Epistemic Foundations of Iterated Admissibility* (Job Market Paper)

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Abstract

How can we justify the play of iteratively admissible strategies in a game as a consequence of the players' rationality? Brandenburger, Friedenberg, and Keisler (2008) models beliefs as lexicographic probability systems toward that end. If rational players never rule out any scenarios, then they will avoid inadmissible (i.e., weakly dominated) strategies. Under this definition of rationality, Brandenburger et al. showed that, when the set of beliefs is complete (i.e., each lexicographic probability system is a possible belief), iteratively admissible strategies will be played if each player is rational, each player thinks the other players are rational, and so on. They leave as an open question whether the condition on interactive beliefs about rationality, called *rationality and common assumption of rationality*, and completeness of beliefs can be satisfied simultaneously. I answer this question in the affirmative. Thus, an epistemic foundation for iterated admissibility is provided.

Keywords: Epistemic game theory, rationality, admissibility, iterated weak dominance, completeness.
JEL Classification: C72, D80.

*After completing this paper, I learned that H. Jerome Keisler had shown the same main result in a preprint dated January 2009. His paper, "Common Assumption of Rationality", can be downloaded from `http://www.math.wisc.edu/%7Ekeisler/`. We have agreed to combine our results in a joint paper to be available shortly.

1 Introduction

Analysis of games typically begins by assuming that players are rational. Moreover, it is assumed, at least implicitly, that the rationality of the players is common knowledge in the sense of Aumann (1976). That is, each player in the game is rational, each player thinks the other players are rational, and so on. It is natural to ask, what is implied by common knowledge of rationality? The first answer to this question was given by Bernheim (1984) and Pearce (1984), who proposed that players should choose *rationalizable* strategies.[1]

With regards to the implications of beliefs on the actions of Bayesian rational players, Monderer and Samet (1989) show that common certainty, i.e., common belief with probability one, has analogous properties to those of common knowledge. Bernheim (1984) and Pearce (1984) show that rationalizable strategies are played if common certainty of rationality is satisfied. It is well known that the set of rationalizable strategies coincides with the set of strategies that survive iterated elimination of strongly dominated strategies. The result that Bayesian rational players may play any strategy that is not *strongly* dominated is crucial to this equivalence. Is there an analogous correspondence between admissible behavior, i.e., avoidance of *weakly* dominated strategies, and some notion of rationality? Presumably, if each player in a game is rational, each player thinks the other players are rational, and so on, then we would hope to obtain the prediction that all players choose iteratively admissible strategies, i.e. strategies that survive iterated elimination of weakly dominated strategies.[2] This paper provides an epistemic justification for iterated admissibility (IA in the sequel) along those lines.

Admissibility is a prima facie reasonable criterion of rational decision making whose roots

[1] See Aumann (1987); Brandenburger and Dekel (1987); Tan and Werlang (1988) for formal results on the relationship between common knowledge of rationality and rationalizability.

[2] It is well known that the order of eliminating weakly dominated strategies matters whereas the order does not matter for elimination of strongly dominated strategies. Iteratively admissible strategies correspond to an order of elimination in which all weakly dominated strategies of all players are deleted simultaneously in each round; this is sometimes called iterated maximal deletion of weakly dominated strategies.

in statistical decision theory go back to Wald (1939). A Bayesian rational player who never rules out any scenario will choose admissible strategies. Such caution has been advocated as a starting point of sensible intellectual inquiry since classical times. Aside from the attractive epistemic foundations of admissible behavior, IA has numerous convenient properties as a solution concept that have long been recognized. Luce and Raiffa (1957, pp. 98–101) informally argue that IA is a consequence of common knowledge of rationality and apply it to obtain the backward induction outcome in the finitely repeated prisoner's dilemma.[3] IA also reflects forward induction reasoning, which refines the equilibria of various signaling games, including the beer-quiche game. Furthermore, IA yields solutions of games that are invariant up to irrelevant transformations of the game tree à la Thompson (1952a) and Dalkey (1953).[4] Therefore it is consistent with von Neumann and Morgenstern's (1944) influential argument that the normal form of a game should capture all strategically relevant information about its extensive form.

IA also gives tighter predictions in games with weak equilibria in weakly dominant strategies. For example, consider costless majority voting games in which, if everyone votes truthfully, only the median voter's choice matters. If each person believes that everyone else votes truthfully, then those beliefs form an equilibrium of the game.[5] However, voters with minority opinions are indifferent between truthful and untruthful voting under these beliefs. Thus, despite the beliefs being in equilibrium, equilibrium actions (that is, truthful voting by all) may not be observed. If rationality incorporates admissibility, each voter will vote truthfully since there is at least one scenario in which she will be the pivotal voter, and in all other scenarios her vote does not matter.

[3] In Luce and Raiffa (1957), any strategy that does not satisfy IA is said to be "dominated in the wide sense". Moulin (1986) explores when IA yields unique predictions of payoffs for all players.

[4] For a more detailed discussion, see Kohlberg and Mertens (1986).

[5] The interpretation of Nash equilibrium as a profile of mutually consistent beliefs is due to Aumann and Brandenburger (1995).

	L	R
T	1, 1	1, 0
B	1, 0	0, 1

Figure 1: Ann is the row player and Bob is the column player.

Unfortunately, there is a fundamental tension between the epistemic foundations of admissible behavior and common certainty of rationality. In order to obtain admissible behavior, we require that players never entirely discount any possibility. However, in being certain of others' rationality, each player completely rules out the scenario that some of her opponents are irrational. Consider the game in Figure 1, which is Example 8 in Samuelson (1992). In order to give an epistemic argument for the strong conclusion that Ann will necessarily play T rather than B (that is, she will play her unique IA strategy), we must suppose that she will put positive weight on the possibility that Bob will play R, even though R will be dominated once B is eliminated. In other words, once Bob is certain that B will not be played, he will not play R. However, we would like Ann, being certain of these facts, to put positive weight on R. It follows that IA cannot obtained as a consequence of common certainty of rationality in this game.

Brandenburger, Friedenberg, and Keisler (2008), hereafter BFK, cleared this hurdle by modeling beliefs as lexicographic probability systems (LPS in the sequel), which were introduced in Blume, Brandenburger, and Dekel (1991a). Loosely speaking, an LPS is a finite sequence of hypotheses based on mutually exclusive premises, which are ordered by their importance to the player. Each hypothesis is represented by a standard Borel probability measure. The most important hypothesis may be called primary, the second most important hypothesis may be called secondary, and so on. The role of "importance" as it relates to decision making is as follows: A player evaluates her actions according to her many hypotheses, starting with the primary hypothesis and subsequently moving on to the next most important hypothesis in the case where no uniquely optimal action can be found under the previous hypotheses. In this context, we may interpret more important hypotheses as *infinitely more likely* hypotheses.

LPSs can express the belief that each player is rational without ruling out the scenario that there are irrational players. Imagine an LPS in which the hypotheses that presume rational players are all more important than hypotheses that do not presume rational players. If a player holds such a belief, it is said that she *assumes* each player is rational (In general, a player assumes an event if it is given probability one in all sufficiently important hypotheses and probability zero in all less important hypotheses). Then the analogue of common knowledge of rationality in the BFK framework, called *rationality and common assumption of rationality* (RCAR in the sequel), can be defined as the conjunction of the following statements.

(1) each player is rational;

(2) each player assumes (1);

(3) each player assumes (1), (2);

and so on...

If (1),..., (m), and (m+1) are simultaneously satisfied, it is said that there is *rationality and mth order assumption of rationality* (RmAR in the sequel). BFK find that, if the set of beliefs is *complete* in the sense that each player considers all beliefs of her opponent to be possible, then RmAR implies that each player will choose an m-admissible strategy (i.e., a strategy that survives m rounds of elimination of inadmissible strategies) and RCAR implies that each player will choose an iteratively admissible strategy. When the set of beliefs is complete, BFK show that RmAR can be satisfied, but leave open the question of whether RCAR can be satisfied. My main result shows that completeness and RCAR can indeed be satisfied simultaneously, providing an epistemic foundation for iterated admissibility.

Section 2 defines the BFK framework of epistemic analysis. Section 3 states the main result. Section 4 discusses the tension between completeness and a condition BFK call continuity as they relate to the existence of RCAR. All proofs are contained in the appendices.

2 Setup

The setup is borrowed verbatim from BFK with a few inconsequential stylistic alterations. The discussion in this paper is limited to finite games of complete information played by Ann and Bob. I follow BFK in adopting this simplifying convention. The analysis easily extends to three or more players. This section is organized as follows. First, we define lexicographic probability systems, which will express beliefs in our model, and related apparati. Second, we define a model of interactive beliefs via the use of types in the style of Harsanyi. Lastly, we formally define *rationality and common assumption of rationality* (RCAR).

2.1 Lexicographic Probability Systems

Fix a Polish space Ω and a compatible metric, where Polish means complete and separable. Let $\mathscr{M}(\Omega)$ denote the set of all Borel probability measures on Ω. We define the set $\mathscr{N}(\Omega)$ of all finite sequences of Borel probability measures on Ω in the following way.

$$\mathscr{N}_n(\Omega) \equiv \overbrace{\mathscr{M}(\Omega) \times \cdot \times \mathscr{M}(\Omega)}^{n \text{ times}}$$

$$\mathscr{N}(\Omega) \equiv \bigcup_{n=1}^{\infty} \mathscr{N}_n(\Omega)$$

We define a Polish topology on $\mathscr{N}(\Omega)$ by following the usual conventions. First, we give $\mathscr{M}(\Omega)$ its weak* topology, which makes it a Polish space. Second, we give $\mathscr{N}_n(\Omega) = \prod_{k=1}^{n} \mathscr{M}(\Omega)$ the product topology. $\mathscr{N}_n(\Omega)$ can be metrized by defining the distance between $(\mu_0, \ldots, \mu_{n-1}), (\nu_0, \ldots, \nu_{n-1}) \in \mathscr{N}_n(\Omega)$ as the maximum of the Prohorov distances between μ_k and ν_k for $k < n$.[6] Finally, we topologize $\mathscr{N}(\Omega) = \bigcup_{n=1}^{\infty} \mathscr{N}_n(\Omega)$ as a disjoint union. This

[6] The Prohorov metric generates the weak* topology on $\mathscr{M}(\Omega)$, the space of component measures.

can be done by defining the distance between any two sequences of unequal lengths to be one. Then $\mathscr{N}(\Omega)$ is a countable union of Polish spaces at uniform distance one from each other. $\mathscr{N}(\Omega)$ is also a Polish space, and we call its members sequential probability systems (SPS in the sequel). SPSs were originally called LPSs in Blume, Brandenburger, and Dekel (1991a). However, I follow BFK in reserving the term LPS for an SPS that satisfies what is called the mutual singularity requirement.

Definition 1 (Lexicographic probability systems)**.** Fix $\mu = (\mu_0, \dots, \mu_{n-1}) \in \mathscr{N}(\Omega)$, for some integer n. Say μ is a *lexicographic probability system (LPS)* if μ is *mutually singular*—that is, for each $j = 0, \dots, n-1$, there are Borel sets U_j in Ω with $\mu_j(U_j) = 1$ and $\mu_j(U_k) = 0$ for $j \neq k$.

Write $\mathscr{L}(\Omega)$ for the set of LPSs and write $\overline{\mathscr{L}}(\Omega)$ for the closure of $\mathscr{L}(\Omega)$ in $\mathscr{N}(\Omega)$. The closure operation is with respect to the topology on $\mathscr{N}(\Omega)$ that was previously defined. We will later make use of the convenient fact that $\overline{\mathscr{L}}(\Omega)$ is Polish since closed subsets of Polish spaces are themselves Polish.

Definition 2. The support of $\mu = (\mu_0, \dots, \mu_{n-1}) \in \mathscr{N}(\Omega)$ is defined as follows:

$$\operatorname{Supp} \mu = \bigcup_{k<n} \operatorname{Supp} \mu_k$$

The sequence μ is *full-support* if $\Omega = \operatorname{Supp} \mu$.

We write $\mathscr{N}^+(\Omega)$ and $\mathscr{L}^+(\Omega)$ for the set of full-support sequences in $\mathscr{N}(\Omega)$ and $\mathscr{L}(\Omega)$, respectively.

Definition 3. Fix an event E and an LPS $\mu = (\mu_0, \dots, \mu_{n-1}) \in \mathscr{N}(\Omega)$. E is *assumed* under μ if and only if there is a j such that:

1. $\mu_j(E) = 1$ for all $j \leq k$;
2. $\mu_j(E) = 0$ for all $j > k$; and

3. if U is open with $U \cap E \neq \varnothing$, then $\mu_j(U \cap E) > 0$ for some j (i.e., μ has full support relative to E).

The above characterization of assumption is from Proposition 5.1 in BFK. Since it is quite convenient, it is adopted as the primary definition of assumption. Intuitively speaking, E is assumed if it is infinitely more likely than its complement.

2.2 RmAR and RCAR

For the remainder of the paper, fix a game $\langle S^a, S^b, \pi^a, \pi^b \rangle$. S^a is Ann's strategy set and π^a is her payoff function. The corresponding objects for Bob are S^b and π^b. The payoffs π^a and π^b are extended in the usual way to take distributions on S^a and S^b as arguments.

Definition 4. An (S^a, S^b)-*based type structure* is a 4-tuple

$$\langle T^a, T^b, \lambda^a, \lambda^b \rangle ,$$

where T^a and T^b are nonempty Polish spaces and $\lambda^a : T^a \to \overline{\mathscr{L}}(S^b \times T^b)$ and $\lambda^b : T^b \to \overline{\mathscr{L}}(S^a \times T^a)$ are Borel measurable.[7] Members of T^a, T^b are called types. Members of $S^a \times T^a \times S^b \times T^b$ are called states (of the world). A type structure is called *lexicographic* if range $\lambda^a \subseteq \mathscr{L}(S^b \times T^b)$ and range $\lambda^b \subseteq \mathscr{L}(S^a \times T^a)$.

We define the lexicographic order $>^L$ between finite sequences of equal lengths as follows. Given n and $(x_0, \ldots, x_{n-1}), (y_0, \ldots, y_{n-1}) \in \mathbb{R}^n$, write $x >^L y$ if there exists $k < n$ such that, for all $j < k$, $x_j = y_j$ and $x_k > y_k$. Write $x \geq^L y$ if $x = y$ or $x >^L y$. Preference maximization under lexicographic beliefs (lexicographic utility maximization) is defined using $>^L$.

[7] BFK define a type structure to be a 6-tuple $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. I elect to omit the strategy spaces S^a and S^b from the definition since the analysis herein limits itself to a fixed game.

Definition 5. A strategy s^a is *optimal* under $(\mu_0, \ldots, \mu_{n-1}) \in \mathcal{L}(S^b \times T^b)$ if

$$\forall r^a \in S^a \quad \left[\pi^a(s^a, \operatorname{marg}_{S^b} \mu_j(s^b))\right]_{j=0}^{n-1} \geq^L \left[\pi^a(r^a, \operatorname{marg}_{S^b} \mu_j(s^b))\right]_{j=0}^{n-1}$$

More compactly, we may write $s^a \in \underset{r^a \in S^a}{\arg\max} \left[\pi^a(r^a, \operatorname{marg}_{S^b} \mu_j(s^b))\right]_{j=0}^{n-1}$.

Definition 6. A strategy-type pair $(s^a, t^a) \in S^a \times T^a$ is *rational* if $\lambda^a(t^a) \in \mathcal{L}^+(S^b \times T^b)$ and s^a is optimal under $\lambda^a(t^a)$.

Definition 7. Fix an event $E \subseteq S^b \times T^b$ and write

$$A^a(E) \equiv \{t \in T^a : \lambda^a(t^a) \text{ assumes } E\}.$$

Definition 8. Let R_1^a be the set of rational strategy-type pairs (s^a, t^a). For finite m, define R_m^a by

$$R_{m+1}^a \equiv R_m^a \cap [S^a \times A^a(R_m^b)].$$

Definition 9. If $(s^a, t^a, s^b, t^b) \in R_{m+1}^a \times R_{m+1}^b$, say there is *rationality and mth-order assumption of rationality (RmAR)* at this state. If $(s^a, t^a, s^b, t^b) \in \bigcap_{m=1}^{\infty} R_m^a \times R_m^b$, say there is *rationality and common assumption of rationality (RCAR)* at this state.

Definition 10. Two type structures $\langle T^a, T^b, \lambda^a, \lambda^b \rangle$ and $\langle T^a, T^b, \kappa^a, \kappa^b \rangle$ are *equivalent* if:

- they have the same type spaces;
- for each $t^a \in T^a$, if either $\kappa^a(t^a)$ or $\lambda^a(t^a)$ belongs to $\mathcal{L}^+(S^b \times T^b)$, then $\kappa^a(t^a) = \lambda^a(t^a)$ (and likewise with a and b reversed)

Loosely speaking, a type structure is called *complete* if it contains all beliefs that can be expressed as LPSs.

Definition 11. A type structure $\langle T^a, T^b, \lambda^a, \lambda^b \rangle$ is *complete* if $\mathscr{L}^+(S^b \times T^b) \subsetneq \text{range}\, \lambda^a$ and $\mathscr{L}^+(S^a \times T^a) \subsetneq \text{range}\, \lambda^b$.

Definition 12. The sets of m-admissible strategies S^a_m and S^b_m are defined as follows. Let $S^a_0 = S^a$ and $S^b_0 = S^b$. We say that a strategy s^a is admissible against $X^b \subseteq S^b$ if s^a is not weakly dominated in S^a against X^b. Then S^a_m is the set of all s^a that are admissible against $S^b_0, \ldots, S^b_{m-1}$. That is, S^a_m is the set of Ann's m-admissible strategies. Let $S^a_\infty \equiv \bigcap_{m=1}^\infty S^a_m$. Then S^a_∞ is Ann's IA set.

3 Main Result

The main result can formally stated as follows.

Theorem 1 (Main Theorem)**.** *There exists a complete lexicographic type structure such that the set of states satisfying RCAR is nonempty.*

Theorem 1 is an immediate corollary of Lemma 1, a more general result that can be used to shed light on the relationship between continuity and completeness.

Lemma 1 (Main Lemma)**.** *Given uncountable Polish spaces T^a and T^b, there exist bijective Borel functions $\lambda^a : T^a \to \mathscr{L}(S^b \times T^b)$ and $\lambda^b : T^b \to \mathscr{L}(S^a \times T^a)$ such that $\langle T^a, T^b, \lambda^a, \lambda^b \rangle$ is a complete lexicographic type structure with a nonempty RCAR set.*

4 Continuity and Completeness

In BFK, a negative result shows that if RCAR is nonempty, then the type structure cannot be simultaneously complete and continuous.

Theorem (Theorem 10.1 in BFK)**.** *RCAR is empty in each complete and continuous type structure.*

Consider the following theorem from descriptive set theory:[8]

Theorem. *Let* $(X, \mathscr{T})$ *be a Polish space,* Y *a second countable space, and* $f : X \to Y$ *a Borel function. Then there is a Polish topology* $\mathscr{T}_f \supseteq \mathscr{T}$ *with* $\mathbf{B}(\mathscr{T}_f) = \mathbf{B}(\mathscr{T})$ *such that* $f : (X, \mathscr{T}_f) \to Y$ *is continuous.*

It will be convenient to let $X = T^a$ and $Y = \overline{\mathscr{L}}(S^b \times T^b)$. Since separability is equivalent to second countability in metrizable spaces, all Polish spaces are second countable as well. Now consider the following paradoxical sequence of facts. By adding some open sets to the topology of T^a, any type-belief mapping λ^a that exists by Lemma 1 can be made continuous while maintaining the "Polish-ness" of T^a. Call these new Polish spaces with the augmented topologies $\widehat{T}^a$ and $\widehat{T}^b$. Once the type-belief mappings are made continuous in this way, Theorem 10.1 in BFK applies and no RCAR state exists. Then Lemma 1 implies that there exists a complete type structure with type spaces $\widehat{T}^a$ and $\widehat{T}^b$ such that RCAR is nonempty.

Rather than continuity, it is the topologies of the spaces T^a and T^b that determine whether RCAR is empty or not. The above theorem makes a mapping continuous by adding some open sets to the existing topology. The primary implication of such additions is that the set of full-support measures shrinks since full-support measures need to assign positive measure to the added open sets. A continuous type structure is often described as a type structure in which neighboring full-support LPSs are associated with neighboring full-support types. However, given that any type structure can be made continuous without changing the type-belief mappings or the topology of the belief space, the desirable economic meaning of "nearness" of types and the appropriate mathematical definition that captures it would seem to require further scrutiny than previously believed.

[8]Kechris (1995, see 13.11, pp. 84).

A Appendix: The Shape of RmAR Sets

It turns out that the shapes of RmAR sets are invariant across all complete type structures associated with the same game. The precise sense in which Ann's RmAR sets are invariant can be expressed using an index set $\mathscr{X}_m^a \subseteq \mathscr{P}(S^a)$. More precisely R_m^a can be written in the form $\bigcup\{X^a \times \tau_m(X^a) : X^a \in \mathscr{X}_m^a\}$. The invariance property states that the appropriate index set $\mathscr{X}_m^a$ is the same in all complete type structures and that, for every $X^a \in \mathscr{X}_m^a$, $\tau_m(X^a) \neq \varnothing$.

Recall that whether or not Ann's state (s^a, t^a) satisfies RmAR is determined by two requirements. First, (s^a, t^a) must be rational in the given type structure $\langle T^a, T^b, \lambda^a, \lambda^b \rangle$. Second, t^a must be a type that assumes the sets $R_1^b, \ldots, R_{m-1}^b$. The interpretation of $t^a \in \tau_m(X^a)$ is that X^a is exactly the set of optimal strategies for t^a, and t^a assumes $R_1^b, \ldots, R_{m-1}^b$.

Consider the sequence $\{\mathscr{X}_m^a : m \in \mathbb{N}_0\}$ (and Bob's equivalent, which is obtained by switching a and b), defined below. It is obvious that $\{\mathscr{X}_m^a : m \in \mathbb{N}_0\}$ and $\{\mathscr{X}_m^b : m \in \mathbb{N}_0\}$ are entirely determined by $\{S_m^a : m \in \mathbb{N}_0\}$ and $\{S_m^b : m \in \mathbb{N}_0\}$, which in turn are entirely determined by the game $\langle S^a, S^b, \pi^a, \pi^b \rangle$.

Definition 13. Let $\mathscr{X}_0^a \equiv \{S_0^a\}$. For each $m \in \mathbb{N}$, define

$$\mathscr{X}_m^a \equiv \left\{ \underset{s^b \in S^b}{\arg\max} \left[\pi^a(s^a, \nu_k)\right]_{k=0}^{m-1} : \nu \in \mathscr{M}^+(S_{m-1}^b) \times \cdots \times \mathscr{M}^+(S_0^b) \right\}.$$

If $X^a \in \mathscr{X}_m^a$, then X^a is exactly the set of Ann's optimal strategies under some $\nu \in \mathscr{M}^+(S_{m-1}^b) \times \cdots \times \mathscr{M}^+(S_0^b)$. The interpretation of ν is as the marginal on S^b of some belief that μ that assumes $R_1^b, \ldots, R_{m-1}^b$. This is view is justified by the fact that $S_{m-1}^b = \operatorname{proj}_{S^b} R_{m-1}^b$ and $S_{m-2}^b = \operatorname{proj}_{S^b}(R_{m-2}^b \setminus R_{m-1}^b), \ldots, S_0^b = \operatorname{proj}_{S^b}(R_0^b \setminus R_1^b)$. Consider any $\mu = (\mu_0, \ldots, \mu_{m-1})$ such that $\mu_0(R_{m-1}^b) = 1$ and $\mu_1(R_{m-2}^b \setminus R_{m-1}^b) = 1, \ldots, \mu_{m-1}(R_0^b \setminus R_1^b) = 1$. Then μ is an LPS, it assumes $R_1^b, \ldots, R_{m-1}^b$, and $\operatorname{marg}_{S^b} \mu \in \mathscr{M}^+(S_{m-1}^b) \times \cdots \times \mathscr{M}^+(S_0^b)$. Let $\mathscr{X}_\infty^a \equiv \bigcap\{\mathscr{X}_m^a : m \in \mathbb{N}\}$. Analogous objects are defined for Bob.

Remark. It is obvious from the definition that, for all $m \in \mathbb{N}$, $\mathscr{X}^a_m \neq \varnothing$ and $\mathscr{X}^a_{m+1} \subseteq \mathscr{X}^a_m$. Since each $\mathscr{X}^a_m \subseteq 2^{S^a}$ and S^a is a finite set, the above implies that $\mathscr{X}^a_\infty \neq \varnothing$.

Lemma 2 (Invariance of RmAR)**.** *In any complete type structure $\langle T^a, T^b, \lambda^a, \lambda^b \rangle$, for all $m \in \mathbb{N}$, there exists a family of pairwise disjoint sets $\{\tau_m(X^a) \subseteq T^a : X^a \in \mathscr{X}^a_m\}$ such that*

$$R^a_m = \bigcup \{X^a \times \tau_m(X^a) : X^a \in \mathscr{X}^a_m\}.$$

Analogous results hold for Bob.

To show my main result, I first show the existence of sets that have the shape of RmAR sets in the sense above and have a nonempty intersection. Lemma 2 provides useful guidelines in that regard. Then I show that there exists a complete type structure whose RmAR sets coincide with those candidates.

Pf. of Lemma 2. We will have the desired result if the following two statements jointly hold for all $m \in \mathbb{N}$.

1. For all $X^a \in \mathscr{X}^a_m$, there exists $t^a \in \operatorname{proj}_{T^a} R^a_m$ such that X^a is exactly the optimal set under $\lambda^a(t^a)$.
2. For all $t^a \in \operatorname{proj}_{T^a} R^a_m$, there exists $X^a \in \mathscr{X}^a_m$ such that X^a is exactly the optimal set under $\lambda^a(t^a)$.

Part 1. Fix $m \in \mathbb{N}$. By definition, for each $X^a \in \mathscr{X}^a_m$, there exists $\nu \in \mathscr{M}^+(S^b_{m-1}) \times \cdots \times \mathscr{M}^+(S^b_0)$ such that X^a is exactly the optimal set under ν. By the proof of Theorem 9.1 BFK, pp. 330–331, there exists $\mu \in \mathscr{L}^+(S^b \times T^b)$ such that μ assumes $R^b_1, \ldots, R^b_{m-1}$ and $\operatorname{marg}_{S^b} \mu = \nu$. It follows that X^a is exactly the optimal set under μ. By completeness, there exists t^a such that $\lambda^a(t^a) = \mu$ since $\mu \in \mathscr{L}^+(S^b \times T^b)$.

Part 2. For all $m \in \mathbb{N}_0$, $S^b_m = \operatorname{proj}_{S^b} R^b_m$ by Theorem 9.1 in BFK. Fix $m \in \mathbb{N}$. Let $t^a \in \operatorname{proj}_{T^a} R^a_m$. Let $\mu = \lambda^a(t^a)$ and let $\nu = \operatorname{marg}_{S^b} \mu$. Denote the set of optimal strategies under μ as X^a. Then μ assumes $R_1^b, \ldots, R_{m-1}^b$. Let $\alpha(k)$ denote the level in μ at which R_k^b is assumed. Then, Theorem 9.1 in BFK implies that $\bigcup_{j \leq \alpha(k)} \operatorname{Supp} \nu_j = S_k^b$. By Proposition 1 in Blume, Brandenburger, and Dekel (1991b), for each $k = 0, \ldots, m-1$, there exists $\widehat{\nu}_{m-1-k} \in \mathscr{M}^+(S_k^b)$ such that the set of optimal strategies under $(\widehat{\nu}_{m-1-k})$ is equal to the set of optimal strategies under $(\nu_0, \ldots, \nu_{\alpha(k)})$. Then the set of optimal strategies under $\widehat{\nu} = (\widehat{\nu}_0, \ldots, \widehat{\nu}_{m-1}) \in \mathscr{M}^+(S_{m-1}^b) \times \cdots \times \mathscr{M}^+(S_0^b)$ is equal to X^a, the set of optimal strategies under ν (equivalently, optimal under $\lambda^a(t^a)$). Therefore $X^a \in \mathscr{X}_m^a$. Analogous arguments hold for Bob. $\square$

B Appendix: Proof of Main Result

Recall that RCAR is the intersection $\bigcap_{m=1}^{\infty}(R_m^a \times R_m^b)$ of an infinite sequence of nested sets. Intuitively, it would be easier to show that RCAR is nonempty, if each RmAR set was large in some sense. If RmAR is large in a probabilistic sense (i.e, given probability one by some full-support measure that is fixed across all values of m), then the difference between RmAR and R$(m+1)$AR is small (i.e., given probability zero by the same measure). Furthermore, since probability measures are countably additive, it follows that the set of states not satisfying RCAR is small in the same sense. Nonemptiness of RCAR is an immediate consequence of this fact.

Unfortunately, we may not directly apply the method of constructing RmAR sets. Since the strategy space is finite, if some strategy s^a is inadmissible, then the RmAR sets will have an empty intersection with the open set $\{s^a\} \times T^a$. Therefore, the RmAR sets cannot be given probability one by a full-support measure if some strategies are inadmissible. Instead, we replace "given probability one by a full-support measure" with "given probability one by a measure having full-support on some open set U" as the appropriate notion of large size.

Our general strategy will be to first construct some sequence of sets that we would like to be the RmAR sets, then show the existence of a complete type structure in which the candidates are indeed the RmAR sets. For each m, the candidate for RmAR will need to be shaped like RmAR as described by Lemma 2 and given probability one by a measure having full-support on the same open set U for all m.

As we have done throughout the paper, we fix the underlying game $\langle S^a, S^b, \pi^a, \pi^b \rangle$. We also fix T^a and T^b and assume that they are uncountable Polish spaces.

Lemma 3. *There exists $\{\tau_1(X^a) : X^a \in \mathscr{X}_1^a\}$, a family of pairwise disjoint uncountable open sets such that $T^a \setminus \bigcup \{\tau_1(X^a) : X^a \in \mathscr{X}_1^a\}$ is uncountable and closed. Analogous objects exist for Bob.*

Pf. of Lemma 3. Since $\mathscr{X}_1^a$ is a finite set, the desired result follows immediately from Lemma 10.

□

For the remainder, fix $\{\tau_1(X^a) : X^a \in \mathscr{X}_1^a\}$, which exists by Lemma 3. Furthermore, fix some full-support Borel probability measures $\phi^a \in \mathscr{M}^+(T^b)$ and $\phi^b \in \mathscr{M}^+(T^a)$.

Lemma 4. *Fix $X^a \in \mathscr{X}_1^a$. There exists a family $\{\tau_m(X^a) : m > 1\}$ of open sets in T^a such that, for all $m \in \mathbb{N}$,*

(i) $\tau_m(X^a) \supseteq \tau_{m+1}(X^a)$;

(ii) $\tau_m(X^a) \setminus \tau_{m+1}(X^a)$ *is an uncountable* ϕ^b*-null set;*

(iii) $\tau_\infty(X^a) \equiv \bigcap_{m=1}^{\infty} \tau_m(X^a)$ *is an uncountable open set; and*

(iv) $\phi^b(\tau_\infty(X^a)) = \phi^b(\tau_m(X^a))$.

Analogous results hold for Bob.

Pf. of Lemma 4. The result follows immediately from Lemma 12. □

For each m, fix the family of sets $\{\tau_m(X^a) : X^a \in \mathscr{X}_1^a\}$ that exists by Lemma 3 and Lemma 4.

Definition 14. For all $m \in \mathbb{N}$, define

$$\begin{aligned}
\widehat{R}^a_0 &\equiv S^a \times T^a \\
\widehat{R}^a_m &\equiv \bigcup \{X^a \times \tau_m(X^a) : X^a \in \mathscr{X}^a_m\} \text{ and} \\
\widehat{R}^a_\infty &\equiv \bigcap_{m=1}^{\infty} \widehat{R}^a_m.
\end{aligned}$$

Analogous objects are defined for Bob.

Lemma 5. *For all $m \in \mathbb{N}_0$,*

(i) $\widehat{R}^a_m \supseteq \widehat{R}^a_{m+1}$*;*

(ii) $\widehat{R}^a_m$ *is an uncountable open set;*

(iii) $\widehat{R}^a_\infty$ *is an uncountable open set; and*

(iv) $\widehat{R}^a_m \setminus \widehat{R}^a_{m+1}$ *is uncountable.*

Analogous results hold for Bob.

Pf. of Lemma 5. The desired results follow from Lemma 4 and the finiteness of S^a. □

Lemma 6. *For any $\nu \in \mathscr{M}^+(S^a)$, $\nu \otimes \phi^b \in \mathscr{L}^+(S^a \times T^a)$ denotes the product measure on $S^a \times T^a$. For all $m \in \mathbb{N}$, if $\mathscr{X}^a_m = \mathscr{X}^a_{m+1}$, then $\widehat{R}^a_m \setminus \widehat{R}^a_{m+1}$ is $(\nu \otimes \phi^b)$-null. Analogous results hold for Bob.*

Pf. of Lemma 6. Fix $m \in \mathbb{N}$. If $\mathscr{X}^a_m = \mathscr{X}^a_{m+1}$, then

$$\widehat{R}^a_m \setminus \widehat{R}^a_{m+1} = \bigcup \{X^a \times [\tau_m(X^a) \setminus \tau_{m+1}(X^a)] : X^a \in \mathscr{X}^a_m\}.$$

By Lemma 4, $\tau_m(X^a) \setminus \tau_{m+1}(X^a)$ is ϕ^b-null. It follows that $X^a \times (\tau_m(X^a) \setminus \tau_{m+1}(X^a))$ is $(\nu \otimes \phi^b)$-null. Then $\widehat{R}^a_m \setminus \widehat{R}^a_{m+1}$ is $(\nu \otimes \phi^b)$-null since it is a finite union of such sets. □

Definition 15. Given any $X^a \in \mathscr{X}^a_1$, let $\Lambda_m(X^a)$ denote the set of all $\mu \in \mathscr{L}^+(S^b \times T^b)$ such that

1. $X^a = \{s^a \in S^a : s^a \text{ is optimal under } \mu\}$; and
2. μ assumes $\widehat{R}^b_1, \ldots, \widehat{R}^b_{m-1}$.

Let $\Lambda_\infty(X^a) \equiv \bigcap_{m=1}^\infty \Lambda_m(X^a)$. Analogous objects are defined for Bob.

Lemma 7. *The following holds for all $m \in \mathbb{N}$:*

(i) For all $X^a \in \mathscr{X}^a_{m+1}$, $\Lambda_m(X^a) \setminus \Lambda_{m+1}(X^a)$ is uncountable; and

(ii) For all $X^a \in \mathscr{X}^a_\infty$, $\Lambda_\infty(X^a)$ is uncountable; and

(iii) For all $X^a \in \mathscr{X}^a_m$, $\Lambda_m(X^a)$ is Borel.

Analogous results hold for Bob.[9]

Proof. Part i. Since $X^a \in \mathscr{X}^a_{m+1}$, it follows that $X^a \in \mathscr{X}^a_m$. By the definition of $\mathscr{X}^a_m$, there must exist $\nu = (\nu_0, \ldots, \nu_{m-1}) \in \mathscr{N}^+(S^b)$ such that X^a is exactly the optimal set of strategies under ν; and $\operatorname{Supp} \nu_k = S^b_{m-1-k}$ for all $k = 0, \ldots, m-1$. By Lemma 13, there exists $\mu = (\mu_0, \ldots, \mu_{m-1}) \in \mathscr{L}^+(S^b \times T^b)$ such that $\operatorname{marg}_{S^b} \mu = \nu$; μ assumes $\widehat{R}^b_k$ at level $m-1-k$ for all $k = 0, \ldots, m-1$; and μ does not assume $\widehat{R}^b_m$. It follows that $\Lambda_m(X^a) \setminus \Lambda_{m+1}(X^a)$ is nonempty. Lemma E.2 in BFK implies that, for any μ, there exists a continuum of $\widehat{\mu}$ such that $\operatorname{marg}_{S^b} \widehat{\mu} = \operatorname{marg}_{S^b} \mu$ and $\widehat{\mu}$ assumes the same sets as μ.

Part ii. There exists an $M \in \mathbb{N}$ such that $\mathscr{X}^b_\infty = \mathscr{X}^b_M$ since $\left\{\mathscr{X}^b_n : n \in \mathbb{N}\right\}$ is a decreasing sequence of nonempty finite sets. By the definition of $\mathscr{X}^a_{M+1}$, there must exist $\nu = (\nu_0, \ldots, \nu_M) \in \mathscr{N}^+(S^b)$ such that X^a is exactly the optimal set of strategies under ν; and $\operatorname{Supp} \nu_k = S^b_{M-k}$ for all $k = 0, \ldots, M$. Now take any $\nu'_0 \in \mathscr{M}^+(S^b)$. By Lemma 6, for all $m \geq M$, $\widehat{R}^b_m \setminus \widehat{R}^b_{m+1}$ is $(\nu \otimes \phi^a)$-null and $\nu \otimes \phi^a \in \mathscr{M}^+(statesofbob)$. It follows, by Lemma 14, that there exists $\mu \in \mathscr{L}^+(S^b \times T^b)$ such $\operatorname{marg}_{S^b} \mu = \nu$ and μ assumes $\widehat{R}^b_m$ for all $m \geq 0$. Therefore $\Lambda_\infty(X^a)$ is nonempty. Lemma E.2 in BFK implies that, for any μ, there exists a continuum of $\widehat{\mu}$ such that $\operatorname{marg}_{S^b} \widehat{\mu} = \operatorname{marg}_{S^b} \mu$ and $\widehat{\mu}$ assumes the same sets as μ.

Part iii. First, the set of all LPSs μ that assume a Borel set is Borel (See Lemma C.3 in BFK, p. 340). Second, the set of all LPSs μ under which a strategy is optimal is Borel since,

[9] I am grateful to H. Jerome Keisler for pointing out an error in an earlier version of this proof.

by the finiteness of the game, optimality is defined by a finite set of inequalities between two continuous real functions of μ (i.e., expected utility under μ). $\Lambda_m(X^a)$ is a set of LPSs μ for which a finite combination of these conditions are satisfied. Therefore it is a finite intersection of Borel sets, which is also a Borel set. □

Lemma 8. *There exists a Borel isomorphism $\lambda^a : T^a \to \overline{\mathscr{L}}(S^b \times T^b)$ such that, for all $m \in \mathbb{N}$,*

(i) For all $X^a \in \mathscr{X}^a_{m+1}$, $\lambda^a \left(\Lambda_m(X^a) \setminus \Lambda_{m+1}(X^a)\right) = \tau_m(X^a) \setminus \tau_{m+1}(X^a)$; and

(ii) For all $X^a \in \mathscr{X}^a_m \setminus \mathscr{X}^a_{m+1}$, $\lambda^a \left(\Lambda_m(X^a)\right) = \tau_m(X^a)$.

Analogous results hold for Bob.

Pf. of Lemma 8. Fix $X^a \in \mathscr{X}^a_1$. For each $m \in \mathbb{N}$, if $X^a \in \mathscr{X}^a_{m+1}$, then $\Lambda_m(X^a) \setminus \Lambda_{m+1}(X^a)$ is uncountable. Consider two cases.

Case 1: There exists a largest M such that $X^a \in \mathscr{X}^a_M$. Then $\Xi(X^a)$ is a partition of $\Lambda_1(X^a)$. In the case that $M = 1$, $\Xi(X^a) = \{\Lambda_1(X^a)\}$. Each member of $\Xi(X^a)$ is uncountable.

$$\Xi(X^a) \equiv \{\Lambda_1(X^a) \setminus \Lambda_2(X^a), \ldots, \Lambda_{M-1}(X^a) \setminus \Lambda_M(X^a), \Lambda_M(X^a)\}$$

The corresponding partition $\Pi(X^a)$ of $\tau_1(X^a)$ is defined below. Each member of $\Pi(X^a)$ is uncountable.

$$\Pi(X^a) \equiv \{\tau_1(X^a) \setminus \tau_2(X^a), \ldots, \tau_{M-1}(X^a) \setminus \tau_M(X^a), \tau_M(X^a)\}$$

$\Xi(X^a)$ and $\Pi(X^a)$ are equinumerous (in particular, have size M) and each member of each partition is an uncountable Borel set. Therefore, by the Borel Schröder-Bernstein Theorem (See 15.7, 15.8 in Kechris, 1995, pp. 90-91), there exists a Borel isomorphism $f : \tau_1(X^a) \to$

$\Lambda_1(X^a)$ such that

$$f(\tau_1(X^a) \setminus \tau_2(X^a)) = \Lambda_1(X^a) \setminus \Lambda_2(X^a)$$
$$f(\tau_2(X^a) \setminus \tau_3(X^a)) = \Lambda_2(X^a) \setminus \Lambda_3(X^a)$$
$$\ldots = \ldots$$
$$f(\tau_{M-1}(X^a) \setminus \tau_M(X^a)) = \Lambda_{M-1}(X^a) \setminus \Lambda_M(X^a)$$
$$f(\tau_M(X^a)) = \Lambda_M(X^a).$$

Case 2: For all $m \in \mathbb{N}$, $X^a \in \mathscr{X}^a_m$. We define the partitions $\Xi(X^a)$ of $\Lambda_1(X^a)$ and $\Pi(X^a)$ of $\tau_1(X^a)$ as follows.

$$\Xi(X^a) \equiv \{\Lambda_1(X^a) \setminus \Lambda_2(X^a), \Lambda_2(X^a) \setminus \Lambda_3(X^a), \ldots, \Lambda_\infty(X^a)\}$$
$$\Pi(X^a) \equiv \{\tau_1(X^a) \setminus \tau_2(X^a), \tau_2(X^a) \setminus \tau_3(X^a), \ldots, \tau_\infty(X^a)\}$$

$\Xi(X^a)$ and $\Pi(X^a)$ are equinumerous (in particular, countably infinite) and each member of each partition is an uncountable Borel set. Therefore, by the Borel Schröder-Bernstein Theorem (See 15.7, 15.8 in Kechris, 1995, pp. 90-91), there exists a Borel isomorphism $f : \tau_1(X^a) \to \Lambda_1(X^a)$ such that

$$f(\tau_1(X^a) \setminus \tau_2(X^a)) = \Lambda_1(X^a) \setminus \Lambda_2(X^a)$$
$$f(\tau_2(X^a) \setminus \tau_3(X^a)) = \Lambda_2(X^a) \setminus \Lambda_3(X^a)$$
$$\ldots = \ldots$$
$$f(\tau_\infty(X^a)) = \Lambda_\infty(X^a).$$

Since $T^a \setminus \bigcup \{\tau_1(X^a) : X^a \in \mathscr{X}^a_1\}$ and $\overline{\mathscr{L}}(S^b \times T^b) \setminus \bigcup \{\Lambda_1(X^a) : X^a \in \mathscr{X}^a_1\}$ are uncountable, we conclude that there exists a Borel isomorphism $f : T^a \to \overline{\mathscr{L}}(S^b \times T^b)$ such that f

satisfies the properties described in the cases above. □

Lemma 9. *Fix a type structure* $\langle T^a, T^b, \lambda^a, \lambda^b \rangle$ *in which* λ^a *and* λ^b *are given by Lemma 8. Then, for all* $m \in \mathbb{N}$, $\widehat{R}^a_m = R^a_m$ *and* $\widehat{R}^b_m = R^b_m$.

Pf. of Lemma 9. First, note that, for all $m \in \mathbb{N}$ and $X^a \in \mathscr{X}^a_m$, $\lambda^a(\tau_m(X^a)) = \Lambda_m(X^a)$. For all $X^a \in \mathscr{X}^a_1$, $\Lambda_1(X^a)$ is the set of all full-support beliefs under which X^a is exactly the set of optimal strategies. By Lemma 2, it follows that

$$R^a_1 = \{X^a \times \tau_1(X^a) : X^a \in \mathscr{X}^a_1\} = \widehat{R}^a_1.$$

An analogous result holds for Bob.

It follows by induction that, for each $m \in \mathbb{N}$, $\tau_{m+1}(X^a)$ is precisely the set of all full-support types such that

1. each type in $\tau_{m+1}(X^a)$ assumes $R^b_1, \ldots, R^b_m$; and
2. X^a is exactly the set of optimal strategies for each type in $\tau_{m+1}(X^a)$.

It follows that $R^a_{m+1} = \left\{X^a \times \tau_{m+1}(X^a) : X^a \in \mathscr{X}^a_{m+1}\right\} = \widehat{R}^a_{m+1}$. □

Pf. of Lemma 1. Since $\bigcap_{m=1}^{\infty}(R^a_m \times R^b_m) = \bigcap_{m=1}^{\infty}(\widehat{R}^a_m \times \widehat{R}^b_m) \neq \varnothing$, there exists an RCAR state in $\langle T^a, T^b, \lambda^a, \lambda^b \rangle$. λ^a and λ^b are isomorphisms, therefore $\langle T^a, T^b, \lambda^a, \lambda^b \rangle$ is a complete type structure. □

C Appendix: Polish Spaces and Assumption

Remark. It is well known that every uncountable subset of a Polish space has cardinality equal to that of the continuum. Throughout this paper, it is implicitly understood that such sets have cardinality $2^{\aleph_0}$.

Lemma 10. *Given any uncountable Polish space X and $n \in \mathbb{N}$, there exists a family of pairwise disjoint uncountable open sets $\{U_1, \ldots, U_n\}$ in X such that $X \setminus \bigcup \{U_1, \ldots, U_n\}$ is uncountable.*

Pf. of Lemma 10. The Cantor-Bendixon Theorem states that each Polish space X can be uniquely written as a disjoint union of a perfect set P and a countable open set C. It is immediate that P is uncountable. Now fix $n+1$ distinct points $x_1, \ldots, x_{n+1} \in P$. All singletons in Polish spaces are closed sets. All Polish spaces are normal topological spaces, i.e., any two disjoint closed sets are separated by open neighborhoods. Therefore there exist disjoint open neighborhoods $U_1, \ldots, U_{n+1}$ of $x_1, \ldots, x_{n+1}$, respectively. By the proof of the Cantor-Bendixon Theorem (see Kechris, 1995, p. 32), $x_k \in P$ if and only if all open neighborhoods of x_k are uncountable. Therefore $U_1, \ldots, U_{n+1}$ are all uncountable open sets. Since $X \setminus \bigcup \{U_1, \ldots, U_n\}$ contains an uncountable set U_{n+1}, it is uncountable. □

Definition 16. The **Cantor space** is $\mathfrak{C} \equiv \{0,1\}^{\mathbb{N}}$ endowed with the product topology, where $\{0,1\}$ is given the discrete topology.

Lemma 11. *Let X be a Polish space and fix a Borel probability measure μ on X. For all uncountable open $U \subseteq X$, there exists an uncountable μ-null Borel set E in U such that $U \setminus E$ is uncountable.*

Pf. of Lemma 11. Let $\mathscr{B}$ denote the Borel sets of X. Let A denote the atoms of μ. A is a countable set. The desired result follows immediately if $\mu(U \setminus A) = 0$. Consider the case when $\mu(U \setminus A) > 0$. The restriction of $\mathscr{B}$ to $U \setminus A$ is denoted $\mathscr{B}|(U \setminus A)$. Then $(U \setminus A, \mathscr{B}|(U \setminus A))$ is an uncountable standard Borel space. The conditional probability $\mu(\cdot|U \setminus A)$ is nonatomic since it excludes the atoms of μ and can be defined using Bayes' Rule. By the isomorphism theorem for Borel measures (see Kechris, 1995, p. 116), there exists a Borel isomorphism $f : U \setminus A \to [0,1]$ such that $\mu(f^{-1}(\cdot)|U \setminus A)$ is the Lebesgue measure on $[0,1]$. The space $[0,1]$ contains an uncountable Borel set that has zero Lebesgue measure, namely the ternary

Cantor set, which is denoted $C_{1/3}$. It follows that $E \equiv f^{-1}(C_{1/3})$ is an uncountable μ-null Borel set. $(U \setminus A) \setminus E$ is uncountable since it is a not μ-null and contains no atoms of μ. □

Lemma 12. *Let X be a Polish space and fix a Borel probability measure μ on X. For all uncountable open $U \subseteq X$, there exists a decreasing sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of U such that, for each $n \in \mathbb{N}$,*

(i) $U_1 = U$;

(ii) $U_n \setminus U_{n+1}$ *is uncountable;*

(iii) $\mu(U_n) = \mu(U)$;

(iv) $\bigcap\{U_n : n \in \mathbb{N}\}$, *denoted* U_∞, *is an uncountable open set; and*

(v) $\mu(U_\infty) = \mu(U)$.

Pf. of Lemma 12. By Lemma 11, there exists an uncountable μ-null Borel set E in U such that $U \setminus E$ is uncountable. By Theorem 13.6 in Kechris (1995), E, being an uncountable Borel set, contains a homeomorphic copy Z of the Cantor space. Therefore, Z can written as the disjoint union $\bigcup\{Z_n : n \in \mathbb{N}\}$, where each Z_n is a compact set containing a homeomorphic copy of Z.[10] Let $U_0 \equiv U$. Now, for each $n \in \mathbb{N}$, define $U_{n+1} \equiv U_n \setminus Z_n$. For all $n \in \mathbb{N}$, Z_n is compact since it is a continuous of image of a compact set. It follows that U_2 is open. An induction argument shows that, for all $n \in \mathbb{N}$, U_n is open. By definition, $U_\infty = U \setminus \bigcup\{Z_n : n \in \mathbb{N}\} = U \setminus Z$. U_∞ is an uncountable open set since $U \setminus Z \supseteq U \setminus E$. It is obvious that $Z, Z_1, Z_2, \ldots$ are all μ-null since they are subsets of E, a μ-null set. Therefore $\mu(U_n) = \mu(U) = \mu(U_\infty)$ for all $n \in \mathbb{N}$. □

Lemma 13. *Let Z be a nonempty finite set and let X be a Polish space. $Y \equiv Z \times X$ is a Polish space. Let $Y_0 \equiv Y$. Let $\{Y_n : n \in \mathbb{N}\}$ be a family of nonempty open subsets of Y*

[10] Let the Cantor space be denoted $\mathfrak{C} = \{0,1\}^{\mathbb{N}}$. Let $\mathfrak{C}_n \equiv \{0\}^{n-1} \times \{1\} \times \mathfrak{C}$. $\{\mathfrak{C}_n : n \in \mathbb{N}\}$ forms a family of pairwise disjoint sets whose union is $\mathfrak{C} \setminus \{0\}^{\mathbb{N}}$. It is obvious that, for all $n \in \mathbb{N}$, $\mathfrak{C}_n$ is homeomorphic to $\mathfrak{C}$. Let $\mathfrak{C}'_1 \equiv \mathfrak{C}_1 \cup \{0\}^{\mathbb{N}}$. $\mathfrak{C}'_1$ is a finite union of compact sets, and therefore is itself compact. Then $\mathfrak{C} = \mathfrak{C}'_1 \cup \bigcup\{\mathfrak{C}_n : n \in \mathbb{N} \setminus \{1\}\}$

such that, for all $n \geq 0$, $Y_n \supsetneq Y_{n+1}$ *and* $\operatorname{proj}_Z Y_n = \operatorname{proj}_Z(Y_n \setminus Y_{n+1})$. *Fix* $m \in \mathbb{N}$. *Given* $\nu = (\nu_0, \ldots, \nu_{m-1}) \in \mathscr{N}^+(Z)$ *such that* $\operatorname{Supp} \nu_{m-1-k} = \operatorname{proj}_Z Y_k$ *for* $k = 0, \ldots, m-1$, *there exists* $\mu = (\mu_0, \ldots, \mu_{m-1}) \in \mathscr{L}^+(Y)$ *such that* $\operatorname{marg}_Z \mu = \nu$; μ *assumes* Y_k *at level* $m-1-k$ *for all* $k = 0, \ldots, m-1$; *and* μ *does not assume* Y_m.

Pf. of Lemma 13. First, for all $n \geq 0$, define $Z_n \equiv \operatorname{proj}_Z Y_n$. Note that, for all $n \geq 0$, Y_n and $Y_n \setminus Y_{n+1}$ are nonempty G_δ sets in Y. By Theorem 3.11 in Kechris (1995), a subspace of Y is Polish if and only if it is a G_δ. Therefore, for all $n \geq 0$, Y_n and $Y_n \setminus Y_{n+1}$ are Polish spaces if given the relative topology.

For any $k \geq 1$, fix some $\rho_k \in \mathscr{M}^+(Y_{m-1-k} \setminus Y_{m-k})$. Define μ_k in the following way:

$$\mu_k(E) \equiv \sum_{z \in Z_{m-1-k}} \nu_k(z) \rho_k(E | (Y_{m-1-k} \setminus Y_{m-k}) \cap (\{z\} \times X)).$$

$(Y_{m-1-k} \setminus Y_{m-k}) \cap (\{z\} \times X)$ is open in $Y_{m-1-k} \setminus Y_{m-k}$ since $\{z\} \times X$ is open in Y. $\{z\} \times X$ is open in Y because $Y = Z \times X$ and Z is finite. The conditional probability involving ρ_k above is well-defined via Bayes' rule since ρ_k is a full-support measure on $Y_{m-1-k} \setminus Y_{m-k}$ and $(Y_{m-1-k} \setminus Y_{m-k}) \cap (\{z\} \times X)$ is open in $Y_{m-1-k} \setminus Y_{m-k}$. It is clear from the definition that μ_k and ρ_k are mutually absolutely continuous. Also, $\operatorname{marg}_Z \mu_k = \nu_k$.

Fix some $\rho_0' \in \mathscr{M}^+(Y_{m-1} \setminus Y_m)$ and $\rho_0'' \in \mathscr{M}^+(Y_m)$. Now fix a nontrivial convex combination of ρ_0' and ρ_0'', say $\rho_0 \equiv \frac{1}{2}\rho_0' + \frac{1}{2}\rho_0''$. It follows that $\rho_0 \in \mathscr{M}^+(Y_{m-1})$, $\rho_0(Y_{m-1} \setminus Y_m) > 0$, and $\rho_0(Y_m) > 0$. Define μ_0 in the following way:

$$\mu_0(E) \equiv \sum_{z \in Z_{m-1}} \nu_0(z) \rho_0(E | Y_{m-1} \cap (\{z\} \times X)).$$

By arguments similar to those in the previous paragraph, it is readily apparent that ρ_0 is well-defined via Bayes' rule. It is clear from the definition that μ_0 and ρ_0 are mutually absolutely continuous. An immediate implication of this fact is that $\mu_0 \in \mathscr{M}^+(Y_{m-1})$,

$\mu_0(Y_{m-1} \setminus Y_m) > 0$, and $\mu_0(Y_m) > 0$. Also, $\operatorname{marg}_Z \mu_0 = \nu_0$.

The following facts are easily verified: $\mu = (\mu_0, \ldots, \mu_{m-1}) \in \mathscr{L}^+(Y)$; $\operatorname{marg}_Z \mu = \nu$; μ assumes Y_k at level $m-1-k$ for all $k = 0, \ldots, m-1$; and μ does not assume Y_m. $\square$

Lemma 14. *Let Z be a nonempty finite set and let X be a Polish space. $Y \equiv Z \times X$ is a Polish space. Let $Y_0 \equiv Y$. Let $\{Y_n : n \in \mathbb{N}\}$ be a family of nonempty open subsets of Y such that, for all $n \geq 0$, $Y_n \supsetneq Y_{n+1}$ and $\operatorname{proj}_Z Y_n = \operatorname{proj}_Z(Y_n \setminus Y_{n+1})$. Let $Y_\infty \equiv \bigcap \{Y_n : n \in \mathbb{N}\}$ be a nonempty open set. Suppose that there exist $\phi \in \mathscr{M}^+(Y)$ and $M \in \mathbb{N}$ such that $\phi(Y_M \setminus Y_\infty) = 0$.*

Given $\nu = (\nu_0, \ldots, \nu_M) \in \mathscr{N}^+(Z)$ such that $\operatorname{Supp} \nu_{M-k} = \operatorname{proj}_Z Y_k$ for $k = 0, \ldots, M$, there exists $\mu = (\mu_0, \ldots, \mu_M) \in \mathscr{L}^+(Y)$ such that $\operatorname{marg}_Z \mu = \nu$; μ assumes Y_k for all $k \geq 0$.

Pf. of Lemma 14. By Lemma 13, there exists $\rho = (\rho_0, \ldots, \rho_M) \in \mathscr{L}^+(Y)$ such that $\operatorname{marg}_Z \mu = \nu$; ρ assumes Y_k at level $M-k$ for all $k = 0, \ldots, M$. Now define $\mu_0 \in \mathscr{M}^+(Y_M)$ as follows.

$$\mu_0(E) \equiv \sum_{z \in Z_M} \nu_0(z) \phi(E | Y_M \cap (\{z\} \times X)).$$

The conditional probability $\phi(\cdot | Y_M \cap (\{z\} \times X))$ is well-defined via Bayes' rule because $\phi(Y_M \cap (\{z\} \times X)) > 0$. It is clear from the definition that $\mu_0 \ll \phi$. For all $k \geq 1$, let $\mu_k \equiv \rho_k$.

The following facts are immediate: $\mu = (\mu_0, \ldots, \mu_M) \in \mathscr{L}^+(Y)$, $\operatorname{marg}_Z \mu = \nu$, and μ assumes Y_k for all $k \leq M$. It remains to verified that μ assumes Y_k for all $k \geq M$. By construction, if $k \geq M$ and $j \geq 1$, then $\mu_j(Y_k) = 0$. Since $\mu_0 \ll \phi$ and $\phi(Y_M \setminus Y_\infty) = 0$, we have $\mu_0(Y_M \setminus Y_\infty) = 0$. Furthermore, $\mu_0(Y_\infty) = 1$ since $\mu_0 \in \mathscr{M}^+(Y_M)$ and $\mu_0(Y_M \setminus Y_\infty) = 0$. It follows that $\mu_0(Y_k) = 1$ for all $k \geq M$ since $Y_k \supset Y_\infty$ for all $k \geq M$. $\square$

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