A Model of Limited Foresight*

Asen Kochov†

November 11, 2009

Job Market Paper

Abstract

The paper models an individual who may not foresee all relevant aspects of an uncertain environment. The model is axiomatic and provides a novel choice-theoretic characterization of the subalgebra of foreseen events. It is proved that all recursive, consequentialist models imply perfect foresight and thus cannot accommodate unforeseen contingencies. In particular, the model is observationally distinct from recursive models of ambiguity. The process of learning implied by dynamic behavior generalizes the Bayesian model and permits the subalgebra of foreseen events to expand over time.

*This work has benefited from the helpful comments of Paulo Barelli and Bart Lipman. I am especially grateful to my advisor Larry Epstein for his support and for many insightful discussions.

†Department of Economics, University of Rochester, Rochester, NY 14627. E-mail: akochov@mail.rochester.edu.

1 Introduction

1.1 Objectives

Contingencies arise that were not foreseen at earlier dates. Individuals adapt their strategies and behavior often reflects the awareness that other, yet unanticipated changes may occur. How does one model such behavior?

The standard approach to dynamic choice postulates an idealized individual who comprehends fully the uncertainty describing her environment. She foresees each contingency that may eventuate and knows the outcomes induced by each action in each state of the world. Plans reflect this knowledge and are consistently implemented over time. In effect, dynamic behavior is reduced to the static choice of an optimal strategy.

This paper develops an axiomatic model of dynamic choice in which the individual may not foresee all relevant aspects of an uncertain environment. Two properties characterize the model.

First, the individual is self-aware and knows that her perception of the environment may be incomplete. The paper provides a choice-theoretic characterization of the subalgebra of foreseen events and shows that awareness induces a nonsingleton set of beliefs over this collection. The multiplicity of priors reflects a preference for robustness or hedging against unanticipated events.

Second, the paper models a forward-looking individual who plans ahead but also adapts to unforeseen contingencies. As time unfolds, her perception of the environment improves and the individual revises her strategy. A novel axiom, *Weak Dynamic Consistency,* assumes that adaptations arise *only when* unanticipated events alter the individual's perception. The axiom characterizes a process of learning which generalizes the Bayesian model and permits the subalgebra of foreseen events to expand over time.

The key to developing the model comes from answering the question: *At any point in time, what behavior would reveal the collection of events foreseen by the individual?* In atemporal settings, Epstein, Marinacci, and Seo [4] and Gilboa and Schmeidler [10] show that models of limited foresight are observationally equivalent to ambiguity averse behavior. Since ambiguity aversion is conceptually distinct from limited foresight, the atemporal setting provides no behaviorally meaningful way to define unforeseen events.

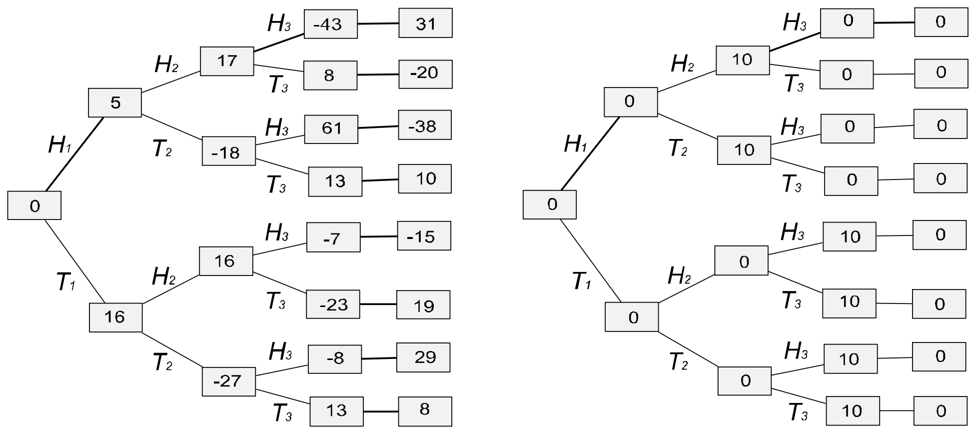

Figure 1: State-contingent payoffs of the actions L (left) and R (right).

This paper provides a choice-theoretic characterization when outcomes unfold over time. The characterization is illustrated in the following example which exhibits the static and dynamic implications of limited foresight and shows how the temporal domain of choice permits the behavioral separation of limited foresight from existing models of uncertainty.

1.2 Motivating Examples

1.2.1 Static Behavior

The individual chooses between two actions L and R whose payoffs unfold over a four-period horizon. A sequence of binary shocks affect outcomes as depicted in Figure 1. For simplicity, consider an individual who is risk-neutral and does not discount the future.[1] Her objective is to maximize cumulative wealth.

The individual who foresees all contingencies and knows the corresponding outcomes deduces correctly that both actions L and R are *effectively certain:* the cumulative payoff of either action is *\$10* across all states of the world. Irrespective of her beliefs, she is then indifferent between L and R

[1] The assumption of no discounting is relaxed in the formal model. For ease of exposition, risk neutrality is maintained throughtout but can be similarly generalized.

and the action which pays *$10* at the time of choice:

$$L \sim_0 10 \text{ and } R \sim_0 10 \tag{1.1}$$

To understand the implications of limited foresight, consider the rankings:

$$L \prec_0 10 \text{ and } R \sim_0 10 \tag{1.2}$$

How does one interpret such behavior? The effectively certain action R entails a ten-dollar bet on the event H_1 paying at period $t = 2$ and an analogous bet on the event T_1 paying at period $t = 3$. The indifference $R \sim_0 10$ suggests that the individual understands the immediate contingencies H_1 and T_1 and sees correctly that the two bets offset one another. If similar indifference holds for all other effectively certain bets on H_1 and T_1, the paper concludes that the events are *subjectively foreseen.*

The outcomes of action L depend, in contrast, on contingencies that resolve in the more distant future. An individual with limited foresight may not see that these distant payoffs offset the more imminent, short-run uncertainty she faces. Thus, she requires a premium captured by the preference $L \prec_0 10$.

The latter ranking cannot be attributed to uncertainty about likelihoods when all events are foreseen. If the individual foresees the possible contingencies and knows the corresponding outcomes, she necessarily exhibits the rankings in (1.1). These hold independently of beliefs, and thus their negation reveals limited foresight.

This possibility to test knowledge of the environment independently of beliefs is specific to the temporal domain of choice and the existence of nontrivial effectively certain actions. If all uncertainty resolves in a single period, the only effectively certain actions are constant acts whose ranking reveals no information about the perception of events. Conversely, any nontrivial act is uncertain and its evaluation requires an assessment of beliefs. Thus, the atemporal domain is too 'small' to separate ambiguity aversion from limited foresight. This conclusion is confirmed by Epstein, Marinacci, and Seo [4], Ghirardato [8], Gilboa and Schmeidler [10] and Mukerji [16].

1.2.2 Dynamic Behavior

To illustrate the implications of limited foresight for dynamic behavior, consider the simple example of an individual whose foresight at time $t = 0$ is limited but who understands the environment perfectly at time $t = 1$. The choices below contrast her posterior and prior valuations of L and R:

$$L \sim_{1,H_1} 10, \qquad L \sim_{1,T_1} 10 \quad \text{and} \quad L \prec_0 10 \tag{1.3}$$

$$R \sim_{1,H_1} 10, \qquad R \sim_{1,T_1} 10 \quad \text{and} \quad R \sim_0 10 \tag{1.4}$$

The rankings in (1.3) and (1.4) reveal two important characteristics of dynamic choice under limited foresight.

The premium required for distant, poorly foreseen bets disappears as time unfolds and the perception of the individual improves. The corresponding rankings (1.3) imply a violation of dynamic consistency and are precluded by the standard approach to dynamic choice. This violation arises as the individual learns aspects of the environment she did not anticipate and could not take into account ex ante.

In (1.4), the indifference $R \sim_0 10$ reveals that the individual understands the immediate contingencies H_1 and T_1. Then, the conditional preferences indicate that she evaluates the action consistently over time.

These examples illustrate the general approach to modeling coherent dynamic behavior when some contingencies are unforeseen: the individual is forward-looking and revises her plans *only when* unanticipated circumstances contradict her perception.

1.2.3 Foresight and Dynamic Consistency

The general approach to modeling dynamic choice assumes that foresight implies intertemporal consistency: an individual who foresees all contingencies, anticipates future behavior and plans ahead. The paper proves that foresight is furthermore necessary: *dynamically consistent, consequentialist and state-independent models of behavior imply perfect foresight* where the latter is defined by the ranking of effectively certain acts.

This result has two important implications. First, recursive models of dynamic behavior in general preclude the ex ante rankings in (1.2) that motivate this paper. Importantly, the premium on R cannot be interpreted as preference for early resolution of uncertainty studied by Kreps and Porteus [15] and Epstein and Zin [6]. Recursive, intertemporal models of ambiguity as in Epstein and Schneider [5] and Klibanoff, Marinacci, and Mukerji [11] similarly preclude such a premium, proving formally that limited foresight and ambiguity aversion are observationally distinct in a temporal domain of choice.

Second, a corollary of this result provides an alternative and equivalent characterization of the collection of foreseen events. In example (1.2), the contingencies H_1 and T_1 foreseen by the individual are revealed by the *static* ranking of effectively certain acts. The paper shows that the *intertemporal consistency* of behavior leads to an alternative characterization. The latter confirms a conjecture by Kreps [13, p.278] that dynamic behavior may reveal the collection of foreseen events and provide a foundation for separating limited foresight from existing models of uncertainty.

1.3 Related Literature

Kreps [13] is the first to model unforeseen contingencies. He takes as primitive a state space S depicting the individual's *incomplete* perception of the environment. By modeling the preference for flexibility, Kreps derives an *extended* state space $\Omega := S \times \Theta$ and interprets the endogenous contingencies θ as *completing* the individual's perception.

Both Kreps [13] and Dekel, Lipman, and Rustichini [3] observe that the model is observationally equivalent to a standard model with an extended state space $S \times \Theta$. In that interpretation, all events are *foreseen* but some contingencies, namely θ, are *unverifiable* by an outside observer.

Gilboa and Schmeidler [10] reinterpret nonadditive models of ambiguity aversion as models of unforeseen contingencies. Like Kreps [13], they derive an extended state space Ω which completes the individual's perception captured by the exogenous state space S. The nonadditivity of their model reflects the behavior of a self-aware person who tries to hedge against unanticipated contingencies.

Epstein, Marinacci, and Seo [4] argue against modeling the individual's perception as an observable primitive. Modifying Kreps' [12] framework of preference for flexibility, they endogenize both the coarse state space S and its completion Ω. However, their model retains the observational equivalence with ambiguity aversion.

This paper takes as primitive a state space Ω describing an uncertain objective environment and derives a coarse state space S describing the individual's subjective perception. In a temporal setting, the paper exhibits behavior that is inconsistent with knowledge of all contingencies within the primitive state space Ω, and thus separates limited foresight from ambiguity aversion.

The identification of foreseen events within a primitive objective environment permits the analysis of dynamic choice and adaptation over time. Kreps [14] advocates the study of adaptive behavior and provides an overview of statistical learning models common in macroeconomics. The latter assume that static choice conforms to expected utility while positing a non-Bayesian updating rule. Kreps [14] and Cogley and Sargent [2] point out the juxtaposition of static rationality and dynamic inconsistency: the individual continually adapts her behavior as if she did not anticipate the states of the world when adaptation occurs, yet static behavior conforms to the standard model of perfect foresight. Dynamic choice in this paper differs in two important aspects. First, intertemporal consistency is related directly to properties of static behavior. The individual who foresees all contingencies plans ahead and anticipates future behavior. Thus, perfect foresight implies dynamic consistency. Conversely, adaptation reveals that some contingencies are unforeseen. This brings forth a second difference. The self-aware individual in this paper knows that her perception of the environment is incomplete and exhibits a preference for robustness, thus departing further from the standard model of choice.

2 Static Model

2.1 Domain

The objective environment is described by a state space Ω and a filtration $\mathcal{F} := \{\mathcal{F}_t\}$ where time varies over a finite horizon $\mathcal{T} = \{0, 1, ..., T\}$. An action taken by the individual induces a real-valued, $\{\mathcal{F}_t\}$-adapted process of outcomes lying in some compact interval M. Call any such process an *act* and denote generic acts by h, h'.

The distinction between acts and actions is important in a model of limited foresight. Actions, such as the purchase of a dividend-paying stock, comprise the individual's domain of choice. Acts are the mathematical representation of actions: they map states of the world into outcomes. The paper assumes that the modeler observes the choice of action and knows the corresponding *objective acts*. Under the standard assumption that each act is induced by a unique action, the observable choice over actions induces a unique preference over acts. The latter is adopted as a primitive of the model. The objective is to infer the individual's perception of the environment as a component of the representation.

Thus, let $\succeq$ denote a preference relation over the set of objective acts $\mathcal{H}$. For any act h, let $\mathcal{F}(h)$ denote the filtration induced by h. Conversely, for any subfiltration $\mathcal{G}$, let $\mathcal{H}_{\mathcal{G}}$ denote the set of $\mathcal{G}$-adapted acts. For any t, $\Pi_{\mathcal{G}_t}$ denotes the partition generating the algebra $\mathcal{G}_t$ and, for any ω, $\mathcal{G}_t(\omega)$ is the atom in $\Pi_{\mathcal{G}_t}$ containing ω. It is assumed that each $\mathcal{F}_t$ is finitely generated, so the corresponding partition is well-defined. A final assumption on the objective environment is that $\mathcal{F}_T = \mathcal{F}_{T-1}$. Thus, the individual lives for another period after she learns all relevant information. The assumption implies that the subset of effectively certain acts is rich. In particular, it generates the objective filtration $\mathcal{F}$.[2]

Generic outcomes in M are denoted by x, x', y. The deterministic act paying x in each period and each state of the world is denoted by $\mathbf{x}$. For any act h and state ω, $h(\omega)$ denotes the deterministic act which pays $h_t(\omega)$ in period t.

2.2 Axiomatic Framework

This section adopts a set of axioms about the preference ordering $\succeq$ defined on the space $\mathcal{H}$ of objective acts. Some axioms have a standard interpretation if the individual has perfect knowledge of the environment. In this case, the subjective perception of the individual and the primitive objective environment coincide. However, if perception is coarse, the axioms make implicit assumptions about how this perception differs from and approximates the objective world. The first two axioms fall in this category and their content is reinterpreted accordingly.

Axiom 1 *(Basic) The preference $\succeq$ is complete and transitive, mixture-continuous and monotone.*

[2] The assumption is redundant in an infinite-horizon model. An extension is currently in progress.

The assumption of complete and transitive preferences models an individual who knows the grand set of actions and understands any choice problem she may face. It is an important aspect of bounded rationality that individuals do not always conceive of all alternatives at their disposal and may consequently exhibit cyclical choices. While interesting, the problem is conceptually distinct from the problem of unforeseen contingencies. To focus on the perception of uncertainty of an otherwise rational individual, the paper maintains the standard assumption of complete and transitive preferences.

Monotonicity is arguably the least contentious axiom of the standard model. It requires that an act h is preferred to h' whenever the outcomes of the former exceed the outcomes of h' in every period and every state of the world. Monotonicity remains compelling even if the individual has an incomplete perception of the objective environment. To take an extreme example, imagine the individual whose perception of the world is trivial: she knows only that 'something' may happen. For any given action, she foresees the worst possible outcome and ranks actions accordingly. Such behavior is (weakly) monotone. Intuitively, the individual may have an incomplete perception of the environment but need not be delusional and prefer an act h' to another act that dominates it.

Axiom 2 *(Convexity) For all $h, h' \in \mathcal{H}$, $h \sim h'$ implies $\alpha h + (1-\alpha)h' \succeq h$.*

To understand Convexity, it is useful to imagine a hypothetical, auxiliary step in which the individual is asked to compare the *subjective mixture* of h and h' to either action. That is, the individual can mix the outcomes of h and h' as *she perceives* them. The subjective mixture 'smooths' outcomes across states foreseen by the individual. Aware that her perception may be incomplete, the individual prefers the mixture. The latter hedges her exposure to contingencies that she fears might be only a coarse approximation to the world. Next, Convexity requires that the actual objective mixture $\alpha h+(1-\alpha)h'$ be preferred to the subjective one. This is because the objective mixture smooths the uncertainty *within* as well as *across* any of the foreseen events.

To state the next axioms, it is necessary to formalize the characterization of *foreseen events* suggested by the introductory examples. As Section 1.2.1 describes, the key is provided by the subset of effectively certain acts.

Definition 1 *An act* $h \in \mathcal{H}$ *is* ***effectively certain*** *if* $h(\omega) \sim h(\omega')$ *for all states of the world* $\omega, \omega' \in \Omega$.

Section 3.3 proves that for all recursive, consequentialist and state-independent models of choice, an effectively certain act h is necessarily indifferent to the constant stream of outcomes $h(\omega)$. Such indifference is intuitive if the individual has perfect foresight. However, as the rankings in (1.2) suggest, it is intuitive *only if* the individual has perfect foresight. In this sense, existing models of dynamic choice implicitly *assume* perfect foresight. This paper does not impose such indifference for all effectively certain acts, but rather takes the subset of acts when indifference holds as revealing the collection of foreseen events. This subset is defined next.

Definition 2 *An effectively certain act* $h \in \mathcal{H}$ *is* ***subjectively certain*** *if for all effectively certain,* $\mathcal{F}(h)$*-adapted acts* h', $h' \sim h'(\omega)$ *for all* $\omega \in \Omega$.

The subjective certainty of an act h requires not only that the act be indifferent to $h(\omega)$, but that similar indifference holds for all other $\mathcal{F}(h)$-adapted, effectively certain acts. This 'robustness' check ensures that subjective certainty is a property of the events in $\mathcal{F}(h)$ and does not depend on the specific outcomes of h. Following the intuition suggested by the introductory examples, let $\mathcal{G}$ denotes the algebra on $\Omega \times \mathcal{T}$ induced by all subjectively certain acts and interpret $\mathcal{G}$ as *the collection of foreseen events,* or simply, *the subjective filtration.* Similarly, refer to any $\mathcal{G}$-adapted act as *subjectively measurable.*[3]

Axiom 3 *(Strong Certainty Independence) For all acts* $h, h' \in \mathcal{H}$ *and effectively certain, subjectively measurable acts* g*:*

$$h \succeq h' \text{ if and only if } \alpha h + (1-\alpha) g \succeq \alpha h' + (1-\alpha) g.$$

To understand Strong Certainty Independence, imagine the mixture of the action L and the subjectively certain action R in example (1.1-1.2). Since R is subjectively measurable, it is constant within the events H_1 and T_1 foreseen by the individual. Hence, it cannot hedge the poorly understood uncertainty *within* these events. Since the action is effectively certain, it also

[3] Lemma 10 in the Appendix shows that $\mathcal{G}$ is necessarily a filtration.

provides no hedging *across* the collection of foreseen events. The conjunction of these arguments motivates Strong Certainty Independence.

To illustrate the next axiom, consider the subjectively measurable act R in the introductory example and the corresponding ranking $R \sim 10$. The action R entails a ten-dollar bet on the event H_1 paying in period $t = 2$ and an analogous bet on the event T_1 paying in period $t = 3$. Suppose one were to delay the payment of these bets by one period. The new act entails bets on the same events H_1 and T_1 but pays in periods $t = 3$ and $t = 4$, respectively. Since both events are foreseen and do not change, Stationarity requires that the ranking remains the same.

Axiom 4 *(Stationarity) For all acts $h, h' \in \mathcal{H}$ and for all outcomes $x \in M$,*

$$(h_0, ..., h_{T-1}, x) \succeq (h'_0, ..., h'_{T-1}, x) \text{ if and only if } (x, h_0, ..., h_{T-1}) \succeq (x, h'_0, ..., h'_{T-1}),$$

whenever the acts on the left (right) are subjectively measurable.

Events in the subjective filtration $\cup_t \Pi_{\mathcal{G}_t}$ correspond to states of the world as *perceived* by the individual. The next axiom is the subjective analogue of the standard monotonicity or state-independence assumption applied to these subjective states.

Axiom 5 *(Subjective Monotonicity) For all acts $h, h' \in \mathcal{H}$ and for all outcomes $x \in M$,*

$$hAx \succeq h'Ax \text{ for all } A \in \cup_t \Pi_{\mathcal{G}_t} \text{ implies } h \succeq h'.$$

Consider an act h whose continuation at some node $\mathcal{F}_t(\omega)$ is nonconstant: $h_\tau(\omega') \neq h_\tau(\omega'')$ for some $\tau > t$ and $\omega', \omega'' \in \mathcal{F}_t(\omega)$. Say that h' *simplifies* h if h' has a constant continuation at $\mathcal{F}_t(\omega)$ and equals h elsewhere. By construction, h' depends on events that are strictly closer in time. The next axiom requires that h' is subjectively measurable whenever h is. Thus, events closer in time are easier to foresee.

Axiom 6 *(Sequentiality) If g' simplifies a subjectively measurable act g, then g' is subjectively measurable.*

Define nullity in the usual way: the event $A \in \mathcal{F}_T$ is $\succeq$-null if $h(\omega) = h'(\omega)$ for all $\omega \in A^c$ implies $h \sim h'$.

Axiom 7 *(Nonnullity) Every nonempty subjectively measurable event is non-null.*

2.3 Representation

2.3.1 Subjective Filtration

The section introduces the class of filtrations used to model the individual's perception of the objective environment $(\Omega, \{\mathcal{F}_t\})$.

Definition 3 *A subfiltration $\{\mathcal{G}_t\}$ of $\{\mathcal{F}_t\}$ is* ***sequentially connected*** *if*

$$\Pi_{\mathcal{G}_t} \backslash \Pi_{\mathcal{F}_t} \subset \Pi_{\mathcal{G}_{t+1}} \text{ for all } t < T.$$

To understand the definition, consider the disjoint events A_1 and A_2, either of which may be realized in period t. The individual foresees their union $A_1 \cup A_2$ but does not foresee the finer contingencies A_1 and A_2. Thus, $A_1 \cup A_2 \in \Pi_{\mathcal{G}_t} \backslash \Pi_{\mathcal{F}_t}$. The definition of a *sequentially connected filtration* requires that if the individual's perception of period t is coarse, she does not foresee any of the *more distant* contingencies within $A_1 \cup A_2$. In particular, $A_1 \cup A_2 \in \Pi_{\mathcal{G}_{t+1}}$. This captures the intuitive requirement that events more distant in time are more difficult to foresee.

It is not difficult to see that any sequentially connected filtration $\{\mathcal{G}_t\}$ is fully determined by the algebra $\mathcal{G}_T$:

$$\mathcal{G}_t = \mathcal{F}_t \cap \mathcal{G}_T \text{ for all } t \in \mathcal{T}. \tag{2.1}$$

Define an algebra to be *sequentially connected*, if it induces a sequentially connected filtration via (2.1). The rest of the paper uses $\mathcal{G}$ interchangeably to denote the filtration and the algebra which generates it.

Sequentially connected filtrations include a number of common and intuitive specifications.

Example 1 *(Fixed Horizon) The individual foresees all events up to some period k:*

$$\mathcal{G}_t = \mathcal{F}_t \text{ for all } t \leq k \text{ and } \mathcal{G}_t = \mathcal{F}_k \text{ for } t > k.$$

More generally, the individual may not foresee the contingencies describing an unlikely event A, but have a better understanding of its complement. Her depth of foresight is then a random variable and the corresponding sequentially connected algebra may be modeled as a *stopping time.*

Example 2 *(Random Horizon) The individual foresees all events up to a stopping time τ, where*

$$\tau : \Omega \to \mathcal{T} \text{ such that } [\tau = k] \in \mathcal{F}_k \text{ for all } k \in \mathcal{T}.$$

The filtration $\{\mathcal{G}_t\}$ induced by the stopping time τ

$$\mathcal{G}_t := \{A \in \mathcal{F}_t : A \cap [\tau = k] \in \mathcal{F}_k \text{ for all } k \in \mathcal{T}\} \text{ for } t \in \mathcal{T}$$

is sequentially connected. Appendix 4.6 shows that sequentially connected filtrations inherit the lattice structure of stopping times. Specifically, the supremum (infimum) of sequentially connected filtrations is sequentially connected.

Sequentially connected filtrations arise as the outcome of a satisficing procedure for simplifying decision trees proposed by Gabaix and Laibson [7] in a setting of objective uncertainty.

Example 3 *(Satisficing) The individual ignores branches of the decision tree whose probability is lower than some threshold $\alpha \in [0, 1]$.*[4]

The Gabaix and Laibson [7] procedure leads to a parsimonious parametrization of sequentially connected filtrations using the single threshold parameter α.

The class of sequentially connected filtrations *excludes* the following subfiltration:

$$\mathcal{G} = \{\mathcal{F}_0, \mathcal{F}_0, ..., \mathcal{F}_0, \mathcal{F}_T\}$$

In this example, the individual *foresees all* possible contingencies ($\mathcal{G}_T = \mathcal{F}_T$) but *delays* the resolution of uncertainty: she believes erroneously that all information is revealed at the terminal node. This filtration does not capture a coarse perception of the environment and is precluded by Definition 3.

[4] Appendix 4.6 provides a detailed translation of the Gabaix-Laibson definition into the setting of this paper.

2.3.2 Subjective Perception of Acts

If the individual has a coarse perception of the state space, the outcomes of many actions depend on contingencies the individual does not foresee. This section introduces a class of mappings which model the individual's perception of such 'nonmeasurable' actions. A function Φ in the class maps each *objective act* h into a *perceived, subjectively measurable act* $\Phi(h)$.

Definition 4 *A continuous, monotone and concave function* $\Phi : \mathcal{H} \to \mathcal{H}_{\mathcal{G}}$ *is a* $\mathcal{G}$***-approximation mapping*** *if:*

(i) Φ *is* $\mathcal{G}$*-additive:* $\Phi(\alpha h + g) = \alpha\Phi(h) + g$*, for all* $h \in \mathcal{H}$ *and* $g \in \mathcal{H}_{\mathcal{G}}$*,*

(ii) Φ *is separable:* $h_t|A = h'_t|A$ *implies* $(\Phi h)_t|A = (\Phi h')_t|A$*,*

for all t *and* $A \in \mathcal{G}_t$*.*

Approximation from below provides a simple example of a $\mathcal{G}$-approximation mapping.

Example 4 *(Approximation From Below) For every* $h \in \mathcal{H}$ *and* $t \in \mathcal{T}$*, let* $(\Phi h)_t$ *be the lower* $\mathcal{G}_t$*-measurable envelope of* h*:*

$$(\Phi h)_t|A = \min_{\omega\in A} h_t(\omega) \text{ for every } t \text{ and } A \in \Pi_{\mathcal{G}_t}.$$

Then, Φ *is a* $\mathcal{G}$*-approximation mapping.*

More generally, the subjective act $(\Phi h)_t$ is bounded by the lower and upper $\mathcal{G}_t$-measurable envelopes of the objective act:

$$\min_{\omega\in A} h_t(\omega) \leq (\Phi h)_t|A \leq \max_{\omega\in A} h_t(\omega)$$

for every act h, period t and event $A \in \Pi_{\mathcal{G}_t}$. Two properties of $\mathcal{G}$-approximation mappings stand out. Concavity models an individual whose perception of nonmeasurable acts is conservative. In particular, the upper envelope is not a $\mathcal{G}$-approximation mapping. Separability requires that the perception of an act within any subjective event A does not depend on the perception of the act outside of A. In conjunction with concavity, separability implies that the mapping Φ captures aversion to uncertainty *within the events* of the subjective filtration.

The properties of Φ imply the following convenient parametrization. The result is a corollary of Gilboa and Schmeidler [9, Theorem 1].

Lemma 1 *Φ is a $\mathcal{G}$-approximation mapping if and only if for every t and every $A \in \Pi_{\mathcal{G}_t}$ there exist a nonempty, closed, convex subset $\mathcal{C}_A$ of $\Delta(A, \mathcal{F}_t)$ such that:*

$$(\Phi h)_t|_A = \min_{p \in \mathcal{C}_A} \int_A h_t \; dp. \tag{2.2}$$

2.3.3 Representation Theorem

The static model of limited foresight is defined next.

Definition 5 *A preference relation $\succeq$ on $\mathcal{H}$ has a **limited foresight representation** $(\mathcal{G}, \Phi, \mathcal{C})$ if it admits a utility function of the form:*

$$V(h) = \min_{p \in \mathcal{C}} \int_\Omega \sum_t \beta^t (\Phi h)_t \; dp, \tag{2.3}$$

where the subjective filtration $\mathcal{G}$ is sequentially connected, Φ is a $\mathcal{G}$-approximation mapping and $\mathcal{C}$ is a closed, convex subset of $\Delta^\circ(\Omega, \mathcal{G}_T)$.[5]

The filtration $\mathcal{G}$ and the mapping Φ characterize the individual's perception of the environment: the contingencies she foresees and her subjective perception of acts. The nonsingleton set of priors $\mathcal{C}$ reflects her awareness that this perception may be incomplete.

Theorem 2 *The preference $\succeq$ satisfies Basic, Convexity, Strong Certainty Independence, Stationarity, Sequentiality and Nonnullity if and only if it has a limited foresight representation $(\mathcal{G}, \Phi, \mathcal{C})$. Moreover, the latter is unique.*

To apply the model, it is necessary to specify the components of the utility function in (2.3). A difficulty is that the definition of a limited foresight model requires that $\mathcal{G}$ is the subjective filtration of the induced preference. This is not easily verifiable. Therefore, the next lemma provides sufficient conditions on the mapping Φ which guarantee that $\mathcal{G}$ is indeed the subjective filtration. Recall that by Lemma 1, a $\mathcal{G}$-approximation mapping Φ may be identified with a collection $\{\mathcal{C}_A\}$ where $\mathcal{C}_A$ is a set of priors and the event A varies over cells of the filtration $\mathcal{G}$.

[5] For any algebra $\mathcal{G}$, $\Delta^\circ(\Omega, \mathcal{G})$ denotes the subset of probability measures p in $\Delta(\Omega, \mathcal{G})$ such that $p(A) > 0$ for all $A \in \mathcal{G}$.

Lemma 3 *If $\mathcal{G}$ is sequentially connected and $\mathcal{C}_A$ has nonempty interior in $\Delta(A, \mathcal{F}_t)$ for every t and every A in $\Pi_{\mathcal{G}_t}$, then $(\mathcal{G}, \{\mathcal{C}_A\}, \mathcal{C})$ is a limited foresight model.*

The next examples provide some tractable specifications of Φ which satisfy the sufficient conditions.

Example 5 *(Approximation From Below) $(\Phi h)_t$ is the lower $\mathcal{G}_t$-measurable envelope of h_t for every t. Then $(\mathcal{G}, \Phi, \mathcal{C})$ is a limited foresight model.*

An important feature of Example 5 is that the model is fully specified by the filtration $\mathcal{G}$ and the set of priors $\mathcal{C}$. A drawback is the 'coarseness' of the approximation mapping. In terms of the representation derived in Lemma 1, each set $\mathcal{C}_A$ in the construction of Φ equals the entire simplex $\Delta(A)$. To provide a less extreme approximation, one may take the set $\mathcal{C}_A$ to be an ϵ-contraction of the simplex around a focal measure p^*:

$$\mathcal{C}_A := \{\epsilon p + (1 - \epsilon)p^* : p \in \Delta(A)\}$$

The corresponding mapping Φ_ϵ is determined by the filtration $\mathcal{G}$ and the single parameter $0 < \epsilon < 1$.

Example 6 *(ϵ-Contamination) For every $t \in \mathcal{T}$ and every $A \in \Pi_{\mathcal{G}_t}$, let $\mathcal{C}_A$ be an ϵ-contraction of the simplex. Then $(\mathcal{G}, \Phi_\epsilon, \mathcal{C})$ is a limited foresight model.*

3 Dynamic Model

3.1 Axiomatic Framework

This section develops the dynamic model of limited foresight. The primitive is an $\mathcal{F}$-adapted process of conditional preferences $\{\succeq_{t,\omega}\}$ where $\succeq_{t,\omega}$ describes the ranking of acts in state ω at time t.

To emphasize learning, the first axiom postulates that the individual knows the realized history of events. That is, she observes and becomes aware of any contingency that takes place.

Axiom 8 *(Consequentialism) For each t and ω and all acts h, h',*

$$h_\tau(\omega') = h'_\tau(\omega') \text{ for all } \tau \geq t \text{ and } \omega' \in \mathcal{F}_t(\omega) \text{ implies } h \sim_{t,\omega} h'.$$

Consequentialism restricts the scope of limited foresight modeled in this paper. The stated indifference requires that any acts whose possible continuations are *objectively* identical are also *perceived* as identical. Intuitively, perception may be incomplete, but not delusional. The individual does not imagine differences when there are none.

The next axiom requires that *tastes* do not depend on time and the state of the world. As in the static model, limited foresight pertains to the perception of uncertainty and has no implications for the evaluation of deterministic acts.

Axiom 9 *(State Independence) For each t and ω,*

$$(x_0, ..., x_{t-1}, y_t, ..., y_T) \succeq_{t,\omega} (x_0, ..., x_{t-1}, y'_t, ..., y'_T) \text{ if and only if}$$

$$(x_0, ..., x_{t-1}, y_t, ..., y_T) \succeq_0 (x_0, ..., x_{t-1}, y'_t, ..., y'_T).$$

To understand the next axiom, consider a simple example when one of three events A_1, A_2 and A_3 can be realized tomorrow. An individual with perfect foresight evaluates all acts consistently:

$$h \succeq_{1,A_i} h' \text{ for all } i \text{ implies } h \succeq_0 h'. \tag{3.1}$$

If she were to learn $A_1 \cup A_2$ at some hypothetical intermediate stage τ, then intertemporal consistency would similarly imply:

$$h \succeq_{1,A_1} h' \text{ and } h \succeq_{1,A_2} h' \text{ implies } h \succeq_{\tau,A_1 \cup A_2} h'. \tag{3.2}$$

Preferences (3.1) and (3.2) reflect an individual who plans consistently. She knows all possible contingencies and anticipates accurately the future choices she is going to make. This knowledge is incorporated in her behavior in the initial period $t = 0$.

To see the implications of limited foresight, consider an individual who initially foresees the events $A_1 \cup A_2$ and A_3 only. Thinking of the future, she contemplates behavior conditional on the events $A_1 \cup A_2$ and A_3 that she foresees. If $\succeq^a_{1,A_1\cup A_2}$ and $\succeq^a_{1,A_3}$ represent these *anticipated preferences*, then:

$$h \ \succeq^a_{1,A_1\cup A_2} \ h' \text{ and } h \ \succeq^a_{1,A_3} \ h' \text{ implies } h \succeq_0 h'. \tag{3.3}$$

As in (3.1) and (3.2), preferences (3.3) describe an individual who is forward-looking and plans ahead. However, the anticipated preferences in (3.3) reflect prior foresight and may differ from *actual* future behavior. Specifically, $\succeq^a_{1,A_1\cup A_2}$ represents behavior if the individual were to learn the event $A_1 \cup A_2$ and *perceive* the world as she does at $t = 0$. In contrast, the preference $\succeq_{\tau,A_1\cup A_2}$ in (3.2) represents behavior if she learnt $A_1 \cup A_2$ and *perceived* the world as she does at $t = 1$.

These perceptions necessarily coincide only when the acts h and h' are subjectively measurable at $t = 0$. Then (3.2) and (3.3) imply:

$$\begin{aligned} h \succeq_{1,A_i} h' \text{ for all } i \quad &\Rightarrow \quad h \succeq_{\tau,A_1\cup A_2} h' \text{ and } h \succeq_{\tau,A_3} h' \\ &\Rightarrow \quad h \succeq^a_{1,A_1\cup A_2} h' \text{ and } h \succeq^a_{1,A_3} h' \\ &\Rightarrow \quad h \succeq_0 h'. \end{aligned}$$

This implication motivates the next axiom. It formalizes the general approach to modeling sophisticated dynamic behavior when some contingencies are unforeseen: the individual is forward-looking and revises her plans *only when* unanticipated circumstances contradict her perception. The axiom is illustrated in the introductory example.

Axiom 10 *(Weak Dynamic Consistency) For each t and ω and for all acts g, g' in $\mathcal{H}_{\mathcal{G}^{t,\omega}}$ such that $g_\tau = g'_\tau$ for all $\tau \leq t$,*

$$g \succeq_{t+1,\omega'} g' \textit{ for all } \omega' \textit{ implies } g \succeq_{t,\omega} g'.$$

3.2 Representation

Assuming that each conditional preference $\succeq_{t,\omega}$ has a limited foresight representation $(\mathcal{G}^{t,\omega}, \Phi^{t,\omega}, \mathcal{C}^{t,\omega})$, this section derives the process of learning over time implied by Consequentialism, State Independence and Weak Dynamic

Consistency. The objective is to characterize how the individual's perception of the environment and her beliefs evolve over time.

The first implication captures a notion of expanding foresight. Thus, for every t and ω, the posterior filtration $\mathcal{G}^{t+1,\omega}$ or 'perception tomorrow' refines the prior filtration $\mathcal{G}^{t,\omega}$ or 'perception today'. To state this formally, let $\mathcal{G}^{t,\omega} \cap \mathcal{F}_{t+1}(\omega)$ denote the restriction of the prior filtration $\mathcal{G}^{t,\omega}$ to the subtree emanating from the event $\mathcal{F}_{t+1}(\omega)$. The latter event is realized and thus known by the individual at period $t+1$ and state ω.[6]

Definition 6 *A process of filtrations $\{\mathcal{G}^{t,\omega}\}$ is* ***refining*** *if $\mathcal{G}^{t+1,\omega}$ refines $\mathcal{G}^{t,\omega} \cap \mathcal{F}_{t+1}(\omega)$ for all t and ω.*

The next implication characterizes the evolution of conditional beliefs $\{\mathcal{C}^{t,\omega}\}$. Some preliminary definitions are necessary. For a set of priors $\mathcal{C}$ on the objective algebra $\mathcal{F}_T$, define the set of Bayesian updates by

$$\mathcal{C}_t(\omega) := \{p(\cdot \mid \mathcal{F}_t(\omega)) : p \in \mathcal{C}\}$$

and define the set of conditional one-step-ahead measures by

$$\mathcal{C}_t^{+1}(\omega) := \{\mathrm{marg}_{\mathcal{F}_{t+1}} p : p \in \mathcal{C}_t(\omega)\}.$$

The following definition generalizes the familiar decomposition of a measure in terms of its conditionals and marginals to the decomposition of a set of measures $\mathcal{C}$. The requirement is studied in Epstein and Schneider [5] who discuss its role for modeling dynamically consistent behavior when the individual has more than a single prior.

Definition 7 *A set $\mathcal{C}$ is $\{\mathcal{F}_t\}$-rectangular if for all t and ω,*

$$\mathcal{C}_t(\omega) = \{\int_\Omega p_{t+1}(\omega')\, dm : p_{t+1}(\omega') \in \mathcal{C}_{t+1}(\omega') \text{ for all } \omega' \text{ and } m \in \mathcal{C}_t^{+1}(\omega)\}.$$

The main feature of Definition 7 is that the decomposition on the right combines a marginal from $\mathcal{C}_t^{+1}(\omega)$ with *any* measurable selection of conditionals. This will typically involve 'foreign' conditionals. If the set $\mathcal{C}$ is a

[6] For every subset A of Ω, an algebra $\mathcal{G}$ on Ω induces the algebra $\mathcal{G} \cap A := \{B \cap A : B \in \mathcal{G}\}$ on A. A filtration $\{\mathcal{G}_t\}$ induces the filtration $\{\mathcal{G}_t\} \cap A := \{\mathcal{G}_t \cap A\}$ on A.

singleton, there are no foreign conditionals and the definition of rectangularity reduces to the standard decomposition of a probability measure.

The next property completes the description of learning under limited foresight. To understand the definition, consider the case when all conditional beliefs $\mathcal{C}^{t,\omega}$ are singleton sets. The definition posits the existence of a 'shadow' probability measure $\mathcal{C}$ defined on the objective algebra $\mathcal{F}_T$. For every t and ω, the conditional belief $\mathcal{C}^{t,\omega}$ is the Bayesian update of the shadow measure, restricted to the respective collection of foreseen events $\mathcal{G}^{t,\omega}$.

Definition 8 *A process* $\{\mathcal{C}^{t,\omega}, \mathcal{G}^{t,\omega}\}$ ***admits a consistent extension*** *if there exists an* $\{\mathcal{F}_t\}$*-rectangular, closed and convex subset* $\mathcal{C}$ *of* $\Delta^\circ(\Omega, \mathcal{F}_T)$ *such that*

$$\mathcal{C}^{t,\omega} = \{\text{marg}_{\mathcal{G}^{t,\omega}} p : p \in \mathcal{C}_t(\omega)\} \textit{ for each } t \textit{ and } \omega.$$

The next theorem shows that Consequentialism, State Independence and Weak Dynamic Consistency are necessary and sufficient for the process of learning described by Definitions 6 and 8.

Theorem 4 *A family* $\{\succeq_{t,\omega}\}$ *of limited foresight preferences satisfies Consequentialism, State Independence and Weak Dynamic Consistency if and only if* $\{\mathcal{G}^{t,\omega}\}$ *is refining and* $\{\mathcal{C}^{t,\omega}, \mathcal{G}^{t,\omega}\}$ *admits a consistent extension* $\mathcal{C}$*.*

The axioms characterizing the dynamic model describe an individual who plans ahead but also adapts to unforeseen contingencies. These properties of behavior are complementary: planning ahead 'leads' to intertemporal consistency while adaptation negates it. Both have a respective representation in the process of learning characterized by Theorem 4.

Intertemporal consistency is reflected in the existence of the extension $\mathcal{C}$ that consists of beliefs defined on the *objective* algebra. The rectangularity of the set implies that the corresponding process of Bayesian updates $\{\mathcal{C}_t(\omega)\}$ defines a 'shadow' dynamically consistent model. Beliefs $\{\mathcal{C}^{t,\omega}\}$ in the limited foresight model are marginals of the latter and inherit many of the properties of the shadow model. This permits the application of a wide range of results from (robust) Bayesian inference to the process $\{\mathcal{C}^{t,\omega}\}$.

Adaptation is reflected in the process of filtrations $\{\mathcal{G}^{t,\omega}\}$ and the corresponding mappings $\{\Phi^{t,\omega}\}$. Changes in these components describe the expanding foresight of the individual and augment the dynamics of Bayesian learning captured by the process of beliefs $\{\mathcal{C}^{t,\omega}\}$. This new source of dynamics may be particularly useful in applications.

To complete the description of the dynamic model, the next theorem proves that the consistent extension $\mathcal{C}$ is unique whenever the individual foresees all one-step-ahead contingencies.

Theorem 5 *If $\mathcal{F}_{t+1}(\omega) \in \mathcal{G}^{t,\omega}$ for all t and ω, the consistent extension $\mathcal{C}$ provided by Theorem 4 is unique.*

3.3 Foresight and Dynamic Consistency

Kreps [13, p.278] conjectured that dynamic behavior may reveal the collection of foreseen events and provide a foundation for separating limited foresight from existing models of uncertainty. The next theorem establishes a close connection between the intertemporal consistency of behavior and the static ranking of effectively certain acts. As a corollary, this section shows that dynamic behavior provides an equivalent and alternative characterization of the process of subjective filtrations $\{\mathcal{G}^{t,\omega}\}$.

Theorem 6 *If a family of preferences $\{\succeq_{t,\omega}\}$ satisfies Consequentialism, State Independence and Dynamic Consistency, then*

$$h(\omega) \sim_0 h(\omega') \text{ for all } \omega, \omega' \in \Omega \text{ implies } h \sim_0 h(\omega). \tag{3.4}$$

Theorem 6 proves that *recursive models* imply *perfect foresight* as defined by the ranking of effectively certain acts. To gain some intuition for this result, imagine that you observe conditional preferences at every node and find that behavior is dynamically consistent. If the latter is also consequentialist, you know that the individual observes and recognizes events that transpire. The consistency of her behavior then 'reveals' that the individual has foreseen all possible changes of the environment and incorporated them into her plans. This implication is captured formally in Theorem 6 where dynamic consistency implies foresight as defined by the ranking of effectively certain acts.

To extend this result to the dynamic model of limited foresight, a definition is necessary. Below, $\{\widehat{\mathcal{G}}^{t,\omega}\}$ denotes an $\mathcal{F}$-adapted process of filtrations. The class of such processes are ordered pointwise.

Definition 9 *A family of posterior preferences is* ***dynamically consistent relative to*** $\{\widehat{\mathcal{G}}^{t,\omega}\}$ *if for each* t *and* ω *and all* $\widehat{\mathcal{G}}^{t,\omega}$*-adapted acts* g, g'*:*

$$g \succeq_{t+1,\omega'} g' \text{ for all } \omega' \in \mathcal{F}_t(\omega) \text{ implies } g \succeq_{t,\omega} g'.$$

The next corollary proves that the process of subjective filtrations $\{\mathcal{G}^{t,\omega}\}$ is the largest refining, regular process relative to which a limited foresight model $\{\succeq_{t,\omega}\}$ is dynamically consistent.[7] The result provides an alternative characterization of the collection of foreseen events based solely on the intertemporal consistency of dynamic behavior.

Corollary 7 *If* $\{\succeq_{t,\omega}\}$ *is a dynamic model of limited foresight, then the process of subjective filtrations* $\{\mathcal{G}^{t,\omega}\}$ *is the largest refining, regular process relative to which* $\{\succeq_{t,\omega}\}$ *is dynamically consistent.*

The equivalence of the static and dynamic characterizations implies that the key to identifying the collection of foreseen events is provided the temporal domain of choice. Within this domain, static and dynamic properties of behavior are linked by Theorem 6 and lead to equivalent characterizations.

4 Appendix

A filtration $\mathcal{G}$ on Ω is identified with the algebra on $\Omega \times \mathcal{T}$ generated by sets of the form $A \times \{t\}$ for $A \in \mathcal{G}_t$ and $t \in \mathcal{T}$. Under this identification, an act h is $\mathcal{G}$-adapted if and only if it is $\mathcal{G}$-measurable as a mapping from $\Omega \times \mathcal{T}$ to the set of real numbers $\mathbb{R}$.

4.1 Proof of Theorem 2

Adopt the arguments in Epstein and Schneider [5, Lemma A.1] to deduce that $\succeq$ has a representation:

$$U(h) = \min_{q \in Q} \sum_t \beta^t \langle q_t, h_t \rangle . \tag{4.1}$$

[7] A subfiltration $\widehat{\mathcal{G}}$ is regular if $\widehat{\mathcal{G}}_T = \widehat{\mathcal{G}}_{T-1}$. A process is regular if each subfiltration is regular. By the richness assumption $\mathcal{F}_T = \mathcal{F}_{T-1}$, all sequentially connected subfiltrations are regular.

Above, Q is a closed, convex subset of $\times_{t\in\mathcal{T}}\Delta(\Omega,\mathcal{F}_t)$. Denote a generic element in Q by $q := (q_t)_{t\in\mathcal{T}}$. For every subset $\mathcal{T}'\subset\mathcal{T}$, $proj_{\mathcal{T}'}(q)$ denotes the vector $(q_t)_{t\in\mathcal{T}'}$. Without loss of generality, set $\beta = 1$ and define $\langle q,h\rangle := \sum_t \langle q_t,h_t\rangle$.

A subfiltration $\mathcal{G}$ of $\mathcal{F}$ defines a subspace of $\times_{t\in\mathcal{T}} ba(\Omega,\mathcal{F}_t)$:

$$diag(\mathcal{G}) := \{q\in\times_{t\in\mathcal{T}} ba(\Omega,\mathcal{F}_t) : \mathrm{marg}_{\mathcal{G}_t} q_{t+1} = \mathrm{marg}_{\mathcal{G}_t} q_t \text{ for all } t<T\}$$

Note that $diag(\mathcal{G})\neq diag(\mathcal{G}')$ whenever $\mathcal{G}_t\neq\mathcal{G}'_t$ for some $t<T$. Equivalently, there exists a bijection between the diagonals of $\times_{t\in\mathcal{T}} ba(\Omega,\mathcal{F}_t)$ and the set of subfiltrations $\mathcal{G}$ such that $\mathcal{G}_T=\mathcal{G}_{T-1}$. Call such subfiltrations *regular*. For a regular subfiltration $\mathcal{G}$, the following lemma establishes a basic duality between the diagonal $diag(\mathcal{G})$ and the set of effectively certain acts in $\mathcal{H}_{\mathcal{G}}$. In view of (4.1), it is convenient to normalize the set of certain acts:

$$\mathcal{H}^c := \{h\in\mathcal{H} : \textstyle\sum_t h_t(\omega)=0 \text{ for all } \omega\in\Omega\}$$

Lemma 8 *$q\in diag(\mathcal{G})$ if and only if $\langle q,h\rangle = 0$ for all $h\in\mathcal{H}_{\mathcal{G}}\cap\mathcal{H}^c$.*

Proof: To prove necessity observe that for any $h\in\mathcal{H}_{\mathcal{G}}$ and $q\in diag(\mathcal{G})$,

$$\langle q,h\rangle = \textstyle\sum_t \langle q_t,h_t\rangle = \sum_t \langle q_T,h_t\rangle = \langle q_T,\sum_t h_t\rangle . \tag{4.2}$$

If $h\in\mathcal{H}_{\mathcal{G}}\cap\mathcal{H}^c$, then $\sum_t h_t(\omega)=0$ for all ω and $\langle q,h\rangle = \langle q_T,\sum_t h_t\rangle = 0$ for all $q\in diag(\mathcal{G})$.

To establish sufficiency, first prove that for any $h\in\mathcal{H}_{\mathcal{G}}$

$$h\in\mathcal{H}^c \text{ if and only if } \langle q,h\rangle = 0 \text{ for all } q\in diag(\mathcal{G}). \tag{4.3}$$

Sufficiency of (4.3) follows (4.2). To see the reverse implication, fix some $h\in\mathcal{H}_{\mathcal{G}}\backslash\mathcal{H}^c$ and without loss of generality suppose that $\sum_t h_t(\omega)>0$ for some ω. Let q_T be a measure in $\Delta(\Omega,\mathcal{G}_T)$ such that $q_T(\mathcal{G}_T(\omega))=1$. Since $h\in\mathcal{H}_{\mathcal{G}}$, $\langle q_T,\sum_t h_t\rangle$ is well-defined and strictly positive. Extend q_T to a vector $q\in diag(\mathcal{G})$ and note that $\langle q,h\rangle = \langle q_T,\sum_t h_t\rangle > 0$ proving the necessity of (4.3).

To complete the proof of the lemma, fix some $q'\notin diag(\mathcal{G})$. It suffices to find an act $h'\in\mathcal{H}_{\mathcal{G}}\cap\mathcal{H}^c$ such that $\langle q',h'\rangle\neq 0$. Since $diag(\mathcal{G})$ is a subspace of $\times_{t\in\mathcal{T}} ba(\Omega, \mathcal{G}_t)$ and any subspace is the intersection of the hyperplanes that contain it, there exists an act $h' \in \mathcal{H}_{\mathcal{G}}$ such that

$$\langle q', h' \rangle \neq 0 \text{ and } \langle q, h' \rangle = 0 \text{ for all } q \in diag(\mathcal{G}).$$

By (4.3), the act h' lies in $\mathcal{H}_{\mathcal{G}} \cap \mathcal{H}^c$, as desired.■

Lemma 9 *For every closed set $Q \subset \times_{t\in\mathcal{T}} \Delta(\Omega, \mathcal{G}_t)$,*

$$Q \subset diag(\mathcal{G}) \textit{ if and only if } \min_{q\in Q} \langle q, h \rangle = 0 \textit{ for all } h \in \mathcal{H}_{\mathcal{G}} \cap \mathcal{H}^c.$$

Proof: Sufficiency follows directly from Lemma 8. To see necessity, suppose there exists some $q' \in Q \backslash diag(\mathcal{G})$. By Lemma 8, there exists $h' \in \mathcal{H}_{\mathcal{G}} \cap \mathcal{H}^c$ such that $\langle q', h' \rangle \neq 0$. Since

$$h' \in \mathcal{H}_{\mathcal{G}} \cap \mathcal{H}^c \text{ if and only if } -h' \in \mathcal{H}_{\mathcal{G}} \cap \mathcal{H}^c,$$

one can choose h' such that $\min_{q\in Q} \langle q, h' \rangle \leq \langle q', h' \rangle < 0$. This establishes a contradiction.■

For any set Q in $\times_{t\in\mathcal{T}} \Delta(\Omega, \mathcal{F}_t)$, there exists a collection Λ of *maximal* filtrations $\mathcal{G}'$ such that $\mathcal{G}'_T = \mathcal{G}'_{T-1}$ and $Q \subset diag(\mathcal{G}')$. In contrast, there may not exist a *largest* filtration containing Q since $diag(\vee_{\mathcal{G}'\in\Lambda}\mathcal{G}')$ is in general a proper subset of $\cap_{\mathcal{G}'\in\Lambda} diag(\mathcal{G}')$. The next lemmas show that under Strong Certainty Independence and Stationarity:

$$Q \subset diag(\vee_{\mathcal{G}'\in\Lambda}\mathcal{G}') = \cap_{\mathcal{G}'\in\Lambda} diag(\mathcal{G}').$$

Recall that

$$\mathcal{H}^* = \{h \in \mathcal{H}^c : h' \sim h'(\omega) \text{ for all } \omega \in \Omega \text{ and all } h' \in \mathcal{H}^c \cap \mathcal{H}_{\mathcal{F}(h)}\},$$

and $\mathcal{G}^*$ is the algebra on $\Omega \times \mathcal{T}$ induced by $\mathcal{H}^*$.

Lemma 10 *The algebra $\mathcal{G}^*$ on $\Omega \times \mathcal{T}$ is a regular filtration on Ω.*

Proof: First prove that for every $h \in \mathcal{H}^c$, the filtration $\mathcal{F}(h)$ is the smallest algebra on $\Omega \times \mathcal{T}$ induced by $\mathcal{H}^c \cap \mathcal{H}_{\mathcal{F}(h)}$. For every $t < T$, the act

$(\mathbf{0}_{-t,-(t+1)}, \mathbf{1}_\Omega, -\mathbf{1}_\Omega) \in \mathcal{H}^c \cap \mathcal{H}_{\mathcal{F}(h)}$ implying that the smallest algebra contains the set $\Omega \times \{t\}$ for every $t \in \mathcal{T}$. Also,

$$\begin{aligned} h &\in \mathcal{H}^c \Rightarrow h_T = -\textstyle\sum_{\tau<T} h_\tau \in \vee_{\tau \leq T-1} \sigma(h_\tau) \Rightarrow \\ \sigma(h_T) &\leq \vee_{\tau \leq T-1} \sigma(h_\tau). \end{aligned}$$

But then

$$\mathcal{F}(h)_T = \vee_{\tau \leq T} \sigma(h_\tau) = [\vee_{\tau \leq T-1} \sigma(h_\tau)] \vee \sigma(h_T) = \vee_{\tau \leq T-1} \sigma(h_\tau) = \mathcal{F}(h)_{T-1}.$$

Conclude that $\mathcal{F}(h)$ is regular and for all events $A \in \mathcal{F}(h)_T$ and payoffs $x \in M$ (in particular $x \neq 0$):

$$(\mathbf{0}_{-(T-1),-T}, xA^c(-x), xA(-x)) \in \mathcal{H}^c \cap \mathcal{H}_{\mathcal{F}(h)},$$

and, since $\mathcal{F}(h)$ is a filtration, for all $t < T$ and $A \in \mathcal{F}(h)_t$,

$$(\mathbf{0}_{-t,-(t+1)}, xA^c(-x), xA(-x)) \in \mathcal{H}^c \cap \mathcal{H}_{\mathcal{F}(h)}.$$

The above inclusions imply that $\mathcal{F}(h)$ is the smallest algebra induced by $\mathcal{H}^c \cap \mathcal{H}_{\mathcal{F}(h)}$.

Finally, fix $h \in \mathcal{H}^*$, $h' \in \mathcal{H}^c \cap \mathcal{H}_{\mathcal{F}(h)}$, and $h'' \in \mathcal{H}^c \cap \mathcal{H}_{\mathcal{F}(h')}$. Since $\mathcal{F}(h') \subset \mathcal{F}(h)$, $h'' \in \mathcal{H}^c \cap \mathcal{H}_{\mathcal{F}(h)}$. By the choice of h, the latter implies $h'' \sim h''(\omega)$ for all ω. Conclude that $h' \in \mathcal{H}^*$. But then

$$\begin{aligned} \mathcal{H}^* &= \cup_{h \in \mathcal{H}^*} [\mathcal{H}^c \cap \mathcal{H}_{\mathcal{F}(h)}] \Rightarrow \\ \mathcal{G}^* &= \vee_{h \in \mathcal{H}^*} \mathcal{F}(h) \end{aligned}$$

Since the supremum of regular filtrations is a regular filtration, the lemma is proved.■

Lemma 11 *$Q \cap diag(\mathcal{G}^*) \neq \varnothing$.*

Proof: Let $Q' = \{(\text{marg}_{\mathcal{G}_t^*} q_t)_{t \in \mathcal{T}} : (q_t)_{t \in \mathcal{T}} \in Q\}$ and define the linear functional

$$\phi : (q_t)_{t>0} \longmapsto (marg_{\mathcal{G}_{t-1}^*} q_t)_{t>0}.$$

Consider the following subdomains of acts

$$\begin{aligned} D_{\mathcal{T} \setminus \{T\}} &: = \{(h_0, ..., h_{T-1}, x_0) \in \mathcal{H}_{\mathcal{G}^*}\}, \\ D_{\mathcal{T} \setminus \{0\}} &: = \{(x_0, h_0, ..., h_{T-1}) : (h_0, ..., h_{T-1}, x_0) \in \mathcal{H}_{\mathcal{G}^*}\} \end{aligned}$$

Under the obvious identification, $D_{\mathcal{T}\setminus\{T\}} = D_{\mathcal{T}\setminus\{0\}}$. The restrictions of $\succeq$ to $D_{\mathcal{T}\setminus\{T\}}$ and $D_{\mathcal{T}\setminus\{0\}}$, respectively, are represented by the following utility functions:

$$\begin{aligned} U_{D_{\mathcal{T}\setminus\{T\}}} &= \min_{q\in proj_{\mathcal{T}\setminus\{T\}}Q'} \textstyle\sum_{t<T} \langle q_t, h_t\rangle \\ U_{D_{\mathcal{T}\setminus\{0\}}} &= \min_{q\in \phi\circ proj_{\mathcal{T}\setminus\{0\}}Q'} \textstyle\sum_{t>0} \langle q_t, h_t\rangle \end{aligned}$$

By Stationarity, $U_{D_{\mathcal{T}\setminus\{T\}}}$ and $U_{D_{\mathcal{T}\setminus\{0\}}}$ represent the same preference relation. [9, Theorem 1] implies that

$$proj_{\mathcal{T}\setminus\{T\}}Q' = \phi\circ proj_{\mathcal{T}\setminus\{0\}}Q' =: K$$

Define the correspondence

$$\psi := \phi\circ proj_{\mathcal{T}\setminus\{0\}}\circ\left(Q'\cap proj^{-1}_{\mathcal{T}\setminus\{T\}}\right) : K \rightrightarrows K$$

Since Q' is closed, ψ is the composition of a continuous function and an upper hemicontinuous correspondence. Thus ψ is upper hemicontinuous. Since Q' is convex, ψ is also convex-valued. By the Kakutani fixed point theorem [1, Corollary 16.51], ψ has a fixed point $q\in\psi(q)$. Equivalently, there exists a point $(q_0, q_1, ..., q_{T-1}, q_T)\in Q'$ such that

$$\begin{aligned} \phi(q_1, ..., q_T) &= (q_0, q_1, ..., q_{T-1}) \\ &\Leftrightarrow \\ marg_{\mathcal{G}^*_{t-1}} q_t &= q_{t-1}, \forall t>1. \blacksquare \end{aligned}$$

Lemma 12 *$\mathcal{H}^c\cap\mathcal{H}_{\mathcal{G}^*} = \mathcal{H}^*$ and $\mathcal{G}^*$ is the largest regular filtration $\mathcal{G}$ such that $Q\subset diag(\mathcal{G})$.*

Proof: By construction, $\mathcal{H}^*\subset\mathcal{H}^c\cap\mathcal{H}_{\mathcal{G}^*}$. To see the reverse inclusion, fix $h\in\mathcal{H}^c\cap\mathcal{H}_{\mathcal{G}^*}$ and let $x\sim(-h)$. Strong Certainty Independence implies that $\frac{1}{2}x+\frac{1}{2}h\sim\frac{1}{2}(-h)+\frac{1}{2}h$. The two indifferences imply

$$\begin{aligned} x &= -\frac{1}{T+1}\max_{q\in Q}\textstyle\sum_t \langle q_t, h_t\rangle, \\ x &= -\frac{1}{T+1}\min_{q\in Q}\textstyle\sum_t \langle q_t, h_t\rangle. \end{aligned}$$

Conclude that for every $h \in \mathcal{H}^c \cap \mathcal{H}_{\mathcal{G}^*}$, $\langle q, h \rangle = \langle q', h \rangle$ for every $q, q' \in Q$. By Lemma 11, there exists a $q \in Q \cap diag(\mathcal{G}^*)$. It follows that $\langle q, h \rangle = 0$ for every $q \in Q$ and so $\mathcal{H}^c \cap \mathcal{H}_{\mathcal{G}^*} \subset \mathcal{H}^*$. Also by Lemma 9, $Q \subset diag(\mathcal{G}^*)$.

If $\mathcal{G}$ is any filtration such that $\mathcal{G}_T = \mathcal{G}_{T-1}$ and $Q \subset diag(\mathcal{G})$, Lemma 9 implies that

$$\mathcal{H}^c \cap \mathcal{H}_{\mathcal{G}} \subset \mathcal{H}^* = \mathcal{H}^c \cap \mathcal{H}_{\mathcal{G}^*}$$

Conclude that $\mathcal{G} \subset \mathcal{G}^*$.■

4.1.1 Properties of the Filtration $\mathcal{G}^*$

Lemma 13 *$\mathcal{G}^*$ is connected.*

Proof: By Lemma 10, $\mathcal{G}^*$ is a filtration. Thus, $\mathcal{G}_t^* \subset \mathcal{G}_T^* \cap \mathcal{F}_t$ for all t. Conversely, fix an event $A \in \mathcal{G}_T^* \cap \mathcal{F}_t$ for some $t \in \mathcal{T}$. Since $\mathcal{G}^*$ is regular by Lemma 10, $\mathcal{G}_{T-1}^* = \mathcal{G}_T^* \ni A$ for $t = T-1$. For $t < T-1$, it suffices to show that $(\mathbf{0}_{-t,-(t+1)}, xAy, (-x)A(-y) \sim \mathbf{0}$ for all $x, y \in M$. By the regularity of $\mathcal{G}^*$, $(\mathbf{0}_{-(T-1),-T}, xAy, (-x)A(-y) \in \mathcal{H}^c \cap \mathcal{H}_{\mathcal{G}^*}$, which implies that $(\mathbf{0}_{-(T-1),-T}, xAy, (-x)A(-y) \sim \mathbf{0}$. Applying Stationarity repeatedly, conclude that $(\mathbf{0}_{-t,-(t+1)}, xAy, (-x)A(-y) \sim \mathbf{0}$.■

Lemma 14 *$\mathcal{G}^*$ is sequentially connected.*

Proof: Fix an event $A \in \Pi_{\mathcal{G}_t} \backslash \Pi_{\mathcal{F}_t}$ for some $t < T$. By way of contradiction, suppose there exists a set $\varnothing \neq B \in \mathcal{G}_{t+1}$ such that $B \subsetneq A$. First, suppose $B \subset C \subsetneq A$ for some $C \in \Pi_{\mathcal{F}_t}$. Since $\mathcal{G}$ is connected, conclude that $B \subsetneq C$. Otherwise, $B = C \in \mathcal{G}_{t+1} \cap \Pi_{\mathcal{F}_t}$ implies that $C \in \Pi_{\mathcal{G}_t}$ contradicting the choice of A. Now take the acts $g = (\mathbf{0}_{-t,-(t+1)}, \mathbf{1}_A, \mathbf{1}_B)$ and $g' = (\mathbf{0}_{-t,-(t+1)}, \mathbf{1}_A, \mathbf{1}_C)$. By construction, $g \in \mathcal{H}_{\mathcal{G}}$ and g' simplifies g at $C \in \Pi_{\mathcal{F}_t}$. By Sequential Thinking, $g' \in \mathcal{H}_{\mathcal{G}}$ and so $C \in \mathcal{G}_{t+1} \cap \Pi_{\mathcal{F}_t}$. Since $\mathcal{G}$ is connected, $C \in \Pi_{\mathcal{G}_t}$ contradicting $C \subsetneq A \in \Pi_{\mathcal{G}_t}$.

Conversely, suppose $B \cap C \neq \varnothing$ and $B \cap C^c \neq \varnothing$ for some $C \in \Pi_{\mathcal{F}_t}$. The acts $g = (\mathbf{0}_{-t,-(t+1)}, \mathbf{1}_A, \mathbf{1}_B)$ is $\mathcal{G}^*$-measurable and $g' = (\mathbf{0}_{-t,-(t+1)}, \mathbf{1}_A, \mathbf{1}_{B \cup C})$ simplifies g at $C \in \Pi_{\mathcal{F}_t}$. By Sequential Thinking, $B \cup C \in \mathcal{G}_{t+1}$. But $B \cup C \in \mathcal{G}_{t+1}$ and $B \in \mathcal{G}_{t+1}$ imply $C \backslash B = B \cup C \backslash B \in \mathcal{G}_{t+1}$ and, by construction, $\varnothing \neq C \backslash B \subsetneq C \subsetneq A$ and $C \in \Pi_{\mathcal{F}_t}$. But then $g' = (\mathbf{0}_{-t,-(t+1)}, \mathbf{1}_A, \mathbf{1}_C)$ simplifies $g = (\mathbf{0}_{-t,-(t+1)}, \mathbf{1}_A, \mathbf{1}_{C \backslash B}) \in \mathcal{H}_{\mathcal{G}}$ at $C \in \Pi_{\mathcal{F}_t}$. As before, conclude that $C \in \Pi_{\mathcal{G}_t}$ contradicting $C \subsetneq A \in \Pi_{\mathcal{G}_t}$.■

4.1.2 Construction of the Approximation Mapping Φ

Lemma 15 *For all acts h, h', payoffs x and y and events $A \in \Pi_{\mathcal{G}^*}$,*

$$hAx \succeq h'Ax \text{ if and only if } hAy \succeq h'Ay$$

Proof: Suppose $hAx \succeq h'Ax$ and note that $(hAy)Ax = hAx$ and $(h'Ay)Ax = h'Ax$. Conclude that $(hAy)Ax \succeq (h'Ay)Ax$. For any $A' \in \Pi_{\mathcal{G}^*}$ and $A' \neq A$,

$$(hAy)A'x = (h'Ay)A'x = yA'x.$$

Conclude that

$$(hAy)A'x \sim (h'Ay)A'x.$$

Thus, $(hAy)A'x \succeq (h'Ay)A'x$ for all $A' \in \Pi_{\mathcal{G}^*}$. By Subjective Monotonicity, $hAy \succeq h'Ay$ as desired.■

For every $A \in \Pi_{\mathcal{G}^*}$, define the preference $\succeq_A$ as

$$h \succeq_A h' \text{ if and only if } hAx_0 \succeq h'Ax_0.$$

By the above lemma, the 'conditional' preference $\succeq_A$ is independent of the choice of x_0. By construction, $\succeq_A$ inherits convexity, monotonicity and mixture-continuity. By Nonnullity, the preference is also nontrivial. By Strong Certainty Independence and by Lemma 15 in turn,

$$\begin{aligned} hAx_0 &\succeq h'Ax_0 \Rightarrow \\ [\alpha h + (1-\alpha)x]A[\alpha x_0 + (1-\alpha)x] &\succeq [\alpha h' + (1-\alpha)x]A[\alpha x_0 + (1-\alpha)x] \Rightarrow \\ [\alpha h + (1-\alpha)x]Ax_0 &\succeq [\alpha h' + (1-\alpha)x]Ax_0 \end{aligned}$$

Conclude that for all $A \in \Pi_{\mathcal{G}^*}$, $\succeq_A$ is a multiple prior preference. For any tuple $(\omega, t) \in \Omega \times \mathcal{T}$, let $A_{\omega,t}$ be the event in $\Pi_{\mathcal{G}_t^*}$ containing ω. By [9, Theorem 1], there exists a set $\mathcal{C}_{A_{\omega,t}} \subset \Delta(A_{\omega,t}, \mathcal{F}_t)$ such that

$$h \succeq_{A_{\omega,t}} h' \text{ if and only if } \min_{q \in \mathcal{C}_{A_{\omega,t}}} \langle q, h_t \rangle \geq \min_{q \in \mathcal{C}_{A_{\omega,t}}} \langle q, h'_t \rangle . \qquad (4.4)$$

Define the mapping $\Phi : \mathcal{H} \to \mathcal{H}$

$$(\Phi h)_t(\omega) = \min_{q \in \mathcal{C}_{A_{\omega,t}}} \langle q, h_t \rangle .$$

By construction,

$$\Phi(hAx_0) = \Phi(h)Ax_0 \sim hAx_0 \text{ for all } A \in \Pi_{\mathcal{G}^*}.$$

By Subjective Monotonicity, $\Phi(h) \sim h$ for all $h \in \mathcal{H}$. It is evident that Φ is an approximation mapping. To conclude the proof of the theorem, for all $h \in \mathcal{H}$, define

$$\begin{aligned} V(h) &= U \circ \Phi(h) \\ &= \min_{q \in Q} \sum_t \langle q_t, \Phi(h)_t \rangle \\ &= \min_{q \in Q} \sum_t \langle q_T, \Phi(h)_t \rangle \\ &= \min_{q \in Q} \left\langle q_T, \sum_t \Phi(h)_t \right\rangle . \end{aligned}$$

The third equality follows from Lemma 12. Finally set

$$\mathcal{C} := marg_{\mathcal{G}_T^*} \circ proj_{\{T\}} Q.$$

The claim that $\mathcal{C}$ is a subset of $\Delta^\circ(\Omega, \mathcal{G}_T^*)$ follows from the following property of multiple-prior preferences.

Lemma 16 *Let $\succeq$ be a multiple-prior preference on $\mathcal{B}(\Omega, \mathcal{F}_T)$ and $\mathcal{C}$ be the set of measures in the representation of $\succeq$. Let Π be any partition such that $\Pi \leq \Pi_{\mathcal{F}_T}$. If every event $A \in \Pi$ is nonnull and if for all payoffs $x \in M$:*

$$hAx \succeq h'Ax \text{ for all } A \in \Pi \text{ implies } h \succeq h'$$

then $p(A) > 0$ for all $A \in \Pi$ and $p \in \mathcal{C}$.

Proof: Suppose by way of contradiction, that $p(A) = 0$ for some $A \in \Pi$. Since A is nonnull, $\max_{q \in \mathcal{C}} q(A) > p(A) = 0$. Fix some payoffs y, y' such that $1 > y > y' > 0$ and note that:

$$U(yA0) = \min_{q \in C}[q(A)y] = y \min_{q \in C} q(A) = 0 = U(y'A0),$$

$$U(yA1) = \min_{q \in C}[(y-1)q(A) + 1] = 1 - (1-y)\max_{q \in C} q(A) >$$

$$> 1 - (1-y')\max_{q \in C} q(A) = U(yA1).$$

Conclude that $yA0 \sim y'A0$ and $yA1 \succ y'A1$ in contradiction of Lemma 15.■

4.1.3 Uniqueness

Uniqueness of the set $\mathcal{C}$ follows from familiar arguments. To prove the uniqueness of the $\mathcal{G}^*$-approximation mapping, take two such mappings Φ, $\widehat{\Phi}$. By separability, for all $A \in \Pi_{\mathcal{G}^*}$, h' and x:

$$\begin{aligned}\widehat{\Phi}(h'Ax) &= \widehat{\Phi}((h'Ax)Ax)A\widehat{\Phi}((h'Ax)A^c x) \\ &= \widehat{\Phi}(h'Ax)A\widehat{\Phi}(x) \\ &= \widehat{\Phi}(h'Ax)Ax\end{aligned}$$

The last equality follows from the fact that $\widehat{\Phi}$ must be identity on $\mathcal{G}^*$-measurable acts. Since $A \in \Pi_{\mathcal{G}^*}$ and $\Phi(h'Ax)$, $\widehat{\Phi}(h'Ax) \in \mathcal{H}_{\mathcal{G}^*}$, there exist payoffs y_Φ, $y_{\widehat{\Phi}}$ such that

$$\begin{aligned}\widehat{\Phi}(h'Ax) &= y_{\widehat{\Phi}}Ax \\ \Phi(h'Ax) &= y_\Phi Ax\end{aligned}$$

Since $\widehat{\Phi}(h'Ax) \sim \Phi(h'Ax)$ and $\succeq$ is strictly increasing on $\mathcal{H}_{\mathcal{G}^*}$, it must be the case that $y_\Phi = y_{\widehat{\Phi}}$. The proof is completed by induction on the number of events $A \in \Pi_{\mathcal{G}^*}$ such that an act $h' \in \mathcal{H}$ is nonconstant.

4.2 An Alternative Formulation

This section describes an alternative formulation of the static model. It shows that the subjective filtration defined by the ranking of effectively certain effects is canonical in the sense of being largest than any alternative definition of foreseen events. Some preliminary definitions are necessary.

Definition 10 *A preference relation $\succeq$ on $\mathcal{H}$ has a* ***regular representation*** $(\mathcal{G}, \Phi, \mathcal{C})$ *if it admits a utility function V of the form (2.3) where $\mathcal{G}$ is regular, the mapping Φ is identity on $\mathcal{H}_\mathcal{G}$ and $\mathcal{C}$ is a closed, convex subset of $\Delta^\circ(\Omega, \mathcal{G}_T)$.*

Definition 11 *A preference relation $\succeq$ on $\mathcal{H}$ has a* ***largest representation*** $(\mathcal{G}, \Phi, \mathcal{C})$ *if it admits a utility function V of the form (2.3) where $\mathcal{G}$ is sequentially connected, Φ is a $\mathcal{G}$-approximation mapping, $\mathcal{C}$ is a closed, convex subset of $\Delta^\circ(\Omega, \mathcal{G}_T)$ and $\mathcal{G}$ is the largest filtration for which a regular representation exists.*

Theorem 17 $(\mathcal{G}, \Phi, \mathcal{C})$ *is a limited foresight representation if and only if it is a largest representation.*

Proof: If $(\mathcal{G}, \Phi, \mathcal{C})$ is a regular representation for $\succeq$, then $\mathcal{H}^c \cap \mathcal{H}_{\mathcal{G}} \subset \mathcal{H}^*$ and so $\mathcal{G} \subset \mathcal{G}^*$. Thus, the limited foresight model is sufficient for a largest representation. Conversely, if a largest representation $(\mathcal{G}^*, \Phi, \mathcal{C})$ exists, then

$$\begin{aligned} \mathcal{G}^* &= \vee_{Q \subset diag(\mathcal{G}')} \mathcal{G}' \Rightarrow \\ \mathcal{H}^c \cap \mathcal{H}_{\mathcal{G}^*} &= \cup_{\{\mathcal{G}': Q \subset diag(\mathcal{G}')\}} \mathcal{H}^c \cap \mathcal{H}_{\mathcal{G}'} \end{aligned}$$

At the same time,

$$\mathcal{H}^* = \cup_{\{\mathcal{G}': Q \subset diag(\mathcal{G}')\}} \mathcal{H}^c \cap \mathcal{H}_{\mathcal{G}'}.$$

To see this note that, $\cup_{\{\mathcal{G}': Q \subset diag(\mathcal{G}')\}} \mathcal{H}^c \cap \mathcal{H}_{\mathcal{G}'} \subset \mathcal{H}^*$. If $h \in \mathcal{H}^*$, then for all $h' \in \mathcal{H}^c \cap \mathcal{H}_{\mathcal{F}(h)}$, $h' \sim \mathbf{0}$. From Lemma 9, conclude that $Q \subset diag(\mathcal{F}(h))$ and so $h \in \cup_{\mathcal{G}': Q \subset diag(\mathcal{G}')} \mathcal{H}^c \cap \mathcal{H}_{\mathcal{G}'}$. Thus $\mathcal{H}^* = \mathcal{H}^c \cap \mathcal{H}_{\mathcal{G}^*}$ and so $\mathcal{G}^* = \mathcal{F}(\mathcal{H}^*)$ as desired.■

4.3 Proof of Lemma 3

Suppose $\succeq$ has a representation $(\mathcal{G}, \{\mathcal{C}_A\}, \mathcal{C})$ such that $\mathcal{C}_A$ has nonempty interior in $\Delta(A, \mathcal{F}_t)$ for all $t \in \mathcal{T}$ and $A \in \Pi_{\mathcal{G}_t}$. Let Q be the set in $\times_t \Delta(\Omega, \mathcal{F}_t)$ that represents $\succeq$ as in (4.1). It suffices to show that for all filtrations $\mathcal{G}'$ such that $\mathcal{G}'_{T-1} = \mathcal{G}'_T$:

$$Q \subset diag(\mathcal{G}') \text{ implies } \mathcal{G}' \leq \mathcal{G}.$$

Equivalently,

$$\forall t < T, \forall B \notin \mathcal{G}_t, \ \exists q \in Q \text{ such that } q_t(B) \neq q_{t+1}(B).$$

For all $t \in \mathcal{T}$ and $A \in \Pi_{\mathcal{G}_t}$, define $\succeq_A$ as in (4.4). The family of preferences $\{\succeq_A\}$ and $\succeq$ satisfy the conditions of [5, Theorem 3.2]. Conclude that:

$$Q = \cup_{\mu \in \mathcal{C}} \{(q_t) : q_t = \int p_A d\mu \text{ for some selection } \{p_A\}_{A \in \Pi_{\mathcal{G}_t}} \text{ s.t. } p_A \in \mathcal{C}_A\}.$$

Fix $t < T$ and $B \notin \mathcal{G}_t$ and any $\mu \in \mathcal{C}$. By the above decomposition of Q, it suffices to find two selections $\{p_A\}_{A \in \Pi_{\mathcal{G}_t}}$ and $\{p'_A\}_{A \in \Pi_{\mathcal{G}_t}}$ such that:

$$\int p_A(B) d\mu \neq \int p'_A(B) d\mu.$$

Since $B \notin \mathcal{G}_t$, there exists $A^* \in \Pi_{\mathcal{G}_t}$ such that $A^* \neq B \cap A^* \neq \varnothing$. Since $\mathcal{C}_{A^*}$ has nonempty interior, there exist p_{A^*} and p'_{A^*} such that $p_{A^*}(B) \neq p'_{A^*}(B)$. Choose any $p_A = p'_A$ for all $A \neq A^*$ and $A \in \Pi_{\mathcal{G}_t}$ to complete the proof of the theorem.

4.4 Proof of Theorem 4

Necessity is standard. To prove sufficiency, first show that $\{\mathcal{G}^{t,\omega}\}$ is refining. For t, ω such that $\mathcal{F}_{t+1}(\omega) \notin \mathcal{G}^{t,\omega}$, the fact that $\mathcal{G}^{t,\omega}$ is sequentially connected implies that

$$\{B \cap \mathcal{F}_{t+1}(\omega) : B \in \mathcal{G}^{t,\omega}\} = \{\varnothing, \mathcal{F}_{t+1}(\omega)\}$$

Thus, $\mathcal{G}^{t+1,\omega}$ refines the trivial filtration $\mathcal{G}^{t,\omega} \cap \mathcal{F}_{t+1}(\omega)$. If $\mathcal{F}_{t+1}(\omega) \in \mathcal{G}^{t,\omega}$, take an act $h \in \mathcal{H}_{\mathcal{G}^{t,\omega}} \cap \mathcal{H}^c$ such that $h_{t'}(\omega') = 0$ whenever $t' < t$ or $\omega' \notin \mathcal{F}_{t+1}(\omega)$. By Consequentialism, it suffices to show that h is indifferent to the constant act $\mathbf{0}$. By construction, $h \sim_{t+1,\omega'} \mathbf{0}$ for all $\omega' \notin \mathcal{F}_{t+1}(\omega)$. Since $h, \mathbf{0} \in \mathcal{H}_{\mathcal{G}^{t,\omega}}$, Weak Dynamic Consistency and Lemma 15 imply that:

$$h \sim_{t+1,\omega} \mathbf{0} \text{ if and only if } h \sim_{t,\omega} \mathbf{0}.$$

The latter is true by the choice of $h \in \mathcal{H}_{\mathcal{G}^{t,\omega}} \cap \mathcal{H}^c$.

To prove that $\{\mathcal{C}^{t,\omega}\}$ admits a consistent extension, construct the set $\mathcal{C}$ recursively. For all ω and $t \geq T-1$, set $\widehat{\mathcal{C}}^{t,\omega} := \mathcal{C}^{t,\omega}$. Fix ω and $t < T-1$ and suppose $\widehat{\mathcal{C}}^{t+1,\omega'}$ has been defined for all ω'. For each ω' such that $\mathcal{F}_{t+1}(\omega') \notin \mathcal{G}^{t,\omega}$, or equivalently, such that $\mathcal{F}_{t+1}(\omega') \subsetneq \mathcal{G}^{t,\omega}(\omega')$, fix a measure $\lambda_{\omega'} \in \Delta^\circ(\mathcal{G}^{t,\omega}(\omega'), \mathcal{F}_{t+1})$ such that $\lambda_{\omega''} = \lambda_{\omega'}$ for all $\omega'' \in \mathcal{G}^{t,\omega}(\omega')$. For each $\mu \in \mathcal{C}^{t,\omega}$ define the measure $\widehat{\mu} = \int_\Omega \widehat{\lambda}_{\omega'} dm$ in $\Delta^\circ(\mathcal{F}_t(\omega), \mathcal{F}_{t+1})$ where

$$m = \text{marg}_{\mathcal{G}^{t,\omega}_{t+1}} \mu \text{ and } \widehat{\lambda}_{\omega'} = \left\{ \begin{array}{l} \lambda_{\omega'} \text{ if } \mathcal{F}_{t+1}(\omega') \notin \mathcal{G}^{t,\omega} \\ \delta_{\omega'} \text{ if } \mathcal{F}_{t+1}(\omega') \in \mathcal{G}^{t,\omega} \end{array} \right. . \tag{4.5}$$

In effect, the constructed measures $\widehat{\mu}$ extend the individual's one-step ahead beliefs at t, ω from the subjectively foreseen events $\mathcal{G}^{t,\omega}_{t+1}$ to all $\mathcal{F}_{t+1}$-measurable subsets of $\mathcal{F}_t(\omega)$. The construction ensures that this set of extensions $M^{t,\omega} := \{\widehat{\mu} : \mu \in \mathcal{C}^{t,\omega}\} \subset \Delta^\circ(\mathcal{F}_t(\omega), \mathcal{F}_{t+1})$ is closed and convex.

Next, let p denote a generic, $\mathcal{F}_{t+1}$-measurable selection from $\omega' \longmapsto \widehat{\mathcal{C}}^{t+1,\omega'}$ and define

$$\widehat{\mathcal{C}}^{t,\omega} = \{\int_\Omega p_{\omega'} d\widehat{\mu}(\omega') : \widehat{\mu} \in M^{t,\omega} \text{ and } p_{\omega'} \in \widehat{\mathcal{C}}^{t+1,\omega'} \text{ for all } \omega'\}. \tag{4.6}$$

From [5, Theorem 3.2], conclude that $\widehat{\mathcal{C}}^{t,\omega}$ is a closed and convex subset of $\Delta^{\circ}(\mathcal{F}_t(\omega), \mathcal{F}_T)$ and $\mathcal{C} := \widehat{\mathcal{C}}^0$ is $\{\mathcal{F}_t\}$-rectangular. In particular,

$$\{\mu(\cdot \mid \mathcal{F}_t(\omega)) : \mu \in \mathcal{C}\} = \widehat{\mathcal{C}}^{t,\omega} \text{ for all } t \text{ and } \omega.$$

To complete the proof, it remains to show that

$$\text{marg}_{\mathcal{G}^{t,\omega}} \widehat{\mathcal{C}}^{t,\omega} := \{\text{marg}_{\mathcal{G}^{t,\omega}} \mu : \mu \in \widehat{\mathcal{C}}^{t,\omega}\} = \mathcal{C}^{t,\omega} \text{ for all } t \text{ and } \omega. \tag{4.7}$$

The next lemmas show that both $\text{marg}_{\mathcal{G}^{t,\omega}} \widehat{\mathcal{C}}^{t,\omega}$ and $\mathcal{C}^{t,\omega}$ admit decompositions similar to (4.6).

Lemma 18 *For all t and ω, the set $marg_{\mathcal{G}^{t,\omega}} \widehat{\mathcal{C}}^{t,\omega}$ admits the decomposition*

$$marg_{\mathcal{G}^{t,\omega}} \widehat{\mathcal{C}}^{t,\omega} = \{\int_{\Omega} p_{\omega'} d\mu : \mu \in marg_{\mathcal{G}^{t,\omega}_{t+1}} M^{t,\omega} \text{ and } p_{\omega'} \in marg_{\mathcal{G}^{t,\omega}} \widehat{\mathcal{C}}^{t+1,\omega'}\}.$$

Proof: By (4.6), all measures in $\widehat{\mathcal{C}}^{t,\omega}$ are of the form $\int_{\Omega} p_{\omega'} d\widehat{\mu}$ where $\widehat{\mu} \in M^{t,\omega}$ and p is an $\mathcal{F}_{t+1}$-measurable selection from $\omega' \longmapsto \widehat{\mathcal{C}}^{t+1,\omega'}$. Since $\mathcal{G}^{t,\omega}$ is sequentially connected, $\text{marg}_{\mathcal{G}^{t,\omega}} p$ is a $\mathcal{G}^{t,\omega}_{t+1}$-measurable selection from $\omega' \longmapsto \text{marg}_{\mathcal{G}^{t,\omega}} \widehat{\mathcal{C}}^{t+1,\omega'}$. Conclude that

$$\text{marg}_{\mathcal{G}^{t,\omega}} \int_{\Omega} p_{\omega'} d\widehat{\mu} = \int_{\Omega} \text{marg}_{\mathcal{G}^{t,\omega}} p_{\omega'} \; d\left(\text{marg}_{\mathcal{G}^{t,\omega}_{t+1}} \widehat{\mu}\right). \blacksquare$$

Lemma 19 *For all t and ω, the set $\mathcal{C}^{t,\omega}$ admits the decomposition*

$$\mathcal{C}^{t,\omega} = \{\int p_{\omega'} dm : m \in marg_{\mathcal{G}^{t,\omega}_{t+1}} \mathcal{C}^{t,\omega} \text{ and } p_{\omega'} \in marg_{\mathcal{G}^{t,\omega}} \mathcal{C}^{t+1,\omega'} \text{ for all } \omega'\}.$$

Proof: For each $\omega' \in \mathcal{F}_t(\omega)$, let $\succeq^a_{t+1,\omega'}$ and $\succeq^a_{t,\omega}$ denote the respective restrictions of $\succeq_{t+1,\omega'}$ and $\succeq_{t,\omega}$ to $\mathcal{H}_{\mathcal{G}^{t,\omega}}$. Since $\mathcal{H}_{\mathcal{G}^{t+1,\omega'}}$ refines $\mathcal{H}_{\mathcal{G}^{t,\omega}}$ for each $\omega' \in \mathcal{F}_t(\omega)$, the corresponding preference $\succeq^a_{t+1,\omega'}$ has a representation

$$U^{t+1,\omega'}(h) = \min_{\mu \in \text{marg}_{\mathcal{G}^{t,\omega}} \mathcal{C}^{t+1,\omega'}} \int \sum_{\tau \geq t'} \beta^{\tau-t-1} h_\tau d\mu$$

Since $\mathcal{G}^{t,\omega}$ is sequentially connected, the mapping $\omega' \longmapsto \succeq^a_{t+1,\omega'}$ is $\mathcal{G}^{t,\omega}_{t+1}$-measurable. Thus the collection of preferences $\succeq^a_{t,\omega}, \succeq^a_{t+1,\omega'}$ for $\omega' \in \mathcal{F}_t(\omega)$

satisfies Consequentialism with respect to the filtration $\mathcal{G}^{t,\omega}$. By State Independence and Lemma 15, the collection of preferences is also dynamically consistent. The claim of the lemma follows from [5, Theorem 3.2].■

Complete the proof of (4.7) by induction. The claim holds trivially for ω and $t \geq T-1$. Fix some ω and $t < T-1$ and suppose the claim has been established for $t+1$. Applying Lemma 19, the induction hypothesis and Lemma 18 in turn, conclude that

$$\begin{aligned} \{\mu(\cdot \mid \mathcal{F}_{t+1}(\omega')) : \mu \in \mathcal{C}^{t,\omega}\} &= \text{marg}_{\mathcal{G}^{t,\omega}} \mathcal{C}^{t+1,\omega'} \\ &= \text{marg}_{\mathcal{G}^{t,\omega}} \widehat{\mathcal{C}}^{t+1,\omega'} \\ &= \text{marg}_{\mathcal{G}^{t,\omega}} \{\mu(\cdot \mid \mathcal{F}_{t+1}(\omega')) : \mu \in \widehat{\mathcal{C}}^{t,\omega}\} \end{aligned} \tag{4.8}$$

Also, by construction,

$$\text{marg}_{\mathcal{G}^{t,\omega}_{t+1}} \mathcal{C}^{t,\omega} = \text{marg}_{\mathcal{G}^{t,\omega}_{t+1}} M^{t,\omega}. \tag{4.9}$$

Properties (4.8) and (4.9) show that $\text{marg}_{\mathcal{G}^{t,\omega}} \widehat{\mathcal{C}}^{t,\omega}$ and $\mathcal{C}^{t,\omega}$ induce the same sets of conditionals and one-step-ahead marginals. By Lemmas 18 and 19, the sets $\text{marg}_{\mathcal{G}^{t,\omega}} \widehat{\mathcal{C}}^{t,\omega}$ and $\mathcal{C}^{t,\omega}$ are uniquely determined by the respective sets of conditionals and marginals. Conclude that $\text{marg}_{\mathcal{G}^{t,\omega}} \widehat{\mathcal{C}}^{t,\omega} = \mathcal{C}^{t,\omega}$.

4.4.1 Uniqueness

Let $\mathcal{C}$ be an $\{\mathcal{F}_t\}$-rectangular subset of $\Delta^\circ(\Omega, \mathcal{F}_T)$ and for each t, ω, let $\mathcal{G}^{t,\omega}$ be a sequentially connected algebra such that $\mathcal{F}_{t+1}(\omega') \in \mathcal{G}^{t,\omega}$ for all $\omega' \in \mathcal{F}_t(\omega)$.

Lemma 20 *A measure μ in $\Delta^\circ(\Omega, \mathcal{F}_T)$ belongs to $\mathcal{C}$ if and only if*

$$\textit{marg}_{\mathcal{G}^{t,\omega}} \mu(\cdot \mid \mathcal{F}_t(\omega)) \in \{\textit{marg}_{\mathcal{G}^{t,\omega}} \mu'(\cdot \mid \mathcal{F}_t(\omega)) : \mu' \in \mathcal{C}\} \textit{ for all } t \textit{ and } \omega.$$

Proof: Sufficiency is immediate. To prove necessity, note the recursive construction of the $\{\mathcal{F}_t\}$-rectangular set in the proof of Theorem 4. An $\{\mathcal{F}_t\}$-rectangular subset contains all measures $\widehat{\mu}$ and only the measures $\widehat{\mu}$ such that

$$\widehat{\mu}(\cdot \mid \mathcal{F}_T(\omega)) \in \{\mu'(\cdot \mid \mathcal{F}_T(\omega)) : \mu' \in \mathcal{C}\} \text{ for } \omega \in \Omega, \text{ and}$$

$$\widehat{\mu}(\cdot \mid \mathcal{F}_t(\omega)) \in \{\text{marg}_{\mathcal{F}_{t+1}}\mu'(\cdot \mid \mathcal{F}_t(\omega)) : \mu' \in \mathcal{C}\} \text{ for } \omega \in \Omega \text{ and } t < T.$$

By construction, the restriction of $\mathcal{F}_T$ to $\mathcal{F}_T(\omega)$ equals $\{\mathcal{F}_T(\omega), \varnothing\}$ which equals $\mathcal{G}^{T,\omega}$ for each $\omega \in \Omega$. By hypothesis, the restriction of $\mathcal{F}_{t+1}$ to $\mathcal{F}_t(\omega)$ is refined by $\mathcal{G}^{t,\omega}$ for each t and ω. Conclude that $\mu \in \mathcal{C}$.■

Fix a closed and convex subset $\mathcal{C}$ of $\Delta(\Omega, \mathcal{F}_T)$ such that

$$\text{clconv}\{\text{marg}_{\mathcal{G}^{t,\omega}}\mu(\cdot \mid \mathcal{F}_t(\omega)) : \mu \in \mathcal{C}' \text{ s.t. } \mu(\mathcal{F}_t(\omega)) > 0\} = \mathcal{C}^{t,\omega}. \tag{4.10}$$

Note that $\mathcal{C}$ must be a subset of $\Delta^\circ(\Omega, \mathcal{F}_T)$. Suppose by way of contradiction that there exists a measure $\mu' \in \mathcal{C}$ such that $\mu'(\mathcal{F}_t(\omega)) = 0$ for some t and ω. Since for all ω', $\mu'(\mathcal{F}_0(\omega')) = \mu'(\Omega) = 1$, conclude that $t > 0$. Let t^* be the largest t' such that $\mu'(\mathcal{F}_{t^*}(\omega)) > 0$. The time t^* exists since $\mu'(\mathcal{F}_0(\omega)) > 0$. By the definition of t^*, $\mu'(\cdot \mid \mathcal{F}_{t^*}(\omega))$ is well-defined and $\mu'(\mathcal{F}_{t^*+1}(\omega) \mid \mathcal{F}_{t^*}(\omega)) = 0$. The latter is impossible since $\mathcal{F}_{t^*+1}(\omega) \in \mathcal{G}^{t^*,\omega}$ and by property (4.10) above

$$\text{marg}_{\mathcal{G}^{t^*,\omega}}\mu'(\cdot \mid \mathcal{F}_{t^*}(\omega)) \in \mathcal{C}^{t^*,\omega} \subset \Delta^\circ(\mathcal{F}_{t^*}(\omega), \mathcal{G}^{t^*,\omega}).$$

Apply Lemma 20 to conclude that $\mathcal{C}$ must be a subset of the closed, convex $\{\mathcal{F}_t\}$-rectangular subset generated by the sets $\mathcal{C}^{t,\omega}$ for $\omega \in \Omega, t \in \mathcal{T}$. From Lemma 20 again, the latter is well-defined and unique.

4.5 Proof of Theorem 6

To prove Theorem 6, take an act h such that $h(\omega) \sim_0 h(\omega')$ for all $\omega, \omega' \in \Omega$. By Consequentialism,

$$h \sim_{T,\omega} h(\omega) \text{ for all } \omega \in \Omega. \tag{4.11}$$

Fix some ω and note that:

$$h_\tau(\omega') = h_\tau(\omega'') \text{ for all } \omega', \omega'' \in \mathcal{F}_{T-1}(\omega) \text{ and } \tau \leq T-1.$$

By State Independence,

$$h(\omega') \sim_{T,\omega} h(\omega'') \text{ for all } \omega', \omega'' \in \mathcal{F}_{T-1}(\omega). \tag{4.12}$$

But then (4.11) and (4.12) imply that for all $\omega' \in \mathcal{F}_{T-1}(\omega)$:

$$h \sim_{T,\omega'} h(\omega') \sim_{T,\omega'} h(\omega).$$

By Dynamic Consistency, $h \sim_{T-1,\omega} h(\omega)$. Proceeding inductively, conclude that $h \sim_0 h(\omega)$.

4.6 Sequentially Connected Filtrations

Say that a filtration $\{\mathcal{G}_t\}$ is connected if

$$\mathcal{G}_t = \mathcal{F}_t \cap \mathcal{G}_T \text{ for all } t \in \mathcal{T}.$$

Proposition 21 *A sequentially connected filtration $\{\mathcal{G}_t\}$ is connected.*

Let $\{\mathcal{G}_t\}$ be sequentially connected. It is evident that $\mathcal{G}_t \subset \mathcal{G}_T \cap \mathcal{F}_t$ for all $t \in \mathcal{T}$. To prove the opposite inclusion, take an event $A \in \mathcal{G}_T \cap \mathcal{F}_t$. If $A \notin \mathcal{G}_t$, then there exists a set $B \in \Pi_{\mathcal{G}_t}$ such that $\varnothing \neq B \cap A \neq B$. Then $A \in \mathcal{F}_t$ implies $B \notin \Pi_{\mathcal{F}_t}$. Conclude that $B \in \Pi_{\mathcal{G}_t} \backslash \Pi_{\mathcal{F}_t}$ and since $\{\mathcal{G}_t\}$ is connected, $B \in \Pi_{\mathcal{G}_T}$. But then $\varnothing \neq B \cap A \neq B$ contradicts the fact that $A \in \mathcal{G}_T$.■

The following proposition shows that sequentially connected filtrations inherit the lattice properties of stopping-times.

Proposition 22 *The class of sequentially connected filtrations is a lattice. It is lattice-isomorphic to the class of sequentially connected algebras.*

First, establish the following distributive law.

Lemma 23 *If $\mathcal{G}$ and $\mathcal{G}'$ are sequentially connected algebras, then*

$$\Pi_{\mathcal{G} \cap \mathcal{F}_t} \vee \Pi_{\mathcal{G}' \cap \mathcal{F}_t} = \Pi_{\mathcal{G} \vee \mathcal{G}'} \wedge \Pi_{\mathcal{F}_t} \textit{ for all } t \in \mathcal{T}.$$

Proof: For any partitions Π and Π', $\Pi \subset \Pi'$ if and only if $\Pi = \Pi'$. Thus, it suffices to show that

$$\Pi_{\mathcal{G} \cap \mathcal{F}_t} \vee \Pi_{\mathcal{G}' \cap \mathcal{F}_t} \subset \Pi_{\mathcal{G} \vee \mathcal{G}'} \wedge \Pi_{\mathcal{F}_t} \text{ for all } t \in \mathcal{T}.$$

Fix some $t \in \mathcal{T}$ and an event $A \in \Pi_{\mathcal{G} \cap \mathcal{F}_t} \vee \Pi_{\mathcal{G}' \cap \mathcal{F}_t}$. By definition of the supremum, $A = B \cap B'$ for some sets $B \in \Pi_{\mathcal{G} \cap \mathcal{F}_t}$ and $B' \in \Pi_{\mathcal{G}' \cap \mathcal{F}_t}$. Since $\mathcal{G}$ is connected, $\Pi_{\mathcal{G} \cap \mathcal{F}_t} \backslash \Pi_{\mathcal{F}_t} \subset \Pi_{\mathcal{G} \cap \mathcal{F}_T} = \Pi_{\mathcal{G}}$. Equivalently,

$$\Pi_{\mathcal{G} \cap \mathcal{F}_t} \subset \Pi_{\mathcal{G}} \cup \Pi_{\mathcal{F}_t}. \tag{4.13}$$

An analogous argument holds for $\mathcal{G}'$. By (4.13), there are two cases to consider:

If $B \in \Pi_{\mathcal{G}}$ and $B' \in \Pi_{\mathcal{G}'}$, then

$$B \cap B' \in (\Pi_{\mathcal{G}} \vee \Pi_{\mathcal{G}'}) \cap \mathcal{F}_t = \Pi_{\mathcal{G}\vee\mathcal{G}'} \cap \mathcal{F}_t \subset \Pi_{\mathcal{G}\vee\mathcal{G}'} \wedge \Pi_{\mathcal{F}_t}.$$

If $B \in \Pi_{\mathcal{F}_t}$ (or $B' \in \Pi_{\mathcal{F}_t}$), then $B \cap B' \in \Pi_{\mathcal{G}\cap\mathcal{F}_t} \vee \Pi_{\mathcal{G}'\cap\mathcal{F}_t} \leq \Pi_{\mathcal{F}_t}$ implies that $B = B \cap B'$. But then

$$B \cap B' = B \in \Pi_{\mathcal{F}_t} \cap \mathcal{G} \subset \Pi_{\mathcal{F}_t} \cap (\mathcal{G} \vee \mathcal{G}') \subset \Pi_{\mathcal{F}_t} \wedge \Pi_{\mathcal{G}\vee\mathcal{G}'}.\blacksquare$$

By Lemma 23, it is enough to prove that the class of connected algebras is a lattice. Take the supremum $\mathcal{G} \vee \mathcal{G}'$ of two sequentially connected algebras $\mathcal{G}$ and $\mathcal{G}'$ and an event $A \in \Pi_{(\mathcal{G}\vee\mathcal{G}')\cap\mathcal{F}_t} \backslash \Pi_{\mathcal{F}_t}$. By Lemma 23,

$$\begin{aligned} \Pi_{(\mathcal{G}\vee\mathcal{G}')\cap\mathcal{F}_t} &= \Pi_{\mathcal{G}\vee\mathcal{G}'} \wedge \Pi_{\mathcal{F}_t} \\ &= \Pi_{\mathcal{G}\cap\mathcal{F}_t} \vee \Pi_{\mathcal{G}'\cap\mathcal{F}_t} \end{aligned}$$

Thus there exist sets $B \in \Pi_{\mathcal{G}\cap\mathcal{F}_t}$ and $B' \in \Pi_{\mathcal{G}'\cap\mathcal{F}_t}$ such that $A = B \cap B'$. If $B \in \Pi_{\mathcal{F}_t}$, then $B' \in \mathcal{F}_t$ implies $A = B \cap B' = B \in \Pi_{\mathcal{F}_t}$ contradicting the choice of $A \notin \Pi_{\mathcal{F}_t}$. A symmetric argument implies $B' \in \Pi_{\mathcal{G}'\cap\mathcal{F}_t} \backslash \Pi_{\mathcal{F}_t}$. Since $\mathcal{G}$ and $\mathcal{G}'$ are sequentially connected, $B \in \Pi_{\mathcal{G}\cap\mathcal{F}_\tau}$ and $B' \in \Pi_{\mathcal{G}'\cap\mathcal{F}_\tau}$ for all $\tau \geq t$. Conclude that $A = B \cap B' \in \Pi_{\mathcal{G}\cap\mathcal{F}_\tau} \vee \Pi_{\mathcal{G}'\cap\mathcal{F}_\tau} = \Pi_{\mathcal{G}\vee\mathcal{G}'} \wedge \Pi_{\mathcal{F}_\tau} = \Pi_{(\mathcal{G}\vee\mathcal{G}')\cap\mathcal{F}_\tau}$ for all $\tau \geq t$.

To show that $\mathcal{G} \wedge \mathcal{G}'$ is sequentially connected, take $A \in \Pi_{\mathcal{G}\cap\mathcal{G}'\cap\mathcal{F}_t} \backslash \Pi_{\mathcal{F}_t}$. Notice that

$$\begin{aligned} \Pi_{\mathcal{G}\cap\mathcal{G}'\cap\mathcal{F}_t} &= \Pi_{(\mathcal{G}\cap\mathcal{F})_t \wedge (\mathcal{G}'\cap\mathcal{F}_t)} = \\ &= \Pi_{\mathcal{G}\cap\mathcal{F}_t} \wedge \Pi_{\mathcal{G}'\cap\mathcal{F}_t}. \end{aligned}$$

But $A \in \Pi_{\mathcal{G}\cap\mathcal{F}_t} \wedge \Pi_{\mathcal{G}'\cap\mathcal{F}_t}$ if and only if for all $B_0 \in \Pi_{\mathcal{G}\cap\mathcal{F}_t} \cup \Pi_{\mathcal{G}'\cap\mathcal{F}_t}$ such that $B_0 \subset A$:

$$A = \cup_{\{B_0,B_1,\ldots,B_k\}} \cup_{B\in\{B_0,B_1,\ldots,B_k\}} B,$$

where the union is taken over all sequences $\{B_0, B_1, ..., B_k\}$ of subsets of A such that consecutive elements intersect and belong alternatively to $\Pi_{\mathcal{G}\cap\mathcal{F}_t}$ and $\Pi_{\mathcal{G}'\cap\mathcal{F}_t}$.

Since $A \in (\Pi_{\mathcal{G}\cap\mathcal{F}_t} \wedge \Pi_{\mathcal{G}'\cap\mathcal{F}_t}) \backslash \Pi_{\mathcal{F}_t}$, there exists a set $B_0 \in (\Pi_{\mathcal{G}\cap\mathcal{F}_t} \cup \Pi_{\mathcal{G}'\cap\mathcal{F}_t}) \backslash \Pi_{\mathcal{F}_t}$ such that $B_0 \subset A$. Fix such a set B_0 and consider a sequence

$\{B_0, B_1, ..., B_k\}$ satisfying the conditions above. For $i < k$, each B_i intersects two disjoint subsets of $\mathcal{F}_t$ and so $B_i \notin \Pi_{\mathcal{F}_t}$. Moreover, if $B_k \in \Pi_{\mathcal{F}_t}$ then $B_k \cap B_{k-1} \neq \varnothing$ implies that $B_k \subset B_{k-1}$. Conclude that A can be written as the union over sequences $\{B_0, B_1, ..., B_k\}$ in $(\Pi_{\mathcal{G}\cap\mathcal{F}_t} \cup \Pi_{\mathcal{G}'\cap\mathcal{F}_t})\backslash\Pi_{\mathcal{F}_t}$. Since $\mathcal{G}$ and $\mathcal{G}'$ are sequentially connected, A can be written as the union over sequences $\{B_0, B_1, ..., B_k\}$ in $\Pi_{\mathcal{G}\cap\mathcal{F}_\tau} \cup \Pi_{\mathcal{G}'\cap\mathcal{F}_\tau}$ for all $\tau \geq t$. Conclude that A must be a subset of some element in $\Pi_{\mathcal{G}\cap\mathcal{F}_\tau} \wedge \Pi_{\mathcal{G}'\cap\mathcal{F}_\tau}$. But since $\Pi_{\mathcal{G}\cap\mathcal{F}_\tau} \wedge \Pi_{\mathcal{G}'\cap\mathcal{F}_\tau}$ is finer than $\Pi_{\mathcal{G}\cap\mathcal{F}_t} \wedge \Pi_{\mathcal{G}'\cap\mathcal{F}_t}$ and $A \in \Pi_{\mathcal{G}\cap\mathcal{F}_t} \wedge \Pi_{\mathcal{G}'\cap\mathcal{F}_t}$, it must be that $A \in \Pi_{\mathcal{G}\cap\mathcal{F}_\tau} \wedge \Pi_{\mathcal{G}'\cap\mathcal{F}_\tau}$ for all $\tau \geq t$.■

Proposition 24 *A stopping time g induces a sequentially connected algebra.*

A stopping time is a function $g : \Omega \to \mathcal{T}$ such that $[g = t] \in \mathcal{F}_t$ for all $t \in \mathcal{T}$. The stopping time g is induces the algebra $\mathcal{G}$:

$$\mathcal{G} := \{A \in \mathcal{F}_T : A \cap [g = t] \in \mathcal{F}_t, \forall t \in \mathcal{T}\}.$$

To see that $\mathcal{G}$ is sequentially connected, first prove that

$$\{A \in \Pi_{\mathcal{F}_t} : A \subset [g = t]\} \subset \Pi_{\mathcal{G}}, \forall t \in \mathcal{T} \tag{4.14}$$

Fix $t \in \mathcal{T}$ and $A \in \Pi_{\mathcal{F}_t}$ such that $A \subset [g = t]$. Since $A \cap [g = t] = A \in \mathcal{F}_t$ and $A \cap [g = t'] = \varnothing \in \mathcal{F}_t$ for all $t' \neq t$, the definition of $\mathcal{G}$ implies $A \in \mathcal{G}$. If $B \subsetneq A \in \Pi_{\mathcal{F}_t}$, then $B \cap [g = t] = B \notin \mathcal{F}_t$ and thus $B \notin \mathcal{G}$. Conclude that $A \in \Pi_{\mathcal{G}}$.

Next fix $t < T$ and take $A \in \Pi_{\mathcal{G}\cap\mathcal{F}_t}\backslash\Pi_{\mathcal{F}_t}$. If $A \cap [g \geq t] \neq \varnothing$, then there exists $A' \subsetneq A$ such that $A' \in \Pi_{\mathcal{F}_t}$ and $A' \cap [g \geq t] \neq \varnothing$. But $[g \geq t] = [g < t]^c \in \mathcal{F}_t$ and $A' \in \Pi_{\mathcal{F}_t}$ imply that $A' \subset [g \geq t]$ and so $A' \in \mathcal{G}$. In turn, $A' \in \mathcal{G} \cap \Pi_{\mathcal{F}_t}$ implies $A' \in \Pi_{\mathcal{G}\cap\mathcal{F}_t}$ contradicting the choice of A. Conclude that $A \subset [g < t]$.

Fix some $t' < t$ such that $A \cap [g = t'] \neq \varnothing$. Since $[g = t'] \in \mathcal{G} \cap \mathcal{F}_t$ and $A \in \Pi_{\mathcal{G}\cap\mathcal{F}_t}$, it must be the case that $A \subset [g = t']$. This implies $A \in \mathcal{F}_{t'}$, for otherwise, $A \cap [g = t'] = A \notin \mathcal{F}_{t'}$ contradicts $A \in \mathcal{G}$. For any $A' \in \Pi_{\mathcal{F}_{t'}}$ and $A' \subset A \subset [g = t']$, equation (4.14) implies that $A' \in \Pi_{\mathcal{G}}$ and so $A' \in \Pi_{\mathcal{G}\cap\mathcal{F}_t}$. Since $A \in \Pi_{\mathcal{G}\cap\mathcal{F}_t}$, it must be the case that $A = A' \in \Pi_{\mathcal{G}}$. But then $A \in \Pi_{\mathcal{G}} \cap \mathcal{F}_{t+1} \subset \Pi_{\mathcal{G}\cap\mathcal{F}_{t+1}}$ as desired.■

The next example translates the Gabaix and Laibson [7] procedure for simplifying decision trees in the setting of this paper and shows that it induces a sequentially connected filtration.

Example 7 *(Satisficing) "Start from the initial node and follow all branches whose probability is greater than or equal to some threshold level α. Continue in this way down the tree. If a branch has a probability less than α, consider the node it leads to, but do not advance beyond that node."*

Thus, let μ be a measure on $(\Omega, \mathcal{F}_T)$ and $\alpha \in [0,1]$ be a threshold level. For each event A and algebra $\mathcal{F}$, define $r_{\mathcal{F}}(A)$ to be the smallest $\mathcal{F}$-measurable superset of A. The collection of events $\{\mathcal{A}_t\}$ is a satisficing procedure if:

$$\begin{aligned} \mathcal{A}_0 &= \{\Pi_{\mathcal{F}_0}\}, \text{ and for all } t > 0 \\ \mathcal{A}_t &= \{A \in \Pi_{\mathcal{F}_t} : r_{\mathcal{F}_{t-1}}(A) \in \mathcal{G}_{t-1} \text{ and } \mu(r_{\mathcal{F}_{t-1}}(A) \mid r_{\mathcal{F}_{t-2}}(A)) \geq \alpha\}. \end{aligned}$$

It is not difficult to see that $\{\mathcal{A}_t\}$ generates a sequentially connected filtration. In fact, the filtration is induced by the stopping time:

$$\begin{aligned} [\tau &= 0] = \varnothing, \text{ and for all } t > 0 \\ [\tau &= t] = \cup \left\{ A \in \Pi_{\mathcal{F}_t} : \mu(A \mid r_{\mathcal{F}_{t-1}}(A)) < \alpha \text{ and } \mu(r_{\mathcal{F}_{t-1}}(A) \mid r_{\mathcal{F}-2}(A)) \geq \alpha \right\}. \end{aligned}$$

References

[1] C. Aliprantis and K. Border. *Infinite Dimensional Analysis*. Springer-Verlag, 2 edition, 1999.

[2] T. Cogley and T. Sargent. Anticipated utility and rational expectations as approximations to Bayesian decision making. *International Economic Review*, 49(1):185–221, February 2008.

[3] E. Dekel, B. Lipman, and A. Rustichini. Recent developments in modeling unforeseen contingencies. *European Economic Review*, 42:523–542, 1998.

[4] L. Epstein, M. Marinacci, and K. Seo. Coarse contingencies and ambiguity. *Theoretical Economics*, 2(4):355–394, 2007.

[5] L. Epstein and M. Schneider. Recursive multiple-priors. *Journal of Economic Theory*, 113:1–31, 2003.

[6] L. Epstein and S. Zin. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica*, 57(4):937–969, 1989.

[7] X. Gabaix and D. Laibson. A boundedly rational decision algorithm. *The American Economic Review*, 90(2):433–438, May 2000.

[8] P. Ghirardato. Coping with ignorance: Unforeseen contingencies and non-additive uncertainty. *Economic Theory*, 17(2):247–276, 2001.

[9] I. Gilboa and D. Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18:141–153, 1989.

[10] I. Gilboa and D. Schmeidler. Additive representations of non-additive measures and the Choquet integral. *Annals of Operations Research*, 52: 43–65, 1994.

[11] P. Klibanoff, M. Marinacci, and S. Mukerji. Recursive smooth ambiguity preferences. *Journal of Economic Theory*, 144:930–976, 2009.

[12] D. Kreps. A representation theorem for preference for flexibility. *Econometrica*, 47:565–576, 1979.

[13] D. Kreps. Static choice and unforeseen contingencies. In P. Dasgupta, D. Gale, O. Hart, and E. Maskin, editors, *Economic Analysis of Markets and Games: Essays in Honor of Frank Hahn*, pages 259–281. MIT Press, Cambridge, MA, 1992.

[14] D. Kreps. Anticipated utility and dynamic choice. In D. Jacobs, E. Kalai, and M. Kamien, editors, *Frontiers of Research in Economic Theory*. Cambridge University Press, 1995.

[15] D. Kreps and E. Porteus. Temporal resolution of uncertainty and dynamic choice theory. *Econometrica*, 46(1):185–200, 1978.

[16] S. Mukerji. Understanding the nonadditive probability decision model. *Economic Theory*, 9:23–46, 1997.