When Optimists Need Credit: Asymmetric Filtering of Optimism and Implications for Asset Prices

Alp Simsek*
MIT

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Abstract

Heterogeneity of beliefs has been suggested as a major contributing factor to the recent financial crisis. This paper theoretically evaluates this hypothesis. Similar to Geanakoplos (2009), I assume that optimists have limited wealth and take on leverage in order to take positions in line with their beliefs. To have a significant effect on asset prices, they need to borrow from traders with moderate beliefs using loans collateralized by the asset itself. Since moderate lenders do not value the collateral as much as optimists do, they are reluctant to lend, providing an endogenous constraint on optimists' ability to leverage and to influence asset prices. I demonstrate that optimism concerning the likelihood of bad events has no or little effect on asset prices because these types of optimism are disciplined by this constraint. Instead, optimism concerning the relative likelihood of good events could have significant effects on asset prices. At the root of this result is the general characterization of the effect of belief heterogeneity that I provide, which shows that optimism is asymmetrically filtered in the sense that the asset is priced according to a mixture of moderate and optimistic beliefs: the moderate beliefs are used to assess the likelihood of default events, while the optimistic beliefs are used to assess the conditional likelihood of non-default events. These results emphasize that what investors disagree about matters for asset prices, to a greater extent than the level of disagreement.

I then use a dynamic extension to show how this type of belief heterogeneity interacts with the speculative component of asset prices identified in Harrison and Kreps (1978). When optimists have limited wealth, belief heterogeneity can lead to speculative asset price "bubbles" but only if it concerns the relative likelihood of good events. The asymmetric filtering characterization shows that the size of the bubble depends on the skewness of belief heterogeneity. This result also shows that bubbles can come to an end because of a shift in belief heterogeneity towards the likelihood of bad events.

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1 Introduction

Belief heterogeneity and optimism have been suggested as contributing factors to the recent financial crisis. Shiller (2005), Reinhart and Rogoff (2008) and Gorton (2008), along with many other commentators, have identified the optimism of a fraction of investors as a potential cause for the increase in housing and complex security prices in the run-up to the crisis. As noted by Geanakoplos (2009), for optimism of a group of traders to have a significant effect on asset prices, they need to leverage their investments by borrowing from traders with more moderate beliefs using loans collateralized by the asset itself (e.g., mortgages, REPOs, or asset purchases on margin). However, as moderate lenders value the collateral (the asset) less than optimists do, they may be reluctant to lend, which represents an endogenous constraint on optimists' ability to leverage and influence asset prices. The purpose of this paper is to understand the implications of this constraint for asset prices. Using an equilibrium model in the asset and the loan market, I show that certain types of optimism, specifically those concerning likelihood of bad events, have no or little effect on asset prices because they are disciplined by financial constraints resulting from belief heterogeneity. Instead, optimism concerning the relative likelihood of good events could have significant effects on asset prices as these types of optimism are unchecked by these constraints.

To isolate the effect of belief heterogeneity on asset prices, I consider a stylized model in which a single asset is traded among risk-neutral traders with heterogeneous prior beliefs about the asset's dividend yield. The model is simplified by assuming that there are only two belief types of traders, optimists and moderates, but each type is allowed to have a general belief distribution over a continuum of future states. Optimists value the asset more, but they do not have sufficient wealth, and they purchase the asset by taking loans from moderate traders. A central feature of the model is that optimists are constrained in their borrowing because loans need to be collateralized. In particular, a debt contract consists of a promise of payment and a pledge of collateral, and loans are non-recourse in the sense that, if the borrower defaults on her promise, the court system enforces the transfer of collateral to the lender while the borrower is not punished beyond the loss of collateral. The loan market is analyzed through a competitive equilibrium notion, collateral equilibrium, developed in Geanakoplos and Zame (1997, 2009). In particular, each debt contract is traded in an anonymous market at competitive equilibrium prices.[1]

This paper considers variants of this basic setup that differ in the types of collateralized contracts that are available for trade. In the baseline version of the model, I restrict attention to non-contingent loans that promise the same amount in all future states, and I also rule out the short selling of the asset. This baseline version is arguably a good starting point, because collateralized loans (e.g., mortgages, REPOs) typically do not have many contingencies in their payoffs; and short selling of many assets other than stocks (and some stocks) is difficult and

[1] As the payment is only enforced by collateral, the lenders pay attention to the value of collateral and they need not know about the identity of the borrower, which ensures the anonymity of the market.

costly.[2]

The main result of this paper characterizes the price of the asset in the baseline model, and shows that the asset is priced according to a mixture of moderate and optimistic beliefs: the moderate beliefs are used to assess the likelihood of default states, while the optimistic beliefs are used to assess the conditional likelihood of non-default states. More precisely, the asset price can be written as

$$p = \frac{1}{1+r} \left(\Pr_{\text{moderate}} \left[v < \bar{v} \right] \ \mathrm{E}_{\text{moderate}} \left[v \mid v < \bar{v} \right] + \Pr_{\text{moderate}} \left[v \geq \bar{v} \right] \ \mathrm{E}_{\text{optimistic}} \left[v \mid v \geq \bar{v} \right] \right), \tag{1}$$

where r is the interest rate on a benchmark asset, the random variable v captures the future value of the asset, and $\bar{v}$ is the default threshold value, that is, collateralized loans in this economy default when the asset value v falls below $\bar{v}$. The notation $\Pr_{\text{moderate}} [v < \bar{v}]$ captures the probability of the event $\{v < \bar{v}\}$ with respect to the moderate beliefs, and $\mathrm{E}_{\text{optimistic}} [v \mid v \geq \bar{v}]$ captures the expected value of the asset conditional on the event $\{v \geq \bar{v}\}$ with respect to the optimistic beliefs.

The expression in (1) illustrates that optimism is *asymmetrically filtered*. In particular, optimism concerning the likelihood of default states (i.e., future states that lead to $v < \bar{v}$) has no effect on the asset price because moderate beliefs are used to assess the likelihood of these states, along with the value of the asset conditional on these states. In contrast, optimism concerning the relative likelihood of non-default states (i.e., future states that lead to $v > \bar{v}$) has a significant effect on the asset price, because optimistic beliefs are used to assess the value of the asset conditional on these states. This asymmetric filtering result is robust to allowing for more general collateralized loans and short selling. In particular, Sections 6 and 7 of this paper show that the asset price in these more general settings can also be represented with an expression similar to (1). While the details of the expressions depend on the type of the contracts available for trade, it remains true that optimism about bad states is filtered more than optimism concerning the relative likelihood of good states.

The intuition for the asymmetric filtering result is related to the asymmetry in the shape of the debt contract payoffs. These contracts make the same full payment in non-default states, but they make losses in default states. Consequently, any disagreement about the probability of default states translates into a disagreement about how to value the debt contracts, which in turn tightens optimists' financial constraints. In contrast, disagreements about the relative likelihood of non-default states do not tighten the financial constraints.

More specifically, collateralized loans always trade at an interest rate with a *spread* over the benchmark interest rate, which compensates the lenders for expected losses in case of default.

[2]Short selling constraints on stocks has recently been the subject of much empirical work, e.g., by Jones and Lamont (2001), D'Avolio (2002), Ofek and Richardson (2002), Lamont and Stein (2004) and Asquith, Pathak, and Ritter (2005). This literature has generally emphasized that, while short selling stocks is not too costly in general, certain stocks (and in certain periods) are quite expensive to short sell. Some of the papers have also noted that the demand for short selling, e.g., as measured by shares sold short, remains low despite the low levels of short selling fees, suggesting that short selling of stocks might be constrained for alternative reasons.

Moreover, in a competitive loan market, the spread on a loan is just enough to compensate the lenders for their expected losses according to their moderate beliefs. Nonetheless, when the disagreements are about the probability of default states, this spread appears too high to optimists. This is because, in this case, optimists assign a lower probability to the default event than moderates do, and thus they find it more likely that they will pay the spread. Therefore, optimists believe they are paying a higher expected interest rate than the benchmark interest rate, which discourages them from leveraging their investment in the asset. This lowers optimists' demand for the asset and leads to an equilibrium price closer to the moderate valuation. In contrast, when the disagreements are about the relative likelihood of non-default states, the spread appears fair to optimists, and they are enticed to leverage their investment more. This increases their demand for the asset and leads to an equilibrium price closer to the optimistic valuation.

The asymmetric filtering characterization of asset prices lends itself to a number of comparative statics results regarding the effect of a change in the *skewness* and the *level* of belief heterogeneity. I use the term right-skewed (resp. left-skewed) optimism to capture the type of optimism concerning the relative likelihood of good states (resp. the likelihood of bad states). I formalize this notion of skewness with a condition on belief distributions such that a distribution with more right-skewed optimism leads to a greater valuation of the asset conditional on good states, while having the same unconditional valuation. An increase in this type of right-skewness of belief heterogeneity unambiguously increases the asset price, because a given level of optimism is filtered less by financial constraints when it is more right-skewed. In contrast, an increase in the level of belief heterogeneity does not offer general robust predictions on the asset price. This is because while an increase in optimists' optimism tends to increase the price, an increase in moderate lenders' pessimism tends to decrease the price through the tightening of financial constraints. However, an increase in belief heterogeneity offers robust predictions once the skewness of the additional belief heterogeneity is also taken into account. In view of the asymmetric filtering result, the asset price increases if the increase in belief heterogeneity is concentrated on non-default states, while it decreases if the increase is concentrated on default states. These results suggest that *what investors disagree about* matters for asset prices, to a greater extent than the level of their disagreement.

Dynamic Extension and Implications for Speculative Bubbles. I consider a dynamic extension of the baseline model to show how the asymmetric filtering result interacts with the speculative component of asset prices identified in Harrison and Kreps (1978). In a dynamic economy in which the identity of optimists changes over time, a speculative phenomenon obtains as the current optimists purchase the asset not only because they believe it will yield greater dividend returns, but also because they expect to make capital gains by selling the asset to future optimists. The asset price exceeds the present discounted valuation of the asset with respect to the beliefs of *any* trader because of the resale option value introduced by the speculative trading motive. As Scheinkman and Xiong (2003) note, this resale option value may be reasonably called a "speculative bubble." This setup is the starting point of the dynamic extension, which introduces the additional element of optimists' financial constraints. The dynamic model reveals that, when optimists need to purchase the asset by borrowing from moderate lenders, belief heterogeneity can lead to speculative asset price bubbles, but only if it concerns the relative likelihood of non-default states. When this is the case, however, the resale option value can increase the size of the speculative component of the asset price considerably because large positions can be financed by credit collateralized by the speculative asset. This is because moderate lenders' valuation, as well as optimists' valuation, features a speculative component. Put differently, in a speculative episode, moderate lenders agree to finance optimists' purchase of the asset by extending large loans because they think, should the optimist default on the loan, they can sell the collateral (the asset) to another optimist in the next period. The asymmetric filtering characterization shows that the size of the bubble depends on the skewness of belief heterogeneity. This result also shows that bubbles can come to an end because of a shift in belief heterogeneity towards the likelihood of default states.

While the baseline model with non-contingent loans and limited short selling is a good starting point, it is important to verify the robustness of the asymmetric filtering characterization to more general settings, especially because allowing for a richer set of contracts introduces new economic forces. I show that slightly modified versions of the asymmetric filtering result continue to apply in two extensions of the baseline model, one in which collateralized loans can be fully contingent and one in which short selling is allowed.

Extension with Contingent Loans. The optimal contingent loan (collateralized by one unit of the asset) is such that optimistic borrowers give up the asset completely if the state realization is below a threshold level, while paying nothing if the state is above the threshold. While this contract is different than a non-contingent loan, which would make a positive fixed payment in states above a threshold level, it has the same feature of making a *fixed payment* (namely, zero dollars) for all relatively good states. Consequently, optimism about the relative likelihood of good states does not lead to a heterogeneity in the valuation of the optimal contingent loans. It follows that these types of optimism do not tighten optimists' financial constraints, and thus they lead to a relatively higher asset price. In contrast, optimism about the relative likelihood of states below the threshold level tightens optimists' financial constraints and leads to a lower asset price.

Furthermore, in the setting with contingent loans, the asset price can exceed even the most optimistic valuation. Intuitively, the ability to fine-tune their borrowing enables optimists to take loans which they perceive to be even more favorable than borrowing at the benchmark interest rate. Consequently, optimists continue to demand the asset even if the price exceeds their valuation (which is calculated according to the benchmark rate). A complementary intuition is that the availability of fully contingent loans enables the splitting of the asset such that each type traders hold the asset in the states which they assign a greater probability. Consequently, the maximum price at which optimists demand the asset is the valuation according to the "upper-envelope" of the moderate and the optimistic probability densities. This upper-envelope valuation is greater than even the most optimistic valuation. This result creates a presumption that finer levels of financial engineering of loans can potentially have a large impact on asset prices.

Extension with Short Selling. This extension is particularly relevant for understanding the data for a fraction of assets that can be short sold (e.g., for the majority of stocks). The asymmetric filtering result continues to apply in this case. In particular, short selling takes away the general overvaluation of the asset, but it remains true that optimism about the relative likelihood of good states leads to a greater asset price than optimism about the likelihood of bad states. To see this, note that moderate traders that wish to short sell the asset need to borrow the asset from other (more optimistic) traders. Thus, short selling also needs to be collateralized, because loans in this economy are non-recourse. It follows that short sellers return the asset to the lenders if the state realization is below a threshold level, but they default on the short contracts if the state is above the threshold. Therefore, short contracts promise the same fixed payment (namely, the value of the posted collateral) in all relatively good states. Consequently, short sellers cannot bet on their pessimism for the relative likelihood of good states, because they are selling a contract that promises the same payment in these states (put differently, they would have to post very high levels of collateral to promise different amounts in relatively good states). Hence, when the belief heterogeneity is about the relative likelihood of good states, the asset price is closer to the optimistic valuation. In contrast, short sellers can more easily bet on their pessimism for the likelihood of bad states. Thus, with these types of belief heterogeneity, the asset price is closer to the moderate valuation.

Related Literature. The closest work to my paper is by Geanakoplos (2009), who considers the determination of leverage and asset prices in a model with two continuation states and traders with a continuum of belief types. In contrast, I consider a model with a continuum of continuation states and traders with two belief types (optimists and moderates). My assumptions are relevant for understanding a range of situations, including the effect of different types of belief disagreements on asset prices, leverage, and the default frequency of equilibrium loans. In particular, while Geanakoplos (2009) illustrates that an increase in belief heterogeneity can decrease asset prices considerably, my paper shows that an increase in the level of belief heterogeneity generally has ambiguous effects on asset prices, and identifies the skewness of belief heterogeneity as an important determinant of asset prices. In the model considered by Geanakoplos (2009), the increase in the level of heterogeneity decreases asset prices because the additional heterogeneity is concentrated on default states. An increase in the level of heterogeneity in that model would rather increase asset prices if the additional heterogeneity were concentrated on good states. Moreover, in the two state model analyzed in Geanakoplos

(2009), loans that are traded in equilibrium are always fully secured with respect to the worst case scenario, i.e., there is no default. This feature makes it impossible to analyze the effect of belief heterogeneity on the default frequency and riskiness of equilibrium loans, which is one of the topics that I consider. In addition, my paper extends the model in Geanakoplos (2009) by allowing for short selling, and characterizes the effect of belief heterogeneity in this more general setting.[3]

My paper is also related to the overvaluation hypothesis, originated by Miller (1977), which posits that belief heterogeneity and limited short selling leads to an overvaluation of an asset (relative to the average valuation of the population) because the asset is held by the most optimistic investors. A central implication of this mechanism is that an increase in investors' belief heterogeneity should increase the asset price. This implication has been emphasized and empirically tested by a growing literature in finance, e.g., Chen, Hong and Stein (2002), Diether, Malloy and Scherbina (2002) and Ofek and Richardson (2003). As noted above, my paper emphasizes the skewness of the heterogeneity as a more robust predictor of the asset price than the level of the heterogeneity. My paper also relates to the literature on speculative asset price bubbles. Harrison and Kreps (1978), Morris (1996) and Scheinkman and Xiong (2002) consider the overvaluation mechanism in a dynamic setting, and they show that belief heterogeneity and short selling restrictions can also lead to a speculative component in asset prices. My paper shows that, when optimists have limited wealth, only certain types of belief heterogeneity can lead to a speculative component in asset prices, and shifts in the type of belief heterogeneity can generate large fluctuations in the speculative component without any apparent change in investors' valuations.

There is a large literature that concerns the plausibility of the heterogeneous priors assumption in financial markets. The market selection hypothesis, which goes back to Alchian (1950) and Friedman (1953), posits that investors with incorrect beliefs should be driven out of the market as they would consistently lose money. Thus, this hypothesis suggests that investors that remain in the long run should have accurate (and common) beliefs. Recent research has emphasized that the market selection hypothesis does not apply for incomplete markets, that is, traders with inaccurate beliefs may have a permanent presence when asset markets are incomplete.[4] Of particular interest for my paper is the work by Cao (2009), who considers a similar economy in which markets are endogenously incomplete because of collateral constraints. Cao (2009) shows that belief heterogeneity in this economy remains in the long run, thus providing theoretical support for my central assumptions. Another strand of literature

[3] Other related papers that concern the endogenous determination of leverage include Bernanke, Gertler and Gilchrist (1996, 1998), Geanakoplos (1997, 2003), Geanakoplos and Zame (1997, 2009), Fostel and Geanakoplos (2008), and Brunnermeier and Pedersen (2009). In addition, a large literature concerns the effect of collateral constraints on asset prices, e.g. Kiyotaki and Moore (1997) and Caballero and Krishnamurty (2001). On the empirical side, a number of recent studies document the variation in leverage and its effect on asset prices (see, for example, Adrian and Shin, 2009).

[4] See, for example, DeLong, Shleifer, Summers and Waldman (1990,1991), Blume and Easley (1992, 2006), Sandroni (2001), Cao (2009).

concerns whether investors' Bayesian learning dynamics would eventually lead to accurate, and thus common, beliefs. Recent work (e.g., by Acemoglu, Chernozhukov and Yildiz, 2009) has emphasized the limitations of Bayesian learning in generating long run agreement.[5]

Outline. The organization of the rest of this paper is as follows. Section 2 introduces the baseline version of the model and defines the collateral equilibrium. Section 3 characterizes the collateral equilibrium and presents the asymmetric filtering result. Section 4 establishes the comparative statics of collateral equilibrium, including the characterization of the effect of the skewness and the level of belief heterogeneity on asset prices. Section 5 introduces the dynamic extension and presents the results for the speculative asset price bubbles. Section 6 considers an extension of the baseline static model in which loans are allowed to be fully contingent and shows that the asymmetric filtering result generalizes to this setting. Section 7 presents an extension with short selling and generalizes the asymmetric filtering result to this setting. Section 8 concludes. The paper ends with several appendices that present the proofs omitted from the main text.

2 Environment and Equilibrium

Consider a two period economy with a single numeraire good in which a continuum of risk neutral traders have endowments in period 0 but they need to consume in period 1. The resources can be transferred between periods by investing either in a risk-free bond, denoted by B, or a risky asset, denoted by A. Bond B is supplied elastically at a normalized price 1 in period 0. Each unit of the bond yields $1+r$ units of the numeraire good in period 1. Asset A is in fixed supply, which is normalized to 1. The asset pays dividend only once (in units of the numeraire good), and it pays it in period 1. The dividend payment of each unit of the asset is denoted by $v(s)$. Taking the set of all possible states as $\mathcal{S} = \left[s^{\min}, s^{\max}\right] \subset \mathbb{R}$, I assume that the function $v : \mathcal{S} \rightarrow \mathbb{R}_{++}$ is strictly increasing and continuously differentiable.[6] I denote the price of the asset by p.

Traders have heterogeneous priors about the return of the asset. In particular, there are two types of traders, *optimists* and *moderates*, respectively denoted by subscript $i \in \{1, 0\}$, with corresponding prior belief about the next period state $s \in \mathcal{S}$ given by the probability distribution F_i. Traders know each others' priors, and thus optimists and moderates agree to disagree. I normalize the population measure of each type of traders to 1, and I let α_i (resp. w_i) denote type i traders' period 0 endowment of the asset (resp. the numeraire good).

[5] For further discussion on the merits of the common prior assumption in economic theory, see Bernheim (1986), Aumann (1986,1998), Varian (1989), Morris (1995), and Gul (1998).

[6] Note that the state space could be equivalently defined as $v(\mathcal{S}) = \left[v\left(s^{\min}\right), v\left(s^{\max}\right)\right]$ over asset payoffs, so the value function $v(\cdot)$ is redundant in this section. Put differently, without loss of generality, the value function can be taken to be the identity function $v(s) = s$. I introduce the value function $v(\cdot)$ because this will considerably simplify the analysis of the dynamic model in Section 5, in which the value function will be endogenously determined.

The asset endowments satisfy $\alpha_0 > 0$ and $\alpha_0 + \alpha_1 = 1$. An economy is denoted by the tuple $\mathcal{E} = (\mathcal{S};\ v(\cdot);\ \{F_i\}_i; \{w_i\}_i;\ \{\alpha_i\}_i)$.

The following definition formalizes the notion of optimism that will be adopted in this paper.

Definition 1 (Optimism Order). *Consider two probability distributions $H, \tilde{H}$ over $\mathcal{S} = [s^{\min}, s^{\max}]$ with density functions $h, \tilde{h}$ that are continuous and positive over $\mathcal{S}$. The distribution $\tilde{H}$ is* more optimistic *than H, denoted by $\tilde{H} \succ_O H$, if $\frac{1-\tilde{H}(s)}{1-H(s)}$ is strictly increasing over $(s^{\min}, s^{\max})$, equivalently, if the following hazard rate inequality is satisfied for all $s \in (s^{\min}, s^{\max})$:*

$$\frac{\tilde{h}(s)}{1-\tilde{H}(s)} < \frac{h(s)}{1-H(s)}. \tag{2}$$

The distribution $\tilde{H}$ is weakly more optimistic than H, denoted by $\tilde{H} \succeq_O H$, if $\frac{1-\tilde{H}(s)}{1-H(s)}$ is weakly increasing over $(s^{\min}, s^{\max})$.

Assumption (O). The probability distributions F_1 and F_0 have density functions f_1, f_0 that are continuous and positive over $\mathcal{S}$, and they satisfy $F_1 \succ_O F_0$.

Note that $F_1 \succ_O F_0$ implies that F_1 dominates F_0 in the first order stochastic sense, since $\frac{1-F_1(s)}{1-F_0(s)} > \frac{1-F_1(s^{\min})}{1-F_0(s^{\min})} = 1$ for each $s > s^{\min}$. But the optimism order in Definition 1 is stronger than first order stochastic dominance. It concerns optimists' relative probability assessment for the *upper-threshold events* $[s, s^{\max}] \subset \mathcal{S}$, and it posits that optimists are increasingly optimistic for these events as the threshold level s is increased. Intuitively, it captures the idea that, the "better" the event, the greater the optimism regarding the event. I adopt this notion of optimism partly because it is intuitive, but also because it will provide much tractability in the subsequent analysis. Note also that the optimism order is weaker than the monotone likelihood ratio property (MLRP), that is, if $\frac{f_1(s)}{f_0(s)}$ is strictly increasing over $\mathcal{S}$, then $F_1 \succ_O F_0$ (cf. Appendix A.1).

Let $E_i[\cdot]$ denote the expectation operator corresponding to type i traders' belief. Assumption (O) also implies $E_0[v(s)] < E_1[v(s)]$, that is, moderate traders value the asset less than optimists. This further implies that moderate traders would like to short sell the asset in this economy, which is ruled out by assumption.

Assumption (S). Asset A cannot be short sold.

This assumption will be maintained for most of the paper (until Section 7). In reality, many assets other than stocks, and also some stocks, are difficult and costly to short sell (see, e.g., Jones and Lamont, 2001).

Given assumption (S), if there were no financial frictions, i.e., if optimists could freely borrow and lend at the going interest rate $1 + r$, they would bid up the price of the asset to the optimistic valuation $\frac{E_1[v(s)]}{1+r}$. However, financial frictions may prevent optimists from increasing the asset price to this level. With financial frictions, in general, the asset will trade at a price in the interval

$$\left[\frac{E_0\left[v\left(s\right)\right]}{1+r}, \frac{E_1\left[v\left(s\right)\right]}{1+r}\right], \tag{3}$$

the exact location being determined by optimists' wealth and the type of the constraints.

2.1 Financial Frictions and Collateral Equilibrium

I introduce financial frictions using a competitive equilibrium notion, *collateral equilibrium*, originally developed by Geanakoplos and Zame (1997, 2009). Suppose that loans in this economy must be secured by collateral owned by the borrower, and the court system enforces the transfer of collateral to the lender in case the borrower does not pay.[7] Suppose also that loans are non-recourse, that is, the borrower does not get further punishment than potential loss of collateral.

More specifically, traders in this economy trade contracts that specify a promise of payment and a pledge of collateral. I make a couple of simplifying assumptions for the types of contracts that are traded in equilibrium. First, I restrict attention to *non-contingent* loans, that is, loans that promise the same payment in all continuation states $s \in \mathcal{S}$. Even though this assumption is restrictive from a theoretical point of view, it is a natural starting point because most non-recourse loans in reality are non-contingent. Second, note that, in principle, both the bond B and the asset A could be used as collateral. For expositional reasons, I suppose that only the asset can be pledged as collateral. This assumption is without loss of generality in this model.[8] Finally, note also that there is one degree of freedom in the contract space, so I normalize the contracts by assuming that each contract pledges one unit of the asset.

Formally, *a unit debt contract*, denoted by $\varphi \in \mathbb{R}_+$, is a promise of φ units of the numeraire good in period 1 by the borrower, collateralized by 1 unit of the asset A (which the borrower owns). In period 1, the borrower defaults on the unit debt contract φ if and only if the collateral value is less than the promised amount. Thus, each contract φ pays

$$\min\left(v\left(s\right), \varphi\right). \tag{4}$$

I analyze the loan market using a competitive equilibrium notion, in particular, each debt

[7] There is a potential question of who holds the collateral throughout the term of the loan contract, i.e. should the collateral be locked in a warehouse, held by the lender, or the borrower. In reality (e.g., in mortgages or REPOs), different variants are used intuitively depending on whether the borrower or the lender benefits more from holding the contract during the loan period. A common aspect of all variants of collateralized lending relationships is that the borrower must own the asset at the time of the loan payment. This aspect is necessary because otherwise the borrower would not have any incentive to pay back the loan and collateral would not enforce payment.

In this model, traders receive no utility from holding the collateral in period 0 therefore the different variants of collateralized lending all amount to the same thing. Therefore, without loss of generality, the borrower is required to own the collateral that she pledges.

[8] More precisely, the equilibrium described below in Theorem 2 continues to be the essentially unique equilibrium in the more general setup in which bonds can also be used as collateral. This is intuitively because optimistic borrowers do not hold any bonds in equilibrium (unless their wealth is more than sufficient to purchase the entire asset supply), and thus they do not use bonds as collateral.

contract φ is traded in an anonymous market at a competitive price $q(\varphi)$. Note that the anonymity of the market is ensured by collateral: each lender knows that repayment is only secured by collateral, and that she will get the payment in (4) regardless of the identity of the borrower in the transaction. I refer to the price of the debt contract, $q(\varphi)$, also as the *loan size*, since this is the amount of that the borrower receives by collateralizing one unit of the asset. Moreover, I define the *interest rate on the loan* as the ratio of the interest payment to the loan size:

$$\frac{\varphi - q(\varphi)}{q(\varphi)}. \tag{5}$$

To define the equilibrium in this model, let $x_i = \left(x_i^A, x_i^B\right) \in \mathbb{R}_+^2$ denote type i traders' asset and bond allocation. Let $z_i : \mathbb{R}_+ \to \mathbb{R}$ denote a Lebesgue measurable function, where $z_i(\varphi)$ represents trader i's position in debt contract φ. Unlike the asset, debt contracts can be short sold (which corresponds to borrowing) but subject to a collateral restriction. In particular, a trader that chooses $z_i(\varphi) < 0$ must pledge $-z_i(\varphi)$ units of the asset as collateral. Given the above description of the economy, the problem of a type i trader can be written as follows:

$$\max_{x_i \geq 0,\ z_i(\cdot)} \quad x_i^A E_i[v(s)] + x_i^B(1+r) + \int_{\mathbb{R}_+} z_i(\varphi) E_i[\min(v(s), \varphi)]\, d\varphi, \tag{6}$$

$$\text{s.t.} \quad p x_i^A + x_i^B + \int_{\mathbb{R}_+} q(\varphi) z_i(\varphi)\, d\varphi \leq w_i + p\alpha_i, \tag{7}$$

$$\int_{\mathbb{R}_+} \max(0, -z_i(\varphi))\, d\varphi \leq x_i^A. \tag{8}$$

Note that, in (6), trader i maximizes her expected payoff with respect to her own beliefs. The inequality in (7) captures her budget constraint, and the inequality in (8) captures the collateral constraint.

Definition 2 (Collateral Equilibrium). *Given an economy $\mathcal{E}$ with assumptions (O) and (S), a collateral equilibrium is a collection of prices $\left(p, [q(\varphi)]_{\varphi \in \mathbb{R}_+}\right)$ and allocations $\left(x_i^A, x_i^B,\ z_i(\cdot)\right)_{i \in \{1,0\}}$ such that the allocation of each type trader $i \in \{1,0\}$ solves Problem (6), and asset and unit debt markets clear, that is, $\sum_{i \in \{1,0\}} x_i^A = 1$ and $\sum_{i \in \{1,0\}} z_i(\varphi) = 0$ for each $\varphi \in \mathbb{R}_+$.*

Even though all unit debt contracts will be priced and available, only some of these contracts will be actually traded, i.e., traders will endogenously choose $z_i(\varphi) = 0$ for some of these contracts. In this sense, the equilibrium will select the debt contracts that will be traded in equilibrium, as characterized in the next section.

3 Characterization of Collateral Equilibrium

This section provides a characterization of collateral equilibrium and presents the main result regarding the effect of belief heterogeneity on the asset price. The equilibrium will intuitively have the form that moderate traders hold the bond and collateralized debt contracts (i.e., they lend to optimists), while optimists make leveraged investments in the asset by selling collateralized debt contracts.

To characterize this collateral equilibrium, it is useful to define the notion of a *quasi-equilibrium*, which is a collection of prices $\left(p, [q(\varphi)]_{\varphi \in \mathbb{R}_+}\right)$ and allocations $\left(x_i^A, x_i^B,\ z_i(\cdot)\right)_{i \in \{1,0\}}$ such that markets clear and the allocation of each type trader $i \in \{1,0\}$ solves Problem (6) with the additional requirement that $z_1(\cdot) \leq 0 \leq z_0(\cdot)$. That is, in a quasi-equilibrium, optimists are restricted only to sell debt contracts and moderate traders are restricted only to buy debt contracts. For expositional reasons, I will first construct a quasi-equilibrium. Theorem 2 below establishes that the constructed quasi-equilibrium corresponds to a collateral equilibrium with the same allocations and the same asset price (and with potentially different debt contract prices). The same theorem also establishes that the asset price in a collateral equilibrium is uniquely determined.

To construct a quasi-equilibrium, consider debt contract prices

$$q(\varphi) = \frac{E_0[\min(v(s), \varphi)]}{1+r} \text{ for each } \varphi \in \mathbb{R}_+, \tag{9}$$

that make the moderate lenders indifferent between investing in the bond and any debt contract $\varphi \in \mathbb{R}_+$. Given the prices in (9) and the asset price $p \geq \frac{E_0[v(s)]}{1+r}$, moderate traders' optimal decision in a quasi-equilibrium is completely characterized: they are indifferent between purchasing the bond and any debt contract, and they always weakly prefer these options to investing in the asset (and strictly so whenever $p > \frac{E_0[v(s)]}{1+r}$). Moreover, market clearing in debt contracts will be automatic, as moderate traders will absorb any supply of debt contracts from optimists.

The quasi-equilibrium asset price and allocations are then determined by optimists' optimal investment decision. I next analyze optimists' problem for a given asset price p, and I then combine this analysis with asset market clearing to solve for the quasi-equilibrium.

3.1 Main Result: Asymmetric Filtering of Optimism

The next result, which is also the main result, characterizes optimists' investment decision. The result establishes that optimists choose to sell a single debt contract $\varphi \in \left[v\left(s^{\min}\right), v\left(s^{\max}\right)\right]$. I refer to a debt contract $\varphi = v(\bar{s}) \in \left[v\left(s^{\min}\right), v\left(s^{\max}\right)\right]$ as a *loan with riskiness* $\bar{s}$, since this contract defaults if and only if the realized state is below $\bar{s}$. The result characterizes the riskiness $\bar{s}$ of the optimal loan for given price p, which in turn shows that the equilibrium price has an asymmetric filtering property.

Theorem 1 (Optimal Contract Choice and Asymmetric Filtering). *Suppose assumptions (O) and (S) hold, debt prices are given by* (9) *and the asset price satisfies*

$p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$. *In a quasi-equilibrium:*

(i) There exists $\bar{s} \in \mathcal{S}$ such that optimists only sell the debt contract $\varphi = v(\bar{s})$, i.e., they borrow according to a single loan with riskiness $\bar{s}$. Optimists' collateral constraint (8) *is binding, i.e., they borrow as much as possible according to the optimal loan. Optimists choose $x_1^B = 0$, i.e., they invest all of their leveraged wealth in the asset A.*

(ii) The riskiness $\bar{s}$ of the optimal loan is characterized as the unique solution to the following equation over $\mathcal{S}$:

$$\begin{aligned} p = p^{opt}(\bar{s}) &\equiv \frac{1}{1+r}\left(\int_{s^{\min}}^{\bar{s}} v(s)\,dF_0 + (1-F_0(\bar{s}))\int_{\bar{s}}^{s^{\max}} v(s)\frac{dF_1}{1-F_1(\bar{s})}\right) \qquad (10) \\ &= \frac{1}{1+r}\left(F_0(\bar{s})\,E_0[v(s) \mid s<\bar{s}] + (1-F_0(\bar{s}))\,E_1[v(s) \mid s \geq \bar{s}]\right). \end{aligned}$$

The riskiness $\bar{s}$ of the optimal loan is decreasing in the price level p.

Suppose instead that asset price satisfies $p \geq \frac{E_1[v(s)]}{1+r}$. If $p = \frac{E_1[v(s)]}{1+r}$, then optimists are indifferent between selling any safe debt contract $\varphi \leq v(s^{\min})$, investing in the asset or investing in the bond. If $p > \frac{E_1[v(s)]}{1+r}$, then optimists do not sell any debt contracts, and they invest all of their wealth in the bond.[9]

I will shortly provide a sketch proof of this result along with an intuition. Before doing so, I note a couple of important aspects of the function $p^{opt}(\bar{s})$. First note that the function $p^{opt}(\bar{s})$ is the inverse demand function: it describes the asset price p for which the riskiness level $\bar{s}$ would be optimal. Assumption (O) implies $p^{opt}(\bar{s})$ is strictly decreasing and continuous (cf. Appendix A.1). Since $p^{opt}(s^{\min}) = \frac{E_1[v(s)]}{1+r}$ and $p^{opt}(s^{\max}) = \frac{E_0[v(s)]}{1+r}$, this further implies that there is a unique solution to Eq. (10), and that the solution is strictly decreasing in p.

Second, note that $p^{opt}(\bar{s})$ also describes the equilibrium asset price conditional on the equilibrium loan riskiness $\bar{s}$. Hence, Theorem 1 is the main result of this paper, as it shows that optimism will be asymmetrically filtered in equilibrium. In particular, the second line of (10) replicates Eq. (1) from Introduction and shows that the expected value of the asset is taken with respect to a mixture of moderate and optimistic beliefs. The moderate belief is used to assess the likelihood of default states $s < \bar{s}$, along with the value of the asset conditional on these states, while the optimistic belief is used to assess the likelihood of non-default states $s > \bar{s}$. Consequently, the function $p^{opt}(\bar{s})$ will "filter" any optimism about the probability of default states, while "incorporating" any optimism about the relative probability of states conditional on no default. The following example describes two scenarios that differ about the type of optimism and illustrates the asymmetric filtering property.

Example 1 (Asymmetric Filtering of Optimism). *Consider the state space $\mathcal{S} = [1/2, 3/2]$ and the value function $v(s) = s$. As the first scenario, suppose moderate traders and optimists*

[9] Optimists' investment decision for $p \leq \frac{E_0[v(s)]}{1+r}$ is omitted, since the equilibrium asset price always satisfies $p > \frac{E_0[v(s)]}{1+r}$ (cf. Theorem 2).

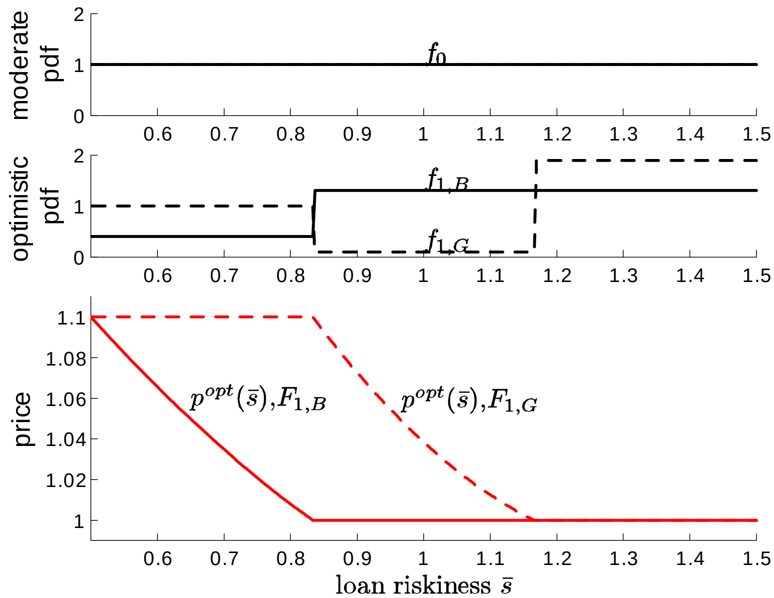

Figure 1: The top two panels display the probability density functions for traders' priors in the two scenarios of Example 1. The bottom panel displays the corresponding curves $p^{opt}(\bar{s})$, the inverse of which gives the optimal loan riskiness $\bar{s}$ for a given price level p.

have the prior distributions F_0 and $F_{1,B}$ with density functions:

$$\begin{aligned} f_0(s) &= 1 \text{ for each } s \in \mathcal{S}, \\ \text{and } f_{1,B}(s) &= \begin{cases} 0.4 & \text{if } s \in \mathcal{S}_B \equiv [2/3 - 1/6, 2/3 + 1/6) \\ 1.3 & \text{if } s \in \mathcal{S}_N \equiv [1 - 1/6, 1 + 1/6) \\ 1.3 & \text{if } s \in \mathcal{S}_G \equiv [4/3 - 1/6, 4/3 + 1/6] \end{cases}, \end{aligned} \tag{11}$$

where $\mathcal{S}_B$, $\mathcal{S}_N$, and $\mathcal{S}_G$ intuitively capture bad, normal and good events, respectively. In words, moderate types find all states equally likely, while optimists are optimistic because they believe that a bad event, that is, a realization around the bad state 2/3, is less likely than a normal or a good event (which they find equally likely).[10]

As an alternative to this scenario, suppose moderate types have the same prior, but opti-

[10] Note that the distributions in (11) do not exactly satisfy the regularity assumption (O). In particular, the density functions are not continuous, and $F_{1,B}$ is only weakly more optimistic than F_0. However, there exists arbitrarily close perturbations of these distributions that satisfy assumption (O). The equilibrium for these arbitrarily close perturbations approximates the equilibrium with the distributions in (11). The examples consider the distributions in (11), because these distributions provide a clearer intuition. The formal results require the stricter assumption (O) for analytical tractability.

mists' prior is changed to the distribution $F_{1,G}$ with density function

$$f_{1,G} = \begin{cases} 1 & \text{if } s \in \mathcal{S}_B \\ 0.1 & \text{if } s \in \mathcal{S}_N \\ 1.9 & \text{if } s \in \mathcal{S}_G \end{cases} . \tag{12}$$

That is, in the alternative scenario, optimists are optimistic not because they think the bad event is less likely, but because they believe the good event is more likely than the normal event. Note also that optimists are equally optimistic in both scenarios, i.e., $E_{1,G}[v(s)] = E_{1,B}[v(s)]$.

The top two panels of Figure 1 display the density functions in the two scenarios described in Example 1, and the bottom panel displays the corresponding $p^{opt}(\bar{s})$ curves. As suggested by the definition in (10)*, in the first scenario, $p^{opt}(\bar{s})$ gradually decreases as $\bar{s}$ is increased over the range $\mathcal{S}_B$. This is because optimism about the probability of default states is filtered. In the second scenario, $p^{opt}(\bar{s})$ is equal to the optimistic valuation as $\bar{s}$ is changed over the same range $\mathcal{S}_B$, because optimism about the relative likelihood of non-default states is not filtered.*

I next present a sketch proof of Theorem 1, which is completed in Appendix A.2, and which will be useful to provide the intuition for the filtering property of $p^{opt}(\bar{s})$.

Sketch proof for Theorem 1. Note that optimists can get an expected *unleveraged return*

$$R^U = \frac{E_1[v(s)]}{p} > 1 + r$$

simply by investing their wealth in the asset, that is, without borrowing. Since $R^U > 1 + r$, if optimists could borrow at interest rate r without constraints, they would leverage this return infinitely by borrowing and investing as much as possible. In the present model with financial constraints, optimists can leverage by borrowing through a loan with riskiness $\tilde{s} \in \mathcal{S}$, which represents a trade-off.

On the one hand, larger and riskier loans (with greater $\tilde{s}$) enable optimists to leverage their unleveraged return R^U more. On the other hand, riskier loans trade at a greater interest rate (5). Thus, optimists that take these loans have to make a greater interest payment in non-default states, which they find particularly likely. More precisely, moderate lenders believe that they are receiving an expected payment $E_0[\min(v(s), v(\tilde{s}))]$ on a loan with riskiness $\tilde{s}$, while optimists believe that they are paying a greater amount $E_1[\min(v(s), v(\tilde{s}))]$ in expectation. Optimists believe they will pay more than the moderates think they will receive, i.e., $E_1[\min(v(s), v(\tilde{s}))] \geq E_0[\min(v(s), v(\tilde{s}))]$, intuitively because optimists believe the loan will default less often. The *expected interest rate* on the loan (as perceived by optimists) is

$$1 + r_1^{exp}(\tilde{s}) \equiv \frac{E_1[\min(v(s), v(\tilde{s}))]}{\frac{1}{1+r}E_0[\min(v(s), v(\tilde{s}))]} = (1+r)\frac{\int_{s^{\min}}^{\tilde{s}} v(s)\,dF_1 + v(\tilde{s})(1 - F_1(\tilde{s}))}{\int_{s^{\min}}^{\tilde{s}} v(s)\,dF_0 + v(\tilde{s})(1 - F_0(\tilde{s}))}. \tag{13}$$

Assumption (O) implies that the expected interest rate $r_1^{exp}(\tilde{s})$ is strictly increasing in $\tilde{s}$. This is intuitively because, the riskier the loan, the greater the measure of states in which there is default, and the greater the disagreement regarding the payoff of the loan.

It follows that, in deciding whether to choose a larger and riskier loan, optimists trade off the ability to leverage more the unleveraged return $R^U = \frac{E_1[v(s)]}{p}$ against the higher expected interest rate $r_1^{exp}(\tilde{s})$. More precisely, the analysis in Appendix A.2 reveals that optimists choose a riskiness level of $\tilde{s} \in \mathcal{S}$ that solves the following leveraged investment problem:

$$\max_{\tilde{s} \in \mathcal{S}} \left(R_1^L(\tilde{s}) \equiv \frac{E_1[v(s)] - E_1[\min(v(s), v(\tilde{s}))]}{p - \frac{1}{1+r} E_0[\min(v(s), v(\tilde{s}))]} \right). \tag{14}$$

Here, $R_1^L(\tilde{s})$ is optimists' rate of return if they borrow as much as they can using a loan with riskiness $\tilde{s}$. Alternatively, and equivalently, it can be viewed as the rate of return from buying the asset on margin. Note that, given the contract prices in (9), $p - \frac{E_0[\min(v(s), v(\tilde{s}))]}{1+r}$ is the downpayment (i.e., the margin) that optimists need to put down to purchase one unit of the asset. Optimists believe that they will get the expected return $E_1[v(s)]$ on the asset, and they expect to make a payment $E_1[\min(v(s), v(\tilde{s}))]$ on their debt, leading to the return expression in (14).

The relation $p = p^{opt}(\bar{s})$ is essentially the first order optimality condition corresponding to the leveraged investment problem (14). The analysis in Appendix A.2 shows that the solution $\bar{s}$ to this equation also solves problem (14), which completes the sketch proof of Theorem 4.

I next use this sketch proof to provide an intuition for why the function $p^{opt}(\bar{s})$ is decreasing in $\bar{s}$ and why it has the filtering property. First consider the former feature, which is equivalent to the statement that the optimal loan riskiness $\bar{s}$ is decreasing in the asset price p. This feature holds because a lower asset price p tilts optimists' trade-off towards riskier and larger loans, by increasing optimists' unleveraged return $R^U = \frac{E_1[v(s)]}{p}$. Intuitively, as the asset price falls towards the moderate valuation, optimists see a greater bargain and they have a greater incentive to leverage this return by taking a riskier (and larger) loan, agreeing to pay a greater expected interest rate $r_1^{exp}(\bar{s})$ at the margin.

To see the intuition for the filtering property of $p^{opt}(\bar{s})$, suppose optimists choose a loan with riskiness $\bar{s}$, and consider how much the price should drop (from the optimistic valuation) to entice optimists to take this loan. Consider this question in the context of Example 1 as the riskiness level $\bar{s}$ is gradually increased over the interval $\mathcal{S}_B$. In the first scenario of Example 1, optimists are optimistic because they find the bad event unlikely. Thus, the scope of disagreement *about the likelihood of default* gradually increases as $\bar{s}$ is increased over the interval $\mathcal{S}_B$. This further implies that optimists' expected interest rate $r_1^{exp}(\bar{s})$ gradually increases (cf. Eq. (13)) as $\bar{s}$ is increased over this interval. Then, the higher $\bar{s}$, the tighter credit appears to optimists, and the more the price should fall to entice optimists to take the loan $\bar{s}$, as illustrated in the bottom panel of Figure 1. Consider instead the second scenario of Example 1 in which optimists are optimistic because they find the good event more likely than the normal event. In this case, as $\bar{s}$ increases over the interval $\mathcal{S}_B$, the scope of disagreement about default probability remains constant because the traders agree about the probability of the states in $\mathcal{S}_B$. Thus, optimists' expected interest rate does not increase, i.e., it remains equal to r. Optimists do compensate the lenders for their potential losses in case of a default. However, as there is agreement about default probability, optimists believe they are paying a fair interest rate in expectation. Consequently, credit appears loose to optimists and the asset price does not need to fall to entice them to take a loan with riskiness $\bar{s} \in \mathcal{S}_B$, as illustrated in Figure 1.

Taken together, these observations provide the intuition for the filtering feature of the function $p^{opt}(\bar{s})$. While optimism tends to increase the asset price, the tightness of financial constraints perceived by optimists, measured by the expected interest rate $r_1^{exp}(\bar{s})$, tends to decrease the asset price. Eq. (13) establishes that the tightness of financial constraints depends on the traders' relative valuations of collateralized loans. The same equation also establishes that, as the loans make losses only in default states, the tightness of financial constraints is effectively determined by the belief disagreements about *the likelihood of default states.* It follows that, optimism about default states does not increase the asset price because this type of optimism comes bundled with tighter financial constraints. On the other hand, optimism about the relative likelihood of non-default states leads to a higher asset price, because this type of optimism is unchecked by financial constraints.[11]

3.2 Asset Market Clearing and Collateral Equilibrium

Theorem 1 characterizes the riskiness $\bar{s}$ of the optimal contract as a function of the asset price p, given the candidate prices (9) for debt contracts. I next consider the market clearing price level p and solve for the equilibrium.

Suppose optimists choose to borrow using a loan with riskiness $\bar{s}$ and consider the price that

[11] A second and complementary intuition for the filtering property of the function $p^{opt}(\bar{s})$ is obtained by rewriting Eq. (14) as

$$\begin{aligned} p &= \frac{E_0\left[\min\left(v\left(s\right),v\left(\bar{s}\right)\right)\right]}{1+r} + \frac{E_1\left[s\right] - E_1\left[\min\left(v\left(s\right),v\left(\bar{s}\right)\right)\right]}{R_1^L\left(\bar{s}\right)} \\ &= \frac{1}{1+r}\left(\int_{s^{\min}}^{\bar{s}} v\left(s\right)dF_0 + v\left(\bar{s}\right)\left(1-F_0\left(\bar{s}\right)\right)\right) + \frac{1}{R_1^L\left(\bar{s}\right)}\int_{\bar{s}}^{s^{\max}}\left(v\left(s\right)-v\left(\bar{s}\right)\right)dF_1. \qquad (15) \end{aligned}$$

This expression shows that a leveraged investment is essentially a joint venture between the lender and the borrower. The debt portion, which is the downside payoff and a fixed payment on the upside, is priced by lenders' moderate beliefs and discounted by lenders' discount rate, as captured by the left hand side term in Eq. (15). The remaining portion of the leveraged investment, which is the upside payoff net of a fixed payment, is priced by borrowers' optimistic beliefs and discounted by their discount rate $R_1^L(\bar{s})$, as captured by the right hand side term in Eq. (15). Even though the borrowers' discount rate $R_1^L(\bar{s})$ is endogenously determined (along with the contract choice), Eq. (15) represents a powerful economic force and it provides a second intuition for its analogue $p = p^{opt}(\bar{s})$, which holds in equilibrium. Since the downside of any leveraged investment is held by lenders, the tightness of the financial constraints are determined by their moderate beliefs for the likelihood of bad states. If the lenders are relatively more pessimistic about bad states, then credit constraints appear tighter to optimists, while lenders' pessimism about the relative likelihood of good states is largely irrelevant for credit constraints.

would clear asset markets. Suppose, for a second, that optimists hold the entire asset supply of 1 units. In this case, optimists would borrow a total of $\frac{1}{1+r} E_0 [\min v(s), v(\bar{s})]$ against their asset holdings, and the maximum wealth that they can obtain in the first period is given by

$$w_1^{\max}(\bar{s}) = w_1 + \frac{1}{1+r} E_0 [\min v(s), v(\bar{s})]. \tag{16}$$

As characterized in the next lemma, the market clearing price depends on the comparison of optimists' maximum wealth, $w_1^{\max}(\bar{s})$, and the total value of assets in the hands of moderate types, $\alpha_0 p$ (which optimists seek to purchase).

Lemma 1. *Suppose optimists borrow using a loan with riskiness $\bar{s}$ as much possible subject to their collateral constraint* (8)*, and invest all of their leveraged wealth in the asset as long as $p < \frac{E_1[v(s)]}{1+r}$. Then, the market clearing asset price is given by*

$$p = p^{mc}(\bar{s}) \equiv \begin{cases} \frac{E_1[v(s)]}{1+r} & \text{if } \frac{w_1^{\max}(\bar{s})}{\alpha_0} > \frac{E_1[v(s)]}{1+r} & \text{[case (i)]} \\ \frac{w_1^{\max}(\bar{s})}{\alpha_0} & \text{if } \frac{w_1^{\max}(\bar{s})}{\alpha_0} \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right] & \text{[case (ii)]} \\ \frac{E_0[v(s)]}{1+r} & \text{if } \frac{w_1^{\max}(\bar{s})}{\alpha_0} \leq \frac{E_0[v(s)]}{1+r} & \text{[case (iii)]} \end{cases}, \tag{17}$$

where $w_1^{\max}(\bar{s})$ (cf. Eq. (16)*) is the maximum wealth optimists can obtain in the first period by borrowing with a loan with riskiness $\bar{s}$.*

In case (i), optimists' maximum wealth (given $\bar{s}$) is sufficiently high that they can purchase α_0 units of the assets in the hands of moderate traders regardless of the price in the interval (3). In this case, optimists are marginal holders of the asset and they are indifferent between investing in the asset and the bond, which implies that the price is given by their valuation $\frac{E_1[v(s)]}{1+r}$. In case (ii), optimists still purchase all α_0 units of the asset from moderate traders. However, their financial constraints are strictly binding, i.e.,

$$w_1^{\max}(\bar{s}) = p\alpha_0,$$

which pins down the asset price. In this case, optimists use all of their leveraged wealth to purchase the asset (and they hold no bonds), but their maximum wealth is not sufficient to bid up the asset price as high as their valuation. In case (iii), optimists' maximum wealth is not sufficient to purchase the units of the asset in the hands of moderate traders regardless of the price in the interval (3). Thus, moderate traders are marginal holders of the asset, and they are indifferent between investing in the asset and the bond. This implies that the asset price in this case is given by the moderate valuation, $\frac{E_0[v(s)]}{1+r}$.

Note that Eq. (17) describes an increasing relation between the asset price and the loan riskiness $\bar{s}$. Intuitively, when optimists take a larger and riskier loan $\bar{s}$, they have greater first period wealth to spend on the asset, which enables them to bid up the asset price more towards the optimistic valuation, $\frac{E_1[v(s)]}{1+r}$.

Combining Theorem 1 and Lemma 1, the equilibrium price and loan riskiness pair, $(p, \bar{s}^*)$,

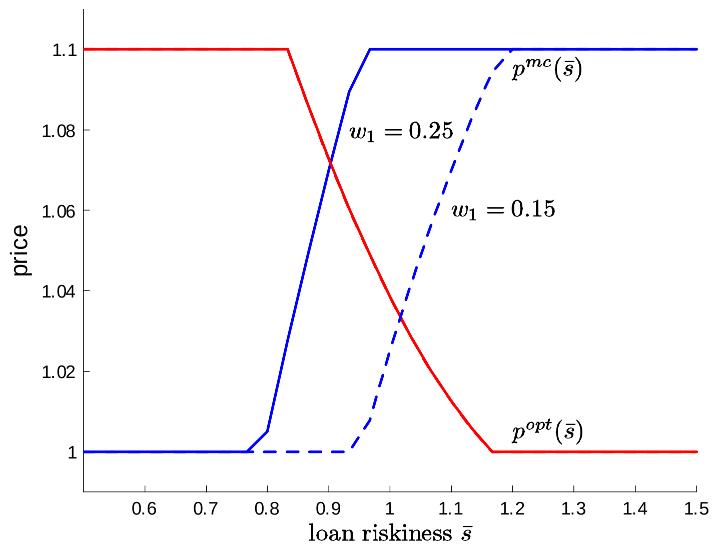

Figure 2: The figure displays the collateral equilibrium, and the response of the equilibrium to a decline in optimistic wealth w_1.

is determined as the unique intersection of the strictly decreasing function $p^{opt}(\bar{s})$ and the weakly increasing function $p^{mc}(\bar{s})$, which provides a graphical characterization as displayed in Figure 2. Appendix A.2 also provides an analytical characterization, which shows that if optimists' endowment w_1 is not too large (in particular, if condition $(A.20)$ in Appendix A.2 holds), then $p^{opt}(\bar{s})$ intersects $p^{mc}(\bar{s})$ in the case (ii) region of Eq. (17). In this case, there is an interior equilibrium with $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$ and $\bar{s}^* \in \left(s^{\min}, s^{\max}\right)$. If instead optimists' endowment is sufficiently large, then the curves intersect in the case (i) region of Eq. (17). In this case, there is a corner equilibrium with $p = \frac{E_1[v(s)]}{1+r}$ and $\bar{s}^* = s^{\min}$.[12]

This analysis completes the characterization of the quasi-equilibrium. The analysis in Appendix A.3 establishes that this quasi-equilibrium is a collateral equilibrium with modified debt contract prices given by:

$$q(\varphi) = \max\left(\frac{E_0\left[\min\left(v(s), \varphi\right)\right]}{1+r}, \frac{E_1\left[\min\left(v(s), \varphi\right)\right]}{R_1^L(\bar{s}^*)}\right). \tag{18}$$

Note that $R_1^L(\bar{s}^*)$ (cf. Eq. (14)) is optimists' expected return on capital in equilibrium. Thus, the expression $\frac{E_1[\min(v(s),\varphi)]}{R_1^L(\bar{s}^*)}$ is optimists' valuation of debt contracts in equilibrium. Unlike in a quasi-equilibrium, optimists can demand debt contracts in a collateral equilibrium. Hence, the price of a debt contract is given by the upper-envelope of the moderate and the

[12] The appendix also shows that $p^{opt}(\bar{s})$ can never intersect $p^{mc}(\bar{s})$ in the case (iii) region of Eq. (17). In particular, the equilibrium prices are always greater than the moderate valuation $\frac{E_0[s]}{1+r}$. Intuitively, prices cannot fall all the way to moderate valuation $\frac{E_0[s]}{1+r}$, since in this case (under assumption (O)) optimists would choose infinite leverage and bid up the asset prices, albeit slightly.

optimistic valuations, as captured by (18). The analysis in Appendix A.3 establishes that optimists' and moderate lenders' allocations continue to be optimal when the prices are given by (18) and when the constraints $z_1(\cdot) \leq 0 \leq z_0(\cdot)$ are relaxed.[13] This further implies that the quasi-equilibrium characterized in this section corresponds to a collateral equilibrium. The following result summarizes this discussion and proves the essential uniqueness of the collateral equilibrium.

Theorem 2 (Existence, Characterization, Essential Uniqueness). *Consider the above described economy with assumptions (O) and (S). There exists a collateral equilibrium in which contract prices are given by* (18)*, moderate types are indifferent between buying bonds and lending to optimists, and optimists make leveraged investments in the asset by borrowing through a single loan with riskiness* $\bar{s}^* \in \mathcal{S}$*. The asset price* p *and riskiness* $\bar{s}^*$ *of loans in this equilibrium are determined as the unique solution to* $p = p^{opt}(\bar{s}) = p^{mc}(\bar{s})$ *over* $\bar{s} \in \mathcal{S}$*.*

In any collateral equilibrium, the asset price, p*, and the price of the optimal debt contract,* $q(v(\bar{s}^*))$*, are uniquely determined. Except for the corner case in which* $p = \frac{E_1[v(s)]}{1+r}$*, traders' allocations,* $(x_i, z_i(\cdot))_i$*, are also uniquely determined. However, prices of the remaining debt contracts,* $q(\varphi)$ *for* $\varphi \neq v(\bar{s}^*)$*, are not uniquely determined.*

In other words, most of the equilibrium is uniquely determined, except for the prices of debt contracts that are not traded in equilibrium. Appendix A.3 establishes that, for each contract $\varphi = v(\tilde{s}) \neq v(s^*)$, there exists a continuum of prices that can support the equilibrium with no-trade in these contracts. This completes the characterization of the collateral equilibrium. In the next section, I use this characterization to analyze the comparative statics of equilibrium.

4 Comparative Statics of Collateral Equilibrium

In this section, I use the characterization in the previous section to derive a number of results about the equilibrium asset price and leverage. I first describe the effect of a change in optimistic wealth and the interest rate. I then turn to the focus of this paper and describe the effect of different types of belief heterogeneity on the asset price.

In addition to the equilibrium loan riskiness $\bar{s}^*$ and the asset price p, I will consider the comparative statics of two more variables. First, I will consider the *leverage ratio* for optimists'

[13] To see the intuition for this result, consider a debt contract φ whose price is higher in the collateral equilibrium relative to the quasi-equilibrium, i.e., suppose

$$\frac{E_1[\min(v(s), \varphi)]}{R_1^L(\bar{s}^*)} > \frac{E_0[\min(v(s), \varphi)]}{1+r}. \tag{19}$$

Note that moderates strictly prefer the bond B to this contract, and thus optimists cannot borrow from moderates by selling this contract. The collateral equilibrium will have the intuitive property that optimists will borrow from moderates. This further implies that any contract φ that satisfies the inequality in (19) will not be traded, i.e., $z_i(\varphi) = 0$ for these contracts. Put differently, even though these contracts will be priced differently in a collateral equilibrium (compared to a quasi-equilibrium), they will not be actually used to transfer resources between traders, and thus they do not affect the equilibrium allocations.

asset purchase, denoted by L. Recall that optimists buy one unit of the asset by paying $p - \frac{E_0[v(s),v(\bar{s}^*)]}{1+r}$ out of their wealth and financing the rest of the purchase, $\frac{E_0[v(s),v(\bar{s}^*)]}{1+r}$, by borrowing from moderates. Thus, the leverage ratio for the asset purchase is given by

$$L \equiv \frac{p}{p - E_0\left[v\left(s\right), v\left(\bar{s}^*\right)\right] / \left(1+r\right)}. \tag{20}$$

The leverage ratio L is an important variable because it has counterparts in real financial markets. In particular, the loan-to-value ratio of a mortgage loan is equal to $1 - \frac{1}{L}$, and the haircut ratio of a REPO loan is equal to $\frac{1}{L}$. Second, I also consider the comparative statics of the asset *overvaluation ratio*, defined as the unique $\theta \in (0, 1]$ such that

$$p = (1-\theta) \frac{E_0\left[v\left(s\right)\right]}{1+r} + \theta \frac{E_1\left[v\left(s\right)\right]}{1+r}. \tag{21}$$

As $\theta \to 0$ (resp. $\theta \to 1$) prices reflect only the moderate (resp. the optimistic) valuation. Intuitively, θ is a measure of overvaluation that controls for the level of optimism. Hence, θ is a theoretically important variable because it captures how much of the optimism in prior beliefs is reflected in the asset price.

4.1 Effect of Optimistic Wealth and Interest Rate

The following result describes the effect of optimists' wealth and the interest rate on the collateral equilibrium.

Theorem 3. *Consider the collateral equilibrium characterized in Theorem 2.*

(i) If optimists' endowment w_1 decreases, then: the asset price p and the overvaluation ratio θ weakly decrease, the loan riskiness $\bar{s}^*$ and the leverage ratio L weakly increase.

(ii) If the interest rate r decreases, then: the asset price p, the loan riskiness $\bar{s}^*$, and the leverage ratio L weakly increase, while the overvaluation ratio θ weakly decreases.

The first part of this theorem shows that a negative wealth shock to optimists decreases the price and the overvaluation of the asset. Intuitively, when optimists have less wealth, they have less financial muscle to bid up the asset price, and thus the price is closer to the moderate valuation. As the price declines towards the moderate valuation, optimists see more of a bargain in the asset price and they are incentivized to leverage their investments more by taking larger and riskier loans, as illustrated in Figure 2. Put differently, the equilibrium price and leverage move in opposite directions in response to a negative wealth shock, and the higher leverage ratio ameliorates the drop in the asset price.

This result is similar to the findings in Geanakoplos (2009), who considers a version of this model with two continuation states. Geanakoplos (2009) demonstrates that, in the two-state model, a negative wealth shock to optimists decreases the price and increases the leverage ratio, and that the response of the leverage ratio dampens the effect of the initial wealth shock. Theorem 3 additionally reveals that a negative wealth shock increases the size and the riskiness of equilibrium loans. This effect is absent from Geanakoplos (2009) because, in the two-state model, loans are always fully secured with respect to the worst case scenario, i.e., there is no default. The loan riskiness in the two-state model does not respond to any shock (since loans never default), and the loan size does not respond to a wealth shock to optimists.

The second part of Theorem 3 establishes the comparative statics of the equilibrium variables with respect to the interest rate. In response to a reduction in the interest rate, the asset price is naturally higher. Theorem 3 also shows that loans become riskier and optimists become more leveraged. Moreover, the overvaluation ratio decreases, that is, a lower interest rate weakens the effect of a given level of optimism on the asset price. Even though the asset price increases, the asset valuations $\frac{E_i[v(s)]}{1+r}$ also increase, and the overvaluation ratio is lower. The intuition for these comparative statics can be gleaned from Eq. (17), which shows that a reduction in the interest rate acts very similar to a negative wealth shock to optimists. A lower interest rate increases the asset price, hence optimists become more stretched to purchase the same amount of asset supply. Financial conditions are effectively tightened, which implies that optimists leverage more and that the overvaluation ratio decreases.

4.2 Effect of Different Types of Belief Heterogeneity

This section describes the effect of different types of belief heterogeneity on the asset price. The following definition formalizes the notion of skewness that is used in subsequent results.

Definition 3 (Skewed Optimism). *Consider two probability distributions $H, \tilde{H}$ over $\mathcal{S} = \left[s^{\min}, s^{\max}\right]$ with density functions $h, \tilde{h}$ that are continuous and positive over $\mathcal{S}$, and consider a continuously differentiable and strictly increasing asset value function $v : \mathcal{S} \to \mathbb{R}_{++}$. The optimism of distribution $\tilde{H}$ about the asset is* weakly more right-skewed *than H, denoted by $\tilde{H} \succeq_R H$, if and only if:*

(a) The distributions yield the same valuation of the asset, that is, $E\left[v(s)\ ;\ \tilde{H}\right] = E\left[v(s)\ ;\ H\right]$.

(b) There exists $s^R \in \mathcal{S}$ such that $\frac{1-\tilde{H}(s)}{1-H(s)}$ is weakly decreasing over $\left(s^{\min}, s^R\right)$ while it is weakly increasing over $\left(s^R, s^{\max}\right)$, which is the case if and only if the hazard rates of $\tilde{H}$ and H satisfy the (weak) single crossing condition:

$$\begin{cases} \frac{\tilde{h}(s)}{1-\tilde{H}(s)} \geq \frac{h(s)}{1-H(s)} \ \text{ if } s < s^R, \\ \frac{\tilde{h}(s)}{1-\tilde{H}(s)} \leq \frac{h(s)}{1-H(s)} \ \text{ if } s > s^R. \end{cases} \tag{22}$$

The optimism of distribution $\tilde{H}$ is weakly more skewed to the right of $\tilde{s} \in S$ *than H, denoted by $\tilde{H} \succeq_{R,\tilde{s}} H$, if the conditions (a)-(b) are satisfied with the additional requirement that $s^R \geq \tilde{s}$.*

To interpret this definition, recall from Definition 1 that the notion of optimism in this paper concerns the probability of upper-threshold events $[s, s^{\max}] \subset \mathcal{S}$. Note that $\tilde{H}$ and H

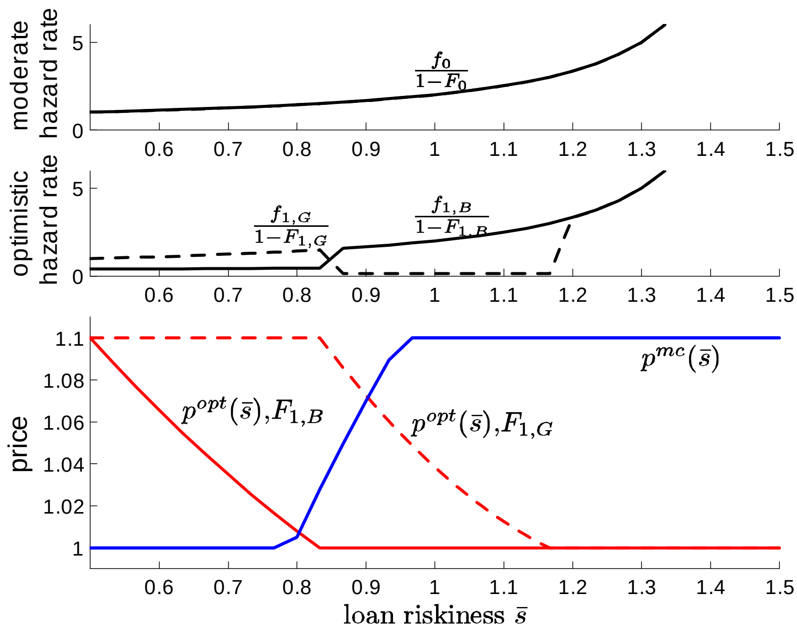

Figure 3: The top two panels display the hazard rates for traders' priors in the two scenarios analyzed in Example 1. The bottom panel plots the corresponding equilibria.

cannot be compared according to the optimism order in Definition 1. In addition, they lead to the same valuation of the asset, therefore it may be reasonable to say that these distributions have the same level of optimism. Note, however, that $\tilde{H}$ assigns increasingly greater relative probability to events $[s, s^{\max}]$ for $s \in \left(s^R, s^{\max}\right)$. Thus, $\tilde{H}$ is more optimistic in this region in the sense of Definition 1. In contrast, H assigns increasingly greater relative probability to events $[s, s^{\max}]$ for $s \in \left(s^{\min}, s^R\right)$. Hence, the optimism of $\tilde{H}$ is right-skewed in the sense that $\tilde{H}$ represents more optimistic beliefs only for relatively good events.

Note that the probability distributions $F_{1,B}$ and $F_{1,G}$ of Example 1 satisfy condition (22). That is, $F_{1,G}$ and $F_{1,B}$ lead to the same valuation for the asset but the optimism of $F_{1,G}$ is weakly more right skewed, as illustrated in Figure 3. The same figure also plots the optimality relation $p^{opt}(\bar{s})$ from Figure 1 together with the market clearing curve $p^{mc}(\bar{s})$, and illustrates that the equilibrium price p and loan riskiness $\bar{s}^*$ are higher when optimists' optimism is more right-skewed. The next result shows that this observation holds for a general class of distributions.

Theorem 4. *Consider the collateral equilibrium characterized in Theorem 2 and let $\bar{s}^*$ denote the equilibrium loan riskiness.*

(i) If optimists' optimism becomes weakly more right-skewed, i.e., if their prior is changed to $\tilde{F}_1$ that satisfies $\tilde{F}_1 \succeq_R F_1$ and $\tilde{F}_1 \succ_O F_0$ (so that assumption (O) continues to hold), then: the asset price p, the overvaluation ratio θ, the loan riskiness $\bar{s}^*$, and the leverage ratio L

weakly increase.

(ii) If moderate traders' optimism becomes weakly more skewed to the left of $\bar{s}^*$, i.e., if their prior is changed to $\tilde{F}_0$ that satisfies $F_0 \succeq_{R,\bar{s}^*} \tilde{F}_0$ and $F_1 \succ_O \tilde{F}_0$, then: the asset price p and the overvaluation ratio θ weakly increase.

I provide a sketch proof of this result, which is completed in Appendix A.4. First observe that, using Eq. (10), the relation $p^{opt}(\bar{s})$ can be rewritten as

$$p^{opt}(\bar{s}) = \frac{E_0[v(s)]}{1+r} + \frac{1}{1+r}(1-F_0(\bar{s}))(E_1[v(s) \mid s \geq \bar{s}] - E_0[v(s) \mid s \geq \bar{s}]). \quad (23)$$

In view of the asymmetric filtering result, the distance of the asset price from the moderate valuation depends on moderate lenders' assessment of the non-default probability, and traders' valuation differences conditional on no default. For part (i), the analysis in Appendix A.4 shows that

$$\tilde{E}_1[v(s) \mid s \geq \bar{s}] \geq E_1[v(s) \mid s \geq \bar{s}] \text{ for each } \bar{s} \in \left(s^{\min}, s^{\max}\right), \quad (24)$$

where $\tilde{E}_i[\cdot]$ denotes the expectation operator with respect to distribution $\tilde{F}_i$. That is, when optimists' optimism becomes more right-skewed, their valuation of the asset conditional on any upper-threshold event increases, even though their unconditional valuation is the same. Using Eq. (24), it follows that the optimality curve $p^{opt}(\bar{s})$ shifts up pointwise. As the market clearing curve $p^{mc}(\bar{s})$ remains constant, the equilibrium asset price p and the loan riskiness $\bar{s}^*$ increase, which further implies the remaining comparative statics. For part (ii), a similar argument carried out in Appendix A.4 shows that the optimality curve $p^{opt}(\bar{s})$ shifts up over the region $\bar{s} \in (0, \bar{s}^*)$. In this case, the market clearing curve $p^{mc}(\bar{s})$ also shifts up over the region $\bar{s} \in (0, \bar{s}^*)$. Intuitively, as lenders' optimism is more skewed to the left of $\bar{s}$, they value the debt contracts more, which enables optimists to borrow more against a loan with a given riskiness. The shift of the market clearing curve $p^{mc}(\bar{s})$, as well as the shift of the optimality curve $p^{mc}(\bar{s})$, tends to increase the asset price, which implies that p increases.[14]

Theorem 4 points to the importance of the *skewness* of belief heterogeneity for the asset price. A natural question is whether the *level* of belief heterogeneity has similar robust predictions regarding the price of the asset. The answer is no, as illustrated in the following example.

Example 2 (Ambiguous Price Effect of Increased Belief Heterogeneity). *Consider the first scenario in Example 1 in which optimists are optimistic because they find the bad event unlikely, i.e., they have the prior $F_{1,B}$. Suppose the moderate and the optimistic beliefs*

[14] Note, however, that the effect on loan riskiness and leverage in this part is ambiguous. Intuitively, the fact that optimists can borrow more against the asset acts similar to a positive wealth shock, which tends to decrease leverage. This effect counters the shift of the $p^{opt}(\bar{s})$ curve which tends to increase leverage, thus the net effect is ambiguous. See Appendix A.4 for the details of this argument.

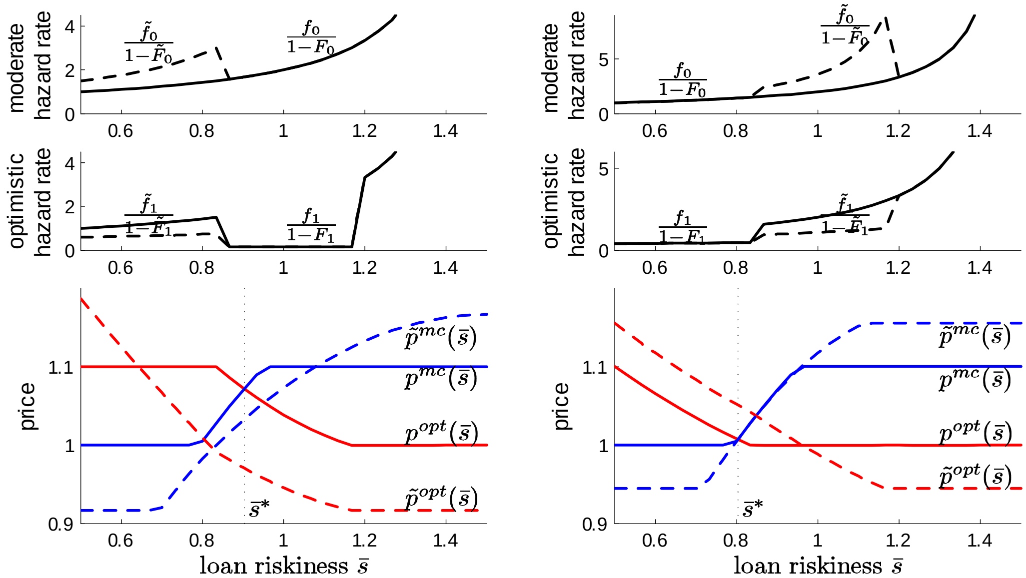

Figure 4: The left panel plots the equilibrium in the first scenario considered in Example 2: the increase in belief heterogeneity is concentrated to the left of state $\bar{s}^*$ and it decreases the asset price. The right panel plots the equilibrium in the second scenario considered in Example 2: the increase in belief heterogeneity is to the right of $\bar{s}^*$ and it increases the asset price.

are changed to $\tilde{F}_0 = F_{0,G}$ *and* $\tilde{F}_1 = F_{1,BG}$ *with density functions given by*

$$f_{0,G} = \begin{cases} 1 & \text{if } s \in \mathcal{S}_B \\ 1 + 0.5 & \text{if } s \in \mathcal{S}_N \\ 1 - 0.5 & \text{if } s \in \mathcal{S}_G \end{cases}, \quad f_{1,BG} = \begin{cases} 0.4 & \text{if } s \in \mathcal{S}_B \\ 1.3 - 0.5 & \text{if } s \in \mathcal{S}_N \\ 1.3 + 0.5 & \text{if } s \in \mathcal{S}_G \end{cases}.$$

That is, moderate traders' prior probability for the normal event increases and their probability for the good event decreases, while the opposite happens to optimists' prior. As the right panel of Figure 4 shows, in this example, the increase in belief heterogeneity leads to an increase in the asset price.

Consider the second scenario in Example 1 in which optimists are optimistic because they find the good event more likely than the normal event, i.e., they have the prior $F_{1,G}$*. Suppose the moderate and the optimistic beliefs are changed to* $\tilde{F}_0 = F_{0,B}$*,* $\tilde{F}_1 = F_{1,GB}$ *with density functions given by*

$$f_{0,B} = \begin{cases} 1 + 0.5 & \text{if } s \in \mathcal{S}_B \\ 1 - 0.25 & \text{if } s \in \mathcal{S}_N \\ 1 - 0.25 & \text{if } s \in \mathcal{S}_G \end{cases}, \quad f_{1,GB} = \begin{cases} 1(1 - 0.5) & \text{if } s \in \mathcal{S}_B \\ 0.1(1 + 0.25) & \text{if } s \in \mathcal{S}_N \\ 1.9(1 + 0.25) & \text{if } s \in \mathcal{S}_G \end{cases}.$$

That is, moderate traders' prior probability for the bad event increases and their relative prob-

ability for good and the normal event remains constant, while optimists' prior probability for the bad event decreases. Figure 4 shows, in this scenario, the increase in belief heterogeneity leads to a decrease in the asset price.

Example 2 illustrates that the increase in belief heterogeneity has no robust predictions on the price of the asset. This is intuitive in view of the asymmetric filtering property $p^{opt}(\bar{s})$. In particular, since both the optimistic and the moderate beliefs play a part in the determination of the asset price, the price tends to increase as optimists become more optimistic, but it also tends to decrease as pessimists become more pessimistic.

I next show that increased belief heterogeneity has robust predictions regarding the asset price if its skewness is also taken into account. As illustrated in Figure 4, the two scenarios of Example 2 differ in the *type* of the increase in belief heterogeneity. In the first case, the belief heterogeneity is concentrated to the left of the default threshold $\bar{s}^*$, and the asset price decreases. In the second case, the belief heterogeneity is concentrated to the right of the default threshold $\bar{s}^*$, and the asset price increases. The next result shows that this is a general property, that is, belief heterogeneity has unambiguous effects on the asset price if it is concentrated to the left, or to the right, of the equilibrium default threshold $\bar{s}^*$.

Theorem 5. *Consider the collateral equilibrium characterized in Theorem 2 and let $\bar{s}^*$ denote the equilibrium loan riskiness, which is also the threshold state below which loans default. Consider a (weak) increase in belief heterogeneity, in particular, suppose beliefs are changed to $\tilde{F}_1$ and $\tilde{F}_0$ that satisfy $\tilde{F}_1 \succeq_O F_1$ and $F_0 \succeq_O \tilde{F}_0$:*

(i) Suppose the increase in belief heterogeneity is concentrated to the right *of $\bar{s}^*$, that is, suppose $\frac{1-\tilde{F}_1(s)}{1-F_1(s)}$ and $\frac{1-\tilde{F}_0(s)}{1-F_0(s)}$ are constant over the set $\left(s^{\min}, \bar{s}^*\right)$. Then the asset price p, the loan riskiness $\bar{s}^*$, and the leverage ratio L weakly increase.*

(ii) Suppose the increase in belief heterogeneity is concentrated to the left *of $\bar{s}^*$, that is, suppose $\frac{1-\tilde{F}_1(s)}{1-F_1(s)}$ and $\frac{1-\tilde{F}_0(s)}{1-F_0(s)}$ are constant over the set $(\bar{s}^*, s^{\max})$. Then the asset price p weakly decreases.*

Theorem 4 suggests that an increase in belief heterogeneity about the relative likelihood of good events increases the asset price while an increase in belief heterogeneity about the likelihood of bad events depresses the price, even if the average valuation among traders is unchanged. For instance, a seemingly neutral news about good events can lead to an increase in asset prices if the news is interpreted differently by different traders. This is because, with this type of news, optimists will be able to leverage more easily and bid up asset prices. In contrast, neutral news about bad events can decrease asset prices by tightening optimists' financial constraints. This result creates a presumption that the media focus on good or bad events may be an important driving force of asset price fluctuations.

Taken together with Theorem 4 and Example 2, Theorem 4 also emphasizes that the skewness of belief heterogeneity matters for the price of the asset, to a greater extent than the level of belief heterogeneity. These results complete the characterization of the effect of different types of belief heterogeneity on the asset price in the baseline model. In the next section, I provide a dynamic extension of this baseline model to analyze the implications of these results for speculative bubbles based on belief heterogeneity. In Section 7, I generalize the asymmetric filtering result to an extension of the baseline model with short selling.

5 Dynamic Model: Financing Speculative Bubbles

In this section, I consider a dynamic extension of the model to show how the collateral constraints interact with the speculative component of asset prices (resulting from the resale option value) identified in Harrison and Kreps (1978). I first describe the basic environment without financial constraints and illustrate that the asset price features a speculative component. I next characterize the dynamic collateral equilibrium and analyze the comparative statics of the speculative component of the asset price.

5.1 Basic Dynamic Environment

Consider an infinite horizon OLG economy in which the periods and generations are denoted by $n \in \{0, 1, ...\}$. There is a continuum of traders in each generation n, who are born in period n and live in periods n and $n+1$. Each trader of generation n has an endowment of the numeraire good in period n, and consumes only in period $n+1$. The resources can be transferred between periods by investing either in the bond B or the asset A. Bond B is supplied elastically at a normalized price 1 in every period. Each unit of the bond yields $1+r$ units of the numeraire good in the next period, and then disappears (i.e., the bond pays dividend only once). Asset A is in fixed supply, which is normalized to 1. The asset yields a_n units of dividends in each period n. Suppose that log dividend follows a random walk, that is, the dividend yield follows the process

$$a_{n+1} = a_n s_{n+1}. \tag{25}$$

Here, s_{n+1} is a random variable with distribution F^{true} which has a density function that is continuous and positive over $\mathcal{S} = \left[s^{\min}, s^{\max}\right] \subset \mathbb{R}_{++}$. Suppose also that $1 \in \mathcal{S}$ and that the mean of s_{n+1} is normalized to 1. In other words, the next period dividend yield fluctuates around the current dividend yield a_n, with expected value equal to a_n.

All young traders in period n observe all past realizations of the dividend yield and the current realization a_n, but they have heterogeneous priors about the next period realization a_{n+1}. In each period n, similar to the static model, there are two types of young traders, *optimists* and *moderates*, respectively with priors F_1 and F_0 about the next period state s_{n+1}.

Assumption ($\mathbf{O}_\mathrm{d}$). Period n young traders' belief distributions F_1 and F_0 for the next period state s_{n+1} have density functions f_1, f_0 that are continuous and positive over $\mathcal{S}$. The moderate belief distribution is given by $F_0 = F^{true}$ while the optimistic distribution satisfies $F_1 \succ_O F_0$.

In addition, traders' beliefs for the random variables s_{n+k}, for $k \geq 2$, are identical and given by the true distribution F^{true}.

One way to interpret this assumption is that all traders know the dividend yield process described in (25), but in every period, some traders (optimists) become optimistic regarding the next period realization.[15] Under assumption ($\mathrm{O_d}$), optimists' expectation for the dividend yields in any future period is given by

$$E_{n,1}\left[a_{n+k}\right] = E_{n,1}\left[a_{n+1}\right] = E_1\left[a_n\right](1+\varepsilon).$$

Here, the parameter

$$\varepsilon \equiv E_{n,1}\left[s_{n+1}\right] - 1 > 0$$

controls optimists' level of optimism (recall that the true distribution has mean equal to 1). Consequently, optimists' present discounted value of the future dividends can be calculated as

$$p_1^{pdv}\left(a_n\right) \equiv \sum_{k=1}^{\infty} \frac{E_{n,1}\left[a_{n+k}\right]}{(1+r)^k} = \frac{a_n(1+\varepsilon)}{r}.$$

Note that moderate traders' present discounted value is given by $p_0^{pdv}\left(a_n\right) = a_n/r$. Thus, optimists' overvaluation of the asset is given by ε/r. Intuitively, optimists expect the next period realization for the dividend yield to be higher, and they expect future dividend yields to fluctuate around this higher (expected) level. This leads to the valuation difference ε/r.

Similar to the two-period model, moderate traders would like to short sell the asset, which is ruled out by assumption (S). Let $\left(w_{i,n}\right)_{i\in\{1,0\}}$ denote type i traders' endowment of the numeraire good, and suppose

$$w_{i,n} = \omega_i a_n, \text{ where } \omega_i \in \mathbb{R}_{++}.$$

That is, young traders' endowments are proportional to the current dividend yield of the asset. This assumption is not essential for the main economic points, but it will provide much tractability in the subsequent analysis.[16] I also assume

$$\omega_0 > \frac{1+\varepsilon}{r-\varepsilon}, \tag{26}$$

which will rule out a corner solution for the asset price by ensuring that young traders' endowments are sufficiently large to cover the expenditure on the asset.

[15] There could be a number of explanations for the source of this type of optimism. As in Scheinkman and Xiong (2003), optimists may be overconfident about a signal they receive about the next period shock. Alternatively, optimists may be simply optimistic about the next period shock, thinking that the current period is special. Reinhart and Rogoff (2008) refer to this type of optimism as "this time it is different syndrome."

[16] I thank Ivan Werning for suggesting this simplification.

This completes the description of the basic elements of the dynamic economy. Note that the economy has a recursive structure. This is because the dividend yield process follows a random walk (cf. Eq. (25)), and young traders' beliefs are formed independently of the past dividend yield realizations (cf. assumption ($\mathrm{O_d}$)). This observation leads to the following lemma, which provides a sufficient statistic for the dynamic economy and simplifies the subsequent notation.

Lemma 2. *Given any history $(a_0, ..., a_{n-1}, a_n)$ of dividend yield realizations, the current dividend yield a_n is a sufficient statistic for the determination of the equilibrium allocations in this economy.*

In view of this lemma, let $a \equiv a_n \in \mathbb{R}_{++}$ denote the current dividend yield, $s \equiv s_{n+1} \in \mathcal{S}$ denote the next period shock, and $p(a)$ denote the current asset price.

5.2 Speculative Bubbles without Financial Constraints

As a benchmark, I first consider the asset price in an economy in which individuals can borrow and lend freely in a competitive loan market. In other words, there exists no limited liability or enforcement problems. Since the aggregate endowment is sufficiently large (cf. condition (26)), some of this endowment will necessarily be invested in the bond, which pins down the equilibrium interest rate as r. In other words, optimists can always borrow at rate $1+r$ from moderate traders. Thus, optimists bid up the asset price until, in equilibrium, the asset price is equal to the present discounted value of the asset with respect to the optimistic priors. That is,

$$p(a) = \frac{1}{1+r}\left(a(1+\varepsilon) + \int_{\mathcal{S}} p(as)\, dF_1\right), \text{ for all } a \in \mathbb{R}_{++}. \tag{27}$$

Eq. (27) provides a recursive characterization of the asset price which can be solved as

$$p(a) = \frac{a(1+\varepsilon)}{r-\varepsilon}. \tag{28}$$

Note that the asset price $p(a)$ is higher than optimists' present discounted valuation of the asset, $p_1^{pdv}(a) = \frac{a(1+\varepsilon)}{r}$. In other words, in equilibrium, the asset price exceeds the present discounted value with respect to all traders' priors. The component of the asset price in excess of the present discounted value of the holder of the asset, $p(a) - p_1^{pdv}(a)$, is what Scheinkman and Xiong (2003) call a speculative "bubble." For the remainder of the analysis, I will also adopt this terminology. I also define

$$\lambda = \frac{p(a) - p_1^{pdv}(a)}{p(a)} = \frac{\varepsilon}{r} \tag{29}$$

as the share of the speculative component of the asset price in the unconstrained economy.

The asset price features a speculative component because optimists hold the asset not only for the higher expected dividend gains in the next period, but also since they are planning to sell the asset to a trader who will be even more optimistic than them in the next period. In view of these expected speculative capital gains, optimists bid up the asset price higher than the present discounted value of dividends.

The expression in (29) also implies that the speculative component could represent a large *fraction* of the asset price, even for a relatively small belief disagreement ε (especially when the interest rate is low). The rationale for this observation is related to a powerful amplification effect: *the dynamic multiplier*. Note that optimists in the next period also expect to make speculative capital gains by selling the asset to yet more optimistic traders in the subsequent period, which increases the price in the next period. But this further increases the valuation of current optimists who are planning to sell to future optimists, increasing the current asset price further. In other words, a high asset price in the next period feeds back into the asset price today, amplifying the effect of heterogeneous beliefs and leading to a large speculative component.

I next incorporate financial constraints into this economy. With financial constraints, the asset price will not necessarily satisfy the recursion in (27). Thus, the expression in (29) will be an upper bound on the share of the speculative component. Rather, the asset price will lie between the optimistic and the moderate valuations, with the exact recursion (and the share of the speculative component) being determined by the type of financial constraints.

5.3 Recursive Collateral Equilibrium

I model financial constraints using the collateral equilibrium notion described in the earlier sections. This subsection extends the definition of the collateral equilibrium in Section 2 to the dynamic setting, and it characterizes this equilibrium by utilizing the analysis in Section 3.

I define the *value function* as the payoff of the asset in the next period,

$$v(a,s) \equiv as + p(as) \text{ for each } s \in \mathcal{S}. \tag{30}$$

Similar to the static model, loans in this economy are non-recourse and non-contingent. Formally, given state $a \in \mathbb{R}_{++}$, *a unit debt contract*, denoted by $\varphi \in \mathbb{R}_+$, is a promise of φ units of the numeraire good in the next period by the borrower collateralized by 1 unit of the asset. In period $n+1$, each contract φ pays $\min(v(a,s), \varphi)$. Each debt contract $\varphi \in \mathbb{R}_+$ is traded in an anonymous market at a competitive price $q(a,\varphi)$.

Let $x_i(a)$ and $z_i(a,\varphi)$ denote trader i's position respectively in the asset and the debt contract φ. The problem of a type i trader in a period with state $a \in \mathbb{R}_{++}$ is given by

$$\max_{\substack{x_i(a)\geq 0,\\ z_i(a,\cdot)}} \quad x_i^A(a)\, E_i[v(a,s)] + x_i^B(a)(1+r) + \int_{\mathbb{R}_+} z_i(a,\varphi)\, E_i[\min(v(a,s),\varphi)]\, d\varphi, \quad (31)$$

$$\text{s.t.} \quad p(a)\, x_i^A(a) + x_i^B(a) + \int_{\mathbb{R}_+} q(a,\varphi)\, z_i(a,\varphi)\, d\varphi \leq \omega_i a,$$

$$\int_{\mathbb{R}_+} \max(0, -z_i(a,\varphi))\, d\varphi \leq x_i^A.$$

Note that the problem of a trader in the dynamic economy is identical to problem (6) in the static economy, except for the fact that the trader in the dynamic economy has no endowments of the asset, i.e., $\alpha_i = 0$ (because the asset is held by the old generation).

Definition 4 (Recursive Collateral Equilibrium). *Under assumptions* (O_d) *and (S), and condition* (26)*, a recursive collateral equilibrium (RCE) is a collection of prices* $\left(p(a), [q(a,\varphi)]_{\varphi\in\mathbb{R}_+}\right)_{a\in\mathbb{R}_{++}}$ *and allocations* $\left((x_i(a),\ z_i(a,\cdot))_{i\in\{1,0\}}\right)_{a\in\mathbb{R}_{++}}$ *such that, for each dividend realization* $a \in \mathbb{R}_{++}$*, the allocation of each trader* $i \in \{1,0\}$ *solves problem* (31)*, and asset and unit debt markets clear, that is,* $\sum_{i\in\{1,0\}} x_i^A(a) = 1$ *and* $\sum_{i\in\{1,0\}} z_i(a,\varphi) = 0$ *for each* $\varphi \in \mathbb{R}_+$.

Note that, given the value function in the next period (cf. Eq. (30)), the economy in the current period is very similar to the static economy analyzed earlier, with the only difference that the asset is not endowed to the current young generation. The next result uses this observation to show that a recursive collateral equilibrium can be constructed based on the analysis in Section 3. Note that a *loan with riskiness* $\bar{s}$ in the dynamic economy is a debt contract $\varphi = v(a,\bar{s})$ that defaults if and only if the next period state is below the threshold level $\bar{s} \in \mathcal{S}$.

Lemma 3 (Constructing the RCE). *Suppose there exists a collection of price and loan riskiness pairs,* $(p(a) \in \mathbb{R}_+, \bar{s}^*(a) \in \mathcal{S})_{a\in\mathbb{R}_{++}}$*, such that for each* $a \in \mathbb{R}_{++}$*, the pair* $(p(a), \bar{s}^*(a))$ *corresponds to the collateral equilibrium characterized in Theorem 2 for the static economy*

$$(\mathcal{S};\ v(a,\cdot);\ \{F_i\}_i;\ \{w_i \equiv \omega_i a\}_i;\ \{\alpha_1 = 0, \alpha_0 = 1\}). \quad (32)$$

Then, there exists a recursive collateral equilibrium in which, for each $a \in \mathbb{R}_{++}$*, optimists make leveraged investments in the asset by borrowing through a single loan with riskiness* $\bar{s}^*(a)$*, and the asset price is given by* $p(a)$.

Lemma 3 reduces the characterization of the RCE to the static case along with the additional requirement that the value function $v(a,\cdot)$ is endogenously determined by Eq. (30). I next use this result to construct a RCE through a fixed point argument. Given that endowments are linear in a, consider price and value functions that are linearly homogeneous in a

and a level of riskiness that is constant in a, that is,

$$p(a) = p_d a \text{ and } \bar{s}^*(a) = \bar{s}_d^* \in \mathcal{S}, \tag{33}$$

where p_d denotes the *price to dividend ratio.* Plugging this conjecture in Eq. (30) also implies a linearly homogeneous value function $v(a, s) = v_d(s \mid p_d)\, a$, where $v_d(\cdot \mid p_d) : \mathcal{S} \to \mathbb{R}$ is defined by

$$v_d(s \mid p_d) = s(1 + p_d). \tag{34}$$

Next note that using Eq. (33) in the characterization of the static equilibrium (cf. Eqs. (10) and (17)) and using linear homogeneity in a, the constants $p_d, \bar{s}_d^*$ are characterized as the collateral equilibrium of the static economy

$$\mathcal{E}(p_d) = \left(\mathcal{S};\ v_d(\cdot \mid p_d);\ \{F_i\}_i;\ \{w_i \equiv \omega_i\}_i;\ \{\alpha_1 = 0, \alpha_0 = 1\}\right). \tag{35}$$

In particular, $(p_d, \bar{s}_d^*)$ is the unique solution to the following equations:[17]

$$p_d = p^{opt}(\bar{s}_d\ ;\ v_d(\cdot \mid p_d)) = p^{mc}(\bar{s}_d\ ;\ v_d(\cdot \mid p_d)). \tag{36}$$

Put differently, the characterization of the RCE is transformed into the characterization of the static equilibrium corresponding to a particular value function $v_d(\cdot \mid p_d)$, which also depends on the price to dividend ratio because the value function incorporates future asset prices.

Next define $(P_d(\tilde{p}_d), S_d(\tilde{p}_d))$ as the static equilibrium for the economy $\mathcal{E}(\tilde{p}_d)$, that is, as the solution to Eq. (36) if the future price to dividend ratio were given by $\tilde{p}_d$. Then, the equilibrium price to dividend ratio, p_d, is a fixed point of the mapping $P_d(\cdot)$. To solve for this fixed point, first note that p_d lies in the following interval:

$$\left[p_d^{\min} \equiv \frac{1}{r}, p_d^{\max} \equiv \frac{1+\varepsilon}{r-\varepsilon}\right].$$

Intuitively, the upper limit corresponds to the unconstrained solution (28), and the lower limit is the price to dividend ratio that would obtain if moderate traders were always the marginal holders of the asset. Second, the analysis in Appendix A.5 shows that the mapping $P_d(\tilde{p}_d)$ is strictly increasing over $\left[p_d^{\min}, p_d^{\max}\right]$ with the boundary conditions

$$P_d\left(p_d^{\min}\right) > p_d^{\min} \text{ and } P_d\left(p_d^{\max}\right) \leq p_d^{\max}. \tag{37}$$

It follows that $P_d(\cdot)$ has a unique fixed point $p_d \in (p_d^{\min}, p_d^{\max}]$, which corresponds to a RCE. The following result summarizes this discussion.

Theorem 6 (Existence and Characterization of RCE). *Under assumptions (O_d), (S)*

[17] The notations $p^{opt}(\cdot\ ;\ v), p^{mc}(\cdot\ ;\ v)$ respectively denote the functions $p^{opt}(\cdot), p^{mc}(\cdot)$ evaluated with the particular value function $v(\cdot)$.

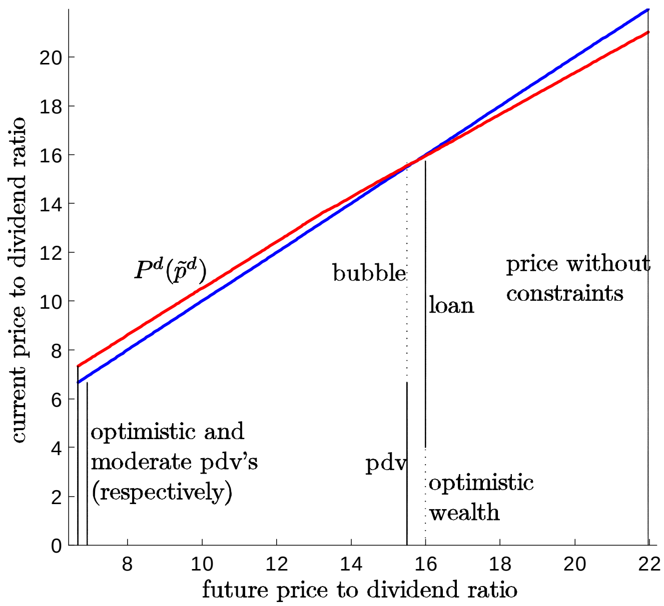

Figure 5: The figure plots the current equilibrium price coefficient, $P_d(\tilde{p}_d)$, that would obtain if the future price coefficient were given by $\tilde{p}_d$. The equilibrium price coefficient, p_d, is the fixed point of this mapping

and condition (26)*, there exists a recursive collateral equilibrium in which equilibrium loan prices are linearly homogeneous in the current dividend realization and the equilibrium loan riskiness is constant, i.e.,* $p(a) \equiv p_d a$ *and* $\bar{s}^*(a) = \bar{s}^*_d$ *for each* $a \in \mathbb{R}_{++}$*. The price to dividend ratio* p_d *is the unique fixed point of the mapping* $P_d : [p_d^{\min}, p_d^{\max}] \to [p_d^{\min}, p_d^{\max}]$*, where* $P_d(\tilde{p}_d)$ *is the collateral equilibrium price of the static economy* $\mathcal{E}(\tilde{p}_d)$ *in* (35)*.*

The next example illustrates this equilibrium and the effect of collateral constraints on the speculative bubble.

Example 3. *Consider the prior distributions* F_0 *and* $F_{1,G}$ *of Example 1 in which the valuation difference for the next period shock is given by* $\varepsilon = E_1[s] - E_0[s] = 0.1$*. Consider the corresponding RCE with interest rate* $r = 0.15$ *and optimistic wealth* $\omega_1 = 4$*. Figure 5 plots the mapping* $P_d(\cdot)$ *and shows that it intersects the 45 degree line exactly once, which corresponds to the equilibrium. The equilibrium price is lower than the unconstrained level, however it is still higher than the present discounted value according to either the moderate or optimistic priors (which are close to each other). In particular, in this example, there is a large speculative component in the asset price despite the financial constraints.*

The figure also illustrates the balance sheet of the optimistic investors. Optimists' down-payment is about 1/4 of the asset price, and they borrow the remaining amount from moderate traders, collateralized against one unit of the asset. In particular, moderate lenders, who cor-

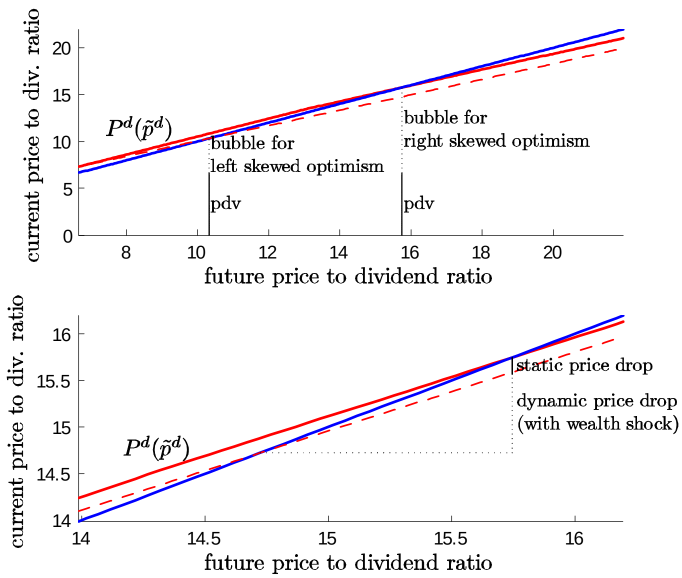

Figure 6: The top panel plots the dynamic equilibrium corresponding to the two scenarios of Example 1, illustrating that an increase in the left-skewness of optimism reduces the speculative bubble. The bottom panel displays the response of the price to a 10% negative wealth shock to optimists, and breaks down the response to a static and a dynamic component.

rectly know the dividend yield process in (25)*, agree to finance about 3/4 of the asset purchase despite the fact that the present discounted value of the asset is less than half of its price.*

The last feature of this example provides insights for how large speculative bubbles can exist when optimists are financially constrained. In this example, lenders have correct priors and they know that the asset price is much higher than the present discounted value. Nonetheless, they agree to finance most of the asset purchase. This is because lenders' incentives are not related to the present discounted value of the collateral, but to the expected value of the collateral, given by $\frac{E_0[v_d(\cdot \mid p_d)]a}{1+r}$. This value is much larger than the present discounted value because it also contains a speculative component (captured by the future price to dividend ratio, p_d). Intuitively, lenders agree to extend large loans because they think that, should the borrower default, they could always sell the collateral to another optimist in the next period.

Put differently, a marked characteristic of this speculative episode is that the bubble raises everybody's valuation, and optimists' and moderates' valuation difference in any period, $(E_1 [v_d (\cdot \mid p_d)] - E_0 [v_d (\cdot \mid p_d)])\, a$, is relatively small. As in the unconstrained case (cf. Section 5.2), a large bubble forms from the accumulation of small valuation differences through the dynamic multiplier. This is perhaps unfortunate, because a small valuation difference makes the financing of bubbles relatively easy, opening the way for large speculative bubbles even when optimists are financially constrained.

Naturally, as the previous sections show, a small valuation difference does not guarantee that financial constraints are lax. Whether financing will actually go through, and the size of the speculative bubble, also depends on a number of other factors, such as optimists' wealth level and the type of belief heterogeneity. For example, consider the equilibrium in Example 3 with the only difference that the optimistic priors are changed to $F_{1,B}$ (defined in Example 1). This prior leads to the same asset valuation, but it is more left-skewed than $F_{1,G}$. The top panel in Figure 6 shows that, in response to this change, the speculative bubble shrinks by about half.

The next result shows that this is a general property, that is, an increase in the right-skewness of optimism unambiguously increases the asset price and the share of the speculative component. To state the result, I define the overvaluation ratio $\theta_d \in (0, 1]$ in the dynamic setting (cf. Eq. (21)) as the unique solution to

$$p_d = (1 - \theta_d) \frac{E_0 [v_d (\cdot \mid p_d)]}{1 + r} + \theta_d \frac{E_1 [v_d (\cdot \mid p_d)]}{1 + r}. \tag{38}$$

I also define the share of the speculative component (i.e., generalization of Eq. (29) to the financially constrained economy) as[18]

$$\lambda_d = \frac{p(a) - p^{pdv}(a)}{p(a)}, \text{ where } p^{pdv}(a) = (1 - \theta_d) p_0^{pdv}(a) + \theta_d p_1^{pdv}(a). \tag{39}$$

The following result establishes that an increase in the right-skewness of belief heterogeneity increases the asset price and the share of the speculative component.

Theorem 7. *Consider the recursive collateral equilibrium characterized in Theorem 6 and let $\bar{s}_d^*$ denote the equilibrium loan riskiness.*

(i) If optimists' optimism becomes weakly more right-skewed, i.e., if their prior is changed to $\tilde{F}_1$ that satisfies $\tilde{F}_1 \succeq_R F_1$ and $\tilde{F}_1 \succ_O F_0$ (so that assumption (O_d) continues to hold), then: the price to dividend ratio p_d, the overvaluation ratio θ_d, the loan riskiness $\bar{s}_d^*$, and the share of the speculative component λ_d weakly increase.

(ii) If moderate traders' optimism becomes weakly more skewed to the left of $\bar{s}_d^*$, i.e., if their prior is changed to $\tilde{F}_0$ that satisfies $F_0 \succeq_{R,\bar{s}_d^*} \tilde{F}_0$ and $F_1 \succ_O \tilde{F}_0$, then: the price to dividend ratio p_d, the overvaluation ratio θ_d, and the share of the speculative component λ_d weakly increase.

In view of the above characterization, Theorem 7 is not surprising, as it follows from its counterpart, Theorem 4, for the static economy. Similarly, the analogues of Theorems 3 and 5 can be obtained for the dynamic economy. While the details of these results are slightly different, Theorem 7 suggests that the general message of Section 4 continues to apply in the present dynamic setting. In particular, the type of belief heterogeneity matters for asset

[18] Unlike the unconstrained case, the marginal holder of the asset is not necessarily an optimist, hence the relevant present discounted value is defined as an average of optimistic and moderate present discounted values, weighted by the overvaluation ratio θ_d.

The share of the speculative premium is independent of the state $a \in \mathbb{R}_{++}$ because the functions $p(a), p_0^{pdv}(a)$, and $p_1^{pdv}(a)$ are linearly homogeneous in a.

prices (and the speculative component of prices), to a greater extent than the level of belief heterogeneity.

One feature worth emphasizing is that, while the comparative statics of the asset price in the dynamic model are similar to the static model, the responses are typically amplified due to the dynamic multiplier. To see this, consider a persistent negative shock that affects optimists' financial conditions but not the fundamental value of the asset, such as a negative wealth shock to optimists or an increase in belief heterogeneity about bad events. This shock has a greater effect on the asset price in the dynamic model than in the static model. In particular, the current price decreases not only because of the negative shock to the financial conditions of the current optimists, but also because of the negative shock to the financial conditions of future optimists. As future optimists will be more constrained, the asset price in the next period will be lower, which reduces the value function (cf. Eq. (30)) and depresses the current asset price. The bottom panel of Figure 6 illustrates this amplification effect by considering the response of the equilibrium in Example 3 to a 10% negative wealth shock to optimists. The figure breaks the resulting drop in the price to dividend ratio to a static component which would obtain if the future value function remained constant, and the dynamic component which is the additional drop that obtains because the value function depends on the future price to dividend ratio. The figure illustrates that the dynamic response of the asset price to optimists' financial conditions is (typically) much stronger than the static response.

6 Collateral Equilibrium with Contingent Loans

The previous sections have characterized the effect of belief heterogeneity on asset prices in a baseline model in which loans are restricted to be non-contingent and short selling is not allowed. While the baseline model is a good starting point, it is important to verify the robustness of the results to more general settings, especially because allowing for a richer set of contracts introduces new economic forces. This section considers an extension of the static model in Section 2 in which debt contracts can be fully contingent on the continuation state $s \in \mathcal{S}$. The analysis in this section establishes two results. First, a version of the asymmetric filtering result (cf. Theorem 1) also applies in this setting with fully contingent loans. Second, when optimists can borrow with fully contingent loans, the asset price does not necessarily lie in the interval (3). In particular, when the optimistic wealth is sufficiently large, the asset price in this setting exceeds even the most optimistic valuation.

6.1 Definition of Equilibrium with Contingent Loans

Formally, a unit contingent debt contract, denoted by $\boldsymbol{\varphi} : \mathcal{S} \rightarrow \mathbb{R}_+$, is a collection of promises of $\boldsymbol{\varphi}(s) \geq 0$ units in each state $s \in \mathcal{S}$, collateralized by 1 unit of the asset.[19] The borrower

[19] I assume that contracts must make non-negative promises in all continuation states, because a negative promise by the borrower (which is essentially a promise by the lender) would not be enforced by the court

defaults on the contract if and only if the value of the asset is less than the promise on the contract. Thus, the contract pays $\min(v(s), \boldsymbol{\varphi}(s))$ units. As before, each debt contract $\boldsymbol{\varphi}$ is traded in an anonymous market at a competitive price $q(\boldsymbol{\varphi})$. Let $\mathcal{D}$ denote the set of all unit debt contracts, and let $z_i : \mathcal{D} \to \mathbb{R}$ denote a measurable function where $z_i(\boldsymbol{\varphi})$ represents trader i's position on debt contract $\boldsymbol{\varphi}$.[20]

Traders solve the following analogue of Problem (6):

$$
\begin{aligned}
\max_{x_i \geq 0, z_i(\cdot)} \quad & x_i^A E_i[v(s)] + x_i^B (1+r) + \int_{\mathcal{D}} z_i(\boldsymbol{\varphi}) E_i[\min(v(s), \boldsymbol{\varphi}(s))] \, d\boldsymbol{\varphi}, \\
\text{s.t.} \quad & p x_i^A + x_i^B + \int_{\mathcal{D}} q(\boldsymbol{\varphi}) z_i(\boldsymbol{\varphi}) \, d\boldsymbol{\varphi} \leq w_i + p\alpha_i, \\
& \int_{\mathcal{D}} \max(0, -z_i(\boldsymbol{\varphi})) \, d\boldsymbol{\varphi} \leq x_i^A.
\end{aligned}
$$

Given this problem and the extended contract space, the equilibrium is defined similarly to Section 2.1.

The characterization of equilibrium closely follows the analysis in Section 3. In particular, consider first a quasi-equilibrium by restricting traders' choices with the constraint $z_1(\cdot) \leq 0 \leq z_0(\cdot)$. To construct a quasi-equilibrium, consider contract prices

$$
q(\boldsymbol{\varphi}) = \frac{E_0[\min(v(s), \boldsymbol{\varphi}(s))]}{1+r}, \tag{40}
$$

which make moderate traders indifferent between purchasing the bond and any debt contract $\boldsymbol{\varphi}$. As in Section 3, the equilibrium will be determined by optimists' optimal contract choice given these prices.

6.2 Asymmetric Filtering with Contingent Loans

First suppose that optimists choose to invest all of their leveraged wealth in the asset, and consider optimists' optimal contract choice in a quasi-equilibrium. The same analysis for Theorem 1 (cf. Appendix A.2) applies and shows that optimists' contract choice is the solution

system in this economy since lenders do not set aside any collateral. This is without loss of generality, because if they wish, lenders can also make promises by selling a separate collateralized debt contract.

[20] The definition of $z_i(\cdot)$ poses a technical problem since the set of all contingent debt contracts, $\mathcal{D}$, is not Lebesgue measurable. This problem can be dealt with by considering an alternative economy $\mathcal{E}^k$ which is identical to the original economy $\mathcal{E}$, except for the fact that the state space is discrete and given by:

$$
\mathcal{S}^k = \left\{ s^{\min} + \frac{i}{(k-1)(s^{\max} - s^{\min})} \right\}_{i=0}^{k-1}.
$$

Note that the set $\mathcal{S}^k$ has finite number of elements, and it approximates the state space $\mathcal{S}$ for sufficiently large k. The set of all debt contracts for the economy $\mathcal{E}^k$ is given by $\mathbb{R}_+^k$, which is Lebesgue measurable. Hence, the contract allocations and the equilibrium is well defined for the economy $\mathcal{E}^k$.

Let p^k denote the asset price for the economy $\mathcal{E}^k$, and consider the limiting price $p = \lim_{k\to\infty} p^k$. The characterization of p^k follows the same steps as in the analysis of this section, and it is straightforward to show that the limiting price p is given by the characterization in Theorem 1.

to the following leveraged investment problem, which is the analogue of Problem (14) for contingent loans:

$$\max_{[\boldsymbol{\varphi}(s)\in[0,v(s)]]_{s\in\mathcal{S}}} \left(R_1^{L,cont}(\boldsymbol{\varphi}) = \frac{E_1[v(s)] - E_1[\min(v(s), \boldsymbol{\varphi}(s))]}{p - \frac{1}{1+r}E_0[\min(v(s), \boldsymbol{\varphi}(s))]} \right). \tag{41}$$

In particular, optimists can be thought of as choosing the level of promise $\boldsymbol{\varphi}(s) \in [0, v(s)]$ for each state $s \in \mathcal{S}$, to maximize the leveraged investment return $R_1^{L,cont}(\boldsymbol{\varphi})$. The solution to problem (41) can be characterized under the following assumption, which is slightly stronger than assumption (O):

Assumption (MLRP). The probability distributions F_1 and F_0 have density functions f_1, f_0 which are continuous and positive over $\mathcal{S}$, and which satisfy the monotone likelihood ratio property: that is, $\frac{f_1(s)}{f_0(s)}$ is strictly increasing over $\mathcal{S}$.

The analysis in Appendix A.6 establishes that, under assumption (MLRP), there exists a *threshold state* $\bar{s} \in \mathcal{S}$ such that optimists sell as much promise as possible for states $s < \bar{s}$, while promising zero units for states $s \geq \bar{s}$ (therefore keeping the asset return in those states). In particular, the optimal contingent debt contract is given by

$$\boldsymbol{\varphi}_{\bar{s}}(s) \equiv \begin{cases} v(s) & \text{if } s < \bar{s} \\ 0 & \text{if } s \geq \bar{s}, \end{cases} \tag{42}$$

for a threshold state $\bar{s} \in \mathcal{S}$. The next result, which is the analogue of Theorem 1 for contingent loans, characterizes the threshold state $\bar{s} \in \mathcal{S}$ of the optimal contract. The result also establishes a maximum level for the asset price such that optimists invest all of their leveraged wealth in the asset if the price is below this level, but they invest their wealth in the bond (and they do not borrow) if the price is above this level. This maximum price level is given by

$$p^{\max} = \frac{1}{1+r}\left(\int_{s^{\min}}^{s^{cross}} v(s)\, dF_0 + \int_{s^{cross}}^{s^{\max}} v(s)\, dF_1 \right), \tag{43}$$

where $s^{cross} \in \mathcal{S}$ is the unique state such that $\frac{f_0(s^{cross})}{f_1(s^{cross})} = 1$. Note that the maximum price level $p^{\max}$ is determined according to the upper envelope of optimistic and moderate probability densities, and thus it exceeds the most optimistic valuation $\frac{E_1[v(s)]}{1+r}$.

Theorem 8 (Asymmetric Filtering with Contingent Loans). *Suppose assumptions (MLRP) and (S) hold, debt prices are given by* (40) *and the asset price satisfies* $p \in \left(\frac{E_0[v(s)]}{1+r}, p^{\max}\right)$, *where* $p^{\max}$ *is given by* (43). *In a quasi-equilibrium:*

(i) There exists $\bar{s} \in [s^{cross}, s^{\max}]$ *such that optimists only sell the debt contract* $\boldsymbol{\varphi}_{\bar{s}}$ *defined in* (42)*. Optimists' collateral constraint is binding, i.e., they borrow as much as possible according to the optimal contract. Optimists choose* $x_1^B = 0$*, i.e., they invest all of their leveraged wealth in the asset A.*

(ii) The threshold state $\bar{s} \in [s^{cross}, s^{\max}]$ *of the optimal contract is characterized as the*

unique solution to:

$$p = p^{opt,cont}(\bar{s}) \equiv \frac{1}{1+r}\left(\int_{s^{\min}}^{\bar{s}} v(s)\, dF_0 + \frac{f_0(\bar{s})}{f_1(\bar{s})}\int_{\bar{s}}^{s^{\max}} v(s)\, dF_1\right). \tag{44}$$

If $p > p^{\max}$, then optimists do not borrow, and they invest all of their wealth in the bond.

Note that the function $p^{opt,cont}(\bar{s})$ is the analogue of the function $p^{opt}(\bar{s})$: it describes the asset price conditional on optimists' choice of the threshold state $\bar{s}$. Moreover, the form of $p^{opt,cont}(\bar{s})$ is very similar to the form of $p^{opt}(\bar{s})$, which suggests that optimism is asymmetrically filtered also in this setting. In particular, optimism about the relative likelihood of states above $\bar{s}$ increases the asset price, while the optimism about the relative likelihood of states below $\bar{s}$ does not increase the price. The intuition for this result can be gleaned from the shape of the optimal debt contract $\boldsymbol{\varphi}_{\bar{s}}$. These contracts make the same payment (namely, zero) in all states above the threshold $\bar{s}$, while they have an increasing payment schedule in the states below the threshold $\bar{s}$. Hence, any optimism about the relative likelihood of good states does not increase optimists' expected interest rate in (45), and thus these types of optimism increase the asset price. However, optimism about the relative likelihood of bad states increases optimists' expected interest rate in (45). Thus, these types of optimism are reflected less in the asset price.[21]

Recall also that $p^{\max} > \frac{E_1[v(s)]}{1+r}$, which shows that optimists demand the asset even if the price is greater than their valuation $\frac{E_1[v(s)]}{1+r}$. This opens the way for the equilibrium price to exceed the optimistic valuation (whenever the optimistic wealth is sufficiently large), as illustrated in Figure 7. To understand why optimists invest in the asset when $p > \frac{E_1[v(s)]}{1+r}$, consider optimists' expected interest rate on the contract $\boldsymbol{\varphi}_{\bar{s}}$, given by:

$$1 + r_1^{exp,cont}(\bar{s}) = \frac{E_1\left[\min\left(v(s), \boldsymbol{\varphi}_{\bar{s}}(s)\right)\right]}{\frac{1}{1+r}E_0\left[\min\left(v(s), \boldsymbol{\varphi}_{\bar{s}}(s)\right)\right]} = (1+r)\frac{\int_{s^{\min}}^{\bar{s}} v(s)\, dF_1}{\int_{s^{\min}}^{\bar{s}} v(s)\, dF_0}. \tag{45}$$

Unlike the expected interest rate with non-contingent loans, $r_1^{exp,cont}(\bar{s})$ is not necessarily greater than r. In particular, the ability to fine-tune their borrowing enables optimists to take loans which they perceive to be even more favorable than borrowing at the benchmark interest rate. Consequently, optimists are enticed to borrow and invest in the asset even if the price exceeds their valuation $\frac{E_1[v(s)]}{1+r}$ (which is calculated according to the benchmark rate). A complementary intuition for this result comes from the form of $p^{\max}$ in (43). The availability of fully contingent loans enables the optimists to split the asset in a way that each type traders hold the asset in the states which they assign a greater probability. Consequently, the maximum price at which optimists demand the asset is given by an upper-envelope of the moderate and the optimistic beliefs, which exceeds the optimistic valuation. This result

[21] This intuition also illustrates the limitation of the asymmetric filtering result when loans are fully contingent. Unlike regular debt contracts, a contingent debt contract, $\boldsymbol{\varphi}_{\bar{s}}$, makes a lower payment in states above $\bar{s}$ relative to states below $\bar{s}$. Hence, if optimists' optimism is changed in a way to assign a lower probability to states below $\bar{s}$, then the asset price increases (unlike the case with non-contingent contracts).

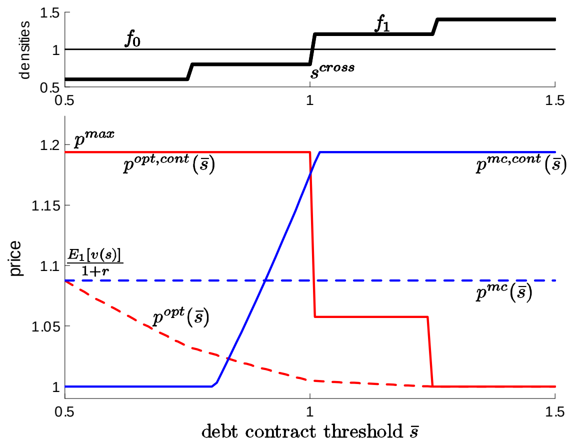

Figure 7: The top panel displays the probability densities. The bottom panel illustrates the equilibrium with and without contingent loans. In this example, the asset price with contingent loans exceeds even the most optimistic valuation.

creates a presumption that finer levels of financial engineering of loans can potentially have a large impact on asset prices.

6.3 Equilibrium Asset Price with Contingent Loans

Similar to Section 6, the equilibrium asset price is determined by combining optimists' optimal contract choice with asset market clearing. In particular, similar to Lemma 1, the asset price also satisfies the market clearing relation given by:

$$p = p^{mc,cont}(\bar{s}) \equiv \begin{cases} p^{\max} & \text{if } \frac{w_1^{\max,cont}(\bar{s})}{\alpha_0} > p^{\max} \\ \frac{w_1^{\max,cont}(\bar{s})}{\alpha_0} & \text{if } \frac{w_1^{\max,cont}(\bar{s})}{\alpha_0} \in (\frac{E_0[v(s)]}{1+r}, p^{\max}] \\ \frac{E_0[v(s)]}{1+r} & \text{if } \frac{w_1^{\max,cont}(\bar{s})}{\alpha_0} \leq \frac{E_0[v(s)]}{1+r} \end{cases}.$$

Here, $w_1^{\max,cont}(\bar{s}) = w_1 + \frac{1}{1+r}\int_{s^{\min}}^{\bar{s}} v(s)\, ds$ denotes optimists' maximum first period wealth given that they choose to borrow with the contingent debt contract $\bar{s}$. The equilibrium asset price p and the threshold level of the optimal contract $\bar{s}^*$ are characterized by considering the intersection of the strictly decreasing curve $p^{opt,cont}(\bar{s})$ and the weakly increasing curve $p^{mc,cont}(\bar{s})$ over the range $\bar{s} \in [s^{cross}, s^{\max}]$. Figure 7 displays the equilibrium with non-contingent and contingent loans. This figure illustrates that the asset price exceeds the optimistic valuation $\frac{E_1[v(s)]}{1+r}$ whenever the optimistic wealth is sufficiently large.

7 Collateral Equilibrium with Short Selling

The previous sections have characterized the effect of belief heterogeneity on asset prices in the baseline model in which short selling is not allowed. While the baseline model is a good starting point, it is important to verify the robustness of the results to more general settings, especially because allowing for a richer set of contracts introduces new economic forces. This section considers an extension of the static model in Section 2 in which short selling is allowed, and establishes that a version of the asymmetric filtering result (cf. Theorem 1) applies in this setting. This extension is also relevant for understanding the data for a fraction of assets that can be short sold (e.g., for the majority of stocks).

I model short selling of the asset symmetrically to the short selling of debt contracts. In particular, short selling also needs to be collateralized. Formally, consider the economy described in Section 2 but suppose that, in addition to the unit debt contracts, there are also collateralized short contracts. A *unit short contract*, denoted by $\psi \in \mathbb{R}_+$, is a promise of $v(s)$ units of the numeraire good conditional on state $s \in \mathcal{S}$, collateralized by ψ units of the bond. In this section, let $q^{short}(\psi)$ denote the price of the short contract $\psi \in \mathbb{R}_+$, and $q^{debt}(\varphi)$ denote the price of the debt contract $\varphi \in \mathbb{R}_+$.

Note that a trader selling the unit short contract can be interpreted as borrowing the asset from a lender, and posting ψ units of the bond as collateral in a margin account. In reality, the lender of the security will require a certain short fee.[22] In the model, the short fee demanded by the lender will be implicitly captured by the price of the short contract ψ, with the lower price corresponding to a higher short fee.

In addition to the availability of the short contracts, I also assume that only a random fraction $\gamma^{short} \in (0,1)$ of traders can sell short contracts, while only a random fraction $\gamma^{debt} \in (0,1)$ can sell debt contracts and leverage. These assumptions are made to simplify the analysis, but they are not unreasonable because short selling in financial markets (and to some extent, leverage) is confined to a small fraction of investors. I denote the short selling ability of a trader with $\eta^{short} \in \{0,1\}$, and the leverage ability with $\eta^{debt} \in \{0,1\}$. Taking the belief heterogeneity also into account, note that there are 8 types of traders, where a type is denoted by $T = \left(i, \eta^{short}, \eta^{debt}\right)$.

Let $z_T^{short}, z_T^{debt} : \mathbb{R}_+ \to \mathbb{R}$ be Lebesgue measurable functions, where $z_T^{short}(\psi)$ and $z_T^{debt}(\varphi)$ capture the position of type T traders on the short contract ψ and the debt contract φ, respectively. The above restriction is formalized by assuming that

$$z_T^{short}(\cdot) \geq 0 \text{ for each } T = \left(\cdot, \eta^{short} = 0, \cdot\right), \text{ and } z_T^{debt}(\cdot) \geq 0 \text{ for each } T = \left(\cdot, \cdot, \eta^{debt} = 0\right). \tag{46}$$

The definition of equilibrium follows closely Definition 2 in Section 2 with minor changes that

[22] For detailed descriptions of the shorting market, see, for example, Jones and Lamont (2001), D'Avolio (2002), and Duffie, Gârleanu and Pedersen (2002).

take into account the availability of short contracts and the additional assumption (46).

To characterize the collateral equilibrium, I first consider a quasi-equilibrium in which optimists are restricted to choose $z^{debt}_{(1,\cdot,\cdot)}(\cdot) \leq 0 \leq z^{short}_{(1,\cdot,\cdot)}(\cdot)$, while moderate traders are restricted to choose $z^{debt}_{(0,\cdot,\cdot)}(\cdot) \geq 0 \geq z^{short}_{(0,\cdot,\cdot)}(\cdot)$. In other words, optimists are not allowed to buy debt contracts (and sell short contracts), while moderates are not allowed to buy short contracts (and sell debt contracts). Similar to before, these restrictions will not be binding in equilibrium and the quasi-equilibrium will correspond to a collateral equilibrium.

To characterize the quasi-equilibrium, I first conjecture an equilibrium of a particular form in which traders are endogenously matched through competitive markets, by certain types of traders buying the contracts sold by certain types of other traders. I then characterize traders' optimal contract choices and solve for the equilibrium level of asset prices.

7.1 Matching of Optimists and Moderates in Debt and Short Markets

Under appropriate parametric restrictions there exists a quasi-equilibrium in which traders take the following position. First, optimists that can leverage, i.e., traders with type $T_1 \equiv (1, \cdot, 1)$, invest all of their wealth in the asset and they leverage as much as possible given their choice of contract φ. Second, optimists that cannot leverage, i.e., traders with type $T_2 \equiv (1, \cdot, 0)$, invest all of their wealth either in the asset or the short contracts sold by type T_3 traders. Third, moderates that can short sell, i.e., traders with type $T_3 \equiv (0, 1, \cdot)$, invest all of their wealth in the bond and they short sell as much as possible given their choice of contract ψ. Fourth, moderates that cannot short sell, i.e., traders with type $T_4 \equiv (0, 0, \cdot)$, invest all of their wealth either in the bond or the debt contracts sold by type T_1 traders.

In other words, type T_1 optimists borrow from type T_4 moderates that cannot short sell, while type T_3 moderates borrow the asset A from type T_2 optimists that cannot leverage. To see the intuition for this matching, note that type T_3 moderates require a greater interest rate than type T_4 moderates to part with their wealth (i.e., to lend), because, in equilibrium, they receive a greater expected return on their wealth (since they have the ability to short sell). This implies that, the debt contracts sold by type T_1 optimists are purchased by type T_4 moderates. A similar reasoning shows that, the short contracts sold by type T_3 moderates are bought by type T_2 optimists.

Given this matching, the characterization of the quasi-equilibrium follows closely the analysis in Section *3*. In particular, consider debt contract prices given by (9), which corresponds to the valuation of type T_4 moderates in this setting. Given these prices and the asset price $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r} \right)$, Theorem 1 continues to apply. That is, type T_1 optimists choose to borrow and leverage with a single loan with riskiness $\bar{s}_{le} \in \mathcal{S}$ that solves $p = p^{opt}(\bar{s}_{le})$.

Similarly, note that in an equilibrium with the above matching, type T_2 optimists must be indifferent between investing in the asset and the short contracts. Type T_2 optimists' expected return from investing in the asset is given by $\frac{E_1[v(s)]}{p}$. Thus, consider short contract prices:

$$q^{short}(\psi) = \frac{1}{\frac{E_1[v(s)]}{p}} E_1 [\min (\psi (1+r), 1)] \text{ for each } \psi \in \mathbb{R}_+. \tag{47}$$

Given the prices in (47), type T_2 optimists absorb any potential supply of short contracts from type T_3 moderates. Hence, the equilibrium in the short contract market is determined by type T_3 moderates' optimal contract choice.

I next characterize type T_3 moderates' optimal contract choice and illustrate that a version of the asymmetric filtering result applies in the present setting with short selling. I then combine this characterization with the characterization of type T_1 optimists' debt contract choice and market clearing for the asset, and solve for the equilibrium price.

7.2 Asymmetric Filtering with Short Selling

Given the prices in (47) and the asset price $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$, it can be seen that type T_3 moderates short sell according to a unit short contract $\psi = \frac{v(\bar{s}_{sh})}{1+r}$ that will default if the realized state is *above* some threshold state $\bar{s}_{sh} \in \mathcal{S}$. This is because, for sufficiently high states, the value of the promised asset will exceed the value of the posted collateral, and the short seller will find it optimal to default. The next result, which is the counterpart of Theorem 1 for short contracts, characterizes the threshold state $\bar{s}_{sh}$ for the optimal short contracta.

Theorem 9 (Asymmetric Filtering with Short Selling). *Suppose assumption (MLRP) holds, short contract prices are given by* (47) *and the asset price satisfies* $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$*. In a quasi-equilibrium:*

(i) There exists $\bar{s}_{sh} \in \mathcal{S}$ *such that moderates that are able to short sell only sell the unit short contract* $\psi = \frac{v(\bar{s}_{sh})}{1+r}$*. These moderates invest all of their wealth in the bond and short sell the asset as much as possible subject to their collateral constraints.*

(ii) The threshold state $\bar{s}_{sh} \in \mathcal{S}$ *of the optimal short contract is characterized as the unique solution to:*

$$p = p^{short}(\bar{s}_{sh}) = \frac{E_1[v(s)]/(1+r)}{1 + F_0(\bar{s}_{sh})\left(\frac{\int_{s_{\min}}^{\bar{s}_{sh}} v(s) dF_1}{\int_{s_{\min}}^{\bar{s}_{sh}} v(s) dF_0} - \frac{\int_{s_{\min}}^{\bar{s}_{sh}} dF_1}{\int_{s_{\min}}^{\bar{s}_{sh}} dF_0}\right)}. \tag{48}$$

Note that $p^{short}(\bar{s}_{sh})$ describes the price for which the short contract with default threshold $\bar{s}_{sh}$ would be optimal. Under assumption (MLRP), this curve is strictly decreasing from the optimistic valuation $\frac{E_1[v(s)]}{1+r}$ towards the moderate valuation $\frac{E_0[v(s)]}{1+r}$. Thus, there is a unique solution to (48).

It can also be seen that the function $p^{short}(\bar{s}_{sh})$ features an asymmetric filtering property. To see this, suppose the belief of moderate traders, F_0, is kept constant and the belief of optimists, F_1, is changed in a way to keep the optimistic valuation $\frac{E_1[v(s)]}{1+r}$ constant. By Eq. (48), the effect of this type of change on $p^{short}(\bar{s}_{sh})$ is characterized by its effect on the expression:

$$\frac{\int_{s^{\min}}^{\bar{s}_{sh}} v(s)\, dF_1}{\int_{s^{\min}}^{\bar{s}_{sh}} v(s)\, dF_0} - \frac{\int_{s^{\min}}^{\bar{s}_{sh}} dF_1}{\int_{s^{\min}}^{\bar{s}_{sh}} dF_0}. \tag{49}$$

By assumption (MLRP), this expression is always positive. Intuitively, both terms in the expression can be thought of as an "average" of the likelihood ratios $\left(\frac{f_1(s)}{f_0(s)}\right)_{s\in[s^{\min},\bar{s}_{sh}]}$, with the term on the left putting relatively greater weight $v(s)$ on the higher likelihood ratios $\frac{f_1(s)}{f_0(s)}$ (corresponding to higher s). This intuition also suggest that a shift of optimism towards the relative likelihood of states above $\bar{s}_{sh}$ decreases the expression in (49). In the most extreme case, if $\frac{f_1(s)}{f_0(s)}$ were constant over $s \in \left[s^{\min}, \bar{s}_{sh}\right]$ (so that all the optimism is concentrated on the relative likelihood of states above $\bar{s}_{sh}$), then the expression in (49) would be equal to zero. By Eq. (48), the function $p^{short}(\bar{s}_{sh})$ negatively depends the expression in (49). It follows that a shift of optimism towards the relative likelihood of states above $\bar{s}_{sh}$ increases $p^{short}(\bar{s}_{sh})$. That is, for any given level of default threshold $\bar{s}_{sh}$ for short contracts, the asset price is higher when optimism is concentrated more on the relative likelihood of good states. This illustrates the asymmetric filtering property of the optimal short contract.

The proof of Theorem 9 is relegated to Appendix A.7, which also provides a precise intuition for the asymmetric filtering property captured by the expression in (48). For a simpler intuition, suppose that the belief heterogeneity is concentrated on the relative likelihood of good states, in particular, states above the threshold level $\bar{s}_{sh}$. Note that short contracts default above the threshold state $\bar{s}_{sh}$. Thus, they pay the same amount $v(\bar{s}_{sh})$, the value of the posted collateral, in these states. Then, using a short contract with threshold $\bar{s}_{sh}$, it is impossible for a moderate trader to bet on her pessimism about the relative likelihood of states above $\bar{s}_{sh}$. Consequently, moderate traders' pessimism about the relative likelihood of good states are not reflected in the asset price. In particular, the asset price in this case is closer to the optimistic valuation, as suggested by the expression in (48). In contrast, suppose belief heterogeneity is concentrated more on the likelihood of bad states, i.e., states below the threshold level $\bar{s}_{sh}$. In this case, moderate traders can bet on their pessimism by selling a short contract with threshold $\bar{s}_{sh}$. Thus, with these types of belief heterogeneity, moderate traders' pessimism is reflected in the asset price. The asset price is closer to the moderate valuation, as suggested by (48).

Put differently, it is easier for moderate traders to bet on their pessimism about the likelihood of bad states than to bet on their pessimism for the relative likelihood of good states. To bet on the latter types of pessimism, moderate traders have to post a very high level of collateral e (equivalently, choose a high default threshold $\bar{s}_{sh}$). Hence, these types of short sales are more difficult to leverage.

7.3 Equilibrium Asset Price with Short Selling

The equilibrium is characterized by type T_1 optimists' and type T_3 moderates' optimal contract choice, along with the market clearing condition for the asset, which I derive next. To simplify the analysis, suppose the parameters are such that the equilibrium asset price satisfies $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$. Note that type T_1 optimists that leverage by using a loan with riskiness $\bar{s}_{le}$ spend a total of

$$\gamma_{le}\left(w_1 + p\alpha_1\right)\frac{p}{p - \frac{E_0[\min(v(s),v(\bar{s}_{le}))]}{1+r}} \tag{50}$$

on the asset. Here, recall that γ_{le} is the fraction of investors that are able to leverage, $w_1 + p\alpha_1$ is the total wealth of optimists, and the second term in (50) is the leverage ratio. Note also that type T_2 optimists make unleveraged investments in the asset and the short contracts sold by type T_3 moderates. In particular, type T_2 optimists' net expenditure on the asset is given by

$$\left(1-\gamma_{le}\right)\left(w_1 + p\alpha_1\right) - W^{short}, \tag{51}$$

where W^{short} denotes the expenditure of type T_2 optimists on the short contracts sold by type T_3 moderates. By market clearing in short contracts, W^{short} is also equal to type T_3 moderates' total revenue from sales of short contracts. The analysis in the appendix shows that this expression has a similar form to the expression in (50). In particular, it is given by

$$W^{short} = \gamma_{sh}\left(w_0 + p\alpha_0\right)\frac{p}{\frac{v(\bar{s}_{sh})}{E_1[\min(v(s),v(\bar{s}_{sh}))]}\frac{E_1(v(s))}{1+r} - p}. \tag{52}$$

The market clearing for the asset implies that the total spending on the asset, that is, the sum of the expressions in (50) and (51), should be equal to the total value of the asset, p. After substituting for W^{short} from the expression in (52) and rearranging terms, the asset market clearing condition can be written as:

$$\gamma_{le}\frac{w_1 + p\alpha_1}{p - \frac{E_0[\min(v(s),v(\bar{s}_{le}))]}{1+r}} + \left(1-\gamma_{le}\right)\frac{w_1 + p\alpha_1}{p} = 1 + \gamma_{sh}\frac{w_0 + p\alpha_0}{\frac{v(\bar{s}_{sh})}{E_1[\min(v(s),v(\bar{s}_{sh}))]}\frac{E_1(v(s))}{1+r} - p}. \tag{53}$$

This expression shows that short selling of the asset is effectively expanding the supply of the asset, as captured by the second term on the right hand side.

The equilibrium tuple $(p, \bar{s}^*_{le}, \bar{s}^*_{sh})$ is then characterized by the optimality conditions $p = p^{opt}\left(\bar{s}_{le}\right) = p^{short}\left(\bar{s}_{sh}\right)$, along with the market clearing condition (53). It can be seen that an increase in the fraction of short sellers, γ_{sh}, decreases the asset price because it increases the effective supply of the asset, as captured by the right hand side of Eq. (53). Conversely, an increase in the fraction of leveraged investors, γ_{le}, increases the asset price because it increases the demand for the asset, as captured by the left hand side of Eq. (53).

In addition, it can be seen that an increase in the right-skewness of optimism tends to increase the asset price. To see the intuition for this effect, first recall that this type of change shifts up the $p^{opt}(\cdot)$ curve. Hence, the debt threshold $\bar{s}_{le}$ increases for any price level $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$. Second, as captured by the expressions (48) and (52), this type of change also tends to shift up the $p^{short}(\cdot)$ curve. Suppose this is the case so that $\bar{s}_{le}$ and $\bar{s}_{sh}$ are both higher for any price level. This means that debt contracts default more often (since they default for states $s < \bar{s}_{le}$) while short contracts default less often (since they default for states $s > \bar{s}_{sh}$). Put differently, optimists leverage more (by choosing larger and riskier loans), while short sellers leverage less (by posting a higher collateral level $\frac{v(\bar{s}_{sh})}{1+r}$ in their short sales). Eq. (53) reflects these forces and shows that the net supply of the asset is decreasing in both $\bar{s}_{le}$ and $\bar{s}_{sh}$. It follows that an increase in the right-skewness of optimism (that increases both $\bar{s}_{le}$ and $\bar{s}_{sh}$) decreases the net supply of the asset for any price level, which leads to a higher equilibrium price p.

8 Conclusion

In this paper, I have theoretically analyzed the effect of belief heterogeneity on asset prices. The central feature of the model is that, to take positions in line with their beliefs, investors need to borrow from traders with different beliefs using collateralized contracts. The lenders do not value the collateral as much as the borrowers do, which represents a constraint on investors' ability to borrow and leverage their investments. I have considered the effect of this constraint on asset prices in a variety of settings that differ in the types of collateralized contracts that are available for trade. In the baseline model, I have restricted attention to non-contingent loans and disallowed short selling, and I have relaxed these restrictions in two extensions of the model. In each of these scenarios, my paper has established that optimism is asymmetrically filtered by endogenous financial constraints. In particular, optimism about the likelihood of bad states has a smaller effect on asset prices than optimism about the relative likelihood of good states. I have also considered a dynamic extension of the model which reveals that the speculative asset price bubbles, identified by Harrison and Kreps (1978), are also asymmetrically filtered by optimists' financial constraints. Taken together, my results suggest that certain economic environments that generate uncertainty (and thus belief heterogeneity) about good states are conducive to asset price increases and speculative bubbles.

The asymmetric filtering characterization of asset prices also emphasizes that what investors disagree about matters for asset prices, to a greater extent than the level of the disagreement. In particular, when optimists are financially constrained, an increase in the level of belief heterogeneity in general has ambiguous effects on asset prices. However, the effect can be characterized once the skewness of the increase is taken into account. Additional belief heterogeneity tends to decrease asset prices when it concerns the likelihood of bad states, but it tends to increase asset prices when it concerns the relative likelihood of good states. A growing empirical literature in finance considers the effect of the level of belief heterogeneity on asset prices and subsequent asset returns (e.g., Chen, Hong and Stein, 2001, Diether, Malloy and Scherbina, 2002, and Ofek and Richardson, 2003). My paper suggests that a fruitful future research direction may be to empirically investigate the effect of the skewness of the belief heterogeneity on asset prices.

A Appendices

A.1 Properties of Optimism Order

This appendix establishes the properties of optimism order (cf. Definition (1)) which have been referred to in the analysis. Consider two probability distributions $H, \tilde{H}$ over $S = \left[s^{\min}, s^{\max}\right] \subset R$ with corresponding density functions $h, \tilde{h}$ that are continuous and positive at each $s \in S$.

I first show that $\frac{1-\tilde{H}(s)}{1-H(s)}$ is strictly increasing at some $s \in S$ if and only if the hazard rate inequality in (2) is satisfied. To see this, consider the derivative of $\frac{1-\tilde{H}(s)}{1-H(s)}$

$$\frac{d}{ds}\frac{1-\tilde{H}(s)}{1-H(s)} = \frac{-\tilde{h}(s)(1-H(s)) + h(s)\left(1-\tilde{H}(s)\right)}{(1-H(s))^2}, \text{ for each } s \in [s^{\min}, s^{\max}),$$

and note that this expression is positive if and only if the hazard rate inequality (2) holds.

I next show that the optimism order is weaker than the monotone likelihood ratio property (MLRP), that is, if $\frac{\tilde{h}(s)}{h(s)}$ is strictly increasing over S, then $\tilde{H} \succ_O H$. To see this, suppose (MLRP) holds and note that this implies, for each $s < s^{\max}$,

$$\frac{\tilde{h}(s)}{h(s)} h(\tilde{s}) < \tilde{h}(\tilde{s}) \text{ for all } \tilde{s} \in (s, s^{\max}).$$

Integrate both sides of this equation over $(s, s^{\max})$ to get

$$\frac{\tilde{h}(s)}{h(s)}(1-H(s)) < \left(1-\tilde{H}(s)\right),$$

which proves the hazard rate inequality (2) and shows that $\tilde{H} \succ_O H$.

I next note the following result, which derives the implications of assumption (O) for the key variables used in the analysis, including the expected payoff of a loan with riskiness $\bar{s}$, $E_i[\min(v(s), v(\bar{s}))]$, the expected interest rate, $r_1^{exp}(\bar{s})$, and the optimality curve, $p^{opt}(\cdot)$.

Lemma 4. *Consider two probability distributions F_1 and F_0 that satisfy assumption (O).*

(i) The expected payoff of a loan with riskiness $\bar{s}$, $E_i[\min(v(s), v(\bar{s}))]$, is strictly increasing in $\bar{s}$.

(ii) Optimists' expected interest rate $r_1^{exp}(\bar{s})$ (cf. Eq. (13)) is strictly increasing in $\bar{s}$. In particular, $r_1^{exp}(\bar{s}) > r_1^{exp}\left(s^{\min}\right) = r$ for each $\bar{s} > s^{\min}$.

(iii) $p^{opt}(\bar{s})$ is continuously differentiable and strictly decreasing, i.e., $\frac{dp^{opt}(\bar{s})}{d\bar{s}} < 0$.

Proof of Lemma 4. Part (i). Note that the derivative of $E_i[\min(v(s), v(\bar{s}))] = \int_{s^{\min}}^{\bar{s}} v(s)\, dF_i(s) + v(\bar{s})(1-F_i(\bar{s}))$ is given by

$$\frac{dE_i[\min(v(s), v(\bar{s}))]}{d\bar{s}} = v(\bar{s}) f_i(\bar{s}) + v'(\bar{s})(1-F_i(\bar{s})) - v(\bar{s}) f(\bar{s}) = v'(\bar{s})(1-F_i(\bar{s})) > 0, \tag{A.1}$$

which completes the proof.

Part (ii). The derivative of $\frac{1+r_1^{exp}(\bar{s})}{1+r} = \frac{E_1[\min(v(s),v(\bar{s}))]}{E_0[\min(v(s),v(\bar{s}))]}$ can be calculated as

$$\begin{aligned}\frac{d}{d\bar{s}}\left(\frac{1+r_1^{exp}(\bar{s})}{1+r}\right) &= \frac{\frac{dE_1[\min(v(s),v(\bar{s}))]}{d\bar{s}} E_0\left[\min\left(v\left(s\right),v\left(\bar{s}\right)\right)\right] - E_1\left[\min\left(v\left(s\right),v\left(\bar{s}\right)\right)\right]\frac{dE_1[\min(v(s),v(\bar{s}))]}{d\bar{s}}}{\left(E_0\left[\min\left(v\left(s\right),v\left(\bar{s}\right)\right)\right]\right)^2} \\ &= \frac{E_0\left[\min\left(v\left(s\right),v\left(\bar{s}\right)\right)\right]\left(1-F_1\left(\bar{s}\right)\right) - E_1\left[\min\left(v\left(s\right),v\left(\bar{s}\right)\right)\right]\left(1-F_0\left(\bar{s}\right)\right)}{\left(E_0\left[\min\left(v\left(s\right),v\left(\bar{s}\right)\right)\right]\right)^2}, \quad (A.2)\end{aligned}$$

where the last line uses Eq. $(A.1)$.

I next claim that

$$\frac{E_1\left[\min\left(v\left(s\right),v\left(\bar{s}\right)\right)\right]}{E_0\left[\min\left(v\left(s\right),v\left(\bar{s}\right)\right)\right]} < \frac{1-F_1\left(\bar{s}\right)}{1-F_0\left(\bar{s}\right)} \text{ for each } \bar{s} \in \left(s^{\min}, s^{\max}\right), \quad (A.3)$$

which, in view of Eq. $(A.2)$, proves that the effective interest rate $1+r_1^{exp}(\bar{s})$ is strictly increasing.

To prove the claim, note that for each $\bar{s} \in \left(s^{\min}, s^{\max}\right)$,

$$\begin{aligned}\frac{E_1\left[\min\left(v\left(s\right),v\left(\bar{s}\right)\right)\right]}{E_0\left[\min\left(v\left(s\right),v\left(\bar{s}\right)\right)\right]} &= \frac{\int_{s^{\min}}^{\bar{s}} v\left(s\right)dF_1 + v\left(\bar{s}\right)\left(1-F_1\left(\bar{s}\right)\right)}{\int_{s^{\min}}^{\bar{s}} v\left(s\right)dF_0 + v\left(\bar{s}\right)\left(1-F_0\left(\bar{s}\right)\right)} \\ &< \frac{\int_{s^{\min}}^{\bar{s}} v\left(s\right)\frac{1-F_1(s)}{1-F_0(s)}dF_0 + v\left(\bar{s}\right)\left(1-F_1\left(\bar{s}\right)\right)}{\int_{s^{\min}}^{\bar{s}} v\left(s\right)dF_0 + v\left(\bar{s}\right)\left(1-F_0\left(\bar{s}\right)\right)} \\ &< \frac{\int_{s^{\min}}^{\bar{s}} v\left(s\right)dF_0\frac{1-F_1(\bar{s})}{1-F_0(\bar{s})} + v\left(\bar{s}\right)\left(1-F_1\left(\bar{s}\right)\right)}{\int_{s^{\min}}^{\bar{s}} v\left(s\right)dF_0 + v\left(\bar{s}\right)\left(1-F_0\left(\bar{s}\right)\right)} = \frac{1-F_1\left(\bar{s}\right)}{1-F_0\left(\bar{s}\right)},\end{aligned}$$

where the first inequality uses the hazard rate inequality (2) and the second inequality uses the fact that $\frac{1-F_1(s)}{1-F_0(s)}$ is strictly increasing. This proves the claim in $(A.3)$ and completes the proof of this part.

Part (iii). Using the definition of $p^{opt}(\bar{s})$ in Eq. (10), note that

$$\begin{aligned}\frac{dp^{opt}\left(\bar{s}\right)}{d\bar{s}} &\equiv \frac{1}{1+r}\left(\begin{array}{c} v\left(\bar{s}\right)f_0\left(\bar{s}\right) + \left(-f_0\left(\bar{s}\right) + f_1\left(\bar{s}\right)\frac{1-F_0(\bar{s})}{1-F_1(\bar{s})}\right)\left(\int_{\bar{s}}^{s^{\max}} v\left(s\right)\frac{dF_1}{1-F_1(\bar{s})}\right) \\ -\frac{1-F_0(\bar{s})}{1-F_1(\bar{s})}v\left(\bar{s}\right)f_1\left(\bar{s}\right)\end{array}\right) \\ &= \frac{-1}{1+r}\left(\frac{f_0\left(\bar{s}\right)}{1-F_0\left(\bar{s}\right)} - \frac{f_1\left(\bar{s}\right)}{1-F_1\left(\bar{s}\right)}\right)\left(1-F_0\left(\bar{s}\right)\right)\left(E_1\left[v\left(s\right) \mid s \geq \bar{s}\right] - v\left(\bar{s}\right)\right), \quad (A.4)\end{aligned}$$

where the first line applies the chain rule and the second line substitutes $E_1\left[v\left(s\right) \mid s \geq \bar{s}\right]$ and rearranges terms. The term, $\left(\frac{f_0(\bar{s})}{1-F_0(\bar{s})} - \frac{f_1(\bar{s})}{1-F_1(\bar{s})}\right)$, in Eq. $(A.4)$ is positive from the hazard rate inequality (2). Since the terms, $\left(1-F_0\left(\bar{s}\right)\right)$ and $\left(E_1\left[v\left(s\right) \mid s \geq \bar{s}\right] - v\left(\bar{s}\right)\right)$, are also positive, it follows that $\frac{dp^{opt}(\bar{s})}{d\bar{s}} < 0$, completing the proof of the lemma.

I next present the final result of this appendix, which uses assumption (O) to derive the effects of an increase in optimists' (moderate traders') optimism on the $p^{opt}(\cdot)$ and $p^{mc}(\cdot)$ curves.

Lemma 5. *Consider two probability distributions F_1 and F_0 that satisfy assumption (O).*

(i) Suppose optimists become weakly more optimistic, i.e., consider their beliefs are changed to $\tilde{F}_1 \succeq_O F_1$. Then:

(i.1) Conditional expectations increase, that is, $\tilde{E}_1\left[v\left(s\right) \mid s \geq \bar{s}\right] \geq E_1\left[v\left(s\right) \mid s \geq \bar{s}\right]$ for each $\bar{s} \in \left[s^{\min}, s^{\max}\right)$.[23]

[23] Throughout the appendices, the notation $\tilde{E}_1\left[s \mid s \geq \bar{s}\right]$ corresponds to the conditional expectation

(i.2) The optimality curve $p^{opt}(\bar{s})$ *shifts up pointwise, that is,*

$$p^{opt}\left(\bar{s}\ ;\ \tilde{F}_1\right) \geq p^{opt}\left(\bar{s}\ ;\ F_1\right) \text{ for each } \bar{s} \in \left[s^{\min}, s^{\max}\right].$$

(i.3) The market clearing curve changes as follows:

$$p^{mc}\left(\bar{s}\ ;\ \tilde{F}_1\right) \begin{cases} = p^{mc}\left(\bar{s}\ ;\ F_1\right) & \text{if } p^{mc}\left(\bar{s}\ ;\ F_1\right) < \frac{E_1[v(s)]}{1+r} \\ \geq p^{mc}\left(\bar{s}\ ;\ F_1\right) & \text{if } p^{mc}\left(\bar{s}\ ;\ F_1\right) = \frac{E_1[v(s)]}{1+r} \end{cases}. \tag{A.5}$$

(ii) Suppose moderate traders become weakly more optimistic, i.e., consider their beliefs are changed to $\tilde{F}_0 \succeq_O F_0$ *(which also satisfies* $F_1 \succeq_O \tilde{F}_0$ *so that assumption (O) continues to hold). Then:*

(ii.1)The optimality curve $p^{opt}(\bar{s})$ *shifts up pointwise, that is,*

$$p^{opt}\left(\bar{s}\ ;\ \tilde{F}_0\right) \geq p^{opt}\left(\bar{s}\ ;\ F_0\right) \text{ for each } \bar{s} \in \left[s^{\min}, s^{\max}\right].$$

(ii.2) The market clearing curve $p^{mc}(\bar{s})$ *shifts up pointwise, that is,*

$$p^{mc}\left(\bar{s}\ ;\ \tilde{F}_1\right) \geq p^{mc}\left(\bar{s}\ ;\ F_1\right) \text{ for each } \bar{s} \in \left[s^{\min}, s^{\max}\right].$$

Proof of Lemma 5. Part (i.1). Define the function $g : S \to R$ with

$$g(\bar{s}) = \tilde{E}_1\left[v(s) \mid s \geq \bar{s}\right] - E_1\left[v(s) \mid s \geq \bar{s}\right]. \tag{A.6}$$

Note that $g(s^{\max}) = 0$, and note also that the statement in the lemma is equivalent to the following claim:

$$g(\bar{s}) \geq 0 \text{ for each } \bar{s} \in [s^{\min}, s^{\max}). \tag{A.7}$$

I will first find an upper bound for the derivative of $g(\bar{s})$ which I will then use to prove the claim in $(A.7)$.

To put an upper bound on the derivative of $g(\bar{s})$, consider first the derivative of the conditional expectation $E_1[v(s) \mid s \geq \bar{s}]$ at some $\bar{s} \in [s^{\min}, s^{\max})$. With some rearrangement, this derivative can be written as

$$\frac{d}{d\bar{s}} E_1\left[v(s) \mid s \geq \bar{s}\right] = \frac{f_1(\bar{s})}{1 - F_1(\bar{s})}\left(E_1\left[v(s) \mid s \geq \bar{s}\right] - v(\bar{s})\right).$$

Using this expression, the derivative of $g(\bar{s})$ can be written as

$$\begin{aligned} g'(\bar{s}) &= \frac{\tilde{f}_1(\bar{s})}{1 - \tilde{F}_1(\bar{s})}\left(\tilde{E}_1\left[v(s) \mid s \geq \bar{s}\right] - v(\bar{s})\right) - \frac{f_1(\bar{s})}{1 - F_1(\bar{s})}\left(E_1\left[v(s) \mid s \geq \bar{s}\right] - v(\bar{s})\right) \\ &= \left(\frac{\tilde{f}_1(\bar{s})}{1 - \tilde{F}_1(\bar{s})} - \frac{f_1(\bar{s})}{1 - F_1(\bar{s})}\right)\left(\tilde{E}_1\left[v(s) \mid s \geq \bar{s}\right] - v(\bar{s})\right) + \frac{f_1(\bar{s})}{1 - F_1(\bar{s})} g(\bar{s}), \end{aligned} \tag{A.8}$$

where the second line follows by rearranging terms and substituting the definition of $g(\bar{s})$ from Eq. $(A.6)$. Note that $\tilde{E}_1[v(s) \mid s \geq \bar{s}] - v(\bar{s}) > g(\bar{s})$ and the first term in Eq. $(A.8)$ is always non-

of optimists according to the belief distribution $\tilde{F}_1$.

positive (since $\tilde{F}_1 \succeq_O F_1$), which provides the following upper bound on the derivative of $g'(\bar{s})$:

$$g'(\bar{s}) \leq \frac{\tilde{f}_1(\bar{s})}{1-\tilde{F}_1(\bar{s})} g(\bar{s}) \text{ for each } \bar{s} \in [s^{\min}, s^{\max}). \tag{A.9}$$

Next, to prove the claim in $(A.7)$, suppose the contrary, that is, suppose there exists $\tilde{s} < s^{\max}$ such that $g(\tilde{s}) < 0$. Consider next

$$\hat{s} = \sup\{\bar{s} \in [\tilde{s}, s^{\max}) \mid g(\bar{s}) \leq g(\tilde{s})\}.$$

Note that $\hat{s}$ exists and that $g(\hat{s}) = g(\tilde{s}) < 0$ by the continuity of the function $g(\cdot)$. This further implies that $\hat{s} \neq s^{\max}$ since $g(s^{\max}) = 0$. Then, Eq. $(A.9)$ applies for $\hat{s}$ and implies $g'(\hat{s}) \leq \frac{\tilde{f}_1(\hat{s})}{1-\tilde{F}_1(\hat{s})} g(\hat{s}) < 0$. This further implies that there exists $\bar{s} \in (\hat{s}, s^{\max})$ such that $g(\bar{s}) < g(\hat{s}) = g(\tilde{s})$, which contradicts the definition of $\hat{s}$. This proves the claim in $(A.7)$ and completes the proof of the first part.

Part (i.2). Note, by Eq. (23), that the optimality curve $p^{opt}(\bar{s})$ can be written as

$$p^{opt}(\bar{s}) = \frac{1}{1+r}\left(E_0[v(s)] + (1-F_0(\bar{s}))\left(E_1[v(s) \mid s \geq \bar{s}] - E_0[v(s) \mid s \geq \bar{s}]\right)\right).$$

Then, using part (i.1) shows that $p^{opt}(\bar{s})$ shifts up pointwise, completing the proof.

Part (i.3). Consider the definition of $w^{\max}(\bar{s})$ in (16) and note that $w^{\max}(\bar{s})$ does not depend on F_1, as it depends on moderate traders' valuation of debt contracts. Eq. $(A.5)$ then follows by the definition of $p^{mc}(\bar{s})$ in (17). Intuitively, the change, $\tilde{F}_1 \succeq F_1$, only affects $p^{mc}(\bar{s})$ by increasing optimists' valuation, thus it only shifts the $p^{mc}(\bar{s})$ curve in case (ii) region of Eq. (17), while it leaves it constant in other cases.

Part (ii.1). Similar to part (i.1) of the lemma, define the function $g_{mix}: S \to R$ with

$$g_{mix}(\bar{s}) = p^{opt}\left(\bar{s} \ ; \ \tilde{F}_0\right) - p^{opt}(\bar{s} \ ; \ F_0).$$

Note that the statement in the lemma is equivalent to the claim:

$$g_{mix}(\bar{s}) \geq 0 \text{ for each } \bar{s} \in \left(s^{\min}, s^{\max}\right). \tag{A.10}$$

I will prove a stronger claim, that

$$\frac{dg_{mix}(\bar{s})}{d\bar{s}} \geq 0 \text{ for each } \bar{s} \in \left(s^{\min}, s^{\max}\right), \tag{A.11}$$

which implies the claim in $(A.10)$ since $g_{mix}\left(s^{\min}\right) = 0$.

To prove the claim in $(A.11)$, note that using Eq. $(A.4)$ and rearranging terms, the derivative of $g_{mix}(\bar{s})$ can be written as

$$\frac{dg_{mix}(\bar{s})}{d\bar{s}} = \frac{1}{1+r}\left[\begin{array}{c} f_0(\bar{s}) - \tilde{f}_0(\bar{s}) \\ -\frac{f_1(\bar{s})}{1-F_1(\bar{s})}\left(\tilde{F}_0(\bar{s}) - F_0(\bar{s})\right) \end{array}\right]\left(E_1[v(s) \mid s \geq \bar{s}] - v(\bar{s})\right). \tag{A.12}$$

Next note that $F_1 \succeq_O \tilde{F}_0 \succeq_O F_0$ implies $\frac{f_0(\bar{s})}{1-F_0(\bar{s})} \geq \frac{\tilde{f}_0(\bar{s})}{1-\tilde{F}_0(\bar{s})} \geq \frac{f_1(\bar{s})}{1-F_1(\bar{s})}$. After rearranging terms, this further implies

$$\frac{f_0(\bar{s}) - \tilde{f}_0(\bar{s})}{\tilde{F}_0(\bar{s}) - F_0(\bar{s})} \geq \frac{\tilde{f}_0(\bar{s})}{1-\tilde{F}_0(\bar{s})} \geq \frac{f_1(\bar{s})}{1-F_1(\bar{s})}.$$

Using this inequality in Eq. $(A.12)$ and noting that $E_1[v(s) \mid s \geq \bar{s}] - v(\bar{s}) \geq 0$ proves the claim in $(A.11)$, completing the proof of the lemma.

Part (ii.2). First note that, applying the argument in part (iii) of Lemma 4 for the distributions $\tilde{F}_0 \succeq_O F_0$ implies

$$\tilde{E}_0[\min(v(s), v(\bar{s}))] \geq E_0[\min(v(s), v(\bar{s}))] \text{ for each } \bar{s} \in \mathcal{S}.$$

By Eq. (16), this further implies $w_1^{\max}\left(\bar{s}\ ;\ \tilde{F}_0\right) \geq w_1^{\max}(\bar{s}\ ;\ F_0)$. Using this inequality and the fact that $\tilde{E}_0[v(s)] \geq E_0[v(s)]$, Eq. (17) implies that $p^{mc}(\bar{s})$ shifts up pointwise, completing the proof.

A.2 Characterization of Quasi-equilibrium

This section completes the analysis of the quasi-equilibrium, by providing the proofs for Theorem 1 and Lemma 1, and by providing an analytical characterization of equilibrium.

Proof of Theorem 1. I prove the theorem in two steps. I first show that optimists borrow with a debt contract $v(\bar{s})$, where $\bar{s}$ solves problem (14). I then characterize the unique solution to this problem and complete the sketch proof provided after the theorem statement.

To prove the first step, first note that optimists' debt contract choice can be restricted to $\varphi \in \left[v\left(s^{\min}\right), v\left(s^{\max}\right)\right]$ without loss of generality, i.e., let $z_i(\varphi) = 0$ for each $\varphi \notin \left[v\left(s^{\min}\right), v\left(s^{\max}\right)\right]$.[24] Then, after applying the change of notation $\varphi = v(\tilde{s})$ and substituting the debt prices from Eq. (9), optimists' problem in a quasi-equilibrium can be written as:

$$\begin{aligned}
\max_{x_1^A \geq 0, [z_1(v(\tilde{s})) \leq 0]_{\tilde{s} \in S}} \quad & x_1^A E_1[v(s)] + \int_{s^{\min}}^{s^{\max}} z_1(v(\tilde{s})) E_1[\min(v(s), v(\tilde{s}))]\, dv(\tilde{s}), \\
\text{s.t.} \quad & p x_1^A + \int_{s^{\min}}^{s^{\max}} z_1(v(\tilde{s})) \frac{E_0[\min(v(s), v(\tilde{s}))]}{1+r} dv(\tilde{s}) \leq w_1 + p\alpha_1, \\
& \int_{s^{\min}}^{s^{\max}} -z_1(v(\tilde{s}))\, dv(\tilde{s}) \leq x_1^A.
\end{aligned}$$

At the optimal solution, the collateral constraints will be binding because, as $p > \frac{E_1[v(s)]}{1+r}$, optimists would always rather use the extra collateral to borrow with the non-default loan $s^{\min}$ (at interest rate $1+r$) and invest in the asset. Then, letting λ denote the Lagrange multiplier for the budget constraint and γ the Lagrange multiplier for the collateral constraint, the first order condition for $z_1(v(\tilde{s}))$ is given by

$$\lambda \frac{E_0[\min(v(s), v(\tilde{s}))]}{1+r} \leq E_1[\min(v(s), v(\tilde{s}))] + \gamma \qquad \text{(A.13)}$$
$$\text{with strict inequality only if } z_1(v(\tilde{s})) = 0.$$

Moreover, the first order condition with respect to x_1^A leads to

$$\gamma = \lambda p - E_1[v(s)].$$

[24] This is because any safe contract with $\varphi < v\left(s^{\min}\right)$ can be replicated by the alternative safe contract $v\left(s^{\min}\right)$ (which has the additional benefit of using less collateral), and any contract $\varphi > v(s^{\max})$ that defaults in all states can be replicated by the contract $v(s^{\max})$.

Plugging this expression for γ into $(A.13)$ yields the following first order condition:

$$R_1^L(\tilde{s}) = \frac{E_1[v(s)] - E_1[\min(v(s), v(\tilde{s}))]}{p - E_0[\min(v(s), v(\tilde{s}))]/(1+r)} \leq \lambda, \tag{A.14}$$
with strict inequality only if $z_1(v(\tilde{s})) = 0$.

This equation implies that optimists choose to borrow with a contract that maximizes $R_1^L(\tilde{s})$. In other words, optimists' debt contract choice solves problem (14), completing the first step of the proof.

As the second step, I show that problem (14).has a unique solution, and I characterize the solution. To this end, consider the derivative of $R_1^L(\tilde{s})$, which can be written as

$$\frac{d}{d\tilde{s}} R_1^L(\tilde{s}) = \frac{1}{p - \frac{E_0[\min(v(s), v(\tilde{s}))]}{1+r}} \left(\frac{R_1^L(\tilde{s})}{1+r}(1 - F_0(\tilde{s})) - (1 - F_1(\tilde{s})) \right). \tag{A.15}$$

Note that

$$R_1^L(s^{\min}) = \frac{E_1[v(s)] - v(s^{\min})}{p - v(s^{\min})/(1+r)} > 0 \text{ and } R_1^L(s^{\max}) = \frac{E_1[v(s)] - E_1[v(s)]}{p - E_0[v(s)]/(1+r)} = 0.$$

Thus, the derivative in $(A.15)$ satisfies the boundary conditions

$$\frac{d}{d\tilde{s}} R_1^L(\tilde{s})|_{\tilde{s}=s^{\min}} > 0 \text{ and } \frac{d}{d\tilde{s}} R_1^L(\tilde{s})|_{\tilde{s}=s^{\max}} < 0. \tag{A.16}$$

Eq. $(A.15)$ also leads to the first order condition

$$\frac{d}{d\tilde{s}} R_1^L(\tilde{s}) = 0 \text{ for } \tilde{s} \in [s^{\min}, s^{\max}) \text{ iff } \frac{R_1^L(\tilde{s})}{1+r} = \frac{1 - F_1(\tilde{s})}{1 - F_0(\tilde{s})}. \tag{A.17}$$

Plugging this first order condition in Eq. (15) and rearranging terms yields $p = p^{opt}(\tilde{s})$. By Lemma 4, $p^{opt}(\tilde{s})$ is strictly decreasing, which implies that there exists exactly one $\bar{s} \in S$ (the solution to $p = p^{opt}(\bar{s})$) that satisfies the first order condition in $(A.17)$. By the boundary conditions in $(A.16)$ and the continuity of $\frac{d}{d\tilde{s}} R_1^L(\tilde{s})$, it follows that $R_1^L(\tilde{s})$ has a unique maximum characterized as the solution to Eq. (10). This establishes the second step, and completes the proof of Theorem 1.

Proof of Lemma 1. Note that, by assumption, optimists' allocation $z_1(\cdot)$ for debt contracts is given by the Dirac-delta function $Z_1(v(\bar{s}))\,\delta(v(\bar{s}))$, where $Z_1(v(\bar{s})) \leq 0$ denotes the mass of the debt contract $v(\bar{s})$ that optimists sell. Note that the budget constraint (7) can be written as:

$$p\left(x_1^A - \alpha_1\right) + x_1^B + q(v(\bar{s}))\, Z_1(v(\bar{s})) = w_1.$$

Note also that optimists' (binding) collateral constraint (8) can be written as $-Z_1(v(\bar{s})) = x_1^A$. Using this expression and substituting for contract prices from Eq. (18), the budget constraint is further simplified to:

$$p\left(x_1^A - \alpha_1\right) + x_1^B = w_1 + \frac{E_0[\min(v(s), v(\bar{s}))]}{1+r} x_1^A. \tag{A.18}$$

Intuitively, the right hand side is optimists' first period wealth, while the left hand side is optimists' first period investments.

Next note that optimists invest all of their first period wealth on the asset when $p < \frac{E_1[v(s)]}{1+r}$. In particular,

$$x_1^B = 0 \text{ unless } p = \frac{E_1[v(s)]}{1+r}. \tag{A.19}$$

Using this observation, Eq. $(A.18)$ characterizes optimists' demand for the asset. Recall also that moderates choose $x_1^A = 0$ except for the corner case $p = \frac{E_0[v(s)]}{1+r}$, which characterizes moderates' demand for the asset. Finally, note that the asset market clearing condition is given by $x_1^A + x_0^A = 1$.

Note that these observations imply $x_1^B = 0$ and $x_1^A = 1$, except for the case $p = \frac{E_0[v(s)]}{1+r}$. Using this observation, the budget constraint in $(A.18)$ can be written as:

$$p\alpha_0 + x_1^B = w_1^{\max}(\bar{s}), \text{ when } p > \frac{E_0[v(s)]}{1+r}.$$

Using this expression and Eq. $(A.19)$, cases (i) and (ii) of the lemma follow. In particular, if $w_1^{\max}(\bar{s}) \geq \frac{E_1[v(s)]}{1+r}\alpha_0$ [case (i)], then the asset price is given by the optimistic valuation $p = \frac{E_1[v(s)]}{1+r}$, and optimists invest some of their first period wealth in the bond, i.e., $x_1^B \geq 0$. If instead $w_1^{\max}(\bar{s}) \in \left(\frac{E_0[v(s)]}{1+r}\alpha_0, \frac{E_1[v(s)]}{1+r}\alpha_0\right)$ [case (ii)], then optimists choose $x_1^B = 0$ and the price is given by $p = \frac{w_1^{\max}(\bar{s})}{\alpha_0} > \frac{E_0[v(s)]}{1+r}$.

The only remaining case is $p = \frac{E_0[v(s)]}{1+r}$, which obtains when $w_1^{\max}(\bar{s}) < \frac{E_0[v(s)]}{1+r}\alpha_0$ [case (iii)]. In this case, optimists choose $x_1^B = 0$, and their choice x_1^A can be solved from Eq. $(A.18)$. This solution satisfies $x_1^A < 1$, hence, asset market clearing implies that $x_0^A > 0$. In this case, some of the asset is held by moderates, which is consistent with moderates' demand because $p = \frac{E_0[v(s)]}{1+r}$. This completes the proof of the lemma.

Analytical Characterization of Equilibrium I next provide an analytical characterization of the quasi-equilibrium described by $p = p^{opt}(\bar{s}) = p^{mc}(\bar{s})$. Note that if optimists' wealth is not too large, in particular, if

$$w_1 < \frac{\alpha_0 E_1[v(s)] - v(s^{\min})}{1+r}, \tag{A.20}$$

then the two curves intersect in the case (ii) region of Eq. (17) and the equilibrium pair $(p, \bar{s}^*)$ is characterized as follows:

$$p = p^{mc}(\bar{s}^*) = \frac{1}{\alpha_0}\left(w_1 + \frac{1}{1+r}E_0[\min(v(s), v(\bar{s}^*))]\right), \tag{A.21}$$

where $\bar{s}^*$ is the unique solution to

$$G(\bar{s}^*) \equiv \alpha_0 \frac{1 - F_0(\bar{s}^*)}{1 - F_1(\bar{s}^*)} \int_{\bar{s}^*}^{s^{\max}} (v(s) - v(\bar{s}^*))\, dF_1 - \alpha_1 E_0[\min(v(s), v(\bar{s}^*))] = w_1(1+r). \tag{A.22}$$

In this case, optimists' take loans with riskiness $\bar{s}^* \in (s^{\min}, s^{\max})$ and the price satisfies $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$.

For future reference, I also show that the function $G(\cdot)$ in Eq. $(A.22)$ is differentiable and strictly decreasing. To see this, note that the term $\frac{1-F_0(\bar{s}^*)}{1-F_1(\bar{s}^*)}$ is strictly decreasing in $\bar{s}^*$, the term $\int_{\bar{s}^*}^{s^{\max}} (v(s) - v(\bar{s}^*))\, dF_1$ is strictly decreasing in $\bar{s}^*$, and $E_0[\min(v(s), v(\bar{s}^*))]$ is strictly increasing in $\bar{s}^*$. It follows that $G(\bar{s}^*)$ is differentiable and is strictly decreasing. This also establishes that there is a unique solution to Eq. $(A.22)$.

If the opposite of condition $(A.20)$ holds, then the two curves intersect in the case (i) region of Eq. (17). In this case, optimists' financial constraints are not binding, they borrow with a safe loan contract (with riskiness $\bar{s}^* = s^{\min}$) and they bid up the asset price to the optimistic valuation, i.e., $p = \frac{E_1[v(s)]}{1+r}$. This analysis also verifies that the two curves never intersect in case (iii) region of Eq. (17), which further implies that $p > \frac{E_0[v(s)]}{1+r}$, completing the analytical characterization of equilibrium.

A.3 Characterization of Collateral Equilibrium

This section completes the characterization of the collateral equilibrium by providing the proof of Theorem 2.

Proof of Theorem 2. As the first step, I show that the prices and allocations in Theorem 2 constitute a collateral equilibrium. I next prove the essential uniqueness of the collateral equilibrium, in particular, I show that the collateral equilibrium asset price is uniquely determined.

Existence of the Collateral Equilibrium. I claim that the allocation in Theorem 2 constitutes a collateral equilibrium.. The analysis for the corner price $p = \frac{E_1[v(s)]}{1+r}$ is straightforward. Therefore, suppose that the asset price satisfies $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$. Lemma 1 has established that the price p clears the asset market given the debt contract choice $\bar{s}^*$. Hence, all that remains to check is that loan market is in equilibrium, that is, debt contract choices are optimal for traders (after relaxing the restrictions $z_0(\cdot) \geq 0 \geq z_1(\cdot)$) and that debt contract markets clear.

I next establish an easy-to-check condition for equilibrium in the loan market, based on traders' bid and ask prices for the loans. Recall that the moderates' rate of return on capital is given by $1+r$, while optimists' rate of return on capital is given by $R_1^L(\bar{s}^*) > 1+r$ (cf. Eq. (14)), where the inequality follows since $p < \frac{E_1[v(s)]}{1+r}$. Given the rates of return $1+r$ and $R_1^L(\bar{s}^*)$, consider traders' bid prices for each debt contract $\varphi \in R_+$, defined as:

$$q_0^{bid}(\varphi) = \frac{E_0[\min(v(s), \varphi)]}{1+r} \text{ and } q_1^{bid}(\varphi) = \frac{E_1[\min(v(s), \varphi)]}{R_1^L(\bar{s}^*)}. \tag{A.23}$$

Note that these are the prices that would make moderates (resp. optimists) indifferent between buying a debt contract φ and pursuing their equilibrium investment strategy. Similarly, consider the ask prices for a debt contract φ that would make the traders indifferent between selling the debt contract φ and pursuing their equilibrium investment strategy. Note that short selling a debt contract requires the trader to also buy the asset to pledge as collateral. Hence, for any contract φ with $z_i(\varphi) \leq 0$, the ask price makes the trader indifferent between buying the asset on margin with a debt contract φ and pursuing the equilibrium investment strategy. Formally, $q_0^{ask}(\varphi)$ and $q_1^{ask}(\varphi)$ are defined as the solutions to:

$$\frac{E_0[v(s)] - E_0[\min(v(s), \varphi)]}{p - q_0^{ask}(\varphi)} = 1+r \text{ and } \frac{E_1[v(s)] - E_1[\min(v(s), \varphi)]}{p - q_1^{ask}(\varphi)} = R_1^L(\bar{s}^*). \tag{A.24}$$

The bid and ask prices in $(A.23)$ and $(A.24)$ can also be used to define the aggregate bid and ask price for the contract φ, given by:

$$q^{bid}(\varphi) = \max_i q_i^{bid}(\varphi) \text{ and } q^{ask}(\varphi) = \min_i q_i^{ask}(\varphi).$$

Note that, if the price of a contract φ is below $q^{bid}(\varphi)$, a trader would demand infinite units of the contract, which would violate market clearing. Similarly, if the price is above $q^{ask}(\varphi)$, a trader would sell infinite units, which would again violate market clearing. Moreover, non-zero trade in a contract requires at least one type of trader to buy the contract and another type of trader to sell, which can happen only if $q^{bid}(\varphi) = q^{ask}(\varphi)$. It follows that the loan market is in equilibrium if and only if debt contract prices and allocations satisfy the following condition:

$$\begin{cases} q^{bid}(\varphi) \leq q(\varphi) \leq q^{ask}(\varphi), \text{ and} \\ q(\varphi) = q^{bid}(\varphi) = q^{ask}(\varphi) \text{ whenever } z_i(\varphi) > 0 \text{ for some } i. \end{cases} \tag{A.25}$$

I next show that the loan market allocation of Theorem 2 satisfies the loan market equilibrium condition $(A.25)$. In particular, I claim:

$$q^{bid}(\varphi) \leq q^{ask}(\varphi) \text{ with equality iff } \varphi = v(\bar{s}^*). \tag{A.26}$$

Note that the debt contract prices of Theorem 2 (cf. Eq. (18)) are chosen such that $q(\varphi) = q^{bid}(\varphi)$. Moreover, the allocations are such that there is trade only for contract $\varphi = v(\bar{s}^*)$. Hence, the claim in $(A.26)$ implies $(A.25)$, which ensures that the loan market is indeed in equilibrium.

Note that the claim in $(A.26)$ is true for all $\varphi \in R_+$, if it is true for all $\varphi \in \left[v\left(s^{\min}\right), v\left(s^{\max}\right)\right]$. To prove the claim for the relevant set of debt contracts, $\varphi = v(\tilde{s})$ for some $\tilde{s} \in S$, first note that

$$q_i^{bid}(v(\tilde{s})) < q_i^{ask}(v(\tilde{s})) \text{ for each } \tilde{s} \in \mathcal{S} \text{ and } i. \tag{A.27}$$

which is straightforward to check. There is a wedge between each type traders' bid and ask prices, intuitively because buying the debt contract has no collateral requirements while selling the debt contract requires the trader to pledge collateral (and thus, the traders' ask price to sell a contract is higher).

Second, note that $\bar{s}^*$ is the unique solution to problem (6) by definition, and thus

$$R_1^L(\bar{s}^*) = \frac{E_1[v(s)] - E_1[\min(v(s), v(\bar{s}^*))]}{p - E_0[\min(v(s), v(\bar{s}^*))]/(1+r)} > \frac{E_1[v(s)] - E_1[\min(v(s), v(\tilde{s}))]}{p - E_0[\min(v(s), v(\tilde{s}))]/(1+r)} \text{ for each } \tilde{s} \neq \bar{s}^*.$$

Using this inequality and the definition of $q_1^{ask}(v(\tilde{s}))$ in $(A.24)$ shows

$$q_1^{ask}(v(\tilde{s})) \geq \frac{E_0[\min(v(s), v(\tilde{s}))]}{1+r} = q_0^{bid}(v(\tilde{s})) \text{ with equality iff } \tilde{s} = \bar{s}^*. \tag{A.28}$$

Third, recall that $\frac{E_1[\min(v(s),v(\tilde{s}))]}{E_0[\min(v(s),v(\tilde{s}))]}$ is equal to 1 for $\tilde{s} = s^{\min}$, and is strictly increasing in $\tilde{s}$. By Eq. $(A.23)$, it follows that $\frac{q_1^{bid}\left(v\left(s^{\min}\right)\right)}{q_0^{bid}\left(v\left(s^{\min}\right)\right)} = \frac{1+r}{R_L(\bar{s})} < 1$, and that $\frac{q_1^{bid}(v(\tilde{s}))}{q_0^{bid}(v(\tilde{s}))}$ is strictly increasing in $\tilde{s}$. Then, there are two cases to consider. As the first case, $\frac{q_1^{bid}(v(\tilde{s}))}{q_0^{bid}(v(\tilde{s}))}$ may never exceed 1, that is, it may be the case that

$$q_1^{bid}(v(\tilde{s})) < q_0^{bid}(v(\tilde{s})) \text{ for each } \tilde{s} \in \mathcal{S}. \tag{A.29}$$

In this case, combining Eqs. $(A.27)$,$(A.28)$ and $(A.29)$ proves the claim in $(A.26)$. The left panel of Figure 8 plots the bid and ask prices in this first case. The figure illustrates that, in this case, the quasi-equilibrium debt prices in (9) and the collateral equilibrium debt prices in (9) are identical.

As the second case, $\frac{q_1^{bid}(v(\tilde{s}))}{q_0^{bid}(v(\tilde{s}))}$ may exceed 1 for sufficiently large 1, i.e., it may be the case that there exists $\tilde{s}^{cross}$ such that

$$\begin{cases} q_1^{bid}(v(\tilde{s})) < q_0^{bid}(v(\tilde{s})) \text{ for all } \tilde{s} < \tilde{s}^{cross}, \\ q_1^{bid}(v(\tilde{s})) \geq q_0^{bid}(v(\tilde{s})) \text{ for all } \tilde{s} \geq \tilde{s}^{cross}. \end{cases} \tag{A.30}$$

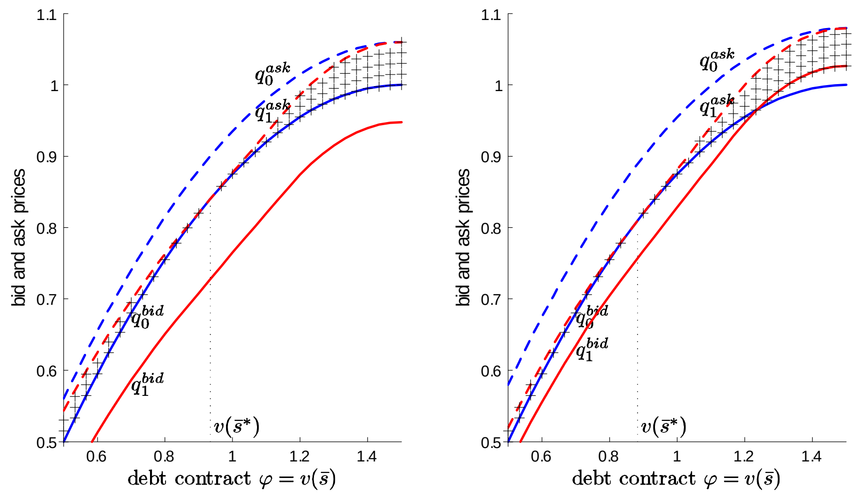

Figure 8: The left panel displays the bid and ask prices for the case in which the inequality in $(A.29)$ holds, and the right panel displays the case in which the inequality in $(A.29)$ fails. The shaded areas display the set of all possible equilibrium debt contract prices in each case.

Note that, in this case, $\tilde{s}^{cross}$ is uniquely defined as the solution to

$$\frac{E_1\left[\min\left(v\left(s\right),v\left(\tilde{s}^{cross}\right)\right)\right]}{E_0\left[\min\left(v\left(s\right),v\left(\tilde{s}^{cross}\right)\right)\right]}=\frac{R_L\left(\bar{s}^*\right)}{1+r}.$$

Moreover, it can be checked that $\frac{E_1[\min(v(s),v(\bar{s}^*))]}{E_0[\min(v(s),v(\bar{s}^*))]} < \frac{R_L(\bar{s}^*)}{1+r}$,[25] which implies $\tilde{s}^{cross} > \bar{s}^*$. It can also be seen that[26]

$$q_0^{ask}\left(v\left(\tilde{s}\right)\right) \geq q_1^{ask}\left(v\left(\tilde{s}\right)\right) \text{ for each } \tilde{s} \geq \tilde{s}^{cross}. \tag{A.32}$$

[25] To see this, consider the expression for $R_1^L\left(\bar{s}^*\right)$ in Eq. (14), which can be rewritten as

$$p-\frac{E_0\left[\min\left(v\left(s\right),v\left(\bar{s}^*\right)\right)\right]}{1+r}=\frac{E_1\left[s\right]}{R_1^L\left(\bar{s}^*\right)}-\frac{E_1\left[\min\left(v\left(s\right),v\left(\bar{s}^*\right)\right)\right]}{R_1^L\left(\bar{s}^*\right)}.$$

Note that $R_1^L\left(\bar{s}^*\right) > \frac{E_1[v(s)]}{p}$ because optimists always have the option of buying the asset without borrowing. Hence, the previous inequality implies $\frac{E_1[\min(v(s),v(\bar{s}^*))]}{1+r} < \frac{E_0[\min(v(s),v(\bar{s}^*))]}{R_1^L(\bar{s}^*)}$, which can be rewritten as $\frac{E_1[\min(v(s),v(\bar{s}^*))]}{E_0[\min(v(s),v(\bar{s}^*))]} < \frac{1+r}{R_1^L(\bar{s}^*)}$.

[26] Note that from the definition of $q_i^{ask}\left(v\left(\tilde{s}\right)\right)$ in $(A.24)$, the inequality in $(A.32)$ is equivalent to

$$\frac{E_0\left[v\left(s\right)\right]-E_0\left[\min\left(v\left(s\right),v\left(\tilde{s}\right)\right)\right]}{1+r} \leq \frac{E_1\left[v\left(s\right)\right]-E_1\left[\min\left(v\left(s\right),v\left(\tilde{s}\right)\right)\right]}{R_1^L\left(\bar{s}^*\right)} \text{ for each } \tilde{s} \geq \tilde{s}^{cross}. \tag{A.31}$$

Recall that $\frac{E_1[v(s)]}{E_0[v(s)]} \geq \frac{E_1[\min(v(s),v(\tilde{s}))]}{E_0[\min(v(s),v(\tilde{s}))]}$, which implies that

$$\frac{E_1\left[s\right]-E_1\left[\min\left(v\left(s\right),v\left(\tilde{s}\right)\right)\right]}{E_0\left[s\right]-E_0\left[\min\left(v\left(s\right),v\left(\tilde{s}\right)\right)\right]} \geq \frac{E_1\left[\min\left(v\left(s\right),v\left(\tilde{s}\right)\right)\right]}{E_0\left[\min\left(v\left(s\right),v\left(\tilde{s}\right)\right)\right]} \geq \frac{R_1^L\left(\bar{s}^*\right)}{1+r},$$

where the last inequality holds for each $\tilde{s} \geq \tilde{s}^{cross}$ in view of the definition of $\tilde{s}^{cross}$. This proves the inequality in $(A.32)$, which in turn shows the inequality in $(A.32)$.

Then, using Eqs. $(A.27)$, $(A.28)$ and $(A.30)$, it follows that $q^{bid}(v(\tilde{s})) \leq q^{ask}(v(\tilde{s}))$ for each $\tilde{s} \leq \tilde{s}^{cross}$, with equality iff $\tilde{s} = \bar{s}^*$. Moreover, using Eqs. $(A.27)$, $(A.28)$, $(A.32)$ and $(A.30)$, it also follows that $q^{bid}(\tilde{s}) < q^{ask}(\tilde{s})$ for each $\tilde{s} \geq \tilde{s}^{cross}$. This completes the proof of claim $(A.26)$, and establishes that the allocation characterized in Theorem 2 is indeed a collateral equilibrium. The right panel of Figure 8 plots the bid and ask prices in this second case. This figure illustrates that, in this case, the quasi-equilibrium debt prices in (9) and the collateral equilibrium debt prices in (9) are not the same, but the difference in prices does not upset the optimality of the debt contract $\bar{s}^*$.

Figure 8 also illustrates that the debt contract prices are not uniquely determined in equilibrium (except for the price of the optimal contract $v(\bar{s}^*)$, which is uniquely determined). In particular, any price function $q(\cdot)$ such that $q(v(\tilde{s})) \in \left[q^{bid}(v(\tilde{s})), q^{ask}(v(\tilde{s}))\right]$ can support the equilibrium allocation in equilibrium. However, the equilibrium allocations in the loan market and the equilibrium asset price p is uniquely determined, as I next prove.

Essential Uniqueness of Collateral Equilibrium. I first prove that the equilibrium asset price p is uniquely determined. Let R_0 and R_1 denote traders' equilibrium rates of return on capital (in the above equilibrium, $R_0 = 1+r$ and $R_1 = R_1^L(\bar{s})$). Since traders always have the option to invest in the bond, R_0 and R_1 are always weakly greater than $1+r$. Moreover, in equilibrium some investors must agree to hold the bond (as can be seen by aggregating the budget constraints (7)), hence either R_0 or R_1 must be equal to $1+r$. Since optimists have a greater valuation of the asset, it can be checked that $R_1 \geq R_0 = 1+r$.

I next claim that, for any given price $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$, optimists' rate of return on capital is uniquely determined as $R_1 = R_1^L(\bar{s})$, and the loan market allocations are uniquely determined as characterized by Theorem 1. To prove this claim, consider optimists' bid and ask prices in $(A.23)$ and $(A.24)$ for an arbitrary price level $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$ and an arbitrary required rate of return R_1 (i.e., replace $R_1^L(\bar{s}^*)$ in these expressions with R_1). Eq. $(A.24)$ shows that $q_1^{ask}(v(\bar{s}))$ increases in the required rate of return (and Eq. $(A.23)$ shows $q_1^{bid}(v(\bar{s}))$ decreases in the required rate of return). It follows that the loan market is at equilibrium for a unique required rate of return such that $q_1^{ask}(v(\tilde{s})) \geq q_0^{bid}(v(\tilde{s}))$ for all $\tilde{s}$ with equality for exactly one state $\bar{s}$ (as displayed in Figure (8)). Then, this rate of return R_1 satisfies:

$$\begin{aligned}
\frac{E_1[v(s)] - E_1[\min(v(s), v(\bar{s}))]}{p - q_0^{bid}(v(\bar{s}))} &= R_1 = \frac{E_1[v(s)] - E_1[\min(v(s), v(\tilde{s}))]}{p - q_1^{ask}(v(\tilde{s}))} \\
&\geq \frac{E_1[v(s)] - E_1[\min(v(s), v(\tilde{s}))]}{p - q_0^{bid}(v(\tilde{s}))},
\end{aligned} \tag{A.33}$$

where the first equality uses the definition of $q_1^{ask}(v(\bar{s}))$ and the fact that $q_1^{ask}(v(\bar{s})) = q_0^{bid}(v(\bar{s}))$, the second equality uses the definition of $q_1^{ask}(v(\tilde{s}))$, and the last inequality follows from the inequality $q_1^{ask}(v(\tilde{s})) \geq q_0^{bid}(v(\tilde{s}))$. The comparison between the first and the last terms in $(A.33)$ shows that $\bar{s} \in S$ solves problem (14). In particular, for any price $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$, the unique loan market allocation is the same as the quasi-equilibrium loan market allocation characterized by Theorem 1, and the unique rate of return that equilibrates the loan market is given by $R_1 = R_1^L(\bar{s})$.

I next prove the uniqueness of the collateral equilibrium price p. For any price $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$, optimists loan market allocation is uniquely determined, and thus their leveraged investment on the asset (i.e., their demand for the asset) is uniquely determined. Combining this with the asset market clearing condition (as characterized by Lemma 1) the equilibrium price p is also uniquely determined.

The above analysis also establishes that the price of the optimal contract, $q\left(v\left(\bar{s}^*\right)\right)$, and the equilibrium allocations, $\left(x_i, z_i\left(\cdot\right)\right)_i$, are uniquely determined for any price $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$. For the corner price $p = \frac{E_1[v(s)]}{1+r}$ (which corresponds to the optimal contract $\bar{s}^* = s^{\min}$), it can be checked that the price of the optimal contract $q\left(v\left(s^{\min}\right)\right)$ is still uniquely determined. However, in this case, the equilibrium allocations are not necessarily unique since optimists may be indifferent among some of the safe contracts $\varphi \leq v\left(s^{\min}\right)$. This establishes the essential uniqueness of the collateral equilibrium and completes the proof of Theorem 2.

A.4 Proofs for Section 4

Proof of Theorem 3. Part (i). Note that p and $\bar{s}^*$ are the solution to the equation $p = p^{mc}\left(\bar{s}\right) = p^{opt}\left(\bar{s}\right)$. In response to a decrease in w_1, the market clearing curve $p^{mc}\left(\bar{s}\right)$ shifts down, i.e., $p^{mc}\left(\bar{s}\right)$ weakly decreases for each $\bar{s} \in S$, while the optimality curve $p^{opt}\left(\bar{s}\right)$ remains intact. It follows that $\bar{s}^*$ weakly increases and p weakly decreases. By Eq. (21), this further implies that θ weakly decreases because the valuations $E_i\left[s\right]$ are unchanged.

To obtain the comparative statics of the leverage ratio, rewrite Eq. (20) as $L = \frac{1}{1 - \frac{E_0[\min(v(s), v(\bar{s}^*))]}{1+r}\frac{1}{p}}$. In response to a decline in w_1, p weakly decreases and $\bar{s}^*$ weakly increases, which implies that $\frac{E_0[\min(v(s), v(\bar{s}^*))]}{p}$ weakly increases. This further implies that L weakly increases, completing the proof of part (i).

Part (ii). First suppose condition $(A.20)$ is violated so that the asset price is $p = \frac{E_1[v(s)]}{1+r}$. The same steps in part (i) show that, in response to a decline in r, p weakly increases and $\bar{s}^* = s^{\min}$, $L = \frac{E_1[v(s)]}{E_1[v(s)] - v(s^{\min})}$ and $\theta = 1$ remain constant, proving the result for this case.

Therefore suppose condition $(A.20)$ holds. In this case, the equilibrium is characterized as the solution to $G\left(\bar{s}^*\right) = \left(1+r\right) w_1$ (cf. Eq. $(A.22)$). Implicitly differentiating this equation with respect to r and evaluating at $\bar{s}^*$ gives

$$G'\left(\bar{s}^*\right) \frac{d\bar{s}^*}{dr} = w_1,$$

which implies that

$$\frac{d\bar{s}^*}{dr} = \frac{w_1}{G'\left(\bar{s}\right)} < 0. \tag{A.34}$$

It follows that $\bar{s}^*$ increases in response to a decrease in r.

To obtain the comparative statics for the asset price, differentiate the expression for the asset price in Eq. $(A.21)$ with respect to r, which gives

$$\frac{dp}{dr} = \frac{1}{\alpha_0}\left(\frac{-E_0\left[\min\left(v\left(s\right), v\left(\bar{s}^*\right)\right)\right]}{\left(1+r\right)^2} + \frac{1}{1+r}\frac{dE_0\left[\min\left(v\left(s\right), v\left(\bar{s}^*\right)\right)\right]}{d\bar{s}^*}\frac{d\bar{s}^*}{dr}\right).$$

The right hand side of this expression is strictly negative because $\frac{dE_0[\min(v(s), v(\bar{s}^*))]}{d\bar{s}^*} > 0$ and $\frac{d\bar{s}^*}{dr} < 0$ (by Eq. $(A.34)$). In particular, $\frac{dp}{dr} < 0$, which implies that the asset price increases in response to a decrease in r.

To obtain the comparative statics for the leverage ratio, note that using Eq. $(A.21)$ to substitute for $\frac{E_0[\min(v(s), v(\bar{s}^*))]}{1+r}$, the leverage ratio in (20) can be rewritten as

$$L = \frac{p}{p - \left(p\alpha_0 - w_1\right)} = \frac{1}{1 - \alpha_0 + \frac{w_1}{p}}.$$

Then, $\frac{dp}{dr} < 0$ implies that $\frac{dL}{dr} < 0$, which in turn implies that leverage ratio increase in response to a decrease in interest rate.

Finally, to obtain the comparative statics for the overvaluation ratio, note that Eq. (21) can be rearranged to give

$$\theta = \frac{(1+r)\,p - E_0\left[v\left(s\right)\right]}{E_1\left[v\left(s\right)\right] - E_0\left[v\left(s\right)\right]}.$$

Hence, θ comoves with $(1+r)\,p = (1+r)\,p^{opt}\left(\bar{s}\right)$. Note that

$$\frac{d}{dr}\left((1+r)\,p^{opt}\left(\bar{s}^*\right)\right) = p^{opt}\left(\bar{s}^*\right) + (1+r)\,\frac{dp^{opt}\left(\bar{s}^*\right)}{d\bar{s}}\frac{d\bar{s}^*}{dr}.$$

This derivative is positive since $\frac{dp^{opt}(\bar{s})}{d\bar{s}} < 0$ (by part (iii) of Lemma 4) and $\frac{d\bar{s}^*}{dr} < 0$ (by Eq. $(A.34)$). This further implies that $\frac{d\theta}{dr} > 0$, that is, θ decreases in response to a decrease in r. This completes the proof of Theorem 3.

Proof of Theorem 4. Part (i). Similar to the proof of Lemma 5, define the function $g : S \to R$ with

$$g\left(\bar{s}\right) = \tilde{E}_1\left[v\left(s\right) \mid s \geq \bar{s}\right] - E_1\left[v\left(s\right) \mid s \geq \bar{s}\right].$$

Note that $g\left(s^{\min}\right) = 0$ since $\tilde{E}_1\left[v\left(s\right)\right] = E_1\left[v\left(s\right)\right]$, and note also that $g\left(s^{\max}\right) = 0$. I claim that

$$g\left(\bar{s}\right) \geq 0 \text{ for all } \bar{s} \in \left(s^{\min}, s^{\max}\right), \tag{A.35}$$

which implies Eq. (24) in the main text. The rest of the proof then follows by the argument provided after Theorem 4. The comparative statics for the leverage ratio follows by the same argument in part (ii) of Theorem 3.

To prove the claim in $(A.35)$, first note that Eq. $(A.8)$ applies also in this setting. Since $\frac{\tilde{f}_1(\bar{s})}{1-\tilde{F}_1(\bar{s})} \leq \frac{f_1(\bar{s})}{1-F_1(\bar{s})}$ over the range $\bar{s} \in \left(s^R, s^{\max}\right)$, the same argument used in the proof of part (i) of Lemma 5 shows that

$$g\left(\bar{s}\right) \geq 0 \text{ for all } \bar{s} \in [s^R, s^{\max}). \tag{A.36}$$

Second, suppose, to reach a contradiction, that there exists $\tilde{s} \in [s^{\min}, s^R)$ with $g\left(\tilde{s}\right) < 0$. Since $g\left(s^{\min}\right) = 0$, this further implies that there exists $\hat{s} \in [s^{\min}, \tilde{s})$ such that $g\left(\hat{s}\right) = 0$ and $g'\left(\hat{s}\right) < 0$, since otherwise, the differentiable function $g\left(\cdot\right)$ could not become negative over the range $[s^{\min}, \tilde{s})$. Considering Eq. $(A.8)$ for $\bar{s} = \hat{s}$ and using $g\left(\hat{s}\right) = 0$ implies

$$g'\left(\hat{s}\right) = \left(\frac{\tilde{f}_1\left(\hat{s}\right)}{1-\tilde{F}_1\left(\hat{s}\right)} - \frac{f_1\left(\hat{s}\right)}{1-F_1\left(\hat{s}\right)}\right)\left(\tilde{E}_1\left[v\left(s\right) \mid s \geq \hat{s}\right] - v\left(\hat{s}\right)\right) \geq 0,$$

where the inequality follows since $\frac{\tilde{f}_1(\hat{s})}{1-\tilde{F}_1(\hat{s})} \geq \frac{f_1(\hat{s})}{1-F_1(\hat{s})}$ (as $\hat{s} < s^R$). Since $g'\left(\hat{s}\right) < 0$ by the choice of $\hat{s}$, the previous displayed inequality yields a contradiction, completing the proof.

Part (ii). Applying the proof of part (i) for distributions $F_0 \succeq_R \tilde{F}_0$ shows that

$$E_0\left[v\left(s\right) \mid s \geq \bar{s}\right] \geq \tilde{E}_0\left[v\left(s\right) \mid s \geq \bar{s}\right] \text{ for each } \bar{s} \in \mathcal{S}. \tag{A.37}$$

Note also that $F_0 \succeq_{R,\bar{s}^*} \tilde{F}_0$ implies $\frac{1-\tilde{F}_0(\bar{s})}{1-F_0(\bar{s})}$ is weakly increasing for $\bar{s} \in \left(s^{\min}, \bar{s}^*\right)$, which further implies $\tilde{F}_0\left(\bar{s}\right) \leq F_0\left(\bar{s}\right)$ over this range. In view of this observation and Eq. $(A.37)$, Eq. (23) implies that

$$p^{opt}\left(\bar{s}\right) \text{ weakly increases for each } \bar{s} \in \left(s^{\min}, \bar{s}^*\right). \tag{A.38}$$

Next consider the effect on the market clearing curve $p^{mc}\left(\bar{s}\right)$. Note that since $F_0 \succeq_{R,\bar{s}^*} \tilde{F}_0$,

$$\frac{\tilde{f}_0\left(\bar{s}\right)}{1-\tilde{F}_0\left(\bar{s}\right)} \leq \frac{f_0\left(\bar{s}\right)}{1-F_0\left(\bar{s}\right)} \text{ for each } \bar{s} \in \left(s^{\min}, \bar{s}^*\right).$$

Then, the same steps as in the proof of part (ii.2) of Lemma 5 applies in this case and shows

$$p^{mc}\left(\bar{s}\right) \text{ weakly increases for each } \bar{s} \in \left(s^{\min}, \bar{s}^*\right). \tag{A.39}$$

Using Eqs. $(A.38)$ and $(A.39)$ along with the fact that $p^{opt}\left(\bar{s}\right)$ is a decreasing relation and $p^{mc}\left(\bar{s}\right)$ is a weakly increasing relation, it follows that p is weakly greater at the new intersection point while the effect on $\bar{s}^*$ is ambiguous.[27] This also implies that θ weakly increases, completing the proof.

Proof of Theorem 5. Part (i). The fact that $\tilde{F}_0$ and F_0 are equally optimistic over $\left(s^{\min}, \bar{s}^*\right)$ implies that $\frac{1-\tilde{F}_0(\bar{s})}{1-F_0(\bar{s})} = \frac{1-\tilde{F}_0\left(s^{\min}\right)}{1-F_0\left(s^{\min}\right)} = 1$, that is, $\tilde{F}_0\left(\bar{s}\right) = F_0\left(\bar{s}\right)$ for each $\bar{s} \in \left(s^{\min}, \bar{s}^*\right)$. Then, Eq. (10) implies

$$p^{opt}\left(\bar{s}\ ;\ \tilde{F}_0, \tilde{F}_1\right) = p^{opt}\left(\bar{s}\ ;\ F_0, \tilde{F}_1\right) \text{ for each } \bar{s} \in \left(s^{\min}, \bar{s}^*\right).$$

By part (i) of Lemma 5, $\tilde{F}_1 \succeq_O F_1$ implies $p^{opt}\left(\bar{s}\ ;\ F_0, \tilde{F}_1\right) \geq p^{opt}\left(\bar{s}\ ;\ F_0, F_1\right)$. Using this along with the previous displayed equation shows Eq. $(A.38)$, that is, $p^{opt}\left(\bar{s}\right)$ weakly increases for each $\bar{s} \in \left(s^{\min}, \bar{s}^*\right)$.

Note also that $\tilde{E}_0\left[v\left(s\right)\right] \leq E_0\left[v\left(s\right)\right]$. Combining this with the fact that $\tilde{F}_0\left(\bar{s}\right) = F_0\left(\bar{s}\right)$ for each $\bar{s} \in \left(s^{\min}, \bar{s}^*\right)$ implies that $p^{mc}\left(\bar{s}\right)$ curve remains constant over $\left(s^{\min}, \bar{s}^*\right)$ except for the fact that its lower bound "extends," that is,

$$p^{mc}\left(\bar{s}\ ;\ \tilde{F}_0, \tilde{F}_1\right) \begin{cases} = p^{mc}\left(\bar{s}\ \right) \text{ if } p^{mc}\left(\bar{s}\right) > \frac{E_0[v(s)]}{1+r} \\ \leq p^{mc}\left(\bar{s}\right) \text{ if } p^{mc}\left(\bar{s}\right) = \frac{E_0[v(s)]}{1+r} \end{cases} \text{ for each } \bar{s} \in \left(s^{\min}, \bar{s}^*\right).$$

Note also that the proof of Theorem 2 establishes that $p = p^{mc}\left(\bar{s}^*\right) > \frac{E_0[v(s)]}{1+r}$. Using this in the previous displayed equation implies that there exists $\varepsilon > 0$ such that $p^{mc}\left(\bar{s}\right)$ remains constant in a neighborhood $\bar{s} \in \left(\bar{s}^* - \varepsilon, s^*\right)$. Combining this with Eq. $(A.38)$ and using the facts that $p^{opt}\left(\cdot\right)$ is a decreasing curve and $p^{mc}\left(\cdot\right)$ is an increasing curve shows that the new intersection point of these curves is weakly to the right of $\bar{s}^*$, which further implies p and $\bar{s}^*$ weakly increase. The comparative statics for the leverage ratio follows by the same argument given for part (ii) of Theorem 3.

Part (ii). First, I claim that F_1 and $\tilde{F}_1$ have the same distribution conditional on any upper-threshold

[27] The effect on $\bar{s}^*$ depends on how much the $p^{mc}\left(\bar{s}\right)$ curve shifts up. If the effect on the $p^{mc}\left(\bar{s}\right)$ curve is strong, perhaps because α_0 is small, then $\bar{s}^*$ may decrease. To see the intuition for this result, suppose many units of the asset are already endowed to optimists, i.e. α_1 is high. Then the increase in lenders' valuation of debt contracts acts similar to a positive wealth shock to optimists (cf. Eq. (16)), because optimists can borrow more against the units they already own. This effect tends to lower the leverage ratio, and if sufficiently strong, it can overcome the effect from the shift of the $p^{opt}\left(\bar{s}\right)$ curve which (similar to part (i)) tends to increase the leverage ratio.

event $[\bar{s}, s^{\max}]$ with $\bar{s} > \bar{s}^*$. That is, for any $\bar{s} \in (\bar{s}^*, s^{\max})$,

$$\frac{f_1(s)}{1 - F_1(\bar{s})} = \frac{\tilde{f}_1(s)}{1 - \tilde{F}_1(\bar{s})} \text{ for each } s \in [\bar{s}, s^{\max}). \tag{A.40}$$

To see this, note that by assumption

$$\frac{1 - \tilde{F}_1(s)}{1 - F_1(s)} = \frac{1 - \tilde{F}_1(\bar{s})}{1 - F_1(\bar{s})} \text{ for each } s \in [\bar{s}, s^{\max}). \tag{A.41}$$

Moreover, taking the derivative of this equation with respect to s implies

$$\frac{f_1(s)}{1 - F_1(s)} = \frac{\tilde{f}_1(s)}{1 - \tilde{F}_1(s)} \text{ for each } s \in [\bar{s}, s^{\max}). \tag{A.42}$$

Using Eqs. $(A.41)$ and $(A.42)$, it follows that, for each $s \in (\bar{s}, s^{\max})$,

$$\frac{f_1(s)}{1 - F_1(\bar{s})} = \frac{f_1(s)}{1 - F_1(s)} \frac{1 - F_1(s)}{1 - F_1(\bar{s})} = \frac{\tilde{f}_1(s)}{1 - \tilde{F}_1(s)} \frac{1 - \tilde{F}_1(s)}{1 - \tilde{F}_1(\bar{s})} = \frac{\tilde{f}_1(s)}{1 - \tilde{F}_1(\bar{s})},$$

proving Eq. $(A.40)$.

Next consider the case in which condition $(A.20)$ is violated so that $\bar{s}^* = s^{\min}$. Using Eq. $(A.40)$ for $\bar{s} = \bar{s}^* + \varepsilon$ and taking $\varepsilon \to 0$ shows that, in this case, F_1 and $\tilde{F}_1$ are the same distribution. By the same argument, F_0 and $\tilde{F}_0$ are also the same distribution. It follows that the asset price p remains constant for this case.

Consider next the case in which condition $(A.20)$ holds so that $\bar{s}^* > s^{\min}$. Consider respectively the shift in the $p^{opt}(\bar{s})$ and $p^{mc}(\bar{s})$ curves. First consider the optimality curve $p^{opt}(\bar{s})$ and note that Eq. $(A.40)$ in Eq. (10) implies

$$p^{opt}\left(\bar{s}\ ;\ \tilde{F}_0, \tilde{F}_1\right) = p^{opt}\left(\bar{s}\ ;\ \tilde{F}_0, F_1\right) \text{ for each } \bar{s} \in (\bar{s}^*, s^{\max}).$$

By part (ii) of Lemma 5, $F_0 \succeq_O \tilde{F}_0$ implies $p^{opt}\left(\bar{s}\ ;\ \tilde{F}_0, F_1\right) \leq p^{opt}(\bar{s}\ ;\ F_0, F_1)$. Using this in the previous displayed equation shows

$$p^{opt}(\bar{s}) \text{ weakly decreases for each } \bar{s} \in (\bar{s}^*, s^{\max}). \tag{A.43}$$

Next consider the market clearing curve $p^{mc}(\bar{s})$. Note that $\bar{s}^* > s^{\min}$ implies $p^{opt}(\bar{s}^*) = p^{mc}(\bar{s}^*) < \frac{E_1[v(s)]}{1+r}$, thus by Eq. $(A.5)$ (cf. part (ii) of Lemma 5), the increase in optimism of F_1 leaves $p^{mc}(\bar{s})$ unchanged in a neighborhood $(\bar{s}^*, \bar{s}^* + \varepsilon)$. Note also that the decrease in optimism of F_0 weakly decreases $p^{mc}(\bar{s})$ downwards pointwise. It follows that

$$p^{mc}(\bar{s}) \text{ weakly decreases for each } \bar{s} \in (\bar{s}^*, \bar{s}^* + \varepsilon). \tag{A.44}$$

Combining Eqs. $(A.43)$ and $(A.44)$ and using the fact that $p^{opt}(\bar{s})$ is a decreasing curve while $p^{mc}(\bar{s})$ is an increasing curve, the asset price p is weakly lower at the new intersection point. This completes the proof of Theorem 5.

A.5 Proofs for Section 5

Proof of Lemma 3. Let the prices $\left(p(a), [q(a,\varphi)]_{\varphi\in\mathbb{R}_+}\right)_{a\in\mathbb{R}_{++}}$ and allocations $\left((x_i(a), z_i(a,\cdot))_{i\in\{1,0\}}\right)_{a\in\mathbb{R}_{++}}$ be such that, for each a, the prices $\left(p(a), [q(a,\varphi)]_{\varphi\in\mathbb{R}_+}\right)$ and allocations $(x_i(a), z_i(a,\cdot))_{i\in\{1,0\}}$ correspond to the collateral equilibrium of the static economy in (32), characterized in Theorem 2. I claim that there exists a RCE corresponding to this allocation.

Note that optimists' problem (31) is equivalent to optimists' problem (6) in this static economy, given prices

$$p = p(a) \text{ and } q(\varphi) = q(a,\varphi) \text{ for each } \varphi \in \mathbb{R}_+. \tag{A.45}$$

Hence optimists' allocations are also optimal in the dynamic economy. Moderate types' problem is slightly different since $\alpha_0 = 1$ in the static economy whereas α_0 is equal to 0 in the dynamic model (as the asset is held by the old generation). Because of this difference, the allocation $(x_0(a), z_0(a,\cdot))$ violates the budget constraint of moderate types in the dynamic model by an amount $p(a)$. However, note that if

$$\tilde{x}_0^B(a) \equiv x_0^B(a) - p(a) \geq 0 \text{ for each } a \in \mathbb{R}_{++}, \tag{A.46}$$

then the allocation $\left(x_0^A(a), \tilde{x}_0^B(a), z_0(a,\cdot)\right)$ satisfies moderate types' budget constraint. When this is the case, it can also be seen that this allocation solves Problem (31) given the prices in $(A.45)$. To see this, note that Eq. $(A.46)$ implies $x_0^B(a) > p(a) > 0$ for the static economy in (32), which further implies that moderate types in the static economy are indifferent between holding bonds and debt contracts. As their budgets and bond holdings are reduced by the same amount $p(a)$, the allocation $\left(x_0^A(a), \tilde{x}_0^B(a), z_0(a,\cdot)\right)$ is optimal for moderate types in the dynamic model, proving our claim.

Hence, if the inequality in $(A.46)$ is satisfied, then the prices $\left(p(a), [q(a,\varphi)]_{\varphi\in\mathbb{R}}\right)_{a\in\mathbb{R}_{++}}$ and allocations $\left((x_1(a),\ z_1(a,\cdot)), \left(x_0^A(a), \tilde{x}_0^B(a),\ z_0(a,\cdot)\right)\right)_{a\in\mathbb{R}_{++}}$ constitute an equilibrium of the dynamic economy. To verify the inequality in $(A.46)$, consider the budget constraint (7) of moderate traders in the corresponding static economy

$$x_0^B(a) + \int_{\mathbb{R}_+} q(a,\varphi) z_0(a,\varphi) d\varphi = \omega_0 a + p(a), \tag{A.47}$$

where the equality holds since $\alpha_0 = 1$ and $x_0^A(a) = 0$ (as moderate types do not invest in the asset in the static equilibrium). Next consider the collateral constraint (8), and note that, using the inequality $q(a,\varphi) \leq p(a)$ (which follows from Eq. (9)), the debt market clearing condition $z_0(a,\varphi) = -z_1(a,\varphi) > 0$ and the asset market clearing condition $x_1^A(a) = 1$, the collateral constraint implies

$$\int_{\mathbb{R}_+} q(a,\varphi) z_0(a,\varphi) d\varphi \leq p(a).$$

Using this inequality in $(A.47)$ implies $\omega_0 a \leq x_0^B(a)$, that is, moderate types always invest in the bond weakly more than the value of their endowments. Using condition (26) and the fact that $p(a)$ is weakly less than the unconstrained level $\frac{1+\varepsilon}{r-\varepsilon}a$, the previous displayed equation implies that the allocation $\tilde{x}_0^B(a)$ in $(A.46)$ is positive, completing the proof of Lemma 3.

Proof of Theorem 6. Most of the argument is included in the main text preceding the theorem statement. The remaining step is to check that the mapping $P_d(\cdot)$ is strictly increasing and that it satisfies the boundary conditions in (37), which implies that $P_d(\cdot)$ has a unique fixed point over $\left[p_d^{\min}, p_d^{\max}\right]$. Recall that $(P_d(\tilde{p}_d), S_d(\tilde{p}_d))$ is the unique solution to the static equilibrium conditions in (36) for the economy $E(\tilde{p}_d)$. I first claim that the loan riskiness $S_d(\tilde{p}_d) \in [s^{\min}, s^{\max})$ is weakly increasing in $\tilde{p}_d$. I then use this claim to prove that $P_d(\cdot)$ is strictly increasing and that it satisfies the boundary conditions.

To prove that $S_d(\tilde{p}_d)$ is weakly increasing, note that there are two sub-cases depending on condition $(A.20)$. Using the value function $v_d(s \mid \tilde{p}_d) = s(1+\tilde{p}_d)$ (cf. Eq. (34)), $\alpha_0 = 1$, and $E_1[v(s)] = 1+\varepsilon$, condition $(A.20)$ can be written as

$$\omega_1 < (1+\tilde{p}_d)\frac{1+\varepsilon - s^{\min}}{1+r}. \tag{A.48}$$

First suppose $\tilde{p}_d$ is sufficiently large that this condition is violated. In this case, by the characterization in the proof of Theorem 2, the loan riskiness $S_d(\tilde{p}_d) = s^{\min}$ is constant. Second, suppose condition $(A.48)$ is satisfied, and thus $S_d(\tilde{p}_d) \in \left(s^{\min}, s^{\max}\right)$ is determined as the unique solution to Eq. $(A.22)$, which can be simplified to

$$\frac{1-F_0(\bar{s})}{1-F_1(\bar{s})}\int_{\bar{s}}^{s^{\max}} (v(s) - v(\bar{s}))\, dF_1 = w_1 \frac{1+r}{1+\tilde{p}_d}.$$

The proof of Theorem 2 shows that the left hand side of this expression is a strictly decreasing function of $\bar{s}$. Since the right hand side is decreasing in $\tilde{p}_d$, it follows that in this case the loan riskiness $S_d(\tilde{p}_d)$ is increasing. Combining this with the first case proves the claim that the loan riskiness $S_d(\tilde{p}_d)$ is weakly increasing in $\tilde{p}_d$.

Next, to show that $P_d(\tilde{p}_d)$ is strictly increasing in $\tilde{p}_d$, note that

$$\begin{aligned} P_d(\tilde{p}_d) &= p^{mc}\left(S_d(\tilde{p}_d)\ ;\ v_d(\cdot \mid \tilde{p}_d)\right) \\ &= \min\left(\frac{E_1[v_d(s \mid \tilde{p}_d)]}{1+r}, \omega_1 + \frac{E_0\left[\min\left(v_d(s \mid \tilde{p}_d), v_d(S_d(\tilde{p}_d) \mid \tilde{p}_d)\right)\right]}{1+r}\right), \end{aligned}$$

where the second equality combines cases (i) and (ii) of Eq. (17) and uses $\alpha_0 = 1$. Substituting the value function $v_d(s \mid \tilde{p}_d) = s(1+\tilde{p}_d)$ (cf. Eq. (34)) and using $E_1[s] = 1+\varepsilon$, the previous equation can be written as

$$P_d(\tilde{p}_d) = \min\left((1+\tilde{p}_d)\frac{1+\varepsilon}{1+r}, \omega_1 + (1+\tilde{p}_d)\frac{E_0\left[\min\left(s, S_d(\tilde{p}_d)\right)\right]}{1+r}\right). \tag{A.49}$$

Since $S_d(\tilde{p}_d)$ is weakly increasing in $\tilde{p}_d$,Eq. $(A.49)$ implies that $P_d(\tilde{p}_d)$ is strictly increasing in $\tilde{p}_d$.

Finally, to show that $P_d(\tilde{p}_d)$ satisfies the boundary conditions in (37), note that Eq. (10) implies

$$P_d(\tilde{p}_d) = p^{opt}\left(S_d(\tilde{p}_d)\ ;\ v_d(\cdot \mid \tilde{p}_d)\right).$$

Using the definition of $p^{opt}(\cdot)$ from Eq. (10) and substituting $v_d(s \mid \tilde{p}_d) = s(1+\tilde{p}_d)$ (cf. Eq. (34)), the previous displayed equation can be written as

$$P_d(\tilde{p}_d) = \frac{1+\tilde{p}_d}{1+r}\left(\int_{s^{\min}}^{S_d(\tilde{p}_d)} s dF_0 + \frac{1-F_0(S_d(\tilde{p}_d))}{1-F_1(S_d(\tilde{p}_d))}\int_{S_d(\tilde{p}_d)}^{s^{\max}} s dF_1\right).$$

Next, consider this expression for $\tilde{p}_d = \frac{1}{r}$ and note that

$$
\begin{aligned}
P_d\left(\frac{1}{r}\right) &= \frac{1+\frac{1}{r}}{1+r}\left(\int_{s^{\min}}^{S_d\left(\frac{1}{r}\right)} sdF_0 + \frac{1-F_0\left(S_d\left(\frac{1}{r}\right)\right)}{1-F_1\left(S_d\left(\frac{1}{r}\right)\right)}\int_{S_d\left(\frac{1}{r}\right)}^{s^{\max}} sdF_1\right) \\
&> \frac{1+\frac{1}{r}}{1+r}\left(\int_{s^{\min}}^{s^{\max}} sdF_0 + \frac{1-F_0\left(s^{\max}\right)}{1-F_1\left(s^{\max}\right)}\int_{s^{\max}}^{s^{\max}} sdF_1\right) = \frac{1+\frac{1}{r}}{1+r}E_0[s] = \frac{1}{r}.
\end{aligned}
$$

Here, the second line replaces $S_d\left(\frac{1}{r}\right)$ in the first line with $s^{\max} > S_d\left(\frac{1}{r}\right)$, and the inequality follows since the expression in the first line is a decreasing function of $S_d\left(\frac{1}{r}\right)$ (by the argument in the proof of part (iii) of Lemma 4). Similarly,

$$
\begin{aligned}
P_d\left(\frac{1+\varepsilon}{r-\varepsilon}\right) &= \frac{1+\frac{1+\varepsilon}{r-\varepsilon}}{1+r}\left(\int_{s^{\min}}^{S_d\left(\frac{1+\varepsilon}{r-\varepsilon}\right)} sdF_0 + \frac{1-F_0\left(S_d\left(\frac{1+\varepsilon}{r-\varepsilon}\right)\right)}{1-F_1\left(S_d\left(\frac{1+\varepsilon}{r-\varepsilon}\right)\right)}\int_{S_d\left(\frac{1+\varepsilon}{r-\varepsilon}\right)}^{s^{\max}} sdF_1\right) \\
&\leq \frac{1}{r-\varepsilon}\left(\int_{s^{\min}}^{s^{\min}} sdF_0 + \frac{1-F_0\left(s^{\min}\right)}{1-F_1\left(s^{\min}\right)}\int_{s^{\min}}^{s^{\max}} sdF_1\right) = \frac{1}{r-\varepsilon}E_1[s] = \frac{1+\varepsilon}{r-\varepsilon},
\end{aligned}
$$

where the second line replaces $S_d\left(\frac{1+\varepsilon}{r-\varepsilon}\right)$ in the first line with $s^{\min} \leq S_d\left(\frac{1+\varepsilon}{r-\varepsilon}\right)$. It follows that $P_d\left(\tilde{p}_d\right)$ satisfies the boundary conditions in (37), completing the proof of Theorem 6.

Proof of Theorem 7. Part (i). Since $\left(P_d\left(\tilde{p}_d\right), S_d\left(\tilde{p}_d\right)\right)$ is the static equilibrium for the economy $E\left(\tilde{p}_d\right)$ and since optimists' optimism becomes weakly more right-skewed, Theorem 4 applies and shows that $P_d\left(\tilde{p}_d\right)$ and $S_d\left(\tilde{p}_d\right)$ weakly increase for each $\tilde{p}_d$. Since p_d is the fixed point of the strictly increasing mapping $P_d\left(\tilde{p}_d\right)$ and since $P_d(\cdot)$ shifts up, it follows that p_d weakly increases.

Next note that $S_d(\cdot)$ is a weakly increasing function (by the proof of Theorem 6) and that the equilibrium price to dividend ratio, p_d, weakly increases. Since $S_d\left(\tilde{p}_d\right)$ also weakly increases for each $\tilde{p}_d$, it follows that the equilibrium loan riskiness, $\bar{s}_d^* = S_d\left(p_d\right)$, weakly increases.

Next consider the overvaluation ratio, θ_d. Plugging in the value function, $v_d\left(\cdot \mid p_d\right) = s\left(1+p_d\right)$ (cf. Eq. (34)), and using $E_0[s] = 1$ and $E_1[s] = 1+\varepsilon$, Eq. (38) can be rewritten as

$$
\frac{p_d}{1+p_d} = \left(1-\theta_d\right)\frac{1}{1+r} + \theta_d\frac{1+\varepsilon}{1+r}.
$$

Since p_d weakly increases, the previous displayed equation further implies that θ_d weakly increases. Moreover, rearranging terms, θ_d can be explicitly solved in terms of the price to dividend ratio as

$$
\theta_d = \frac{1}{\varepsilon}\frac{p_d r - 1}{1+p_d}. \tag{A.50}
$$

Finally note that Eq. (39) can be written as

$$
\begin{aligned}
1-\lambda_d = \frac{p^{pdv}(a)}{p(a)} &= \frac{1}{p_d}\left(\left(1-\theta_d\right)\frac{1}{r} + \theta_d\frac{1+\varepsilon}{r}\right) \\
&= \frac{1}{p_{dr}}\left(1+\theta_d\varepsilon\right) = \frac{1+1/r}{1+p_d},
\end{aligned}
$$

where the last equality substitutes θ_d from Eq. $(A.50)$. It follows that $\lambda_d = 1 - \frac{1+1/r}{1+p_d}$, which implies that the share of the speculative component, λ_d, and the price to dividend ratio, p_d, move in the same direction. In particular, since p_d weakly increases, λ_d also weakly increases, completing the proof for the first part.

Part (ii). Since $(P_d(\tilde{p}_d), S_d(\tilde{p}_d))$ is the static equilibrium for the economy $E(\tilde{p}_d)$ and since optimists' optimism becomes weakly more skewed to the left of $\bar{s}_d^*$, Theorem 4 applies and shows that $P_d(\tilde{p}_d)$ and $S_d(\tilde{p}_d)$ weakly increase for each $\tilde{p}_d$. Since p_d is the fixed point of the strictly increasing mapping $P_d(\tilde{p}_d)$ and since $P_d(\cdot)$ shifts up, it follows that p_d weakly increases. The same steps for part (i) apply and show that θ_d and λ_d also weakly increase, completing the proof of Theorem 7.

A.6 Proofs for Section 6

Proof of Theorem 8. I first claim that the solution to the leveraged investment problem (41) is given by the solution to equation $p = p^{opt,cont}(\bar{s})$. Second, I claim that the optimal leveraged return, $R_1^{L,cont}(\boldsymbol{\varphi}_{\bar{s}} \mid p)$, is greater than $1+r$ if and only if $p < p^{\max}$. This implies that optimists make a leveraged investment in the asset if and only if $p < p^{\max}$, which completes the proof of the theorem.

To prove the first claim, note that, without loss of generality, the debt contracts can be restricted to the set such that $\varphi(s) \in [0, v(s)]$ for each $s \in S$. The same steps in the proof of Theorem 1 show that $z_1(\boldsymbol{\varphi}) < 0$ for a debt contract φ in this set if and only if φ solves Problem (41). To characterize the solution to this problem, note that the derivative of $R_1^{L,general}(\boldsymbol{\varphi})$ with respect to $\varphi(s)$ is given by:

$$\frac{\partial R_1^{L,cont}(\boldsymbol{\varphi})}{\partial \boldsymbol{\varphi}(s)} = \frac{-f_0(s)}{p - \frac{1}{1+r}\int_{s^{\min}}^{s^{\max}} \boldsymbol{\varphi}(s)\, dF_0} \left(\frac{f_1(s)}{f_0(s)} - \frac{R_1^{L,cont}(\boldsymbol{\varphi})}{1+r} \right).$$

From this expression and assumption (MLRP), it follows that the derivative $\frac{\partial R_1^{L,cont}(\boldsymbol{\varphi})}{\partial \boldsymbol{\varphi}(s)}$ satisfies a cutoff property. In particular, there exists a threshold state $\bar{s} \in S$ such that

$$\frac{\partial R_1^{L,cont}(\boldsymbol{\varphi})}{\partial \boldsymbol{\varphi}(s)} \begin{cases} > 0 \text{ for each } s < \bar{s} \\ < 0 \text{ for each } s > \bar{s} \end{cases}.$$

Consequently, the optimal level of promise for each state $s \in S \setminus \{\bar{s}\}$ has a corner solution. Hence, the solution φ to Problem (41) has the form in Eq. (42), except potentially a Lebesgue measure zero of states. In particular, the contract specified in Eq. (42) is one optimal solution to Problem (41). Moreover, the threshold state $\bar{s} \in S$ is characterized as the solution to $\frac{f_1(\bar{s})}{f_0(\bar{s})} = \frac{R_1^{L,cont}(\boldsymbol{\varphi})}{1+r}$, which after using the form in Eq. (42) can be written as

$$\frac{\frac{1}{1+r}\int_{\bar{s}}^{s^{\max}} v(s)\, dF_1}{p - \frac{1}{1+r}\int_{s^{\min}}^{\bar{s}} v(s)\, dF_0} = \frac{f_1(\bar{s})}{f_0(\bar{s})}.$$

Rearranging this expression shows that the threshold state is characterized as the unique solution to $p = p^{opt,cont}(\bar{s})$, completing the proof of the first claim.

To prove the second claim, fix a price level p, consider the corresponding optimal threshold $\bar{s}(p)$, and note that the optimal leveraged return is given by:

$$R_1^{L,cont}\left(\boldsymbol{\varphi}_{\bar{s}(p)} \mid p\right) = \frac{E_1[v(s)] - \int_{s^{\min}}^{\bar{s}(p)} v(s)\, dF_1}{p^{opt,cont}(\bar{s}(p)) - \frac{1}{1+r}\int_{s^{\min}}^{\bar{s}(p)} v(s)\, dF_0} = (1+r)\frac{f_1(\bar{s}(p))}{f_0(\bar{s}(p))}. \tag{A.51}$$

Here, the first equality uses the fact that $p = p^{opt,cont}(\bar{s}(p))$ (since $\bar{s}(p)$ is optimal), and the second equality uses Eq. (44). Next, note that $p^{\max} = p^{opt,cont}(s^{cross})$ (cf. Eqs. (43) and (44)), and thus s^{cross} is the optimal threshold corresponding to price level $p^{\max}$. Using Eq. $(A.51)$, this further implies that $R_1^{L,cont}(\boldsymbol{\varphi}_{s^{cross}} \mid p^{\max}) = 1 + r$. Since $R_1^{L,cont}\left(\boldsymbol{\varphi}_{\bar{s}(p)} \mid p\right)$ is decreasing in p, it follows that the optimal leveraged return is greater than $1 + r$ if and only if $p < p^{\max}$. Hence, optimists borrow and invest in the asset if $p < p^{\max}$, but they invest in the bond (and do not borrow) if $p > p^{\max}$. This completes the proof of the theorem.

A.7 Proofs for Section 7

This appendix provides a proof of Theorem 9 in Section 7, which is useful to understand short sellers' trade-off and to provide the intuition for the filtering property of $p^{short}(\bar{s}_{sh})$. Similar steps as in the derivation of Theorem 1 show that the default threshold $\bar{s}_{sh}$ for short contracts solves the following problem, which is the analogue of leveraged investors' problem (14):

$$\max_{\tilde{s} \in \mathcal{S}} R_0^{short}(\tilde{s}) = \frac{v(\tilde{s}) - E_0[\min(v(s), v(\tilde{s}))]}{\frac{v(\tilde{s})}{1+r} - E_1[\min(v(s), v(\tilde{s}))] / \frac{E_1[v(s)]}{p}}. \tag{A.52}$$

Eq. (48) corresponds to the first order condition for this problem, and under assumption (MLRP), the unique solution to this equation corresponds to the solution to problem $(A.52)$, completing the sketch proof of Theorem 9.

To interpret problem $(A.52)$, note that $R_0^{short}(\tilde{s})$ is the return of short sellers from selling one unit of the short contract $\psi = \frac{v(\tilde{s})}{1+r}$. More specifically, short sellers receive

$$q^{short}\left(\frac{v(\tilde{s})}{1+r}\right) = E_1[\min(v(s), v(\tilde{s}))] / \frac{E_1[v(s)]}{p} \tag{A.53}$$

from the sale of this contract, and they use this amount towards meeting the collateral requirement. However, they need to post a total of $\psi = \frac{v(\tilde{s})}{1+r}$ units of the numeraire good as collateral. Thus, they pay the difference (the denominator of $(A.52)$) out of their wealth. In the next period, short sellers receive $v(\tilde{s})$ from the collateral that they have posted, and they expect to pay $E_0[\min(v(s), v(\tilde{s}))]$ on the promises they have made. This is because, short sellers return the asset if the realized state is below $\tilde{s}$, but they default on the short contract if the realized state is above $\tilde{s}$. In the latter scenario, short sellers lose only the collateral that they have posted, which is worth $v(\tilde{s})$. Hence, short sellers' payment has the same exact form as a debt contract with default threshold $\tilde{s}$, and thus their expected payment is given by $E_0[\min(v(s), v(\tilde{s}))]$.

Problem $(A.52)$ captures the essential trade-off that short sellers are facing. Note that short sellers with moderate beliefs expect to the pay the interest rate $r_0^{short}(\tilde{s})$ on a short contract, defined by:

$$1 + r_0^{short}(\tilde{s}) \equiv \frac{E_0[\min(v(s), v(\tilde{s}))]}{E_1[\min(v(s), v(\tilde{s}))] / \frac{E_1[v(s)]}{p}} = \frac{E_0[\min(v(s), v(\tilde{s}))]}{E_1[\min(v(s), v(\tilde{s}))]} \frac{E_1[v(s)]}{p}. \tag{A.54}$$

This expression further implies that $r_0^{short}(\tilde{s}) < r$ for the equilibrium short contract $\tilde{s} = \bar{s}_{sh}$. Intuitively, short sellers sell the short contract and buy the bond with the proceeds, making an expected net return $r - r_0^{short}(\bar{s}_{sh})$. Moreover, under assumption (MLRP) this return is increasing in the short threshold $\tilde{s}$. This is because, the higher $\tilde{s}$, the less often the short contract defaults, and the greater portion of the asset the short sellers effectively sell. On the other hand, problem $(A.52)$ shows that a higher threshold $\tilde{s}$ requires short sellers to post a greater level of collateral $\frac{v(\tilde{s})}{1+r}$. This restricts short sellers' ability to leverage the net return $r-r_0^{short}(\bar{s}_{sh})$. It follows that, when choosing $\bar{s}_{sh}$, short sellers trade off greater leverage against a lower net return. This trade-off is resolved by problem $(A.52)$, and leads to the optimal short contract characterized by (48).

I next provide the intuition for why the function $p^{short}(\bar{s}_{sh})$ is decreasing in the default threshold $\bar{s}_{sh}$, and why it has the asymmetric filtering property. Consider first the former statement, which is equal to saying that the default threshold $\bar{s}_{sh}$ for the optimal short contract is decreasing in the asset price. Note that, by Eq. $(A.54)$, a higher price p increases the wedge $r-r_0^{short}(\bar{s}_{sh})$ that short sellers expect to make. This incentivizes short sellers to leverage more, by choosing a lower default threshold $\bar{s}_{sh}$. Intuitively, as prices are higher, short sellers see a greater bargain in short selling and they leverage their short sales more.

Consider next the intuition for the filtering property of $p^{short}(\bar{s}_{sh})$. To understand this property, suppose the equilibrium default level is given by $\bar{s}_{sh}$, and consider how high the prices should be (relative to the moderate valuation) to entice short sellers to choose this default threshold. If the belief heterogeneity is concentrated on states below $\bar{s}_{sh}$, then Eq. $(A.54)$ reveals that the return wedge $r-r_0^{short}(\bar{s}_{sh})$ expected by short sellers is higher. Thus, prices need not be too high to entice short sellers to choose the default threshold level $\bar{s}_{sh}$. Consequently, with these types of belief heterogeneity concerning bad states, prices are closer to the moderate valuation, which implies that optimism about the likelihood of bad states is filtered. In contrast, suppose the belief heterogeneity is concentrated more on the relative likelihood of states above $\bar{s}_{sh}$. In this case, Eq. $(A.54)$ implies that the return wedge $r-r_0^{short}(\bar{s}_{sh})$ expected by short sellers is lower. Then, short sellers are enticed to choose the threshold level $\bar{s}_{sh}$ only if prices are sufficiently higher than the moderate valuation, which further implies that optimism about the relative likelihood of good states is filtered less.

Next consider the total expenditure on short sales, W^{short}, referred to in the main text. Note that $\frac{v(\bar{s}_{sh})}{1+r}-q^{short}\left(\frac{v(\tilde{s})}{1+r}\right)$ is the amount of wealth short sellers need to allocate to sell one unit of the short contract $\psi=\frac{v(\bar{s}_{sh})}{1+r}$. Short sellers are type T_3 moderate traders, who have a total wealth of $\gamma_{sh}(w_0+p\alpha_0)$. Thus, the total number of short contracts $\frac{v(\bar{s}_{sh})}{1+r}$ sold by short sellers is given by $\frac{\gamma_{sh}(w_0+p\alpha_0)}{\frac{v(\bar{s}_{sh})}{1+r}-q^{short}\left(\frac{v(\tilde{s})}{1+r}\right)}$. The total expenditure on short sales is then given by:

$$W^{short}=\frac{\gamma_{sh}(w_0+p\alpha_0)}{\frac{v(\bar{s}_{sh})}{1+r}-q^{short}\left(\frac{v(\tilde{s})}{1+r}\right)}q^{short}\left(\frac{v(\tilde{s})}{1+r}\right).$$

Substituting for $q^{short}\left(\frac{v(\tilde{s})}{1+r}\right)$ from Eq. $(A.53)$, and rearranging terms Combining this with the previous displayed equation and rearranging terms implies yields the expression (52) for W^{short}.

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