigcup_{i\in N}\big([v_i, v_i^*(f)] \times \prod_{j\neq i}[0,\overline{x}_j]\big)$. If $|v^*(f) - v| \leq \xi$ where $\xi > 0$ is very small,

$$\begin{aligned}\int_{A(v)} (x_i - v_i) f(x)dx &\leq f_H \sum_{j\in N}(v_j^*(f) - v_j) \prod_{k\in N} \overline{x}_k \\ &\leq \xi n f_H \prod_{k\in N} \overline{x}_k\end{aligned}$$

If $v^*(f) \in A(v)$, $\min_{j\in N}(y_j - v_j) \geq 2\eta$ and $|v^*(f) - v| \leq \xi$, we have

$$\begin{aligned}\int_{A(v)} (x_i - v_i)\tilde{f}(x)dx - \int_{A(v)} (x_i - v_i) f(x)dx &= \int_{A(v)} (x_i - v_i)(\tilde{f}(x) - f(x))dx \\ &= \frac{\varepsilon}{|f|+|g|} \int_{A(v)} (x_i - v_i)(g(x) - f(x))dx \\ &\geq \frac{\varepsilon}{|f|+|g|}\Big((\tilde{y}_i - v_i) - \Big(\xi n f_H \prod_{k\in N} \overline{x}_k\Big)\Big).\end{aligned}$$

If $j \in J$ and $|v^*(f) - v| \leq \xi$ where $\xi > 0$ is very small, then

$$\int_{A(v)} (x_j - v_j)\tilde{f}(x)dx - \int_{A(v)} (x_j - v_j) f(x)dx \geq \frac{\varepsilon}{2(|f|+|g|)}(\tilde{y}_j - v_j^*(f)).$$

Let $w(t) = v^*(t;\tilde{f}) - v^*(t;f)$. Since ODE (1) is continuous in the parameters, for all

$\zeta > 0$, there exists $\varepsilon > 0$ such that $|w(t)| \le \zeta$ for all $t \in [0, T]$. Suppose that T and t are very large so that $|v^*(f) - v^*(t; f)| \le \xi$. For $j \in J$, $w_j'(t)$ is estimated as follows:

$$\begin{aligned}
w_j'(t) &= \lambda \int_{A(v^*(t;\tilde{f}))} (x_j - v_j^*(t;\tilde{f}))\tilde{f}(x)dx - \lambda \int_{A(v^*(t;f))} (x_j - v_j^*(t;f))f(x)dx \\
&= \lambda \int_{A(v^*(t;\tilde{f}))} (x_j - v_j^*(t;\tilde{f}))\tilde{f}(x)dx - \lambda \int_{A(v_j^*(t;\tilde{f})))} (x_j - v_j^*(t;\tilde{f})))f(x)dx \\
&\quad + \lambda \int_{A(v^*(t;\tilde{f}))} (x_j - v_j^*(t;\tilde{f}))f(x)dx - \lambda \int_{A(v^*(t;f))} (x_j - v_j^*(t;f))f(x)dx \\
&\ge \frac{\lambda\varepsilon}{2(|f|+|g|)}(\tilde{y}_j - v_j^*(f)) - \lambda \int_{A(v^*(t;f))\cap A(v^*(t;\tilde{f}))} w_j(t)f(x)dx \\
&\quad - \lambda \int_{A(v^*(t;f))\setminus(A(v^*(t;f))\cap A(v^*(t;\tilde{f})))} (x_j - v_j^*(t;f))f(x)dx \\
&\ge \frac{\lambda\varepsilon}{2(|f|+|g|)}(\tilde{y}_j - v_j^*(f) - \zeta) - \lambda\zeta\xi \sum_{k\in N}\prod_{l\ne k}\overline{x}_l - \lambda\xi n f_H \prod_{k\in N}\overline{x}_k.
\end{aligned}$$

Therefore when $\xi > 0$ is sufficiently small, $w_j'(t)$ is bounded away from zero:

$$w_j'(t) \ge \frac{\lambda\varepsilon}{4(|f|+|g|)}(\tilde{y}_j - v_j^*(f) - \zeta).$$

This implies that for small $\varepsilon > 0$ and large t, $v_j^*(t;\tilde{f}) > v^*(f)$ for all $j \in J$. Then the similar method to Step 3 in the proof of Proposition 10 shows that $v^*(t;\tilde{f})$ converges to a Pareto efficient allocation in X.

B.9 Proof of Proposition 10

Let $f_L = \inf_{x\in X} f(x) > 0$, $f_H = \sup_{x\in X} f(x)$, and $\overline{x}_i = \max\{x_i \,|\, x \in X\}$ for $i \in N$. Assumption 4 ensures existence of these values. Let $A = \{x \in X \,|\, x \ge v^*\}$, $I = \{i \in N \,|\, x_i = v_i^* \text{ for all } x \in A\} \subseteq N$, and $J = N \setminus I$. Suppose that there exists $x \in X$ which Pareto dominates v^*, thereby $J \ne \emptyset$.

Step 1: We show that I is nonempty. If there is no such player, there exist $y(1), \ldots, y(n)$ such that $y(j) \in A$ and $y_j(j) > v_j^*$ for all $j \in N$. This implies that $y = \frac{1}{n}\sum_{j\in N} y(j)$ strictly Pareto dominates v^*. Since X is convex, y also belongs to A. This contradicts the weak Pareto efficiency of v^* shown in Proposition 4.

Step 2: Next we show that if v^* is not Pareto efficient in X, and $i \in I$, then $x_i \le v_i^*$ for all $x \in X$.

Let i be the player in I. Suppose that there exists $y \in X$ with $y_i > v_i^*$. Since X is convex, $\alpha y + (1-\alpha)x \in X$ for all $0 \le \alpha \le 1$ and $x \in X$. Since we assumed that there exists $x \in X$ which Pareto dominates v^*, $x_j > v_j^*$ for $j \in J$. Then there exists $\alpha > 0$ such that $\alpha y + (1-\alpha)x \ge v^*$, and $\alpha y_j + (1-\alpha)x_j > v_j^*$ for some j. By Step 1, we must have

$x_i = v_i^*$. Therefore, $\alpha y_i + (1-\alpha)x_i > v_i^*$, which contradicts the fact that $i \in I$.

Step 3: Finally we show that $v^*(t)$ converges to a Pareto efficient allocation in X as $t \to \infty$.

By convexity of X, one can find $y_j, \bar{y}_j$ ($j \in J$) such that $v_j^* < y_j < \bar{y}_j$, and $\prod_{i\in I}[v_i^* - \varepsilon, v_i^*] \times \prod_{j\in J}[y_j, \bar{y}_j]$ is contained in X for small $\varepsilon > 0$. Let $\varepsilon \in (0, 1/2)$ be sufficiently small such that $\varepsilon \leq \dfrac{2f_L \prod_{j\in J}(\bar{y}_j - y_j)}{f_H \prod_{j\in J} \overline{x}_j}$. Since $v^*(t)$ converges to v^* as $t \to \infty$, there exists $\bar{t}$ such that $\max_{i\in N}\{v_i^* - v_i^*(t)\} \leq \varepsilon$ whenever $t \geq \bar{t}$. Let $Y(t) = \prod_{i\in I}[v_i^*(t), v_i^*] \times \prod_{j\in J}[y_j, \bar{y}_j] \subseteq A(t)$.

We have $A(t) \subseteq \prod_{i\in I}[v_i^*(t), v_i^*] \times \prod_{j\in J}[0, \overline{x}_j]$ since there is no $x \in A(t)$ with $x_i > v_i^*$. By equation (1), for $i \in I$,

$$\begin{aligned}
v_i^{*\prime}(t) &= \lambda \int_{A(t)} \big(x_i - v_i^*(t)\big) d\mu \\
&\leq \lambda \int_{\prod_{i'\in I}[v_{i'}^*(t), v_{i'}^*]} \big(x_i - v_i^*(t)\big) \int_{\prod_{j\in J}[0,\overline{x}_j]} f_H \prod_{j\in J} dv_j \prod_{i'\in I} dv_{i'} \\
&\leq \frac{1}{2}\lambda f_H (v_i^* - v_i^*(\bar{t})) \prod_{i'\in I}(v_{i'}^* - v_{i'}^*(t)) \prod_{j\in J} \overline{x}_j
\end{aligned}$$

for all $t \geq \bar{t}$. On the other hand, for $j \in J$,

$$\begin{aligned}
v_j^{*\prime}(t) &= \lambda \int_{A(t)} \big(x_j - v_j^*(t)\big) d\mu \\
&\geq \lambda \int_{Y(t)} (y_j - v_j^*) d\mu \\
&= \lambda (y_j - v_j^*) \mu(Y(t)) \\
&\geq \lambda f_L (y_j - v_j^*) \prod_{i\in I}(v_i^* - v_i^*(t)) \prod_{j'\in J}(\bar{y}_{j'} - y_{j'}).
\end{aligned}$$

Then for $i \in I$ and $j \in J$,

$$\begin{aligned}
\frac{v_i^{*\prime}(t)}{v_j^{*\prime}(t)} \cdot \frac{v_j^* - v_j^*(\bar{t})}{v_i^* - v_i^*(\bar{t})} &\leq \frac{f_H (v_i^* - v_i^*(\bar{t}))(v_j^* - v_j^*(\bar{t})) \prod_{j\in J} \overline{x}_j}{2 f_L \prod_{j'\in J}(\bar{y}_{j'} - y_{j'})} \\
&\leq \frac{(v_i^* - v_i^*(\bar{t}))(v_j^* - v_j^*(\bar{t}))}{\varepsilon} \\
&\leq \varepsilon \leq \frac{1}{2}
\end{aligned}$$

for all $t \geq \bar{t}$. Therefore,

$$\frac{v_i^{*\prime}(\bar{t})}{v_j^{*\prime}(\bar{t})} \leq \frac{v_i^* - v_i^*(\bar{t})}{2\big(v_j^* - v_j^*(\bar{t})\big)}$$

holds for all $t \geq \bar{t}$. This inequality implies

$$v_i^*(t) - v_i^*(\bar{t}) \leq \frac{v_i^* - v_i^*(\bar{t})}{2\big(v_j^* - v_j^*(\bar{t})\big)}\big(v_j^*(t) - v_j^*(\bar{t})\big)$$

for all $t \geq \bar{t}$. By letting $t \to \infty$ in the above inequality, we have $0 < v_i^* - v_i^*(\bar{t}) \leq \big(v_i^* - v_i^*(\bar{t})\big)/2$, a contradiction. Hence v^* is strictly Pareto efficient in X.

B.10 Proof of Proposition 12

First, we define the notion of the edge of the Pareto frontier. Suppose that w is Pareto efficient in X, and $w_i > 0$ for all $i \in X$. Let us denote an $(n-1)$-dimensional subspace orthogonal to w by $D = \{z \in \mathbb{R}^n \mid w \cdot z = 0\}$. For $\xi > 0$, let D_ξ be an $(n-1)$-dimensional disk defined as

$$D_\xi = \{z \in D \mid |z| \leq \xi\},$$

and let S_ξ be its boundary. We say that a Pareto efficient allocation w in X is *not* located at the edge of the Pareto frontier of X if there is $\xi > 0$ such that for all vector $z \in D_\xi$ there is a scalar $\alpha > 0$ such that $\alpha(w+z)$ is Pareto efficient in X. We denote this Pareto efficient allocation by $w_z \in X$.

Let $B_\varepsilon(y) = \{x \in X \mid |w - x| \leq \varepsilon\}$ for $y \in X$ and $\varepsilon > 0$. We denote the volume of $B_\varepsilon(y)$ by $V_\varepsilon(y)$, and the volume of the n-dimensional ball with radius ε by V_ε. Note that $\min_{y \in X} V_\varepsilon(y) > 0$ by Assumption 4. Let g be a continuous density function on an n-dimensional ball centered at $0 \in \mathbb{R}^n$ with radius ε, assumed to take zero on the boundary of the ball. Let $\tilde{f}$ be the uniform density function on X. For a Pareto efficient allocation y, we define a probability density function f_y on X by

$$f_y(x) = \eta \tilde{f}(x) + (1-\eta) g(y-x) \frac{V_\varepsilon}{V_\varepsilon(y)}$$

where $\eta > 0$ is small. Note that $f_y(x)$ is uniformly bounded above and away from zero in x and y.

For $z \in D_\xi$, let $\tilde{\varphi}(z)$ be the limit of the solution of ODE (1) with density f_{w_z}, and define a function φ from D_ξ to D by $\varphi(z) = \tilde{\varphi}(z) + \delta w \in D$ for some $\delta \in \mathbb{R}$. By the form of ODE (1), the solution of (1) with density f_{w_z} is continuously deformed if z changes continuously. Since w is not at the edge of the Pareto frontier, $\tilde{\varphi}(z)$ is also Pareto efficient in X and comes close to w if ξ, ε, and η are small. Therefore $\varphi(z)$ is a continuous function. The rest of the proof consists of two steps.

Step 1: We show that for any $\xi > 0$, there exist $\varepsilon > 0$ and $\eta > 0$ such that $|\varphi(z) - z| \leq \xi$ for all $z \in D_\xi$. If a density function has a positive value only in $B_\varepsilon(y)$ for some y in the Pareto frontier of X, then the barycenter of $A(t)$ is always contained in $B_\varepsilon(y)$. In such a case, the limit allocation with density f_y belongs to $B_\varepsilon(y)$. As $\eta \to 0$,

f_y approaches the above situation. Therefore, for sufficiently small $\eta > 0$, the distance between the limit allocation and y is smaller than 2ε. For $y = w_z$ and letting ε very small, we have $|\varphi(z) - z| \leq \xi$. Since D_ξ is compact, such we can take such small $\varepsilon > 0$ and $\eta \to 0$ uniformly.

Step 2: We show that there is $z \in D_\xi$ such that $\varphi(z) = 0$. Let $\psi(z) = z - \varphi(z)$. By Step 1, $\psi(z)$ belongs to D_ξ for all $z \in D_\xi$. By Brouwer's fixed point theorem, there exists $z \in D_\xi$ such that $\psi(z) = z$. Therefore there exists $z \in D_\xi$ such that $\varphi(z) = 0$.

Hence for $z \in D_\xi$ such that $\varphi(z) = 0$, the limit allocation with density f_{w_z} coincides with w.

B.11 Proof of Proposition 13

Let $v^0(t;\lambda)$ be the solution of (1) for $\rho = 0$. Fix any $t \in [0,T]$. Recall that $v^0(t;\alpha\lambda) = v^0(\alpha t;\lambda)$ for all $\alpha > 0$. Since we defined as $\lim_{\lambda\to\infty} v^0(t;\lambda) = v^*(t;0,\infty)$, there exists $\bar\lambda^1 > 0$ such that

$$\begin{aligned} \left|v^*(t;0,\infty) - v^0(t;\lambda)\right| &= \left|v^*(t;0,\infty) - v^0(\lambda t;1)\right| \\ &\leq \varepsilon/2 \end{aligned} \tag{B.7}$$

for all $\lambda \geq \bar\lambda^1$.

Since the right hand side of ODE (10) is continuous in ρ, λ, and uniformly Lipschitz continuous in v, the unique solution $v^*(t;\rho,\lambda)$ is continuous in ρ, λ for all $t \in [0,T]$. Recall that $v^*(t;\rho,\alpha\lambda) = v^*(\alpha t;\rho/\alpha,\lambda)$ for all $\alpha > 0$. Therefore by continuity in ρ, there exists $\bar\lambda^2 > 0$ such that

$$\begin{aligned} \left|v^*(t;\rho,\lambda) - v^0(t;\lambda)\right| &= \left|v^*(\lambda t;\rho/\lambda,1) - v^0(\lambda t;1)\right| \\ &\leq \varepsilon/2 \end{aligned} \tag{B.8}$$

for all $\lambda \geq \bar\lambda^2$. By adding (B.7) and (B.8), we obtain the desired inequality for $\bar\lambda = \max\{\bar\lambda^1, \bar\lambda^2\}$.

B.12 Proof of Proposition 14

Let $v(t)$ be the solution of ODE (10). The proof consists of five steps.

Step 1: We show that for any $t > 0$, $\mu(A(t)) \to 0$ as $\lambda \to \infty$. If not, there exist a positive value $\varepsilon > 0$ and an increasing sequence $(\bar\lambda_k)_{k=1,2,\ldots}$ such that $\mu(A(t)) \geq \varepsilon$ for all $\bar\lambda_k$. Since X is compact and f is bounded from above, there exists $\eta > 0$ such that

$\mu\big(A(v(t)+(\eta,\dots,\eta))\big) \geq \varepsilon/2$. In fact, since

$$\begin{aligned}\mu\big(A(v(t)) \setminus A(v(t)+(\eta,\dots,\eta))\big) &\leq \sum_{i\in N} \mu\Big([v_i(t), v_i(t)+\eta] \times \prod_{j\neq i}[0,\overline{x}_j]\Big) \\ &\leq f_H \sum_{i\in N} \eta \prod_{j\neq i} \overline{x}_j,\end{aligned}$$

we have $\mu\big(A(v(t)+(\eta,\dots,\eta))\big) \geq \varepsilon/2$ for $\eta = \dfrac{\varepsilon}{2f_H \sum_{i\in N}\prod_{j\neq i}\overline{x}_j}$. For this η, the integral in ODE (10) is estimated as

$$\begin{aligned}\int_{A(t)} \big(x_i - v_i(t)\big)d\mu &\geq \int_{A(v(t)+(\eta,\dots,\eta))} \big(x_i - v_i(t)\big)d\mu \\ &\geq \int_{A(v(t)+(\eta,\dots,\eta))} \eta d\mu \\ &\geq \eta\varepsilon/2.\end{aligned}$$

By ODE (10),

$$v_i'(t) \geq -\rho\overline{x}_i + \bar{\lambda}_k \eta\varepsilon/2,$$

which obviously grows infinitely as $\bar{\lambda}_k$ becomes large. This contradicts compactness of X.

Step 2: We compute the direction of $\int_{A(t)}\big(x_i - v_i(t)\big)d\mu$ in the limit as $\lambda \to \infty$. By Step 1, the boundary of X contains all accumulation points of $\{v_i(t) \mid \lambda > 0\}$ for fixed $t > 0$. Fix an accumulation point $v^*(t)$. There exists an increasing sequence $(\lambda_k)_{k=1,2,\dots}$ with $v^*(t) = \lim_{k\to\infty} v(t)$. By Assumption 5, there exists a unit normal vector of X at $v^*(t)$, which we denote by $\alpha \in \mathbb{R}_{++}$.

Step 1 implies that $v(t)$ is very close to the boundary of X when λ_k is very large. By smoothness of the boundary of X, $A(t)$ looks like a polyhedron defined by convex hull of $\{v(t), v(t)+(z_1(t),0,\dots,0), v(t)+(0,z_2(t),0,\dots,0),\dots,v(t)+(0,\dots,0,z_n(t))\}$ where $z_i(t)$'s are positive length of edges such that the last n vertices are on the boundary of X. This vector $z(t)$ is parallel to $(1/\alpha_1,\dots,1/\alpha_n)$. Let $r(t)$ be the ratio between the length of $z(t)$ and $(1/\alpha_1,\dots,1/\alpha_n)$, i.e., $r(t) = z_1(t)\alpha_1 = \dots = z_n(t)\alpha_n$.

Since density f is bounded from above and away from zero, distribution μ looks almost uniform on $A(t)$ if λ_k is large. Then the integral $\int_{A(t)}\big(x_i - v_i(t)\big)d\mu$ is almost parallel to the vector from $v(t)$ to the barycenter of the polyhedron, namely, $z(t)/(n+1)$. Therefore $\int_{A(t)}\big(x_i - v_i(t)\big)d\mu$ is approximately parallel to $(1/\alpha_1,\dots,1/\alpha_n)$ when λ_k is large.

Step 3: We show that $\sum_{i\in N}\alpha_i v_i'(t) \geq 0$ for large λ. Let $(\lambda_k)_{k=1,2,\dots}$ be the sequence defined in Step 2. For large λ_k, $A(t)$ again looks like a polyhedron with the uniform distribution. By Step 2, the ODE near $v_i(t)$ is written as

$$v_i'(t) = -\rho v_i(t) + \lambda_k \frac{z_i(t)}{n+1} \cdot \mu(A(t)). \tag{B.9}$$

Note that $v_i(t)$ is close to $v_i^*(t)$ and $\mu(A(t))$ is order n of the length of $z(t)$. By replacing the above equation by $r(t)$, ODE (B.9) approximates

$$r'(t) = \rho a - \lambda_k b r(t)^{n+1} \tag{B.10}$$

for some constants $a, b > 0$. Since $r(t)$ is large when t is small, the above ODE shows that $r(t)$ is decreasing in t. Therefore $\mu(A(t))$ is also decreasing in t. For large λ_k, this implies that

$$\alpha \cdot v'(t) = \sum_{i \in N} \alpha_i v_i'(t) \geq 0.$$

Step 4: We show that the Nash product is nondecreasing if λ is large. By ODE (B.9), we have

$$\alpha_i v_i'(t) = -\rho \alpha_i v_i(t) + \beta \tag{B.11}$$

where $\beta = \lambda_k \mu(A(t))/(n+1)$ independent of i. Let us assume without loss of generality that $\alpha_1 v_1'(t) \geq \cdots \geq \alpha_n v_n'(t)$. Then we must have $1/\alpha_1 v_1(t) \geq \cdots \geq 1/\alpha_n v_n(t)$.

Let $L(t) = \sum_{i \in N} \ln v_i(t)$ be a logarithm of the Nash product. Then $L'(t) = \sum_{i \in N} v_i'(t)/v_i(t)$. By Chebyshev's sum inequality,

$$\begin{aligned} L'(t) &= \sum_{i \in N} \frac{v_i'(t)}{v_i(t)} \\ &\geq \frac{1}{n} \Big(\sum_{i \in N} \alpha_i v_i'(t) \Big) \Big(\sum_{i \in N} \frac{1}{\alpha_i v_i(t)} \Big) \geq 0. \end{aligned}$$

Hence, $L(t)$ is nondecreasing if λ_k is large. Moreover, equality holds if and only if $\alpha_1 v_1'(t) = \cdots = \alpha_n v_n'(t)$ or $\alpha_1 v_1(t) = \cdots = \alpha_n v_n(t)$.

Step 5: We show that $v(t)$ converges to a point in the Nash set as $\lambda \to \infty$. Step 4 shows that $L'(t)$ converges to zero as $\lambda \to \infty$. Then $\alpha_1 v_1'(t) = \cdots = \alpha_n v_n'(t)$ or $\alpha_1 v_1(t) = \cdots = \alpha_n v_n(t)$ in the limit of $\lambda \to \infty$. The former case implies $v_i'(t) = 0$ for all $i \in N$ by Step 3. Then ODE (B.11) shows that the latter case holds. Therefore the latter case always holds in the limit of $\lambda \to \infty$. This implies that the boundary of X at $v^*(t)$ is tangent to the hypersurface defined by "Nash product $= \prod_{i \in N} v_i^*(t)$." Hence any accumulation point $v^*(t)$ belongs to the Nash set.

Since we assumed that the Nash set consists of isolated points, $v^*(t)$ is isolated. If $v(t)$ does not converge to $v^*(t)$, there is $\delta > 0$ such that for any $\bar{\lambda}$ there exists $v(t)$ with $\lambda \geq \bar{\lambda}$.

Let $\delta > 0$ be small such that there is no point in the Nash set in $\{x \in X \,|\, |v^*(t) - x| \leq \delta\}$. Since $v(t)$ is continuous with respect to λ, for any $\bar{\lambda}$, there exists $\lambda > \bar{\lambda}$ such that $\delta/2 \leq |v^*(t) - v(t)| \leq \delta$. Since $\{x \in X \,|\, \delta/2 \leq |v^*(t) - x| \leq \delta\}$ is compact, $v(t)$ must have an accumulation point in this set. This contradicts the fact that any accumulation point is contained in the Nash set. Furthermore, $v^*(t)$ does not depend on t since $v^*(t)$ is continuous in t.

B.13 Proof of Proposition 15

(Sketch of proof): The ODE (B.10) is approximated by a linear ODE, which has a solution converging to v^* with an exponential speed. Therefore for large λ, $r(t)$ is approximated by $r(t) = \left(\frac{\rho a}{\lambda b}\right)^{\frac{1}{n+1}}$. Since $\mu(A(t))$ is proportional to $r(t)^n$, $\mu(A(t)) = c\lambda^{-\frac{n}{n+1}}$ for a constant $c > 0$. the probability that players reach an agreement before time $-(T-s)$ is

$$1 - e^{-\int_{T-s}^{T} \mu(A(t))\lambda dt} = 1 - e^{-sc\lambda^{\frac{1}{n+1}}},$$

which converges to one as $\lambda \to \infty$.

B.14 Proof of Proposition 16

By equation (12), $v_i(\frac{t}{\Delta t})$ is a nondecreasing sequence. Since X is bounded and convex, $v_i(\frac{t}{\Delta t})$ converges to a Pareto efficient allocations as $\Delta t \to 0$. Let $v^*(\frac{t}{\Delta t})$ be the solution of equation (12), and $v^* = \lim_{\Delta t \to 0} v^*(\frac{t}{\Delta t})$ for $t > 0$.

The proof proceeds basically on the same route as that in Proposition 6. Let $f_h(\frac{t}{\Delta t}), f_L(\frac{t}{\Delta t})$, be defined as in the proof of Proposition 6. Then a parallel argument to Step 1 shows an inequality analogous to (B.6): For large $\bar{t}$,

$$\frac{\pi(\Delta t) f_L(\frac{\bar{t}}{\Delta t})}{n(n+1)} \Big(\frac{\alpha \cdot (v^* - v^*(\frac{t}{\Delta t})) - \eta}{\alpha_i}\Big) \prod_{j \in N} \Big(\frac{\alpha \cdot (v^* - v^*(\frac{t}{\Delta t})) - \eta}{\alpha_j}\Big)$$
$$\leq v_i^*\Big(\frac{t}{\Delta t} + 1\Big) - v_i^*\Big(\frac{t}{\Delta t}\Big) \leq \frac{\pi(\Delta t) f_H(\frac{\bar{t}}{\Delta t})}{n(n+1)} \Big(\frac{\alpha \cdot (v^* - v^*(\frac{t}{\Delta t})) + \eta}{\alpha_i}\Big) \prod_{j \in N} \Big(\frac{\alpha \cdot (v^* - v^*(\frac{t}{\Delta t})) + \eta}{\alpha_j}\Big),$$

where notations are the same as in the proof of Proposition 6. Therefore we have $\lim_{\Delta t \to 0} \dfrac{v_j^*(\frac{t}{\Delta t} + 1) - v_j^*(\frac{t}{\Delta t})}{v_i^*(\frac{t}{\Delta t} + 1) - v_i^*(\frac{t}{\Delta t})} = \dfrac{\alpha_i}{\alpha_j}$. Similar computations as in Step 2 show an approximation for small Δt

$$\Big(v_i^* - v_i^*\Big(\frac{t}{\Delta t} + 1\Big)\Big) - \Big(v_i^* - v_i^*\Big(\frac{t}{\Delta t}\Big)\Big) \approx -\frac{\pi(\Delta t) f_m(\frac{\bar{t}}{\Delta t})}{n(n+1)} \Big(\prod_{j \neq i} \frac{\alpha_i}{\alpha_j}\Big) \Big(v_i^* - v_i^*\Big(\frac{t}{\Delta t}\Big)\Big)^{n+1},$$

where $f_m(\frac{t}{\Delta t})$ is the average density in $A(\frac{t}{\Delta t})$. Then we can show that

$$\lim_{\Delta t \to 0} \alpha_i \Big(v_i^* - v_i^*\Big(\frac{t}{\Delta t}\Big)\Big) \cdot \Big(\frac{\pi(\Delta t)t}{\Delta t}\Big)^{\frac{1}{n}} = \Big(\frac{n+1}{f(v^*)n^{n+1}} \prod_{j \in N} \alpha_j\Big)^{\frac{1}{n}}.$$

Here we used the fact that Δt is very small when compared to $\pi(\Delta t)$ if Δt is small, to ignore the constant derived from an initial condition.

A similar computation to Step 3 shows that

$$\lim_{\Delta t \to 0} p(t) \cdot \Big(\frac{\pi(\Delta t)t}{\Delta t}\Big) = \frac{n+1}{n^2},$$

and thus the limit expected duration is $D(\infty) = \frac{n^2}{n^2+n+1}$.

B.15 Proof of Proposition 17

(Sketch of proof): The approximated ODE (B.10) for large t in the proof of Proposition 14 is rearranged as follows:

$$\lambda^{\frac{1}{n}} r'(t) = \lambda^{\frac{1}{n}} \rho a - b \cdot \big(\lambda^{\frac{1}{n}} r(t)\big)^{n+1}$$

If $\lambda^{\frac{1}{n}} \rho \to 0$, this ODE is approximated as

$$\lambda^{\frac{1}{n}} r'(t) \approx -b\big(\lambda^{\frac{1}{n}} r(t)\big)^{n+1}$$

which yields $r(t) = O\Big(\frac{1}{(\lambda t)^{\frac{1}{n}}}\Big)$. This is the same case with $\rho = 0$. On the other hand, If $\lambda^{\frac{1}{n}} \rho \to \infty$, the ODE is approximated as

$$\begin{aligned} \lambda^{\frac{1}{n}} r'(t) &= \big((\lambda^{\frac{1}{n}} \rho a)^{\frac{1}{n+1}} - b^{\frac{1}{n+1}} (\lambda^{\frac{1}{n}} r(t))\big) \\ &\quad \cdot \big((\lambda^{\frac{1}{n}} \rho a)^{\frac{n}{n+1}} + (\lambda^{\frac{1}{n}} \rho a)^{\frac{n-1}{n+1}} \cdot b^{\frac{1}{n+1}} (\lambda^{\frac{1}{n}} r(t)) + \cdots + b^{\frac{n}{n+1}} (\lambda^{\frac{1}{n}} r(t))^n\big) \\ &\approx (\lambda^{\frac{1}{n}} \rho a) - (\lambda^{\frac{1}{n}} \rho a)^{\frac{n}{n+1}} \cdot b^{\frac{1}{n+1}} (\lambda^{\frac{1}{n}} r(t)) \end{aligned}$$

which implies $r(t) = \big(\frac{\rho a}{\lambda b}\big)^{\frac{1}{n+1}} - O(e^{-t})$. This corresponds to the case with $\rho > 0$.

B.16 Proof of Proposition 23

By symmetry, $v_1^*(t) = v_2^*(t)$ and $v^* = (1/2, 1/2)$. Let $z(t) = v_i^* - v_i^*(t)$. Suppose that t is large and $z(t)$ is small, so that $z(t) \leq \frac{1-a}{2(1+a)}$. It is straightforward to see that an agreement is reached after negotiation with a costly transfer if and only if realized allocation $x \in X$ is in the triangle $T_1 \cup T_2 \cup T_3$ shown in Figure 17, where the slopes of the line segments are $-a, a, 1/a, -1/a$, respectively, from southeast to northwest.

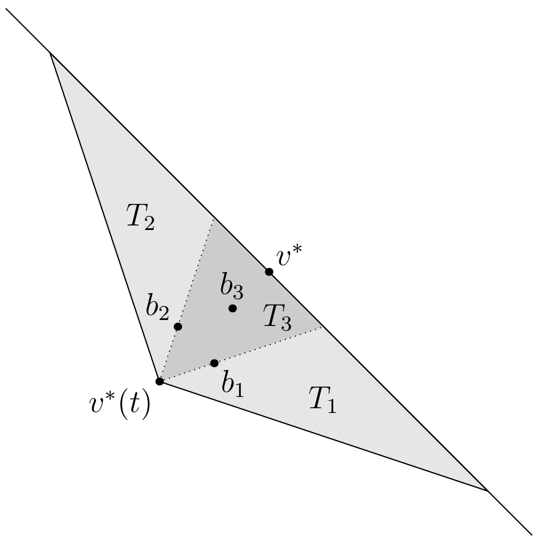

Figure 17: The set of realized allocations that the players accept

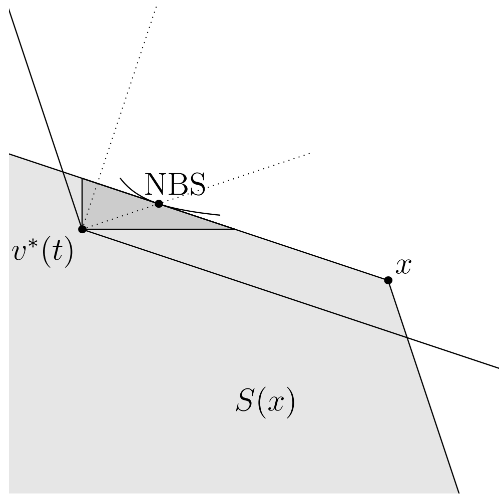

Figure 18: The set of feasible allocations when $x \in T_1$ is realized

Suppose that realized allocation x belongs to the triangle T_1 in Figure 17. Then the set $S(x)$ of feasible allocations is described in Figure 18. Since the disagreement point is at $v^*(t)$, the Nash bargaining solution (NBS) is located on the borderline between T_1 and T_3. Therefore the ex post distribution of payoff profiles on agreement has a mass on the line segment between T_1 and T_3, and the barycenter b_1 of the mass is the intersection point between the line segment and the line drawn through the barycenter of T_1 with slope $-a$. The symmetric argument applies to the case of $x \in T_2$, and the barycenter b_2 of the mass on the borderline between T_2 and T_3 is computed correspondingly.

If x belongs to T_3, the Nash bargaining solution is x itself. The the barycenter b_3 of the set of ex post payoff profiles conditional on the realized allocation x being contained in T_3 is exactly the barycenter of T_3. A computation shows that

$$b_1 = v(t) + \Big(\frac{2}{3(1+a)}, \frac{2a}{3(1+a)}\Big) z(t), \quad b_2 = v(t) + \Big(\frac{2a}{3(1+a)}, \frac{2}{3(1+a)}\Big) z(t),$$

$$b_3 = v(t) + \Big(\frac{2}{3}, \frac{2}{3}\Big) z(t),$$

$$\mu(T_1) = \mu(T_2) = \frac{8a}{1-a^2} z(t)^2, \quad \mu(T_3) = \frac{2(1-a)}{1+a} z(t)^2.$$

Therefore the barycenter of the entire set of ex post payoff profiles is computed as a convex combination of b_1, b_2, b_3. By ODE (1),

$$
\begin{aligned}
z'(t) &= -v_1'(t) \\
&= -\lambda\big((b_1 - v(t))\mu(T_1) + (b_2 - v(t))\mu(T_2) + (b_3 - v(t))\mu(T_3)\big) \\
&= -\lambda \cdot \frac{8(1+a^2)}{3(1-a^2)} z(t)^3.
\end{aligned}
$$

Since $p(t) = \mu(T_1) + \mu(T_2) + \mu(T_3) = \dfrac{4(1+a)}{1-a} z(t)^2$,

$$
\begin{aligned}
p'(t) &= \frac{8(1+a)}{1-a} z(t) z'(t) \\
&= \lambda \cdot \frac{4(1+a^2)}{3(1+a)^2} p(t)^2.
\end{aligned}
$$

Therefore the constant r defined in Section 4.2.2 is $\frac{4(1+a^2)}{3(1+a)^2}$. By Theorem 4, the limit duration is

$$
\begin{aligned}
D(\infty) &= \frac{1}{1 + \left(\frac{4(1+a^2)}{3(1+a)^2}\right)^{-1}} \\
&= \frac{4 + 4a^2}{7 + 6a + 7a^2}.
\end{aligned}
$$

References

Abdelaziz, F. and S. Krichen (2007): "Optimal stopping problems by two or more decision makers: a survey," *Computational Management Science*, 4, 89–111.

Albrecht, J., A. Anderson, and S. Vroman (2010): "Search by committee," *Journal of Economic Theory*, 145, 1386–1407.

Alpern, S. and S. Gal (2009): "Analysis and design of selection committees: a game theoretic secretary problem," *International Journal of Game Theory*, 38, 377–394.

Alpern, S., S. Gal, and E. Solan (2010): "A sequential selection game with vetoes," *Games and Economic Behavior*, 38, 1–14.

Ambrus, A. and J. Burns (2010): "Gradual bidding in ebay-like auctions," Mimeo.

Ambrus, A. and S.-E. Lu (2010a): "A continuous-time model of multilateral bargaining," Mimeo.

——— (2010b): "Legislative bargaining with long finite horizons," Mimeo.

Bergemann, D. and J. Välimäki (2011): "Efficient Search by Committee," Mimeo.

Calcagno, R. and S. Lovo (2010): "Preopening and Equilibrium Selection," Mimeo.

Cho, I.-K. and A. Matsui (2011): "A Search Theoretic Foundation of Nash Bargaining Soluiton," Mimeo.

Coddington, E. A. and N. Levinson (1955): *Theory of Ordinary Differential Equations*, McGraw-Hill.

Compte, O. and P. Jehiel (2004): "Bargaining over Randomly Generated Offers: A new perspective on multi-party bargaining," Mimeo.

——— (2010): "Bargaining and Majority Rules: A Collective Search Perspective," *Journal of Political Economy*, 118, 189–221.

Ferguson, T. S. (1989): "Who solved the secretary problem?" *Statistical Science*, 4, 282–296.

——— (2005): "Selection by Committee," in *Advances in Dynamic Games*, ed. by A. S. Nowak and K. Szajowski, Birkhäuser Boston, vol. 7 of *Annals of the International Society of Dynamic Games*, 203–209.

Gomes, A., S. Hart, and A. Mas-Colell (1999): "Finite Horizon Bargaining and the Consistent Field," *Games and Economic Behavior*, 27, 204–228.

Herrero, M. J. (1989): "The Nash Program: Non-Convex Bargaining Problems," *Journal of Economic Theory*, 49, 266–277.

Imai, H. and H. Salonen (2011): "A characterization of a limit solution for finite horizon bargaining problems," *International Journal of Game Theory*, forthcoming.

Ishii, Y. and Y. Kamada (2011): "The Effect of Correlated Inertia on Coordination," Mimeo.

Kamada, Y. and M. Kandori (2009): "Revision Games," Mimeo.

——— (2011): "Asynchronous Revision Games," Mimeo.

Kamada, Y. and N. Muto (2011a): "Delay in Multi-Agent Search with Time Costs," Mimeo.

——— (2011b): "Multi-Agent Search with Deadline," Mimeo.

——— (2011c): "Multi-Agent Search with Recall," Mimeo.

Kamada, Y. and T. Sugaya (2010a): "Asynchronous Revision Games with Deadline: Unique Equilibrium in Coordination Games," Mimeo.

——— (2010b): "Policy Announcement Game: Valence Candidates and Ambiguous Policies," Mimeo.

Lippman, S. A. and J. J. McCall (1976): "The Economics of Job Search: A Survey," *Economic Inquiry*, 14, 155–189.

Ma, C.-T. A. and M. Manove (1993): "Bargaining with Deadlines and Imperfect Player Control," *Econometrica*, 61, 1313–1339.

Maschler, M., G. Owen, and B. Peleg (1988): "Paths leading to the Nash set," in *The Shapley value: Essays in Honor of Lloyd S. Shapley*, ed. by A. E. Roth, Cambridge University Press, 321–330.

McCall, J. J. (1970): "Economics of Information and Job Search," *The Quarterly Journal of Economics*, 84, 113–126.

Moldovanu, B. and X. Shi (2010): "Search Committees," Mimeo.

Mortensen, D. T. (1970): "Job Search, the Duration of Unemployment, and the Phillips Curve," *The American Economic Review*, 60, 847–862.

Nash, J. F. (1953): "Two-Person Cooperative Games," *Econometrica*, 21, 128–140.

Osborne, M. J. and A. Rubinstein (1990): *Bargaining and Markets*, Academic Press.

Rogerson, R., R. Shimer, and R. Wright (2005): "Search-Theoretic Models of the Labor Market: A Survey," *Journal of Economic Literature*, 43, 959–988.

Romm, A. (2011): "Reputation in Revision Games," Mimeo.

Rubinstein, A. (1982): "Perfect Equilibrium in a Bargaining Model," *Econometrica*, 50, 97–109.

Sakaguchi, M. (1973): "Optimal Stopping in Sampling from a Bivariate Distribution," *Journal of the Operations Research Society of Japan*, 16, 186–200.

——— (1978): "When to Stop: Randomly Appearing Bivariate Target Values," *Journal of the Operations Research Society of Japan*, 21, 45–58.

Stinchcombe, M. B. (1992): "Maximal Strategy Sets for Continuous-Time Game Theory," *Journal of Economic Theory*, 56, 235–265.

Wilson, C. A. (2001): "Mediation and the Nash bargaining solution," *Review of Economic Design*, 6, 353–370.