Competing Auctioneers *

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Abstract

We study a model where multiple sellers with limited supply strategically choose both the price formation rule and the allocation rule. We show that inefficiency in this setting arises *both* because sellers withhold the good *and* because they sell to lower valuation buyers despite the presence of higher valuation buyers. This contrasts with the findings of the literatures on monopolistic and competing sellers, which suggest that the only form of inefficiency that arises is from sellers choosing to withhold the good. Both types of inefficiencies vanish as the number of sellers in the market grow large: it is an equilibrium for all sellers to offer efficient mechanisms in the limit.

Keywords: efficiency, optimal auctions, competition

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1 Introduction

A central concern of micro-economic theory is the allocative efficiency of market settings: will scarce resources/goods be allocated (sold) to the agents who value them the most? At a basic level, the answer is well understood. We expect perfect competition among sellers to result in allocative efficiency and monopolies to result in inefficiency. The case of oligopoly hinges on the type of competition between the sellers, e.g. Bertrand vs. Cournot competition. Bertrand competition results in allocative efficiency for any number of identical sellers, Cournot competition converges to efficiency as the number of sellers grows large.

However, at a more nuanced level, the answer is less clear. The literatures on monopolistic and competing sellers suggest that in market settings, inefficiencies result from sellers choosing to withhold the good. For example, the classical welfare loss from monopolyis measured as the social cost of the seller withholding the good. However, there is another form of inefficiency. A seller could choose to 'misallocate' the good, i.e. allot the good to a low value buyer despite higher value buyers being present. We construct a model of competing auctioneers, each with limited quantity, where the sellers are free to chose any auction form. We show that in this setting, both forms of inefficiency will be seen in equilibrium.

A long literature beginning with Bertrand (1883) and Edgeworth (1925) has looked to characterize outcomes in settings of competition among capacity constrained sellers. However, that line of work usually specifies an exogenous allocation rule and/or limits the price formation rule (e.g. sellers can only post a price). These rules generally do not allow the seller to misallocate the good, he can only withhold it if he chooses. Thus they assume away the second form of inefficiency that we identify. We allow sellers to strategically announce auctions, hence rendering both the allocation rule and the price formation rule endogenous. In this setting, we investigate qualitative features of the auctions chosen by sellers in equilibrium.

A model of competing auctioneers is also appealing from a positive standpoint. There are several settings where sellers use formal auctions to sell the good: the search-engine services Google, Yahoo! and Microsoft use auctions to sell space for 'sponsored links' on their search results pages to a common pool of advertisers. Even when sellers do not formally use auctions, sellers may be strategically choosing both the rationing rule and the pricing rule: for example, airlines pricing their tickets through yield management systems. Equilibria in models of competing auctioneers may shed light on the outcome in such settings.

The theoretical literature on auctions has little to say about how a seller should design an auction when he faces competition. The key difficulty is that fixing the mechanisms offered by competitors, a seller's choice of mechanism will determine which buyers participate in his mechanism. In other words, both the number of buyers participating in the mechanism and the distribution of valuations among these visiting buyers is endogenously determined. Monopolist mechanism design is easier since both are exogenously specified.

Further, little is known about what auctions will emerge in equilibrium among multiple sellers with limited supply, or whether the resulting allocations are efficient. The slender literature is bookended by seminal results on the two limiting cases. The first, Myerson (1981), considers the case of a monopoly seller with limited supply and determines that he should employ an efficient auction with an appropriately chosen reserve price, under what are thought of as weak assumptions on the distribution of buyers' valuations. The second, McAfee (1993), considers the case of 'large markets,' i.e. as the number of sellers goes to infinity. He determines that it is an equilibrium for all sellers to employ a second price auction with no reserve, regardless of the distribution of buyers' valuations.[1] The intermediate case of oligopoly was considered first by Burguet & Sakovics (1999). However, this and subsequent papers confine sellers to using a second price auction with reserve, i.e. inefficiencies result solely from the seller withholding the good. Thus, it is assumed that the optimal auction form for a monopolist seller and in the case of perfect competition must also be chosen in the case of oligopoly. We show that this assumption is only justified under strong additional conditions.

1.1 Overview and Discussion of Results

We assume two sellers each with a single unit selling to a common pool of n buyers whose private valuations (types) are i.i.d. from a common distribution F. Sellers simultaneously announce mechanisms, i.e. a function from a vector of reports to an allocation rule and a pricing rule.[2] Buyers then simultaneously choose which seller to visit based on both their own private valuation and the sellers' announcements– each buyer can participate in the mechanism of only one seller. Thus, in considering the design of his mechanism, the seller must take into account that his choice of mechanism will impact both, the distribution of buyer types who appear at his mechanism, *and* the outside option of each buyer.

We make the following contributions to understanding equilibria among competing sellers:

1. We establish the existence of a symmetric equilibrium among sellers in our setting. (Theorem 2)

[1] In order to better contrast with our own results, a discussion of the related literature is deferred to Section 4.

[2] As a caveat, we should point out that this class of mechanisms is not fully general in that, for example, we do not allow a seller to announce mechanisms which are contingent on his competition's announced mechanism. A formal definition of allowed mechanisms in our model, and some discussion of what this definition includes and excludes is in Section 2.2.

2. We study the best response of a seller when the other seller offers a quasi-efficient mechanism. By quasi-efficient, we mean inefficiencies result solely from the seller withholding the good.[3] We identify sufficient (and almost necessary in a sense made precise) conditions for the best response to be quasi-efficient. These conditions amount to requiring the distribution F over valuations to be 'close' to uniform. (Theorem 3)

3. a) We show that these conditions are also sufficient for there to exist an equilibrium in quasi-efficient mechanisms.

   b) We show that if these conditions are violated, there may be no equilibrium in quasi-efficient mechanisms. In particular, sellers may not sell the good to the buyer with the highest valuation, and may instead sell to a buyer with a lower valuation. (Proposition 1)

4. Finally, we show that as the number of sellers m goes to infinity, both sorts of inefficiencies disappear- sellers offer second price auctions with no reserve: hence we recover the results of McAfee (1993) and Peters & Severinov (1997) in our more general setting. (Proposition 2)

The key insight driving our results is about how a seller should best respond when the other seller announces a second price auction with reserve. Intuition from the standard theory of price competition suggests an efficient mechanism, e.g. a second price auction with no reserve price. The literature on monopolist sellers suggests a quasi-efficient mechanism, e.g. a second price mechanism with a reserve price. On closer inspection however, both intuitions fail in this setting.

For a monopolist seller, lowering the reserve price r has two opposing effects– including more low valuation buyers while decreasing the price at which the good is sold in some scenarios.

Instead, suppose there are two sellers, and suppose sellers 1 and 2 have announced second price auctions with reserve prices $r_1 \leq r_2$ respectively. To understand the effect of a seller's choice of reserve price, we must first understand which buyers visit each seller:

- Buyers with valuations in the range $[0, r_1]$ are excluded by both sellers. Therefore their visit decisions are irrelevant.

- Buyers with valuations in the range $[r_1, r_2]$ are excluded by seller 2. Therefore they exclusively visit seller 1.

- Consider a buyer with valuation slightly larger than r_2. Suppose this buyer visited seller 2. Clearly, even if she won the good, she would make very close to zero surplus. Therefore she must strictly prefer to visit seller 1. By continuity, there exists a $v_I \geq r_2$ such that all buyers with valuations in the interval $[r_2, v_I)$ to strictly prefer seller 1.

[3] See Definition 3 for a formal definition in our setting.

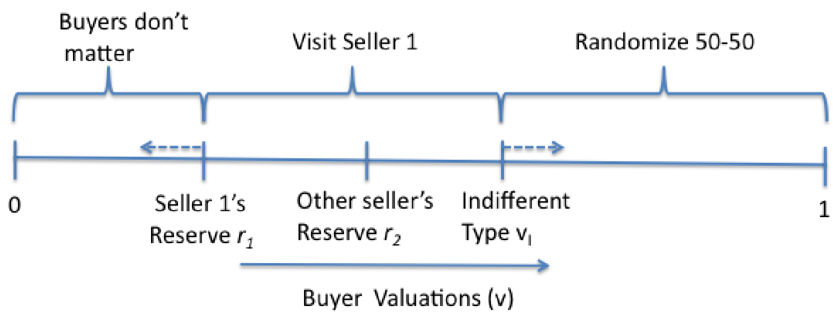

FIGURE 1: Buyer Visit Decisions

- All buyers with valuations larger than v_I are indifferent between the two sellers, and choose to visit with either seller with equal probability.

Fixing the reserve price announced by (say) seller 2, seller 1's reserve price determines not only the marginal excluded buyer (r_1), but also the marginal 'high valuation' buyer who exclusively visits his mechanism (v_I). A lower reserve price induces low value buyers to visit as in the monopolist case. However, it also causes more high valuation buyers to exclusively visit seller 1 rather than randomize across the two sellers: see Figure 1 for buyer visit decisions and Figure 2 for resulting buyer surpluses. The best response for seller 1 among the class of second price mechanisms must choose a reserve price to trade off these two effects: this analysis was done in Burguet & Sakovics (1999).

However this does not answer whether a second price mechanism is seller 1's best response to seller 2's announcement.[4] To build intuition for why not, consider the following thought experiment. Fix the mechanism announced by seller 2 (second price auction with reserve r_2) and the highest valuation type (v_I) who exclusively visits seller 1. Buyers with valuation larger than v_I behave as above, i.e they split across the two sellers; while buyers with a lower valuation exclusively visit seller 1. Is a second price auction with appropriate reserve the only mechanism seller 1 can announce to induce these visit decisions from buyers? No! Any mechanism announced by seller 1 which (1) is efficient above v_I, (2) offers exactly enough surplus to a buyer with valuation v_I, and (3) offers weakly more surplus to buyers with valuation below v_I will induce the same visit decisions among buyers.

A non-quasi-efficient mechanism may produce higher revenue than a second price auction with reserve. Why? Loosely speaking, second price mechanisms are revenue maximizing for a monopo-

[4]When discussing best-responses of a seller, we will maintain the convention that seller 2 is the 'incumbent' seller and seller 1 is the responding seller.

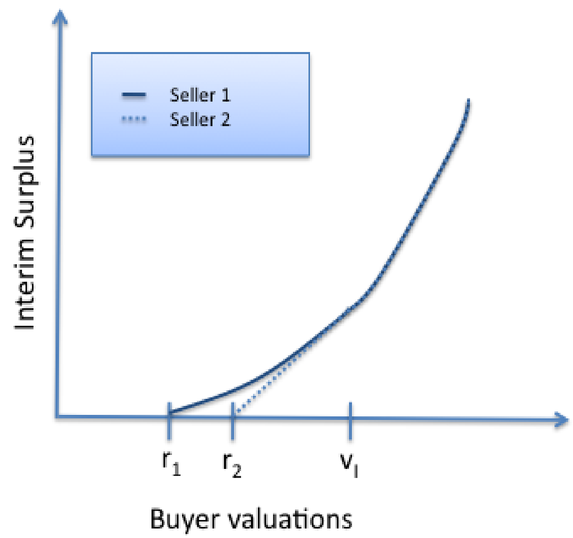

FIGURE 2: Buyer Expected Surpluses

list. Therefore, they leave less surplus for buyers: by assumption, seller 1 is a monopolist seller to buyers with valuations $[0, v_I]$. As a result, restricting attention to second price mechanisms may require a sub-optimally low reserve price in order to provide this marginal buyer (v_I) with enough surplus. Offering a non-quasi-efficient mechanism may better trade-off the task of extracting revenue from low types $[0, v_I)$, while providing enough surplus to high types (v_I). Stated alternately, the cost to the seller of 'misallocating' the good may be less than allocating correctly but with a low reserve price.

Figure 3 shows the buyer surpluses when seller 1 pools buyers on interval $[r_1', v_I]$, $r_1' > r_1$, and is efficient on the interval $[v_I, 1]$. Due to the pooling, the surplus curve on the interval $[r_1', v_I]$ is linear rather than strictly convex. This pooling allows him to (profitably) exclude buyers in the interval $[r_1, r_1']$, while maintaining buyer v_I's expected surplus.

The analysis below clarifies when restricting attention to quasi-efficient mechanisms is without loss of generality.

1.2 Technical Contribution

A technical contribution of this paper is to overcome an obstacle that previous authors have assumed away. In the case of (finite) competition, given the mechanisms announced by other sellers, the choice of mechanism by one seller changes the distribution of buyers who visit each. Therefore the outside option of each buyer is determined by the seller's own choice of mechanism. This renders

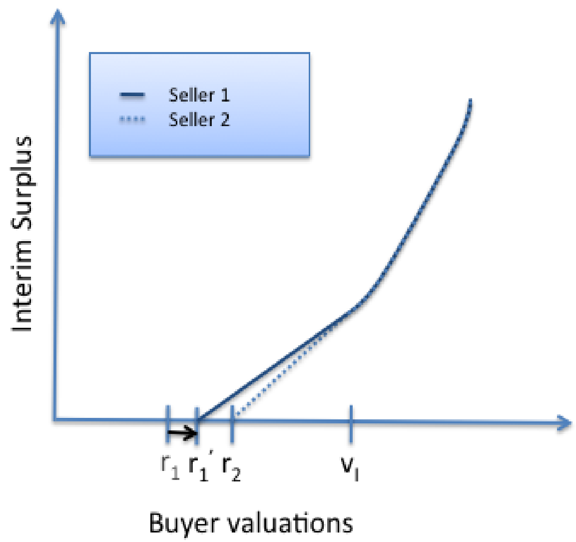

FIGURE 3: New Expected Surpluses

standard approaches to mechanism design untenable.[5] There are two key insights which make analysis in this setting feasible:

First, the literature on mechanism design with Bayes-Nash equilibrium as the solution concept normally defines mechanisms in the interim space. Each buyer is offered an expected allocation as a function of his report, the expectation being taken over the reports of other buyers. This requires the distribution of types visiting the seller be fixed, whereas in the case of competing sellers the distribution is endogenous. Hence, this approach is a non-starter. The other standard approach is to represent a mechanisms as an allocation rule and a price rule for each possible vector of reports by the buyers. This is distribution-free but intractable for analysis in a Bayesian setting. We show how to specify the allocation rule in a distribution free manner, without resorting to explicitly specifying the allocation rule for each profile of buyer reports. We thus identify a class of mechanisms which are both sufficiently general, yet easy to work with.

We then show how a part of the problem of a seller selecting his best response to other sellers' mechanism(s) can be formulated as a linear program. The steps that lead to the key qualitative insight of our paper are therefore relatively transparent.

[5]Our results do not, therefore, follow as a consequence of the results on contracting with type-dependent outside options, e.g. Rochet & Stole (2002) and Jullien (2000).

1.3 Organization

As regards the organization of the rest of this paper: Section 2 lays out the model we study and our solution concept. Section 3 then considers the best response of a seller to second price auction with reserve. It then discusses the implications for equilibria among sellers. Section 4 discusses more thoroughly this paper's relation to the literature. Section 5 concludes, and also discusses various desirable extensions to this model and pitfalls thereof. Most proofs are relegated to the appendix.

2 Model and Preliminaries

2.1 The Market

There is a set $M = \{1, 2, \ldots m\}$ of risk neutral sellers each looking to sell 1 unit of an indivisible good. All sellers have no value for the good. For most of this paper $m = 2$. There are n risk neutral buyers, each wanting at most 1 unit. Buyer i has a private valuation v_i for the good, valuations are i.i.d. draws from a common knowledge distribution with density f and cdf F over a compact support V. We normalize the support of the distribution to be a subset of $[0, 1]$.

The game we study proceeds thus:

1. Sellers compete by simultaneously offering 'auctions'- these are direct revelation mechanisms as in the monopolist case. A precise definition of what sellers can announce is in Section 2.2.

2. Buyers then simultaneously choose which seller to visit. At the time of choosing, each buyer knows her own valuation and the mechanisms announced by each seller. Each buyer can visit at most one seller.[6]

3. Each seller then executes his announced mechanism with the pool of buyers who chose to visit him.

Our equilibrium concept is weak-perfect Bayesian equilibrium. Since sellers have no private information in our setting, this amounts to requiring that : (1) each buyer visits the seller that gives her the highest interim expected surplus (calculating interim expected surplus will involve taking expectations over other buyers' choice of seller in equilibrium), and (2) a seller considering deviating takes into account the equilibrium in the sub-game among buyers that will result given his deviation.

To rule out multiple equilibria in the sub-game among buyers, we make assumptions to make this market 'anonymous.' Firstly, sellers are constrained to announce anonymous mechanisms,

[6]See Section 5 for difficulties raised by relaxing this assumption.

i.e. the outcome in a mechanism can depend only on the reports of the buyers and not on their identities. Further, we restrict attention to symmetric strategies by buyers, i.e. the buyer's choice of which seller to visit depends only on her valuation, not her identity.

Below, we describe each of the three components in further detail, i.e. which mechanisms a seller can offer, how buyers choose between sellers and our solution concept.

2.2 Pure Strategies of Sellers

A key step in this paper is to define what a 'pure strategy' of a seller is. We restrict sellers to announce mechanisms that are not contingent on the choices of the other.

Any number of buyers $0, 1, \dots n$ might chose to visit a seller. A mechanism offered by a seller can therefore be summarized as : $(Q^k, P^k)_{k=1}^n$, where:[7]

$$\forall k \leq n: \quad Q^k\colon V^k \to [0,1]^k, \qquad P^k\colon V^k \to \Re^k.$$

Here $Q^k(v_1, \dots, v_k)$ is the (possibly stochastic) allocation rule when k agents show up at the auction, and the profile of reported types is $(v_1, \dots v_k)$, and P^k is the associated expected price each agent must pay when that profile realizes.

We further require that these mechanisms satisfy some basic properties:

1. Feasibility: Feasibility requires that for any profile of reports, at most 1 unit of the good is sold, i.e. for all k and any profile of reports $\mathbf{v} \in V^k$, $\sum_{i=1}^k Q_i^k(\mathbf{v}) \leq 1$.

2. Anonymity: Sellers are constrained to use mechanisms where the outcomes depend only on the vector of reports, not on the identity of the buyers submitting them. Formally, for any $k \leq n$, $\bar{v} \in V^k$ vector of reports, and permutation σ; $Q^k(\sigma(\bar{v})) = \sigma(Q^k(\bar{v}))$ and $P^k(\sigma(\bar{v})) = \sigma(P^k(\bar{v}))$.

3. Interim Incentive compatibility: For any choice of mechanism by the other seller and distribution over number and valuations of buyers visiting the mechanism that result, it is a Bayes-Nash equilibrium for each buyer to submit his true value.

4. Individual Rationality: For any choice by the other sellers and distribution over number and valuations of buyers visiting the mechanism that result, each buyer gets non-negative expected surplus in equilibrium.

[7]Under our assumptions, this is the complete space of mechanisms the seller can offer, see McAfee (1993).

This space of feasible mechanisms is hard to work with. In particular, requirements 3 (and 4) are problematic since the set of all distributions (over number and valuation of buyers visiting) that can result from valid mechanism choices by the other seller is hard to determine.[8]

In the standard problem of mechanism design by a monopolist, the distribution of buyer types is fixed. Risk-neutrality and the rent-extraction formula allow us to express buyer surpluses and seller profit solely as a function of the *interim* allocation rule $a(\cdot)$. By interim allocation rule, we mean each type's probability of getting allotted, taking expectations over the reports of other buyers. Tractability springs from working directly with the space of (feasible) interim allocation rules.

In this setting however, the interim allocation probabilities offered by a seller will depend on the *equilibrium* distribution of types visiting the seller. Therefore, unlike the case of a monopolist seller, a mechanism cannot be summarized by just the interim allocation probabilities it awards each type (and the surplus of the lowest type), because it depends on the mechanism announced by the other seller(s).

We therefore further restrict the space of pure mechanisms available to sellers. Most allocation rules one can conceive of come down to an order over the type space, such that at each profile, the highest type(s) according to this order get allotted. For example, the standard second price auction allots to the highest bid and breaks ties randomly.

One can define a *hierarchical allocation rule* more generally as:[9]

Definition 1 (Hierarchical Allocation Rule) *Given a space of types T, a hierarchical allocation rule is an order $\succeq$ on T, and a designated element $\underline{t}$ such that:*

1. *At any profile of types allocate to the unique highest type according to the order $\succeq$ if it is larger than $\underline{t}$: formally at $(t_1, \ldots, t_k)$ allot t_i if*

$$t_i \succeq \underline{t},$$
$$t_i \succ t_j \quad \text{for all } j \neq i.$$

2. *If there are multiple highest types in a profile larger than $\underline{t}$, randomize over them: formally,*

[8] We do not restrict sellers to announcing dominant strategy implementable mechanisms. The solution concept for buyers' choices in the sub-game among buyers is Bayes-Nash. Indeed, 'most' sub-games will not be dominance solvable. It would be incongruous to require a more exacting solution concept from sellers.

[9] The term hierarchical allocation rule is due to Border (1991).

at $(t_1, \ldots, t_k)$, if $I \subseteq \{1, 2, \ldots, k\}$ such that

$$t_i \sim t_{i'} \succeq \underline{t} \quad \text{for all } i, i' \in I,$$
$$t_i \succ t_j \quad \text{for all } i \in I, j \notin I,$$

then allot each buyer in I with probability $\frac{1}{|I|}$.

3. *If all types are less than the designated element $\underline{t}$, then retain the object: formally, retain the object if for all i, $t_i \prec \underline{t}$.*

For example, in our type space (valuations in $[0, 1]$):

Example 1 *The standard order on real numbers with $\underline{t} = 0.5$ corresponds to an efficient allocation rule with reserve price 0.5.*

Example 2 *The trivial order (all types are equal) with $\underline{t} = 0.5$ corresponds to a posted price of 0.5 (with randomization/ rationing if there is more than 1 buyer).*

We identify the set of pure strategy allocation rules available to a seller with the set of hierarchical allocation rules. We further restrict the hierarchy to be a coarsening of the standard order on the reals.[10] Formally:

Definition 2 *We define the pure strategy allocation rules available to each seller as:*

$$\mathcal{A} = \{(\succeq, \underline{v}) |\, \underline{v} \in \Re, \succeq \ \text{ is an order on } [0,1],\ v \geq v' \Rightarrow v \succeq v'\}. \tag{1}$$

The space of 'pure' mechanisms will be identified with space of pure strategy allocation rules

$$\mathcal{M} = \mathcal{A}. \tag{2}$$

A mixed strategy for a seller is a probability distribution over $\mathcal{M}$, the space of all mixed strategies is denoted $\Delta\mathcal{M}$.[11] Note that if sellers choose mixed mechanisms, buyers choose between sellers *after* the uncertainty resolves. For example if each seller announces a second price auction with a stochastic reserve; each seller must also reveal what reserve price they drew from the given distribution *before* buyers decide which seller to visit.

[10] By standard arguments, incentive compatibility requires that conditional on restricting to the space of hierarchical allocation rules, the order $\succeq$ be concordant with the standard ordering on reals.

[11] We gloss over measurability, topologies etc. here— for details see Appendix B.

This set of pure strategies is clearly more restrictive than the space of all feasible mechanisms $(Q^k, P^k)_{k=1}^n$ outlined above. However it can be argued that this is set is the 'correct' set of mechanisms to consider. This is due to an observation that follows easily from Border (1991). See also (Manelli & Vincent 2008) for a related result.

Observation 1 *Fix a finite type space T, number of agents n and distribution over types F. Each agent i has a type $t_i \in T$ that is an i.i.d. draw according to F. Let $\mathbb{A} = \{a | a\colon T \to [0,1], a \text{ feasible}\}$ be the space of all feasible interim allocation rules in this setting. $\mathbb{A}$ is compact and convex. Let ρ map hierarchical allocation rules $\mathcal{A}$ to their induced interim allocation rule $\mathbb{A}$. Then:*

1. *The range of ρ is exactly the set of extreme points of $\mathbb{A}$.*
2. *If F has full support on T then ρ is a bijection from $\mathcal{A}$ to extreme points of $\mathbb{A}$.*

As a result, the pure strategies $\mathcal{M}$ we allow a seller are the extreme points of the set of mechanisms allowed a seller in the standard monopolist case. Defintion 2 therefore rules out two classes of mechanisms from the original class of mechanisms we identified:

1. Mechanisms which condition on the number of buyers participating, e.g. a second price auction where the reserve price depends on the number of buyers visiting.
2. Mechanisms where the uncertainty over the order and reserve price resolves *after* buyers choose between sellers, e.g. a second price auction with a stochastic reserve.

We rule both out for the same (technical) reason. Given the set of pure strategies as currently defined, we are able to show that there is a 'unique' resulting buyer visit decision for any choice of mechanisms by the sellers (Theorem 1). Therefore the outcome in any sub-game following seller announcements is determined. If we extend sellers' strategies to allow either of the two above, we are unable to establish uniqueness.

However, it can be shown that if the other seller announces a mechanism from $\mathcal{M}$, the best response in $\mathcal{M}$ is also a best response when the responding seller can select any mechanism in the more general space, i.e. $\mathcal{M}$ is closed under best response– therefore the equilibria we identify remain equilibria in the more general setting.

We define a mechanism that is efficient conditional on the sale of the good as a quasi-efficient mechanism. Formally:

Definition 3 *A mechanism $\mathbf{m} \in \mathcal{M}$ is said to be quasi-efficient if $\mathbf{m} = (\succeq, \underline{v})$ such that*

$$\forall v, v' \in [\underline{v}, 1] : \quad v > v' \Rightarrow v \succ v'.$$

A mixed strategy mechanism is said to be quasi-efficient if it is a randomization over pure strategy quasi-efficient mechanisms.

Standard auctions, such as second price auctions with a reserve, are quasi-efficient mechanisms.

A mechanism is said to be efficient on an interval $[v_1, v_2]$ if conditional on allotting to a buyer with valuation in $[v_1, v_2]$, it allots to the highest valuation buyer in that range. It is said to be efficient at v if types above v are always allotted over types below v in any profile where both are present.

DEFINITION 4 *A mechanism* $\mathbf{m} \in \mathcal{M}$ *is said to be efficient on* $[v_1, v_2]$ *if* $\mathbf{m} = (\succeq, \underline{v})$ *such that*

$$\forall v, v' \in [v_1, v_2] : \quad v > v' \Rightarrow v \succ v'.$$

A mechanism $\mathbf{m} \in \mathcal{M}$ *is said to be efficient at* v *if* $\mathbf{m} = (\succeq, \underline{v})$ *such that*

$$\forall v'' < v < v' : v'' \prec v'.$$

A mixed strategy mechanism is said to be efficient on $[v_1, v_2]$ *(respectively, at* v*) if it is a randomization over pure strategy mechanisms which are efficient at* $[v_1, v_2]$ *(respectively, at* v*).*

2.3 Sub-Game Among Buyers

After sellers announce the mechanisms they will employ, buyers must choose between the sellers. Sellers have no private information, and their choice of mechanisms is common knowledge before buyers must choose. It is therefore proper to refer to the buyer choice stage as the *sub-game among buyers*.

Define

$$\Theta = \{\theta |\ \theta \colon V \to \Delta^{|M|}\}.$$

A $\theta \in \Theta$ is called the *visit decision* of the buyers. We can write θ as

$$\theta = (\theta_j)_{j=1}^m, \quad \theta_j : V \to [0, 1],$$

such that $\sum_j \theta_j(v) = 1$ for all $v \in V$. This is to be interpreted as saying that a buyer with valuation v visits seller j with probability $\theta_j(v)$.

We can treat buyers who visit seller $j' \neq j$ as reporting a value of 0 at seller j.[12] Given θ,

[12]This simplifies calculations since this way we do not need to consider multinomial distribution over the number of buyers visiting a seller. This approach is feasible due to our assumption that sellers cannot

therefore, each seller j will see n reports which are i.i.d. draws from the 'effective distribution:'

$$F_j(v) = 1 - \int_v^1 \theta_j(v) f(v) dv. \tag{3}$$

This distribution will have an atom at 0 even though the population distribution of buyer types is atomless: this atom corresponds to the ex-ante probability with which a buyer visits some other seller. This 'effective distribution' is extremely convenient as a representation– the number of agents visiting the mechanism is now exogenously specified (n), and therefore buyer surpluses and seller revenues are much easier to compute.

Let $a_j(v)$ be the interim allocation probability of a buyer of valuation v if he visits seller j (taking expectation over the types and visit decisions of other buyers, equivalently n buyers with i.i.d. draws from F_j). Given the space of mechanisms which we restrict the sellers to, a_j must be non-decreasing.

Observation 2 *Let the interim expected payment of a bidder with valuation* 0 *who shows up at seller j be* 0*. The standard rent extraction formula still applies in this setting (see for example McAfee (1993)). Therefore:*

$$p_j(v) = va_j(v) - \int_0^v a_j(x) dx. \tag{4}$$

The interim expected surplus of a buyer with valuation v from visiting seller j therefore is:

$$s_j(v) = \int_0^v a_j(x) dx. \tag{5}$$

Our solution concept, weak perfect Bayesian equilibrium requires that agents' actions are sequentially rational given beliefs, and that their beliefs are correct. In our setting, sellers have no private information. Therefore, weak perfect Bayesian equilibrium requires that buyers' visit decisions in each buyer sub-game constitute a Bayes-Nash equilibrium of that sub-game. Fix a sub-game resulting from seller reports. If θ is a symmetric Bayes-Nash equilibrium for the sub-game, we say that θ is sequentially rational. Formally:

Definition 5 *Fix the mechanisms offered by each seller, and buyer visit decisions θ. Let $s_j(\cdot)$ be the resulting expected surplus function offered by seller j. θ is said to be sequentially rational if:*

$$\forall v : \quad \theta_j(v) > 0 \iff j \in \arg\max_{k \in M} s_k(v).$$

Further, it can be shown that given a mechanism announced by each seller, there is a 'unique'

announce mechanisms which condition on the number of buyers who visit their mechanism.

sequentially rational θ. Formally:

THEOREM 1 *Suppose each seller j announces mechanism m_j. A sequentially rational visit decision for buyers θ exists. Further, let θ and θ' be two (distinct) sequentially rational visit decisions by the buyers. Let $a_j(v)$ and $a'_j(v)$ be the interim allocation probability of a buyer of valuation v at seller j given visit decisions by buyers θ and θ' respectively. Then:*

1. *$\forall j$, $a_j = a'_j$ almost everywhere.*
2. *For each seller j, expected revenue under θ = expected revenue under θ'.*

Thus, given the mechanisms announced by the sellers, the (expected) outcome in the sub-game among buyers is determined. We will refer without ambiguity to *the* implied visit decision θ, since all buyers and sellers are indifferent over all sequentially rational θ's.

SKETCH OF PROOF The crux of the argument goes thus- ignore buyers that are served at neither seller. Suppose θ and θ' are two sequentially rational buyer visit decisions which differ on a set of valuations of non-zero probability (if they differ only on a set of 0 probability, clearly they will result in the same buyer surpluses and seller revenues). Now suppose θ causes buyers to visit seller 1 with a lower ex-ante probability than θ'. This will make seller 1 more attractive to all buyers (since there is 'less' competition for the good) relative to θ'. For similar reasons, seller 2 must be less attractive to all buyers, since there is more competition for the good, relative to θ'. This should intuitively cause more (respectively less) buyers to visit 1 (respectively 2) in θ than in θ'. This contradicts our claim that θ causes buyers to visit seller 1 with a lower ex-ante probability than θ'.

Having argued that both candidate sub-game equilibria θ and θ' must have buyers visit sellers with the same ex-ante probability, one could ask whether buyers could randomize differently across mechanisms while maintaining the same ex-ante probability of visiting any seller. We show that this cannot be the case. To see why, consider again the case where both sellers announce second price auctions with reserve prices $r_1 \leq r_2$. Recall that we suggested that in the sub-equilibrium, there would exist some $v_I \geq r_2$, such that all buyers with valuations in the interval $[r_1, v_I)$ would visit seller 1 exclusively, while buyers with valuations in $[v_I, 1]$ would randomize $50 - 50$ across the two mechanisms. Is there some other way for buyers with valuations in $[v_I, 1]$ to randomize, while maintaining indifference, and such that conditional on a buyer having value in $[v_I, 1]$, he visits each seller with probability 0.5?

The answer is no (up to sets of probability 0). Let $[v, v'] \subseteq [v_I, 1]$ be a sub-interval that visits (say) seller 1 with probability > 0.5. Then buyers in this interval could not be indifferent across the two mechanisms: recall that by revenue equivalence, i.e. (4), the rents that accrue to a type is the integral of allocation to types below it. If buyers are randomizing differently than 0.5, then rents in this interval grow at different rates, contradicting our original hypothesis.

Finally one could ask, given the above sketch, why a stronger statement of the form, "any two sub-game equilibria θ, θ' must differ only on a set of measure 0" is not true. Suppose both sellers announce the same posted price p: if multiple buyers visit a seller, the seller uniformly picks one of these buyers. Each buyer with valuation $[p, 1]$ randomizing $50 - 50$ is an equilibrium in the sub-game. However, a visit decision where all buyers in the interval $[p, v]$ visit seller 1 and buyers in the interval $(v, 1]$ visit seller 2, such that v solves

$$F(v) - F(p) = 1 - F(v) = (1 - F(p))/2,$$

is also clearly an equilibrium. These two visit decisions differ on the entire set of valuations $[p, v]$, but clearly they are outcome indifferent in terms of expected surplus of buyers in either equilibrium and expected revenue of sellers.

Appendix B.1 provides a formal proof.

2.4 Solution Concept

Our equilibrium concept is weak perfect Bayesian equilibrium. We restrict attention to equilibria where buyers play symmetric strategies. We shall therefore call our solution concept Buyer symmetric weak perfect Bayesian equilibrium (BSWPBE). Sellers have no private information in our model, and there are only 2 stages, a simultaneous move stage for sellers followed by a simultaneous move stage for buyers. Therefore, in this model, BSWPBE can be defined thusly:

Definition 6 (Buyer Symmetric Weak Perfect Bayesian Equilibrium) *We say that a mechanism choice* $\mathbf{m_j} \in \Delta\mathcal{M}$ *by each seller* j*, and the associated visit decisions by buyers* θ *constitute a buyer symmetric weak perfect Bayesian equilibrium if:*

1. θ *satisfies sequential rationality of buyers given the mechanisms chosen by each seller.*
2. *A unilateral deviation by a seller* j *to mechanism* $\mathbf{m_j}'$ *with the new induced visit decision* θ' *gives weakly less expected revenue to seller* j*.*

Among the papers mentioned before, both Burguet & Sakovics (1999) and Peters & Severinov (1997) consider this solution concept, but restrict sellers to announcing second price mechanisms with their only strategic choice being the reserve price. McAfee (1993) allows sellers to announce mechanisms from a more general class of mechanisms. However, in McAfee's (1993) solution concept, when a seller j considered a deviation he *did not* take into account the effect on other sellers.

Thus when considering a deviation, seller j assumed the interim expected surplus of each type at other sellers would stay the same if he deviated. The seller anticipated the visit decision by buyers to his mechanism and expected revenue under this assumption. Since McAfee wanted to study large markets, this assumption was tenable– a seller in a large market should have a negligible impact on other sellers. However, as McAfee himself points out, this assumption is inappropriate for small markets.

THEOREM 2 *There exists a Buyer Symmetric Weak Perfect Bayesian Equilibrium where each seller chooses the same mechanism* $\mathbf{m} \in \Delta\mathcal{M}$.

3 Competing Sellers

In this section, we consider first the best response of seller 1 when seller 2 announces a quasi-efficient mechanism. We provide sufficient conditions on the population distribution of buyers for seller 1's best response to be quasi-efficient. These conditions are stronger than the standard monotone hazard rate conditions required by Myerson (1981). We then show that these conditions are sufficient for an equilibrium in quasi-efficient mechanisms to exist. Finally, we show that both sources of inefficiency (sellers choosing to withhold the good and sellers 'misallocating the good) vanish as the number of sellers grows.

3.1 Best Response to Quasi-efficient mechanisms

Suppose seller 2 announces a mechanism $\mathbf{m} \in \mathcal{M}$ from the class we identified above. How should seller 1 best respond? Stated alternately, how should one approach the problem of mechanism design under competition?

As we pointed out above, the key difficulty is that fixing the mechanism chosen by seller 2, seller 1's choice of mechanism affects the distribution of types visiting his mechanism. Seller 1, therefore, needs to choose his best response taking into account how this distribution changes as a function of the mechanism he chooses. We show how seller 1 should best respond to a quasi-efficient mechanism.

THEOREM 3 *Suppose seller* 2 *announces a quasi-efficient mechanism with reserve* r_2. *Seller* 1*'s best response is quasi-efficient if the distribution of buyers' types satisfies increasing hazard rate, i.e.*

$$\frac{f(v)}{1-F(v)} \text{ is weakly increasing in } v, \tag{IHR}$$

and decreasing density, i.e.

$$f(v) \text{ is weakly decreasing in } v. \tag{DD}$$

Sketch of Proof This is the key result in our paper, and the proof is a reasonably involved, so we provide an overview to orient the reader. The proof proceeds in 4 main steps.

1. In earlier discussions we stated that when if a buyer is indifferent between 2 quasi-efficient mechanisms, he must randomize $50-50$ between the two in equilibrium. Lemma 1 proves this.

2. Next, consider the best response of seller 1 *among* the class of quasi-efficient mechanisms with reserve $r_1 \leq r_2$. Lemma 2 shows that if the distribution of types satisfies the increasing hazard rate property, (IHR), then seller 1's best response among this class must have a negative virtual value, i.e. $r_1 - \frac{1-F(r_1)}{f(r_1)} \leq 0$.

3. The next step is the heart of the proof. If seller 1 announces a quasi-efficient mechanism with reserve $r_1 < r_2$, there must exist a $v_I > r_2$ such that all buyers with valuations in $[r_1, v_I)$ solely visit seller 1, while all buyers with valuations in $[v_I, 1]$ are indifferent between the two sellers. Fixing this first point of indifference v_I, is seller 1 being quasi-efficient on $[0, v_I]$ revenue maximizing? Lemma 3 answers this question. It shows that this question can be formulated as one of monopolist auction design, with the added constraint that valuation v_I gets enough surplus to keep her indifferent between both sellers. It then shows (IHR) and (DD) are sufficient to guarantee that seller 1's choice of mechanism will be quasi-efficient on $[0, v_I]$.

4. The final step of this proof shows that under the same conditions, i.e. (IHR) and (DD), seller 1's best response will be efficient on $[v_I, 1]$. (Lemma 4)

Proof: As suggested above, the proof proceeds in 4 steps.

Step 1 Recall that by Theorem 1, for any choice of mechanisms by the two sellers, $\mathbf{m}_1$ and $\mathbf{m}_2$, the visit decisions by each type of buyer θ will be essentially unique.

Therefore, given sellers' choices $\mathbf{m}_1$ and $\mathbf{m}_2$, we can compute the effective distribution of valuations faced by each seller j, F_j, and the surplus a buyer of type v can expect if he visits seller j, $s_j(v)$. Computing F_j might not be easy. The following lemma partially alleviates this problem: it says that if for some interval $[\underline{v}, \bar{v}]$, if $s_1(v) = s_2(v)$ at both end points, and both mechanisms are efficient on the interval, then buyers with valuations in that interval must visit each seller with equal probability (i.e. regardless of what the mechanisms do elsewhere). Versions of this result can be found in Peters & Severinov (1997) and Burguet & Sakovics (1999).

Lemma 1 *Suppose the sellers announce mechanisms* $\mathbf{m}_1$ *and* $\mathbf{m}_2$*, and the resulting buyer visit decisions are* θ*. Let* $s_j : V \rightarrow \Re_+$ *be the resulting interim surplus offered by seller* j*. Suppose there exists* $[\underline{v}, \bar{v}]$*, such that:*

$$s_1(v) = s_2(v), \quad \textit{for } v = \underline{v}, \bar{v},$$
$$\textit{and } \mathbf{m}_1, \mathbf{m}_2 \textit{ are efficient on } [\underline{v}, \bar{v}].$$

Then, $\theta_j(v) = 0.5$ *for* $v \in [\underline{v}, \bar{v}], j = 1, 2.$

Step 2 Consider seller 1's best response among the class of quasi-efficient mechanisms. If seller 1 announces quasi-efficient mechanism with reserve $r_1 < r_2$, then there will exist $v_I > r_2$ such that

$$\theta_1(v) = \begin{cases} 1 & v \in [0, v_I] \\ 0.5 & v > v_I. \end{cases} \tag{6}$$

Alternately if seller 1 announces a quasi-efficient mechanism with reserve price $r_1 > r_2$, then there will exist $v_I > r_1$ such that:

$$\theta_1(v) = \begin{cases} 0 & v \in [0, v_I] \\ 0.5 & v > v_I. \end{cases} \tag{7}$$

Here, v_I solves

$$\int_{r_1}^{v_I} a_1(v)dv = \int_{r_2}^{v_I} a_2(v)dv,$$

which can be further simplified (since both mechanisms are second price mechanisms) to:

$$r_1 < r_2 \Rightarrow \quad \int_{r_1}^{v_I} F_1^{n-1}(v)dv = (v_I - r_2)F_2^{n-1}(v_I). \tag{8}$$

$$r_1 > r_2 \Rightarrow \quad (v_I - r_1)F_1^{n-1}(v_I) = \int_{r_2}^{v_I} F_2^{n-1}(v)dv. \tag{9}$$

The next lemma shows that seller 1's best response among quasi-efficient mechanisms will always have a reserve r_1 that has a non-positive virtual valuation.

Lemma 2 *Suppose the distribution of buyer valuations is* f, F *and satisfies the monotone hazard rate condition (IHR). Further, suppose seller* 2 *announces a quasi-efficient mechanism with reserve* r_2*. Then seller* 1*'s best response among quasi-efficient mechanisms that have reserve* $r_1 \leq r_2$ *must* *satisfy:*

$$r_1 - \frac{1-F(r_1)}{f(r_1)} \leq 0. \tag{10}$$

Fix $v_I \geq r_2$, and assume that $\theta_1(v) = 1$ for all $v \in [0, v_I]$. Consider a feasible allocation rule $a_1(\cdot)$ such that:

$$a_1 \text{ is efficient on } [v_I, 1], \tag{11}$$

$$\int_0^v a_1(t)dt \geq \int_0^v a_2(t)dt = (v - r_2)F_2^{n-1}(v) \quad \forall\, v \in [r_2, v_I], \tag{12}$$

$$\int_0^{v_1} a_1(t)dt = (v_I - r_2)F_2^{n-1}(v_I). \tag{13}$$

Any such mechanism will result in the same buyer visit decision θ as (6)– therefore the virtual revenue from values $[v_1, 1]$ is fixed regardless of how seller 1 treats buyers with valuations in $[0, v_I]$. Therefore, fixing v_I, and requiring (11-13) above, one could ask whether a_1 being quasi-efficient is revenue maximizing (and if not, what is). Having thus found the revenue maximizing mechanism for each (feasible) $v_I \in [r_2, 1]$, we can simply maximize over v_I to find the optimal mechanism that can result in a 'cutoff' θ of type (6).

We now proceed to:

1. Show that (IHR) and (DD) are sufficient for the revenue maximizing mechanism on $[0, v_I]$ to be quasi-efficient (Lemma 3 below).
2. Show that they are also sufficient conditions for restricting attention to 'cutoff' θ's of type (6) or (7) (Lemma 4).

Thus, the lemmata below conclude the proof of Theorem 3.

Step 3

Lemma 3 *Suppose seller* 2 *announces a second price auction with reserve price* r*. Consider the best response* $\mathbf{m}_1$ *of seller* 1 *subject to:*

1. $\mathbf{m}_1$ *is efficient at* v_I *(this is a relaxation of constraint (11)).*
2. *Constraints (12) and (13), i.e. all buyers of value less than* v_I *go to seller* 1*, and* v_I *is the lowest valuation indifferent between the two sellers.*

Then, $\mathbf{m}_1$ *will be quasi-efficient on* $[0, v_I]$ *if the distribution of types satisfies (IHR) and (DD).*

Proof: By (the proof of) Theorem 1, $\theta(v)$ for $v \in [v_I, 1]$ does not depend on seller 1's choice of mechanism for $v \in [0, v_I]$ as long as $\mathbf{m}_1$ is efficient at v_I and (12) and (13) are satisfied. Therefore, fixing v_I and $\mathbf{m}_1$ on $[v_I, 1]$ fixes θ. The effective distribution F_2 can thus be computed:

$$F_2(0) = 1 - \int_{v_I}^{1} \theta_2(v) f(v) dv,$$

$$F_2(v) = \begin{cases} F_2(0) & v \leq v_I \\ 1 - \int_v^1 \theta_2(t) f(t) dt & v > v_I, \end{cases}$$

$$s_2(v) = \begin{cases} 0 & v \leq r_2 \\ (v - r_2)(F_2(v_I))^{n-1} & v \in [r_2, v_I]. \end{cases}$$

Any type $v \in [r_2, v_I]$ visiting seller 2 will pay the reserve price if he wins (since types in that interval visit seller 2 with probability 0), type v's surplus from visiting seller 2 is just his surplus contingent on winning $(v - r_2)$ times his probability of winning.

Fixing the mechanism seller 1 announces on $[v_I, 1]$, the 'optimal' mechanism for seller 1 to announce on $[0, v_I]$ is:

$$\max_a \int_0^{v_I} \left(v - \frac{1 - F_1(v)}{f_1(v)} \right) a(v) f_1(v) dv \tag{Opt}$$

$$\text{s.t.} \quad \int_0^{v_I} a(v) dv = (v_I - r_2) F_2^{n-1}(v_I)$$

$$\int_0^{v} a(x) dx \geq (v - r_2) F_2^{n-1}(v) \quad \forall v \in [r, v_I]$$

$a(v)$ increasing in v

a feasible

Next, a is feasible if and only if:

$$\forall v < v_I : \quad \int_v^{v_I} a(t) f(t) dt \leq \frac{(F_1^n(v_I) - F_1^n(v))}{n}, \tag{14}$$

$$\int_{v_I}^{1} a(t) f_1(t) dt = \frac{1 - F_1^n(v_I)}{n}, \tag{15}$$

$$\forall v > v_I : \quad \int_v^{1} a(t) f_1(t) dt \leq \frac{1 - F_1^n(v)}{n}. \tag{16}$$

To see this, note that $\mathbf{m}_1$ being efficient at v_I requires (15). Therefore a is feasible and efficient around v_I if and only if (14 -16): see Border (1991) or Appendix A.

Further, note that the objective function of (Opt) can be rewritten as: [13]

$$\int_0^{v_I}\left(v-\frac{1-F(v)}{f(v)}\right)a(v)f(v)dv+\left(\int_{v_I}^1(1-\theta_1(t))f(t)dt\right)(v_I-r)F_2^{n-1}(v_I).$$

Clearly the second term is a constant (only depends on f, v_I and a *above* v_I). Therefore we can rewrite (Opt) as:

$$\max_a \int_0^{v_I}\left(v-\frac{1-F(v)}{f(v)}\right)a(v)f(v)dv \tag{ConsOpt}$$
$$\text{s.t. } (12),(13),(14),\ a(v) \text{ increasing in } v.$$

Relax the inequalities (12) in ConsOpt. The relaxed program becomes:

$$\max_a \int_0^{v_I}\left(v-\frac{1-F(v)}{f(v)}\right)a(v)f(v)dv \tag{RConsOpt}$$
$$\text{s.t. } (13),(14),\ a(v) \text{ increasing in } v.$$

The Lagrangian relaxation of (13) in (RConsOpt) is:

$$\max_a \int_0^{v_I}\left(v-\frac{1-F(v)}{f(v)}\right)a(v)f(v)dv-\lambda\left(\int_0^{v_I}a(v)dv+s_2(v_I)\right) \tag{LConsOpt}$$
$$\text{s.t. } (14),\ a(v) \text{ increasing in } v. \tag{17}$$

The objective function of (LConsOpt) can be rewritten as

$$\int_0^{v_I}\left(v-\frac{1-F(v)}{f(v)}-\frac{\lambda}{f(v)}\right)a(v)f(v)dv.$$

If the solution is quasiefficient, then $\lambda=(r_1-\frac{1-F(r_1)}{f(r_1)})f(r_1)$ where r_1 is the lowest type allotted by seller 1. By Lemma 2, $(r_1-\frac{1-F(r_1)}{f(r_1)})$ is non-positive.

Further, our program is similar to Myerson's (1981) program, with the standard 'virtual valuation' $v-\frac{1-F(v)}{f(v)}$ replaced by $v-\frac{1-F(v)}{f(v)}-\frac{\lambda}{f(v)}$. Therefore for the solution to be quasi efficient on $[0,v_I]$ we require that $v-\frac{1-F(v)}{f(v)}-\frac{\lambda}{f(v)}$ be non-decreasing in v, $v\in[0,v_I]$. Note that $\frac{1-F(v)}{f(v)}$ and $f(v)$ decreasing in v is sufficient for this. Finally note that since $r_1\le r_2$ by hypothesis, it is easy to verify that this solution to (RConsOpt) is feasible in (ConsOpt) and therefore optimal, hence completing the proof. □

Note that necessary and sufficient conditions for quasi efficiency are weaker- they only require

[13]The slightly involved calculations are in Appendix C.3.

that $v - \frac{1-F(v)}{f(v)} - \frac{\lambda}{f(v)}$ is weakly increasing for the appropriate λ. The stronger conditions of monotone hazard rate, decreasing density are listed in the statement of the lemma since these are the weakest conditions on the primitives of the problem (λ is endogenous).

If this 'adjusted virtual valuation' is not non-decreasing in v, extra ironing will be required (and hence the mechanism will not be quasi efficient). We do not explicitly characterize the solution of (Opt) in this case. [14]

Therefore, by Lemma 3 we can conclude that when conditions (IHR, DD) hold, seller 1's best response must be quasi-efficient below some v_I, where v_I is the lowest type indifferent between the two mechanisms.

STEP 4 The final step is to show that both sellers want high types to show up at their auctions, i.e. there will exist a v_I above which seller 1's mechanism will be efficient.

LEMMA 4 *Suppose seller* 2 *announces a quasi-efficient mechanism. Suppose further that the distribution of buyer types satisfies both (IHR,DD). Let the best response of seller* 1 *be such that the lowest type indifferent between the two sellers is* v_I. *Then this mechanism must be efficient on* $[v_I, 1]$.

Theorem 3 now follows from the following argument. Let $v_I \geq \max\{r_1, r_2\}$ be the lowest type that is indifferent between both sellers given seller 1's best response. All types in $[\min\{r_1, r_2\}, v_I)$ must strictly prefer seller $\arg\min\{r_1, r_2\}$.

- If $r_1 < r_2$, all buyers with valuations in $[r_1, v_I)$ exclusively visit seller 1. Then by Lemma 3, as long as the distribution of buyer types satisfies both conditions (IHR, DD), the revenue maximizing mechanism $\mathbf{m}(v)$ for seller 1 such that v_I is the lowest valuation buyer indifferent between the two sellers is quasi-efficient on $[0, v_I]$.
- If $r_1 > r_2$, all buyers with valuations in $[r_2, v_I)$ exclusively visit seller 2. Therefore, the design of seller 1's mechanism on $[r_1, v_I]$ does not matter since no buyer with value $\leq v_I$ visits seller 1.

By Lemma 4, as long as the distribution of buyer types satisfies (IHR, DD) the mechanism must be efficient on $[v_I, 1]$. Therefore, seller 1's mechanism must be quasi-efficient. □

[14] A technique from Pai & Vohra (2009) can be used to solve (Opt) even if (IHR, DD) are violated. The basic idea is to discretize the type space. The discrete analog of (Opt) is a linear program. The solution to this program can be characterized by a primal-dual method. Finally, it can be shown that the limit of the solution for finer and finer discretizations solves (Opt).

We recycle ideas from this proof to show that the best response to a randomized quasi-efficient mechanism will also be quasi-efficient when (IHR, DD) are satisfied.

Corollary 1 *Suppose seller* 2 *announces a mixed strategy quasi-efficient mechanism. Seller* 1*'s best response is quasi-efficient if the distribution of buyers' types satisfies both the increasing hazard rate (IHR) and the decreasing density (DD) conditions.*

3.2 Characterizing Equilibria

So far, we have only characterized properties of the best response map, i.e. qualitative features of seller 1's best response to seller 2's offered mechanism. We now characterize the implications on equilibrium among sellers in our setting.

Characterizing whether equilibria are quasi-efficient is an easy consequence of Corollary 1.

Proposition 1 *Suppose the distribution among buyer types f, F is such that both $\frac{f(v)}{1-F(v)}$ is increasing and $f(v)$ is decreasing in v. Then there exists a symmetric equilibrium among sellers in which each seller offers a quasi-efficient mechanism.*

If the distribution among buyers does not satisfy increasing hazard rate (IHR) and decreasing density (DD), then the equilibrium may be in non-quasi-efficient mechanisms.

Burguet & Sakovics (1999), among others, restrict sellers to announcing second price auctions, with the only strategic variable available being their reserve price. As we pointed out earlier, the justification for restricting sellers to this class of mechanisms is far from obvious.

Proposition 1 suggests that this restriction (and hence the results) are valid in a more general setting *if* the distribution of buyer types satisfies both the monotone hazard rate and decreasing density properties. If the distribution of buyer types violates these conditions, these results need not hold. The analysis of Theorem 3 sheds light on why these conditions are not just sufficient but *almost* necessary.

3.3 Large Markets

In our oligopolistic setting, existence of an equilibrium in quasi-efficient mechanisms depends on the distribution over buyer types satisfying certain requirements. In contrast, McAfee's (1993) large market result showed that each seller announcing a second price auction with no reserve was an equilibrium regardless of the distribution of types. One might wonder why the former requires (strong) distributional assumptions, while the latter requires none at all.

Proposition 2 reconciles these two results, suggesting that as the number of sellers grows large, the pooling implied by our equilibria disappears. This also bridges the gap in the results of:

1. Peters & Severinov (1997) who use the same equilibrium concept as us but restrict sellers to announcing second price auctions with reserve.

2. McAfee (1993) who allows sellers to announce more general mechanisms but uses the 'large market' solution concept we described above.

Our result shows that the result of all sellers announcing reserve second price auctions without a reserve is an $\epsilon-$ equilibrium in our setting. Further ϵ vanishes as the number of sellers goes to infinity. This result holds *exactly* in the limit. In any finite market, if the distributional conditions specified are not met, then there may be no (exact) equilibria in quasi-efficient mechanisms.

PROPOSITION 2 *Suppose m sellers, and suppose sellers 2 through m all announce a quasi-efficient mechanism with reserve 0. Then, the difference in expected revenue for seller 1 from announcing his best response and announcing a quasi-efficient mechanism with no reserve vanishes as $m \to \infty$.*

4 Related Literature

CAPACITY CONSTRAINED SELLERS As we pointed out in the Introduction, there is a long and distinguished literature on competition among sellers with capacity constraints. The seminal work of Kreps & Scheinkman (1983) considers a sequential game where sellers first choose quantity, and then simultaneously announce prices. The outcome in this setting resembles the standard Cournot equilibrium. However, they exogenously specify how buyers 'split' across sellers given the prices they announce ('efficient rationing'). There is disagreement in the literature on whether this rationing rule is well founded, notably from Davidson & Deneckere (1986). They show that the results of Kreps & Scheinkman (1983) are sensitive to the choice of allocation rule.

Moldovanu, Sela & Shi (2008), analyze a setting where sellers first simultaneously choose quantity, and then simultaneously sell the good by second price auctions with no reserve price. The price formation rule is thus endogenous.

COMPETING AUCTIONEERS To our knowledge, the original paper on competing auctioneers is McAfee (1993). His paper considers a similar model to ours, but concerns itself with the equilibria of 'large' markets. He shows that there is an equilibrium in large markets in which all sellers announce efficient mechanisms. However, in his solution concept, a deviating seller only considers how his deviation will affect the distribution of types visiting his mechanism, but assumes that it will not affect the distribution of types visiting other sellers. We can think of this as mechanism design when buyers have a fixed type dependent outside option, the outside option being determined in equilibrium. McAfee notes that this assumption is only suitable for large markets.

Peters & Severinov (1997) also consider 'large' markets, and in their model a deviating seller correctly considers the impact of his deviation on other sellers. However, they restrict sellers to second price mechanisms. Burguet & Sakovics (1999) explicitly compute an equilibrium among two sellers when both are restricted to using second price (or more generally quasi-efficient) mechanisms, and buyer valuations are uniformly distributed. Further they showed that equilibria in this setting were necessarily in mixed strategies. Virag (2007) shows that if the distribution of buyer valuations is shifted to the right, i.e. the support is $[a, b]$ where a is sufficiently larger than 0, there can be a pure strategy equilibrium in quasi-efficient mechanisms. He shows further results similar to ours for the more restrictive case of two buyers.

Principal-Agent Models The list of papers attempting to endogenize buyer participation is too long to enumerate here- we restrict attention to a few key papers. Jullien (2000) considers the problem of contracting between a principal and an agent when the agent has a type dependent outside option, and characterizes the optimal mechanism. Rochet & Stole (2002) consider a variant when both valuation and outside option are the agents' private information– and solve it under the assumption that these two components of type are independently determined. Neither of these results apply to the problem of sale to multiple agents since they ignore the additional feasibility restrictions which selling a limited supply good to multiple agents entail. To our knowledge, the problem of optimal auction design when buyers have an exogenously specified type-dependent outside option is unsolved.

Common Agency The literature on common agency, studies (both simultaneous and sequential) contracting between multiple principals and an agent for related activities/ sale. A large portion of this literature discusses the failure of the 'standard' revelation principle, identifies the correct extended type space to study in this setting, and discusses the resulting difficulties for mechanism design– see Peters (2001) for a central result in this setting. We skirt these difficulties by allowing our buyers visit only one seller.

5 Conclusion

We displayed a model of competing capacity constrained sellers. Sellers compete in this model by strategically offering auctions, buyers then strategically choose which seller to visit. We provided sufficient conditions on the distribution of buyer valuations under which sellers would choose quasi-efficient mechanisms in equilibrium. We further indicated that these sufficient conditions were almost necessary: if the distribution on buyer valuations 'strongly' violated these conditions, we would see non-quasi-efficient mechanisms in equilibrium. We thus suggest that in oligopolistic settings, there may be an additional source of inefficiency that the literature does not consider: namely, from sellers choosing to sell the good to a lower valuation buyer when higher valuation buyers are present.

5.1 Discussion

A summary of our formal results was provided in Section 1.1, and will not be repeated here. However it is now appropriate to comment on a few issues that might not have been obvious ex-ante.

Uniqueness of Buyer Visits Theorem 1 showed that, given the mechanisms announced by sellers, there is a unique outcome in the sub-game among buyers. It should be clear that unique outcomes in the sub-game among buyers is key to analyzing competition among sellers. If there were multiple possible outcomes, a seller would be unable to predict which one would be realized. Hence it would be ambiguous how, for example, a seller considering deviating should account for buyer behavior were he to deviate.

A very attractive feature of our model, therefore, is that uniqueness is implied by the Buyer Symmetric Weak perfect Bayesian equilibrium solution concept. Neither additional information or co-ordination devices among buyers, nor any equilibrium refinements are required.

How demanding are the sufficient conditions? Our main result, Theorem 3, says that a seller's best response to a quasi-efficient mechanism is quasi-efficient if the distribution on buyer valuations satisfies both the increasing hazard rate and decreasing density conditions. One might wonder if most 'interesting' distributions are covered in this class, hence rendering our key qualitative insight a technicality. Quite to the contrary, the two conditions (IHR) and (DD) taken together are quite strong: increasing hazard rate rules out distributions where the density decreases 'sharply' in any interval (any increasing density is fine). Formally if the density f is differentiable, our condition requires that $f' \in [-\frac{f^2}{1-F}, 0]$, a fairly restrictive requirement. This is also the sense in which, in the Introduction, we referred to our conditions as requiring that the distribution F be 'close' to uniform.

What we mean by 'almost necessary' Lemma 3 formally shows that necessary conditions on distribution f, F over buyers' valuations are $v - \frac{1-F(v)}{f(v)} - \frac{c}{f(v)}$ increasing in v. Lemma 2 shows that c must be negative. This is the sense in which we mean our conditions are almost necessary. However we are unable to analytically determine c. Thus we are unable to quantify the gap between necessary and 'almost necessary'.

5.2 Extensions

The extensions we can envisage divide neatly into two categories: (1) using this model of competing sellers in a larger game, and (2) relaxing modeling assumptions dictated by tractability.

Capacity Choice stage In our model sellers have 1 costless unit each. We are interested in what sellers' incentives to produce costly units are in such a setting. Would the resulting outcomes more closely mirror Bertrand or Cournot competition? This would resolve a long-standing question from the Kreps & Scheinkman (1983) - Davidson & Deneckere (1986) debate, namely what sort of rationing rules should we 'expect' to see, and what upstream incentives does this have for capacity choice among firms.

Dynamics A model of oligopolistic capacity constrained sellers in dynamic settings would shed light on economically important industries such as airlines and hotels (current models almost uniformly assume a monopolist seller).

Airline ticketing systems display both the kinds of inefficiency we discuss. Even the airline's lowest fare will ration some buyers completely. Further by varying availability across low and high fare buckets, airlines control the amount of 'misallocations.' One might ask how these two kinds of rationing play out over time in a strategic setting. In particular, this would be a nice complement to the literature on dynamic pricing which focuses on only the former.

Definition of Pure Strategies Buyers in our setting choose mechanisms *after* any randomization by the seller resolves. Sellers therefore cannot announce, say, a random reserve price. This restriction allows us to establish uniqueness of buyer visit decisions in the buyer sub-game (Theorem 1). We conjecture that Theorem 1 remains true even if buyer had to choose *before* the randomization resolved. We further conjecture that if the space of 'pure' strategies in our model is identified with *all* points in the polytope identified by Border, rather than just the corner points as currently assumed, then a 'pure strategy' Nash equilibrium exists. Were this true, our remaining results would carry over to this setting.

Contingent Mechanisms One may ask what would happen if sellers could announce mechanisms that require agents to report not just their own valuations but also the mechanisms offered by the other sellers– in other words, if sellers could announce mechanisms which were contingent on the announcement of the other seller. While correctly specifying the space of all feasible mechanisms in this case is non-trivial, nonetheless, as McAfee points out, any pure strategy equilibrium in our original game remains an equilibrium in this extended game: this occurs because a deviating seller believes that other sellers are not deviating, and thus the deviating seller does not expect to learn anything from asking buyers about the mechanisms employed by other sellers. However, expansion of the sellers' strategy space could introduce other equilibria.

Visiting more than 1 seller This would make the model intractable with current methods, since a buyer would have to take into account his probability of winning in multiple auctions and discount that appropriately (for an apples-to-apples comparison we must continue to assume that buyers have unit demand). Attempts to enlarge the space of feasible mechanisms by allowing the outcome at a mechanism to be contingent on the outcome at other mechanisms (e.g. a seller announces a mechanism where buyers can visit and costlessly withdraw if they get the good at a better price from another seller) cannot be analyzed by the techniques in this paper.

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A Border Constraints

In private values mechanism design, it is generally assumed that buyers and the seller are risk-neutral. Due to risk-neutrality and the rent-extraction formula, given a mechanism announced by a seller, a buyer of type v only cares about his interim expected probability of winning the good $a(v)$. Further, the seller's profit can also be expressed solely as a function of this interim expected probability $a(\cdot)$. Therefore, the only reason one might care about how the good is allotted profile by profile is to ensure feasibility, i.e. that the seller does not sell more than his supply. However, working with mechanisms where the allocation rule is specified profile by profile can be extremely cumbersome.

For example, in Myerson (1981), the optimal auction is characterized while leaving this feasibility constraint implicit. The solution is easily described if the distribution of types meets a regularity condition. In the event that the distribution does not satisfy this condition, an opaque 'ironing' procedure is provided. While this approach works perfectly in this case, it is unclear how to proceed when buyers' types are multi-dimensional, there are additional constraints etc.

Border (1991) provides necessary and sufficient conditions which determine whether an interim allocation rule a is feasible. A 'quick and dirty' summary is provided here. For further details, we refer interested readers to the original paper.

Suppose there are n agents. Each has a type from a measurable space of types T; agents' types are i.i.d. w.r.t. some common probability distribution π over T. Let $a : T \to [0, 1]$ be a measurable function. Recall that a is a feasible interim allocation rule if there exists (an anonymous) profile by profile allocation rule

$$q : T^n \to \Delta^n,$$

where Δ^n is the n-dimensional simplex; and

$$\forall t \in T : \ a(t) = \int_{T^n - 1} q_1(t, t^{n-1}) d\pi^{n-1}(t^{n-1}).$$

Border simplifies the burden of checking that a given a is feasible. First he shows that a is feasible if and only if

$$\forall S \subseteq T : \quad \int_{t \in S} a(t) d\pi(t) \leq \frac{1 - (\pi(T \backslash S)^n}{n}. \tag{Border}$$

Further, he shows that if this condition is violated, it must be violated on a set where a is 'large'. In effect checking feasibility of a reduces to checking a 1-parameter family of inequalities:

$$\forall \alpha \in [0, 1] : \quad \int_{\{t | a(t) \geq \alpha\}} a(t) d\pi(t) \leq \frac{1 - (\pi(\{t | a(t) < \alpha\})^n}{n}. \tag{Border2}$$

The constraints (Border2) are very convenient 'standard' auction design, where agents' types are just their valuations, i.e. $T \subseteq \Re_+$. This is because incentive compatibility requires that a be increasing in valuation. For example, when the types are the agents' valuations $T = [0,1]$, and if π has cdf F, (Border2) implies that a is feasible if and only if:

$$\forall v : \quad \int_v^1 a(t)dF(t) \leq \frac{1 - F^n(v)}{n} \tag{18}$$

(18) is the constraint we will use in our proofs in Section 3.

Fix $[0,1]$ as the type space, and let F have full support on $[0,1]$. Let $\mathbf{m} = (\succeq, r)$ be the mechanism selected by the seller. It can be shown that $\mathbf{m}$ is efficient at $v > r \Rightarrow$ (18) binds at v.

We have two further asides for those with an eye for extentions:

The constraints can be easily extended when the seller has $k > 1$ identical units of a good for sale to agents with unit demand. They simply require that:

$$\forall v : \quad \int_v^1 a(t)f(t)dt \leq c_{k,F,n}(v),$$

$$n\, c_{k,F,n}(v) = \sum_{j=1}^{k-1} j \binom{n}{j} (1 - F(v))^j F^{n-j}(v) + \sum_{j=k}^{n} k \binom{n}{j} (1 - F(v))^j F^{n-j}(v),$$

i.e. $= \left(\sum_{j=1}^{k-1} j \times \text{ Probability of } j \text{ agents in } [v,1] \right) + \left(k \times \text{ Probability of } \geq k \text{ agents in } [v,1] \right)$.

Border (2007) extends these conditions to when each agent i has a type drawn from a different distribution F_i. Let a_i be the interim probability with which agent i gets the good. They require that:[15]

$$\forall (v_i)_{i=1}^n \in [0,1]^n : \quad \sum_{i=1}^{n} \int_{v_i}^1 a_i(v)f(v)dv \leq 1 - \prod_{i=1}^{n} F_i(v_i) \tag{19}$$

These constraints are used to show that the problem of a profit-maximizing broker in a Myerson-Satterthwaite setting is a simple variant of the problem of a profit maximing-seller in a Myersonian setting (buyers and sellers in a Myerson-Satterthwaite setting can be thought of as agents with types drawn from 2 different distributions).

[15]The constraints can be extended to the case when the seller has multiple units for sale analogously.

B Proofs from Section 2

B.1 Proof of Theorem 1

The proof will proceed in two parts. First we will show existence of an equilibrium in the subgame among buyers. We are unaware of any 'off-the-shelf' result proving existence. We then prove 'uniqueness' (in the sense of Theorem 1).

Existence will be shown for the more general case of m sellers since that the extra generality does not add much notational burden. Before the proof, some preliminaries are in order.

A Meta-game Fix the mechanisms offered by each seller, the total number of buyers n and the population distribution of buyer valuations f, F.

An equilibrium in the buyer subgame is equivalently an equilibrium in a meta-game defined thus:

1. The players are identified with the valuations $v \in V$.
2. The set of pure strategies for each player v is to visit one of the sellers 1 through m, and is thus identified with the set of sellers M; a mixed strategy can be defined analogusly as ΔM.
3. Each player moves simultaneously.

Define a (mixed-) strategy profile as $\theta = (\theta_j)_{j=1}^m, \theta_j : V \to [0,1]$, where $\theta_j(v)$ is the probability with which player v plays strategy j. Clearly, $\sum_j \theta_j(v) = 1$ for all $v \in V$.

Given θ, the payoffs of each v from strategy j is the interim expected surplus (in the original game) of a buyer of type v when he visits seller j given all other buyers are using visit decisions θ. Given that seller j uses mechanism $\mathbf{m}$ and buyer visit decisions are θ_j, recall that the 'effective distribution' of buyers visiting seller j is

$$F_j(v, \theta) = 1 - \int_v^1 \theta_j(v) f(v) dv.$$

The interim probability of allocation $a_j(\cdot, \theta_j)$ at seller j is thus determined, and hence by Observation 2, the expected payment of a type v,

$$p_j(v, \theta) = v a_j(v, \theta) - \int_0^v a_j(t, \theta) dt.$$

Therefore the expected surplus of each type v,

$$s_j(v, \theta_j) = \int_0^v a_j(t, \theta) dt.$$

By observation the payoffs of the players are absolutely continuous in v, i.e. $s_j(v, \theta)$ is absolutely continuous in v for all j, θ, and further since a_j must be increasing in v, s_j must be convex in v.

Observation 3 *In any such metagame, for any profile of strategies θ, the expected payoff of players from any strategy j, $s_j(v, \theta)$ is Lipschitz continuous in v, and convex in v.*

Existence The proof of existence is based on techniques from the literature on 'Crowding Games,' particularly Milchtaich (2000).

Define the (pure strategy) best response correspondence for a player v at strategy profile θ as:

$$BR(v, \theta) = \{j \in M | s_j(v, \theta) = \max_{k \in M} s_k(v, \theta)\}.$$

The mixed strategy best response correspondence $coBR$ can be defined analogously:

$$coBR(v, \theta) = \{\pi \in \Delta M | \, supp(\pi) \subseteq BR(v, \theta)\}.$$

Define the correspondence $\Gamma : \theta(\cdot) \mapsto coBR(\cdot, \theta)$. If we can show this map has a fixed point we are done. We will use the Glicksberg-Fan fixed point theorem to this end.

Let $\Theta = \{\theta | \theta : V \to \Delta M\}$. This is a subset of the linear space $\Phi = \{\phi | \phi : V \to \Re^M\}$. Note that Θ is clearly convex. Further, endowing the space Φ with the product topology, note that Θ is a compact subset of Φ (Tychonoff's Theorem).

The correspondence Γ is convex-valued by definition ($coBR$ is convex valued). Therefore we are left to show that Γ has a closed graph.

Lemma 5 *Let $\{\theta^{(l)}\}_{l \geq 0}, \{\phi^{(l)}\}_{l \geq 0} \subseteq \Theta$, $\theta^{(l)} \to \theta^{(0)}$. Then $\phi^{(l)} \in \Gamma(\theta^{(l)})$ for all $l > 1$ and $\phi^{(l)} \to \phi^{(0)}$ implies $\phi^{(0)} \in \Gamma(\theta^{(0)})$.*

Proof: We shall prove something slightly stronger, i.e. that the above statement is true for the map $\Lambda : \theta(\cdot) \mapsto BR(\cdot, \theta)$. The result then follows trivially since $coBR$ is the convex-hull of BR.

Suppose $\theta^{(l)} \to \theta^{(0)}$ (pointwise) and $\phi^{(l)} \in BR(\theta^{(l)})$, with $\phi^{(l)} \to \phi^{(0)}$ (pointwise)

Therefore, for a.e. $v \in V$, $\exists n_v$ s.t. $\forall l > n_v$, $\phi^{(0)}(v) \in BR(v, \theta^{(l)})$. This implies that for a.e. $v \in V$, $\exists n_v$ s.t. $\forall l > n_v$ $s_{\phi^{(0)}(v)}(v, \theta^{(l)}) \geq \max_{k \in M} s_k(v, \theta^{(l)})$. It follows by an application of Fatou's Lemma that for a.e. $v \in V$, $s_{\phi^{(0)}(v)}(v, \theta^{(0)}) \geq \max_{k \in M} s_k(v, \theta^{(0)})$. Therefore $\phi^{(0)} \in \Gamma(\theta^{(0)})$. □

In summary, the correspondence Γ is a non-empty convex-valued self map on a compact, convex subset of a locally convex Hausdorff space. Further Γ has a closed graph by Lemma 5. The existence of an equilibrium in this game, and therefore a sequentially rational, symmetric visit decision for buyers θ, follows from the Glicksberg-Fan fixed point theorem– see e.g. Aliprantis & Border (2006, Corollary 17.55).

UNIQUENESS The statement of the theorem only concerns the buyer surpluses and seller expected revenues that result from buyer visit decisions. Therefore, by a slight abuse of terminology, we shall refer to the 'uniqueness' of buyer visit decisions in the sub-game, when we actually only mean that the equilibrium outcome, i.e. buyer surpluses and seller expected revenues are unique.

Since buyers choose mechanisms after any randomizations in sellers' strategies have resolved, it is sufficient to show that there is a 'unique' buyer visit decision θ resulting from any choice of pure strategy mechanisms by the sellers $\mathbf{m}_1$ and $\mathbf{m}_2$.

Before we prove uniqueness, a few simple observations are in order. The proof proceeds in the reverse order of the sketch provided in body of the paper: First, fix any set of valuations $\bar{V} \subseteq V$. We show that there is a unique sequentially rational θ (if any) such that exactly buyers with valuation in $\bar{V}$ are indifferent between the 2 sellers. Then we show that there is a unique feasible $\bar{V}$.

OBSERVATION 4 *For any mechanism $\mathbf{m}$, any distribution of types visiting the mechanism F, and any interval $[\underline{v}, \bar{v}]$, the resulting expected surplus $s(\cdot)$ offered each type satisfies the following:*

1. *If $\mathbf{m}$ is efficient on $[\underline{v}, \bar{v}]$, then $s(\cdot)$ is strictly convex $\iff$ $F(\bar{v}) - F(\underline{v}) > 0$, i.e. only if types in the interval visit with positive probability. Otherwise, $s(\cdot)$ is a straight line in the interval, with slope $a(\underline{v})$.*
2. *If $\mathbf{m}$ is nowhere efficient on $[\underline{v}, \bar{v}]$, $s(\cdot)$ is a straight line on the interval.*

PROOF: Follows by observing the Border conditions. □

OBSERVATION 5 *Let θ be any buyer visit decision (not necessarily sequentially rational); and let $s_j(\cdot)$ be the expected surplus curve of buyers at seller j. Then the set of valuations V can be partitioned into intervals $\{[\underline{v}_i, \bar{v}_i]\}_{i=1}^k$, $\bar{v}_i = \underline{v}_{i+1}$, where for each i, $\exists \bar{M}_i \subseteq M$ such that:*

$$j \in \bar{M}_i \iff \forall v \in (\underline{v}_i, \bar{v}_i), j \in \arg\max_{k \in M} s_k(v)$$

$$\forall i : \quad \bar{M}_i \neq \bar{M}_{i+1}$$

PROOF: The former follow from continuity of the surplus offered by each mechanism (recall Observation 3). We can always define the partition such that the latter is true. □

OBSERVATION 6 *If $s_j(v) = s_{j'}(v)$ for $v \in [\underline{v}, \bar{v}]$, then $a_j(v) = a_{j'}(v)$ for a.e. $v \in [\underline{v}, \bar{v}]$.*

PROOF: Follows since for all v, $s_j(v) = \int_0^v a_j(t)dt$ where $a_j(\cdot)$ is the interim expected allocation probability (see Observations 3, 5). □

The next lemma essentially says this: when a single seller j is strictly preferred over the others by valuation v, sequential rationality of buyers requires $\theta_j(v) = 1$, i.e. buyers with valuation v exclusively visit seller j. However sequential rationality appears to have no bite if there are multiple sellers over which a valuation is indifferent. Therefore one could ask if there exists multiple θ's that preserve this indifference. We show that θ is 'unique' in the sense of Theorem 1.

LEMMA 6 *Let θ, θ' be two sequentially rational visit rules (with resulting interim expected surpluses s, s') such that there exists an interval $[\underline{v}, \bar{v}]$, and a $\bar{M} \subseteq M$, satisfying:*

$$j \in \bar{M} \iff \forall v \in (\underline{v}, \bar{v}), j \in \arg\max_{k \in M} s_k(v) \iff \forall v \in (\underline{v}, \bar{v}), j \in \arg\max_{k \in M} s'_k(v) \tag{20}$$

$$\forall j \in M, v \notin [\underline{v}, \bar{v}] \quad \theta_j(v) = \theta_{j'}(v) \tag{21}$$

Then,

1. *If all members of $\bar{M}$ are efficient on the interval $[\underline{v}, \bar{v}]$ then $\theta_j(v) = \theta'_j(v) = \frac{1}{|\bar{M}|}$ for almost all $v \in [\underline{v}, \bar{v}]$, $j \in \bar{M}$.*

2. *If $M' \subsetneq \bar{M}$ efficient on $[\underline{v}, \bar{v}]$, and the rest are pooled on $[\underline{v}, \bar{v}]$ implies $\theta_j(v) = \theta_{j'}(v) = 0$ for a.e. $v \in [\underline{v}, \bar{v}], j \in M'$ and $\int_{\underline{v}}^{\bar{v}} (\theta_j(v) - \theta'_j(v))f(v)dv = 0$ for $j \in \bar{M} \backslash M'$.*

PROOF:

1. By Observation 6, $a_j(v) = a_{j'}(v)$ for $v \in [\underline{v}, \bar{v}]$, $j, j' \in \bar{M}$. By the relevant Border condition, we have that $a_j(v) = F_j^{n-1}(v)$.[16] $a_j(v) = a_{j'}(v)$ for $v \in [\underline{v}, \bar{v}]$ therefore requires that $f_j(v) = f_{j'}(v)$ a.e. $\Rightarrow \theta_j(v)f(v) = \theta_{j'}(v)f(v) a.e.$. Hence $\theta_j(v) = \frac{1}{\bar{M}}$ a.e.

2. The first part, i.e. $v \in [\underline{v}, \bar{v}], j \in M' \Rightarrow \theta_j(v) = \theta_{j'}(v) = 0$ a.e. follows from Observation 4– if not, then the implied surplus curve would be strictly convex on the interval, whereas the surplus curves for sellers in $\bar{M} \backslash M'$ must be linear; thus contradicting our supposition that $s_j = s_{j'}$ for all $j, j' \in \bar{M}$.

   The latter, i.e. $\int_{\underline{v}}^{\bar{v}} (\theta_j(v) - \theta'_j(v))f(v)dv = 0$ for $j \in \bar{M} \backslash M'$ follows by noting that if j pools in an interval $[\underline{v}, \bar{v}]$, the relevant Border condition implies that $a_j(v) = \frac{F_j^n(\bar{v}) - F_j^n(\underline{v})}{n(F_j(\bar{v}) - F_j(\underline{v}))}$ for all $v \in [\underline{v}, \bar{v}]$. Therefore any θ' that maintains the total probability of a type in $[\underline{v}, \bar{v}]$ visiting seller j will keep a_j constant. The result follows. □

[16] We have implicitly assumed that the distribution is atomless- the result goes through regardless.

We shall discuss the case of 2 sellers from hereon in since that is all our theorem requires, and generalizing to $m > 2$ buyers is not for 'free' from this point on.

The previous lemma said that keeping θ fixed everywhere else, there was no 'wiggle' room on how buyers randomized on an interval on which they were indifferent among multiple sellers. We can strengthen this lemma further. Let $\bar{V} \subseteq V$ be such that there exist multiple sequentially rational θ's such that a buyer with valuation v is indifferent between the two sellers if and only if $v \in \bar{V}$. Then buyers and sellers are indifferent between any two such subgame equilibria θ's.

LEMMA 7 *Let $\bar{V} \subseteq V$. Let θ, θ' be two sequentially rational visit rules (with resulting interim expected surpluses s_j, s'_j) such that:*

$$s_1(v) = s_2(v) \iff v \in \bar{V},$$
$$s'_1(v) = s'_2(v) \iff v \in \bar{V}.$$

Then buyers and sellers are both indifferent between θ and θ' (i.e. 1, 2 of Theorem 1 are satisfied).

PROOF: The proof of this lemma builds on the proof of Lemma 6 (we state and prove them separately for purely expository purposes).

STEP 1: By Observation 5, $\bar{V}$ must partition V into I intervals $[v_i, v_{i+1}]$ such that in the interior of each interval either $s_1 > s_2$, $s_1 < s_2$ or $s_1 = s_2$.

STEP 2: By convexity of the surplus curves, it cannot be the case that $s_1 > s_2$ on two successive intervals, or $s_1 < s_2$ on two successive intervals. Further, $s_1 = s_2$ on two successive intervals is not possible by definition.

Let $I_= \subseteq I$ be the intervals on which $s_1 = s_2$ (these are determined by $\bar{V}$). Let $I_=^+$ be the intervals immediately following. If for each of these intervals, we determine whether $s_1 \gtrless s_2$ or vice-versa, then $s_1 \gtreqless s_2$ is determined for all intervals. Fix an interval $[\underline{v}, \bar{v}]$ in $I_=^+$.

To determine whether $s_1 \gtrless s_2$ on an interval in $I_=^+$ it suffices to look at the mechanisms on the interval $[\underline{v}, \bar{v}]$. We can now categorize case by case: 8

1. Both mechanisms are efficient on $[\underline{v}, \bar{v}]$: Not possible. $s_1(\underline{v}) = s_2(\underline{v})$ by supposition, and (say) $s_1 > s_2$ on $(\underline{v}, \bar{v})$. Then sequential rationality $\theta_1 = 1, \theta_2 = 0$ on the interval. Therefore by Observation 4, s_1 is strictly convex while s_2 is a straight line on $(\underline{v}, \bar{v})$. Therefore $s_1(\bar{v}) \neq s_2(\bar{v})$.
2. At least one mechanism is not efficient on $[\underline{v}, \bar{v}]$: Clearly both mechanisms cannot be inefficient at $\underline{v}$: if both mechanisms are inefficient at $\underline{v}$ then buyers must continue to be indifferent for a neighborhood around $\underline{v}$.
   - Mechanism j is efficient at $\underline{v}$ and mechanism j' is not: then $s_j > s_{j'}$ on this interval.

- If both mechanisms are efficient at $\underline{v}$, then the mechanism that is inefficient at $\bar{v}$ (say j) will be lower, i.e. $s_j < s_{j'}$. There must be exactly one such mechanism since if both are also efficient at $\bar{v}$, then the mechanism that is lower will have a linear surplus on the interval $[\underline{v}, \bar{v}]$, causing a contradiction.

Hence, θ is determined for intervals $I \backslash I_=$ by sequential rationality and the argument above. By Lemma 6, θ is 'determined' on $I_=$ (there is a degree of flexibility when both sellers pool on an interval, as explained in 2 of Lemma 6). 1 and 2 of Theorem 1 follow. □

We can strengthen this lemma by observation. Let $\bar{V}$ be the points at which the surpluses cross, i.e. $s_1(v) - s_2(v)$ changes from being strictly positive (negative) to zero or strictly negative (positive), or zero to non-zero. Knowing these (at most countable) $\bar{V}$ 'determines' θ.

Lemma 8 *Let $\bar{V} \subseteq V$. Let θ, θ' be two sequentially rational visit rules (with resulting interim expected surpluses s_j, s'_j) such that:*

$$sign(s_1(v^-) - s_2(v^-)) \neq sign(s_1(v^+) - s_2(v^+)) \iff v \in \bar{V}, \tag{22}$$
$$sign(s'_1(v^-) - s'_2(v^-)) \neq sign(s'_1(v^+) - s'_2(v^+)) \iff v \in \bar{V}.$$

Then buyers and sellers are both indifferent between θ and θ' (i.e. 1, 2 of Theorem 1 are satisfied).[17]

Proof: Suppose without loss of generality that seller 1's reserve price r_1 is lower than seller 2's reserve r_2. Let $\bar{V} = \{v_i \,|\, i \in I\}$, s.t. $v_i < v_{i+1}$. Note that $v_0 = r_1$ by definition. $\bar{V}$ partitions the interval $[r_1, 1]$ into sub-intervals $[v_i, v_{i+1}]$. By an abuse of notation we will identify i with the interval $[v_{i-1}, v_i]$. The set of intervals I can be partitioned into three subsets $I_1, I_=$ andI_2 depending as whether $s_1 \gtreqless s_2$ on the interval. We know that interval 1 is in I_1. If we can show that the membership of each interval is determined, we are done. By the proof of Lemma 7 we know that given an interval is in $I_=$, we know the membership of the following interval based on how the two mechanisms treat types in the interval. Therefore it is sufficient if we can show how to determine whether the interval following interval in I_1 is in $I_=$ or I_2.

Let $[\underline{v}, \bar{v}]$ be an interval following an interval in I_1. We can once again categorize case by case:

1. Both mechanisms are efficient on $[\underline{v}, \bar{v}]$ then $[\underline{v}, \bar{v}] \in I_=$ (the reason why $\notin I_2$ is the same as in the proof of the previous lemma).
2. At least one mechanism is not efficient on $[\underline{v}, \bar{v}]$:

[17] By $v^-(v^+)$ we mean a neighbourhood to the left (right) of v. The expressions $sign(s_1(v^-) - s_2(v^-))$ etc. are well defined due to Observation 5.

- Mechanism 1 pools around $\underline{v}$ then $[\underline{v}, \bar{v}] \in I_2$: since $s_1(\underline{v}) = s_2(\underline{v})$ but $s_1(\underline{v}^-) > s_2(\underline{v}^-)$, $a_(\underline{v}^-) = s_1'(\underline{v}^-) < s_2'(\underline{v}^-) = a_2(\underline{v}^-)$. Since 1 pools around $\underline{v}$, $a_1(\underline{v}^+) = a_1(\underline{v}^-) < a_2(\underline{v}^-) < a_2(\underline{v}^+)$, where the last inequality follows by monotonicity of the as. Therefore $s_1(\underline{v}^+) < s_2(\underline{v}^+)$.
- Mechanism 1 is efficient on $[\underline{v}, \bar{v}]$ and mechanism 2 is not: then $[\underline{v}, \bar{v}] \in I_=$.
- Neither mechanism is efficient on $[\underline{v}, \bar{v}]$, both mechanisms are efficient at $\underline{v}$: then if both mechanisms behave the same on $[\underline{v}, \bar{v}]$, $[\underline{v}, \bar{v}] \in I_=$ else $[\underline{v}, \bar{v}] \in I_2$.

Since we know that interval 1 belongs to I_1, we can now inductively determine the membership of each interval in I. □

The following lemma shows something we might have suspected: that for any pair of mechanisms announced by the sellers, there is a unique $\bar{V}$ (i.e. any sub-game equilibrium θ, with the resulting interim expected surpluses s_1, s_2 satisfies (22)).

LEMMA 9 *There is a unique $\bar{V} \subseteq V$ such that for any sequentially rational visit rule, with resulting interim expected surpluses s_1, s_2,*

$$sign(s_1(v^-) - s_2(v^-)) \neq sign(s_1(v^+) - s_2(v^+)) \iff v \in \bar{V}.$$

PROOF: We deal with the proof by dividing into 2 cases, namely (1) sellers 1 and 2 announce mechanisms that are quasi-efficient and (2) sellers 1 and 2 announce mechanisms that are not quasi-efficient. While the proof of (1) is a special case of (2), it helps to build intuition.

QUASI-EFFICIENT MECHANISMS Suppose seller 1 (respectively 2) announces a quasi-efficient mechanism with reserve r_1 (respectively r_2). Without loss of generality suppose $r_1 \leq r_2$. By Lemma 1 and Lemma 6, there exists $v_I \in [r_2, 1]$ s.t.

$$\theta_1(v) > \theta_2(v) \quad \forall v < v_I$$
$$\theta_1(v) = \theta_2(v) \quad \forall v > v_I$$

Suppose there are two such sub-game equilibria $v_I < v_I'$. We show that these are mutually incompatible. Let us denote by a_j and a_j' the interim allocation probabilities resulting at seller j under the equilibria corresponding to v_I and v_I' respectively.

However, for $v \in [r_1, v_I]$,

$$a_1(v) = F_1^{n-1}(v) = \left(F(v) + \frac{1 - F(v_I)}{2}\right)^{n-1},$$
$$a_1'(v) = F_1'^{n-1}(v) = \left(F(v) + \frac{1 - F(v_I')}{2}\right)^{n-1}.$$

Similarly for $v \in [r_2, v_I]$,

$$a_2(v) = F_2^{n-1}(v) = \left(\frac{1 + F(v_I)}{2}\right)^{n-1},$$
$$a_2'(v) = F_2'^{n-1}(v) = \left(\frac{1 + F(v_I')}{2}\right)^{n-1}.$$

Clearly $a_1 \geq a_1'$ for $[r_1, v_I]$; similarly $a_2 \leq a_1'$ for $[r_1, v_I]$. Therefore it cannot be the case that $s_1(v_I) = s_2(v_I)$ and $s_1'(v_I') = s_2'(v_I')$.

Not Quasi-efficient Suppose each seller j announces mechanism $\mathbf{m}_j = (\succeq_j, r_j)$. Without loss of generality suppose that $r_1 \leq r_2$. Let θ be a sequentially rational equilibrium for buyers in the sub-game, with associated interim surpluses s_1, s_2. Let $\bar{V} \subseteq V$ be the subset (at most countable) satisfying (22). Let $\bar{V} = \{r_1 = v_0, v_1, v_2, \ldots, \bar{v} = 1\}$. We are agnostic as to whether this is finite or countable.

Let $\pi_j(v) = \int_v^1 \theta_j(v) f(v) dv$, by definition, $\pi_1(v) + \pi_2(v) = 1 - F(v)$. Further, by definition $F_j(v) = 1 - \pi_j(v)$.

So suppose some θ' resulting in $\bar{V}' = \{v_1', v_2', \ldots\}$; $\bar{V} \neq \bar{V}'$. Let the associated interim allocations be a_1', a_2', surpluses be s_1', s_2', define π_1', π_2' as above, and let effective distribution be F_1', F_2'.

Claim Suppose $\pi_1'(r_1) = \pi_1(r_1) + \delta > \pi_1(r_1)$. For any $v > r_1$, if

$$\int_{r_1}^v \theta_1'(t) f(t) dt > \int_{r_1}^v \theta_1(t) f(t) dv, \tag{23}$$

then for the smallest $v_k' > v$

$$\pi_1'(v_k') < \pi_1(v_k') \tag{24}$$

If we show the above claim, the Lemma follows since it implies either $\int_1^v \theta_1'(t) f(t) dt \leq \int_1^v \theta_1(t) f(t) dv$ (contradiction) or $\pi_1'(1) < \pi(1)$ (contradiction).

We shall show this claim for $v = v_1$. The same argument follows for recursively for larger v.

Case 1 Both mechanisms are efficient at v_1. Then clearly $a_1' < a_1$ and $a_2' > a_2$ on $[r_1, v_1]$. But then $v_1' < v_1$. Note $\theta_1 = 1$ on $[r_1, v_1]$, $\theta_1' \leq 1$ on $[r_1, v_1]$. Therefore (23) is violated by definition of π

Case 2 At least 1 mechanism is not efficient at v_1. If $a_1' < a_1$ and $a_2' > a_2$ on $[r_1, v_1]$, clearly $v_1' < v_1$, and we are done: (23) is violated. So suppose not. We shall show for the case $a_1' \not< a_1$ on $[r_1, v_1]$. The case $a_2' \not> a_2$ proceeds analogously.

First, $\mathbf{m}_1$ cannot be efficient at v_1. Further, if $v_1' \leq v_1$, clearly $a_1' \not< a_1$ is impossible. Therefore $v_1' > v_1$. Suppose $v_1' \in [v_1, v_2]$ (the notation if not is slightly more involved).

Suppose $\mathbf{m}_1$ pools on $[\underline{v}, \bar{v}]$ such that $\underline{v} \leq v_1 \leq \bar{v}$. It should be easy to observe that:

1. $a_1' < a_1$ for $[r_1, \underline{v}]$. Which implies that $a_1' > a_1$ on $[\underline{v}, v_1]$
2. $\pi_1'(\underline{v}) > \pi_1(\underline{v}) \Rightarrow F_1'(\underline{v}) = F_1(\underline{v}) - \delta$. Therefore for $a_1' > a_1$, it must be the case that $F(\min(v_1', \bar{v})) - F(v_1) > \delta$. Why? Because:

$$a_1'(v_1) > a_1(v_1) \Rightarrow \frac{F_1'^n(\bar{v}) - F_1'^n(\underline{v})}{n(F_1(\bar{v}) - F_1(\underline{v})} > \frac{F_1^n(\bar{v}) - F_1^n(\bar{v})}{n(F_1(\bar{v}) - F_1(\underline{v})}$$
$$\Rightarrow \frac{(F_1'(\underline{v}) + F(\min(v_2', \underline{v})) - F(\underline{v}))^n - F_1'^n(\underline{v})}{(F(\min(v_2', \underline{v})) - F(\underline{v}))} > \frac{(F_1(\underline{v}) + (F(v_1) - F(\underline{v}))^n - F_1^n(\underline{v})}{(F(v_1) - F(\underline{v}))}$$

Therefore, by observation if $F(\min(v_2', \underline{v})) - F(v_1) \leq \delta$, $a_1'(v_1) < a_1(v_1)$.

But then $\pi_1'(v_1') < \pi_1(v_1')$. □

B.2 Proof of Theorem 2

Recall that the set of pure strategies available to each seller is

$$\mathcal{M} = \{(\succeq, r) | r \in V \text{ and } v \succeq v' \iff v \geq v'\}.$$

Fix the distribution of buyer types π. Identify each element $\mathbf{m} \in \mathcal{M}$ with the interim allocation rule $a : V \to [0, 1]$ that results from $\mathbf{m}$ given n buyers with types i.i.d. according to π.

Let the space of feasible a's, i.e. a's satisfying (Border) be denoted $\mathbb{A}$. Treat $\mathbb{A}$ as a subset of $L_\infty(\pi)$ i.e. the π essentially-bounded functions on V. By Border (1991, Lemma 5.4), $\mathbb{A}$ is $\sigma(L_\infty, L_1)$ compact, i.e. compact in the weak* topology.[18] Denote the subset corresponding to the space of pure strategies by $\mathbb{A}_\mathcal{M}$.

[18]Strictly speaking, an element of L_∞ is not a measurable function but an equivalence class of measurable functions, where two functions are equivalent if they only differ only on a set of π-measure zero. However two interim allocation rules that are equivalent w.r.t π will produce the same interim expected buyer surpluses

Lemma 10 *The set of pure strategies $\mathbb{A}_{\mathcal{M}}$ is a weak*-closed subset of $\mathbb{A}$ and therefore weak* compact.*

Proof: $\mathbb{A}$ is a weak* compact subset of L_∞. To show that the space of pure strategies $\mathbb{A}_{\mathcal{M}} \subseteq \mathbb{A}$ is closed, it is sufficient to show that it contains every accumulation point. By definition of the weak* topology, $a_j \to_{w^\star} a$ if and only if $< a_j, f > \to < a, f >$ for all $f \in L_1$ (the dual space of L_∞).

Suppose that $a_j \to_{w^\star} a$ s.t. $a_j \in \mathbb{A}_{\mathcal{M}}$ for all j. $a \in \mathbb{A}$ since $\mathbb{A}$ is compact. Denote indicator functions on a set $V' \subseteq V$, $\chi_{[v,1]}$. By Border (1991, Lemma 5.2), if $a \in \mathbb{A}_{\mathcal{M}}$ then $< a, \chi_{[v,1]} >= \frac{1-F^n(v)}{n}$ at the set of points of efficiency of a, and a is a.e. constant on intervals where this does not bind.

Suppose that $a \notin \mathbb{A}_{\mathcal{M}}$. Then a must be non constant on an interval $[\underline{v}, \bar{v}]$ s.t. $< a, \chi_{[\underline{v},1]} > \neq \frac{1-F^n(\underline{v})}{n}$. But this implies that $< a, \chi_{[\underline{v},1]} > < \frac{1-F^n(\underline{v})}{n}$ (a must belong to $\mathbb{A}$). If $a_j \to_{w^\star} a$, and $< a, \chi_{[\underline{v},1]} > < \frac{1-F^n(\underline{v})}{n}$, then there exists J such that for all $j > J$, $< a_j, \chi_{[\underline{v},1]} > < \frac{1-F^n(\underline{v})}{n}$. Therefore each a_j must be a.e. constant on some interval $[\underline{v}, \bar{v}_j]$. Clearly therefore there exists some function $f \in L_1$ s.t. $< a_j, f > \not\to_{w^\star} < a, f >$. □

Next, denote a mixed strategy of a seller is a measure μ on the set of pure strategies, and denote the space of mixed strategies $\mathcal{P}$. $\mathcal{P}$ is clearly convex.

Lemma 11 *$\mathcal{P}$ is weak* compact.*

Proof: $\mathbb{A}_{\mathcal{M}}$ is compact and Hausdorff. Therefore the space of continuous real valued functions on $\mathbb{A}$, denoted $C(\mathbb{A})$ with the topology generated by the sup norm is a Banach space. Further, space of regular countably additive measures with the topology generated by the total variation norm is the dual space (Dunford & Schwartz 1988, Theorem IV.6.3). Therefore by Alaoglu's theorem the closed unit-ball of rca measures is weak* compact. Finally note that the space $\mathcal{P}$ is a weak*-closed subset of the unit ball. □

Let $\rho : \mathcal{M}^2 \to \Re^2_+$ be each seller's expected revenue as a function of the mechanisms chosen by both, and the resulting sequentially rational θ. ρ is continuous in the product topology on $\mathcal{M}^2$. Therefore, we can extend ρ to the space of mixed strategies in the obvious manner $\rho : \mathcal{P}^{|M|} \to \Re^{|M|}_+$. Therefore the payoff function is a continuous function on a compact, locally convex topological vector space. By Berge maximum theorem (Aliprantis & Border 2006, Theorem 17.31), the self-correspondence

$$br : (\mu_j)_{j=1}^m \mapsto \times_{j=1}^m \{\arg\max_{\mu \in \mathcal{P}} (\rho_j(\mu, \mu_{-j}))\}$$

---

and expected seller revenues, and therefore will be indistinguishable by buyers and sellers. Therefore this is the appropriate topology on $\mathbb{A}$. In the analysis below, by an element $a \in \mathbb{A}$ we imply this equivalence class, and by an abuse of notation we shall say that $\mathbb{A}$ is Hausdorff. The reader should not be confused.

is non-empty, convex and upper hemi-continuous. An equilibrium in mixed strategies exists by the Glicksberg Fan theorem (see Aliprantis & Border (2006, Corollary 17.55)). □

C Proofs from Section 3.1

C.1 Proof of Lemma 1

The proof will proceed in two steps. First we will claim that $s_1(v) = s_2(v)$ for almost all $v \in [\underline{v}, \bar{v}]$.

So suppose not. Then by Observation 5 there must be an interval $(\underline{v}', \bar{v}')$ such that $s_1 = s_2$ at the boundary points, but (without loss of generality) $s_1 > s_2$ in the interior. But then sequential rationality requires that $\theta_1(v) = 1$. By Observation 4 therefore, $s_1(\cdot)$ will be strictly convex in the interval, while $s_2(\cdot)$ will be a straight line. However this contradicts the fact that $s_1 = s_2$ at the end points while $s_1 > s_2$ in the interior.

Therefore $s_1(v) = s_2(v)$ for $v \in [\underline{v}, \bar{v}]$. Let the buyer visit decisions be θ.

$$\begin{aligned}
& s_1(v) = s_2(v) \quad \forall v \in [\underline{v}, \bar{v}] \\
\Rightarrow \quad & s_1(\underline{v}) + \int_{\underline{v}}^{v} a_1(t)dt = s_2(\underline{v}) + \int_{\underline{v}}^{v} a_2(t)dt \quad \forall v \in [\underline{v}, \bar{v}] \\
\Rightarrow \quad & a_1(v) = a_2(v) \quad \forall v \in [\underline{v}, \bar{v}] \\
\Rightarrow \quad & \left(F_1(\underline{v}) + \int_{\underline{v}}^{v} \theta_1(t)f(t)dt\right)^{n-1} = \left(F_2(\underline{v}) + \int_{\underline{v}}^{v} \theta_2(t)f(t)dt\right)^{n-1} \quad \forall v \in [\underline{v}, \bar{v}] \\
\Rightarrow \quad & \theta_1(v) = \theta_2(v) = 0.5 \quad \forall v \in [\underline{v}, \bar{v}].
\end{aligned}$$

□

C.2 Proof of Lemma 2

In this case, the visit probabilities of buyers will be

$$\theta_1(v) = \begin{cases} 1 & v \in [0, v_I] \\ 0.5 & v > v_I. \end{cases} \tag{25}$$

The effective distributions F_1 and F_2 can be computed from (25). Further, v_I solves:

$$\int_{r'}^{v_I} F_1^{n-1}(v)dv = (v_I - r)F_2^{n-1}(v_I). \tag{26}$$

Seller 1's expected revenue is :

$$
\begin{aligned}
&\int_{r'}^{1} \big(v - \frac{1-F_1(v)}{f_1(v)}\big) f_1(v) F_1^{n-1}(v) dv \\
=&\int_{r'}^{v_I} \big(v - \frac{1-F(v)-(1-F(v_I))/2}{f(v)}\big) f(v) F_1^{n-1}(v) dv \;+\; \frac{1}{2}\int_{v_1}^{1} \big(v - \frac{1-F(v)}{f(v)}\big) f(v) F_1^{n-1}(v) dv \\
=&\frac{1-F(v_I)}{2}\int_{r'}^{v_I} F_1^{n-1}(v) dv \;+\; \int_{r'}^{v_I} \big(v - \frac{1-F(v)}{f(v)}\big) f(v) F_1^{n-1}(v) dv \\
&+\; \frac{1}{2}\int_{v_1}^{1} \big(v - \frac{1-F(v)}{f(v)}\big) f(v) F_1^{n-1}(v) dv \\
=&\frac{1-F(v_I)}{2}(v_I - r) F_1^{n-1}(v_I) \;+\; \int_{r'}^{v_I} \big(v - \frac{1-F(v)}{f(v)}\big) f(v) F_1^{n-1}(v) dv \\
&+\; \frac{1}{2}\int_{v_1}^{1} \big(v - \frac{1-F(v)}{f(v)}\big) f(v) F_1^{n-1}(v) dv.
\end{aligned}
$$

The last equality follows from (26).

We need to show that the derivative of this w.r.t r' is negative whenever r' has a positive virtual valuation. Therefore, differentiating w.r.t. r', we need to show (we shall denote $\frac{dv_I}{dr'}$ by v_I'):

$$
\begin{aligned}
& 0 \geq v_I' \frac{1-F(v_I)}{2} F_1^{n-1}(v_I) + (v_I - r)\frac{d}{dr'}\left(\frac{1-F(v_I)}{2} F_1^{n-1}(v_I)\right) - \big(r - \frac{1-F(r)}{f(r)}\big) f(r) F_1^{n-1}(r) \\
& \quad + \frac{v_I'}{2}\big(v_I - \frac{1-F(v_I)}{f(v_I)}\big) f(v_I) F_1^{n-1}(v_I) dv + \int_{r'}^{v_I} \big(v - \frac{1-F(v)}{f(v)}\big) f(v) \frac{dF_1^{n-1}(v)}{dr} dv \\
\iff & 0 \geq (v_I - r)\frac{d}{dr'}\left(\frac{1-F(v_I)}{2} F_1^{n-1}(v_I)\right) - \big(r - \frac{1-F(r)}{f(r)}\big) f(r) F_1^{n-1}(r) \\
& \quad + \frac{v_I'}{2} v_I f(v_I) F_1^{n-1}(v_I) + \int_{r'}^{v_I} \big(v - \frac{1-F(v)}{f(v)}\big) f(v) \frac{dF_1^{n-1}(v)}{dr} dv \\
\iff & 0 \geq (v_I - r)\frac{d}{dr'}\left(\frac{1-F(v_I)}{2} F_1^{n-1}(v_I)\right) \qquad \text{(since } r \text{ has +ve virtual value)} \\
& \quad + \frac{v_I'}{2} v_I f(v_I) F_1^{n-1}(v_I) + \int_{r'}^{v_I} \big(v - \frac{1-F(v)}{f(v)}\big) f(v) \frac{dF_1^{n-1}(v)}{dr} dv \\
\iff & 0 \geq (v_I - r)\left(\frac{-f(v_I)}{2} v_I' F_1^{n-1}(v_I) + \frac{1-F(v_I)}{2} \frac{dF_1^{n-1}(v)}{dr}\right) \\
& \quad + \frac{v_I'}{2} v_I f(v_I) F_1^{n-1}(v_I) + \int_{r'}^{v_I} \big(v - \frac{1-F(v)}{f(v)}\big) f(v) \frac{dF_1^{n-1}(v)}{dr} dv \\
\iff & 0 \geq (v_I - r)\left(F_1^{n-1}(v_I) - \frac{1-F(v_I)}{2}(n-1) F_1^{n-2}(v_I)\right) \text{(dividing through out by } -f(v_I)v_I'/2 \text{)} \\
& \quad - v_I F_1^{n-1}(v_I) + \int_{r'}^{v_I} \big(v - \frac{1-F(v)}{f(v)}\big) f(v)(n-1) F_1^{n-2}(v_I) dv \\
\iff & 0 \geq (v_I - r)\left(F_1^{n-1}(v_I) - \frac{1-F(v_I)}{2}(n-1) F_1^{n-2}(v_I)\right) \qquad \text{(since } v \geq v - \frac{1-F(v)}{f(v)} \text{)} \\
& \quad - v_I F_1^{n-1}(v_I) + \int_{r'}^{v_I} v f(v)(n-1) F_1^{n-2}(v_I) dv \\
\iff & 0 \geq (v_I - r)\left(F_1^{n-1}(v_I) - \frac{1-F(v_I)}{2}(n-1) F_1^{n-2}(v_I)\right) - v_I F_1^{n-1}(v_I) \\
& \quad + v_I F_1^{n-1}(v_I) - r' F_1^{n-1}(r') - \int_{r'}^{v_I} F_1^{n-1}(v) dv \\
\iff & 0 \geq (v_I - r)\left(F_1^{n-1}(v_I) - \frac{1-F(v_I)}{2}(n-1) F_1^{n-2}(v_I)\right) - v_I F_1^{n-1}(v_I) \\
& \quad + v_I F_1^{n-1}(v_I) - r' F_1^{n-1}(r') - (v_I - r) F_1^{n-1}(v_I) \\
\iff & 0 \geq (v_I - r)\left(-\frac{1-F(v_I)}{2}(n-1) F_1^{n-2}(v_I)\right) - r' F_1^{n-1}(r')
\end{aligned}
$$

But the right hand side here is clearly negative.

We are finally left to show that an unsubstantiated claim in the 5^{th} line that $(-\frac{f(v_I)}{2} v_I'))$ is indeed negative. Differentiating (26) w.r.t r', we have:

$$
\begin{aligned}
&F_1^{n-1}(v_I)v_I' - F_1^{n-1}(r') + \int_{r'}^{v_I} \frac{dF_1^{n-1}(v)}{dr'}dv = F_1^{n-1}(v_I)v_I' + (v_I - r)\frac{dF_2^{n-1}(v_I)}{dr'} \\
\Rightarrow &\int_{r'}^{v_I} \frac{dF_1^{n-1}(v)}{dr'}dv = F_1^{n-1}(r') + (v_I - r)\frac{dF_2^{n-1}(v_I)}{dr'} \\
\Rightarrow &\int_{r'}^{v_I} (n-1)F_1^{n-1}(v)(-\frac{f(v_I)}{2}v_I')dv = F_1^{n-1}(r') + (v_I - r)(n-1)F_2^{n-2}(v_I)\frac{f(v_I)}{2}v_I' \\
\Rightarrow &(-\frac{f(v_I)}{2}v_I')\left(\int_{r'}^{v_I} (n-1)F_1^{n-1}(v)dv + (v_I - r)(n-1)F_2^{n-2}(v_I)\right) = F_1^{n-1}(r')
\end{aligned}
$$

However the right hand side here is clearly positive, therefore $(-\frac{f(v_I)}{2}v_I'))$ must be positive.

C.3 Objective function of (ConsOpt)

Note that

$$
\begin{aligned}
&\int_0^{v_I} \left(v - \frac{1 - F_1(v)}{f_1(v)}\right)a(v)f_1(v)dv \\
=&\int_0^{v_I} \left(v - \frac{\int_v^1 \theta_1(t)f(t)dt}{\theta_1(v)f(v)}\right)a(v)\theta_1(v)f(v)dv \\
=&\int_0^{v_I} \left(v - \frac{\int_v^1 \theta_1(t)f(t)dt}{f(v)}\right)a(v)f(v)dv \qquad (\theta_1(v) = 1 \text{ for } v \in [0, v_I]) \\
=&\int_0^{v_I} \left(v - \frac{\int_v^{v_I} f(t)dt + \int_{v_I}^1 \theta_1(t)f(t)dt}{f(v)}\right)a(v)f(v)dv \\
=&\int_0^{v_I} \left(v - \frac{\int_v^1 f(t)dt + \int_{v_I}^1 (\theta_1(t) - 1)f(t)dt}{f(v)}\right)a(v)f(v)dv \\
=&\int_0^{v_I} \left(v - \frac{1 - F(v)}{f(v)} + \frac{\int_{v_I}^1 (1 - \theta_1(t))f(t)dt}{f(v)}\right)a(v)f(v)dv \\
=&\int_0^{v_I} \left(v - \frac{1 - F(v)}{f(v)}\right)a(v)f(v)dv + \int_0^{v_I} \frac{\int_{v_I}^1 (1 - \theta_1(t))f(t)dt}{f(v)}a(v)f(v)dv \\
=&\int_0^{v_I} \left(v - \frac{1 - F(v)}{f(v)}\right)a(v)f(v)dv + \left(\int_{v_I}^1 (1 - \theta_1(t))f(t)dt\right)\int_0^{v_I} a(v)dv \\
=&\int_0^{v_I} \left(v - \frac{1 - F(v)}{f(v)}\right)a(v)f(v)dv + \left(\int_{v_I}^1 (1 - \theta_1(t))f(t)dt\right)(v_I - r)F_2^{n-1}(v_I),
\end{aligned}
$$

where the last equality follows from constraint (13).

C.4 Proof of Lemma 4

By hypothesis, seller 2 has announced the mechanism $((\geq, r), 0)$. Suppose seller 1 announces a mechanism $((\geq, r'), 0)$, i.e. a quasi-efficient mechanism with reserve price r'. We know that the resulting θ will be according to (6) if $r' < r$ and according to (7) if $r' \geq r$. Suppose the former without loss of generality, and further suppose r', r such that the solution (26) is v_I.

We will show that seller 1's mechanism $((\geq, r'), 0)$ gets him more revenue than any other mechanism $((\succeq, r''), 0)$ with the same 'first point of indifference', v_I.

So suppose not, suppose $\succeq$ pools agents on some interval $[\underline{v}, \bar{v}]$, $\underline{v} > v_I$, and is the standard order otherwise. The resulting θ must be:

$$\theta_1(v) = \begin{cases} 1 & v \in [r'', v_I) \\ 0.5 & v \in [v_I, v_1) \\ 0 & v \in [v_1, v_2) \\ 1 & v \in [v_2, v_3) \\ 0 & v \in [v_3, v_4) \\ 0.5 & v \in [v_4, 1]. \end{cases} \tag{27}$$

where:

$$v_1 \leq \underline{v} \leq v_2 \leq v_3 \leq \bar{v} \leq v_4.$$
$$\frac{F(v_3) - F(v_2)}{F(v_4) - F(v_1)} = \frac{1}{2}.$$

The former set of inequalities follow by construction, while the latter follows the fact that buyers with valuations $[v_I, v_1]$ must be indifferent to either seller. Therefore by Lemma 1, $F_1(v_1) = F_2(v_1)$. Similarly, by construction, buyers with valuations $[v_4, 1]$ must be indifferent to either seller. Therefore, similarly $F_1(v_4) = F_2(v_4)$. Combining, we have

$$\begin{aligned} & F_1(v_4) - F_1(v_1) = F_2(v_4) - F_2(v_1) \\ \Rightarrow & \int_{v_1}^{v_4} \theta_1(v) f(v) dv = \int_{v_1}^{v_4} \theta_2(v) f(v) dv \\ \Rightarrow & \int_{v_1}^{v_4} (1 - 2\theta_j(v)) f(v) dv = 0 \qquad j = 1, 2 \\ \Rightarrow & \int_{v_1}^{v_4} \theta_j(v) f(v) dv = \frac{F(v_4) - F(v_1)}{2}. \end{aligned}$$

Further, $(v_j)_{j=1}^4$ must solve:

$$(\underline{v} - v_1)a_1 + (v_2 - \underline{v})\bar{a} = \int_{v_1}^{v_2} F_2^{n-1}(v)dv, \tag{28}$$

$$\bar{a} = \frac{F_1^n(v_4) - F_1^n(v_1)}{n(F_1(v_4) - F_1(v_1))} = F_2^{n-1}(v_2), \tag{29}$$

$$(\bar{v} - v_3)\bar{a} + (v_4 - \bar{v})a_2 = \int_{v_3}^{v_4} F_2^{n-1}(v)dv, \tag{30}$$

where:

$$a_1 = F_1^{n-1}(v_1); a_2 = F_1^{n-1}(v_4).$$

Note that the expected revenue of seller 1 is:

$$\int_0^1 \left(v - \frac{1 - F_1(v)}{f_1(v)}\right) f_1(v)a_1(v)dv, \tag{31}$$

where a_1 and F_1 are both determined by seller 1's choice of mechanism.

By observation, seller 1 changing from $((\geq, r')0) \to ((\succeq, r'), 0)$ by pooling types in $[\underline{v}, \bar{v}]$ will only change the revenue on segment $[v_1, v_4]$. Denote by g_1, G_1 the distribution of buyer types that results when seller 1 announces $((\geq, r'), 0)$. It is therefore sufficient to show that:

$$\int_{v_1}^{v_4} \left(v - \frac{1 - F_1(v)}{f_1(v)}\right) f_1(v)a_1(v)dv \leq \int_{v_1}^{v_4} \left(v - \frac{1 - G_1(v)}{g_1(v)}\right) g_1(v)G_1^{n-1}(v)dv, \tag{32}$$

where v_1 through v_4 solve (28- 30). Wlog, fix $\underline{v} > v_I$. At $\bar{v} = \underline{v}$, the inequality above is satisfied as equality. Therefore, treating v_1 through v_4 as implicit functions of $\bar{v}$ (fixing $\underline{v}$) it is sufficient to show that

$$\frac{d}{d\bar{v}} \int_{v_1}^{v_4} \left(v - \frac{1 - F_1(v)}{f_1(v)}\right) f_1(v)a_1(v)dv \leq \frac{d}{d\bar{v}} \int_{v_1}^{v_4} \left(v - \frac{1 - G_1(v)}{g_1(v)}\right) g_1(v)G_1^{n-1}(v)dv,$$

for any $\bar{v} \geq \underline{v}$.

Differentiating (28) w.r.t $\bar{v}$, we have:

$$\frac{da_1}{d\bar{v}}(\underline{v} - v_1) + \frac{d\bar{a}}{d\bar{v}}(v_2 - \underline{v}) = \int_{v_1}^{v_2} \frac{dF_2^{n-1}(v)}{d\bar{v}} dv. \tag{33}$$

Similarly differentiating (30) w.r.t. $\underline{v}$, we have:

$$\frac{d\bar{a}}{d\bar{v}}(\bar{v} - v_3) + \frac{da_2}{d\bar{v}}(v_4 - \bar{v}) + (\bar{a} - a_2) = \int_{v_3}^{v_4} \frac{dF_2^{n-1}(v)}{d\bar{v}} dv. \tag{34}$$

Further, since we know θ_1 from (27), we have

$$F_1(v) = \begin{cases} F_1(v_1) & v \in [v_1, v_2) \\ F_1(v_4) - (F(v_3) - F(v)) & v \in [v_2, v_3) \\ F_1(v_4) & v \in [v_3, v_4]. \end{cases} \tag{35}$$

$$a_1(v) = \begin{cases} a_1 = F_1^{n-1}(v_1) & v \in [v_1, \underline{v}) \\ \bar{a} & v \in [\underline{v}, \bar{v}] \\ a_2 = F_1^{n-1}(v_4) & v \in (\bar{v}, v_4] \end{cases} \tag{36}$$

Substituting (35), (36) into (31), we have:

$$\begin{aligned} &\int_{v_1}^{v_4} \left(v - \frac{1 - F_1(v)}{f_1(v)}\right) f_1(v) a_1(v) dv \\ =&\big(v_1(1 - F_1(v_1))a_1 - v_4(1 - F_1(v_4))a_2\big) + \underline{v}(1 - F_1(v_1))(\bar{a} - a_1) + \bar{v}(1 - F_1(v_4))(a_2 - \bar{a}). \end{aligned}$$

Differentiating wrt to $\bar{v}$:

$$\begin{aligned} &-\left(v_1 - \frac{1 - F_1(v_1)}{f_1(v_1)}\right) f_1(v_1) a_1 \frac{dv_1}{d\bar{v}} + v_1(1 - F_1(v_1))\frac{da_1}{d\bar{v}} + \left(v_4 - \frac{1 - F_1(v_4)}{f_1(v_4)}\right) f_1(v_4) a_2 \frac{dv_4}{d\bar{v}} \\ &- v_4(1 - F_1(v_4))\frac{da_2}{d\bar{v}} - \underline{v} f_1(v_1)\frac{dv_1}{d\bar{v}}(\bar{a} - a_1) + \underline{v}(1 - F_1(v_1))\left(\frac{d\bar{a}}{d\bar{v}} - \frac{da_1}{d\bar{v}}\right) \\ &+ (1 - F_1(v_4))(a_2 - \bar{a}) - \bar{v} f_1(v_4)\frac{dv_4}{d\bar{v}}(a_2 - \bar{a}) + \bar{v}(1 - F_1(v_4))\left(\frac{da_2}{d\bar{v}} - \frac{d\bar{a}}{d\bar{v}}\right). \end{aligned} \tag{37}$$

Differentiating the right hand side of (32) we have:

$$\left(v_4 - \frac{1 - G_1(v_4)}{g_1(v_4)}\right) g_1(v_4) G_1^{n-1}(v_4)\frac{dv_4}{d\bar{v}} - \left(v_1 - \frac{1 - G_1(v_1)}{g_1(v_1)}\right) g_1(v_1) G_1^{n-1}(v_1)\frac{dv_1}{d\bar{v}}.$$

Further, by construction, $G_1(v_j) = F_1(v_j)$ and $g_1(v_j) = f_1(v_j)$ for $j = 1$ and $j = 4$. Therefore substituting into (37) and cancelling appropriately, we are left to show that the following expression is negative.

$$v_1(1 - F_1(v_1))\frac{da_1}{d\bar{v}} + \underline{v}(1 - F_1(v_1))\left(\frac{d\bar{a}}{d\bar{v}} - \frac{da_1}{d\bar{v}}\right) \tag{38}$$

$$-v_4(1 - F_1(v_4))\frac{da_2}{d\bar{v}} + (1 - F_1(v_4))(a_2 - \bar{a}) + \bar{v}(1 - F_1(v_4))\left(\frac{da_2}{d\bar{v}} - \frac{d\bar{a}}{d\bar{v}}\right) \tag{39}$$

$$- \underline{v} f_1(v_1)\frac{dv_1}{d\bar{v}}(\bar{a} - a_1) - \bar{v} f_1(v_4)\frac{dv_4}{d\bar{v}}(a_2 - \bar{a}). \tag{40}$$

Substituting (33) into (38), we see that (38)=

$$(1 - F_1(v_1))\Big(v_2\frac{d\bar{a}}{d\bar{v}} - \int_{v_1}^{v_2}\frac{dF_2^{n-1}(v)}{d\bar{v}}dv\Big).$$

Similarly, substituting (34) into (39) we have (39) =

$$(1 - F_1(v_4))\Big(-v_3\frac{d\bar{a}}{d\bar{v}} - \int_{v_3}^{v_4}\frac{dF_2^{n-1}(v)}{d\bar{v}}dv\Big).$$

However

$$F_2(v) = \begin{cases} (F(v) + \frac{1-F(v_1)}{2}) & v \in [v_1, v_2] \\ F_2(v_2) & v \in (v_2, v_3) \\ (F(v) + \frac{1-F(v_4)}{2}) & v \in [v_3, v_4] \end{cases}$$

Note that:

$$\begin{aligned} -\int_{v_1}^{v_2}\frac{dF_2^{n-1}(v)}{d\bar{v}}dv =& f_1(v_1)\frac{dv_1}{d\bar{v}}\int_{v_1}^{v_2}(n-1)F_2^{n-2}(v)dv, \\ -\int_{v_3}^{v_4}\frac{dF_2^{n-1}(v)}{d\bar{v}}dv =& f_1(v_4)\frac{dv_4}{d\bar{v}}\int_{v_3}^{v_4}(n-1)F_2^{n-2}(v)dv. \end{aligned}$$

Since $f(v)$ is decreasing,

$$\begin{aligned} f_1(v_1)\int_{v_1}^{v_2}(n-1)F_2^{n-2}(v)dv &\geq \int_{v_1}^{v_2}(n-1)F_2^{n-2}(v)f_1(v)dv \\ &= F_2^{n-1}(v_2) - F_2^{n-1}(v_1) \\ &= \bar{a} - a_1 \\ f_1(v_4)\int_{v_3}^{v_4}(n-1)F_2^{n-2}(v)dv &\leq \int_{v_1}^{v_2}(n-1)F_2^{n-2}(v)f_1(v)dv \\ &= F_2^{n-1}(v_4) - F_2^{n-1}(v_3) \\ &= a_2 - \bar{a} \end{aligned}$$

Further $\frac{da_1}{d\bar{v}}$ and $\frac{d\bar{a}}{d\bar{v}}$ must be of opposite sign due to (33). Further (34) requires that atleast one of $\frac{d\bar{a}}{d\bar{v}}$ and $\frac{da_2}{d\bar{v}}$ is non-negative. This combined with (41) implies that $\frac{d\bar{a}}{d\bar{v}}$ and $\frac{dv_4}{d\bar{v}}$ must both be non-negative while $\frac{dv_1}{d\bar{v}}$ must be non positive. Noting that $\frac{dv_1}{d\bar{v}} \leq 0$, $\frac{dv_4}{d\bar{v}}$,

$$\begin{aligned} -\int_{v_1}^{v_2}\frac{dF_2^{n-1}(v)}{d\bar{v}}dv &\leq \frac{dv_1}{d\bar{v}}(\bar{a} - a_1), \\ -\int_{v_3}^{v_4}\frac{dF_2^{n-1}(v)}{d\bar{v}}dv &\leq \frac{dv_4}{d\bar{v}}(a_2 - \bar{a}). \end{aligned}$$

Therefore collecting terms, we need to show that the following expression is negative:

$$\begin{aligned}
&(1-F_1(v_1))\bigg(v_2\frac{d\bar{a}}{d\bar{v}}+\frac{dv_1}{d\bar{v}}(\bar{a}-a_1)\bigg)\\
&(1-F_1(v_4))\bigg(-v_3\frac{d\bar{a}}{d\bar{v}}+\frac{dv_4}{d\bar{v}}(a_2-\bar{a})\bigg)\\
&-\underline{v}f_1(v_1)\frac{dv_1}{d\bar{v}}(\bar{a}-a_1)-\bar{v}f_1(v_4)\frac{dv_4}{d\bar{v}}(a_2-\bar{a}).
\end{aligned}$$

Differentiating $\bar{a}$ w.r.t $\bar{v}$ (29), we see that:

$$\frac{d\bar{a}}{d\bar{v}}=\frac{f_1(v_4)\frac{dv_4}{d\bar{v}}(a_2-\bar{a})+(\bar{a}-a_1)\frac{dv_1}{d\bar{v}}f_1(v_1)}{F_1(v_4)-F_1(v_1)} \tag{41}$$

Substituting back and collecting terms we are left to show that the following expression is negative:

$$\begin{aligned}
&f_1(v_1)\frac{dv_1}{d\bar{v}}(\bar{a}-a_1)\bigg(\frac{v_2(1-F_1(v_1))}{F_1(v_4)-F_1(v_1)}-\frac{v_3(1-F_1(v_4))}{F_1(v_4)-F_1(v_1)}+\frac{(1-F_1(v_1))}{f_1(v_1)}-\underline{v}\bigg)\\
+&f_1(v_4)\frac{dv_4}{d\underline{v}}(a_2-\bar{a})\bigg(\frac{(1-F_1(v_1))v_2}{F_1(v_4)-F_1(v_1)}-\frac{(1-F_1(v_4))v_3}{F_1(v_4)-F_1(v_1)}+\frac{(1-F_1(v_4))}{f_1(v_4)}-\bar{v}\bigg)
\end{aligned}$$

Therefore it is sufficient to show

$$\begin{aligned}
\frac{v_2(1-F_1(v_1))}{F_1(v_4)-F_1(v_1)}-\frac{v_3(1-F_1(v_4))}{F_1(v_4)-F_1(v_1)}+\frac{(1-F_1(v_1))}{f_1(v_1)}-\underline{v}\geq 0\\
\frac{(1-F_1(v_1))v_2}{F_1(v_4)-F_1(v_1)}-\frac{(1-F_1(v_4))v_3}{F_1(v_4)-F_1(v_1)}+\frac{(1-F_1(v_4))}{f_1(v_4)}-\bar{v}\leq 0
\end{aligned}$$

However since $\frac{f}{1-F}$ is increasing and $f(v)$ is decreasing,

$$\begin{aligned}
\frac{v_2(1-F_1(v_1))}{F_1(v_4)-F_1(v_1)}-\frac{v_3(1-F_1(v_4))}{F_1(v_4)-F_1(v_1)}&\geq v_2-\frac{1-F(v_2)}{f(v_2)}\\
\frac{v_2(1-F_1(v_1))}{F_1(v_4)-F_1(v_1)}-\frac{v_3(1-F_1(v_4))}{F_1(v_4)-F_1(v_1)}&\leq v_3-\frac{1-F(v_3)}{f(v_3)}\\
\underline{v}-\frac{(1-F_1(v_1))}{f_1(v_1)}&\leq \underline{v}-\frac{1-F(\underline{v})}{f(\underline{v})}\\
\bar{v}-\frac{(1-F_1(v_4))}{f_1(v_4)}&\geq \bar{v}-\frac{1-F(\bar{v})}{f(\bar{v})}
\end{aligned}$$

While the latter two inequalities are obvious, to see the former 2, note that the left hand side of each=

$$\begin{aligned}
&\frac{v_2(1-F_1(v_1))}{F_1(v_4)-F_1(v_1)} - \frac{v_3(1-F_1(v_4))}{F_1(v_4)-F_1(v_1)}\\
=&\frac{1}{F_1(v_4)-F_1(v_1)}(v_2(1-F_1(v_1)) - v_3(1-F_1(v_4)))\\
=&\frac{1}{F_1(v_3)-F_1(v_2)}(v_2(1-F_1(v_2)) - v_3(1-F_1(v_3))\\
=&\frac{1}{\int_{v_2}^{v_3} f_1(v)dv}(\int_{v_2}^{v_3}(v - \frac{1-F_1(v)}{f_1(v)}f_1(v)dv)\\
=&\frac{\int_{v_2}^{v_3}(v-\frac{1-F_1(v)}{f_1(v)})f_1(v)dv}{\int_{v_2}^{v_3} f_1(v)dv}.
\end{aligned}$$

The result now follows since $\underline{v} \leq v_2 \leq v_3 \leq \bar{v}$. □

C.5 Proof of Corollary 1

Suppose seller 1 announces a non-quasi-efficient mechanism $\mathbf{m}_1 = (\succ, r_1)$. Seller 2 announces a mixed-strategy quasi-efficient mechanism implies he only randomizes over reserve prices. Let r_2 be any reserve price in the support.

First suppose $r_1 < r_2$. By the proof of Lemma 2, r_1 must have non-positive virtual value (if not, then decreasing r_1 increases revenue even for non-quasi-efficient mechanisms). Suppose therefore r_1 has non-positive virtual value. Let v_I be the lowest valuation larger than r_1 such that a buyer with valuation v_I is indifferent between both sellers. By proof of Lemma 3, a mechanism with reserve r_1 that is efficient on $[r_1, v_I]$ produces weakly more revenue than $\mathbf{m}_1$. By Lemma 4, the best response $\mathbf{m}_1$ must be efficient on $[v_I, 1]$.

For $r_1 > r_2$, the best response must be efficient on $[v_1, 1]$ by Lemma 4.

Therefore, by the proof of Theorem 3, a quasi-efficient mechanism with reserve price r_1 is a better response to a quasi-efficient mechanism with reserve r_2 if f, F satisfies (IHR),(DD). □

C.6 Proof of Proposition 2

It is sufficient to show that the difference in revenue between a non-quasi-efficient mechanims with reserve price r_1 and a quasi-efficient mechanism with reserve price $r \to 0$ as number of sellers $m \to \infty$. The Proposition then follows from Peters & Severinov (1997, Theorem 4).

The proof replicates portions of the argument of the proof of Lemma 4. Instead of showing that a certain term is –ve, we shall show that it vanishes as number of sellers $m \to \infty$.

Suppose seller 1 announces $\mathbf{m}_1 = (\succeq, r_1)$.Let v_I be the first point of indifferent between sellers 1 and sellers 2 through m.

Suppose $\succeq$ pools agents on some interval $[\underline{v}, \bar{v}]$, $\underline{v} > v_I$, and is the standard order otherwise. The resulting θ must be:

$$\theta_1(v) = \begin{cases} 0 & v \leq v_I) \\ \frac{1}{m} & v \in [v_I, v_1) \\ 0 & v \in [v_1, v_2) \\ 1 & v \in [v_2, v_3) \\ 0 & v \in [v_3, v_4) \\ \frac{1}{m} & v \in [v_4, 1]. \end{cases} \tag{42}$$

where:

$$v_1 \leq \underline{v} \leq v_2 \leq v_3 \leq \bar{v} \leq v_4.$$
$$\frac{F(v_3) - F(v_2)}{F(v_4) - F(v_1)} = \frac{1}{m}.$$

By symmetry, sellers j in 2 through m must have the same θ_j and a_j. We will refer to this as θ_2 to economize on notation.

Further, $(v_j)_{j=1}^4$ must solve:

$$(\underline{v} - v_1)a_1 + (v_2 - \underline{v})\bar{a} = \int_{v_1}^{v_2} F_2^{n-1}(v)dv, \tag{43}$$

$$\bar{a} = \frac{F_1^n(v_4) - F_1^n(v_1)}{n(F_1(v_4) - F_1(v_1))} = F_2^{n-1}(v_2), \tag{44}$$

$$(\bar{v} - v_3)\bar{a} + (v_4 - \bar{v})a_2 = \int_{v_3}^{v_4} F_2^{n-1}(v)dv, \tag{45}$$

where:

$$a_1 = F_1^{n-1}(v_1); a_2 = F_1^{n-1}(v_4).$$

Note that the expected revenue of seller 1 is:

$$\int_0^1 \left(v - \frac{1 - F_1(v)}{f_1(v)}\right) f_1(v)a_1(v)dv, \tag{46}$$

where a_1 and F_1 are both determined by seller 1's choice of mechanism.

By observation, seller 1 changing from $(\geq, r_1) \to (\succeq, r_1)$ by pooling types in $[\underline{v}, \bar{v}]$ will only change the revenue on segment $[v_1, v_4]$. Denote by g_1, G_1 the distribution of buyer types that results when seller 1 announces $(\geq, r')$. It is therefore sufficient to show that:

$$\lim_{m\to\infty}\int_{v_1}^{v_4}\left(v-\frac{1-F_1(v)}{f_1(v)}\right)f_1(v)a_1(v)dv-\int_{v_1}^{v_4}\left(v-\frac{1-G_1(v)}{g_1(v)}\right)g_1(v)G_1^{n-1}(v)dv=0 \tag{47}$$

where v_1 through v_4 solve (43- 45). Fix $\underline{v} > v_I$. At $\bar{v} = \underline{v}$, (47) is true for any m. Therefore, treating v_1 through v_4 as implicit functions of $\bar{v}$ (fixing $\underline{v}$) it is sufficient to show that

$$\lim_{m\to\infty}\frac{d}{d\bar{v}}\int_{v_1}^{v_4}\left(v-\frac{1-F_1(v)}{f_1(v)}\right)f_1(v)a_1(v)dv-\frac{d}{d\bar{v}}\int_{v_1}^{v_4}\left(v-\frac{1-G_1(v)}{g_1(v)}\right)g_1(v)G_1^{n-1}(v)dv=0$$

for any $\bar{v} \geq \underline{v}$.

Differentiating (43) w.r.t $\bar{v}$, we have:

$$\frac{da_1}{d\bar{v}}(\underline{v}-v_1)+\frac{d\bar{a}}{d\bar{v}}(v_2-\underline{v})=\int_{v_1}^{v_2}\frac{dF_2^{n-1}(v)}{d\bar{v}}dv. \tag{48}$$

Similarly differentiating (45) w.r.t. $\underline{v}$, we have:

$$\frac{d\bar{a}}{d\bar{v}}(\bar{v}-v_3)+\frac{da_2}{d\bar{v}}(v_4-\bar{v})+(\bar{a}-a_2)=\int_{v_3}^{v_4}\frac{dF_2^{n-1}(v)}{d\bar{v}}dv. \tag{49}$$

Further, since we know θ_1 from (42), we have

$$F_1(v)=\begin{cases}F_1(v_1) & v\in[v_1,v_2)\\ F_1(v_4)-(F(v_3)-F(v)) & v\in[v_2,v_3)\\ F_1(v_4) & v\in[v_3,v_4].\end{cases} \tag{50}$$

$$a_1(v)=\begin{cases}a_1=F_1^{n-1}(v_1) & v\in[v_1,\underline{v})\\ \bar{a} & v\in[\underline{v},\bar{v}]\\ a_2=F_1^{n-1}(v_4) & v\in(\bar{v},v_4]\end{cases} \tag{51}$$

Substituting (50), (51) into (46), we have:

$$\begin{aligned}&\int_{v_1}^{v_4}\left(v-\frac{1-F_1(v)}{f_1(v)}\right)f_1(v)a_1(v)dv\\ =&\big(v_1(1-F_1(v_1))a_1-v_4(1-F_1(v_4))a_2\big)+\underline{v}(1-F_1(v_1))(\bar{a}-a_1)+\bar{v}(1-F_1(v_4))(a_2-\bar{a}).\end{aligned}$$

Differentiating wrt to $\bar{v}$:

$$-\left(v_1-\frac{1-F_1(v_1)}{f_1(v_1)}\right)f_1(v_1)a_1\frac{dv_1}{d\bar{v}}+v_1(1-F_1(v_1))\frac{da_1}{d\bar{v}}+\left(v_4-\frac{1-F_1(v_4)}{f_1(v_4)}\right)f_1(v_4)a_2\frac{dv_4}{d\bar{v}}$$
$$-v_4(1-F_1(v_4))\frac{da_2}{d\bar{v}}-\underline{v}f_1(v_1)\frac{dv_1}{d\bar{v}}(\bar{a}-a_1)+\underline{v}(1-F_1(v_1))\left(\frac{d\bar{a}}{d\bar{v}}-\frac{da_1}{d\bar{v}}\right)$$
$$+(1-F_1(v_4))(a_2-\bar{a})-\bar{v}f_1(v_4)\frac{dv_4}{d\bar{v}}(a_2-\bar{a})+\bar{v}(1-F_1(v_4))\left(\frac{da_2}{d\bar{v}}-\frac{d\bar{a}}{d\bar{v}}\right). \quad (52)$$

Differentiating the right hand side of (47) we have:

$$\left(v_4-\frac{1-G_1(v_4)}{g_1(v_4)}\right)g_1(v_4)G_1^{n-1}(v_4)\frac{dv_4}{d\bar{v}}-\left(v_1-\frac{1-G_1(v_1)}{g_1(v_1)}\right)g_1(v_1)G_1^{n-1}(v_1)\frac{dv_1}{d\bar{v}}.$$

Further, by construction, $G_1(v_j)=F_1(v_j)$ and $g_1(v_j)=f_1(v_j)$ for $j=1$ and $j=4$. Therefore substituting into (52) and cancelling appropriately, we are left to show that the following expression vanishes.

$$v_1(1-F_1(v_1))\frac{da_1}{d\bar{v}}+\underline{v}(1-F_1(v_1))\left(\frac{d\bar{a}}{d\bar{v}}-\frac{da_1}{d\bar{v}}\right) \quad (53)$$
$$-v_4(1-F_1(v_4))\frac{da_2}{d\bar{v}}+(1-F_1(v_4))(a_2-\bar{a})+\bar{v}(1-F_1(v_4))\left(\frac{da_2}{d\bar{v}}-\frac{d\bar{a}}{d\bar{v}}\right) \quad (54)$$
$$-\underline{v}f_1(v_1)\frac{dv_1}{d\bar{v}}(\bar{a}-a_1)-\bar{v}f_1(v_4)\frac{dv_4}{d\bar{v}}(a_2-\bar{a}). \quad (55)$$

Since $f_1(v_1)=\frac{1}{m}f(v_1)$, $f(v_4)=\frac{1}{m}f(v_4)$, and $\underline{v},\bar{v}\in V=[0,1]$, $\bar{a},a_1,a_2\leq 1$, term (55) clearly vanishes as $m\to\infty$.

Substituting (48) into (53), we see that (53)=

$$(1-F_1(v_1))\left(v_2\frac{d\bar{a}}{d\bar{v}}-\int_{v_1}^{v_2}\frac{dF_2^{n-1}(v)}{d\bar{v}}dv\right). \quad (56)$$

Similarly, substituting (49) into (54) we have (54) =

$$(1-F_1(v_4))\left(-v_3\frac{d\bar{a}}{d\bar{v}}-\int_{v_3}^{v_4}\frac{dF_2^{n-1}(v)}{d\bar{v}}dv\right). \quad (57)$$

$$F_2(v)=\begin{cases}F_2(v_4)-\frac{F(v_2)-F(v)}{m-1}-\frac{F(v_4)-F(v_3)}{m-1} & v\in[v_1,v_2]\\ F_2(v_2) & v\in(v_2,v_3)\\ 1-\frac{F(v_4)-F(v)}{m-1}-\frac{1-F(v_4)}{m} & v\in[v_3,v_4]\end{cases}$$

Therefore, again, as $m\to\infty$,

1. $v_2 - v_3 \to 0$ (since $F(v_3) - F(v_2) = \frac{F(v_4)-F(v_1)}{m} \leq \frac{1}{m} \to 0$).
2. $(1 - F_1(v_1)) \to (1 - F_1(v_4))$ since $F_1(v_1) = F_1(v_4) + F(v_3) - F(v_2)$.
3. $(1 - F_1(v_4)) = 1 - \frac{1-F(v_4)}{m} \to 0$.
4. $\frac{dF_2^{n-1}(v)}{dv} \to 0$.

Therefore (56), (57) $\to 0$. □