Competitive Contagion in Networks

Sanjeev Goyal* Michael Kearns†

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Abstract

We introduce and develop a framework for the study of competition between firms who have budgets to "seed" the initial adoption of their products by consumers located in a social network. The payoffs to the firms are the eventual number of adoptions of their product through a competitive stochastic diffusion process in the network. This framework yields a very rich class of competitive strategies, which depend in subtle ways on the stochastic dynamics of adoption, the relative budgets of the players, and the underlying structure of the social network.

We identify a general property of the adoption dynamics — namely, decreasing returns to local adoption — for which the inefficiency of resource use at equilibrium (the *Price of Anarchy*) is uniformly bounded above, across all equilibria and networks. We also show that if this property is even slightly violated, the Price of Anarchy can be unbounded, thus yielding sharp threshold behavior for a broad class of dynamics.

We also introduce a new notion, the *Price of Budgets*, that measures the extent that imbalances in player budgets can be amplified at equilibrium. We again identify a general property of the adoption dynamics — namely, proportional local adoption between competitors — for which the (pure) Price of Budgets is uniformly bounded above, across all equilibria and all networks. We show that even a slight departure from this property can lead to unbounded Price of Budgets, again yielding sharp threshold behavior for a broad class of dynamics.

*Faculty of Economics and Christ's College, University of Cambridge.

†Department of Computer and Information Science, University of Pennsylvania. Portions of this research were conducted while the author was visiting the Faculty of Economics and Christ's College, University of Cambridge.

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1 Introduction

The role of social networks in shaping individual choices has been brought out in a number of studies over the years.[1] In the past, the deliberate use of such social influences by external agents was hampered by the lack of good data on social networks. In recent years, data from on-line social networking sites along with other advances in information technology have created interest in ways that firms and governments can use social networks to further their goals.[2]

In this work, we study competition between firms who use their resources to maximize product adoption by consumers located in a social network. [3] The social network may transmit information about products, and adoption of products by neighbors may have direct consumption benefits. The firms, denoted *Red* and *Blue*, know the graph which defines the social network, and offer similar or interchangeable products or services. The two firms simultaneously choose to allocate their resources on subsets of consumers, i.e., to *seed* the network with initial adoptions. The stochastic dynamics of local adoption determine how the influence of each player's seeds spreads throughout the graph to create new adoptions.

A distinctive feature of our framework is the examination of a rather general class of local influence processes. We decompose the dynamics into two parts: a *switching function* f, which specifies the probability of a consumer switching from non-adoption to adoption as a function of the fraction of his neighbors who have adopted *either* of the two products Red and Blue; and a *selection function* g, which specifies, conditional on switching, the probability that the consumer adopts (say) Red as function of the fraction of adopting neighbors who have adopted Red. Each firm seeks to maximize the total number of consumers who adopt its product. Broadly speaking, the switching function captures "stickiness" of the (interchangeable) products based on their local prevalence, and the selection function captures preference for firms based on local market share.

This framework yields a very rich class of competitive strategies, which depend in subtle ways on the dynamics, the relative budgets of the players, and the structure of the social network (Section 4 gives some warm-up examples illustrating this point). Here we focus on understanding two general features of equilibrium: first, the efficiency of resource use by the players (Price of Anarchy) and second, the role of the network in amplifying ex-ante resource differences between the players (Price of Budgets).

Our *first* set of results concern efficiency of resource use by the players. For a fixed graph and fixed local dynamics (given by f and g), and budgets of K_R and K_B seed infections for the players, let (S_R, S_B) be the sets of seed infections that maximize the joint expected

[1] See e.g., Coleman (1966) on doctors' prescription of new drugs, Conley and Udry (2005) and Foster and Rosenzweig (1995) on farmers' decisions on crop and input choice , and Feick and Price (1987), Reingen et al. (1984), and Godes and Mayzlin (2004) on brand choice by consumers.

[2] The popularity of terms such as *word of mouth communication*, *viral marketing*, *seeding the network* and *peer-leading* interventions is an indicator of this interest.

[3] Our model may apply to other settings of competitive contagion, such as between two fatal viruses in a population.

infections (payoffs) $\Pi_R(S_R, S_B) + \Pi_B(S_R, S_B)$ subject to $|S_R| = K_R$ and $|S_B| = K_B$, and let $\Pi_R(\sigma_R, \sigma_B) + \Pi_B(\sigma_R, \sigma_B)$ be the *smallest* joint payoff across all Nash equilibrium, where σ_R and σ_B are mixed strategies obeying the budget constraints. The *Price of Anarchy* (or PoA) is then defined as the ratio of the former divided by the latter, that is,

$$\frac{\Pi_R(S_R, S_B) + \Pi_B(S_R, S_B)}{\Pi_R(\sigma_R, \sigma_B) + \Pi_B(\sigma_R, \sigma_B)}.$$

Our first main result, Theorem 1, shows that if the switching function f is concave and the selection function g is linear, then the PoA is uniformly bounded above by 4, across all equilibria and across all networks. The main proof technique we employ is the construction of coupled stochastic dynamical processes that allow us to demonstrate that, under the assumptions on f and g, the departure of one player can only benefit the other player. This in turn lets us argue that players can successfully defect to the maximum social welfare solution and realize a significant fraction of its payoff, thus implying they must also do so at equilibrium.

Our next main result, Theorem 2, shows that even a small departure away from concavity of the switching function f can lead to arbitrarily high PoA. This result is obtained by constructing a family of layered networks whose dynamics effectively compose f many times, thus amplifying its convexity. Equilibrium and large PoA are enforced by the fact that despite this amplification, the players are better off coordinating and playing near each other unilaterally, even though they would be much better off elsewhere together. Taken together, our PoA upper and lower bounds permit us to exhibit simple parametric classes of dynamics yielding sharp threshold behavior. For example, if the switching function if $f(x) = x^r$ for $r > 0$ and the selection function g is linear, then for all $r \leq 1$ the PoA is at most 4, while for any $r > 1$ it can be unbounded.

Our *second* set of results are about the effects of networks and dynamics on budget differences across the players. We introduce and study a new quantity called the *Price of Budgets* (PoB). For any fixed graph, local dynamics, and initial budgets, with $K_R \geq K_B$, let (σ_R, σ_B) be the Nash equilibrium that *maximizes* the quantity

$$\frac{\Pi_R(\sigma_R, \sigma_B)}{\Pi_B(\sigma_R, \sigma_B)} \times \frac{K_B}{K_R}$$

among all Nash equilibria (σ_R, σ_B); this quantity is just the ratio of the final payoffs divided by the ratio of the initial budgets. The resulting maximized quantity is the Price of Budget, and it measures the extent to which the larger budget player can obtain a final market share that exceeds their share of the initial budgets.

Theorem 4 shows that if the switching function is concave and the selection function is linear, then the (pure strategy) PoB is bounded above by 2, uniformly across all networks. The proof imports elements of the proof for the PoA upper bound, and additionally employs a method for attributing adoptions back to the initial seeds that generated them.

Our next result, Theorem 5, shows that even a slight departure from linearity in the selection function can yield unbounded PoB. The proof again appeals to network structures that amplify the nonlinearity of g by self-composition, which has the effect of "squeezing out" the player with smaller budget. Combining the PoB upper and lower bounds again allows us to exhibit simple parametric forms yielding threshold behavior: for instance, if f is linear and g is from the well-studied Tullock contest function family (discussed later), which includes linear g and therefore bounded PoB, even an infinitesimal departure from linearity can result in unbounded PoB.

Related Literature: Our paper contributes to the study of competitive strategy in network environments. We build a framework which combines ideas from economics (contests, competitive seeding and advertising) and computer science – uniform bounds on properties of equilibria, as in the Price of Anarchy – to address a topical and natural question. The Tullock contest function was introduced in Tullock (1980); for an axiomatic development see Skaperdas (1996). For early and influential studies of competitive advertising, see Butters (1977) and Grossman and Shapiro (1984). The Price of Anarchy (PoA) was introduced in Koutsoupias and Papadimitriou (1999). The tension between equilibrium and Nash efficiency is a recurring theme in economics; for a general result on the inefficiency of Nash equilibria, see Dubey (1986).

More specifically, we contribute to the study of influence in networks. This has been an active field of study in the last decade, see e.g., Ballester, Calvo-Armengol and Zenou (2006); Bharathi, Kempe and Salek (2007); Galeotti and Goyal (2010); Kempe, Kleinberg, and Tardos (2003, 2005); Chasparis and Shamma (2010); Vetta (2002). There are three elements in our framework which appear to be novel: one, we consider a fairly general class of adoption rules at the individual consumer level which correspond to different roles which social interaction can potentially play (existing work typically considers specific local dynamics), two, we study competition for influence in a network (existing work has mainly focused on the case of single player seeking to maximize influence), and three, we introduce and study the notion of Price of Budgets as a measure of how networks amplify budget differences. Finally, to the best of our knowledge, our results on the relationship between the dynamics and qualitative features of the strategic equilibrium are novel.

2 Model

2.1 Graph, Allocations, and Seeds

We consider a 2-player game of competitive adoption on a (possibly directed) graph G over n vertices. G is known to the two players, whom we shall refer to as R(ed) and B(lue). We shall also use R, B and U(ninfected) to denote the state of a vertex in G, according to whether it is currently infected by one of the two players or uninfected. The two players simultaneously choose some number of vertices to initially seed; after this seeding, the stochastic dynamics of local adoption (discussed below) determine how each player's seeds spread throughout G to create adoptions by new nodes. Each player seeks to maximize his (expected) number of eventual adoptions. [4]

More precisely, suppose that player $p = R, B$ has *budget* $K_p \in \mathcal{N}_+$; Each player p chooses an allocation of budget across the n vertices, $a_p = (a_{p1}, a_{p2}, ..., a_{pn})$, where $a_{pj} \in \mathcal{N}_+$ and $\sum_{j=1}^{n} a_{pj} = K_p$. Let A_p be the set of allocations for player p, which is their pure strategy space. A mixed strategy for player p is a probability distribution σ_p on A_p. Let $\mathcal{A}_p$ denote the set of probability distributions for player p. Prior to the contagion process on G, the two players choose their strategies (σ_R, σ_B). Consider any realized initial allocation (a_R, a_B) for the two players. Let $V(a_R) = \{v | a_{vR} > 0\}$, $V(a_B) = \{v | a_{vB} > 0\}$ and let $V(a_R, a_B) = V(a_R) \cup V(a_B)$. A vertex v becomes initially infected if one or more players assigns a seed to infect v. If both players assign seeds to the same vertex, then the probability of initial infection by a player is proportional to the seeds allocated by the player (relative to the other player). More precisely, fix any allocation (a_R, a_B). For any vertex v, the initial state s_v of v is in $\{R, B\}$ if and only if $v \in V(a_R, a_B)$. Moreover, $s_v = R$ with probability $a_{vR}/(a_{vR} + a_{vB})$, and $s_v = B$ with probability $a_{vB}/(a_{vR} + a_{vB})$.

Following the allocation of seeds, the stochastic contagion process on G determines how these R and B infections generate new adoptions in the network. We consider a discrete time model for this process. The state of a vertex v at time t is denoted $s_{vt} \in \{U, R, B\}$, where U stands for Uninfected, R stands for infection by R, and B stands for infection by B.

2.2 Stochastic Updates: The Switching-Selection Model

We assume there is an *update schedule* which determines the order in which vertices are considered for state updates. The primary simplifying assumption we shall make about this schedule is that once a vertex is infected, it is never a candidate for updating again. [5]

Within this constraint, we allow for a variety of behaviors, such as randomly choosing an uninfected vertex to update at each time step, or updating all uninfected vertices simultaneously at each time step. We can also allow for an *immunity* property — if a vertex is exposed once to infection and remains uninfected, it is never updated again. Update schedules may also have finite termination times or conditions. Note that a schedule which perpetually updates uninfected vertices will eventually cause any connected G to become entirely infected, but we allow for considerably more general schedules. In cases where we need some specific property to hold for the update schedule, we shall discuss it in the appropriate place.

For the stochastic update of an uninfected vertex v, we will primarily consider what we shall call the *switching-selection* model. In this model, updating is determined by the application of two functions to v's local neighborhood: $f(x)$ (the *switching* function), and $g(y)$ (the *selection* function). More precisely, let α_R and α_B be the fraction of v's neighbors infected by R and B, respectively, at the time of the update, and let $\alpha = \alpha_R + \alpha_B$ be the total fraction of infected neighbors. The function f maps α to the interval $[0, 1]$ and g maps $\alpha_R/(\alpha_R + \alpha_B)$ (the relative fraction of infections that are R) to $[0, 1]$. These two functions determine the

[4] Throughout the paper, we shall use the terms *infection* and *adoption* interchangeably.

[5] This assumption can be relaxed considerably, at the expense of complicating some of our proofs.

stochastic update in the following fashion:

1. With probability $f(\alpha)$, v becomes infected by either R or B; with probability $1 - f(\alpha)$, v remains in state U.

2. If it is determined that v becomes infected, it becomes infected by R with probability $g(\alpha_R/(\alpha_R + \alpha_B))$, and infected by B with probability $g(\alpha_B/(\alpha_R + \alpha_B))$.

We think of the switching function as specifying how rapidly adoption increases with the fraction of neighbors who have adopted (i.e. the stickiness of the interchangeable products or services), regardless of their R or B value; while the selection function specifies the probability of infection by each firm in terms of the local relative market share split. We assume $f(0) = 0$ (infection requires exposure), $f(1) = 1$ (full neighborhood infection forces infection), and f is increasing (more exposure yields more infection); and $g(0) = 0$ (players need some local market share for infection), $g(1) = 1$. Note that since the selection step above requires that an infection take place, we also have $g(y) + g(1 - y) = 1$, which implies $g(1/2) = 1/2$. In Section 4 we shall provide some economic motivation for this formulation and also illustrate with specific parametric families of functions f and g. We also discuss more general models for the local dynamics at a number of places in the paper. The appendix also illustrates how these switching and selection functions f-g may arise out of optimal decisions made by consumers located in social networks.

2.3 Payoffs and Equilibrium

Given a graph G and an initial allocation of seeds (a_R, a_B), the dynamics described above yield a stochastic number of eventual infections for the two players. Let χ_p, $p = R, B$ denote this random variable for R and B, respectively, at the termination of the dynamics. Given strategy profile (σ_R, σ_B), the payoff to player $p = R, B$ is $\Pi_p(\sigma_R, \sigma_B) = \mathbf{E}[\chi_p|(\sigma_R, \sigma_B)]$. Here the expectation is over any randomization in the player strategies in the choice of initial allocations, and the randomization in the stochastic updating dynamics. A Nash equilibrium is a profile of strategies (σ_R, σ_B) such that σ_p maximizes player p's payoff given the strategy σ_{-p} of the other player.

2.4 Price of Anarchy and Price of Budgets

For a fixed graph G and stochastic update dynamics, and budgets K_R, K_B, the *maximum payoff* allocation is the allocation (a_R^*, a_B^*) obeying the budget constraints that maximizes $\mathbf{E}[\chi_R + \chi_B|(a_R, a_B)]$. For the same fixed graph, update dynamics and budgets, let (σ_R, σ_B) be the Nash equilibrium strategies that *minimize* $\mathbf{E}[\chi_R + \chi_B|(\sigma_R, \sigma_B)]$ among all Nash equilibria — that is, the Nash equilibrium with the smallest joint payoff. Then the *Price of Anarchy* (or PoA) is defined to be

$$\frac{\mathbf{E}[\chi_R + \chi_B|(a_R^*, a_B^*)]}{\mathbf{E}[\chi_R + \chi_B|(\sigma_R, \sigma_B)]}$$

The Price of Anarchy is a measure of the inefficiency in resource use created by competition between the two players.

We also introduce and study a new quantity called the *Price of Budgets* (PoB). The PoB measures the extent to which networks can warp or amplify inequality in the budgets. Thus for any fixed graph G and stochastic update dynamics, and initial budgets K_R, K_B, with $K_R \geq K_B$, let (σ_R, σ_B) be the Nash equilibrium that *maximizes* the ratio

$$\frac{\Pi_R(\sigma_R, \sigma_B)}{\Pi_B(\sigma_R, \sigma_B)} \times \frac{b_B}{b_R}$$

among all Nash equilibria. Similarly, we can define the Price of Budgets when $K_B \geq K_R$. The resulting maximized ratio is the Price of Budget, and it measures the extent to which the larger budget player can obtain a final market share that exceeds their share of the initial budgets.

3 Local Dynamics: Motivation and Examples

In this section, we provide some motivation for, and examples of, the decomposition of the local update dynamics into a switching function f and a selection function g. We view the switching function as representing how contagious a product or service is, regardless of which competing party provides it; and we view the selection function as representing the extent to which a firm having majority local market share favors its selection in the case of adoption. We illustrate the richness of this model by examining a variety of different mathematical choices for the functions f and g, and discuss examples from the domain of technology adoption that might qualitatively match these forms. Finally, to illustrate the scope of this formulation, we also discuss examples of update dynamics that cannot be decomposed in this way.

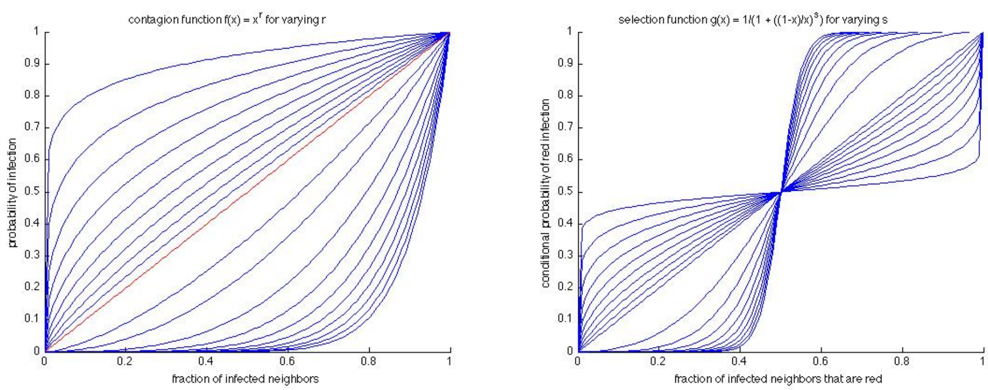

Figure 1: Left: Plots of $f(x) = x^r$ for varying choices of r, including $r = 1$ (linear, red line), $r < 1$ (concave), and $r > 1$ (convex). Right: Plots of $g(y) = 1/(1 + ((1 - y)/y)^s)$ for varying choices of s, including $s = 1$ (linear, red line), $r < 1$ (equalizing), and $r > 1$ (polarizing).

A fairly broad class of dynamics is captured in the following parametric family of functions. Let the switching function be given by

$$f(x) = x^r \;\; r \geq 0$$

and let the selection function be given by

$$g(y) = 1/(1 + ((1-y)/y)^s) \;\; s \geq 0.$$

Regarding this form for f, for $r = 1$ we have linear adoption. For $r < 1$ we have f concave, corresponding to cases in which the probability of adoption rises quickly with only a small fraction of adopting neighbors, but then saturates or levels off with larger fractions of adopting neighbors. In contrast, for $r > 1$ we have f convex, which at very large values of r can begin to approximate threshold adoption behavior — the probability of adoption remains small until some critical fraction of neighbors has adopted, then rises rapidly. See Figure 1.

Regarding this form for g, which is known as the *Tullock contest function* (Tullock (1980)), for $s = 1$ we have a *voter* model in which the probability of selection is proportional to local market share. For $s < 1$ we have what we shall call an *equalizing* g, by which we mean that selection of the minority party in the neighborhood is favored relative to the linear voter model $g(y) = y$; and for $s > 1$ we have a *polarizing* g, meaning that the minority party is disfavored relative to the linear model. As s approaches 0, we approach the completely equalizing choice $g \equiv 1/2$, and as s approaches infinity, we approach the completely polarizing *winner-take-all* g. See Figure 1.

These parametric families of switching and selection functions will play an important role in illustrating our general results. We now discuss a variety of economic examples which are (qualitatively) covered by these families of functions.

- *Online Social Networking Services (Facebook, Google+, MySpace, Friendster):* Here adoption probabilities grow slowly with a small fraction of adopting neighbors, since there is little value in using social networking services if none of your friends are using them; thus a convex switching function f ($r > 1$) might be a reasonable model. However, given that it is currently difficult or impossible to export friends and other settings from one service to another, there are strong platform effects in service selection, so a polarizing or even winner-take-all selection function g ($s > 1$) is appropriate.

- *Televisions, Desktop Computers:* Televisions and computers are immediately useful without the need for adoption by neighbors. The adoption by neighbors serves mainly as a route for information sharing about value of the product. The information value of more neighbors adopting a product is falling with adoption and so a concave f is appropriate. Compared to social networking services, the platform effects are lower here, and so a linear or equalizing g is appropriate.

- *Mobile Phone Service Providers (Verizon, T-Mobile, AT&T):* Mobile phone service was immediately useful upon its introduction without adoption by neighbors, since one could always call land lines, thus arguing for a concave f. Since telephony systems need to be interoperable, platform effects derive mainly from marketing efforts such as "Friends and Family" programs, and thus are extant but perhaps weak, suggesting a strongly equalizing g.

In the proofs of some of our results, it will sometimes be convenient to use a more general adoption function formulation with some additional technical conditions that are met by our switching-selection formulation. We will refer to this general, single-step model as the *generalized adoption function* model. In this model, if the local fractions of Red and Blue neighbors are α_R and α_B, the probability that we update the vertex with an R infection is $h(\alpha_R, \alpha_B)$ for some *adoption function* h with range $[0, 1]$, and symmetrically the probability of B infection is thus $h(\alpha_B, \alpha_R)$. Let us use $H(\alpha_R, \alpha_B) = h(\alpha_R, \alpha_B) + h(\alpha_B, \alpha_R)$ to denote the total infection probability under h. Note that we can still always decompose h into a two-step process by defining the switching function to be $f(\alpha_R, \alpha_B) = H(\alpha_R, \alpha_B)$ and defining the selection function to be $g(\alpha_R, \alpha_B) = h(\alpha_R, \alpha_B)/(h(\alpha_R, \alpha_B) + h(\alpha_B, \alpha_R))$ (the infection-conditional probability that R wins the infection). The switching-selection model is thus the special case of the generalized adoption function model in which $H(\alpha_R, \alpha_B)$ is a function of only $\alpha_R + \alpha_B$, and $g(\alpha_R, \alpha_B)$ is a function of only $\alpha_R/(\alpha_R + \alpha_B)$.[6]

4 Equilibria in Networks: Some Examples

This section shows that our framework yields a very rich class of competitive strategies, which depend in subtle ways on the dynamics, the relative budgets of the players and the structure of the social network.

Price of Anarchy: Suppose that budgets of the firms are $K_R = K_B = 1$, and the update rule is such that all vertices are updated only once. The network contains two connected components with 10 vertices and 100 vertices, respectively. In each component there are 2 influential vertices, each of which is connected to the other 8 and 98 vertices, respectively. So in component 1, there are 16 directed links while in component 2 there are 196 directed links in all.

Suppose that the switching function and the selection function are both linear, $f(x) = x$ and $g(y) = y$. Then there is a unique equilibrium in which players place their seeds on distinct influential vertices of component 2. The total infection is then 100 and this is the maximum number of infections possible with 2 seeds. So here the PoA is 1.

[6]While the decomposition in terms of a switching function and a selection function accommodates a fairly wide range of adoption dynamics there are some cases which are ruled out. Consider the choice $h(x, y) = x(1 - y^2)$; it is easily verified that $h(x, y)$ is competitive, and simple calculus reveals that the total adoption probability of adoption $H(x, y)$ is increasing in x and y. But $H(x, y)$ clearly cannot be expressed as a function of the form $f(x+y)$. Similarly, it is easy to construct an adoption function that is not only not decomposable, but violates monotonicity. Imagine consumers that prefer to adopt the majority choice in their neighborhood, but will only adopt once their local neighborhood market is sufficiently settled in favor of one or the other product. The probability of total adoption may then be higher with $x = 0.2$, $y = 0$ as compared to $x = y = 0.4$.

Let us now alter the switching function such that $f(1/2) = \epsilon$ for some $\epsilon < 1/2$ (keeping $f(1) = 1$, as always), but retain the selection to be $g(y) = y$. Now there also exists an equilibrium in which the firms locate on the influential vertices of component 1. In equilibrium payoffs to each player are equal to 5. Observe that for $\epsilon < 1/25$, a deviation to the other component is not profitable: it yields an expected payoff equal to $\epsilon \times 100$, and this is strictly smaller than 5. Since it is still possible to infect component 2 with 2 seeds, the PoA is 10. Here inefficiency is created by a coordination failure of the players.

Finally, suppose there is only one component with 110 vertices, with 2 influential vertices and 108 vertices receiving directed links. Then equilibrium under both switching functions considered above will involve firms locating at the 2 influential vertices and this will lead to infection of all vertices. So the PoA is 1, irrespective of whether the switching function is linear $f(x) = x$ or whether $f(1/2) < 1/25$.

We have shown for a fixed network, updating rule and selection function, variations in the switching function can generate large variations in the PoA. Similarly, for fixed update rule and switching and selection functions, a change in the network yields very different PoA.

Theorem 1 provides a set of sufficient conditions on switching and selection function, under which the PoA is uniformly bounded from above. Theorem 3 shows how even small violations of these conditions can lead to arbitrarily high PoA.

Price of Budgets: Suppose that budgets of the firms are $K_R = 1$, $K_B = 2$ and the update rule is such that all vertices are updated only once. The network contains 3 influential vertices, each of which has a directed link to all the other $n-3$ vertices, respectively. So there are $(n-3)3$ links in all. Let $n >> 3$.

Suppose the switching function and selection function are both linear, i.e., $f(x) = x$ and $g(y) = y$. There is a unique equilibrium and in this equilibrium, players will place their resources on distinct influential vertices. The (expected) payoffs to player R are $n/3$, while the payoff to player B are $2n/3$. So the PoB is equal to 1.

Next, suppose the switching function is convex with $f(2/3) = 1/25$, and the selection function $g(y)$ is as in Tullock (1980). Suppose the two players place their resources on the three influential vertices. The payoffs to R are $g(1/3)n$, while firm B earns $g(2/3)n$. Clearly this is optimal for firm B as any deviation can only lower payoffs. And, it can be checked that a deviation by firm R to one of the influential vertices occupied by player B will yield a payoff of $n/100$ (approximately). So the configuration specified is an equilibrium so long as $g(1/3) \geq 1/100$. The PoB is now (approximately) 50.

Finally, suppose the network consists of ℓ equally-sized connected components. In each component, there is 1 influential vertex which has a directed link to each of the $(n/\ell) - 1$ other vertices. In equilibrium each player locates on a distinct influential vertex, *irrespective* of whether the switching function is convex or concave and whether the Tullock selection function is linear ($s = 1$) or whether it is polarizing ($s > 1$).

These examples show that for fixed network and updating rule, variations in the switching and selection functions generate large variations in PoB. Moreover, for fixed switching and selection functions the payoffs depend crucially on the network.

Theorem 4 provides a set of sufficient conditions on the switching and selection function, under which the PoB is uniformly bounded. Theorem 5 shows how even small violations of these conditions can lead to arbitrarily high PoB. Theorem 6 illustrates the role of concavity of the switching function in shaping the PoB.

5 Results: Price of Anarchy

We first state and prove a theorem providing general conditions in the switching-selection model under which the Price of Anarchy is bounded by a constant that is independent of the size and structure of the graph G. The simplest characterization is that f being any concave function (satisfying $f(0) = 0$, $f(1) = 1$ and f increasing), and g being the linear voter function $g(y) = y$ leads to bounded PoA; but we shall see the conditions allow for certain combinations of concave f and nonlinear g as well. we then prove a lower bound showing that the concavity of f is required for bounded PoA in a very strong sense. A slight departure from concavity can lead to unbounded POA.

5.1 PoA: Upper Bound

We find it useful to state and prove our theorems using the generalized adoption model formulation described in the previous section, but with some additional conditions on h that we now discuss. If $h(\alpha_R, \alpha_B)$ (respectively, $h(\alpha_B, \alpha_R)$) is the probability that a vertex with fractions α_R and α_B of R and B neighbors is infected by R (respectively, B), we say that the total infection probability $H(\alpha_R, \alpha_B) = h(\alpha_R, \alpha_B) + h(\alpha_B, \alpha_R)$ is *additive in its arguments* (or simply *additive* if H can be written $H(\alpha_R, \alpha_B) = f(\alpha_R + \alpha_B)$ for some increasing function f — in other words, h permits interpretation as a switching function. We shall say that h is *competitive* if $h(\alpha_R, \alpha_B) \leq h(\alpha_R, 0)$ for all $\alpha_R, \alpha_B \in [0, 1]$. In other words, a player always has equal or higher infection probability in the absence of the other player.

Observe that the switching-selection formulation always satisfies the additivity property. Moreover, in the switching-selection formulation, if g is linear, the competitiveness condition becomes $h(x, y) = f(x + y)(x/(x + y)) \leq f(x) = h(x, 0)$ or $f(x + y)/(x + y) \leq f(x)/x$; this condition is satisfied by the concavity of f. We will later see that the following theorem also applies to certain combinations of concave f and nonlinear g.

The first theorem can now be stated.

Theorem 1 *If the adoption function $h(\alpha^R, \alpha^B)$ is competitive and H is additive in its arguments, then Price of Anarchy is at most 4 for any graph G.*

Proof: We establish the theorem via a series of lemmas and inequalities that can be summarized as follows. Let (S_R^*, S_B^*) be an initial allocation of infections that gives the maximum joint payoff, and let (S_R, S_B) be a pure[7] Nash equilibrium with S_R being the larger set of seeds,

[7] The extension to mixed strategies is straightforward and omitted.

so $K_R = |S_R^*| = |S_R| \geq K_B = |S_B^*| = |S_B|$. We first establish a general lemma (Lemma 1) that implies that the set S_R^* alone (without S_B^* present) must yield payoffs close to the maximum joint payoff (Corollary 1). The proof involves the construction of a coupled stochastic process technique we employ repeatedly in the paper. We then contemplate a unilateral defection of the Red player to (S_R^*, S_B). Another coupling argument (Lemma 2) establishes that the total payoffs for both players under (S_R^*, S_B) must be at least those for the Red player alone under $(S_R^*, \emptyset)$. This means that under (S_R^*, S_B), one of the two players must be approaching the maximum joint infections. If it is Red, we are done, since Red's equilibrium payoff must also be this large. If it is Blue, Lemma 1 implies that Blue could still get this large payoff even under the departure of Red. Next we invoke Lemma 2 to show that total eventual payoff to both players under (S_R, S_B) must exceed this large payoff accruing to Blue, proving the theorem.

Lemma 1 *Let A_R and A_B be any sets of seed vertices for the two players. Then if h is competitive and H is additive,*

$$\mathbf{E}[\chi_R|(A_R, \emptyset)] \geq \mathbf{E}[\chi_R|(A_R, A_B)]$$

and

$$\mathbf{E}[\chi_B|(\emptyset, A_B)] \geq \mathbf{E}[\chi_B|(A_R, A_B)].$$

Proof: We provide the proof for the first statement involving χ_R; the proof for χ_B is identical. We introduce a simple *coupled simulation* technique that we shall appeal to several times throughout the paper. Consider the stochastic dynamical process on G under two different initial conditions: both A_R and A_B are present (the *joint* process, denoted (A_R, A_B) in the conditioning in the statement of the lemma); and only the set A_R is present (the *solo Red* process, denoted $(A_R, \emptyset)$). Our goal is to define a new stochastic process on G, called the *coupled process*, in which the state of each vertex v will be a pair $< X_v, Y_v >$. We shall arrange that X_v faithfully represents the state of a vertex in the joint process, and Y_v the state in the solo Red process. However, these state components will be correlated or a coupled in a deliberate manner. More precisely, we wish to arrange the coupled process to have the following properties:

1. At each step, and for any vertex state $< X_v, Y_v >$, $X_v \in \{U, R, B\}$ and $Y_v \in \{U, R\}$.
2. Projecting the states of the coupled process onto either component faithfully yields the respective process. Thus, if $< X_v, Y_v >$ represents the state of vertex v in the coupled process, then the $\{X_v\}$ are stochastically identical to the joint process, and the $\{Y_v\}$ are stochastically identical to the solo Red process.
3. At each step, and for any vertex state $< X_v, Y_v >$, $X_v = R$ implies $Y_v = R$.

Note that the first two properties are easily achieved by simply running *independent* joint and solo Red processes. But this will violate the third property, which yields the lemma, and thus we introduce the coupling.

For any vertex v, we define its initial coupled process state $< X_v, Y_v >$ as follows: $X_v = R$ if $v \in A_R$, $X_v = B$ if $v \in A_B$, and $X_v = U$ otherwise; and $Y_v = R$ if $v \in A_R$, and $Y_v = U$ otherwise. It is easily verified that these initial states satisfy Properties 1 and 3 above, thus encoding the initial states of the two separate processes.

Now consider the first vertex v to be updated in the coupled process. Let α_v^R denote the fraction of v's neighbors w such that $X_w = R$, and α_v^B the fraction such that $X_w = B$. Note that by the initialization of the coupled process, α_v^R is also equal to the fraction of $Y_w = r$ (which we denote $\tilde{\alpha}_v^R$).

In the joint process, the probability that v is updated to R is $h(\alpha_v^R, \alpha_v^B)$, and to B is $h(\alpha_v^B, \alpha_v^R)$. In the solo Red process, the probability that v is updated to R is $h(\alpha_v^R, 0)$, which by competitiveness is greater than or equal to $h(\alpha_v^R, \alpha_v^B)$.

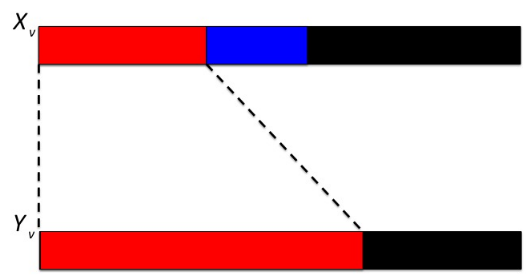

Figure 2: Illustration of the coupled dynamics defined in the proof of Lemma 1. In the update dynamic for X_v (top line), the probabilities of Red and Blue updates are represented by disjoint line segments of length $h(\alpha_v^R, \alpha_v^B)$ and $h(\alpha_v^B, \alpha_v^R)$ respectively. By competitiveness, the Red segment has length less than $h(\alpha_v^R, 0)$, which is the probability of Red update of Y_v (bottom line). The dashed red lines indicate this inequality. Thus by the arrangement of the line segments we enforce the invariant that $X_v = R$ implies $Y_v = R$.

We can thus define the update dynamics of the coupled process as follows: pick a real value z uniformly at random from $[0, 1]$. Update the state $< X_v, Y_v >$ of v as follows:

- X_v update: If $z \in [0, h(\alpha_v^R, \alpha_v^B))$, update X_v to R; if $z \in [h(\alpha_v^R, \alpha_v^B), h(\alpha_v^R, \alpha_v^B) + h(\alpha_v^B, \alpha_v^R)]$, update X_v to B; otherwise, update X_v to U. Note that the probabilities X_v are updated to R and B exactly match those of the joint process, as required by Property 2 above. See Figure 2.

- Y_v update: If $z \in [0, h(\alpha_v^R, 0)]$, update Y_v to R; otherwise, update Y_v to U. The probability Y_v is updated to R is thus exactly $h(\alpha_v^R, 0)$, matching that in a solo Red process. See Figure 2.

Since by competitiveness, $z \in [0, h(\alpha_v^R, \alpha_v^B))$ implies $z \in [0, h^R(\alpha_v^R, 0)]$, we ensure Property 3. Thus in subsequent updates we shall have $\alpha^R \leq \tilde{\alpha}^R$. Thus as long as $h(\alpha^R, \alpha^B) \leq h(\tilde{\alpha}^R, 0)$ we can continue to maintain the invariant. These inequalities follow from competitiveness and the additivity of H.

Since Properties 2 and 3 hold on an update-by-update basis in any run or sample path of the coupled dynamics, they also hold in expectation over runs, yielding the statement of the lemma. ∎(Lemma 1)

Corollary 1 *Let A_R and A_B be any sets of seeded nodes for the two players. Then if the adoption function $h(\alpha^R, \alpha^B)$ is competitive and H is additive,*

$$\mathbf{E}[\chi_R + \chi_B | (A_R, A_B)] \leq \mathbf{E}[\chi_R | (A_R, \emptyset)] + \mathbf{E}[\chi_B | (\emptyset, A_B)].$$

Proof: Follows from linearity of expectation applied to the left hand side of the inequality, and two applications of Lemma 1. ∎(Corollary 1)

Let (S_R^*, S_B^*) be the maximum joint payoff seed sets. Let (S_R, S_B) be any (pure) Nash equilibrium, with S_R having the larger budget. Corollary 1 implies that one of $\mathbf{E}[\chi_R | (S_R^*, \emptyset)]$ and $\mathbf{E}[\chi_B | (\emptyset, S_B^*)]$ is at least $\mathbf{E}[\chi_R + \chi_B | (S_R^*, S_B^*)]/2$; so assume without loss of generality that $\mathbf{E}[\chi_R | (S_R^*, \emptyset)] \geq \mathbf{E}[\chi_R + \chi_B | (S_R^*, S_B^*)]/2$. Let us now contemplate a unilateral defection of the Red player from S_R to S_R^*, in which case the strategies are (S_R^*, S_B). In the following lemma we show that *total* number of eventual adoptions for the two players is larger than adoptions accruing to a single player under solo seeding.

Lemma 2 *Let A_R and A_B be any sets of seeded nodes for the two players. If H is additive,*

$$\mathbf{E}[\chi_R + \chi_B | (A_R, A_B)] \geq \mathbf{E}[\chi_R | (A_R, \emptyset)].$$

Proof: We employ a coupling argument similar to that in the proof of Lemma 1. We define a stochastic process in which the state of a vertex v is a pair $< X_v, Y_v >$ in which the following properties are obeyed:

1. At each step, and for any vertex state $< X_v, Y_v >$, $X_v \in \{R, B, U\}$ and $Y_v \in \{R, U\}$.
2. Projecting the state of the coupled process onto either component faithfully yields the respective process. Thus, if $< X_v, Y_v >$ represents the state of vertex v in the coupled process, then the $\{X_v\}$ are stochastically identical to the joint process (A_R, A_B), and the $\{Y_v\}$ are stochastically identical to the solo Red process $(A_R, \emptyset)$.
3. At each step, and for any vertex state $< X_v, Y_v >$, $Y_v = R$ implies $X_v = R$ or $X_v = B$.

We initialize the coupled process in the obvious way: if $v \in A_R$ then $X_v = R$, if $v \in A_B$ then $X_v = B$, and $X_v = U$ otherwise; and if $v \in A_R$ then $Y_v = R$, and $Y_v = U$ otherwise. Let us fix a vertex v to update, and let α_v^R, α_v^B denote the fraction of neighbors w of v with $X_w = R$ and $X_w = B$ respectively, and let $\tilde{\alpha}_v^R$ denote the fraction with $Y_w = R$. Initially we have $\alpha_v^R = \tilde{\alpha}_v^R$.

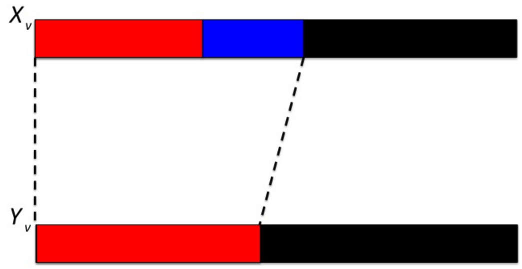

Figure 3: Illustration of the coupled dynamics defined in the proof of Lemma 2. In the update dynamic for X_v (top line), the probabilities of Red and Blue updates are represented by line segments of length $h(\alpha_v^R, \alpha_v^B)$ and $h(\alpha_v^B, \alpha_v^R)$ respectively. By additivity of H, together these two segments are greater than $h(\alpha_v^R, 0)$ which is the probability of Red update of Y_v (bottom line). This inequality is represented by the dashed black lines.

On the first update of v in the joint process (A_R, A_B), the total probability infection by either R or B is $H(\alpha_v^R, \alpha_v^B) = h(\alpha_v^R, \alpha_v^B) + h(\alpha_v^B, \alpha_v^R)$. In the solo process $(A_R, \emptyset)$, the probability of infection by R is $h(\alpha_v^R, 0) \leq h(\alpha_v^R, 0) + h(0, \alpha_v^R) = H(\alpha_v^R, 0) \leq H(\alpha_v^R, \alpha_v^B)$ where the last inequality follows by the additivity of H.

We thus define the update dynamics in the coupled process as follows: pick a real value z uniformly at random from $[0, 1]$. Update $< X_v, Y_v >$ as follows:

- X_v update: If $z \in [0, h(\alpha_v^R, \alpha_v^B))$, update X_v to R; if $z \in [h(\alpha_v^R, \alpha_v^B), h(\alpha_v^R, \alpha_v^B) + h(\alpha_v^B, \alpha_v^R)] \equiv [h(\alpha_v^R, \alpha_v^B), H(\alpha_v^R, \alpha_v^B)]$, update X_v to B; otherwise update X_v to U. See Figure 3.

- Y_v update: If $r \in [0, h(\alpha_v^R, 0))$, update Y_v to R; otherwise update Y_v to U. See Figure 3.

It is easily verified that at each such update, the probabilities of R and B updates of X_v are exactly as in the joint (A_R, A_B) process, and the probability of an R update of Y_v is exactly as in the solo $(A_R, \emptyset)$ process, thus maintaining Property 2 above. Property 3 follows from the previously established fact that $h(\alpha_v^R, 0) \leq H(\alpha_v^R, \alpha_v^B)$, so whenever Y_v is updated to R, X_v is updated to either R or B.

Notice that since $h(\alpha_v^R, 0) \geq h(\alpha_v^R, \alpha_v^B)$ by competitiveness, for the overall theorem (which requires competitiveness of h) we *cannot* ensure that $Y_v = R$ is always accompanied by $X_v = R$. Thus the Red infections in the solo process may exceed those in the joint process, yielding $\tilde{\alpha}_v^R > \alpha_v^R$ for subsequent updates. To maintain Property 3 in subsequent updates we thus require that $\tilde{\alpha}_v^R \leq \alpha_v^R + \alpha_v^B$ implies $h(\tilde{\alpha}_v^R, 0) \leq H(\tilde{\alpha}_v^R, 0) \leq H(\alpha_v^R, \alpha_v^B)$ which follows from the additivity of H. Also, notice that since the lemma holds for every fixed A_R and A_B, it also holds in expectation for mixed strategies. ∎(Lemma 2)

Continuing the analysis of a unilateral defection by the Red player from S_R to S_R^*, we have thus established

$$\begin{aligned}\mathbf{E}[\chi_R + \chi_B|(S_R^*, S_B)] &= \mathbf{E}[\chi_R|(S_R^*, S_B)] + \mathbf{E}[\chi_B|(S_R^*, S_B)] \\ &\geq \mathbf{E}[\chi_R|(S_R^*, \emptyset)] \\ &\geq \mathbf{E}[\chi_R + \chi_B|(S_R^*, S_B^*)]/2\end{aligned}$$

where the equality is by linearity of expectation, the first inequality follows from Lemma 2, and the second inequality from Corollary 1. Thus one of $\mathbf{E}[\chi_R|(S_R^*, S_B)]$ and $\mathbf{E}[\chi_B|(S_R^*, S_B)]$ must be at least $\mathbf{E}[\chi_R + \chi_B|(S_R^*, S_B^*)]/4$.

If $\mathbf{E}[\chi_R|(S_R^*, S_B)] \geq \mathbf{E}[\chi_R+\chi_B|(R^*, B^*)]/4$, then since (S_R, S_B) is Nash, we must also have $\mathbf{E}[\chi_R|(S_R, S_B)] \geq \mathbf{E}[\chi_R + \chi_B|(S_R^*, S_B^*)]/4$, and the theorem is proved. Thus the remaining case is where $\mathbf{E}[\chi_B|(S_R^*, S_B)] \geq \mathbf{E}[\chi_R+\chi_B|(S_R^*, S_B^*)]/4$. But Lemma 1 has already established that $\mathbf{E}[\chi_B|(\emptyset, S_B)] \geq \mathbf{E}[\chi_B|(S_R^*, S_B)]$, and from Lemma 2 we have $\mathbf{E}[\chi_R + \chi_B|(S_R, S_B)] \geq \mathbf{E}[\chi_B|(\emptyset, S_B)]$. Combining, we have the following chain of inequalities:

$$\mathbf{E}[\chi_R + \chi_B|(S_R, S_B)] \geq \mathbf{E}[\chi_B|(\emptyset, S_B)] \geq \mathbf{E}[\chi_B|(S_R^*, S_B)] \geq \mathbf{E}[\chi_R + \chi_B|(S_R^*, S_B^*)]/4$$

thus establishing the theorem. ∎(Theorem 1)

Recall that the switching-selection formulation in which f is concave and g is linear satisfies the hypothesis of the Theorem above. But Theorem 1 also provides more general conditions for bounded PoA in the switching-selection model. For example, suppose we consider switching functions of the form $f(x) = x^r$ for $r \leq 1$ (thus yielding concavity) and selection functions of the Tullock contest form $g(y) = 1/(1 + ((1 - y)/y))$, as discussed in Section 2.2. Letting a and b denote the local fraction of Red and Blue neighbors for notational convenience, this leads to an adoption function of the form $h(a, b) = (a + b)^r/(1 + (b/a)^s)$. The condition for competitiveness is

$$h(a, 0) - h(a, b) = a^r - (a + b)^r/(1 + (b/a)^s) \geq 0.$$

Dividing through by $(a + b)^r$ yields

$$(a/(a + b))^r - 1/(1 + (b/a)^s) = 1/(1 + (b/a))^r - 1/(1 + (b/a)^s) \geq 0.$$

Making the substitution $z = b/a$ and moving the second term to the right-hand side gives

$$1/(1 + z)^r \geq 1/(1 + z)^s.$$

Thus competitiveness is equivalent to the condition $1 + z^s \geq (1 + z)^r$ for all $z \geq 0$. Consider the choice $r = 1/2$. In this case the condition becomes $1 + z^s \geq \sqrt{1 + z}$, or $(1 + z^s)^2 \geq 1 + z$. Expanding gives $1 + 2z^s + z^{2s} \geq 1 + z$. It is easily seen this inequality is obeyed for all $z \geq 0$ provided $s \in [1/2, 1]$, since then for all $z \geq 1$ we have $z^{2s} \geq z$, and for all $z \leq 1$ we have $z^s \geq z$. More generally, if we let $r = 1/k$ for some natural number k, the competitiveness condition becomes $(1+z^s)^k \geq 1+z$. The smallest power of z generated by the left-hand side is z^s and the largest is z^{sk}. As long as $sk \geq 1$, $z^{sk} \geq z$ for $z \geq 1$, and as long as $s \leq 1$ then $z^s \geq z$ for $z \leq 1$. Thus any $s \in [1/k, 1]$ yields competitiveness. In other words, the more concave f is (i.e. the larger k is), the more non-linearly equalizing g can be (i.e. the smaller s can be) while maintaining competitiveness. By Theorem 1 we have thus shown:

Corollary 2 *Let the switching function be $f(x) = x^r$ for $r \leq 1$ and the selection function be $g(y) = 1/(1 + ((1-y)/y)^s)$. Then as long as $s \in [r, 1]$, the Price of Anarchy is at most 4 for any graph G.*

5.2 PoA: Lower Bound

We now show that concavity of the switching function is required in a very strong sense — essentially, even a a slight departure from convexity leads to unbounded PoA. As a first step in this demonstration, it is useful to begin with a simpler result showing that the PoA is unbounded if the switching function is permitted to violate concavity to an arbitrary extent.

Theorem 2 *Fix $\alpha^* \in (0, 1)$, and let the switching function f be the threshold function $f(x) = 0$ for $x < \alpha^*$, and $f(x) = 1$ for $x \geq \alpha^*$. Let the selection function be linear $g(y) = y$. Then for any value $V > 0$, there exists G such that the Price of Anarchy in G is greater than V.*

Proof: Let m be a large integer, and set the initial budgets of both players to be $\alpha^* m/2$. The graph G will consist of two components. The first component C_1 consists of two layers; the first layer has m vertices and the second n_1 vertices, and there is a directed edge from every vertex in the first layer to every vertex in the second layer. The second component C_2 has the same structure, but with m vertices in the first layer and n_2 in the second layer. We let $n_2 >> n_1 >> m$. For concreteness, let us choose an update schedule that updates each vertex in the second layers of the two components exactly once in some fixed ordering (the same result holds for many other updating schedules).

It is easy to see that the maximum joint profit solution is to place the combined $\alpha^* m$ of seeds of the two players in the first layer C_2, in which case the number of second-layer infections will be n_2 since $f(\alpha^*) = 1$. Any configuration which places at least one infection in each of the two components will not cause any second-layer infections, since then the threshold of f will not be exceeding in either component.

It is also easy to see that both players placing all their infections in the first layer of C_1, which will result in n_1 infections in the second layer since the threshold is exceeded, is a Nash equilibrium. Any defection of a player to C_2, or to layer 2 of C_1, causes the threshold to no longer be exceeded in either component. Thus the PoA here is n_2/n_1, which can be made arbitrarily large. Note that the maximum joint infections solution is also a Nash equilibrium — we are exploiting the worst-case (over Nash) nature of the PoA here (as will all our lower bounds). ∎(Theorem 2)

Thus, a switching function strongly violating concavity can lead to unbounded PoA even with a linear selection function. But it turns out that functions even *slightly* violating concavity also cause unbounded PoA — as we shall see, network structure can *amplify* small amounts of convexity.[8]

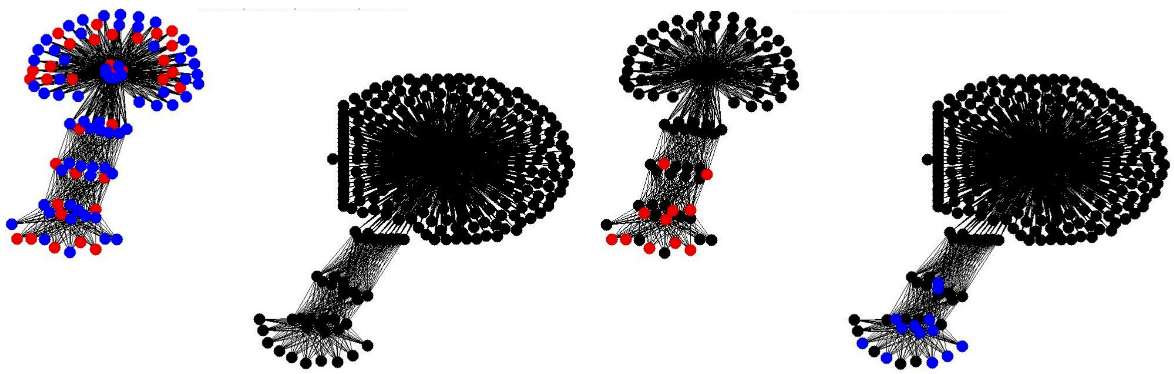

Figure 4: Illustration of convexity amplification in the Price of Anarchy lower bound of Theorem 3, under convex switching function $f(x) = x^3$ and linear selection function g. Left: Two-component, directed, layered "flower" graph, with the right flower having many more petals than the left. In one equilibrium, both players play in the first layer of the stem of the smaller flower. The convexity of f does not enter the dynamics, since at each update an entire successive layer is infected, quickly reaching the petals. There is also a better equilibrium (not shown), in which the players play in the first layer of the larger flower, and again there is complete infection, but with larger payoffs. Right: However, if the two players locate in different components, layers are not fully infected and the convexity of f is amplified via composition in successive layers, damping out the infection rate quickly.

Theorem 3 *Let the switching function be $f(x) = x^r$ for any $r > 1$, and let the selection function be linear $g(y) = y$. Then for any $V > 0$, there exists a graph G for which the Price of Anarchy is greater than V.*

Proof: The idea is to create a layered, directed graph whose dynamics rapidly amplify the convexity of f. Taking two such *amplification gadgets* of differing sizes yields an equilibrium in which the players coordinate on the smaller component, while the maximum joint payoffs solution lies in the larger component. The construction of the proof is illustrated in Figure 4.

The amplification gadget will be a layered, directed graph with ℓ_i vertices in the ith layer and N layers total. There are directed edges from every vertex in layer i to every vertex in layer $i+1$, and no other edges. Let the two players have equal budgets of k, and define $\alpha = 2k/\ell_1$ — thus, α is the fraction of layer 1 the two players could jointly infect.

[8] The theorem which follows considers the family $f(\alpha) = \alpha^r$, but can be generalized to other choices of convex f as well.

Let us consider what happens if indeed the two players jointly infect $2k$ vertices in the first layer, and the update schedule proceeds by updating each successive layer $2, \ldots, N$. Since every vertex in layer 2 has every vertex in layer 1 as a directed neighbor, and no others, the expected fraction of layer 2 that is infected is $f(\alpha) = \alpha^r$. Inductively, the expected fraction of layer 3 that is infected is thus $f(f(\alpha)) = \alpha^{r^2}$. In general, the expected fraction of layer i that is infected is $\alpha^{r^{i-1}}$, and by the linearity of g the two players will split these infections. Here, we note that the actual path of infections will be stochastic; this stochastic path is well approximated by the expected infections, if layers are sufficiently large. Throughout this proof we will use this approximation (which relies on an appeal to the strong law of large numbers).

Now let $\alpha = \beta_1 + \beta_2$, and let us instead place $\beta_1 \ell_1$ seeds at layer 1 and $\beta_2 \ell_1$ at layer i. The total number of infections expected at layer i now becomes $\beta_1^{r^{i-1}} \ell_i + \beta_2 \ell_1$. By the convexity of the function $f^{(i-1)}(x) = x^{r^{i-1}}$, this will be smaller than $\alpha^{r^{i-1}} \ell_i = (\beta_1 + \beta_2)^{r^{i-1}} \ell_i$ as long as $\beta_2 \ell_1 < \beta_2^{r^{i-1}} \ell_i$, or $\ell_i > \ell_1 / \beta_2^{r^{i-2}}$. Also, notice that the smallest nonzero deviation requires $\beta_2 \ell_1 \geq 1$, or $\beta_2 \geq 1/\ell_1$. Thus as long as $\ell_i \geq \ell_1^{r^{i-1}}$, the total fraction of infections generated by placing $\beta_1 \ell_1$ seeds at layer 1 and $\beta_2 \ell_1$ at layer i will be less than by placing all in layer 1. Furthermore, by the linearity of g, any individual player who effects such a unilateral deviation will suffer.

Note that we can make the final, Nth, layer arbitrarily large. In particular, if we choose $\ell_i = \ell_1^{r^{i-1}}$ as specified above for all $2 \leq i \leq N-1$, and choose $\alpha^{r^{N-1}} \ell_N >> \sum_{i=1}^{N-1} \alpha^{r^{i-1}} \ell_i$, the total expected number of infections conditioned on both players playing in the first layer will be dominated by the $\alpha^{r^{N-1}} \ell_N$ expected infections in the final layer.

Now consider a graph consisting of *two* disjoint amplification gadgets G_1 and G_2 that are exactly as described above, but differ only in the sizes of their final Nth layers — $\ell_N(1)$ for G_1 and $\ell_N(2)$ for G_2, where we will choose $\ell_N(2) >> \ell_N(1)$. Consider a configuration where all seeds are in the first layer of G_1. We have already argued above that no deviation to later layers of G_1 can be profitable. Now let us consider a unilateral deviation of the Red player from G_1 to the first layer of G_2. Since Red alone infects now only infects a fraction $\alpha/2$ of the ℓ_1 vertices in the first layer of G_2, the expected final number of Red infections will be approximately $(\alpha/2)^{r^{N-1}} \ell_N(2)$, compared with $\alpha^{r^{N-1}} \ell_N(1)/2$ for not deviating from G_1. Thus as long as $(\alpha/2)^{r^{N-1}} \ell_N(2) \leq \alpha^{r^{N-1}} \ell_N(1)/2$, or $\ell_N(2)/\ell_N(1) \leq 2^{r^{N-1}-1}$, this deviation is unprofitable for Red. More generally, if Red unilaterally divides its $(\alpha/2)\ell_1$ resources by placing a fraction β_1 of them in the first layer of G_1 and a fraction $\beta_2 = 1 - \beta$ of them in the first layer of G_2, its expected payoff is

$$[(1 + \beta_1)(\alpha/2)]^{r^{N-1}} \ell_N(1) \frac{\beta_1}{1 + \beta_1} + [\beta_2(\alpha/2)]^{r^{N-1}} \ell_N(2).$$

The first term of this sum represents the share of the final layer of G_1 that Red obtains given that Blue is playing entirely in this component, while the second term represents the uncontested infections Red wins in G_2. This expression can be written as

$$\alpha^{r^{N-1}} \left[\left(\frac{1 + \beta_1}{2} \right)^{r^{N-1}} \ell_N(1) \frac{\beta_1}{1 + \beta_1} + \left(\frac{1 - \beta_1}{2} \right)^{r^{N-1}} \ell_N(2) \right]$$

which for the choice $\ell_N(2) = 2^{r^{N-1}} \ell_N(1)/2$ becomes

$$\alpha^{r^{N-1}} \left[\left(\frac{1+\beta_1}{2} \right)^{r^{N-1}} \ell_N(1) \frac{\beta_1}{1+\beta_1} + (1-\beta_1)^{r^{N-1}} \ell_N(1)/2 \right].$$

For any $0 < \beta_1 < 1$, both terms inside the brackets above are exponentially damped and result in suboptimal payoff for Red. Thus the best response choices are given by the extremes $\beta_1 = 1$ and $\beta_1 = 0$, which both yield expected payoff $\ell_N(1)/2$ for Red. (Note that by choosing $\ell_N(2)$ slightly smaller above, we can force $\beta_1 = 1$ to be a strict best response.)

However, the maximum joint payoffs solution (as well as the best, as opposed to worst Nash equilibrium) is for *both* players to initially infect in the first layer of G_2, in which case the total payoff will be approximately $\alpha^{r^{N-1}} \ell_N(2)$. The Price of Anarchy is thus

$$\frac{\alpha^{r^{N-1}} \ell_N(2)}{\alpha^{r^{N-1}} \ell_N(1)} = \frac{\ell_N(2)}{\ell_N(1)} \geq 2^{r^{N-1}-1}$$

by the choice of $\ell_N(2)$ above. Thus by choosing the number of layers N as large as needed, the Price of Anarchy exceeds any finite bound V. ∎(Theorem 3)

Combining Theorem 1 and Theorem 3, we note that for $f(x) = x^r$ and linear g we obtain the following sharp threshold result:

Corollary 3 *Let the switching function be $f(x) = x^r$, and let the selection function be linear, $g(y) = y$. Then:*

- *For any $r \leq 1$, the Price of Anarchy is at most 4 for any graph G;*
- *For any $r > 1$ and any V, there exists a graph G for which the Price of Anarchy is greater than V.*

6 Results: The Price of Budgets

As we did for the PoA in Section 5, in this section we derive sufficient conditions for bounded PoB, and show that even slight violations of these conditions can lead to unbounded PoB.

6.1 PoB: Upper Bound

As in the PoA analysis, it will be technically convenient to return to the generalized adoption function model. Recall that for PoA, competitiveness of h and additivity of H were needed to prove upper bounds, but we didn't require that the implied selection function be linear. Here we introduce that additional requirement, and prove that the (pure strategy) PoB is bounded.

Theorem 4 *Suppose the adoption functions $h(\alpha^R, \alpha^B)$ is competitive, that H is additive in its arguments, and that implied selection function is linear: $g(\alpha_R, \alpha_B) = h(\alpha_R, \alpha_B)/(h(\alpha_R, \alpha_B) + h(\alpha_B, \alpha_B)) = \alpha_R/(\alpha_R + \alpha_B)$. Then the pure strategy [9] Price of Budgets is at most 2 for any graph G.*

Proof: The proof borrows elements from the proof of Theorem 1, and introduces the additional notion of tracking or attributing indirect infections generated by the dynamics to specific seeding strategies.

Consider any pure Nash equilibrium given by seed sets S_R and S_B in which $|S_R| = K > |S_B| = L$. For our purposes the interesting case is one in which $\mathbf{E}[\chi_R|(S_R, S_B)] \geq \mathbf{E}[\chi_B|(S_R, S_B))]$ and so $\mathbf{E}[\chi_R|(S_R, S_B)] \geq \mathbf{E}[\chi_R + \chi_B|(S_R, S_B)]/2$. Since the adoption function is competitive and additive, Lemma 1 implies that $\mathbf{E}[\chi_R|(S_R, \emptyset)] \geq E[\chi_R|(S_R, S_B)]$ — that is, the Red player only benefits from the departure of the Blue player.

Let us consider the dynamics of the solo Red process given by $(S_R, \emptyset)$. We first introduce a faithful simulation of these dynamics that also allows us to attribute subsequent infections to exactly one of the seeds in S_R; we shall call this process the *attribution simulation* of $(S_R, \emptyset)$. Thus, let $S_R = \{v_1, \ldots, v_K\}$ be the initial Red infections, and let us label v_i by R_i, and label all other vertices U. All infections in the process will also be assigned one of the K labels R_i in the following manner: when updating a vertex v, we first compute the fraction α_v^R of neighbors whose current label is one of $R_1, \ldots, R_K$, and with probability $H(\alpha_v^R, 0) = h(\alpha_v^R, 0) + h(0, \alpha_v^R)$ we decide that an infection will occur (otherwise the label of v is updated to U). If an infection occurs, we simply choose an infected neighbor of v uniformly at random, and update v to have the same label (which will be one of the R_i). It is easily seen that at every step, the dynamics of the $(S_R, \emptyset)$ process are faithfully implemented if we drop label subscripts and simply view any label R_i as a generic Red infection R. Furthermore, at all times every infected vertex has only one of the labels R_i. Thus if we denote the expected number of vertices with label R_i by $\mathbf{E}[\chi_{R_i}|(S_R, \emptyset)]$, we have $\mathbf{E}[\chi_R|(S_R, \emptyset)] = \sum_{i=1}^{K} \mathbf{E}[\chi_{R_i}|(S_R, \emptyset)]$. Let us assume without loss of generality that the labels R_i are sorted in order of decreasing $\mathbf{E}[\chi_{R_i}|(S_R, \emptyset)]$.

We now consider the payoff to the Blue player under a unilateral defection from S_B to the set $\hat{S}_B = \{v_1, \ldots, v_L\} \subset S_R$ — that is, the L "most profitable" initial infections in S_R. Our goal is to show that the Blue player must enjoy roughly the same payoff from these L seeds as the Red player did in the solo attribution simulation.

Lemma 3 $\mathbf{E}[\chi_B|(S_R, \hat{S}_B)] \geq \frac{1}{2} \sum_{i=1}^{L} \mathbf{E}[\chi_{R_i}|(S_R, \emptyset)] \geq \frac{L}{2K} \mathbf{E}[\chi_R|(S_R, \emptyset)]$.

Proof: The second inequality follows simply from $\mathbf{E}[\chi_R|(S_R, \emptyset)] = \sum_{i=1}^{K} \mathbf{E}[\chi_{R_i}|(S_R, \emptyset)]$, established above, and fact that the vertices in S_R are ordered in decreasing profitability. For the first inequality, we introduce coupled attribution simulations for the two processes $(S_R, \emptyset)$

[9]The theorem actually holds for any equilibrium in which the player with the larger budget plays a pure strategy; the player with smaller budget may always play mixed. It is easy to find cases with such equilibria. The theorem also holds for general mixed strategies under certain conditions — for instance, when both f and g are linear and the larger budget is an integer multiple of the smaller.

(the solo Red process) and $(S_R, \hat{S}_B)$. For simplicity, let us actually examine $(S_R, \emptyset)$ and $(S_R - \hat{S}_B, \hat{S}_B)$; the latter joint process is simply the process $(S_R, \hat{S}_B)$, but in which the contested seeded nodes in $\hat{S}_B$ are all won by the Blue player. (The proof for the general $(S_R, \hat{S}_B)$ case is the same but causes the factor of 1/2 in the lemma.)

The coupled attribution dynamics are as follows: as above, in the solo Red process, for $1 \leq i \leq L$, the vertex v_i in S_R is initially labeled R_i, and all other vertices are labeled U. In the joint process, the vertex v_i is labeled B_i for $i \leq L$ (corresponding to the Blue invasions of S_R), while for $L < i \leq K$ the vertex v_i is labeled R_i as before. Now at the first update vertex v, let α_v^R be the fraction of Red neighbors in the solo process, and let $\tilde{\alpha}_v^R$ and $\tilde{\alpha}_v^B$ be the fraction of Red and Blue neighbors, respectively, in the joint process.

Note that initially we have $\alpha_v^R = \tilde{\alpha}_v^R + \tilde{\alpha}_v^B$. Thus by additivity H, the total probabilities of infection $H(\alpha_v^R, 0)$ and $H(\tilde{\alpha}_v^R, \tilde{\alpha}_v^B)$ in the two processes must be identical. We thus flip a common coin with this shared infection probability to determine whether infections occur in the coupled process. If not, v is updated to U in both processes. If so, we now use a coupled attribution step in which we pick an infected neighbor of v at random and copy its label to v in *both* processes. Thus if a label with index $i \leq L$ is chosen, v will be updated to R_i in the solo process, and to B_i in the joint process; whereas if $L < i \leq K$ is chosen, the update will be to R_i in both processes. It is easily verified that each of the two processes faithfully implement the dynamics of the solo and joint attribution processes, respectively.

This coupled update dynamic maintains two invariants: infections are always matched in the two processes, thus maintaining $\alpha_v^R = \tilde{\alpha}_v^R + \tilde{\alpha}_v^B$ for all v and every step; and for all $i \leq L$, every R_i attribution in the solo Red process is matched by a B_i attribution in the joint process, thus establishing the lemma. ∎(Lemma 3)

Thus, by simply imitating the strategy of the Red player in the L most profitable resources, the Blue player can expect to infect $(1/2)(L/K)$ proportion of infections accruing to Red in isolation. Since (S_R, S_B) is an equilibrium, the payoffs of Blue in equilibrium must also respect this inequality. ∎(Theorem 4)

6.2 PoB: Lower Bound

We have already seen that concavity of f and linearity of g lead to bounded PoA and PoB, and that even slight deviations from concavity can lead to unbounded PoA. We now show that fixing f to be linear (which is concave), slight deviations from linearity of g towards polarizing g can lead to unbounded PoB, for similar reasons as in the PoA case: graph structure can amplify a slightly polarizing g towards arbitrarily high punishment of the minority player.

Theorem 5 *Let the switching function be linear, $f(x) = x$, and let the selection function be of Tullock contest form, $g(y) = 1/(1 + ((1 - y)/y)^s)$, where $s > 1$. Then for any $V > 0$, there exists a graph G for which the Price of Budgets is greater than V.*

Proof: As in the PoA lower bound, the proof relies on a layered amplification graph, this time amplifying punishment in the selection function rather than convexity in the switching function. The graph will consist of two components, C_1 and C_2.

Let us fix the budget of the Red player to be 3, and that of the Blue player to be 1 (the proof generalizes to other unequal values). C_1 is a directed, layered graph with $k+1$ layers. The first layer has 4 vertices, and layers 2 through k have $n >> 4$ vertices, while layer $k+1$ has n_1 vertices, where we shall choose $n_1 >> n$, meaning that payoffs in C_1 are dominated by infections in the final layer.

The second component C_2 is a 2-layer directed graph, with 1 vertex in the first layer and n_2 in the final layer, and all directed edges from layer 1 to 2. We will eventually choose $n_2 << n_1$, so that C_1 is the much bigger component. We choose an update rule in which each layer is updated in succession and only once.

Consider the configuration in which Red places its 3 infections in the first layer of C_1, and Blue places its 1 infection in the first layer of C_2. We shall later show that this configuration is a Nash equilibrium. In this configuration, the expected payoff to Red is approximately $\sum_{i=2}^{k}(3/4)n + (3/4)n_1$ by linearity of f; notice that the selection function does not enter since the players are in disjoint components. Similarly, the expected payoff to Blue is n_2. In this configuration, the ratio of Red and Blue expected payoffs is thus at least $(3/4)n_1/n_2$, whereas the initial budget ratio is 1/3. So the PoB for this configuration is at least $n_1/(4n_2)$.

We now develop conditions under which this configuration is an equilibrium. It is easy to verify that red is playing a best response. Moving vertices to later layers of C_1 lowers Red's payoff, since $n >> 4$ and f is linear. Finally, moving infections to invade the first layer of C_2 will lower Red's payoff as long as, say, $(1/4)n_1$ (Red's current payoff per initial infection in the final layer of C_1) exceeds n_2 (the maximum amount Red could get in C_2 by full defection), or $n_1 >> 4n_2$.

We now turn to deviations by Blue. Moving the solo Blue initial infection to the second layer of C_2 is clearly a losing proposition. So consider deviations in which Blue moves to vertices in component 1. If he moves to the lone unoccupied vertex in layer 1 of C_1, his payoff is approximately:

$$\sum_{i=2}^{k} g^{(i)}(1/4)n + g^{(k+1)}(1/4)n_1 = \sum_{i=2}^{k} \frac{(1/4)^{s^i}}{(1/4)^{s^i} + (3/4)^{s^i}} n + \frac{(1/4)^{s^{k+1}}}{(1/4)^{s^{k+1}} + (3/4)^{s^{k+1}}} n_1$$

Similarly, if Blue directly invades a Red vertex, Blue's payoff is approximately

$$\chi = \sum_{i=2}^{k} \frac{(1/3)^{s^i}}{(1/3)^{s^i} + (2/3)^{s^i}} n + \frac{(1/3)^{s^{k+1}}}{(1/3)^{s^{k+1}} + (2/3)^{s^{k+1}}} n_1$$

Since in both cases Blue's payoff is being exponentially dampened at each successive layer, it is easy to see that the second deviation is more profitable. Finally, Blue may choose a vertex in a later layer of C_1, but again by $n >> 4$ and the linearity of f, this will be suboptimal.

Thus as long as we arrange that n_2 — Blue's payoff without deviation — exceeds χ above, we will have ensured that no player has an incentive to deviate from the specified strategy configuration. Let us scale n_1 up as large as necessary to have χ dominated by the term involving n_1, and now set n_2 to equal that term:

$$n_2 = \frac{(1/3)^{s^{k+1}}}{(1/3)^{s^{k+1}} + (2/3)^{s^{k+1}}} n_1$$

in order to satisfy the equilibrium condition. The ratio n_1/n_2, which we have already shown above lower bounds the PoB, is thus a function that is increasing exponentially in k for any fixed $s > 1$. Thus by choosing k sufficiently large, we can force the PoB larger than any chosen value. This completes the proof. ∎(Theorem 5)

Combining Theorem 4 and Theorem 5, we note that for linear f and Tullock g, we obtain the following sharp threshold result, which is analogous to the PoB result in Corollary 3.

Corollary 4 *Let the switching function f be linear, and let the selection function g be Tullock, $g(y) = 1/(1 + ((1-y)/y)^s)$, where $s > 1$. Then:*

- *For $s = 1$, the Price of Budgets is at most 2 for any graph G;*
- *For any $s > 1$ and any V, there exists a graph G for which the Price of Budgets is greater than V.*

In fact, if we permit a slight generalization of our model, in which certain vertices in the graph are "hard-wired" to adopt only one or the other color (so there is no use for the opposing player to seed them), unbounded PoB also holds in the Tullock case for $s < 1$ (equalizing). So in this generalization, linearity of g is required for bounded PoB.

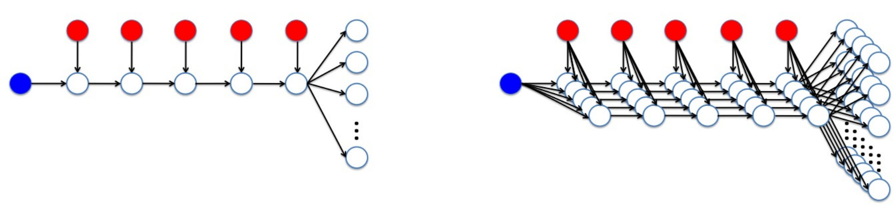

Figure 5: Illustration of the construction in the proof of Theorem 6.

We have thus shown that even when the switching function is "nice" (linear), even slight punishment in the selection function can lead to unbounded PoB. Recall that we require switching and selection functions to be 0 (1, respectively) on input 0 (respectively) and increasing, and additionally that $g(1/2) = 1$. The following theorem shows that if f is allowed to be a sufficiently convex function, then the PoB is again unbounded for *any* selection function. This establishes the importance of concavity of f for both bounded PoA and PoB.

Theorem 6 *Let the switching f satisfy $f(1/2) = 0$ and $f(1) = 1$, and let the selection function g satisfy $g(1/2) = 1$. Then for any value $V > 0$, there exists G such that the Price of Budgets is greater than V.*

Proof: Let the Blue player have 1 initial infection and the Red player have $K \geq 2$ (the proof can be generalized to any unequal initial budgets, which we comment on below). Consider the directed graph shown in the left panel of Figure 5, where we have arranged the 1 Blue and K Red seeded nodes in a particular configuration. Aside from the initially infected vertices, this graph consists of a directed chain of K vertices, whose final vertex then connects to a large number $N >> K$ of terminal vertices. Let us update each vertex in the chain from left to right, followed by the terminal vertices.

Let us first compute the expected payoffs for the two players in this configuration. First, note that since $f(1) = 1$, it is certain that every vertex in the chain will be infected in sequence, followed by all of the terminal vertices; the only question is which player will win the most. By choosing $N >> K$ we can ignore the infections in the chain and just focus on the terminal vertices, which will be won by whichever player infects the final chain vertex. It is easy to see that the probability this vertex is won by Blue is $1/2^K$, since Blue must "beat" a competing Red infection at every vertex in the chain. Thus the expected payoffs are approximately $N/2^K$ for Blue and $N(1 - 1/2^K)$ for Red. If this configuration were an equilibrium, the PoB would thus be $2^K/K$, which can be made as large as desired by choosing K large enough.

However, this configuration is not an equilibrium — clearly, either player would be better off by simplifying initially infecting the final vertex of the chain, thus winning all the terminal vertices. This is fixed by the construction shown in the right panel of Figure 5, where we have replicated the chain and terminal vertices M times, but have only the original $K + 1$ seeded nodes as common "inputs" to all of these replications. Notice that now if either player defects to an uninfected vertex, neither player will receive *any* infections in any of the other replications, since now there is a missing "input infection" and reaching the terminal vertices requires all $K + 1$ input infections since $f(1/2) = 0$ (each chain vertex has two inputs, and if either is uninfected, the chain of infections halts). Similarly, if either player attempts to defect by invading the seeded nodes of the other player, there will be no payoff for either player in any of the replications. Thus the most Blue can obtain by defection is N (moving its one infection to the final chain vertex of a single replication), while the most Red can obtain is KN (moving all of their infections to the final chain vertices of K replications. The equilibrium requirements are thus $M(N/2^K) > N$ for Blue, and $MN(1 - 1/2^k) > KN$ for Red. The Blue requirement is the stronger one, and yields $M > 2^K$. The PoB for this configuration is the same as for the single replication case, and thus if we let K be as large as desired and choose $M > 2^K$, we can make the PoB exceed any value. ■(Theorem 6)

It is worth noting that even if the Blue player has $L > 1$ seeded nodes, and we repeat the construction above with chain length $K + L - 1$, but with Blue forced to play at the beginning of the chain, followed by all the Red infections, the argument and calculations above are unchanged: effectively, Blues L seeded nodes are no better than 1 infection, because they are simply causing a chain of $L-1$ Blue infections before than facing the chain of K Red inputs. In fact, even if we let $L >> K$, Blue's payoff will still be a factor of $1/2^K$ smaller than Red's. Thus in some sense the theorem shows that if f is sufficiently convex, not only is the PoB unbounded, but the much smaller initial budget may yield arbitrarily higher payoffs!

7 Concluding Remarks

We have developed a general framework for the study of competition between firms who use their resources to maximize adoption of their product by consumers located in a social network. This framework yields a very rich class of competitive strategies, which depend on subtle ways on the dynamics, the relative budgets of the players and the structure of the social network. We identified properties of the dynamics of local adoption under which resource use by players is efficient or inefficient. Similarly, we identified adoption dynamics for which networks neutralize or accentuate ex-ante resource difference across players.

There are a number of other questions which can be fruitfully investigated within our framework. We assumed that players' budgets are exogenously given. In many contexts, the budget may itself be a decision variable. It is important to understand if endogenous budgets would aggravate or mitigate the problem of high PoA. Similarly, large network advantages from resources (reflected in high PoB) create an incentive to increase budgets, and may be self-neutralizing. To see how endogenous budgets can have a big impact consider the case where switching function is concave and selection function is linear. Suppose a player can purchase one unit of resource at cost $c = 1/2$. The final payoffs to a player are then equal to the number of adoptions less the seed expenditures. When the network is a star with a single hub, it is an equilibrium for both players to buy $n/2$ units and locate at the hub, thus yielding a payoff 0. By contrast, the joint payoffs are maximized with one unit of resource and yields total joint payoffs $n - 1/2$. The PoA is unbounded.

Similarly, the order of moves can have a big impact in certain networks. Suppose Red moves first and has budget 1, while Blue moves second and has budget 2. The switching function and selection function are linear and a consumer is perpetually active until he adopts. In the ring network, Red will earn 1, while Blue will earn $n-2$. The PoB can be made arbitrarily large by raising the value of n. By contrast, in the star network, the PoB is equal to 1, irrespective of the order of moves of the players.

Other interesting directions include algorithmic issues such as computing equilibria and best responses in our framework, and how their difficulty depends on the switching and selection functions; and the multi-stage version of our game, in which the two firms may gradually spend their seed budgets in a way that depends on the evolving state of the network.

8 Appendix: Consumer Decision Problem

In this section, we illustrate how the switching and selection functions f-g may arise out of optimal decisions made by consumers located in social networks. Information sharing about products and direct advantages accruing from adopting compatible products are two important ingredients of local social influence.

Example 1: Information Sharing: Consumers are looking for a good whose utility depends on its quality; the quality is known or easily verified upon inspection (such products are referred to as 'search' goods), but its availability may not be known. Examples of such products are televisions and desk-top computers. Consumers search on their own and they also get information from their friends and neighbors. Suppose for simplicity that the consumer talks to one friend before making his decision. As he runs into friends at random, other things being equal, the probability of adopting a product is equal to the probability of meeting someone who has adopted it. This probability is in turn given by the fraction of neighbors who have adopted the product. This corresponds to the case where f and g are both linear.

Example 2: Information Sharing and Payoff Externalities. Consumers are choosing between goods whose utility depends on how many other consumers have adopted the same product. Prominent examples include social networking sites. Suppose products offer stand alone advantage v, and a adoption related reward which is equal to # Reds or # Blues. Consumer picks neighbors at random. If neighbor is Red or Blue, then consumer becomes aware of product market. There is a small cost (relative to v) at which he can ask all his neighbors about their status. He then compared the adoption rates of the two products and given the payoffs benefits to being in a larger (local) network, the consumer selects the more popular product. This situation gives rise to an f which is increasing and concave in the fraction of adopters, while g is polarizing (close to a winner-take-it all).

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