# Robust Conditional Predictions in Dynamic Games: An Application to Sovereign Debt[*]

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**Abstract**

Dynamic policy games feature a wide range of equilibria. The goal of this paper is to provide a methodology to obtain robust predictions. We characterize outcomes that are consistent with a subgame perfect equilibrium conditional on the observed history. We focus on a model of sovereign debt, although our methodology applies to other settings, such as models of capital taxation or monetary policy. As a starting point, we show that the Eaton and Gersovitz (1981) model features multiple equilibria—indeed, multiple Markov equilibria—when debt is sufficiently constrained. We focus on predictions for bond yields or prices. We show that the highest bond price is independent of the history, while the lowest is strictly positive and does depend on past play. We show that previous period play is a sufficient statistic for the set of bond prices. The lower bound on bond prices rises when the government avoids default under duress.

# 1 Introduction

Following Kydland and Prescott (1977) and Calvo (1978) the literature on optimal government policy without commitment has formalized these situations by employing dynamic game theory, finding interesting applications for capital taxation (e.g. Chari and Kehoe, 1990, Phelan and Stacchetti, 2001, Farhi et al., 2012), monetary policy (e.g. Ireland, 1997, Chang, 1998, Sleet, 2001) and sovereign debt (e.g. Calvo, 1988, Eaton and Gersovitz, 1981, Chari and Kehoe, 1993, Cole and Kehoe, 2000). This research has helped us understand the distortions introduced by the lack of commitment and the extent to which governments can rely on a reputation for credibility to achieve better outcomes.

One of the challenges in applying dynamic policy games is that these settings typically feature a wide range of equilibria with different predictions over outcomes. For example, for "good" equilibria the government may achieve, or come close to achieving, the optimum with commitment, while there are "bad" equilibria where this is far from the case, and the government may be playing the repeated static best response. In studying dynamic policy games, which of these equilibria should we employ? One approach is to impose refinements, such as various renegotiation proofness notions, that select an equilibrium or significantly reduce the set of equilibria. Unfortunately, no consensus has emerged on the appropriate refinements.

Our goal is to overcome the challenge multiplicity raises by providing predictions in dynamic policy games that are not sensitive to any equilibrium selection. The approach we offer involves making predictions for future play that depend on past play. The key idea is that, even when little can be said about the *unconditional* path of play, quite a bit can be said once we *condition* on past observations. To the best of our knowledge, this simple idea has not been exploited as a way of deriving robust implications from the theory. Formally, we introduce and study a concept which we term "equilibrium consistent outcomes": outcomes of the game in a particular period that are consistent with a subgame perfect equilibrium, conditional on the observed history.

Although it will be clear that the notions we propose and results we derive are general and apply to any dynamic policy game, we develop them for a specific application, using a model of sovereign debt along the lines of Eaton and Gersovitz (1981). This model constitutes a workhorse in international economics. In the model, a small open economy faces a stochastic stream of income. To smooth consumption, a benevolent government can borrow from international debt markets, but lacks commitment to repay. If it defaults on its debt, the only punishment is permanent exclusion from financial markets; it can

never borrow again.[1]

Given that our approach tries to overcome the challenges of multiplicity, as a starting point we first ensure that there is multiplicity in the first place. We show that in the standard Eaton and Gersovitz (1981) model, restrictions on debt, often adopted in the quantitative sovereign-debt literature, imply the existence of multiple equilibria. Our multiplicity relies on the existence of autarky as another Markov equilibrium. This result may be of independent interest, since it implies that rollover crises are possible in this setting. The quantitative literature on sovereign debt following Eaton and Gersovitz (1981) features defaults on the equilibrium path, but to shocks to fundamentals.[2] Another strand literature studies self-fulfilling debt crises following the models in Calvo (1988) and Cole and Kehoe (2000). Our results suggest that crises, defined as episodes where the interest rates are very high but not due to fundamentals, are a robust feature in models of sovereign debt.

Given multiplicity, our main result provides a characterization of equilibrium consistent outcomes in any period (debt prices, debt issuance, and default decisions). Aided by this characterization, we obtain bounds for equilibrium consistent debt prices that are history dependent. The highest equilibrium consistent price is the best Markov equilibria and, thus, independent of past play. The lowest equilibrium consistent price is strictly positive and depends on past play. In our baseline case, due to the recursive nature of equilibria, only the previous period play matters and acts as a sufficient statistic for the set of equilibrium consistent prices.

In our sovereign debt application, equilibrium consistent debt prices improve whenever the government avoids default under duress. In particular, if the country just repaid a high amount of debt, or did so under harsh economic conditions, for example, when output was low, the lowest equilibrium consistent price is higher. The choice to repay under these conditions reveals an optimistic outlook for bond prices that narrows down the set of possible equilibria for the continuation game. This result captures the idea that reputation is built for the long run by short-run sacrifices.

We apply our results to study the probability of a rollover debt crises. As we argued above, rollover debt crises may occur on the equilibrium path for any fundamentals. However, the probability of a rollover crisis, after a certain history, may be constrained. We derive these constraints, showing that rollover crises are less likely if the borrower

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[1]The key features are lack of commitment, a time inconsistency problem, and an infinite horizon that creates reputational concerns in the sense of trigger-strategy equilibria. These features are shared by all dynamic policy games, such as applications to capital taxation and monetary policy.

[2]A recent exception is Stangebye (2014) that studies the role of non fundamental shocks in sovereign crises in a model as in Eaton and Gersovitz (1981).

has recently made sacrifices to repay. This result may be contrasted with Cole and Kehoe (2000). In their setting the potential for rollover crises induces the government to lower debt below a threshold that rules rollover crises out. Thus, the government's efforts have no effect in the short run, but payoff in the long run. In our model, an outside observer will witness that rollover crises are less likely immediately after an effort to repay.

In the Eaton and Gersovitz (1981) setting, all defaults are assumed to be punished by autarky. This may be seen as a feature of the game, or as a restriction to focus on the subset of equilibria with this property. In both cases, it may be viewed as somewhat ad hoc. In Section 5, we relax these assumptions. This captures an important alternative set of models from the literature of sovereign debt, when defaults are excusable, in the sense of Grossman and Huyck (1989). We have also assumed that the government can only borrow, but not save; in the context of excusable defaults, we also relax this assumption. The worst subgame perfect equilibrium remains autarky. We provide a characterization of equilibrium consistent outcomes and prices. As one might expect, the same principle applies: equilibrium consistent debt prices improve as the amount of debt just repaid increases, or if the conditions under which debt was repaid are less favorable. Repaying under duress rules out negative outlooks on prices.

The fact that the last period is a sufficient statistic may seem surprising. This result is a direct expression of robustness: the expected payoff rationalizing a decision may have been realized for histories that have not occurred. When income is continuous, any particular history has probability zero, so the realized expected payoff rationalizing past behavior can always been expected for those realizations that did not materialize.[3]

To better understand the role of robustness in determining equilibrium consistent outcomes, we then move to the case where there are no randomization devices available and output is discrete. In a two period example, we provide a simple argument for why we obtain the sufficiency result in the continuous income case and we characterize how history impacts on the set of equilibrium consistent prices. We show that history can be summarized in a unique variable, a minimum value of utility that makes previous decisions rational. Intuitively, the restrictions that past decisions impose on current prices (and policies) can be decomposed into two terms. The first term is the value that the government receives in the realized history times the probability of that history. The second term is the value of the best equilibrium, off the equilibrium path. The link between current prices and past decisions depends on the strength (probability) and the length (discounting) of the link between current and past periods.

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[3]This intuition was first introduced by Gul and Pearce (1996) to show that Forward induction has much less predictive power as a solution concept if there are correlating devices.

We relate equilibrium consistent outcomes to Robust Bayesian Analysis. In Bayesian Analysis, the econometrician has a prior over fundamental parameters and derives a posterior after observing the data; in Robust Bayesian analysis the econometrician is uncertain about the prior and derives a *set* of posteriors from the set of possible priors. Given equilibrium multiplicity, in a dynamic policy game, the Bayesian econometrician has a prior over outcomes. In many contexts, it may be hard to form a prior; thus, uncertainty about the prior is a natural assumption. We show that the set of equilibrium consistent outcomes is the support of the posterior for a Bayesian that only assume that the data was generated by some subgame perfect equilibrium and is agnostic with respect to the prior over equilibrium outcomes.

**Literature Review.**   Our paper relates to the literature on credible government policies; the seminal contributions in that literature are Chari and Kehoe (1990) and Stokey (1991).[4] These two papers adapt the techniques developed in Abreu (1988) to dynamic policy games. Although these papers provide a full characterization of equilibria, they do not attempt to derive robust predictions across these equilibria, as we do here.

The two papers more closely related to our work are Angeletos and Pavan (2013) and Bergemann and Morris (2013). The first paper obtains predictions that hold across every equilibrium in a global game with an endogenous information structure. The second paper obtains restrictions over moments of observable endogenous variables that hold across every possible information structure in a class of coordination games. Our paper relates to them in that we obtain predictions that hold across all equilibria. Our results are weaker than Angeletos and Pavan (2013) because our predictions are regarding the equilibrium set. But it is also true that our problem has the additional challenge of being a (repeated) dynamic complete information game. The latter is precisely the root of weaker predictions.

The literature of sovereign debt[5] has evolved in several directions. One direction, the quantitative literature on sovereign debt, focuses on a model where asset markets are incomplete and there is limited commitment for repayment, following Eaton and Gersovitz (1981), to study the quantitative properties of spreads, debt capacity, and business cycles. The aim of this strand of the literature is to account for the observed behavior of the data. The seminal contributions in this literature are Aguiar and Gopinath (2006) and Arellano (2008) which study economies with short term debt. Long term debt was introduced

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[4]Atkeson (1991) extends the approach to the case with a public state variable. Phelan and Stacchetti (2001) and Chang (1998) extend the approach to study models where individual agents hold stocks (capital and money respectively).

[5]For a review see Eaton and Fernandez (1995), Aguiar and Amador (2013a).

by Hatchondo and Martinez (2009), Arellano and Ramanarayanan (2012), Chatterjee and Eyigungor (2012). The quantitative literature of sovereign debt has already been successful in explaining the most salient features in the data.[6] Our paper shares with this literature the focus on a model along the lines of Eaton and Gersovitz (1981) but rather than characterizing a particular equilibrium, it tries to study predictions regarding the set of equilibria.

Another direction of the literature focuses in equilibrium multiplicity, and in particular, in self fulfilling debt crises. The seminal contribution is Calvo (1988). Cole and Kehoe (2000) introduce self-fulfilling debt crises in a full-fledged dynamic model where the equilibrium selection mechanism is a sunspot that is realized simultaneously with output. Lorenzoni and Werning (2013) study equilibrium multiplicity in a dynamic version of Calvo (1988). Our paper studies multiplicity but in the Eaton and Gersovitz (1981) setting; the crucial difference between the setting in Calvo (1988) and the one Eaton and Gersovitz (1981) is that in the latter the government issues debt (with commitment) and then the price is realized. This implies that equilibrium multiplicity is coming from the multiplicity of beliefs regarding continuation equilibria. Stangebye (2014) also studies multiplicity in a setting as in Eaton and Gersovitz (1981), but focuses on a Markov equilibrium.

Another strand of the literature studies the risk sharing agreement between international debt holders and a sovereign with some primitive contracting frictions. Worrall (1990) studies an economy with limited commitment. Atkeson (1991) studies an economy with limited commitment and moral hazard and finds that capital outflows during bad times are a feature of the optimal contract. The model we use to discuss excusable defaults is closely related to these two models.[7]

**Outline.** The paper is structured as follows. Section 2 introduces the model. Section 3 studies equilibrium multiplicity in our model of sovereign borrowing. Section 4 characterizes equilibrium consistent outcomes. Section 5 characterizes equilibrium consistent outcomes in a model with savings and excusable defaults. Section 6 discusses the characterization of equilibrium consistent outcomes when there are correlating devices available after debt is issued and when income is discrete. Section 8 concludes.

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[6]Other examples in this literature are Yue (2010), Bai and Zhang (2012), Pouzo and Presno (2011), Borri and Verdelhan (2009), D Erasmo (2008), Bianchi et al. (2012).

[7] Hopenhayn and Werning (2008) study optimal financial contracts when there is private information regarding the outside option and limited commitment to repayment and find that there is default along the equilibrium path. Aguiar and Amador (2013b) exploit this approach to study the optimal repayment of sovereign debt when there are bonds of different maturities.

# 2 A Model of Sovereign Debt

Our model of sovereign debt follows Eaton and Gersovitz (1981). Time is discrete and denoted by $t \in \{0, 1, 2, ....\}$. A small open economy receives a stochastic stream of income denoted by $y_t$. Income follows a Markov process with c.d.f. denoted by $F(y_{t+1} \mid y_t)$. The government is benevolent and seeks to maximize the utility of the households. It does so by selling bonds in the international bond market. The household evaluates consumption streams according to

$$\mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

where $\beta < 1$ and $u$ is increasing and strictly concave. The sovereign issues short term debt at a price $q_t$. The budget constraint is

$$c_t = y_t - b_t + q_t b_{t+1}.$$

Following Chatterjee and Eyigungor (2012) we assume that the government cannot save

$$b_{t+1} \geq 0.$$

This helps focus our discussion on debt and may implicitly capture political economy constraints that make it difficult for governments to save, as modeled by Amador (2013).

There is limited enforcement of debt. Therefore, the government will repay only if it is more convenient to do so. We assume that the only fallout of default is that the government will remain in autarky forever after. We also do not introduce exogenous costs of default. As we will show below, our assumptions are sufficient for autarky to be an equilibrium. If the government cannot save, and there are no output costs of default, if the government expects a zero bond price for its debt now and in every future period, then it will default its debt. To guarantee multiplicity we need to introduce conditions to guarantee that best Markov equilibrium, the one usually studied in the literature of sovereign debt, has a positive price of debt. In Section 5 we characterize subgame perfect equilibrium and equilibrium consistent outcomes when the government can save and when defaults do not need to be punished.

**Lenders.** There is a competitive fringe of risk neutral investors that discount the future at rate $r > 0$. This implies that the price of the bond is given by

$$q_t = \frac{1 - \delta_t}{1 + r}$$

where $\delta_t$ if the default probability on bonds $b_{t+1}$ issued at date $t$.

**Timing.** The sequence of events within a period is as follows. In period $t$, the government enters with $b_t$ bonds that it needs to repay. Then income $y_t$ is realized. The government then has the option to default $d_t \in \{0,1\}$. If the government does not default, the government runs an auction of face value $b_{t+1}$. Then, the price of the bond $q_t$ is realized. Finally, consumption takes place, and is given by $c_t = y_t - b_t + q_t b_{t+1}$. If the government decides to default, consumption is equal to income, $c_t = y_t$. The same is true if the government has ever defaulted in the past. We adopt the convention that if $d_t = 1$ then $d_{t'} = 1$ for all $t' \geq t$.

## 2.1 Dynamic Game: Notation and Definitions

In this section we describe the basic notation for the dynamic game setup.

**Histories.** An income history is a vector $y^t = (y_0, y_1, \ldots, y_t)$ of all income realizations up to time $t$. A history is a vector $h^t = (h_0, h_1, \ldots, h_{t-1})$, where $h_t = (y_t, d_t, b_{t+1}, q_t)$ is the description of all realized values of income and actions, and $h = h'h''$ is the append operator. A partial history is an initial history $h^t$ concatenated with a part of $h_t$. For example, $h = \left(h^t, y_t, d_t, b_{t+1}\right)$ is a history where we have observed $h^t$, output $y_t$ has been realized, the government decisions $(d_t, b_{t+1})$ have been made, but market price $q_t$ has not yet been realized. We will denote these histories $h = h_-^{t+1}$. The set of all partial histories (initial and partial) is denoted by $\mathcal{H}$, and $\mathcal{H}_g \subset \mathcal{H}$ are those where the government has to make a decision; i.e., $h = \left(h^t, y_t\right)$. Likewise, $\mathcal{H}_m \subset \mathcal{H}$ is the set of partial histories where investors set prices; i.e., $h_-^{t+1} = \left(h^t, y_t, d_t, b_{t+1}\right)$.

**Outcomes.** An outcome path is a sequence of measurable functions[8]

$$x = \left(d_t\left(y^t\right), b_{t+1}\left(y^t\right), q_t\left(y^t\right)\right)_{t \in \mathbb{N}}$$

The set of all outcomes is denoted by $\mathcal{X}$. To make explicit that the default, bond policies and prices are the ones associated with the path $x$, sometimes we will write

$$\left(d_t^x\left(y^t\right), b_{t+1}^x\left(y^t\right), q_t^x\left(y^t\right)\right)_{t \in \mathbb{N}}.$$

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[8]For our baseline case, where after default the government is permanently in autarky, the functions have the restriction that bond issues and prices are not defined after a default has been observed: $b_{t+s+1}\left(y^t y^s\right) = q_{t+s}\left(y^t y^s\right) = \varnothing$ for all $y^s$ and $y^t$ such that $d_t\left(y^t\right) = 1$.

An outcome $x_t$ (the evaluation of a path at a particular period) is a description of the government's policy function and market pricing function at time $t$ where the functions in $x_t$ are $d_t : Y \to \{0,1\}$, $b_{t+1} : Y \to \mathbb{R}_+$, and $q_t : Y \to \mathbb{R}_+$. Our focus will be on a shifted outcome, $x_{t-} \equiv (q_{t-1}, d_t(\cdot), b_{t+1}(\cdot))$. The reason to do this is that the prices in $q_{t-1}$ will only be a function of the next period default decision.

**Strategies.** A strategy profile is a complete description of the behavior of both the government and the market, for any possible history. Formally, a strategy profile is defined as a pair of measurable functions $\sigma = (\sigma_g, q_m)$, where $\sigma_g : \mathcal{H}_g \to \{0,1\} \times \mathbb{R}_+$ and $q_m : \mathcal{H}_m \to \mathbb{R}_+$. The government decision will usually be written as

$$\sigma_g\left(h^t, y_t\right) = \left(d_t^{\sigma_g}\left(h^t, y_t\right), b_{t+1}^{\sigma_g}\left(h^t, y_t\right)\right)$$

so that $d_t^{\sigma_g}(\cdot)$ and $b_{t+1}^{\sigma_g}(\cdot)$ are the default decision and bond issuance decision for strategy $\sigma_g$. $\Sigma_g$ is the set of all strategies for the government, and $\Sigma_m$ is the set of market pricing strategies. $\Sigma = \Sigma_g \times \Sigma_m$ is the set of all strategy profiles. Given a history $h^t$, we define the continuation strategy induced by $h^t$ as

$$\sigma_{|h^t}\left(h^s\right) = \sigma\left(h^t h^s\right).$$

Every strategy profile $\sigma$ generates an outcome path $x := x(\sigma)$.[9] Given a set $S \subseteq \Sigma$ of strategy profiles, we denote $x(S) = \cup_{\sigma \in S} x(\sigma)$ for the set of outcome paths of profiles $\sigma \in S$.

**Payoffs.** For any strategy profile $\sigma \in \Sigma$, we define the continuation at $h^t \in \mathcal{H}_g$

$$V(\sigma \mid h^t) = \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} \beta^s \left[ d_s u(y_s - b_s + q_s b_{s+1}) + (1 - d_s) u(y_s) \right] \right\}$$

where $(y_s, d_s, b_{s+1}, q_s)$ are on the path $x = x(\sigma_{|h^t})$.[10]

**Definition 2.1.** A strategy profile $\sigma = (\sigma_g, q_m)$ constitutes a subgame perfect equilibrium (SPE) if and only if, for all partial histories $h^t \in \mathcal{H}_g$

$$V(\sigma \mid h^t) \geq V(\sigma_g', q_m \mid h^t) \text{ for all } \sigma_g' \in \Sigma_g, \tag{2.1}$$

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[9]It can be defined recursively as follows: at $t = 0$ jointly define $(d_0(y_0), b_1(y_0), q_1(y_0)) \equiv (d_0^{\sigma_g}(y_0), b_1^{\sigma_g}(y_0), q_m(y_0, b_1^{\sigma_g}(y_0)))$ and $h^1 = (y_0, d_0(y_0), b_1(y_0), q_1(y_0))$. For $t > 0$, we define $(d_t(y^t), b_{t+1}(y^t), q_t(y^t)) \equiv (d_t^{\sigma_g}(h^t, y_t), b_1^{\sigma_g}(h^t, y_t), q_m(h^t, y_t))$ and $h^{t+1} = (h^t, y_t, d_t(y^t), b_{t+1}(y^t), q_t(y^t))$

[10]The utility of a strategy profile that specifies negative consumption is $-\infty$.

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and for all histories $h_-^{t+1} = (h^t, y_t, d_t, b_{t+1}) \in \mathcal{H}_m$

$$q_m\left(h_-^{t+1}\right) = \frac{1}{1+r} \int (1 - d^{\sigma_g}(h^{t+1}, y_{t+1})dF(y_{t+1} \mid y_t). \tag{2.2}$$

That is, the strategy of the government is optimal given the pricing strategy of the lenders $q_m(\cdot)$, and likewise $q_m(\cdot)$ is consistent with the default policy generated by $\sigma_g$. The set of all subgame perfect equilibria is denoted as $\mathcal{E} \subset \Sigma$.

# 3   Multiple Equilibria in Sovereign Debt Markets

This section characterizes the best and worst equilibrium prices in the dynamic game laid out in the previous section and discusses the scope for multiplicity of equilibria. For any history $h_-^{t+1}$ we consider the highest and lowest prices

$$\bar{q}(h_-^{t+1}) := \max_{\sigma \in \mathcal{E}} q_m\left(h_-^{t+1}\right)$$

$$\underline{q}(h_-^{t+1}) := \min_{\sigma \in \mathcal{E}} q_m\left(h_-^{t+1}\right).$$

The best and worst equilibria turn out to be Markov equilibria and we find conditions for multiplicity. The worst SPE price is zero and the best SPE price is the one of the Markov equilibrium that is characterized in the literature of sovereign debt as in Arellano (2008) and Aguiar and Gopinath (2006). Thus, our analysis may be of independent interest, providing conditions under which there are multiple Markov equilibria in a sovereign debt model along the lines of Eaton and Gersovitz (1981).[11] The importance of this result is that it opens up the possibility of confidence crises in models as in Eaton and Gersovitz (1981). Thus, confidence crises are not necessarily a special feature of the timing in Calvo (1988) and Cole and Kehoe (2000) but a robust feature in most models of sovereign debt.

The lowest price $\underline{q}(h_-^{t+1})$ will be attained by a fixed strategy for all histories $h_-^{t+1}$. It will deliver the utility level of autarky for the government. Thus, the lowest price is associated with the worst equilibrium, in terms of welfare. Likewise, the highest price $\bar{q}(h_-^{t+1})$ is associated with a, different, fixed strategy for all histories (the maximum is attained by the same $\sigma$ for all $h_-^{t+1}$) and delivers the highest equilibrium level of utility for the government. Thus, the highest price is associated with the best equilibrium in terms of welfare.

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[11]Our result complements the results in Auclert and Rognlie (2014); their paper shows uniqueness in the Eaton and Gersovitz (1981) when the government can save.

## 3.1 Lowest Equilibrium Price and Worst Equilibrium

We start by showing that, after any history $h_{-}^{t+1}$, the lowest subgame perfect equilibrium price is equal to zero.

**Proposition 3.1.** *Denote by* **B** *the set of assets for the government. Under our assumption of* **B** $\geq 0$, *the lowest SPE price is equal to zero*

$$\underline{q}(h_{-}^{t+1}) = \underline{q}(y_t, b_{t+1}) = 0$$

*and associated with a Markov equilibrium that achieves the worst level of welfare.*

Whenever the government confronts a price of zero for its bonds in the present period and expects to face the same in all future periods, it is best to default. There is no benefit from repaying. The proof is simple. We need to show that defaulting after every history is a subgame perfect equilibrium. Because the game is continuous at infinity, we need to show that there are no profitable one shot deviations when the government plays that strategy. Note first that, if the government is playing a strategy of always defaulting, it is effectively in autarky. In a history $h_{-}^{t+1}$ with income $y_t$ and debt $b_t$, the payoff of such a strategy is

$$u(y_t) + \frac{\beta}{1 - \beta} \mathbb{E}_{y'|y_t} u(y').$$

Note also that, a one shot deviation involving repayment today has associated utility of

$$u(y_t - b_t) + \frac{\beta}{1 - \beta} \mathbb{E}_{y'|y_t} u(y').$$

Thus, as long as $b_{t+1}$ is non-negative, a one shot deviation of repayment is not profitable. So, autarky is an SPE with an associated price of debt equal to zero.

The equilibrium does not require conditioning on the past history, i.e. it is a Markov equilibrium. Notice, as well, that we have not yet introduced sunspots. Thus, multiplicity does not require sunspots. Sunspots may act as a coordinating device to select a particular continuation equilibrium. We introduce sunspots in Section 6.

Things are different when the government is allowed to save before default and the punishment is autarky, including exclusion from saving. Under this combination of assumptions, the government might want to repay small amounts of debt to maintain the option to save in the future. As a result, autarky is no longer an equilibrium and a unique Markov equilibrium prevails, as shown by Auclert and Rognlie (2014).

A similar result holds when there are output costs of default. The sufficient condition for multiplicity will be that for the government is dominant to default on any amount of

debt that it is allowed to hold, for all $b \in \mathbf{B}$. With default costs, the value of defaulting is lower. Thus, we need to increase the static gain of defaulting for any history. A sufficient condition would then be that $\mathbf{B} > 0$. The lower bound on debt will be increasing in the magnitude of the output costs of default.

## 3.2  Highest Equilibrium Price and Best Equilibrium

We now characterize the best subgame perfect equilibrium and show that it is the Markov equilibrium studied by the literature of sovereign debt. To find the worst equilibrium price, it was sufficient to use the definition of equilibrium and the one shot deviation principle. To find the best equilibrium price it will be necessary to find a characterization of equilibrium prices. Denote by $\overline{W}(y_t, b_{t+1})$ the highest expected equilibrium payoff if the government enter period $t + 1$ with bonds $b_{t+1}$ and income in $t$ was $y_t$. The next lemma provides a characterization of equilibrium outcomes.

**Lemma 3.1.** $x_{t-} = (q_{t-1}, d_t (\cdot), b_{t+1} (\cdot))$ *is a subgame perfect equilibrium outcome at history* $h^t_-$ *if and only in the following conditions hold:*

    *a. Price is consistent*

$$q_{t-1} = \frac{1}{1 + r^*}(1 - \int d_t(y_t)dF(y_t \mid y_{t-1})), \tag{3.1}$$

    *b. IC government*

$$(1 - d(y_t)) \left[ u(y_t - b_t + \overline{q}(y_t, b_{t+1})b_{t+1}) + \beta \overline{W}(y_t, b_{t+1}) \right] + d(y_t)V^d(y_t) \geq V^d(y_t). \tag{3.2}$$

The proof is omitted. Condition (3.1) states that the price $q_{t-1}$ needs to be consistent with the default policy $d_t(\cdot)$. Condition (3.2) states that a policy $d_t (\cdot), b_{t+1} (\cdot)$ is implementable in an SPE if it is incentive compatible given that following the policy is rewarded with the best equilibrium and a deviation is punished with the worst equilibrium. The argument in the proof follows Abreu (1988). These two conditions are necessary and sufficient for an outcome to be part of an SPE.[12]

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[12]Note that at any history (even on those *inconsistent* with equilibria) SPE policies are a function of only one state: the debt that the government has to pay at time $t$ ($b_t$). There are two reasons for this. First, the stock of debt summarizes the physical environment. Second, the value of the worst equilibrium only depends on the realized income.

**Markov Equilibrium.**   We now characterize the Markov equilibrium that is usually studied in the literature of sovereign debt. The value of a government that has the option to default is given by

$$\overline{W}(y_-, b) = \mathbb{E}_{y|y_-} \left[ \max \left\{ \overline{V}^{nd}(b, y), V^D(y) \right\} \right]. \tag{3.3}$$

This is the expected value of the maximum between not defaulting $\overline{V}^{nd}(b, y)$ and the value of defaulting $V^D(y)$. The value of not defaulting is given by

$$\overline{V}^{nd}(b, y) = \max_{b' \geq 0} u(y - b + q(y, b')b') + \beta \overline{W}(y, b'). \tag{3.4}$$

That is, the government repays debt, obtains a capital inflow (outflow), and from the budget constraint consumption is given by $y - b + q(y, b')b'$; next period has the option to default $b'$ bonds. The value of defaulting is

$$V^d(y) = u(y) + \beta \frac{\mathbb{E}_{y'|y} u(y')}{1 - \beta}, \tag{3.5}$$

and is just the value of consuming income forever. These value functions define a default set

$$D(b) = \left\{ y \in Y : \overline{V}^{nd}(b, y) < V^d(y) \right\}. \tag{3.6}$$

A Markov Equilibrium (with state $b, y$) is a: set of policy functions $(c(y, b), d(y, b), b'(y, b))$, a bond price function $q(b')$ and a default set $D(b)$ such that: $c(y, b)$ satisfies the resource constraint; taking as given $q(y, b')$ the government bond policy maximizes $\overline{V}^{nd}$; the bond price $q(y, b')$ is consistent with the default set

$$q(y, b') = \frac{1 - \int_{D(b')} dF(y' \mid y)}{1 + r}. \tag{3.7}$$

The next proposition states that the best Markov equilibrium is the best subgame perfect equilibrium.

**Proposition 3.2.** *The best subgame perfect equilibrium is the best Markov equilibrium.*

From lemma 3.1, the value of the best equilibrium is the expectation with respect to $y_t$, given $y_{t-1}$, and is given by

$$\max_{d_t, b_{t+1}} (1 - d_t) \left[ u(y_t - b_t + \bar{q}(y_t, b_{t+1}) b_{t+1}) + \beta \overline{W}(y_t, b_{t+1}) \right] + d_t V^d(y_t).$$

Note that this is equal to the left hand side of (3.3). The key assumption for the best

subgame perfect equilibrium to be the best Markov equilibrium is that the government is punished with permanent autarky after a default. We shall relax this assumption in Section 5, where we consider the possibility of excusable defaults. Excusable defaults allow for greater risk sharing, which improves welfare.

## 3.3 Multiplicity

Given that the worst equilibrium is autarky, a sufficient condition for multiplicity of Markov equilibria will be any condition that guarantees that the best Markov equilibria has positive debt capacity, a standard situation in quantitative sovereign debt models. In general some debt can be sustained as long as there is enough of a desire to smooth consumption. This will motivate the government to avoid default, at least for small debt levels. The following proposition provides a simple sufficient condition for this to be the case.

Define $\mathcal{V}^{nd}(b, y; B, \frac{1}{1+r})$ as the value function when the government faces the risk free interest rate $q = \frac{1}{1+r}$ and some borrowing limit $B$ as in a standard Bewley incomplete market model. The government has the option to default. This value is not an upper bound on the possible values of the borrower because default introduces state contingency and might be valuable. Our next proposition, however, establishes conditions under which default does not take place.

**Proposition 3.3.** *Suppose that for all $b \in [0, B]$ and all $y \in \mathbf{Y}$, the*

$$\mathcal{V}^{nd}(b, y; B, \frac{1}{1+r}) \geq u(y) + \beta \mathbb{E}_{y'|y} V^d(y'). \tag{3.8}$$

*Then there exist multiple Markov equilibria.*

If the government is confronted with $q = \frac{1}{1+r}$ for $b \leq B$ condition (3.8) ensures that it will not want to default after any history. This justifies the risk free rate for $b \leq B$. A SPE can implicitly enforce the borrowing limit $b \leq B$ by triggering to autarky and setting $q_t = 0$ if ever $b_{t+1} > B$. Since the debt issuance policy is optimal given the risk free rate, we have constructed an equilibrium. This proves there is at least one SPE sustaining strictly positive debt and prices. The best equilibrium dominates this one and is Markov, as shown earlier, so it follows that there exists at least one strictly positive Markov equilibrium. Finally, note that we only require checking this condition (3.8) for small values of $B$. However, the existence result then extends an SPE over the entire $\mathbf{B} = [0, \infty)$. Indeed, it is useful to consider small $B$ and take the limit, this then requires checking only a local condition. The following example illustrates this condition.

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**Example.** Suppose there are two income shocks $y_L$ and $y_H$ that follow a Markov chain (a special case is the i.i.d. case). Denote by $\lambda_i$ the probability of transitioning from state $i$ to state $j \neq i$. We will construct an equilibrium where debt is risk free, the government goes into debt $B$ and stays there as long as income is low, and repays debt and remains debt free when income is high. Conditional on not defaulting, this bang bang solution is optimal for small enough $B$. To investigate whether default is avoided, we must compute the values

$$v_{BL} = u(y_L + (R-1)B) + \beta\left(\lambda_L v_{BH} + (1 - \lambda_L)v_{BL}\right)$$
$$v_{BH} = u(y_H - RB) + \beta\left(\lambda_H v_{0L} + (1 - \lambda_H)v_{0H}\right)$$
$$v_{0L} = u(y_L + B) + \beta\left(\lambda_L v_{BH} + (1 - \lambda_L)v_{BL}\right)$$
$$v_{0H} = u(y_H) + \beta\left(\lambda_H v_{0L} + (1 - \lambda_H)v_{0H}\right)$$

where $R = 1 + r$. Write the solution to this system as a function of $B$. To guarantee that the government does not default in any state, we need to check that $v_{BL}(B) \geq v^{aut}$, $v_{BH}(B) \geq v^{aut}$, $v_{0L}(B) \geq v_L^{aut}$ and $v_{0H}(B) \geq v_H^{aut}$ (some of these conditions can be shown to be redundant).

**Lemma 3.2.** *A sufficient condition for $v_{BL} \geq v^{aut}, v_{BH} \geq v^{aut}, v_{0L} \geq v_L^{aut}, v_{0H} \geq v_H^{aut}$ to hold for some $B > 0$ is $v'_{BL}(0) > 0, v'_{BH}(0) > 0$. When $\lambda_H = \lambda_L = 1$ this simplifies to $\beta u'(y_L) > Ru'(y_H)$.*

Note that the simple condition with $\lambda_H = \lambda_L = 1$ is met whenever $u$ is sufficiently concave or if $\beta$ is sufficiently close to 1. These conditions ensure that the value from consumption smoothing is high enough to sustain debt.

## 3.4   Equilibrium Consistency: Focus on Outcomes

The following example provides further intuition how a the best Markov equilibrium operates, and at the same time helps us to make the point why we focus on predictions about the set of equilibria. A standard property in sovereign debt models is that higher debt implies a higher default probability. That is, if $b_1 \leq b_2$, then $\delta(b_1) \leq \delta(b_2)$. Actually, this is a property of Markov equilibrium. If the government wants to default with lower debt $b_1$, then it will also want to default with higher debt,

$$\begin{aligned} V^d(y) &\geq u(y - b_1 + q(y, b')b') + \beta\overline{W}(y, b') \\ &\geq u(y - b_2 + q(y, b')b') + \beta\overline{W}(y, b'). \end{aligned}$$

A key step is that the continuation value $\overline{W}(y, b')$ does not depend on $b$. However, if we consider all subgame perfect equilibrium, this property will not necessarily hold. Different levels of debt, $b_1$ versus $b_2$, may be associated with different continuation equilibria and therefore, different default probabilities.

# 4 Equilibrium Consistent Outcomes

This section contains the main result of the paper, a characterization of equilibrium consistent outcomes. We work with the baseline case where income is a continuous random variable. After this characterization we turn to our attention to the implications for bond prices.

## 4.1 Equilibrium Consistency: Definitions

We first define the notion of consistent histories.

**Definition 4.1.** (*Consistency*) A history $h$ is consistent with (or generated by) an outcome path $x$ if and only if $d_s = d_s^x(y^s)$, $b_{s+1} = b_{s+1}^x(y^s)$ and $q_s = q_s^x(y^s)$ for all $s < l(h)$ (where $l(h)$ is the length of the history).

If a history $h$ is consistent with an outcome path $x$ we denote it as $h \in \mathcal{H}(x)$. Intuitively, consistency of a history with an outcome means that, given the path of exogenous variables, the endogenous observed variables coincide with the ones that are generated by the outcome.

**Definition 4.2.** A history $h$ is consistent with strategy profile $\sigma \iff h \in \mathcal{H}(x(\sigma))$.[13]

If a history $h$ is consistent with a strategy $\sigma$ we denote it as $h \in \mathcal{H}(\sigma)$. Intuitively, a history is consistent with a strategy if the history is consistent with the outcome that is generated by the strategy. Given a set $S \subseteq \Sigma$ of strategy profiles, we use $x(S) = \cup_{\sigma \in S} x(\sigma)$ to denote the set of outcome paths of profiles $\sigma \in S$. The inverse operator for $\mathcal{H}(\cdot)$ are respectively $X(\cdot)$ for the outcomes consistent with history $h$. We use $\Sigma(h)$ to denote the strategy profiles consistent with $h$. For a given set of strategy profiles $S \subseteq \Sigma$, we write $\mathcal{H}(S) = \cup_{\sigma \in S} \mathcal{H}(\sigma)$ as the set of $S-$consistent histories. When $S = \mathcal{E}$, we call $\mathcal{H}(\mathcal{E})$ the

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[13]Remember that each strategy $\sigma$ generates an outcome path $x := x(\sigma)$. It can be defined recursively as follows: at $t = 0$ jointly define $(d_0(y_0), b_1(y_0), q_1(y_0)) \equiv \left(d_0^{\sigma_g}(y_0), b_1^{\sigma_g}(y_0), q_m\left(y_0, b_1^{\sigma_g}(y_0)\right)\right)$ and $h^1 = (y_0, d_0(y_0), b_1(y_0), q_1(y_0))$. For $t > 0$, we define $(d_t(y^t), b_{t+1}(y^t), q_t(y^t)) \equiv \left(d_0^{\sigma_g}(h^t, y_t), b_1^{\sigma_g}(h^t, y_t), q_m(h^t, y_t)\right)$ and $h^{t+1} = (h^t, y_t, d_t(y^t), b_{t+1}(y^t), q_t(y^t))$.

set of equilibrium consistent histories. The set of equilibria consistent with history $h$ is defined as $\mathcal{E}_{|h} := \mathcal{E} \cap \Sigma(h)$.[14]

**Definition 4.3. (S$-$ consistent outcomes)** An outcome path $x = (d_t(\cdot), b_{t+1}(\cdot), q_t(\cdot))_{t \in \mathbb{N}}$ is $S-$ consistent with history $h^t \iff \exists \sigma \in S \cap \Sigma(h^t)$ such that $x = x(\sigma)$. If $S = \mathcal{E}$ we say $x$ is equilibrium consistent with history $h^t$, and we denote it as "$x \in x\left(\mathcal{E}_{|h}\right)$".

The expected autarky continuation is

$$\mathbb{V}^d(y) \equiv \int u(y') dF(y' \mid y),$$

and the autarky utility (conditional on defaulting) is simply

$$V^d(y) \equiv u(y) + \beta \mathbb{V}^d(y). \tag{4.1}$$

The continuation utility (conditional on not defaulting) of a choice $b'$ given bonds $(b, y)$ is

$$\overline{V}^{nd}(b, y, b') = u\left(y - b + b'\overline{\mathbf{q}}(y, b') b'\right) + \beta \overline{W}(y, b'), \tag{4.2}$$

where $\overline{q}(b')$ is the bond price schedule under the best continuation equilibrium (the Markov equilibrium that we just characterized), if $y_t = y$ and the bonds to be paid tomorrow are $b_{t+1} = b'$. Recall that

$$\overline{V}^{nd}(b, y) = \max_{\tilde{b} \geq 0} \overline{V}^{nd}(b, y, \tilde{b}). \tag{4.3}$$

## 4.2 Equilibrium Consistency: Characterization

Suppose that we have observed so far $h^t_- = \left(h^{t-1}, y_{t-1}, d_{t-1}, b_t\right)$ an equilibrium consistent history (where price at time $t$ has not yet been realized), and we want to characterize the set of *shifted* outcomes $x_{t-} = (q_{t-1}, d_t(\cdot), b_{t+1}(\cdot))$ consistent with this history[15]. Theorem 4.1 provides a full characterization of the set of equilibrium consistent outcomes $x_{t-}\left(\mathcal{E}_{|h^t_-}\right)$, showing that past history only matters through the opportunity cost of not defaulting at $t-1$, $u(y_{t-1}) - u(c_{t-1})$.

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[14]This notation is useful to precisely formulate questions such as: "Is the observed history the outcome of some subgame perfect equilibria?" In our notation "$h \in \mathcal{H}(\mathbf{SPE})$".

[15]An outcome in period $t$ was given by $x_t = (d_t^x(\cdot), b_{t+1}^x(\cdot), q_t^x(\cdot))$; the policies and prices of period $t$. $x_{t-}$ has the policies of period $t$ but the prices of period $t-1$. The focus in $x_{t-}$ as opposed to $x_t$ simplifies the characterization of equilibrium consistent outcomes.

**Proposition 4.1** (Equilibrium Consistent Outcomes). *Suppose* $h^t_- = \left(h^{t-1}, y_{t-1}, d_{t-1}, b_t\right)$ *is an equilibrium consistent history, with no default so far. Then* $x_{t-} = \left(q_{t-1}, d_t\left(\cdot\right), b_{t+1}\left(\cdot\right)\right)$ *is equilibrium consistent with* $h^t_-$ *if and only in the following conditions hold:*

*a. Price is consistent*

$$q_{t-1} = \frac{1}{1+r}(1 - \int d_t(y_t)dF(y_t \mid y_{t-1})), \tag{4.4}$$

*b. IC government*

$$(1 - d(y_t))\left[u(y_t - b_t + \bar{q}(y_t, b_{t+1})b_{t+1}) + \beta\overline{W}(y_t, b_{t+1})\right] + d(y_t)V^d(y_t) \geq V^d(y_t), \tag{4.5}$$

*c. Promise keeping*

$$\beta\left[\int_{d_t=0}\overline{V}^{nd}\left(b_t, y_t, b_{t+1}(y_t)\right)dF(y_t \mid y_{t-1}) + \int_{d_t=1}V^d\left(y_t\right)dF(y_t \mid y_{t-1})\right] \geq$$
$$\left[u\left(y_{t-1}\right) - u\left(y_{t-1} - b_{t-1} + q_{t-1}b_t\right)\right] + \beta\mathbb{V}^d(y_{t-1}). \tag{4.6}$$

**Proof.** See Appendix. □

If conditions (a) through (c) hold, we write simply

$$\left(q_{t-1}, d_t\left(\cdot\right), b_{t+1}\left(\cdot\right)\right) \in \mathbb{ECO}\left(b_{t-1}, y_{t-1}, b_t\right),$$

where $\mathbb{ECO}$ stands for "equilibrium consistent outcomes".

As we mentioned in Section 3, conditions (4.4) and (4.5) in Proposition 4.1 provide a characterization of the set of SPE outcomes. Condition (4.4) states that the price $q_{t-1}$ needs to be consistent with the default policy $d_t(\cdot)$. Condition (4.5) states that a policy $d_t\left(\cdot\right), b_{t+1}\left(\cdot\right)$ is implementable in an SPE if it is incentive compatible given that following the policy is rewarded with the best equilibrium and a deviation is punished with the worst equilibrium. The argument in the proof follows Abreu (1988)[16].

Equilibrium consistent outcomes are characterized by an additional condition, (4.6), which is the main contribution of this paper. This condition characterizes how past observed history (if *assumed* to be generated by an *equilibrium* strategy profile) introduces restrictions on the set of equilibrium consistent policies. In our setting, condition (4.6) will guarantee that the government's decision at $t-1$ of not defaulting was optimal.

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[16]This is the argument in Chari and Kehoe (1990) and Stokey (1991)

That is, on the path of some SPE profile $\hat{\sigma}$, the incentive compatibility constraint from government's utility maximization in $t-1$ is

$$u\left(c_{t-1}\right) + \beta V\left(\hat{\sigma} \mid h^t\right) \geq u\left(y_{t-1}\right) + \beta \mathbb{V}^d(y_{t-1}), \tag{4.7}$$

where $V\left(\hat{\sigma} \mid h^t\right)$ is the continuation value of the equilibrium, as defined before. One interpretation of (4.7) is that the net present value (with respect to autarky) that the government must expect from not defaulting, must be greater (for the choice to have been done optimally) than the opportunity cost of not defaulting: $u\left(y_{t-1}\right) - u\left(c_{t-1}\right)$. This must be true for *any* SPE profile that could have generated $h^t_-$.

The intuition for why (4.6) is necessary for equilibrium consistency is as follows. Notice that the previous inequality also holds for the case the continuation equilibrium is actually the best continuation equilibrium. Therefore, for any equilibrium consistent policy $(d\left(\cdot\right), b'\left(\cdot\right))$ it has to be the case that

$$V\left(\hat{\sigma} \mid h^t\right) = \int_{y_t:d_t(y_t)=1} V^d(y_t)dF(y_t \mid y_{t-1})+$$

$$\int_{y_t:d(y_t)=0} \left[u\left(y_t - b_t + b'\left(y\right)\hat{q}_m\left(h^t, y_t, d_t, b_{t+1}\left(y_t\right)\right)\right) + \beta V\left(\hat{\sigma} \mid h^{t+1}\right)\right] dF(y_t \mid y_{t-1})$$

$$\leq \int_{y_t:d_t(y_t)=1} V^d\left(y_t\right) dF(y_t \mid y_{t-1}) + \int_{y_t:d(y_t)=0} \overline{V}^{nd}\left(b_t, y_t, b_{t+1}\right) dF(y_t \mid y_{t-1}). \tag{4.8}$$

Equations (4.7) and (4.8) imply

$$\left[u\left(y_{t-1}\right) - u\left(y_{t-1} - b_{t-1} + q_{t-1}b_t\right)\right] + \beta \underline{\mathbb{V}}^d(y_{t-1})$$

$$\leq \beta \left[\int_{d_t=0} \overline{V}^{nd}\left(b_t, y_t, b_{t+1}(y_t)\right) dF(y_t \mid y_{t-1}) + \int_{d_t=1} V^d(y_t)dF(y_t \mid y_{t-1})\right]. \tag{4.9}$$

This is condition (4.6). So if the policies do not satisfy (4.6), they are not part of an SPE that generated the history $h^t_-$; in other words, there is no SPE consistent with $h^t_-$ with policies $(d_t\left(\cdot\right), b_{t+1}\left(\cdot\right))$ for period $t$.

We also show that this condition is sufficient, so if $(d_t\left(\cdot\right), b_{t+1}\left(\cdot\right))$ satisfy conditions (4.4), (4.5), and (4.6), we can always find at least one SPE profile $\hat{\sigma}$ that would generate $x_{t-}$ on its equilibrium path. Even after a long history the sufficient statistics to forecast the outcome $x_{t-}$ are

$$\left(b_{t-1}, b_t, y_{t-1}\right).$$

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Thus, effectively

$$\mathbb{ECO}(h_-^t) = \mathbb{ECO}(b_{t-1}, y_{t-1}, b_t).$$

This result may seem surprising, but it is where robustness of the analyst (uncertainty about the equilibrium selection) is expressed. Because income $y$ is a continuous random variable, any promises (in terms of expected utility) that rationalized past choices are "forgotten" each period; the reason is that the outside observer needs to take into account that promises *could* have been be realized in states that did not occur.

Finally, notice that even though the outside observer is using just a small fraction of the history, the set of equilibrium consistent outcomes exhibits history dependence beyond that of the set of SPE. The set of equilibrium consistent outcomes is a function variables $(b_{t-1}, y_{t-1}, b_t)$. Thus, there is a role for past actions to signal future behavior. In contrast the set of subgame perfect equilibria after any history only depends on the Markovian states $y_{t-1}, b_t$.[17]

## 4.3 Equilibrium Consistent Prices

Aided with the characterization of equilibrium consistent outcomes in Proposition 4.1, we will characterize the set of equilibrium debt prices that are consistent with the observed history $h_-^t = \left( h^{t-1}, y_{t-1}, d_{t-1}, b_t \right)$. The highest equilibrium consistent price solves

$$\overline{q} \left( h_-^t \right) = \max_{(\hat{q}, d_t(\cdot), b_{t+1}(\cdot))} \hat{q}$$

subject to

$$(\hat{q}, d_t(\cdot), b_{t+1}(\cdot)) \in \mathbb{ECO}(b_{t-1}, y_{t-1}, b_t).$$

The lowest equilibrium consistent price solves

$$\underline{q} \left( h_-^t \right) = \min_{(\hat{q}, d_t(\cdot), b_{t+1}(\cdot))} \hat{q} \tag{4.10}$$

subject to

$$(\hat{q}, d_t(\cdot), b_{t+1}(\cdot)) \in \mathbb{ECO}(b_{t-1}, y_{t-1}, b_t).$$

**Highest Equilibrium Consistent Price.** The highest equilibrium consistent price is the one of the Markov Equilibrium that we characterized in Section 2. Note that the expected

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[17]Notice that this role contrasts the dependence of the quantitative literature for sovereign debt that follows Eaton and Gersovitz (1981) as in Arellano (2008) and Aguiar and Gopinath (2006) where the fact that a country has just repaid a high quantity of debt, does not affect the future prices that will obtain.

value of the incentive compatibility constraint (4.5), is the value of the option to default $W(y, b')$, in the Markov Equilibrium. The promise-keeping will be generically not binding for the best equilibrium (given that the country did not default). For these two reasons, the best equilibrium consistent price is the one obtained with the default policy and bond policy that maximize the value of the option. Thus,

$$\bar{q}\left(h^t_-\right) = \overline{\mathbf{q}}\left(y_{t-1}, b_t\right). \tag{4.11}$$

**Lowest Equilibrium Consistent Price.** Our focus will be on the characterization of the lowest equilibrium consistent price. Note that the lowest SPE price is zero. This follows because default is implementable after any history if we do not take into account the promise keeping constraint (4.6). On the contrary, we will show that lowest equilibrium consistent price is positive, for every equilibrium history. Furthermore, because the set of equilibrium consistent outcomes after history $h^t_-$ depends only on $(b_{t-1}, y_{t-1}, b_t)$, it holds that

$$\underline{q}\left(h^t_-\right) = \underline{\mathbf{q}}\left(b_{t-1}, y_{t-1}, b_t\right). \tag{4.12}$$

From (4.11) and (4.12), the set of equilibrium consistent prices will be

$$q_t \in \left[\underline{\mathbf{q}}\left(b_{t-1}, y_{t-1}, b_t\right), \overline{\mathbf{q}}\left(y_{t-1}, b_t\right)\right]. \tag{4.13}$$

Proposition 4.2 establishes the main result of this subsection: a full characterization of $\underline{\mathbf{q}}\left(b, y, b'\right)$ (we drop time subscripts) as a solution to a convex minimization program, which can be reduced to a one equation/one variable problem.

**Proposition 4.2.** *Suppose $(b, y, b')$ are such that $\overline{V}^{nd}\left(b, y, b'\right) > V^d\left(y\right)$ (i.e., not defaulting was feasible under the best continuation equilibrium). Then there exists a constant $\gamma = \gamma\left(b, y, b'\right) \geq 0$, such that*

$$\underline{\mathbf{q}}\left(b, y, b'\right) = \frac{1 - \int \underline{d}\left(y'\right) dF\left(y' \mid y\right)}{1 + r},$$

*where*

$$\underline{d}\left(y'\right) = 0 \iff \overline{V}^{nd}\left(b', y'\right) \geq V^d\left(y'\right) + \gamma \text{ for all } y' \in Y;$$

*$\gamma$ is the minimum solution to the equation:*

$$\beta \int_{\Delta^{nd} \geq \gamma} \Delta^{nd} d\hat{F}\left(\Delta^{nd}\right) = u\left(y\right) - u\left(y - b + \frac{1 - \hat{F}\left(\gamma \mid y\right)}{1 + r} b'\right) \tag{4.14}$$

*where $\Delta^{nd} \equiv \overline{V}^{nd}\left(b', y'\right) - V^d\left(y'\right)$ and $\hat{F}\left(\Delta^{nd}\right)$ its conditional cdf. If $dF\left(\cdot\right)$ is absolutely con-*

*tinuous, then $\gamma$ is the unique solution to equation 4.14.*

The proof is in the appendix. We provide a sketch of the argument. First, note that, by choosing the bond policy of the best equilibrium, all of the constraints imposed by equilibrium consistency are relaxed because the value of no default increases. So, finding the lowest ECO price will amount to finding the default policy that yields the lowest price and is consistent with equilibrium. Second, notice that the promise keeping constraint needs to be binding in the optimum. If not, the minimization problem has as its only constraint the incentive compatibility constraint, and the minimum price is zero (with a policy of default in every state). But, if the price is zero, the promise keeping constraint will not be satisfied. Third, notice that the incentive compatibility constraint will not be binding. Intuitively, imposing default is not costly in terms of incentives, and for the lowest equilibrium consistent price, we want to impose default in as many states as possible.

From these observations, note that the tradeoff of the default policy of the lowest price will be: imposing defaults in more states will lower the price at the expense of a tighter promise keeping constraint. This condition pins down the states where the government defaults; as many defaults as possible, but not so many that no default in the previous period was not worth the effort. This, implies that the policy is pinned down by

$$\underline{d}\left(y'\right) = 0 \iff \overline{V}^{nd}\left(b',y'\right) \geq V^d\left(y'\right) + \gamma$$

where $\gamma$ is a constant to be determined. This constant solves a single equation: is the minimum value such that the promise keeping holds with equality, with the optimal bond policy, evaluated at the best continuation

$$\beta \int_{\Delta^{nd} \geq \gamma} \Delta^{nd} d\hat{F}(\Delta^{nd} \mid y) = u(y) - u(y - b + \frac{1 - \hat{F}(\gamma \mid y)}{1 + r}b'). \tag{4.15}$$

*Remark* 4.1. Note that the best equilibrium default policy at $t$

$$d(y_t) = 0 \iff \overline{V}^{nd}(b_t, y_t) \geq V^d(y_t).$$

On the contrary, the default policy of the lowest equilibrium consistent price is

$$d(y_t) = 0 \iff \overline{V}^{nd}(b_t, y_t) \geq V^d(y_t) + \gamma$$

where $\gamma$ is the constant that solves (4.15) and depends on $(b_{t-1}, y_{t-1}, b_t)$. The default policy is shifted to create more defaults, to lower the price, but not so many that the

promise-keeping was not satisfied (i.e., we cannot rationalize previous choices).

*Remark* 4.2. Notice that by focusing on equilibrium consistent outcomes uncovers a novel tension that is not present in SPE. At a particular history $h^t_-$, implementing default is not costly because it is always as good as the worst equilibrium. However, implementing default today lowers the prices that the government was expecting in the past and makes it harder to rationalize a particular history.

The next Proposition describes how the set of equilibirum consistent prices changes with the history of play.

**Proposition 4.3.** *Let* $\underline{q}(b, y, b')$ *be the lowest* $\mathbb{ECO}(b, y, b')$ *price. It holds that*

a. $\underline{q}(b, y, b')$ *is decreasing in* $b'$,

b. $\underline{q}(b, y, b')$ *is increasing in* $b$,

c. *For every equilibrium* $(b, y, b')$, $-b + b'\underline{q}(b, y, b') \leq 0$,

d. *If income is i.i.d.,* $\underline{q}$ *is decreasing in* $y$, *and so is the set* $Q = \left[\underline{q}(b, y, b'), \bar{q}(y, b')\right]$.

First, note that as in the best equilibrium, the lowest equilibrium consistent price is decreasing in the amount of debt issued $b'$. The intuition is that higher amounts of debt issued imply a more relaxed promise keeping constraint. In other words, the past choices of the government can be rationalized with a lower price. A similar intuition holds for $b$; if the country just repaid a high amount of debt (i.e., made an effort for repaying), past choices are rationalized by higher prices.

Second, note that if there is a positive capital inflow with the lowest equilibrium consistent price, it implies that

$$u(y) - u\left(y - b + b'\underline{q}(b, y, b')\right) < 0.$$

Intuitively, the country is not making any effort in repaying the debt. Therefore, it need not be the case that the country was expecting high prices for debt in the next period. Mathematically, when there is a positive capital outflow with the lowest equilibrium consistent price, $\gamma$ is infinite. This implies that $\frac{1-\hat{F}(\gamma)}{1+r^*} = \underline{q}(b, y, b') = 0$, which contradicts a positive capital inflow.

Finally, because there are no capital inflows with the lowest equilibrium consistent price, repaying debt at this price will become more costly as income is lower; this due to

the concavity of the utility function.[18] Mathematically, because of concavity,

$$u\left(y\right) - u\left(y - b + b'\underline{q}\left(b, y, b'\right)\right),$$

is[19] increasing as income decreases, and therefore, the promise keeping constraint tightens as income decreases. Note that, in the non i.i.d. case, this property will not hold, because, even though the burden of repayment is higher, the value of repayment in terms of the continuation value can be increasing.

## 4.4 Interpretation: Robust Bayesian Analysis

In this subsection we provide an interpretation of the set of equilibrium consistent outcomes relating to robust Bayesian analysis. Appendix E provides a more formal connection.

In Bayesian analysis, the econometrician has a prior over the set of fundamental parameters $\Theta$;[20] here will be denoted by $\pi(\theta)$. In addition, because of equilibrium multiplicity, she also has a prior $p(x)$ over the set of outcomes $\mathcal{X}$.[21] Using data (in our case, in the form of a history) and these priors, she obtains a posterior. Suppose that she is interested in the (posterior) mean of a particular statistic $T(x, \theta)$. Conditional on the data, her prediction is

$$\mathbb{E}^{p,\pi}\left[T(x, \theta) \mid \text{data}\right].$$

There are many situations where the econometrician will not want to favor one equilibrium against another one; that is, there is uncertainty with respect to the prior $p \in \mathcal{P}$. Then, there is a whole range of posterior means of the statistic that is given (for a fixed $\theta$, or a degenerate prior over $\Theta$) by

$$\left[\min_p \mathbb{E}^p\left[T\left(\theta, x\right) \mid \text{data}\right], \max_p \mathbb{E}^p\left[T\left(\theta, x\right) \mid \text{data}\right]\right].$$

Our focus in on priors over *equilibrium outcomes*, but we are be agnostic about the partic-

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[18]This observation is used in the literature of sovereign debt. For example, to show that default occurs in bad times, as in Arellano (2008), or to show monotonicity of bond policies with respect to debt, as in Chatterjee and Eyigungor (2012).

[19]The change in this expression will depend on the sign of $u\left(y\right) - u\left(y - b + b'\frac{1-\hat{F}(\gamma)}{1+r^*}\right)$, that is positive due to the result of no capital inflows with the lowest equilibrium consistent price.

[20]In our model, discount factor, parameters of the utility function, volatility of output, etc

[21]For example, Eaton and Gersovitz (1981) select a Markov equilibrium. Chatterjee and Eyigungor (2012) choose an equilibrium with arbitrary probability of crises in their study of optimal maturity of debt. These would be examples of degenerate priors; in other words, there is a particular equilibrium selection.

ular prior (i.e. equilibrium selection). We characterized the set of equilibrium consistent (with history $h$) debt prices

$$\left[ \min_{x \in x\left(\mathcal{E}_{|h}\right)} q_t^x, \max_{x \in x\left(\mathcal{E}_{|h}\right)} q_t^x \right].$$

This interval characterizes the support of the posterior over prices, when the only assumption is that the observed history is part of an SPE.

# 5 Extensions: Excusable Defaults and Savings

In this section we discuss how we characterize equilibrium consistent outcomes in a common different setting for the literature of sovereign debt: we do not restrict that a default needs to trigger a punishment and, we allow for savings. This will break the connection between the best SPE and the Markov equilibrium that we characterized in Section 2. However, autarky will still remain as the worst equilibrium. Given the best SPE values and prices, characterizing equilibrium consistent outcomes will follow the case in Section 4.

## 5.1 Excusable Defaults

The setting where we do not impose that defaults *need* to be punished with financial exclusion is similar to the one in Atkeson (1991) and Worrall (1990).[22] For the moment, assume that the government cannot save. Denote by $\overline{W}^E(y_-, b)$ the expected value of the best equilibrium if the government starts with bonds $b$. The following proposition characterizes equilibrium consistent outcomes in this case.

**Proposition 5.1 (ECO, excusable defaults).** *Suppose $h_-^t = (h^{t-1}, y_{t-1}, d_{t-1}, b_t)$ is an equilibrium consistent history. Then $x_{t-} = (q_{t-1}, d_t (\cdot), b_{t+1} (\cdot))$ is an equilibrium consistent outcome at $h_-^t$ if and only if the following conditions hold:*

*a. Price is consistent*

$$q_{t-1} = \frac{1}{1+r}(1 - \int d_t(y_t)dF(y_t \mid y_{t-1})), \tag{5.1}$$

*b. IC government*

$$u(y_t - b_t(1 - d(y_t))) + \overline{q}^E(y_t, b_{t+1}(y_t))b_{t+1}(y_t)) + \beta\overline{W}^E(y_t, b_{t+1}(y_t)) \geq V^d(y_t), \tag{5.2}$$

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[22]In our case, we restrict the contract to be one where the face value can be chosen, but can either be defaulted or repaid in full.

*c. Equilibrium consistency*

$$\beta \int \left[ u(y_t - b_t(1 - d(y_t)) + \bar{q}^E(y_t, b_{t+1}(y_t))b_{t+1}(y_t)) + \beta \overline{W}^E(y_t, b_{t+1}(y_t)) \right] dF(y_t) \geq$$

$$u(y_{t-1}) - u\left(y_{t-1} - b_{t-1}(1 - d(y_{t-1})) + q_{t-1}b_{t+1}(y_t)\right) + \beta \underline{W}. \tag{5.3}$$

If conditions (a) through (c) hold, we write simply

$$(q_{t-1}, d_t(\cdot), b_{t+1}(\cdot)) \in \mathbb{ECO}_E(b_{t-1}, y_{t-1}, b_t)$$

where $\mathbb{ECO}$ stands for "equilibrium consistent outcomes" and the subscript $E$ stands for the case of excusable defaults.

As in Section 4, conditions (5.1) and (5.2) characterize the set of SPE policies. The first condition (5.1) is again that the price has to be consistent with the default policy. The second condition (5.2) is the incentive compatibility for the government. The difference between (5.2) and the incentive compatibility of Proposition 4.1 that was given by

$$(1 - d(y_t)) \left[ u(y_t - b_t + \overline{q}(y_t, b_{t+1})b_{t+1}) + \beta \overline{W}(y_t, b_{t+1}) \right] + d(y_t) V^d(y_t) \geq V^d(y_t)$$

comes from the fact that defaults are not required to be punished. On the equilibrium path, defaults are excusable in the sense of Grossman and Huyck (1989);[23] off the equilibirum path they are punished with autarky, the worst equilibrium.

The intuition of condition (5.2) is similar to the incentive compatibility in Proposition 4.1 in Section 4. If in a history $h^t_- = \left(h^{t-1}, y_{t-1}, d_{t-1}, b_t\right)$ a default decision and bond issue decision wants to be implemented, it must be the case that it is weakly better than any deviation. Following Abreu (1988), any SPE can be implemented with strategies that impose the worst punishment in case of deviation; and, as in Proposition 4.1, we reward following the policy with the best equilibrium. This implies that $d_t, b_{t+1}$ is implementable if

$$u(y_t - b_t(1 - d_t) + \overline{q}(y_t, b_{t+1})b_{t+1}) + \beta \overline{W}(h^t_-, d_t, b_{t+1}) \geq$$

$$\max_{\tilde{d}, \tilde{b}'} u(y_t - b_t(1 - \tilde{d}) + \underline{q}(y_t, \tilde{b}')\tilde{b}') + \beta \underline{W}(h^t_-, \tilde{d}, \tilde{b}') \tag{5.4}$$

where $\overline{W}, \underline{W}$ denote the best and worst continuation equilibria. The value of the best

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[23]The reason why defaults are part of the equilibrium path is that they introduce stay contingency for the country and are also expected by the borrowers, so they will make zero profits on average.

equilibrium is $\overline{\mathbb{W}}^E(y_t, b_{t+1})$. Because the worst equilibrium is autarky with a price of debt equal to zero ($\underline{q}(y_t, \tilde{b}') = 0$), the right hand side of (5.4) is equal to $V^d(y_t)$. Condition (5.2) follows. Again, conditions (5.1) and (5.2) are necessary and sufficient to characterize SPE outcomes.

Equilibrium consistent outcomes are characterized by an additional condition (5.3). The right hand side of (5.3) is the opportunity cost from not taking the best deviation last period. The left hand side specifies the expected value of the policy under the best equilibrium. The reason why conditions (5.1)-(5.3) are necessary and sufficient is the same as before.

The lowest equilibrium consistent price will solve

$$\underline{q}^E\left(h_-^{t+1}\right) = \min_{(\hat{q}, d_t(\cdot), b_{t+1}(\cdot))} \hat{q}$$

where

$$(\hat{q}, d_t(\cdot), b_{t+1}(\cdot)) \in \mathbb{ECO}_E(b_{t-1}, y_{t-1}, b_t)$$

The intuition of the solution to this program is similar the intuition that we had before. The bond policy will be the one of the best equilibrium, and the default policy will be tilted towards more defaults, but not so many that the previous choices cannot be rationalized. Again, the highest equilibrium consistent price will be $\overline{q}^E$, the best subgame perfect equilibrium price.

Comparative statistics similar to the ones of Corollary 4.3 also hold. If the government issues more debt $b_{t+1}$, the lowest equilibrium consistent price decreases. If the government has not defaulted debt, $d(y_{t-1}) = 0$, an increase in $b_t$ increases $\underline{q}^E$. Also, after any history, the government cannot receive positive capital inflow in every continuation equilibrium. In other words, of the government receives a positive capital inflow, the lowest equilibrium consistent price is zero.

## 5.2 Best SPE

Note that the characterization of equilibrium consistent outcomes will use as input the best equilibrium price $\overline{q}^E(b_t)$ and the value function of the best equilibrium $\overline{\mathbb{W}}^E(y_t, b_{t+1}(y_t))$.[24]

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[24]Note that these ones will not be the ones of the Markov equilibrium that we characterized in Section 2. The reason is that now, the government is allowed to default, on the equilibrium path, without a punishment. A Markov equilibrium with states $b, y$ would imply that the government will default every debt that it has acquired. Therefore, there has a to be a price keeping constraint. An alternative approach is one as in Atkeson (1991) or Worrall (1990) that uses instead of $b, y$ as a state variable, the funds that the government has after repayment, in our notation $y - (1 - d(y))b$. With this state variable, an approach as in Abreu et al. (1990) can be used to obtain the best equilibrium value and the policies. We develop an alternative

In this subsection we discuss how we characterize them.

**Best Equilibrium Price**  Taking as given $\overline{W}^E(y_t, b_{t+1}(y_t))$, the price function $\overline{q}^E$ solves the following functional equation

$$\overline{q}^E(y_{t-1}, b_t) = \max_{d_t(y_t), b_{t+1}(y_t)} q$$

$$u(y_t - b_t(1 - d_t(y_t)) + \overline{q}(y_t, b_{t+1}(y_t))b_{t+1}(y_t)) + \beta \overline{W}^E(y_t, b_{t+1}(y_t)) \geq V^d(y_t)$$

$$q = \frac{1}{1+r}\left(1 - \int d_t(y_t)dF(y_t \mid y_{t-1})\right)$$

A solution to the operator is guaranteed due to the monotonicity of the operator and because the set of continuous and weakly decreasing functions endowed with the sup norm is a complete metric space.

**Best Equilibrium Value**  Notice that we just obtained the best price taking as given the best equilibrium value for debt. Suppose now that we know the best price. The best equilibrium will be the equilibrium with highest expected value that meets the incentive compatibility and the price consistency constraint. It is given by

$$\overline{W}^E(y_{t-1}, b_t) = \mathbb{E}_{y_t \mid y_{t-1}}\left[\overline{W}^{E'}(y_{t-1}, y_t, b_t)\right]$$

where $\overline{W}^{E'}(y_{t-1}, y_t, b_t)$ solves

$$\overline{W}^{E'}(y_{t-1}, b_t, y_t) = \max_{d_t(y_t), b_{t+1}(y_t)}\left\{u(y_t - b_t(1 - d_t(y_t)) + \overline{q}^E(y_t, b_{t+1}(y_t))b_{t+1}(y_t))\right.$$

$$\left. + \beta \overline{W}^E(y_t, b_{t+1}(y_t))\right\}$$

subject to

$$\overline{q}(y_{t-1}, b_t) = \frac{1}{1+r}\left(1 - \int d_t(y_t)dF(y_t \mid y_{t-1})\right) \tag{5.5}$$

Note that, constraint (5.5), is the one that makes sure that the amount lent, will be defaulted with the best equilibrium default rule.

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approach where we iterate directly in the set of equilibrium payoffs and prices. This follows Abreu et al. (1990), but it is still not the same.

## 5.3 Excusable Defaults and Savings

The most general characterization of SPE allows the government to save, and does not impose any exogenous punishment if it defaults. We can show that the worst equilibrium price for debt is zero; autarky is the worst SPE. Subgame perfect equilibrium outcomes are characterized by

$$q_{t-1} = \frac{1}{1+r}(1 - \int d_t(y_t)dF(y_t \mid y_{t-1}))$$

$$u(y_t - b_t(1 - d(y_t))) + \bar{q}^{ES}(y_t, b_{t+1}(y_t))b_{t+1}(y_t)) + \beta\overline{W}^{ES}(y_t, b_{t+1}(y_t)) \geq$$

$$\max_{\tilde{d}, \tilde{b}'} u(y_t - b_t(1 - \tilde{d}) + \underline{q}(y_t, \tilde{b}')\tilde{b}') + \beta\underline{W}^{ES}(y_t, \tilde{b}').$$

Then, the worst SPE price for the case of savings and excusable defaults will be zero. So the characterization of equilibrium consistent outcomes is analogous to the one in Proposition 5.1 without the restriction that $b \geq 0$.

# 6 Sunspots

We are now interested in adding a sunspot variable. Adding a sunspot that is realized together with output adds nothing to the analysis. Effectively, output could already acting as a random coordination device. Thus, the interesting question is to add a sunspot variable after the bond issuance, but before the price is determined. As we shall see, conditional on any single realization, the set of equilibrium consistent outcomes then coincides with the set of subgame perfect equilibria. Despite this we can obtain relevant history dependent predictions.

In this section we do three things. First, we characterize what we term are equilibrium consistent distributions. Those are distributions over prices that consistent with a subgame perfect equilibrium given history. Second, aided with this characterization we obtain bounds of the expectation over prices that hold across all equilibrium. This provides a way to obtain set identification of the set of structural parameters in our particular application. Finally, we provide an intuitive application of our results, and we find a bound on the probability of a non-fundamental debt crises; by crisis we mean an event where the realized price falls below a given threshold $\hat{q}$, which we treat as a parameter.

## 6.1 Equilibrium consistent distributions

Denote the sunspot by $\zeta_t$, realized after the bond issue of the government but before the price $q_t$; i.e, a sunspot is realized after $h_-^t$. Without loss of generality[25] we will assume $\zeta_t \sim$ Uniform $[0, 1]$ i.i.d. over time. If we assume that the game is on the equilibrium path of some subgame perfect equilibrium, then the government strategy before the realization of the sunspot was optimal; that is

$$\int \left[ u\left(y_t - b_t + q_t\left(\zeta_t\right) b_{t+1}\right) + \beta v\left(\zeta_t\right)\right] d\zeta_t \geq V^d\left(y_t\right).$$

The government ex-ante preferred to pay the debt and issue bonds $b_{t+1}$ than to default, where $q\left(\zeta_t\right)$ and $v\left(\zeta_t\right)$ are the market price and continuation equilibrium value conditional on the realization of the sunspot $\zeta_t$. The main difference in the characterization of equilibrium consistent distributions, is that now we cannot rely on the best continuation price, because it might not be realized.

**Best continuation.** Define the maximum continuation value function $\overline{v}\left(b, q\right)$ given bonds $b$ and price $q$ as

$$\overline{v}\left(b, q\right) = \max_{\sigma \in \mathbf{SPE}(b)} V\left(\sigma \mid b_0 = b\right)$$

subject to

$$\frac{\mathbb{E}(1 - d(y_0))}{1 + r} = q$$

This gives the best possible continuation value if we start at bonds $b$ and we restrict prices to be equal to $q$. The following proposition provides a method to compute the function $\overline{v}\left(b, q\right)$ and show that is non-decreasing and concave in $q$.

**Proposition 6.1.** *For all $q \in [0, \overline{\mathbf{q}}(b)]$ the maximum continuation value $\overline{v}\left(b, q\right)$ solves*

$$\overline{v}\left(b, q\right) = \max_{\delta(\cdot) \in [0,1]^Y} \int \left\{ \delta\left(y\right) V^d\left(y\right) + \left[1 - \delta\left(y\right)\right] \overline{V}^{nd}\left(b, y\right) \right\} dF\left(y\right)$$

*subject to*

$$q = \frac{1}{1 + r}\left(1 - \int \delta\left(y\right) dF\left(y\right)\right) \tag{6.1}$$

*Furthermore, is $\overline{v}\left(b, q\right)$ non-decreasing and concave in $q$.*

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[25]This is because of robustness: we will try to map all equilibria that can be contingent on the randomizing device, and hence as long as the random variable remains absolutely continuous, any time dependence in $\zeta_t$ can be replicated by time dependence on the equilibrium itself.

**Proof.** See Appendix C. □

The fact that the function is non-decreasing in $q$ follows from the fact that better prices are associated with better continuation equilibrium, as well as higher contemporaneous consumption (since $b_{t+1} \geq 0$). This follows from the fact that defaults are punished but when the government does not default, it obtains the best continuation equilibrium (under the strategy associated with value $\bar{v}(b_{t+1}, q_t)$). Concavity, follows from the the fact that $\bar{v}(b,q)$ solves a linear programming problem. We use both properties to obtain sharper characterizations of the set of equilibrium consistent distributions and to obtain testable predictions.

**Main result.** For a given equilibrium $\sigma$ at history $h = (h^t, y_t, d_t, b_{t+1})$ the equilibrium price distribution is defined by

$$\Pr(q \in A) = \Pr(\zeta_t : q^\sigma(h, \zeta_t) \in A)$$

Let $\mathcal{Q}(h)$ be the family of price distributions from history consistent equilibria. The following Proposition provides a characterization of this family.

**Proposition 6.2.** *Suppose* $h = (h^t, y_t, d_t, b_{t+1})$ *is equilibrium consistent. Then, a distribution* $P \in \Delta(\mathbb{R}_+)$ *is an equilibrium consistent price distribution; i.e.* $P \in \mathcal{Q}(h)$ *if and only if* **Supp** $(P) \subseteq [0, \overline{\mathbf{q}}(b_{t+1})]$ *and*

$$\int \{u(y_t - b_t + qb_{t+1}) + \beta\bar{v}(b_{t+1}, q)\} dP(q) \geq V^d(y_t) \tag{6.2}$$

*and hence* $\mathcal{Q}(h) = \mathcal{Q}(b_t, y_t, b_{t+1})$.

**Proof.** See Appendix D. □

Condition 6.2 parallels equation (4.6) in Proposition 4.1. There are some differences. We are now characterizing distribution over prices consistent with a decision of not defaulting $d_t = 0$ and issuing debt $b_{t+1}$. Note that we are taking an expectation with respect to $q$: the government does not know what particular price will be realized after it chooses a particular policy. The following two corollaries provide intuition regarding the set of equilibrium consistent distributions $\mathcal{Q}(b_t, y_t, b_{t+1})$.

**Corollary 6.1.** *The set of equilibrium price distributions* $\mathcal{Q}(b_t, y_t, b_{t+1})$ *is non-increasing (in set order sense) with respect to* $b_t$; *if income is i.i.d, it is non-decreasing in* $y_t$.

**Proof.** See Appendix D. □

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The intuition of this comparative statistics is again coming from the revealed preference argument. If the government repaid a higher amount of debt, then the distribution of prices that they could be expecting needs to shift towards higher prices. If the set does not change, then there will be some distribution that will be inconsistent with equilibrium because it will violate the promise keeping constraint. Let $P' \trianglerighteq P$ denote the relationship "$Q'$ first order stochastically dominates $Q$". The next corollary provides a partial ordering in $\mathcal{Q}$.

**Corollary 6.2.** *Suppose that*

$$P \in \mathcal{Q}(b_t, y_t, b_{t+1}) \text{ and } P' \in \Delta([0, \overline{\mathbf{q}}(b_{t+1})])$$

*If $P' \trianglerighteq P$ ( i.e. it first order stochastically dominates P), then $P' \in \mathcal{Q}(b_t, y_t, b_{t+1})$.*

**Proof.** See Appendix D. □

Once that a distribution is consistent with equilibrium, any distribution that first order stochastically dominates it will be an equilibrium consistent distribution. Intuitively, higher prices give both higher consumption and higher continuation equilibrium values for the government, since both are weakly increasing in the realizations of debt price $q_t$.

## 6.2 Expectations of Equilibrium Consistent Distributions

The main application of the analysis in this section if to obtain bounds over expectations equilibrium outcomes. In particular we will focus on bounds over equilibrium consistent prices. The set of equilibrium consistent expected prices is given by

$$E(b_t, y_t, b_{t+1}) = \{a \in \mathbb{R}_+ : \ a = \mathbb{E}_P(q) \text{ for some } P \in \mathcal{Q}(b_t, y_t, b_{t+1})\}$$

where $\mathbb{E}_P(q) \equiv \int q \, dP$. The following Proposition shows that in fact, the set of expected values is identical to the set of equilibrium consistent prices when there are no sunspots.

**Proposition 6.3.** *Suppose history $h = (h^t, y_t, d_t = 0, b_{t+1})$ is equilibrium consistent. Then the set of expected prices is equal to the set of prices without sunspots $Q(b_t, y_t, b_{t+1})$; i.e.*

$$E(b_t, y_t, b_{t+1}) = \left[\underline{\mathbf{q}}(b_t, y_t, b_{t+1}), \overline{\mathbf{q}}(b_{t+1})\right]$$

*Moreover, if $b_{t+1} > 0$ then the minimum expected value is achieved uniquely at the Dirac distribution $\hat{P}$ that assigns probability one to $q = \underline{\mathbf{q}}(b_t, y_t, b_{t+1})$.*

**Proof.** See Appendix D  ☐

The result comes from the concavity of the value function $\overline{v}\left(b_{t+1}, \hat{q}\right)$ and the fact that $\underline{q}\left(\cdot\right)$ is the minimum price that satisfies:

$$u\left(y_t - b_t + \underline{q} b_{t+1}\right) + \beta \overline{v}\left(b_{t+1}, \underline{q}\right) = V^d\left(y_t\right) \tag{6.3}$$

We showed concavity in Proposition 6.1. The equality at $q = \underline{q}\left(\cdot\right)$ follows from the strict monotonicity in $q$ of the left hand side expression: if the inequality was strict, then we can find a lower equilibrium consistent price, which contradicts the definition of $\underline{q}\left(\cdot\right)$. Therefore, the integrand in 6.2 is bigger than $V^d\left(y_t\right)$ only when $q \geq \underline{q}\left(b_t, y_t, b_{t+1}\right)$. Concavity of $\overline{v}\left(b, q\right)$ and Jensen's inequality then imply that for any distribution $P \in \mathcal{Q}\left(b_t, y_t, b_{t+1}\right)$ :

$$u\left(y_t - b_t + \mathbb{E}_P\left(q\right) b_{t+1}\right) + \beta \overline{v}\left(b_{t+1}, \mathbb{E}_P\left(q\right)\right) \geq \int \left\{ u\left(y_t - b_t + q b_{t+1}\right) + \beta \overline{v}\left(b_{t+1}, q\right)\right\} dP\left(q\right)$$

$$\geq V^d\left(y_t\right)$$

and therefore $\mathbb{E}_P\left(q_t\right) \geq \underline{q}\left(b_t, y_t, b_{t+1}\right)$.

The previous Proposition actually provides testable implications for the model. In particular, it yields a necessary and sufficient moment condition for equilibrium consistency at histories $h = \left(h^t, y_t, d_t, b_{t+1}\right)$,

$$\mathbb{E}_{q_t}\left\{ u\left(y_t - b_t + b_{t+1}\right) + \beta \overline{v}\left(b_{t+1}, q_t\right) - V^d\left(y_t\right) \mid h\right\} \geq 0 \tag{6.4}$$

The bounds that we just derived yields moment inequalities that are easier to check

$$\mathbb{E}_{q_t}\left\{ q_t \mid h\right\} \in \left[\underline{q}\left(b_t, y_t, b_{t+1}\right), \overline{q}\left(b_{t+1}\right)\right] \tag{6.5}$$

Aided with these bounds we can perform estimation of the structural set of parameters as in Chernozhukov et al. (2007).

## 6.3  Probability of Crises

Our goal will be to infer the maximum probability (across equilibria) that the government assigns to the market setting a price $q\left(\zeta\right) = \hat{q}$; i.e., a financial crises. Formally,

$$\underline{P}\left(\hat{q}\right) \equiv \max_{P \in \mathcal{Q}\left(b_t, y_t, b_{t+1}\right)} \Pr_P\left(q \leq \hat{q}\right) \tag{6.6}$$

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where $\Pr_P (q \leq \hat{q}) := \int_0^{\hat{q}} dP(q)$. These bounds are independent of the nature of the sunspots (i.e. the distribution of sunspots, its dimensionality, and so on), in the same way as the set of correlated equilibria does not depend on the actual correlating devices.[26] Furthermore this bound will yield a necessary condition for a distribution to be an element in $\mathcal{Q}(b_t, y_t, b_{t+1})$.

**Upper bound on $\Pr(q = 0)$**  To construct the maximum equilibrium consistent probability that $q_t = 0$, we make the promise keeping constraint be as relaxed as possible. We do this by considering continuation equilibria with two properties: first, assign the best continuation equilibria if $q \neq 0$ (i.e, under price $\overline{q}(y_t, b_{t+1})$). Second, note that autarky is the best continuation equilibria feasible with $q = 0$; if the government receives a price of zero, in equilibrium, it will default with probability one in the continuation equilibrium[27]. Let $p_0 = \underline{P}(\hat{q} = 0)$. The IC constraint is now

$$p_0 \left[ u(y_t - b_t + b_{t+1} \times 0) + \beta \mathrm{V}^d(y_t) \right] + (1 - p_0) \left[ \overline{V}^{nd}(b_t, y_t, b_{t+1}) \right] \geq V^d(y_t).$$

Then

$$p_0 = \frac{\Delta^{nd}(b_t, y_t, b_{t+1})}{\Delta^{nd}(b_t, y_t, b_{t+1}) + u(y_t) - u(y_t - b_t)} < 1,$$

where $\Delta^{nd}(\cdot)$ denotes the maximum utility difference between not defaulting and defaulting (under the best equilibrium)

$$\Delta^{nd}(b_t, y_t, b_{t+1}) \equiv \overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t)$$

Thus, the probability of $q = 0$ is bounded away from 1 from an ex-ante perspective (i.e. before the sunspot is realized, but after the government decision). So we obtain a history dependent bound on the probability of a financial crises.

**Upper bound for general $\hat{q} < \underline{q}(b_t, y_t, b_{t+1})$.**  Let $p = \underline{P}(\hat{q})$. Using the same strategy as before, to get the less tight the incentive compatibility constraint for the government we need to

  a. for $\zeta : q(\zeta) > \hat{q}$, we consider equilibria that assign the best continuation equilibria,

  b. maximize equilibrium utility for $q : q \leq \hat{q}$.

---

[26] As long as out interest is in characterizing all correlated equilibria.

[27] The default decision in equilibrium needs to be consistent with the price: a price of zero is only consistent with default in every state of nature. And we assume that after default the government is in autarky forever.

Thus

$$p \leq \underline{P}(\hat{q}) = \frac{\Delta^{nd}(b_t, y_t, b_{t+1})}{V^d(y_t) - [u(y_t - b_t + \hat{q}b_{t+1}) + \beta\overline{v}(b_t, \hat{q})] + \Delta^{nd}(b_t, y_t, b_{t+1})}.$$

Note that this is not an innocuous constraint only when the right hand side is less than 1. This happens only when

$$u(y_t - b_t + \hat{q}b_{t+1}) + \beta\overline{v}(b_t, \hat{q}) \geq V^d(y_t).$$

This constraint holds when and this holds if $\hat{q} \geq \underline{\mathbf{q}}(b_t, y_t, b_{t+1})$ The last inequality comes from the characterization of $\underline{\mathbf{q}}(b_t, y_t, b_{t+1})$. The following Proposition summarizes the results of this section:

**Proposition 6.4.** *Take an equilibrium consistent history $h = (h^t, y_t, d_t, b_{t+1})$ and let $\Delta^{nd} = \overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t)$. For any $\hat{q} < \underline{\mathbf{q}}(b_t, y_t, b_{t+1})$*

$$\underline{P}(\hat{q}) = \frac{\Delta^{nd}}{\Delta^{nd} - [u(y_t - b_t + \hat{q}b_{t+1}) + \beta\overline{v}(b_t, \hat{q}) - u(y_t) - \beta\mathbb{V}^d]} < 1$$

*For any $\hat{q} \geq \underline{\mathbf{q}}(b_t, y_t, b_{t+1})$, $\underline{P}(\hat{q}) = 1$.*

In Proposition 6.4 we find the ex-ante probability (before $\zeta_t$ is realized) of observing $q_t = \hat{q}$ is less than $\underline{P}(\hat{q}) < 1$ for any equilibrium consistent outcome. Note that if the income realization is such that $\overline{V}^{nd}(b_t, y_t) = V^d(y_t)$ (i.e. under the best continuation equilibrium, the government was indifferent between defaulting or not, and still did not default), then $\underline{P}(\hat{q}) = 0$ for any $\hat{q} < \underline{\mathbf{q}}(b_t, y_t, b_{t+1}) = \overline{\mathbf{q}}(y_t, b_{t+1})$, which implies that at such income levels, even with these kind of correlating devices available, only $q = \overline{\mathbf{q}}(y_t, b_{t+1})$ is the equilibrium consistent price. We also show that any price $q \in \left[\underline{\mathbf{q}}(\cdot), \overline{\mathbf{q}}(\cdot)\right]$ could be observed with probability 1, since they are part of the path of a pure strategy SPE profile. When adding sunspots, any price in $[0, \overline{\mathbf{q}}(\cdot)]$ can be observed ex-post, and since the econometrician has no information about the realization of the sunspot (or the particular equilibrium selection and use of the correlating device) any price is feasible ex ante. However, before more information is realized, the econometrician can place bounds on how likely different prices are, which can not be 1, so that the government incentive constraint is satisfied.

Aided with the characterization of Proposition 6.4 we find a restriction satisfied by equilibrium consistent distributions: they stochastically dominate $\underline{P}$, in the first order sense. Note that it is a cumulative distribution function on $q$: it is a non-increasing, right-

continuous function with range $[0,1]$, hence implicitly defining a probability measure over debt prices

**Corollary 6.3.** *The distribution $\underline{P}(\cdot)$ is the maximum lower bound (in the FOSD sense) of the set equilibrium consistent distributions; i.e. for every $P \in \mathcal{Q}(b_t, y_t, b_{t+1})$ we have $P \trianglerighteq \underline{P}$, and if $P'$ is some other lower bound, then $P' \trianglerighteq \underline{P}$. Moreover, $\underline{P} \notin \mathcal{Q}(b_t, y_t, b_{t+1})$*

# 7 Discrete Income

Now, we study the case where there are no sunspots at all. For simplicity in the notation, focus in the case where income is i.i.d. Our main result in the previous section was a characterization of equilibrium consistent outcomes. In this result, the fact that the last opportunity cost is a sufficient statistic for equilibrium consistent outcomes (and prices as a consequence) is somewhat surprising: the outside observer is only using observations from the last period to make an inference, even though she has a whole history of data available. However, as we will see below in a simple two period example, this result is a direct expression of Robustness: the econometrician needs to take into account that the expected payoff that rationalized a particular decision, could have been realized only in histories that did not occur.

Suppose that we observe $(\bar{y}_0, b_0, q_0, b_1, \bar{y}_1, b_2)$. Denote $h^1 = \bar{y}_0, b_0, q_0, b_1$. We will show that

$$\underline{q}_1(\bar{y}_0, b_0, q_0, b_1, \bar{y}_1, b_2) = \underline{q}_1(b_1, \bar{y}_1, b_2)$$

when the probability $p(y_1) = 0$, capturing the main feature of the case with continuous income, that any realization has zero probability. In order to do this, note that, because $\bar{y}_0, b_0, q_0, b_1$ is an equilibrium history, there is a continuation value function such that

$$\sum_i p(y_{1i})V_0(y_{1i}, b_1) \geq \frac{1}{\beta}\left[u(\bar{y}_0) - u(\bar{y}_0 - b_0 + q_0 b_1)\right] + \underline{W} \tag{7.1}$$

where $V_0(y_{1i}, b_1)$ is the continuation value function of a continuation equilibrium strategy after history $\bar{y}_0, b_0, q_0, b_1, y_{1i}$. Note also that $\bar{y}_1, b_2$ and the decision not to default are part of an SPE. This implies that the following constraint has to hold

$$u(\bar{y}_1 - b_1 + q_1(h^1, \bar{y}_1, b_2)b_2) + \beta\sum_i p(y_{2i})V_1(y_{2i}, b_2) \geq u(\bar{y}_1) + \beta\underline{W} \tag{7.2}$$

Note, also, it has to be the case that

$$V_0(\bar{y}_1, b_1) = (\bar{y}_1 - b_1 + q_1(h^1, \bar{y}_1, b_2)b_2) + \beta \sum_i p(y_{2i}) V_1(y_{2i}, b_2) \tag{7.3}$$

That is, on the equilibrium path, the promised continuation needs to coincide with the continuation that we observe in history $(\bar{y}_0, b_0, q_0, b_1, \bar{y}_1, b_2)$. Finally, it also needs to be that the case that

$$V_1(y_{2i}, b_2) \in [\underline{W}, \overline{W}(y_{2i}, b_2)] \tag{7.4}$$

$$V_1(y_{2i}, b_2) \in [\underline{W}, \overline{W}(y_{2i}, b_2)] \tag{7.5}$$

where we abuse notation slightly for the continuation value sets. Now, the lowest equilibrium consistent price solves

$$\min_{\{V_0(y_{1i}, b_1)\}_{y_{1i}}, \{V_1(y_{2i}, b_2)\}_{y_{2i}}} q \tag{7.6}$$

subject to (7.1), (7.2), (7.3), (7.4), and (7.5). Our objective is to show that if $p(\bar{y}_1) = 0$, the constraint (7.1) is not binding, and therefore, the solution will not depend on $(\bar{y}_0, b_0, q_0)$. To solve (7.6), we want to relax the constraint (7.1) as much as possible. So we pick the continuation value function

$$V_0(y_{1i}, b_1) = \begin{cases} V_0(\bar{y}_1, b_1) & \text{for } y_{1i} = \bar{y}_1 \\ \overline{W}(y_{1i}, b_1) & \text{for } y_{1i} \neq \bar{y}_1 \end{cases}$$

where $V_0(\bar{y}_1, b_1)$ is free at the moment. Because the histories $(\bar{y}_0, b_0, q_0, b_1, \text{for } y_{1i} \neq \bar{y}_1)$ are not realized, it could have been the case that the best continuation followed. The outside observer cannot neglect this possibility. Then, by adding and subtracting $p(\bar{y}_1)\overline{V}(\bar{y}_1, b_1)$, we can rewrite the left hand side of (7.1) as

$$\sum_i p(y_{1i}) V_0(y_{1i}, b_1) = p(\bar{y}_1) \left[ V_0(\bar{y}_1, b_1) - \overline{V}(\bar{y}_1, b_1) \right] + \sum_i p(y_{1i})\overline{W}(y_{1i}, b_1) \tag{7.7}$$

Plugging (7.7) in (7.1)

$$p(\bar{y}_1) \left[ V_0(\bar{y}_1, b_1) - \overline{V}(\bar{y}_1, b_1) \right] + \sum_i p(y_{1i})\overline{W}(y_{1i}, b_1) \geq \frac{1}{\beta} \left[ u(\bar{y}_0) - u(\bar{y}_0 - b_0 + q_0 b_1) \right] + \underline{W} \tag{7.8}$$

where $\overline{V}(\bar{y}_1, b_1)$ is the value of not default in the best equilibrium when bonds are $b_1$ and income is $\bar{y}_1$. So, when income is continuous, $p(\bar{y}_1) = 0$. So, the constraint will not be

binding if

$$u(\bar{y}_0 - b_0 + q_0 b_1) + \beta \sum_i p(y_{1i}) \overline{W}(y_{1i}, b_1) \geq u(\bar{y}_0) + \beta \underline{W}$$

holds. And this holds because $\bar{y}_0, b_0, q_0, b_1$ is an SPE history where the government did not default.

If income is discrete, then $b_1, \bar{y}_1, b_2$ will not be sufficient statistics to summarize history. The intuition is that the future policies affect previous decisions, because the particular realized history does not have probability zero. Define

$$\mathbf{oc}_0 = u(\bar{y}_0) - u(\bar{y}_0 - b_0 + q_0 b_1)$$

This is the opportunity cost of not defaulting. Rearrange (7.8), such that

$$V_0(\bar{y}_1, b_1) \geq \frac{1}{p(\bar{y}_1)} \left[ \frac{1}{\beta} \mathbf{oc}_0 + \underline{W} - \overline{W}(b_1) \right] + \overline{V}(\bar{y}_1, b_1)$$

If this constraint binds, the lowest equilibrium consistent price is

$$\underline{q}_1(\bar{y}_0, b_0, q_0, b_1, \bar{y}_1, b_2)$$

with full history dependence.

Whether it will bind or not, depends on the following. First, it depends on the past opportunity cost: if in the past, the government passed on default under very harsh circumstances, then the continuation value needs to be higher. Second, it depends on the strength of the link between current and past decision. If the government discounts more the future, or the history is less likely, then the constraint is less likely to be binding.

# 8 Conclusion and Discussion

Dynamic policy games have been extensively studied in macroeconomic theory to increase our understanding on how the outcomes that a government can achieve are restricted by its lack of commitment. One of the challenges in studying dynamic policy games is equilibrium multiplicity. Our paper acknowledges equilibrium multiplicity, and for this reason focuses on obtaining predictions that hold across all equilibria. To do this, we conceptually introduced and characterized equilibrium consistent outcomes. We did so under different settings, and we found that the assumption that a history was generated by the path of a subgame perfect equilibrium puts restrictions on current policies, and therefore on observables. In addition, we found intuitive conditions under which

past decisions place restrictions on future policies; if the past decision occurred far away in time or in a history where the current history had low probability of occurrence, then it is less likely that a particular past decision influences current policies. In the extreme case that every particular history has probability zero, the restrictions of past decisions in current outcomes die out after one period. At first glance, this is surprising; but as we showed in the paper, this a direct consequence of robustness.

As we discussed in the text, equilibrium consistency is a general principle. Even though we focus on a model of sovereign debt that follows Eaton and Gersovitz (1981), our results generalize to other dynamic policy games. An example is the model of capital taxation as in Chari and Kehoe (1990). In that model, the entrepreneur invests and supplies labor, then the government taxes capital, and finally, the entrepreneur receives a payoff. The worst subgame perfect equilibrium is one where the government taxes all the capital. Note that, if the government has been consistently abstaining from taxing capital, then as outside observers we can rule out that the government will tax all capital. Past behavior, and the sole assumption of equilibrium, is giving information to the outside observer about future outcomes.

We think equilibrium consistency might have applications beyond policy games. The reason is that the sole assumption of equilibrium yields testable predictions. For example, the literature of risk sharing studies barriers to insurance and tries to test among different economics environments. Two environments that have received a lot of attention are Limited Commitment and Hidden Income. To test these two environments, a property of the efficient allocation with limited commitment is exploited: lagged consumption is a sufficient statistic of current consumption. If this hypothesis is rejected, then hidden income is favored in the data. However, the test is rejecting two hypotheses at the same time: efficiency and limited commitment. Our approach could, in principle, be suitable for a test that is tractable and robust to equilibrium multiplicity.

Over the course of the paper, we have been silent with respect to optimal policy. An avenue of future research is to relate equilibrium consistent outcomes and forward reasoning in dynamic games. Our conjecture is that, the set of equilibrium consistent outcomes will be intimately related with the set of outcomes if there is common knowledge of strong certainty of rationality. The reason is that, in the model of sovereign debt that we studied, the outside observer and the lenders have the same information set. Even in the motivating example, equilibrium consistent outcomes and outcomes when the solution concept is strong certainty of rationality are the same. In that case, our results have a different interpretation: the government is choosing the history to manage the expectations of the public.

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# A   Appendix

**Proof. (Lemma 3.2 )** Note that we can rewrite the system of Bellman equations as

$$A.v(B) = u(B)$$

Thus, a condition in primitives is

$$v'(0) = A^{-1}u'(0) \geq 0$$

For the special case where $\lambda = 1$, note that

$$
\begin{aligned}
v_{BH} &= \frac{1}{1-\beta^2}\left(u(y_H - RB) + \beta u(y_L + B)\right)\\
v_{0L} &= u(y_L + B) + \beta v_{BH}
\end{aligned}
$$

Then, $v'_{BH}(0) > 0$ implies that $v'_{0L}(0) > 0$. A sufficient condition is $\beta u'(y_L) > Ru'(y_H)$. The intuition is that, the government is credit constrained in the low state, with no debt, and is willing to tradeoff and have lower consumption in the high state.   □

**Proof. (Proposition 4.1).** (**Necessity,** $\Longrightarrow$) If $(d\,(\cdot)\,, b'\,(\cdot))$ is SPE - consistent, there exists an SPE profile $\hat{\sigma}$ such that $h^t \in \mathcal{H}\,(\hat{\sigma})$ and

$$d\,(y_t) = d^{\hat{\sigma}}_t\left(h^t, y_t\right) \text{ and } b'\,(y) = b^{\hat{\sigma}}_{t+1}\left(h^t, y_t, d = 0\right)$$

That is, there exists a SPE that generated the history $h^t_-$, specifies the contingent policy $d\,(\cdot)\,, b'\,(\cdot)$ in period $t$, and satisfies conditions (4.4) to (4.6). Because $\hat{\sigma}$ is an SPE, using the results of Abreu et al. (1990) we know that if $d\,(y) = 0$ at $h^t = \left(h^t_-, q_{t-1}\right)$ then

$$u\left(y_t - b_t + b'\,(y_t)\,q^{\hat{\sigma}}_m\left(h^t, d_t = 0, b'\,(y_t)\right)\right) + \beta W\left(\hat{\sigma} \mid h^{t+1}\right) \geq u\,(y_t) + \beta \mathbb{V}^d(y_t) \quad \text{(A.1)}$$

By definition of best continuation values and prices

$$W\left(\hat{\sigma} \mid h^{t+1}\right) \leq \overline{W}\,(y_t, b'\,(y_t)) \text{ and } q^{\hat{\sigma}}_m\left(h^t, d_t = 0, b'\,(y_t)\right) \leq \overline{q}\,(y_t, b'\,(y_t)) \quad \text{(A.2)}$$

Because $b'\,(y_t) \geq 0$ (no savings assumption), and $u\,(\cdot)$ is strictly increasing, we can plug in (A.2) into (A.1) to conclude that

$$u\left(y_t - b + b'\,(y_t)\,\overline{q}\,(y_t, b'\,(y_t))\right) + \beta \overline{W}\,(y_t, b'\,(y_t)) \geq$$

$$u\left(y - b_t + b'\left(y_t\right)q^{\hat{\sigma}}_m\left(h^t, d_t = 0, b'\left(y_t\right)\right)\right) + \beta W\left(\hat{\sigma} \mid h^{t+1}\right)$$

Proving condition (4.5). Further, since $\hat{\sigma}$ generated the observed history, past prices must be consistent with policy $(d\left(\cdot\right), b'\left(\cdot\right))$. Formally:

$$q_{t-1} = q^{\hat{\sigma}}_m\left(h^{t-1}, y_{t-1}, d_{t-1}, b_t\right) = \frac{1}{1+r^*}(1 - \int_{y_t \in Y} d^{\hat{\sigma}}\left(h^t, y_t\right) dF(y_t \mid y_{t-1}))$$

$$= \frac{1}{1+r}(1 - \int_{y_t \in Y} d(y_t) dF(y_t \mid y_{t-1}))$$

proving also condition (4.4). Condition (4.6) is the same as condition (4.5) but at $t-1$, using the usual promise keeping accounting. Formally, if $\hat{\sigma}$ is SPE and $h^t \in \mathcal{H}\left(\hat{\sigma}\right)$ then the government's default and bond issue decision at $t-1$ was optimal given the observed expected prices

$$u(\underbrace{y_{t-1} - b_{t-1} + b_t q_{t-1}}_{=c_{t-1}}) + \beta W\left(\hat{\sigma} \mid h^t\right) \geq u\left(y_{t-1}\right) + \beta \mathbb{V}^d(y_{t-1})$$

Using the recursive formulation of $W\left(\cdot\right)$ we get the following inequality:

$$W\left(\hat{\sigma} \mid h^t\right) = \int_{y_t: d(y_t)=0} \left[u(y_t - b_t + b'(y_t)q^{\hat{\sigma}}_m(h^t, y_t, d_t = 0, b'(y_t))) + W\left(\hat{\sigma} \mid h^{t+1}\right)\right] dF(y_t \mid y_{t-1})$$

$$+ \int_{y_t: d(y_t)=1} \left[u\left(y_t\right) + \beta \mathbb{V}^d(y_t)\right] dF(y_t \mid y_{t-1})$$

$$\leq \int_{y_t: d(y_t)=0} \left[u\left(y_t - b_t + b'(y_t)\bar{q}(b'(y_t))\right) + \overline{W}(b'(y_t))\right] dF(y_t \mid y_{t-1})$$

$$+ \int_{y_t: d(y_t)=1} \left[u\left(y_t\right) + \beta \mathbb{V}^d(y_t)\right] dF(y_t \mid y_{t-1})$$

From the previous two inequalities, we show (4.6).

(**Sufficiency,** $\Longleftarrow$) We need to construct a strategy profile $\sigma \in SPE$ such that $h^t_- \in \mathcal{H}\left(\sigma\right)$ and $d\left(\cdot\right) = d^\sigma_t\left(h^t, \cdot\right)$ and $b'\left(\cdot\right) = b^\sigma_{t+1}\left(h^t, \cdot\right)$. Given that $h^t_- \in \mathcal{H}\left(SPE\right)$, we know there exists some SPE profile $\hat{\sigma} = \left(\hat{\sigma}_g, \hat{q}_m\right)$ that generated $h^t_-$. Let $\bar{\sigma}\left(b, y\right)$ be the best continuation SPE (associated with the best price $\bar{q}\left(\cdot\right)$) when $y_t = y$ and $b_{t+1} = b$. Let $\sigma^{aut}$ be the strategy profile for autarky (associated with $q_m = 0$ for all continuation histories). Also, let $h^{t+1}\left(y_t\right) = \left(h^t, y_t, d\left(y_t\right), b'\left(y_t\right), \bar{q}\left(y_t, b'\left(y_t\right)\right)\right)$ be the continuation history at $y_t = y$ and the policy $(d\left(\cdot\right), b'\left(\cdot\right))$ if the government faces the best possible prices. Define $(h^s, y_s) \prec h^t$ as the histories that precede $h^t$ and are not equal to $h^t$. That is, if we truncate

$h^t$ to period $s$, we obtain $h^s$. Denote $(h^s, y_s) \nprec h^t$ as the histories that do not precede $h^t$. The symbol $\preceq$ denotes, histories that precede and can be equal. Construct the following strategy profile $\sigma = (\sigma_g, q_m)$ :

$$
\sigma_g(h^s, y_s) = \begin{cases} \hat{\sigma}_g(h^s, y_s) & \text{for all } (h^s, y_s) \prec h^t \\ \sigma^{aut}(y_s) & \text{for all } s < t \text{ and } (h^s, y_s) \nprec h^t \\ d_t(h^t, y_t) = d(y_t) \text{ and } b_{t+1}(h^t, y_t) = b'(y_t) & \text{for } (h^t, y_t) \text{ for all } y_t \\ \overline{\sigma}_g(b_{s+1}, y_s)(h^s, y_s) & \text{for all } h^s \succeq h^{t+1}(y_t) \\ \sigma^{aut}(y_s) & \text{for all } s > t, \ h^s \nsucceq h^{t+1}(y_t) \end{cases}
$$

and

$$
q_m(h^s, y_s, d_s, b_{s+1}) = \begin{cases} \hat{q}_m(h^s, y_s, d_s, b_{s+1}) & \text{for all } (h^s, y_s) \prec h^t \\ 0 & \text{for all } s < t \text{ and } (h^s, y_s) \nprec h^t \\ \overline{q}(y_s, b'(y_s)) & \text{for all } h^s \succeq (h^t, y_t, d(y_t), b'(y_t)) \\ 0 & \text{for all } h^s \nsucceq (h^t, y_t, d(y_t), b'(y_t)) \end{cases}
$$

By construction $h^t_- \in \mathcal{H}(\sigma)$. This is because, $\sigma = \hat{\sigma}_g$ for histories $(h^s, y_s) \preceq h^t$. Also, the strategy $\sigma$, prescribes the policy $(d(\cdot), b'(\cdot))$ on the equilibrium path. Now we need to show that the constructed strategy profile is indeed an SPE. For this, we will use the one deviation principle. See that for all histories with $s > t$ the continuation profile is an SPE (by construction); it prescribes the best continuation equilibrium, that is a SPE by definition. Now, we need to show that at $h^t$ this is indeed an equilibrium. This comes from the second constraint, the incentive compatibility constraint

$$
(1 - d(y_t)) \left[ u(y_t - b_t + \overline{q}(y_t, b_{t+1}((y_t)))b_{t+1}(y_t)) + \beta \overline{W}(y_t, b_{t+1}((y_t))) \right]
$$

$$
+ d(y_t) V^d(y_t) \geq V^d(y_t)
$$

Note also that the default policy at $t - 1$ was consistent with $\sigma$ (and is an equilibrium) and that $q_{t-1}$ is consistent with the policy $(d(\cdot), b'(\cdot))$. The promise keeping constraint (4.6) translates into the exact incentive compatibility constraint for profile $\sigma$, showing that the default decision at $t - 1$ was indeed optimal given profile $\sigma$. The "price keeping" (4.4) constraint also implies that $q_{t-1}$ was consistent with policy $(d(\cdot), b'(\cdot))$. The final step in sufficiency is to show that, $s < t - 1$ (that is $h^s \prec h^t$). Note that, because $y$ is absolutely continuous, the particular $y$ that is realized, has zero probability. So, the expected value

of this new strategy is the same

$$W(\hat{\sigma} \mid h^s) = W(\sigma \mid h^t)$$

for all $h^s \prec h^t$ with $s < t - 1$; the probability of the realization of $h^t$, is zero. All this together implies that $\sigma$ is indeed an SPE and generates history $h^t_-$ on the equilibrium path, proving the desired result. □

**Proof.** (**Proposition** 4.2) By Proposition 4.1, we can rewrite program (4.10) as,

$$\underline{q}(b,y,b') = \min_{q,d(\cdot)\in\{0,1\}^Y, b''(\cdot)} q$$

subject to

$$q = \frac{1 - \int d(y')dF(y' \mid y)}{1+r} \tag{A.3}$$

$$(1 - d(y')) \left( \overline{V}^{nd}(b',y',b''(y')) - V^d(y') \right) \geq 0 \tag{A.4}$$

and

$$\beta \int \left[ d(y') V^d(y') + (1 - d(y')) \overline{V}^{nd}(b',y',b''(y')) \right] dF(y' \mid y) - \beta \mathbb{V}^d(y) \geq u(y) - u(y - b + b'q) \tag{A.5}$$

First, note that we can relax the constraint (A.4) and (A.5) by choosing

$$b''(y') = \operatorname*{argmax}_{\hat{b}\geq0} \overline{V}^{nd}\left(b',y',\hat{b}\right)$$

Second, define the set $R(b') = \{y' \in Y : \overline{V}^{nd}(b',y') \geq V^d(y')\}$ to be the set of income levels for which the government does not default, under the best continuation equilibrium. Note that, if $y' \notin R(b')$, it implies that no default is not equilibrium feasible for any continuation equilibrium (it comes from the fact that (A.4) is a necessary condition for no default). The minimization problem can now be written as

$$\underline{q}(b,y,b') = \min_{q,d(\cdot)\in\{0,1\}^Y} q$$

subject to

$$q = \frac{1 - \int d(y') dF(y' \mid y)}{1+r}$$

$$(1 - d(y')) \left[ \overline{V}^{nd}(b',y') - V^d(y') \right] \geq 0 \text{ for all } y' \in R(b') \tag{A.6}$$

$$d(y') = 1 \text{ for all } y' \notin R(b') \tag{A.7}$$

$$\beta \int \left[ d(y') V^d(y') + (1 - d(y')) \overline{V}^{nd}(b', y') \right] dF(y') - \beta \mathbb{V}^d(y) \geq u(y) - u(y - b + b'q)$$

As a preliminary step, we need to show that this problem has a non-empty feasible set. For that, choose the default rule that makes all constraints be less binding: i.e. $d(y') = 0 \iff \overline{V}^{nd}(b', y') \geq V^d(y')$. This corresponds to the best equilibrium policy. If this policy is not feasible, then the feasible set is empty. Under this default policy, the one of the best equilibrium, the price $q$ is equal to the best equilibrium price $q = \overline{\mathbf{q}}(y, b')$. The feasible set is non-empty if and only if

$$\beta \int \left[ d(y') V^d(y') + (1 - d(y')) \overline{V}^{nd}(b', y') \right] dF(y' \mid y) - \beta \mathbb{V}^d(y) \geq u(y) - u(y - b + b'\overline{\mathbf{q}}(y, b'))$$

$$u(y - b + b'\overline{\mathbf{q}}(y, b')) + \beta \mathbb{W}(b') \geq u(y) + \beta \mathbb{V}^d(y) \iff$$

$$\overline{V}^{nd}(b, y, b') \geq V^d(y)$$

where $\mathbb{W}(y, b')$ is the value of the option of defaulting $b'$ bonds; this is the initial assumption of this proposition. Also, note that

$$\mathbb{V}^d(y) = \int \left[ d(y') V^d(y') + (1 - d(y')) V^d(y') \right] dF(y' \mid y)$$

So, we can rewrite the promise keeping constraint as

$$\beta \int (1 - d(y')) \left[ \overline{V}^{nd}(b', y') - V^d(y') \right] dF(y') \geq u(y) - u(y - b + b'q) \tag{A.8}$$

We focus on a relaxed version of the problem. We will allow the default rule to be $d(y') \in [0, 1]$ for all $y'$. Given the state variables $(b, y, b')$ the relaxed problem is a convex minimization program in the space $(q, d(\cdot)) \in \left[ 0, \frac{1}{1+r} \right] \times \mathbb{D}(Y)$, where

$$\mathbb{D}(Y) \equiv \{ d : Y \rightarrow [0, 1] \text{ such that } d(y') = 1 \text{ for all } y' \notin R(b') \}$$

is a convex set of default functions. Also, include the constraint for prices

$$q \geq \frac{1 - \int d(y') dF(y' \mid y)}{1 + r}$$

The intuition for this last constraint is that $d(y') = 1$ has to be feasible in the relaxed

problem. The Lagrangian

$$\mathcal{L}\left(q, \delta\left(\cdot\right)\right) = q + \mu \left(-q + \frac{1 - \int d\left(y'\right) dF\left(y' \mid y\right)}{1 + r}\right) +$$

$$\lambda \left(u\left(y\right) - u\left(y - b + b'q\right) - \beta \int \left(1 - d\left(y'\right)\right) \left[\overline{V}^{nd}\left(b', y'\right) - V^{d}\left(y'\right)\right] dF\left(y' \mid y\right)\right)$$

The optimal default rule $d\left(\cdot\right)$ must minimize the Lagrangian $\mathcal{L}$ given the multipliers $\left(\mu, \lambda\right)$ (where $\mu, \lambda \geq 0$). Notice that for $y' \in R\left(b'\right)$ any $d \in [0, 1]$ is incentive constraint feasible, and

$$\frac{\partial \mathcal{L}}{\partial d\left(y'\right)} = \left(-\frac{\mu}{1 + r} + \lambda \beta \left[\overline{V}^{nd}\left(b', y'\right) - V^{d}\left(y'\right)\right]\right) dF\left(y' \mid y\right)$$

So, because it is a linear programming program, the solution is in the corners (and if it is not in the corners, it has the same value in the interior), then the values of $y'$ such that the country does not default are given by

$$d\left(y'\right) = 0 \iff \lambda \Delta^{nd} > \frac{\mu}{\beta\left(1 + r\right)} \tag{A.9}$$

Note that $\lambda > 0$ in the optimum. Suppose not; then $d\left(y'\right) = 1$ for all $y' \in Y$ satisfies the IC and the price constraint. Then, the minimum price is

$$q \geq \frac{1 - 1}{1 + r}$$

So, the minimizer will be zero, $q = 0$. But, this will not meet the promise keeping constraint. Formally,

$$\beta \int V^{d}\left(y'\right) dF\left(y' \mid y\right) - \beta \mathbb{V}^{d}(y) - u\left(y\right) + u\left(y - b\right) =$$

$$= \beta \left(\mathbb{V}^{d}(y) - \mathbb{V}^{d}(y)\right) + u\left(y - b\right) - u\left(y\right) = u\left(y - b\right) - u\left(y\right) < 0$$

This implies $\lambda > 0$. Note that, $\lambda > 0$ implies that $\underline{q}\left(b, y, b'\right) > 0$. Define

$$\gamma \equiv \frac{\mu}{\lambda \beta\left(1 + r\right)}$$

From (A.9)

$$d\left(y'\right) = 0 \iff \Delta^{nd} \geq \gamma \iff \overline{V}^{nd}\left(b', y'\right) \geq V^{d}\left(y'\right) + \gamma$$

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as we wanted to show. Aided with this characterization, from the promise keeping constraint we have an equation for $\gamma$ as a function of the states

$$\beta \int_{\overline{V}^{nd}(b',y') \geq V^d(y') + \gamma} \left[ \overline{V}^{nd}(b',y') - V^d(y') \right] dF(y' \mid y) = u(y) - u(y - b + b'q) \quad \text{(A.10)}$$

where

$$q = \frac{\Pr\left(\overline{V}^{nd}(b',y') \geq V^d(y') + \gamma\right)}{1 + r} \quad \text{(A.11)}$$

Define

$$\Delta^{nd}(y') := \overline{V}^{nd}(b',y') - V^d(y')$$

So,

$$q = \frac{\hat{F}\left(\Delta^{nd}(y') \geq \gamma\right)}{1 + r}$$

where $\hat{F}$ is the probability distribution of $\Delta^{nd}(y')$. The **last step in the proof** involves showing that the solution is well defined. Define the function

$$G(\gamma) = \beta \int_{\Delta^{nd} \geq \gamma} \Delta^{nd} d\hat{F}\left(\Delta^{nd} \mid y\right) - u(y) + u\left(y - b + b'\frac{1 - \hat{F}(\gamma \mid y)}{1 + r}\right)$$

First, note that $G$ is weakly decreasing in $\gamma$, that $G(0) > 0$ (from the assumption $\overline{V}^{nd}(b',y') - V^d(y') > 0$) and $\lim_{\gamma \to \infty} G(\gamma) = u(y - b) - u(y) < 0$. Second, note that $G$ is right continuous in $\gamma$. These two observations imply that we can find a minimum $\gamma : G(\gamma) \geq 0$. If income is an absolutely continuous random variable, then $G(\cdot)$ is strictly decreasing and continuous, implying the existence of a unique $\gamma$ such that $G(\gamma) = 0$. This determines the solution to the price minimization problem. $\qquad\square$

# B Characterization of $\overline{v}(b,q)$

Define the equilibrium value correspondence as

$$\mathcal{E}(b) = \left\{ (v,q) \in \mathbb{R}_2 : \exists \sigma \in SPE(b) : \begin{cases} v = \mathbb{E}\left\{ \sum_{t=1}^{\infty} u\left(c^{\sigma}\left(h^t\right)\right) \right\} \\ q = \frac{1}{1+r}\left(1 - \int d^{\sigma}(y_0) \, dF(y_0)\right) \end{cases} \right\}$$

The set $\mathcal{E}(b)$ has the values and prices that can be obtained in a subgame perfect equilibrium. We need to find a policy that keeps the promise for prices, for one period.

**Enforceability.**     Take a bounded, compact valued correspondence $W : \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$. We will drop the dependence on $d$, and we will bear in mind that after default the government is not in the market.

**Definition B.1.** Given $b \geq 0$, a government strategy $(d(\cdot), b'(\cdot))$ is enforceable in $W(b)$ if we can find a pair of functions $v(y)$ and $q(y)$ such that

a. $(v(y), q(y)) \in W(b'(y))$ for all $y \in Y$

b. For all $y \in Y$, the policy $(d(y), b'(y))$ solves the problem

$$V^{v(\cdot),q(\cdot)}(b,y) = \max_{\hat{d} \in \{0,1\}, \hat{b} \geq 0} (1 - \hat{d}) \left\{ u\left[y - b + q(y)\hat{b}\right] + \beta v(y) \right\} + \hat{d} \left\{ u(y) + \beta \underline{\mathbb{V}}^d \right\}$$

We will refer to the pair $(v(\cdot), q(\cdot))$ as the enforcing values of policy $(d(y), b'(y))$ and we will write $(d(\cdot), b'(\cdot)) \in \mathbf{E}(W)(b)$.

**Definition B.2.** Given a correspondence $W : \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$, we define the generating correspondence $B(W) : \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$ as

$$B(W)(b) = \left\{ (v,q) \in \mathbb{R}^2 : \exists\, (d(\cdot), b'(\cdot)) \in \mathbf{E}(W)(b) : \begin{cases} v = \mathbb{E}\left\{ V^{v(\cdot),q(\cdot)}(b,y) \right\} \\ q = \frac{1}{1+r}(1 - \int d(y)) \end{cases} \right\}$$

**Definition B.3.** A correspondence $W(\cdot)$ is self-generating if for all $b \geq 0$ we have $W(b) \subseteq B(W)(b)$

**Theorem B.1.** *Any bounded, self-generating correspondence gives equilibrium values: i.e. if $W(b) \subseteq B(W)(b)$ for all $b \geq 0$, then $W(b) \subseteq \mathcal{E}(b)$*

**Proof.** The proof follows Abreu et al. (1990) and is constructive; we provide a sketch of the argument. Take any pair $(v_{-1}, q_{-1}) \in W(b)$. We need to construct a subgame perfect equilibrium strategy profile $\sigma \in \mathbf{SPE}(b)$. Since $W(b) \subseteq B(W)(b)$ we know we can find functions $(d_0(y_0), b_1(y_0))$ and values $(v_0(y_0), q_0(y_0)) \in W(b)$ for any $b \geq 0$ such that $(d_0(y_0), b_1(y_0))$ is in the argmax of $V^{v_0(\cdot),q_0(\cdot)}(\cdot)$ and

$$v_{-1} = \mathbb{E}_0 \left\{ V^{v_0(\cdot),q_0(\cdot)}(y,b) \right\}$$

and

$$q_{-1} = \frac{1}{1+r} \left\{ 1 - \int d_0(y_0)\, dF(y_0) \right\}$$

Define

$$\sigma_g \left( h^0 \right) = (d_0 \left( y_0 \right), b_1 \left( y_0 \right))$$

and

$$\sigma_m \left( h^0_- \right) = q_0$$

where $h^0_- = (b_0, q_{-1})$. Because $(v_0 \left( y_0 \right), q_0 \left( y_0 \right)) \in W \left( b_1(y_0) \right)$ and $W$ is self-generating, we know that for any realization of $y_0$, we can find policy functions $(d_1 \left( y_1 \right), b_2 \left( y_1 \right))$ and values $(v_1 \left( y_1 \right), q_1 \left( y_1, b_2 \left( y_1 \right) \right)) \in B \left( W \right) \left( b_2 \left( y_1 \right) \right)$ such that $(d_1 \left( y_1 \right), b_2 \left( y_1 \right))$ is in the argmax of $V^{v_1(\cdot), q_1(\cdot)} \left( \cdot \right)$ and

$$v_0 \left( y_0 \right) = \mathbb{E} \left( V^{v_1(\cdot), q_1(\cdot)} \left( \cdot \right) \right),$$

$$\sigma_m \left( h^1_- \right) = q_1 \left( y_1, b_2 \right) = \frac{1}{1+r} \left( 1 - \int d_1 \left( y_1 \right) \right)$$

Also define

$$\sigma_g \left( h^2_- \right) = (d_1 \left( y_1 \right), b_2 \left( y_1 \right))$$

is clear to see that strategy profiles $\sigma_m$ and $\sigma_g$ defined for all histories of type $h^1_0$ and $h^2_0$ satisfy the first constraints of being a subgame perfect equilibrium. Doing it recursively for all finite $t$, we can then prove by induction (same as APS original proof) that this profile forms a SPE with initial values $(v_0, q_0)$ as we stated. The finiteness of the value function is guaranteed because the set $W$ is bounded. There are no one shot deviations by construction. □

**Theorem B.2.** *The correspondence $\mathcal{E} \left( b \right)$ is the biggest correspondence (in the set order) that is a fixed point of B. That is, $\mathcal{V} \left( \cdot \right)$ satisfies:*

$$B \left( \mathcal{E} \right) \left( b \right) = \mathcal{E} \left( b \right) \tag{B.1}$$

*for all $b \geq 0$, and if another operator $W \left( \cdot \right)$ also satisfies condition B.1, then $W \left( b \right) \subseteq \mathcal{E} \left( b \right)$ for all $b \geq 0$.*

**Proof.** Is sufficient to show that $\mathcal{E} \left( b \right)$ is itself self-generating. As in APS, we start with any strategy profile $\sigma = (\sigma_g, \sigma_m)$ and the values associated with it $(v_0, q_0)$ with initial debt $b$. From the definition of SPE, we know that the policy $d_1 \left( y_1 \right) = d^{\sigma_g} \left( h^1, y_1 \right)$ and $b' \left( y_1 \right) = b_2^{\sigma_g} \left( h^1, y_1 \right)$ is implementable with functions $q \left( y_1, \hat{b} \right) = q_m^{\sigma} \left( y_1, d \left( y_1 \right), b' \left( y_1 \right) \right)$ and $v \left( y_1, \hat{b} \right) = V \left( \sigma \mid h^2 \left( y_1, \hat{b} \right) \right)$, where $h^2 \left( y_1, \hat{b} \right) \equiv \left( h^1, y_1, d_1 \left( y_1 \right), b' \left( y_1 \right), q \left( y_1, \hat{b} \right) \right)$. Moreover, because $\sigma$ is an SPE strategy profile, it means it also is a subgame perfect equi-

libria for the continuation game starting with initial bonds $b = \hat{b}$, and hence

$$\left(v\left(y_1, \hat{b}\right), q\left(y_1, \hat{b}\right)\right) \in \mathcal{V}\left(\hat{b}\right).$$

This then means that $(v_0, q_0) \in B\left(\mathcal{V}\right)(b)$, and hence $\mathcal{V}\left(\cdot\right)$ is a self-generating correspondence. $\qquad\square$

**Bang Bang Property**   Now we are going to relate the APS characterization with the characterization in the main text .First, notice that the singleton set $\{(v,q)\} = \{(\underline{\mathbb{V}}^{aut}, 0)\}$ (corresponding to the autarky subgame perfect equilibria) is itself self-generating, and hence an equilibrium value. Let $(v, q) = \left(\overline{\mathbb{V}}(b), \overline{\mathbf{q}}(b)\right)$ denote the expected utility and debt price associated with the best equilibrium.

**Proposition B.1.** *Let $(d\left(\cdot\right).b'\left(\cdot\right))$ be an enforceable policy on $\mathcal{V}(b)$ (i.e. they are part of a subgame perfect equilibrium). Then, it can be enforced by the following continuation value functions:*

$$v\left(y, \hat{d}\right) = \begin{cases} \overline{\mathbb{V}}\left(b'\left(y\right)\right) & \text{if } d\left(y\right) = 0 \text{ and } \hat{d} = b'\left(y\right) \\ \underline{\mathbb{V}}^d & \text{otherwise} \end{cases} \tag{B.2}$$

*and*

$$q\left(y, \hat{d}\right) = \begin{cases} \overline{\mathbf{q}}\left(b'\left(y\right)\right) & \text{if } d\left(y\right) = 1 \text{ and } \hat{d} = b'\left(y\right) \\ 0 & \text{otherwise} \end{cases} \tag{B.3}$$

**Proof.** Notice that the functions $v\left(\cdot\right), q\left(\cdot\right)$ satisfy the restriction $\left(v\left(y, \hat{d}\right), q\left(y, \hat{d}\right)\right) \in \mathcal{E}\left(\hat{d}\right)$ for all $\hat{b}$. Since $(d\left(\cdot\right), b'\left(\cdot\right))$ are enforceable, there exist functions $(\hat{v}\left(\cdot\right), \hat{q}\left(\cdot\right))$ such that for all $y : d\left(y\right) = 0$ we have

$$u\left[y - b + \hat{q}\left(y, b'\left(y\right)\right) b'\left(y\right)\right] + \beta \hat{v}\left(y, b'\left(y\right)\right) \geq u\left[y - b + \hat{q}\left(y, \hat{b}\right)\hat{b}\right] + \beta \hat{v}\left(y, \hat{b}\right) \tag{B.4}$$

for all $\hat{b} \geq 0$. Now, because the left hand side argument is an equilibrium value (since it is generated by an equilibrium policy), its value must be less than the best equilibrium value for the government, characterized by $q = \overline{\mathbf{q}}\left(b'\left(y\right)\right)$ and $v = \overline{\mathbb{V}}^{nd}\left(b'\left(y\right)\right)$ (that is, the best equilibrium from tomorrow on, starting at a debt value of $\hat{b} = b'\left(y\right)$. This means that

$$V^{nd}\left(b, y, b'\left(y\right)\right) \equiv u\left[y - b + \overline{\mathbf{q}}\left(y, b'\left(y\right)\right) b'\left(y\right)\right] + \beta \overline{\mathbb{V}}\left(b'\left(y\right)\right) \geq$$

$$\geq u\left[y - b + \hat{q}\left(y, b'\left(y\right)\right) b'\left(y\right)\right] + \beta \hat{v}\left(y, b'\left(y\right)\right) \tag{B.5}$$

On the other side, we also have that autarky is the worst equilibrium value (since it coincides with the min-max payoff) which implies

$$u\left[y - b + \hat{q}\left(y, \hat{b}\right)\hat{b}\right] + \beta\hat{v}\left(y, \hat{b}\right) \geq u\left(y\right) + \beta\underline{V}^d \text{ for all } \hat{b} \geq 0 \qquad \text{(B.6)}$$

Combining B.4 with the inequalities given in B.5 and B.6 we get

$$u\left[y - b + \overline{\mathbf{q}}\left(y, b'\left(y\right)\right)b'\left(y\right)\right] + \beta\overline{V}\left(b'\left(y\right)\right) \geq u\left(y\right) + \beta\underline{V}^d \qquad \text{(B.7)}$$

which is the enforceability constraint (conditional on not defaulting) of the proposed functions $(v, q)$ in equations B.2 and B.3. To finish the proof, we need to show that if it is indeed optimal to choose $d\left(y\right) = 0$ under the functions $(\hat{v}\left(\cdot\right), \hat{q}\left(\cdot\right))$, then it will also be so under functions $(v\left(\cdot\right), q\left(\cdot\right))$. This is readily given by condition B.7, since punishment of defaulting coincides with the value of deviating from bond issue rule $\hat{b} = b'\left(y\right)$. Hence, $(v\left(\cdot\right), q\left(\cdot\right))$ also enforce $(d\left(\cdot\right), b'\left(\cdot\right))$. $\qquad \square$

This proposition greatly simplifies the characterization of implementable policies. Remember the definitions of the objects

$$V^{nd}\left(b, y, b'\right) \equiv u\left(y - b + \overline{\mathbf{q}}\left(b'\right)b'\right) + \beta\overline{V}\left(b'\right)$$

as the expected lifetime utility under the best continuation equilibrium for any choice of debt $b'$, and

$$V^d\left(y\right) \equiv u\left(y\right) + \beta\underline{V}^d$$

as the expected lifetime utility of autarky.

**Corollary B.1.** *A policy $(d\left(\cdot\right), b'\left(\cdot\right))$ is enforceable on $\mathcal{E}\left(b\right)$ if and only if $d\left(y\right) = 0$ implies*

$$V^{nd}\left(b, y, b'\left(y\right)\right) \geq V^d\left(y\right)$$

# C   Computing $\overline{v}\left(b, q\right)$

The function $\overline{v}\left(b, q\right)$ gives the highest expected utility that a government can obtain if they raised debt at price $q$ and issued $b$ bonds[28]. This is the Pareto frontier in the set of equilibrium values. We now discuss how we compute $\overline{v}\left(b, q\right)$, which can be redefined

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[28]Because this is the best equilibrium given a price $\hat{q}$ it does not depend on the amount of debt repaid; we are not characterizing equilibrium consistent outcomes.

using the equilibrium correspondence:

$$\overline{v}(b,q) := \max\{v : \exists \hat{q} \geq 0 \text{ such that } (v,\hat{q}) \in \mathcal{E}(b) \text{ and } \hat{q} \leq q\} \tag{C.1}$$

Note that we focus in a relaxes version, where we replace the equality $\hat{q} = q$ by the inequality $\hat{q} \leq q$. We will show that this constraint is binding. The proof of Proposition ?? follows from the next three Lemmas.

**Lemma C.1 (Characterization of $\overline{v}$).** *For all $q \in [0, \overline{\mathbf{q}}(b))$ the maximum continuation value $\overline{v}(b,q)$ solves*

$$\overline{v}(b,q) = \max_{\delta(\cdot)\in[0,1]^Y} \int \left\{ \delta(y) V^d(y) + [1 - \delta(y)] \overline{V}^{nd}(b,y) \right\} dF(y) \tag{C.2}$$

*subject to*

$$q \geq \frac{1}{1+r}\left(1 - \int \delta(y) dF(y)\right) \tag{C.3}$$

*where the constraint C.3 is always binding for all $q > 0$.*

**Proof.** Take an enforceable policy $(\delta(\cdot), b'(\cdot))$ such that $\frac{1}{1+r}(1 - \int \delta(y) dF(y)) = q$. By definition, there must exist functions $(\hat{v}(y,b'), \hat{q}(y,b')) \in \mathcal{E}(b')$ such that for all $y$

$$(\delta(y), b'(y)) \in \underset{(\delta,b')}{\operatorname{argmax}} \; \delta V^d(y) + (1-\delta)\left\{u\left[y - b + \hat{q}(y,b')b'\right] + \beta\hat{v}(y,b')\right\}$$

with the right hand side value (at the optimum) being the ex ante value of the policy. We show in Proposition B.1 that (1) any enforceable policy can also be enforced by the "bang-bang values"

$$\hat{v}(y,b') = \begin{cases} \overline{V}(b'(y)) & \text{if } b' = b'(y) \\ \underline{V}^d & \text{otherwise} \end{cases} \quad \text{and} \quad \hat{q}(y,b') = \begin{cases} \overline{\mathbf{q}}(b'(y)) & \text{if } b' = b'(y) \\ 0 & \text{otherwise} \end{cases}$$

and (2) the continuation value is maximized at this values, since

$$\delta(y) V^d(y) + [1 - \delta(y)]\left\{u\left[y - b + \hat{q}(y,b'(y))b'(y)\right] + \beta\hat{v}(y,b'(y))\right\} \leq$$

$$\delta(y) V^d(y) + [1 - \delta(y)]\left\{u\left[y - b + \overline{\mathbf{q}}(b'(y))b'(y)\right] + \beta\overline{V}[b'(y)]\right\} =$$

$$\underbrace{=}_{\text{by def.}} \delta(y) V^d(y) + [1 - \delta(y)] V^{nd}(b,y,b'(y)) \tag{C.4}$$

Therefore, an enforceable policy $(\delta(\cdot), b'(\cdot))$ policy can generate (conditional on $y$) a

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value given by equation C.4. Therefore, we can write the problem of finding the biggest continuation value consistent with a default price less than $q$ as

$$\bar{v}(b,q) = \max_{(\delta(\cdot),b'(\cdot))} \int \left\{ \delta(y) V^d(y) + [1 - \delta(y)] V^{nd}(b,y,b'(y)) \right\} dF(y)$$

subject to the incentive constraint:

$$V^{nd}(b,y,b'(y)) \geq V^d(y) \text{ for all } y : \delta(y) = 0$$

and that its associated price is less than $q$:

$$\frac{1}{1+r} \left( 1 - \int \delta(y) dF(y) \right) \leq q$$

Finally, notice that $b'(y)$ only enters the problem through the term $V^{nd}(b,y,b'(y))$, and that making this object as large as possible makes both (1) the objective function bigger and (2) the constraints less binding (since it only enters through the incentive compatibility constraint). Therefore, we choose $b'(y)$ to solve

$$\overline{V}^{nd}(b,y) = \max_{b' \geq 0} V^{nd}(b,y,b'(y))$$

showing then the desired result. Finally, note that $\bar{v}(b,q)$ is weakly increasing in $q$, and that if we remove the price constraint, then the agent would choose the default rule to get price $\overline{q}(b)$ (the one associated with the best equilibrium), so for $q < \overline{q}(b)$ this constraint must be binding. □

*Remark* C.1. See that this is a linear programming problem in $\delta(\cdot)$, which we will see is easy to solve. If tractable, this Lemma will help us mapping the boundaries of the equilibrium correspondence $\mathcal{E}(b)$ for any given $q$.

The following proposition solves the programming problem shown in Lemma C.1, reducing it to solving a problem in one equation in one unknown.

**Lemma C.2.** *Given* $(b,q)$ *there exist a constant* $\gamma = \gamma(b,q)$ *such that*

$$\bar{v}(b,q) = \int \left[ \hat{\delta}(y) V^d(y) + (1 - \hat{\delta}(y)) \overline{V}^{nd}(b,y) \right] dF(y)$$

*where*

$$\hat{\delta}(y) = 0 \iff \overline{V}^{nd}(b,y) \geq V^d(y) + \gamma \text{ for all } y \in Y$$

*and $\gamma$ is the (maximum) solution to the single variable equation:*

$$\frac{1}{1+r} \Pr \left\{ y \ : \ \overline{V}^{nd}(b,y) \geq V^d(y) + \gamma \right\} = q$$

*Moreover, $\gamma$ is also the Lagrange multiplier of constraint C.3 in program C.4, so that $\frac{\partial \overline{v}(b,q)}{\partial q} = \gamma(b,q)$.*

**Proof.** Using the Lagrangian in the relaxed program of letting $\delta(y) \in [0,1]$ for all output levels for which no-default is feasible; i.e. for all $y \in D(b) \equiv \left\{ y : \overline{V}^{nd}(b,y) \geq V^d(y) \right\}$. The Lagrangian (without the corner conditions for $\delta$) is

$$\mathcal{L} = \int \left[ \delta(y) V^d(y) + (1 - \delta(y)) \overline{V}^{nd}(b,y) \right] dF(y) +$$

$$+ \int \mu(y) [1 - \delta(y)] \left[ \overline{V}^{nd}(b,y) - V^d(y) \right] dF(y) +$$

$$+ \lambda \left( q(1+r) - 1 + \int \delta(y) dF(y) \right)$$

so that at a $y : \overline{V}^{nd}(y) > V^d(y)$

$$\frac{\partial \mathcal{L}}{\partial [\delta(y)]} = \left[ -\overline{V}^{nd}(b,y) + V^d(y) + \lambda \right] dF(y) \Longrightarrow \hat{\delta}(y) = \begin{cases} 0 & \text{if } \overline{V}^{nd}(b,y) \geq V^d(y) + \lambda \\ 1 & \text{otherwise} \end{cases}$$

Defining $\gamma \equiv \lambda$ we get the desired result, using the binding property of constraint for prices. $\qquad\square$

**Lemma C.3 (Concavity of $\overline{v}$).** *The function $\overline{v}(b,q) = \max \{v : \exists \hat{q} \leq q \text{ such that } (v,\hat{q}) \in \mathcal{E}(b)\}$ is concave in $q$.*

**Proof.** From Lemma C.1 we know that the feasible set of the program in that Lemma is convex, having a linear objective function and an affine restriction. Take $q_0, q_1 \in [0, \overline{q}(b)]$ and $\lambda \in [0,1]$. We need to show that

$$\overline{v}(b, \lambda q_0 + (1 - \lambda) q_1) \geq \lambda \overline{v}(b, q_0) + (1 - \lambda) \overline{v}(b, q_1)$$

Let $G[\delta(\cdot)] = \int \left[ \delta(y) V^d(y) + (1 - \delta(y)) \overline{V}^{nd}(b,y) \right] dF(y)$ be the objective function of the maximization in C.2. Let $\delta_0(y)$ be one of the solutions for the program when $q = q_0$, and likewise $\delta_1(y)$ be one of the solutions of the relaxed program when $q = q_1$. Define

$$\delta_\lambda(y) = \lambda \delta_0(y) + (1 - \lambda) \delta_1(y)$$

Clearly this is not a feasible default policy as it is, since $\delta_\lambda$ may be in $(0,1)$, but it is feasible in the relaxed program of Lemma C.1. Note that it is feasible when $q = q_\lambda :=$ $\lambda q_0 + (1-\lambda) q_1$, since

$$\frac{1}{1+r} \left( 1 - \int \delta_\lambda (y) \, dF (y) \right) = \lambda \frac{1}{1+r} \left( 1 - \int \delta_0 (y) \, dF (y) \right) + ..$$

$$+ (1-\lambda) \frac{1}{1+r} \left( 1 - \int \delta_0 (y) \, dF (y) \right) \leq \lambda q_0 + (1-\lambda) q_1 = q_\lambda$$

Therefore, the optimal continuation value at $q = q_\lambda$ must be greater than the objective function evaluated at $\delta_\lambda$. The reason is that the optimum will be at a corner even in the relaxed problem. Then

$$\bar{v} (b, q_\lambda) \geq G [\delta_\lambda (\cdot)] \underbrace{=}_{(a)} \lambda G [\delta_0 (\cdot)] + (1-\lambda) G [\delta_1 (\cdot)] \underbrace{=}_{(b)} \lambda \bar{v} (b, q_0) + (1-\lambda) \bar{v} (b, q_1)$$

using in $(a)$ the fact that $G [\delta (\cdot)]$ is an affine function in $\delta (\cdot)$ and in $(b)$ the fact that both $\delta_0 (\cdot)$ and $\delta_1 (\cdot)$ are the optimizers at $q_0$ and $q_1$ respectively. This concludes the proof. $\square$

# D  Sunspot Proofs

**Proof of Proposition ??. Necessity** $(\Longrightarrow)$: Suppose there is an equilibrium strategy $\sigma$ such that $h \in \mathcal{H} (\sigma)$. This implies that the government decided optimally not to default at period $t$; i.e.

$$\int_0^1 [u (y_t - b_t + q^\sigma (h, \zeta) b_{t+1}) + \beta V^\sigma (h, \zeta)] \, d\zeta \geq u (y_t) + \beta \underline{V}^d \tag{D.1}$$

Since $\sigma$ is a SPE, we have that for all sunspot realizations $\zeta \in [0, 1]$ we must have

$$(V^\sigma (h, \zeta), q^\sigma (h, \zeta)) \in \mathcal{E} (b_{t+1})$$

using the self-generation characterization of $\mathcal{E} (b)$. This further implies two things:

a. $q^\sigma (h, \zeta) \in [0, \bar{q} (b_{t+1})]$ (i.e. it delivers equilibrium prices)

b. $V^\sigma (h, \zeta) \leq \bar{v} (b_{t+1}, q^\sigma (h, \zeta))$ (because $\bar{v}$ is the maximum possible continuation value with price $q = q^\sigma (h, \zeta)$ )

The price distribution given by $\sigma$ can be defined by a measure $P$ over measurable sets $A \subseteq \mathbb{R}_+$ as

$$P(A) = \int_0^1 \mathbf{1}\{q^\sigma(h,\zeta) \in A\}\, d\zeta = \Pr\{\zeta : q^\sigma(h,\zeta) \in A\}$$

Note that numeral (1) shows that $\mathbf{Supp}(P) \subseteq [0, \overline{\mathbf{q}}(b_{t+1})]$. To show **??**, we change integration variables in D.1 and using the definitions above and properties (1) and (2), we get

$$\int \left[u(y_t - b_t + \hat{q}b_{t+1}) + \beta\overline{v}(b_{t+1},\hat{q})\right] dP(\hat{q}) \geq$$

$$\int_0^1 \left[u(y_t - b_t + q^\sigma(h,\zeta)b_{t+1}) + \beta V^\sigma(h,\zeta)\right] d\zeta \geq u(y_t) + \beta\underline{V}^d$$

Proving the desired result.

**Sufficiency** ($\Longleftarrow$). Suppose that $P$ is an equilibrium consistent distribution with cdf $F_P$. Let

$$\sigma^*(b,q) \in \underset{\sigma \in SPE(b_{t+1})}{\mathrm{argmax}}\; V^\sigma\left(h^0\right) \text{ s.t. } q_0^\sigma \leq q$$

i.e. it is a strategy that achieves the continuation value $\overline{v}(b,q)$. As we showed before, the constraint in this problem will be binding. Because $h^t$ is equilibrium consistent, we know there exist an equilibrium profile $\hat{\sigma}$ such that $h \in \mathcal{H}(\hat{\sigma})$. For histories $h'$ successors of histories $h^{t+1} = \left(h^t, d_t, \hat{b}_{t+1}, \zeta_t, \hat{q}_t\right)$ we define the profile $\sigma$ as

$$\sigma(h') = \begin{cases} \sigma^d(h') & \text{if } d_t = 1, \hat{b}_{t+1} \neq b_{t+1} \text{ or } \hat{q}_t \notin [0, \overline{\mathbf{q}}(b_{t+1})] \\ \sigma^*(b_{t+1}, \hat{q}_t)(h' \sim h^{t+1}) & \text{otherwise} \end{cases}$$

and for histories $h' = \left(h^t, d_t = 0, b_{t+1}, \zeta_t\right)$ let

$$q^\sigma\left(h^t, d_t, b_{t+1}, \zeta_t\right) = F_P^{-1}(\zeta_t)$$

where $F_P(q) = P(\hat{q})$ is the cumulative distribution function of distribution $P$ and $F_P^{-1}(\zeta) = \inf\{x \in \mathbb{R} : F_P(q) \geq \zeta\}$ its inverse. It will be optimal to not default at $t$ (if we follow strategy $\sigma$ for all successor nodes) if

$$\int_0^1 \left[u\left(y_t - b_t + F_P^{-1}(\zeta)b_{t+1}\right) + \beta V^\sigma(b_{t+1},\zeta)\right] d\zeta \geq u(y_t) + \beta\underline{V}^d \underset{(\mathbf{a})}{\Longleftrightarrow}$$

$$\int \left[ u \left( y_t - b_t + \hat{q} b_{t+1} \right) + \beta \bar{v} \left( b_{t+1}, \hat{q} \right) \right] dP \left( \hat{q} \right) \geq u \left( y_t \right) + \beta \underline{V}^d \qquad \text{(D.2)}$$

using the classical result that $F_P^{-1} \left( \zeta \right) =_d P$ if $\zeta \sim$ Uniform $[0,1]$ and the fact that $V^\sigma \left( h' \right) = V \left( \sigma^* \left( h' \right) \right) = \bar{v} \left( b_{t+1}, q_t \right)$ from the definition of $\sigma$. Conditions D.1 is satisfied, and **Supp** $\left( P \right) \subseteq [0, \bar{q} \left( b_{t+1} \right)]$ imply that, if the government follows profile $\sigma$, then $h$ is also on the path of $\sigma$, and $\sigma$ is indeed a Nash equilibrium at such histories (because both $\sigma^d$ and $\sigma^* \left( b_{t+1}, \hat{q} \right)$ are subgame perfect profiles). Finally, for histories $h' \not\succ h^t$ define $\sigma \left( h' \right) = \hat{\sigma} \left( h' \right)$. Therefore, $\sigma \left( h' \right)$ is itself a subgame perfect equilibrium profile (since it is a Nash equilibrium at every possible history) and generates $h = \left( h^t, d_t = 0, b_{t+1} \right)$ on its path. $\qquad \square$

**Proof of Corollary ??.** This comes from the fact that the function

$$U \left( P \right) = \int \left\{ u \left( y_t - b_t + \hat{q} b_{t+1} \right) + \beta \bar{v} \left( b_{t+1}, \hat{q} \right) \right\} dP \left( q \right)$$

is strictly increasing in $y_t$ and strictly decreasing in $b_t$, and the set can be rewritten as

$$Q \left( b_t, y_t, b_{t+1} \right) = \left\{ P \in \Delta \left( [0, \bar{q}] \right) : U \left( P \right) \geq V^d \left( y_t \right) \right\}$$

$\qquad \square$

**Proof of Corollary 6.2.** The function $H \left( q \right) := u \left( y_t - b_t + q b_{t+1} \right) + \beta \bar{v} \left( b_{t+1}, q \right)$ is strictly increasing in $q$. Therefore, if $P' \trianglerighteq P$ and $P \in Q \left( b_t, y_t, b_{t+1} \right)$ then $\int H \left( q \right) dP' \geq \int H \left( q \right) dP \geq V^d \left( y_t \right)$. Using Proposition **??** together with assumption (1) gives the result. $\qquad \square$

It also has a greatest element,

$$\bar{P} \left( q \in A \right) = \begin{cases} 1 & \text{if } \bar{q} \left( b_{t+1} \right) \in A \\ 0 & \text{otherwise} \end{cases}$$

i.e. $\bar{P}$ is the Dirac measure over the best price $q = \bar{q} \left( b_{t+1} \right)$. It also has an infimum, with respect to the first order stochastic dominance, given by the Lebesgue-stjeljes measure associated with the cdf $\underline{P} \left( \cdot \right)$ we characterize in section 3 below. However, this infimum distribution is not an equilibrium distribution.

**Proof of Proposition 6.3.** We already know that $\max E \left( b_t, y_t, b_{t+1} \right) = \bar{q} \left( b_{t+1} \right)$ since the Dirac distribution $\bar{P}$ over $q = \bar{q} \left( b_{t+1} \right)$ is equilibrium feasible. In the same way, we also know that the Dirac distribution $\hat{P}$ that puts probability 1 to $q = \underline{q} \left( b_t, y_t, b_{t+1} \right)$ is also equilibrium consistent; it corresponds to a case where both investors and the government

ignore the realization of the correlated device, and the characterization of $\underline{q}(\cdot)$ is exactly the only price that satisfies

$$u\left(y_t - b_t + \underline{q}(b_t, y_t, b_{t+1}) \ b_{t+1}\right) + \beta \bar{v}\left(b_{t+1}, \underline{q}(b_t, y_t, b_{t+1})\right) = V^d(y_t)$$

and hence satisfies the conditions of Proposition **??**. Lemma C.3 shows that $\bar{v}(b, q)$ is a concave function in $q$, which together with the fact that $u$ is strictly concave and $b' > 0$ implies that the function

$$H(q) := u(y_t - b_t + qb_{t+1}) + \beta \bar{v}(b_{t+1}, q)$$

is strictly concave in $q$. For any distribution $P \in \mathcal{Q}(b_t, y_t, b_{t+1})$, let $\mathbb{E}_P(q) = \int \hat{q} dP(\hat{q})$. Jensen's inequality then implies that

$$u(y_t - b_t + \mathbb{E}_P(q)b_{t+1}) + \beta \bar{v}(b_{t+1}, \mathbb{E}_P(q)) \underbrace{\geq}_{(1)} \int [u(y_t - b_t + \hat{q}b_{t+1}) + \beta \bar{v}(b_{t+1}, \hat{q})] dP(\hat{q}) \geq$$

$$\underbrace{\geq}_{(2)} V^d(y_t)$$

with strict inequality in $(1)$ if $P$ is not a Dirac distribution. Then, the definition of $\underline{q}(b_t, y_t, b_{t+1})$ implies that for any distribution $P \in \mathcal{Q}(b_t, y_t, b_{t+1})$ we have

$$\mathbb{E}_P(q) \geq \underline{q}(b_t, y_t, b_{t+1})$$

and therefore the minimum expected value is exactly $\underline{q}(b_t, y_t, b_{t+1})$, which is achieved uniquely at the Dirac distribution $\hat{P}$ (because of strict concavity of $u(\cdot)$). Finally, knowing that $E$ is naturally a convex set, we then get that

$$E(b_t, y_t, b_{t+1}) = \left[\min_{P \in \mathcal{Q}(b_t, y_t, b_{t+1})} \int \hat{q} dP(\hat{q}), \max_{P \in \mathcal{Q}(b_t, y_t, b_{t+1})} \int \hat{q} dP(\hat{q})\right]$$

$$= \left[\underline{q}(b_t, y_t, b_{t+1}), \bar{q}(b_t, y_t, b_{t+1})\right]$$

as we wanted to show. □

**Proof of Proposition 6.4.** Upper bound for general $\hat{q} < \underline{q}(b_t, y_t, b_{t+1})$ Here we replicate the same strategy: let $p = \Pr(\zeta : q(\zeta) \leq \hat{q})$. Using the same strategy as before, to get the less binding incentive compatibility constraint for the government we need to maximize equilibrium utility for $\zeta : q(\zeta) \leq \hat{q}$ for $\zeta : q(\zeta) > \hat{q}$, we consider equilibria that assign

the best continuation equilibria (to make the incentive constraint of the government as flexible as possible).

For (2) we just follow the same thing we did for the case where $\hat{q} = 0$ and consider the continuation equilibria where $q(\zeta) = \overline{\mathbf{q}}(b_{t+1})$ and $v(\zeta) = \overline{\mathbf{V}}(b_{t+1})$ (the fact that this corresponds to an actual equilibria is easy to check). For (1), we see that focusing on equilibria that have support $q(\zeta) \in \{\hat{q}, \overline{\mathbf{q}}(b_{t+1})\}$ make the government incentive constraint as flexible as possible, since utility of the government is increasing in $\hat{q}$ and moreover, $\overline{v}(b, \hat{q})$ (the biggest continuation utility consistent with $q \leq \hat{q}$) is also increasing in $\hat{q}$ as we saw before. Therefore, if $p$ is the maximum such probability, we must have

$$p\left[u\left(y_t - b_t + \hat{q}b_{t+1}\right) + \beta\overline{v}\left(b_t, \hat{q}\right)\right] + (1-p)V^{nd}\left(b_t, y_t, b_{t+1}\right) \geq V^d\left(y_t\right) \iff$$

$$p \leq \frac{\Delta^{nd}\left(b_t, y_t, b_{t+1}\right)}{V^d\left(y_t\right) - \left[u\left(y_t - b_t + \hat{q}b_{t+1}\right) + \beta\overline{v}\left(b_t, \hat{q}\right)\right] + \Delta^{nd}\left(b_t, y_t, b_{t+1}\right)}$$

See that this is not an innocuous constraint only when the right hand side is less than 1. This happens only when

$$u\left(y_t - b_t + \hat{q}b_{t+1}\right) + \beta\overline{v}\left(b_t, \hat{q}\right) \geq V^d\left(y_t\right)$$

As we argued

$$\hat{q} \geq \underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)$$

where the last inequality comes from the characterization of $\underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)$. $\qquad\square$

**Proof of Corollary 6.3.** $\underline{P}$ as defined in equation 6.6 cannot be an equilibrium consistent price: this implies that the Lebesgue-stjeljes measure associated with $\overline{P}(\cdot)$ has the property that $\mathbf{Supp}(\underline{P}) = \left[0, \underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)\right]$ and $\underline{P}(q=0) = p_0 > 0$, which implies that

$$\int \left\{u\left(y_t - b_t + \hat{q}b_{t+1}\right) + \beta\overline{v}\left(b_{t+1}, \hat{q}\right)\right\} d\underline{P}\left(\hat{q}\right) \quad < \quad u\left(y_t - b_t + \underline{\mathbf{q}}\left(\cdot\right)b_{t+1}\right) + \beta\overline{v}\left(b_{t+1}, \underline{\mathbf{q}}\left(\cdot\right)\right)$$

$$= \quad V^d\left(y_t\right)$$

where the last equation comes from the definition of $\underline{\mathbf{q}}(\cdot)$ and the function $H(\hat{q}) \equiv u\left(y_t - b_t + \hat{q}b_{t+1}\right) + \beta\overline{v}\left(b_{t+1}, \hat{q}\right)$ is strictly increasing in $\hat{q}$. $\qquad\square$

# E    A connection to Robust Bayesian Analysis

We make a formal connection between equilibrium consistent outcomes and Robust Bayesian analysis. The main result is that, if the econometrician assumes the data generating process stems from a SPE of the game, then the set of equilibrium consistent outcomes essentially comprises the set of predictions a Bayesian econometrician can make, for *any* equilibrium Bayesian model (any prior over equilibrium outcomes).

## E.1    Robust Bayesian Analysis

Based in the principles of Robust Bayesian statistics (see Berger et al. (1994)), we study the inferences that can be drawn from the observed data (a particular history $h$), which are *not* sensitive to the particular modeling assumptions (e.g., prior distribution chosen), across a given class of statistical models. Given that equilibrium multiplicity is a well-known problem of infinite horizon dynamic games, an econometrician must specify not only the physical environment for the economy, but also the equilibrium (or family of equilibria) on which they will focus their attention. Formally, the econometrician will try to draw inferences over:

a. **Fundamental parameters** $\theta \in \Theta$. These are parameters that fully describe the *physical environment of the economy* (examples are: The process for output $F(y_t \mid y_{t-1})$, the utility function $u(c_t)$, discount factor $\beta \in (0,1)$, and international interest rate $r$.

b. **Endogenous parameters** $\alpha \in A$. These are parameters that given a physical description of the economy parametrize the stochastic process for the endogenous variables $\mathbf{x}(\alpha) = (d_t, b_{t+1}, q_t)$. These parameters comes from the equilibrium refinement (single valued or set valued) chosen by the econometrician.

For example, in the Eaton and Gersovitz (1981) setting it amounts to the following. The process for income $\{y_t\}_{t\in\mathbb{N}}$ can be an AR(1) process

$$\log y_t = \rho \log y_{t-1} + \epsilon_t$$

where $y_t$ is output, and $\epsilon_t \sim_{i.i.d} N(0, \sigma_\epsilon^2)$. The utility function is $u(c) = c^{1-\gamma}/(1-\gamma)$ with $\gamma > 0$. Hence, the fundamental parameters in this economy are

$$\theta := \left(\rho, \sigma_\epsilon^2, \gamma, \beta, r^*\right)$$

The econometrician assumes that agents behave according to a particular rule, that relates

exogenous variables with endogenous variables. The literature of sovereign debt focuses in the best perfect Markov equilibrium (with the restriction that after default there is a period in autarky). A special case is the equilibrium we covered in Section 2. So, $\mathbf{x}_\theta(\alpha) =$ best Markov equilibrium.

**Bayesian vs Frequentist.** In a *frequentist approach*, parameters $(\theta, \alpha)$ are estimated (by calibration or some other statistical procedure) to best fit the observed historic data. In this section we will focus on the *Bayesian approach* where the econometrician (or outside observer) has a prior distribution for the parameters $(\theta, \alpha)$ and given data obtains a posterior. Our aim is to study inferences of Bayesian statistical models that hold any prior with support over equilibrium outcomes.

**Definition E.1.** A **conditional model** $(m_\theta)_{\theta \in \Theta}$ is a family of triples

$$m_\theta = \{\mathcal{A}_\theta, (\alpha \to \mathbf{x}_\theta(\alpha) \in X), Q_\theta \in \Delta(A_\theta)\}$$

where $\mathcal{A}_\theta = (A_\theta, \Sigma_\theta)$ is the (measurable) space of *process parameters* $\alpha \in A_\theta$; $\alpha \to \mathbf{x}_\theta(\alpha)$ is the mapping that assigns to every parameter $\alpha$ a particular stochastic process $\mathbf{x}_\theta(\alpha)$ for the variables $(d_t, b_{t+1}, q_t)_{t \in \mathbb{N}}$ given an exogenous process for $y_t$; and $Q_\theta \in \Delta(A_\theta)$ is the $\Sigma_\theta$−measurable prior over $\alpha \in A_\theta$. w.l.o.g. we restrict attention to models where $Q_\theta$ is a full-support probability measure; i.e., $\text{supp}(Q_\theta) = A_\theta$.[29] A conditional model $m_\theta$ is **parametric** if $A_\theta \subseteq \mathbb{R}^{k_m}$ (i.e., it has a finite-dimensional parameter space).

**Definition E.2.** A *Bayesian model* (or *specification*) is a pair

$$m = \{(m_\theta)_{\theta \in \Theta}, p(\theta)\}$$

where $(m_\theta)_{\theta \in \Theta}$ is a conditional model and $p(\theta) \in \Delta(\Theta)$ is a prior over fundamental parameters.

For the rest of this section, we will study Bayesian models conditional on a known fundamental parameter $\theta$, fixing the physical environment (and hence dropping the dependence on $\theta$). Once we condition on a particular value of the fundamental parameters, there is only uncertainty about the process followed by endogenous variables $x$. Given $\theta$, one can map a particular model solely in terms of the probability distribution it implies over outcomes. Namely, given a conditional model $m = \{\mathcal{A}, \alpha \to \mathbf{x}(\alpha), Q \in \Delta(A)\}$, we

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[29]If this is not the case, the econometrician can work with an equivalent model, setting the parameter space to $\hat{A}_\theta = \text{supp}(Q_\theta)$.

can define the implied measure over outcomes as

$$\mathbf{Q}_m \left( B \subseteq \mathcal{X} \right) = Q \left\{ \alpha \in A : \mathbf{x} \left( \alpha \right) \in B \right\}$$

We will refer to $\mathbf{Q}_m(\cdot)$ as $m$'s **associated prior**.

**Definition E.3.** $m$ (with associated prior $\mathbf{Q}_m$) is a conditional **equilibrium model** if

$$\mathbf{Q}_m \left( x \in x \left( \mathcal{E}_{|h} \right) \right) = 1$$

; i.e., $m$ assigns probability 1 to the process coming from a subgame perfect equilibrium profile.

The class of conditional equilibrium models is written as $\mathcal{M}_{\mathcal{E}}$. Also, given an equilibrium consistent history $h$, we write

$$\mathcal{M}_{\mathcal{E}} \left( h \right) = \left\{ m : \mathbf{Q}_m \left( x \left( \mathcal{E}_{|h} \right) \right) = 1 \text{ and } \mathbf{Q}_m \left( \mathcal{X} \left( h \right) \right) > 0 \right\}$$

i.e., the family of equilibrium models that assign positive probability to history $h$.[30]

## E.2   Main Result

In the following proposition we will study the inferences a Bayesian econometrician makes conditional on a given fundamental parameter $\theta$. It states the main result of this section, showing that the set of equilibrium consistent outcome paths $x \left( \mathcal{E}_{|h} \right)$ is essentially the union of all paths that have positive probability conditional on the observed history $h$, across all Bayesian equilibrium models.

**Proposition E.1.** *Given an equilibrium consistent history $h \in \mathcal{H} \left( \mathcal{E} \right)$*

  *a. The set of equilibrium consistent outcome paths satisfies:*

$$x \left( \mathcal{E}_{|h} \right) = \{ x \in X : \exists m \in \mathcal{M}_{\mathcal{E}} \left( h \right) \text{ and } \alpha \in supp \left( Q \left( \cdot \mid h \right) \right) \text{ such that } x = \mathbf{x} \left( \alpha \right) \} \tag{E.1}$$

  *b. For any measurable function $T : X \to \mathbb{R}$*

$$\bigcup_{m \in \mathcal{M}_{\mathcal{E}}(h)} \int T \left( \mathbf{x} \left( \alpha \right) \right) dQ \left( \alpha \mid h \right) = ch \, T \left( x \left( \mathcal{E}_{|h} \right) \right) \tag{E.2}$$

---

[30]A more general definition for which the results of the next section hold is to ask that for $\mathcal{X} \left( h \right)$ to be in the **support** of $\mathbf{Q}_m$.

**Restrictions on support.** First, note that (E.1) states that the outcomes that are equilibrium consistent after history $h$ are the outcomes such that, there is a equilibirum conditional model that puts positive support on the parameters that maps into that outcome given the history. So, it formalizes the relation between a conditional equilibrium model and the set of equilibrium consistent outcomes given a history.

**Bounds on statistics.** Second, note that (E.2) can be rewritten it in terms of the associated prior over outcomes $\mathbf{Q}$

$$\bigcup_{m \in \mathcal{M}_{\mathcal{E}}(h)} \int T(x) \, d\mathbf{Q}_m(x \mid h) \subseteq \left[ \inf_{x \in x\left(\mathcal{E}_{|h}\right)} T(x), \ \sup_{x \in x\left(\mathcal{E}_{|h}\right)} T(x) \right]$$

with equality if ch $T\left(x\left(\mathcal{E}_{|h}\right)\right)$ is a closed set. Bayesian statisticians worry about the effect that the choice of the prior has for their inferences. To overcome this sensitivity, they choose a statistic $T$ and report the interval of possible expected values of $T$ under the posteriors in a family of priors $f \in \mathcal{F}$. For the case where $T\left(x\left(\mathcal{E}_{|h}\right)\right)$ is a compact set, we have that the set of all posterior expectations (conditional on $h$ and $\theta$) is identical to the interval $\left[\underline{T}(h), \overline{T}(h)\right]$, where $\underline{T}(h)$ and $\overline{T}(h)$ are, respectively, the minimum and maximum values of the set $\left\{T(x) : x \in x\left(\mathcal{E}_{|h}\right)\right\}$. The most important application of Proposition E.1 is when we take $y^t$ and $T_{y^t}(x) \equiv q_t^x(y^t)$. In this case, condition E.2 helps us characterize the set of all expected values of bond prices $q_t$ across all equilibrium Bayesian models as:

$$\bigcup_{m \in \mathcal{M}_{\mathcal{E}}(h)} \int q_t^{\mathbf{x}(\alpha)}(y^t) \, dQ(\alpha \mid h) = \left[\underline{q}_t, \overline{q}_t\right]$$

This is the interval characterized in Section 4.

## E.3 Further Results

In this section we study models that are based on small perturbations on equilibrium profiles. Our focus will be on "$\epsilon-$ equilibrium models".

**Definition E.4.** Model $m$ is an $\epsilon-$**equilibrium model** if $\mathbf{Q}_m(x \in x(\mathcal{E})) \geq 1 - \epsilon$.

For a given (non-equilibrium) model $m$, we define

$$\mathbf{Q}_m^{\mathcal{E}}(B \subseteq \mathcal{X}) = \frac{\mathbf{Q}_m(B \cap x(\mathcal{E}))}{\mathbf{Q}_m(x(\mathcal{E}))}$$

as the equilibrium conditional prior. We will show that when $\epsilon \to 0$, the posterior moments calculated with $\epsilon-$ equilibrium models converge to the posterior means under their equilibrium conditional priors, and hence converge to elements in ch $T\left(x\left(\mathcal{E}_{|h}\right)\right)$.

**Proposition E.2.** *Take an equilibrium history $h$ and a family of models $(m_\epsilon)_{\epsilon \in (0,1)}$ (with a common parameter space) with associated priors $(\mathbf{Q}_\epsilon)_{\epsilon \in (0,1)}$ such that*

    *a. $m_\epsilon$ is an $\epsilon-$equilibrium model for all $\epsilon \in (0,1)$*

    *b. There exist $\underline{p} > 0$ such that for all $\epsilon$, $\mathbf{Q}_\epsilon(\mathcal{X}(h)) > \underline{p}$*

*Then, for any bounded and measurable function $T : X \to \mathbb{R}$ we have*

$$\left| \int T(x) \, d\mathbf{Q}_\epsilon(x \mid h) - \int T(x) \, d\mathbf{Q}_\epsilon^{\mathcal{E}}(x \mid h) \right| \leq \epsilon \frac{(\overline{T} - \underline{T})}{\underline{p}} \tag{E.3}$$

*where $\overline{T} = \sup T(x)$ and $\underline{T} = \inf T(x)$. This implies that as $\epsilon \to 0$*

$$\left| \int T(x) \, d\mathbf{Q}_\epsilon(x \mid h) - \int T(x) \, d\mathbf{Q}_\epsilon^{\mathcal{E}}(x \mid h) \right| \to 0$$

**Proof.** See Appendix. $\qquad\qquad\square$

Notice that for all $\epsilon > 0$, the prior $\mathbf{Q}_\epsilon^{\mathcal{E}}(\cdot)$ is an equilibrium prior, since by construction it assigns probability one to the set of equilibrium consistent outcomes. Proposition E.1 then implies that

$$\int T(x) \, d\mathbf{Q}_\epsilon^{\mathcal{E}}(x \mid h) \in \text{ch } T\left(x\left(\mathcal{E}_{|h}\right)\right)$$

**Proof.** **(Proposition E.1)** **Step 1**. Showing the first statement **(1)**. We first show if $x \in x\left(\mathcal{E}_{|h}\right)$ i.e. if $x$ is equilibrium consistent at history $h$, we can construct an equilibirum model $m$ and $\alpha$ in the conditional support such that $x = \mathbf{x}(\alpha)$. We construct it as follows: the possible values for the parameter $\alpha$ are $A = \{1\}$. The mapping is such that $\mathbf{x}(\alpha = 1) = x$. The measure $Q$ is simply $Q(\alpha = 1) = 1$. Since $x \in x\left(\mathcal{E}_{|h}\right)$ we know there is an equilibrium $\sigma$ that is consistent with $x$ after $h$. Hence $m_x$ is an equilibrium model. Also, according to our model $\Pr(x \in X(h)) = 1$, and hence $dQ(x \mid h) = \Pr(x \mid h) = 1 > 0$, finishing the proof. For the **converse**, take an equilibrium model $m \in \mathcal{M}_{\mathcal{E}}$ such that $\alpha \in \text{supp}(Q(\cdot \mid h))$ such that $x = \mathbf{x}(\alpha)$. We will show that $x \in x\left(\mathcal{E}_{|h}\right)$. Using Bayes rule, the posterior distribution $Q(\alpha \mid h)$ after observing the history $h$

$$dQ(\alpha \mid h) = \begin{cases} \frac{dQ(\alpha)}{\int_{\hat{\alpha}:h \in \mathcal{H}(x_{\hat{\alpha}})} dQ(\hat{\alpha})} & \text{if } h \in \mathcal{H}(\mathbf{x}(\alpha)) \\ 0 & \text{if } h \notin \mathcal{H}(\mathbf{x}(\alpha)) \end{cases}$$

The prior was putting probability zero over non equilibrium outcomes, so the posterior has to be zero. This implies that $\alpha \in \text{supp}\left(Q\left(\cdot \mid h\right)\right) \iff h \in \mathcal{H}\left(\mathbf{x}\left(\alpha\right)\right) = \mathcal{H}\left(\sigma_\alpha\right)$ for some $\sigma_\alpha \in \mathcal{E}_{\mid h}$ (since $m$ is an equilibrium model). Therefore $x = x\left(\sigma_\alpha\right) \in x\left(\mathcal{E}_{\mid h}\right)$ finishing the proof.

**Step 2.** For **(2)**, first define $\underline{T} := \inf_{x \in x\left(\mathcal{E}_{\mid h}\right)} T\left(x\right)$ and $\overline{T} := \sup_{x \in x\left(\mathcal{E}_{\mid h}\right)} T\left(x\right)$. Take any equilibrium model $m$. Fix the history $h$. The expected value of $T\left(\cdot\right)$ under $Q\left(\cdot \mid h\right)$ is:

$$\mathbb{E}^{Q_\theta}\left\{T\left(x_\alpha\right) \mid h\right\} = \int_{\alpha \in A} T\left(\mathbf{x}\left(\alpha\right)\right) dQ\left(\alpha \mid h\right) = \int_{\alpha \in \text{supp}\left(Q\left(\cdot \mid h\right)\right)} T\left(\mathbf{x}\left(\alpha\right)\right) dQ\left(\alpha \mid h\right)$$

using in the second equality the definition of support, that was restricted without loss generality. Using equality E.1 we know that for all $\alpha \in \text{supp}\left(Q\left(\cdot \mid h\right)\right)$ we have $\mathbf{x}\left(\alpha\right) \in x\left(\mathcal{E}_{\mid h}\right)$ and hence

$$\underline{T} \leq T\left(\mathbf{x}\left(\alpha\right)\right) \leq \overline{T} \text{ for all } \alpha \in \text{supp}\left(Q\left(\cdot \mid h\right)\right)$$

Each of the inequalities are strict unless $\underline{T} \in T\left(x\left(\mathcal{E}_{\mid h}\right)\right)$ and $\overline{T} \in T\left(x\left(\mathcal{E}_{\mid h}\right)\right)$ respectively, showing that $\mathbb{E}^Q\left\{T\left(\mathbf{x}\left(\alpha\right)\right) \mid h\right\} \in \left[\underline{T}, \overline{T}\right]$. We now need to show that it holds for every value in the convex hull. For any $\lambda \in \text{ch } T\left(x\left(\mathcal{E}_{\mid h}\right)\right)$ there exist an equilibrium model $m_\lambda$ such that $\mathbb{E}^{Q_\theta}\left\{T\left(\mathbf{x}\left(\alpha\right)\right) \mid h\right\} = \lambda$. First, suppose $\lambda \in \left(\underline{T}, \overline{T}\right)$. If $\lambda \in T\left(x\left(\mathcal{E}_{\mid h}\right)\right)$, we can specify model $m$ as in the proof of **(1)** creating a model that assigns prob. 1 to $x : T\left(x\right) = \lambda$. If not, we know there exist equilibrium outcomes $x_1, x_2 \in x\left(\mathcal{E}_{\mid h}\right)$ and a number $\gamma \in \left(0, 1\right)$ such that

$$\lambda = \gamma T\left(x_1\right) + \left(1 - \gamma\right) T\left(x_2\right)$$

In this case, define $m_\lambda$ with $A = \{1, 2\}$, with mapping $\alpha = 1 \rightarrow x_1$ and $\alpha = 2 \rightarrow x_2$ and measure

$$Q^\lambda = \begin{cases} \alpha = 1 & \text{with prob. } \gamma \\ \alpha = 2 & \text{with prob. } 1 - \gamma \end{cases}$$

is easy to check that $\mathbb{E}^{Q^\lambda}\left\{T\left(\mathbf{x}\left(\alpha\right)\right) \mid h\right\} = \lambda$. To finish the proof, we need to show the existence of such models on the cases when $\underline{T} \in \text{ch } T\left(x\left(\mathcal{E}_{\mid h}\right)\right)$ and $\overline{T} \in \text{ch } T\left(x\left(\mathcal{E}_{\mid h}\right)\right)$. In those cases, the construction from when $\lambda \in \left(\underline{T}, \overline{T}\right)$ applies. $\qquad \square$

**Proof.** (of **Proposition** E.2) By Bayes rule:

$$\mathbf{Q}_\epsilon \left( B \mid h \right) \equiv \frac{\mathbf{Q}_\epsilon \left( B \cap \mathcal{X} \left( h \right) \right)}{\mathbf{Q}_\epsilon \left( \mathcal{X} \left( h \right) \right)}$$

which obviously implies that $\mathbf{Q}_\epsilon \left( \mathcal{X} \left( h \right) \mid h \right) = 1$. Thus, to calculate $\mathbb{E}^{\mathbf{Q}_\epsilon} \left\{ T \mid h \right\}$, we can just integrate over $\mathcal{X} \left( h \right) \subseteq \mathcal{X}$ to calculate the integral:

$$\int T \left( x \right) d\mathbf{Q}_\epsilon \left( x \mid h \right) = \int_{\mathcal{X}(h) \cap x(\mathcal{E})} T \left( x \right) d\mathbf{Q}_\epsilon \left( x \mid h \right) + \int_{\mathcal{X}(h) \cap (\mathcal{X} \sim x(\mathcal{E}))} T \left( x \right) d\mathbf{Q}_\epsilon \left( x \mid h \right)$$

As previously defined, $x \left( \mathcal{E}_{\mid h} \right) \equiv \mathcal{X} \left( h \right) \cap x \left( \mathcal{E} \right)$ is the set of equilibrium consistent outcomes with $h$, and denote $x \left( \sim \mathcal{E}_{\mid h} \right) := \mathcal{X} \left( h \right) \cap \left( \mathcal{X} \sim x \left( \mathcal{E} \right) \right)$ as the outcomes consistent with $h$ and **not** consistent with any subgame perfect strategy profile. Using these new definitions together with Bayes rule formula for $\mathbf{Q}_\epsilon \left( \cdot \mid h \right)$ we get

$$\int T \left( x \right) d\mathbf{Q}_\epsilon \left( x \mid h \right) = \int_{x \left( \mathcal{E}_{\mid h} \right)} T \left( x \right) \frac{d\mathbf{Q}_\epsilon \left( x \right)}{\mathbf{Q}_\epsilon \left( \mathcal{X} \left( h \right) \right)} + \int_{x \left( \sim \mathcal{E}_{\mid h} \right)} T \left( x \right) \frac{d\mathbf{Q}_\epsilon \left( x \right)}{\mathbf{Q}_\epsilon \left( \mathcal{X} \left( h \right) \right)} \tag{E.4}$$

We now study the equilibrium conditional measure $Q_n^{\mathcal{E}} \left( \cdot \right)$. Applying Bayes rule and the definition of $Q_n^{\mathcal{E}}$ we get

$$\mathbf{Q}_\epsilon^{\mathcal{E}} \left( B \mid h \right) := \frac{\mathbf{Q}_\epsilon^{\mathcal{E}} \left( B \cap \mathcal{X} \left( h \right) \right)}{\mathbf{Q}_\epsilon^{\mathcal{E}} \left( \mathcal{X} \left( h \right) \right)} \underbrace{=}_{\text{by def.}} \frac{\mathbf{Q}_\epsilon \left( B \cap \mathcal{X} \left( h \right) \cap x \left( \mathcal{E} \right) \right) / \mathbf{Q}_\epsilon \left( x \left( \mathcal{E} \right) \right)}{\mathbf{Q}_\epsilon \left( \mathcal{X} \left( h \right) \cap x \left( \mathcal{E} \right) \right) / \mathbf{Q}_\epsilon \left( x \left( \mathcal{E} \right) \right)} =$$

$$= \frac{\mathbf{Q}_\epsilon \left( B \cap x \left( \mathcal{E}_{\mid h} \right) \right)}{\mathbf{Q}_\epsilon \left( x \left( \mathcal{E} \right) \right)}$$

and hence

$$\mathbf{Q}_\epsilon \left( B \cap x \left( \mathcal{E}_{\mid h} \right) \mid h \right) \underbrace{=}_{\text{by def.}} \frac{\mathbf{Q}_\epsilon \left( B \cap x \left( \mathcal{E}_{\mid h} \right) \right)}{\mathbf{Q}_\epsilon \left( \mathcal{X} \left( h \right) \right)} = \frac{\mathbf{Q}_\epsilon \left( x \left( \mathcal{E}_{\mid h} \right) \right)}{\mathbf{Q}_\epsilon \left( \mathcal{X} \left( h \right) \right)} \mathbf{Q}_\epsilon^{\mathcal{E}} \left( B \mid h \right) \tag{E.5}$$

It will be also useful to define the non-equilibrium conditional measure

$$\mathbf{Q}_\epsilon^{\sim \mathcal{E}} \left( B \right) \equiv \frac{\mathbf{Q}_\epsilon \left( B \cap \left( X \sim x \left( \mathcal{E} \right) \right) \right)}{\mathbf{Q}_\epsilon \left( X \sim x \left( \mathcal{E} \right) \right)}$$

for which we get, using Bayes rule:

$$\mathbf{Q}_\epsilon\left(B \cap x\left(\sim \mathcal{E}_{|h}\right) \mid h\right) = \frac{\mathbf{Q}_\epsilon\left(x\left(\sim \mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}\mathbf{Q}_\epsilon^{\sim\mathcal{E}}\left(B \mid h\right) \tag{E.6}$$

Thus we can rewrite the conditional measure $d\mathbf{Q}_\epsilon\left(x \mid h\right)$ as

$$d\mathbf{Q}_\epsilon\left(x \mid h\right) = \begin{cases} \frac{\mathbf{Q}_\epsilon\left(x\left(\mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}d\mathbf{Q}_\epsilon^{\mathcal{E}}\left(x \mid h\right) & \text{if } x \in x\left(\mathcal{E}_{|h}\right) \\ \frac{\mathbf{Q}_\epsilon\left(x\left(\sim\mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}d\mathbf{Q}_\epsilon^{\sim\mathcal{E}}\left(x \mid h\right) & \text{if } x \in x\left(\sim \mathcal{E}_{|h}\right) \\ 0 & \text{elsewhere} \end{cases} \tag{E.7}$$

Using E.7, we then rewrite E.4 as

$$\int T\left(x\right)d\mathbf{Q}_\epsilon\left(x \mid h\right) = \frac{\mathbf{Q}_\epsilon\left(x\left(\mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}\int_{x\left(\mathcal{E}_{|h}\right)} T\left(x\right)d\mathbf{Q}_\epsilon^{\mathcal{E}}\left(x \mid h\right) +$$

$$+\frac{\mathbf{Q}_\epsilon\left(x\left(\sim \mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}\int_{x\left(\sim\mathcal{E}_{|h}\right)} T\left(x\right)d\mathbf{Q}_\epsilon^{\sim\mathcal{E}}\left(x \mid h\right)$$

so that

$$\int T\left(x\right)d\mathbf{Q}_\epsilon\left(x \mid h\right) - \int T\left(x\right)d\mathbf{Q}_\epsilon^{\mathcal{E}}\left(x \mid h\right) =$$

$$= \left[\frac{\mathbf{Q}_\epsilon\left(x\left(\mathcal{E}_{|h}\right)\right) - \mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}\right]\int T\left(x\right)d\mathbf{Q}_\epsilon^{\mathcal{E}}\left(x \mid h\right) +$$

$$+\frac{\mathbf{Q}_\epsilon\left(x\left(\sim \mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}\int T\left(x\right)d\mathbf{Q}_\epsilon^{\sim\mathcal{E}}\left(x \mid h\right) =$$

$$\frac{\mathbf{Q}_\epsilon\left(x\left(\sim \mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}\left(\int T\left(x\right)d\mathbf{Q}_\epsilon^{\sim\mathcal{E}}\left(x \mid h\right) - \int T\left(x\right)d\mathbf{Q}_\epsilon^{\mathcal{E}}\left(x \mid h\right)\right) \tag{E.8}$$

using in the last equation the fact that $\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right) = \mathbf{Q}_\epsilon\left(x\left(\mathcal{E}_{|h}\right)\right) + \mathbf{Q}_\epsilon\left(x\left(\sim \mathcal{E}_{|h}\right)\right)$. See that since $x\left(\sim \mathcal{E}_{|h}\right) \subseteq X \sim x\left(\mathcal{E}\right)$, then

$$\mathbf{Q}_\epsilon\left(x\left(\sim \mathcal{E}_{|h}\right)\right) \le \mathbf{Q}_\epsilon\left(X \sim x\left(\mathcal{E}\right)\right) = 1 - \mathbf{Q}_\epsilon\left(x\left(\mathcal{E}\right)\right) \le \epsilon$$

using in the last inequality the fact that $m_\epsilon$ is an $\epsilon-$equilibrium model for all $\epsilon \in \left(0,1\right)$.

Also, because $T$ is bounded, we get that

$$\int T(x)\,d\mathbf{Q}_\epsilon^{\sim\mathcal{E}}(x\mid h) - \int T(x)\,d\mathbf{Q}_\epsilon^{\mathcal{E}}(x\mid h) \le \sup_{x\in X} T(x) - \inf_{x\in X} T(x) = \overline{T} - \underline{T} < \infty$$

Taking absolute values on both sides of E.8, we get

$$\left|\int T(x)\,d\mathbf{Q}_\epsilon(x\mid h) - \int T(x)\,d\mathbf{Q}_\epsilon^{\mathcal{E}}(x\mid h)\right| =$$

$$= \frac{\mathbf{Q}_\epsilon\left(x\left(\sim\mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon(\mathcal{X}(h))}\left|\int T(x)\,d\mathbf{Q}_\epsilon^{\sim\mathcal{E}}(x\mid h) - \int T(x)\,d\mathbf{Q}_\epsilon^{\mathcal{E}}(x\mid h)\right| \le$$

$$\le \epsilon\frac{\overline{T} - \underline{T}}{\underline{p}}$$

using also the assumption that $\mathbf{Q}_\epsilon(\mathcal{X}(h)) \ge \underline{p}$ for all $\epsilon \in (0,1)$, proving the desired result. $\qquad\square$