What Model for Entry in First-Price Auctions? A Nonparametric Approach*

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Abstract

We develop a nonparametric approach that allows one to discriminate among alternative models of entry in first-price auctions. Three models of entry are considered: Levin and Smith (1994), Samuelson (1985), and a new model in which the information received at the entry stage is imperfectly correlated with valuations. We derive testable restrictions that these three models impose on the quantiles of active bidders' valuations, and develop nonparametric tests of these restrictions. We implement the tests on a dataset of highway procurement auctions in Oklahoma. Depending on the project size, we find no support for the Samuelson model, some support for the Levin and Smith model, and somewhat more support for the new model.

1 Introduction

A robust and well-documented feature of many real-world auctions is that not all bidders who are eligible to submit a bid choose to do so, suggesting that entry into the auction may be costly. In this paper, we develop nonparametric approaches that will allow the empirical researcher to discriminate among different models of entry.

Most of the empirical auctions literature to date is based on the theoretical work of Levin and Smith (1994) (LS hereafter). In their model, potential bidders are initially uninformed about their valuations of the good, but may become informed and submit a bid at a cost. In

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equilibrium, the potential entrants randomize their entry decisions and earn zero expected profit.

Several empirical papers, most of them recent, estimated variants of this model. Bajari and Hortacsu (2003) have studied entry and bidding in eBay auctions, within a common value framework. A Bayesian estimation method is implemented using a dataset of mint and proof sets of US coins. The magnitude of the entry cost is estimated, and expected seller revenues are simulated under different reserve prices. Athey, Levin, and Seira (2004) estimate a model of bidding in timber auctions with costly entry. The entry cost is assumed to be private information of the potential bidders, who sort into the pool of entrants based on their draws of the entry cost.

Li and Zheng (2005) study entry and bidding for lawn mowing contracts using the LS model. To our knowledge, it is the first paper in the literature that utilizes the number of planholders as a measure of potential competition in highway procurement. Li and Zheng (2005) propose and implement a Bayesian estimation method and use their structural estimates to investigate the effect of restricting potential competition on the expected revenue. In addition, Li (2005) develops a general parametric approach for auctions with entry. Krasnokutskaya and Seim (2006) study bid preference programs and bidder participation using California data. Their paper also uses the LS model, and as in Athey, Levin, and Seira (2004), the focus is on asymmetric equilibria. Bajari, Hong, and Ryan (2004) propose a parametric likelihood-based estimation strategy in the presence of multiple equilibria, and apply it to highway procurement auctions, using the LS model.

An alternative model of entry was developed in Samuelson (1985) (S hereafter). In this model, bidders make their entry decisions *after* they have learned their valuations. The entry cost is interpreted solely as the cost of preparing a bid, and bidders choose to enter if their valuations exceed a certain cutoff. The set of entrants is therefore a selected sample, biased towards bidders with higher valuations. We are not aware of any published work applying this model to data.[1]

Both LS and S models are stylized to capture the amount of information available to bidders at the entry stage: no information is available in LS, while the information is perfect in S. These polar assumptions lead to drastically differing policy implications. One of the most important and well-studied policy instruments in auctions is the reserve price. In a seminal paper, Riley and Samuelson (1981) show that, when the entry costs are null, the optimal policy for the seller is to set the reserve price above the level that he would be willing to accept. Moreover, the optimal reserve price does not depend on the number of potential bidders N. In the S model, while the optimal reserve price is also above the seller's willingness to accept, it *increases* with N. LS, on the other hand, reach a striking

[1] In a recent working paper, Xu (2007) adopts Samuelson's model to estimate the entry cost in Michigan highway procurement auctions.

conclusion that it is optimal to set the reserve price *at* the maximal willingness to accept level.

Given that policy implications are so different, it is important to be able to discriminate between these models empirically. We build on the insight in Haile, Hong, and Shum (2003) (HSS hereafter) and propose to use exogenous variation in N as a basis for such a test. Let $F(v)$ denote the cumulative distribution function (CDF) of valuations, and let $F^*(v|N)$ denote the CDF for those potential bidders that have submitted a bid. The CDF $F^*(v|N)$ is a crucial parameter whose behavior across N allows us to discriminate among the alternative models of entry. Following the approach of Guerre, Perrigne, and Vuong (2000) (GPV hereafter) we show that this distribution can be nonparametrically identified in both models if the number of potential bidders and all bids in each auction are observed. We show that, while $F^*(v|N)$ does not depend on N in the LS model, it does in the S model. The intuition here is simply that, in the S model, the valuations of active bidders are truncated by the entry cutoffs $v^*(N)$ that depend on N, but all share the same parent distribution across N.

Noting that

$$1 - F(v^*(N)) = p(N),$$

where $p(N)$ is the probability of submitting a bid, we can write the following formula for the parent distribution

$$F(v) = p(N) F^*(v|N) + 1 - p(N).$$

Since the l.h.s. doesn't depend on N, we obtain a stark testable restriction for $F^*(v|N)$ and $p(N)$.

It is not too hard to show that this restriction implies a stochastic dominance ordering for $F^*(v|N)$:

$$F^*(v|N) \geq F^*\left(v|N'\right) \text{ for } N' > N. \tag{1.1}$$

In other words, as N becomes larger, the distribution become more tilted towards bidders with higher valuations. This is of course an intuitive implication of selective entry. It is also trivially satisfied by the LS model, with equality signs for all N.

In this paper, we also propose a generalized model that allows for selective entry but dispenses with the stark assumption that potential bidders perfectly know their valuations at the entry stage as in S, thus sharing with the LS model a costly valuation discovery stage. It formally nests the LS model. This model, which we refer to as the *generalized* LS (GLS hereafter), is as follows. At the entry stage, the potential bidders each observe a private signal correlated with their yet unknown valuation of the good. Based on this private signal, a bidder may learn the valuation upon incurring an entry cost k. The bidder who entered will only bid if the valuation exceeds the reserve price. The signals may be informative about the valuations, however unlike in the S model, they are not perfectly informative. Both LS and S models can be viewed as its limit cases: the LS model corresponds to uninformative signals, while the S model corresponds to perfectly informative signals.

Models similar to GLS have been looked at in the literature. Hendricks, Pinkse, and Porter (2003) estimate a model of bidding for off-shore oil. They sketch a model of entry that is in some respects similar to ours, but with a common-value component. The focus of their paper is however not on entry but on testing an equilibrium model of bidding. The model is also outlined in the concluding section of Ye (2005).

To implement the tests, we follow the approach of GPV and show that the distribution of entrants' valuations can be nonparametrically identified from the data if N and all bids in each auction are observed. This enables us to develop a nonparametric quantile-based test of selective entry in the spirit of Haile, Hong, and Shum (2003).

Although our approach shares with Haile, Hong, and Shum (2003) the basic idea that exogenous variation in the number of bidders can be used for testing the information environment of the game, there is a number of important differences. Haile, Hong, and Shum (2003) consider a different model in which bidders' valuations may have a common component. They propose a test for common values based on the variation in the number of *actual* bidders, while we test for selective entry using the variation in the number of *potential* bidders.

Our approach is also different in the implementation in that we use a direct quantile estimation method. The method is easy to implement, does not require the computation of pseudo values of GPV, and also allows arbitrary form of dependence on covariates. This last feature is particularly important since the method of covariate control in Haile, Hong, and Shum (2003) is not applicable in the setting with entry considered in this paper.[2]

In our empirical application, we use a dataset of auctions conducted by the Oklahoma Department of Transportation (ODOT). In addition to all winning and losing bids and certain project characteristics, we also observe the number of firms that obtained construction plans, a variable that can serve as a reasonable proxy for the number of potential bidders. We argue that, because the qualification process essentially selects bidders based on working capital requirements, the number of planholders may be assumed to be exogenous. The empirical results are somewhat mixed, but we do have a number of findings. First, the S model is robustly rejected. Second, there is some support for the LS model, but somewhat more support for the GLS model.

[2] See our discussion in Section 4.4.

2 Three models of entry and their testable restrictions

2.1 The LS and S models of entry

The LS and S models share a common structure. There is an entry stage in which N potential bidders contemplate entry into the auction. At the auction stage, a bidding game transpires among those bidders that have entered. The auction is first-price sealed bid, possibly with a reserve price r. Only the bidders with valuations above the reserve price actually submit bids. We call them *actual* bidders. We assume the Independent Private Values (IPV) environment. The bidders' valuations are distributed according to the CDF $F(\cdot)$ that has support $[\underline{v}, \bar{v}]$, a corresponding density $f(\cdot)$ positive on the support. Entry is costly; only the bidders that have incurred the entry cost k can bid in the auction.

The two models differ in the information available at the entry stage. The LS model assumes that no information is available. Upon incurring the entry cost, the bidders learn their valuations and proceed to the bidding stage. Only the entrants with $v \geq r$ submit a bid. Levin and Smith characterize a symmetric perfect-Bayesian equilibrium of this game in which bidders submit a bid with probability $p \in [0,1]$. The equilibrium value of p as a function of N is denoted as $p(N)$. The equilibrium is characterized in the following proposition. We assume that the reserve price is binding, but the result carries over with minor changes to the case when it is not binding.

Proposition 1 ***(Levin and Smith, 1994; Milgrom, 2004)*** *A symmetric equilibrium is characterized by the probability of submitting a bid $p(N)$ and bidding strategy $B_N(v)$. The ex-ante equilibrium profit from bidding is equal to*

$$\Pi_N(p(N)) = \int_r^{\bar{v}} (1 - F(v))(1 - p(N) + p(N) F^*(v|N))^{N-1} \, dv. \tag{2.1}$$

The equilibrium distribution of active bidders valuations is given by

$$F^*(v|N) = \frac{F(v) - F(r)}{1 - F(r)}$$

and does not depend on N. If k is smaller than the equilibrium profit when all rivals enter, i.e. $k < \Pi_N(1)$, then $p(N) = 1$. Otherwise $p(N) \in (0,1)$ is determined from the zero expected profit equation

$$k = \Pi_N(p(N)). \tag{2.2}$$

There is a qualification to be added to the above proposition, as well as to similar results for other models. Throughout the paper, we assume away the uninteresting case of the entry cost so large that there is no entry, $p(N) = 0$. The equilibrium bidding strategy $B_N(v)$ is explicitly derived in LS. The LS model has the following implications (we will show later in the paper that these implications are testable). First, since the profit function in (2.1) is decreasing in the rival bidding probability p as well as in the number of potential rivals N, we can see that the equilibrium probability of submitting a bid is at least non-increasing,

$$p(N) \geq p\left(N'\right) \;\; \forall N < N', \tag{2.3}$$

with *strict* inequality if N' is sufficiently large. Second, the distribution of entrants valuations coincides with the distribution of potential bidders valuations. The CDF of valuations conditional on entry $F^*(v|N)$ has the support $[r, \bar{v}]$ and is independent of N,

$$F^*(v|N) = F^*\left(v|N'\right) \;\; \forall N, N'. \tag{2.4}$$

In the S model, the potential bidders know their valuations already at the entry stage. In any symmetric equilibrium, a bidder whose valuation is at the lower end of the support, $v = \underline{v}$, is unable to win with a positive probability, and will not enter. Samuelson shows that there is a cutoff $v^*(N)$ such that a bidder strictly prefers to enter if and only if $v > v^*(N)$, so that the equilibrium probability of entry is $p(N) = 1 - F(v^*(N))$. Note that, since $v^*(N) \geq r$, in the S model this is the same as the probability of submitting a bid. The equilibrium is formally characterized in the following proposition.

Proposition 2 (Samuelson (1985)) *The bidding stage has a unique symmetric equilibrium, in which the bidding strategy $B_N(v)$ is an increasing and continuous function. The profit at the bidding stage of the marginal entrant with valuation $v^*(N)$ is given by $(v^*(N) - r)(1 - p(N))^{N-1}$, where $p(N) = 1 - F(v^*(N))$ is the probability of bidding. The cutoff $v^*(N)$ is determined by the requirement that bidder with valuation $v^*(N)$ makes zero expected profit:*

$$k = (v^*(N) - r)(1 - p(N))^{N-1}. \tag{2.5}$$

There is always entry with probability less than 1, i.e. $v^*(N) \in (r, \bar{v} - k)$.

The S model shares with the LS model restriction (2.3) that bidding probabilities are non-increasing (they must actually be strictly decreasing in the S model). But it implies a different restriction for the distribution of active bidders valuations $F^*(v|N)$. For $v > v^*(N)$,

$$\begin{aligned} F^*(v|N) &= \frac{F(v) - F(v^*(N))}{1 - F(v^*(N))} \\ &= \frac{F(v) - (1 - p(N))}{p(N)} \end{aligned} \tag{2.6}$$

where we used the fact that the entry probability $p(N)$ is equal to $1 - F(v^*(N))$. Since the distribution F does not depend on N, a manipulation of (2.6) leads to the following restriction of the S model:

$$p(N) F^*(v|N) + 1 - p(N) = p(N') F^*(v|N') + 1 - p(N') \ \ \forall N, N'. \tag{2.7}$$

2.2 The generalized model of entry (GLS)

A model of selective entry proposed in this paper occupies a middle ground between S and LS. Specifically, it shares with S the assumption that information about the valuation is available at the bidding stage, but dispenses with the stark assumption that this information is perfect. The game begins with the entry stage in which N potential risk-neutral bidders obtain preliminary estimates (signals) S_i of their true values V_i; it is assumed that this information is available to them for free. Upon observing S_i, a bidder may expend an entry cost k, which results in observing the true value V_i and entering the auction. Only the bidders that have learned V_i are eligible to submit a bid in the auction. Moreover, only those with valuations at or above the reserve price r submit a bid.

We assume that the pairs (V_i, S_i) are identically and independently distributed across potential bidders $i = 1, ..., N$ and are drawn from distribution $F(v, s)$ with density $f(v, s)$, positive on $[\underline{v}, \overline{v}] \times [0, 1]$. For convenience, we assume that the marginal distribution of the signals is uniform on $[0, 1]$. Since the informational content of signals is preserved under a monotone transformation, this assumption is without loss of generality.

The entry stage is followed by the bidding stage. Active bidders draw their values V_i, and then simultaneously and independently submit sealed bids. Active bidders do not know the number of active bidders, only the number of potential bidders N. The good is awarded to the highest bidder who pays its bid. We assume that the signals are informative and that higher signals are "good news". Formally, we assume affiliation, in the sense of Milgrom and Weber (1982).

Assumption 1 *For each bidder i, the variables (V_i, S_i) are affiliated: $\partial^2 \log f(v.s) / \partial v \partial s \geq 0$.*

Note that both the LS and S models can be viewed as limit cases of the GLS. The LS model corresponds to signals being independent of the valuations; this would effectively purify the mixed-strategy equilibrium. It is formally nested within the GLS model. The S model corresponds to the other extreme, namely the signals and valuations being perfectly correlated. Although it is not formally nested since we assume existence of a joint density $f(v.s)$, it can be informally viewed as a limiting case.

A symmetric equilibrium of the GLS model can be characterized in a manner similar to the LS model. Once again, we assume that the reserve price is binding, but the result carries over with minor changes to the case when it is not binding.

Proposition 3 *A symmetric equilibrium is characterized by a signal cutoff $\bar{s}(N)$ such that only those potential bidders for whom $S_i \geq \bar{s}(N)$ choose to enter. The equilibrium probability of submitting a bid is*

$$p(N) = \Pr\{S_i \geq \bar{s}(N), V_i \geq r\} \tag{2.8}$$

and the distribution of active bidders valuations is

$$F^*(v|N) = \Pr\{V_i \leq v | S_i \geq \bar{s}(N), V_i \geq r\}.$$

For any bidder i with signal $S_i = s$, the ex-ante equilibrium profit from bidding is equal to

$$\Pi_N(p(N), s) = \int_r^{\bar{v}} (1 - F(v|s))(1 - p(N) + p(N) F^*(v|N))^{N-1} dv. \tag{2.9}$$

The bidding strategy is given by

$$B_N(v) = v - \frac{\int_r^v (1 - p(N) + p(N) F^*(\tilde{v}|N))^{N-1} d\tilde{v}}{(1 - p(N) + p(N) F^*(v|N))^{N-1}}. \tag{2.10}$$

If the entry cost k is so small that $\bar{s}(N) = 0$, all potential bidders enter. If k is moderate so that $\bar{s}(N) \in (0,1)$ then the bidder with signal $\bar{s}(N)$ is indifferent between entering or not and the equilibrium cutoff $\bar{s}(N)$ is determined from

$$k = \Pi_N(p(N), \bar{s}(N)). \tag{2.11}$$

where $p(N)$ depends on $\bar{s}(N)$ through (2.8). *For any N, $N' > N$ such that $\bar{s}(N), \bar{s}(N') > 0$, we have $\bar{s}(N') > \bar{s}(N)$, and therefore $p(N') < p(N)$.*

The proof of Proposition 3 parallels the argument in Milgrom (2004) and is sketched in the Appendix.

A common restriction of the three models is that the probability of submitting a bid is decreasing in N, i.e. the restriction (2.3). But the restriction on active bidders' CDF $F^*(v|N)$ is different from either LS or S. To derive this condition, note that $F^*(v|N)$ is equal to $\Pr\{V_i \leq v | S_i \geq \bar{s}(N), V_i \geq r\}$. The affiliation Assumption 1 implies that the probability $\Pr\{V_i \leq v | S_i \geq \bar{s}, V_i \geq r\}$ is non-decreasing in $\bar{s}$ (Theorem 23 in Milgrom and Weber (1982)). Since the cutoff $\bar{s}(N)$ is non-decreasing in N, the $F^*(v|N)$'s are tilted towards bidders with higher valuations:

$$F^*(v|N) \geq F^*\left(v|N'\right) \ \forall N < N' \tag{2.12}$$

Note that this restriction is implied by restrictions (2.4) of the LS model as well as restriction (2.7) of the S model, but is clearly weaker.

3 Nonparametric identification of $p(N)$ and $F^*(v|N)$

This section deals with nonparametric identification of the distribution $F^*(v|N)$. It presents a unified treatment for all models - LS, S, and GLS. It is assumed that the econometrician can observe all the bids and therefore also the number of active bidders n. An important additional information that is assumed to be also available is the number of potential bidders N. In other words, we assume that the data generating process identifies $p(N)$ and $G^*(\cdot|N)$ where $p(N) = E[n|N]/N$ is the probability of submitting a bid and $G^*(b|N)$ is the distribution of entrants' bids conditional on N.

Bidders' valuations are not directly observable, but can be recovered from the first-order conditions following the approach of GPV. Consider first-order equilibrium conditions of the bidding game. A bidder with value v who submits a bid b has a probability of winning over a given rival equal to $1 - p(N) + p(N)G^*(b|N)$, i.e. it is complementary to the probability that the rival submits a bid times the probability that this bid is above b. Since there are $N-1$ identical rivals, it follows by independence that the probability of winning is $(1 - p(N) + p(N)G^*(b|N))^{N-1}$, and the expected profit is

$$\widetilde{\Pi}(b, v) = (b - v)(1 - p(N) + p(N)G^*(b|N))^{N-1}.$$

Writing out the first-order condition, i.e. taking the derivative of $\widetilde{\Pi}(b, v)$ with respect to b and setting it equal to 0, gives the inverse bidding strategy

$$\xi_N(b) = b + \frac{1 - p(N) + p(N)G^*(b|N)}{(N-1)p(N)g^*(b|N)}. \tag{3.1}$$

We can see that ξ_N is identified from the observables, and its inverse, the bidding strategy $B_N(v)$, is also identified. Then the distribution of entrants' valuations $F^*(v|N)$ is identified according to

$$F^*(v|N) = G^*(B_N(v)|N).$$

The identification of $p(N)$ and $F^*(v|N)$ is sufficient for the purposes of testing. We now discuss the identification of the models' primitives. Consider first the LS model. If the reserve price is absent or non-binding, then the distribution of valuations is identified since it is equal to $F^*(v|N)$. The entry cost is also identified from the zero profit condition, provided that $p(N) \in (0,1)$. The presence of a binding reserve price makes identification of both $F(\cdot)$ and the entry cost problematic. First,

$$F(v) = (1 - F(r)) F^*(v|N) + F(r),$$

so we need to know $F(r)$ in order to identify the distribution of valuations $F(v)$ for $v \geq r$.[3] Second, from (2.2), the entry cost is given by

$$k = \int_r^{\bar{v}} (1 - F(v)) (1 - p(N) + p(N) F^*(v|N))^{N-1} dv$$

so once again we need to know $F(v)$ for $v \geq r$ (i.e. know $F(r)$) in order to identify the entry cost.[4] We cannot identify $F(r)$ from the knowledge of bidding probabilities $p(N)$ unless the probabilities of bidding are equal for two distinct values of N. This is because $p(N)$ is generally a product of the probability of entry and $1 - F(r)$, so that unless the probability of entry is 1, we cannot identify $F(r)$. Intuitively, identification fails because we generally cannot distinguish between not bidding due to no entry and due to entering but drawing the valuation below the reserve price r. Note however that this fact does not interfere with the identification of $F^*(v|N)$ as is needed for our tests.

In the S model, the entry cost is identified from equation (2.5), regardless of whether the reserve price is binding or not.[5] However, the distribution of valuations is only identified for $v \geq v^*(\underline{N})$ since $v^*(\underline{N}) = p(\underline{N})$.

Finally, in the model with selective entry that we propose, the primitives for nonparametric identification are the entry cost k and $F(v|s)$, the distribution of valuations conditional on s (recall that the signals can be normalized to have a uniform $[0,1]$ distribution). Neither $F(v|s)$ nor k is nonparametrically identified. The reason is that the data generating process only reveals the distribution of bidders' valuations, i.e. $F^*(v|N) = \Pr\{V_i \leq v | S_i \geq \bar{s}(N), V_i \geq r\}$, but not $F(v|s)$. The knowledge of $F(v|s)$ would also be needed to identify the entry cost according to (2.11).

4 Econometric implementation

In what follows, we allow for auctions heterogeneity by introducing the vector of auctions specific covariates x. We assume now that the distribution of valuations can change from auction to auction depending on the value of x and is denoted by $F(v|x)$. Similarly,

[3] Paarsch (1997) was first to recognize the truncating effect of the reserve price.

[4] In particular, the restriction that the entry cost is constant in N is in general not identifiable.

[5] In a recent working paper, Xu (2007) develops a nonparametric estimator of the entry cost for the S model.

the distribution of valuations conditional on entry is now denoted as $F^*(v|N,x)$, and the probability of submitting a bid as $p(N,x)$. The model selection is also conditional on x, i.e. different models may be true for different values of x.

4.1 Hypotheses

The previous section shows that the distributions of valuations conditional on bidding are identified for the three alternative models, and therefore in principle, model selection tests can be formulated in terms of $F^*(v|N,x)$ or equivalently, in terms of quantiles of this distribution, as suggested in Haile, Hong, and Shum (2003). Define

$$Q^*(\tau|N,x) \equiv F^{*-1}(\tau|N,x)$$

to be the τ-th quantile of the distribution of entrants' valuations. Assume that N varies between the lower bound $\underline{N}$ and the upper bound $\bar{N}$. In terms of the quantiles, the testable restriction of the LS model is

$$H_{LS}: Q^*(\tau|\underline{N},x) = ... = Q^*\left(\tau|\bar{N},x\right),\ \forall\tau \in [0,1],$$

while the GLS model implies the restriction

$$H_{GLS}: Q^*(\tau|\underline{N},x) \leq ... \leq Q^*\left(\tau|\bar{N},x\right),\ \forall\tau \in [0,1]. \tag{4.1}$$

The testable restriction (2.7) of the Samuelson model can also be expressed using the quantiles of the distributions of observable bids as follows. First, by the definition and since F has a compact support, for any $\tau \in [0,1]$, $F(Q(\tau|x)|x) = \tau$. Next, for those quantiles of $F(\cdot)$ that correspond to valuations above the cutoffs $v^*(N)$, i.e. for $\tau > 1 - p(N,x)$, equation (2.6) implies

$$F^*(Q(\tau|x)|N,x) = \frac{\tau - (1 - p(N,x))}{p(N,x)},$$

which in turn implies that

$$Q(\tau|x) = Q^*\left(\frac{\tau - (1 - p(N,x))}{p(N,x)}|N,x\right). \tag{4.2}$$

Define a function

$$\alpha(\tau,N,x) = \frac{\tau - (1 - p(N,x))}{p(N,x)}.$$

The quantiles in the left-hand side of (4.2) do not depend on N because they correspond to the distribution of *potential* bidders' valuations, and we then have that $Q^*(\alpha(\tau,N,x)|N,x)$

must be constant across N's for all $\tau > 1 - p\left(\bar{N}, x\right)$:

$$H_S : Q^*\left(\alpha\left(\tau, \underline{N}, x\right)|\underline{N}, x\right) = \ldots = Q^*\left(\alpha\left(\tau, \bar{N}, x\right)|\bar{N}, x\right),\ \forall \tau > 1 - p\left(\bar{N}, x\right).$$

The restriction in H_S is limited to a particular range of τ's. A similar restriction, however with $\tau \in [0, 1]$, can be obtained from (2.7) directly. Define a function

$$\beta\left(\tau, N, x\right) = 1 - \frac{p\left(\bar{N}, x\right)}{p\left(N, x\right)}\left(1 - \tau\right).$$

Note that, since $p\left(\bar{N}, x\right) \leq p\left(N, x\right)$, $0 \leq \beta\left(\tau, N, x\right) \leq 1$ for all $\tau \in [0, 1]$, and therefore can be interpreted as a legitimate transformation of the quantile order τ.[6] The condition in (2.7) implies that for all N,

$$F^*\left(v|N, x\right) = \beta\left(F^*\left(v|\bar{N}, x\right), N, x\right),$$

and, by the same argument as before, we obtain the following restriction in terms of the transformed quantiles:

$$H_S' : Q^*\left(\beta\left(\tau, \underline{N}, x\right)|\underline{N}, x\right) = \ldots = Q^*\left(\beta\left(\tau, \bar{N}, x\right)|\bar{N}, x\right),\ \forall \tau \in [0, 1].$$

From the practical point of view, testing H_S is similar to testing H_S'; however, the last one does not require truncation of τ's. Therefore, we focus only on H_S'. Note also that because $\beta\left(\tau, N, x\right)$ is decreasing in N, the restrictions under H_S and H_S' are consistent with the restriction of GLS (4.1) on the quantiles $Q^*\left(\tau|N, x\right)$ without the transformation β, but are stronger.

In this paper, we consider independent testing of H_{LS}, H_{GLS}, and H_S' against their corresponding unrestricted alternatives. In addition, we also consider testing whether the entry probabilities $p\left(N, x\right)$ are non-increasing in N. The null hypothesis, for a given value of x, is

$$H_p : 1 > p\left(\underline{N}, x\right) \geq \ldots \geq p\left(\bar{N}, x\right) > 0,$$

and it is also tested against its corresponding unrestricted alternative.

The last test is of independent interest. The fact that the equilibrium probabilities of submitting a bid decline in the number of potential bidders is probably a common feature of many other models of entry. Whenever a model with costly entry is brought to explain why some potential bidders do not bid, one must confront an alternative explanation. Namely, following Paarsch (1997), even if there is no entry cost, non-participation may still

[6] While any other fixed value of N can be used in the place of $\bar{N}$ in the definition of β, the choice $N = \bar{N}$ ensures that β takes on values in the zero-one interval.

be explained by the fact that some bidders draw their valuations below the reserve price. But in that case, the probability of bidding is equal to $\Pr\{V_i \leq r\}$ and therefore does not depend on the number of potential bidders. Viewed this way, the above hypothesis H_p is a testable restriction of costly versus costless entry.

4.2 Formal assumptions

We assume that a sample of L auctions is available, and index each auction by $l = 1, .., L$. In each auction l, we observe the set of (anonymous) potential bidders indexed by $i = 1, ..., N_l$, where $N_l \in \mathcal{N} = \left\{\underline{N}, ..., \bar{N}\right\}$ is the number of potential bidders. Each auction is characterized by a covariates vector $x_l \in \mathcal{X}$. Corresponding to each potential bidder there is a latent valuation V_{il}. Each model provides a structural link between the potential bids b_{il} of all potential bidders and their valuations V_{il}: $b_{il} = B(V_{il}|N_l, x_l)$. The model also provides a link between signals and the entry decisions $y_{il} \in \{0, 1\}$, where $y_{il} = 1$ if the potential bidder decided to enter and zero otherwise. Of course, the potential bids of those who have not entered remain latent, and only the entrants' bids are observable. We make the following assumptions concerning the data generating process.

Assumption 2 **(a)** $\{(N_l, x_l) : l = 1, \ldots, l\}$ *are i.i.d.*

(b) *The marginal PDF of x_l, φ, is strictly positive, continuous on its compact support $\mathcal{X} \subset \mathbb{R}^d$, and admits at least $R \geq 2$ continuous and bounded partial derivatives on $Interior(\mathcal{X})$.*

(c) *The distribution of N_l conditional on x_l, $\pi(N|x)$, has support $\mathcal{N} = \left\{\underline{N}, ..., \bar{N}\right\}$ for all $x \in \mathcal{X}$, $\underline{N} \geq 2$.*

(d) *V_{il} and N_l are independent conditional on x_l.*

(e) $\{V_{il} : i = 1, \ldots N_l; l = 1, \ldots, L\}$ *are i.i.d. conditional on (N_l, x_l)*

(f) *For all $x \in \mathcal{X}$, the density of valuations $f(\cdot|\cdot)$ is strictly positive and bounded away from zero on its support, a compact interval $[\underline{v}(x), \overline{v}(x)] \subset \mathbb{R}_+$, and admits at least R continuous and bounded partial derivatives its interior.*

(g) *$\pi(N|\cdot)$ admit at least $R \geq 2$ continuous bounded derivatives on $Interior(\mathcal{X})$ for all $N \in \mathcal{N}$.*

(h) *The entry probability conditional on (N, x), $p(N, x)$, is strictly positive for all $N \in \mathcal{N}$ and $x \in \mathcal{X}$, and $p(N, \cdot)$ admits at least $R \geq 2$ continuous derivatives bounded away from 0 on an open subset $\mathcal{X}^\dagger \in Interior(\mathcal{X})$ and all $N \in \mathcal{N}$.*

Assumption 2(a) is the usual iid assumption on the data generating process for the covariates. Assumptions 2(b), (f), and smoothness of functions in (g) and (h) are standard in the nonparametric auctions literature (see, for example, GPV). Assumption 2(c) defines the support of the distribution of N_l conditional on the covariates. Assumption 2(d) is one of the most important assumptions; it asserts that in the number of potential bidders N is exogenous conditional on $x_l = x$, which allows us to use the variation in N for the purpose of testing. In Section 6, we explain why this assumption is plausible in the context of our empirical application. Assumption 2(e) is the IPV assumption.

4.3 Estimation of quantiles

In this section, we present our nonparametric estimation method for $Q^*(\tau|N,x)$. Our estimation method is based on the fact that, since the bidding strategies are increasing, the quantiles of valuations $Q^*(\tau|N,x)$ and bids

$$\begin{aligned} q^*(\tau|N,x) &= G^{*-1}(\tau|N,x) \\ &\equiv \inf\{b : G^*(b|N,x) \geq \tau\} \end{aligned}$$

are linked through the (inverse) bidding strategy,

$$Q^*(\tau|N,x) = \xi(q^*(\tau|N,x)|N,x)$$

Since both $\xi(\cdot|N,x)$ and $q^*(\tau|N,x)$ and can be estimated nonparametrically, we are lead to a natural plug-in estimator

$$\hat{Q}^*(\tau|N,x) = \hat{\xi}(\hat{q}^*(\tau|N,x)|N,x). \tag{4.3}$$

The nonparametric estimators for $\hat{\xi}$ and $\hat{q}^*$ are constructed as follows. Recalling that the inverse bidding strategy $\xi(\cdot|N,x)$ is given by

$$\xi(b|N,x) = b + \frac{1 + p(N,x) - p(N,x)\, G^*(b|N,x)}{(N-1)p(N,x)\, g^*(b|N,x)},$$

our estimator $\hat{\xi}(\cdot|N,x)$ is obtained by replacing $p(N,x)$, $G^*(b|N,x)$, and $g^*(b|N,x)$ with nonparametric estimators, $\hat{p}(N,x)$, $\hat{G}^*(b|N,x)$, and $\hat{g}^*(b|N,x)$. The conditional quantile $q^*(\tau|N,x)$ is estimated by inverting the nonparametric estimator for the CDF $\hat{G}(b|N,x)$:

$$\hat{q}^*(\tau|N,x) = \inf\left\{b : \hat{G}(b|N,x) \geq \tau\right\}.$$

The transformation $\beta(\tau, N, x)$ can be similarly estimated by using the estimators $\hat{p}(N, x)$,

$$\hat{\beta}(\tau, N, x) = 1 - \frac{\hat{p}(\bar{N}, x)}{\hat{p}(N, x)}(1 - \tau),$$

and the transformed quantiles estimated as $\hat{Q}^*\left(\hat{\beta}(\tau, N, x)|N, x\right)$.

Our nonparametric estimators for the required input functions $g^*(b|N, x)$, $G^*(b|N, x)$, and $p(N, x)$ are based on the kernel method. Specifically, we use the following estimators:

$$\begin{aligned}
\hat{\pi}(N|x) &= \frac{\sum_{l=1}^{L} 1\{N_l = N\} \prod_{k=1}^{d} K\left(\frac{x_{kl} - x_k}{h}\right)}{\sum_{l=1}^{L} \prod_{k=1}^{d} K\left(\frac{x_{kl} - x_k}{h}\right)}, \text{ and} \\
\hat{p}(N, x) &= \frac{\sum_{l=1}^{L} n_l 1\{N_l = N\} \prod_{k=1}^{d} K\left(\frac{x_{kl} - x_k}{h}\right)}{\sum_{l=1}^{L} 1\{N_l = N\} \prod_{k=1}^{d} K\left(\frac{x_{kl} - x_k}{h}\right)},
\end{aligned}$$

where h is the bandwidth parameter, K is a kernel function satisfying Assumption 3 in the Appendix, and $n_l = \sum_{i=1}^{N_l} y_{il}$ is the number of actual bidders in auction l. Since the probability of observing N conditional on x and the probability of submitting a bid conditional on N and x can be written as $\pi(N|x) = E[1\{N_l = N\}|x]$ and $p(N, x) = E[n|N, x]/N$, their estimators are standard nonparametric regression estimators.

In Proposition 4 in the Appendix we also show that the estimator of bid submission probability $\hat{p}(N, x)$ is asymptotically normal and derive its asymptotic variance

$$V_p(N, x) = \left(\int K(u)^2 du\right)^d \frac{p(N, x)(1 - p(N, x))}{N\pi(N|x)\varphi(x)}.$$

Moreover, we show that the estimators $\hat{p}(N, x)$ are asymptotically independent for any distinct $N, N' \in \{\underline{N}, ...\bar{N}\}$ and $x, x' \in \mathcal{X}^{\dagger}$.

The conditional bid densities and distributions are estimated by a kernel method, with an adjustment needed to account for a random number of observations within each auction. We estimate first the *expected* number of bid observations that correspond to N-bidder auctions in the sample with covariates x as

$$\hat{e}(N, x) = \hat{p}(N, x)\hat{\pi}(N|x) NL.$$

The proposed estimators of g^* and G^* are

$$\begin{aligned}
\hat{g}^*(b|N, x) &= \frac{\sum_{l=1}^{L} \sum_{i=1}^{N_l} y_{il} 1\{N_l = N\} K\left(\frac{b_{il} - b}{h}\right) \prod_{k=1}^{d} K\left(\frac{x_{kl} - x_k}{h}\right)}{h^{d+1}\hat{e}(N, x)\hat{\varphi}(x)}, \\
\hat{G}^*(b|N, x) &= \frac{\sum_{l=1}^{L} \sum_{i=1}^{N_l} y_{il} 1\{N_l = N\} 1(b_{il} \leq b) \prod_{k=1}^{d} K\left(\frac{x_{kl} - x_k}{h}\right)}{h^{d}\hat{e}(N, x)\hat{\varphi}(x)}.
\end{aligned}$$

where $\hat{\varphi}(x)$ is the standard multivariate kernel density estimator. The estimators $\hat{g}^*$ and $\hat{G}^*$ are essentially standard nonparametric conditional density and CDF estimators with the number of bids observations $\sum_{l=1}^{L}\sum_{i=1}^{N_l} y_{il} 1\{N_l = N\}$ replaced by its estimated expected value $\hat{e}(N, x)$.

In the Appendix, we prove that the estimators $\hat{Q}^*(\tau|N, x)$ and $\hat{Q}^*\left(\hat{\beta}(\tau, N, x)|N, x\right)$ are asymptotically normal. Specifically, we prove that, under certain technical but standard econometric assumptions,

$$\sqrt{Lh^{d+1}}\left(\hat{Q}^*(\tau|N, x) - Q^*(\tau|N, x)\right)$$

is asymptotically normal with mean zero and variance

$$V_Q(N, \tau, x) = \left(\int K(u)^2\, du\right)^{d+1} \frac{(1 - p(N, x)(1 - \tau))^2}{(N-1)^2 N p^3(N, x)\, g^{*3}(q^*(\tau|N, x)|N, x)\pi(N|x)\varphi(x)}.$$

A consistent estimator $\hat{V}_Q(N, \tau, x)$ can be obtained by replacing the functions $p(N, x)$, $q^*(\tau|N, x)$ etc. by their estimators. We also show that

$$\sqrt{Lh^{d+1}}\left(\hat{Q}^*\left(\hat{\beta}(\tau, N, x)|N, x\right) - Q^*(\beta(\tau, N, x)|N, x)\right)$$

converges in distribution to a normal random variable with mean zero and variance $V_Q(N, \beta(\tau, N, x), x))$. Moreover, for any distinct $N, N' \in \left\{\underline{N}, ...\bar{N}\right\}$, $\tau, \tau' \in \Upsilon$, and $x, x' \in \mathcal{X}^\dagger$, the estimators $\hat{Q}^*(\tau|N, x)$ are asymptotically independent, as are the estimators $\hat{Q}^*\left(\hat{\beta}(\tau, N, x)|N, x\right)$. These results are contained in Proposition 5 in the Appendix.

4.4 Comparison with the estimation method of Haile, Hong, and Shum (2003)

It is interesting to compare our estimator $\hat{Q}^*(\tau|N, x)$ with that of HHS, who present their method in a model different from ours in that they allow for common value effects. their method is semiparametric and in the present context, reduces to the following.

One begins with removing the effect of covariates on valuations by performing a preliminary regression. The main assumption in HHS is that the valuations depend on covariates additively, with the mean valuation specified by some function $\Gamma(x_l; \theta)$ that depends on x and a finite-dimensional parameter θ:

$$v_{il} = \Gamma(x_l; \theta) + \varepsilon_{il}. \tag{4.4}$$

The error term ε_{il} is mean-zero and distributed independently of x_l with CDF $F_\varepsilon(\cdot)$. It is also assumed that the reserve price has the same additive form, $r_l = r_0 + \Gamma(x_l; \theta)$. One can always write the bidding strategy as the valuation minus the markup, $B(v_{il}|N_l, x_l) = v_{il} - m(v_{il}, x_l, N_l)$. HSS show that in their setting, the markup $m(v_{il}, x_l, N_l)$ does not depend on x_l: $m(v_{il}, x_l, N_l) = m_0(\varepsilon_{il}, N_l)$. It is then easy to show that the bids regression takes the additive form

$$b_{il} = \alpha(N_l) + \Gamma(x_l; \theta) + \varepsilon'_{il}, \tag{4.5}$$

where $\alpha(N_l) = E[m_0(\varepsilon_{il}, N_l)|N_l]$ and $\varepsilon'_{il} = \varepsilon_{il} - m_0(\varepsilon_{il}, N_l) + \alpha(N_l)$. The parameter θ can then be estimated by any nonlinear regression method, and the bids "homogenized" according to $\hat{b}_{il} = b_{il} - \Gamma\left(x_l; \hat{\theta}\right)$.

Next, the inverse bidding strategy $\xi(b|N)$ is estimated nonparametrically, in essentially the same way as in our method, and a pseudo sample of bids (cf. GPV) is formed according to $\hat{v}_{il} = \hat{\xi}(b_{il}|N_l)$, trimmed appropriately to avoid biasing boundary effects. Lastly, the quantiles $Q^*(\tau|N)$ are estimated as sample quantiles of the trimmed sample. If desired, the conditional quantiles can be estimated as $\hat{Q}^*(\tau|N, x) = \Gamma\left(x; \hat{\theta}\right) + \hat{Q}^*(\tau|N)$.

This "homogenization" is not feasible in our setting, since even if one assumes the additive form (4.4), the markup $m(v_{il}, x_l, N_l)$ in general depends on x_l. This is because, unlike in the models considered in HSS, in our case the equation for the bidding strategy (2.10) also contains the entry probability $p(N, x)$, an unknown function of x. Therefore the regression (4.5) produces inconsistent estimates of θ. Our method effectively changes the order of steps of the HSS estimator. Unlike HHS, our treatment of covariates is fully nonparametric. We first nonparametrically estimate the quantiles of bids $q(\tau|N, x)$ and then insert them into the estimator $\hat{\xi}(\cdot|N, x)$ to obtain our conditional quantile estimator $\hat{Q}^*(\tau|N, x) = \hat{\xi}(\hat{q}(\tau|N, x)|N, x)$.

4.5 Tests

In view of the results of Section 4.3, quantile restrictions derived from the LS, S, and GLS models can be tested in a standard manner as equality or inequality constraints. We implement the tests using a finite set of τ's from $(0, 1)$ interval, $\Upsilon = \{\tau_1, \tau_2, \ldots, \tau_k\}$. The tests of H_{LS} and H'_S are based on the corresponding statistics $T^{LS}(x)$ and $T^S(x)$ that measure deviations of the estimated quantiles $\hat{Q}^*(\tau|N, x)$ and $\hat{Q}^*\left(\hat{\beta}(\tau, N, x)|N, x\right)$ from their means:[7]

$$T^{LS}(x) = Lh^{d+1} \min_{\mu \in \mathbb{R}} \sum_{N \in \mathcal{N}, \tau \in \Upsilon} \frac{\left(\hat{Q}^*(\tau|N, x) - \mu\right)^2}{\hat{V}_Q(N, \tau, x)},$$

[7] Recall that $Q^*(\tau|N, x)$ is the same for all N under H_{LS}, and $Q^*(\beta(\tau, N, x)|N, x)$ is the same for all N under H'_S.

$$T^S(x) \quad = \quad Lh^{d+1} \min_{\mu \in \mathbb{R}} \sum_{N \in \mathcal{N}, \tau \in \Upsilon} \frac{\left(\hat{Q}^* \left(\hat{\beta}(\tau, N, x) | N, x\right) - \mu\right)^2}{\hat{V}_Q \left(N, \hat{\beta}(\tau, N, x), x\right)}.$$

By the results of Proposition 5, $\hat{Q}^*(\tau|N,x)$ and $\hat{Q}^*\left(\hat{\beta}(\tau,N,x)|N,x\right)$ are approximately normal in the large samples and independent across N; therefore, the LS model is rejected at level α for the auctions with covariates' values x whenever $T^{LS}(x) > \chi^2_{(\#\mathcal{N}-1)k,1-\alpha}$, where $\chi^2_{(\#\mathcal{N}-1)k,1-\alpha}$ denotes the $1-\alpha$ quantile of the chi-square distribution with degrees of freedom $(\#\mathcal{N}-1)k$, where $\#\mathcal{N}$ denotes the number of elements in $\mathcal{N}$. Similarly, one rejects H_S' if $T^S(x) > \chi^2_{(\#\mathcal{N}-1)k,1-\alpha}$. Note that due to asymptotic normality and independence of the quantile estimators, $T^{LS}(x)$ and $T^S(x)$ can be also viewed as likelihood ratio (LR) statistics.

We now turn to testing of the GLS model. In this case, the distance or LR statistic is

$$T^{GLS}(x) = \min_{y_{\underline{N},\tau} \leq \ldots \leq y_{\bar{N},\tau}, \tau \in \Upsilon} Lh^{d+1} \sum_{N \in \mathcal{N}, \tau \in \Upsilon} \frac{\left(\hat{Q}^*(\tau|N,x) - y_{N,\tau}\right)^2}{\hat{V}_Q(N,\tau,x)}, \tag{4.6}$$

and one rejects the null of GLS in favor of the general alternative when $T^{GLS}(x)$ takes on large values.

The GLS model does not determine uniquely the null distribution of the $T^{GLS}(x)$ statistic; however, we show in Proposition 6 in the Appendix that the probability of type I error is maximized when all inequalities are replaced with the equalities, i.e. the same restrictions as in the LS model. This proposition also shows that, under the restriction of the LS model, the statistic $T^{GLS}(x)$ is asymptotically equivalent to the random variable defined by

$$\mathcal{T}^{GLS}(x) = \sum_{\tau \in \Upsilon} \min_{\left\{\mu:\, \Omega^{1/2}(\tau,x)\mu \leq 0\right\}} \|Z_\tau - \mu\|^2. \tag{4.7}$$

where $Z_\tau \sim N(0, I_{\#\mathcal{N}-1})$ and independent across τ's, $\Omega(\tau,x) = RV_Q(\tau,x)R'$, $V_Q(\tau,x) = diag\left(V_Q(\underline{N},\tau,x), \ldots, V_Q(\bar{N},\tau,x)\right)$, and R is the $(\#\mathcal{N}-1) \times (\#\mathcal{N})$ differencing matrix

$$R = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}.$$

Consequently, a test that rejects H_{GLS} when $T^{GLS}(x) > c(x)_{GLS,1-\alpha}$, where $c(x)_{GLS,1-\alpha}$ is the $1-\alpha$ quantile of the distribution of $\mathcal{T}^{GLS}(x)$, has asymptotic size α.

The distribution of $\mathcal{T}^{GLS}(x)$ depends on asymptotic variances of the quantile estimators; however, the critical values can be simulated as follows. First, from the estimated asymptotic variances $\hat{V}_Q(\underline{N}, \tau, x), \ldots, \hat{V}_Q(\bar{N}, \tau, x)$ construct the matrices $\hat{V}_Q(\tau, x)$ and $\hat{\Omega}(\tau, x)$ for $\tau_1, \ldots, \tau_k$. Second, for $m = 1, \ldots, M$, generate independent $N(0, I_{\#\mathcal{N}-1})$ vectors $Z_{\tau_1,m} \ldots, Z_{\tau_k,m}$, and compute $\hat{\mathcal{T}}^{GLS}_m(x)$ as defined in (4.7), but with Z_τ replaced by $Z_{\tau,m}$ and Ω replaced with $\hat{\Omega}$. The simulated critical value for a test with asymptotic size α, say $\hat{c}(x)_{GLS,1-\alpha}$, is then computed as the $1-\alpha$ sample quantile of $\left\{\hat{\mathcal{T}}^{GLS}_m(x) : m = 1, \ldots, M\right\}$. One should reject the null of GLS when $T^{GLS}(x) > \hat{c}(x)_{GLS,1-\alpha}$.

Testing whether the entry probabilities are non-increasing in N, the H_p hypothesis, can be performed similarly to testing H_{GLS}. Define

$$T^p(x) = \min_{y_{\underline{N}} \geq \ldots \geq y_{\bar{N}}} Lh^d \sum_{N \in \mathcal{N}} \frac{(\hat{p}(N, x) - y_N)^2}{\hat{V}_p(N, x)}.$$

One should reject H_p whenever $T^p(x) > c(x)_{p,1-\alpha}$, where $c(x)_{p,1-\alpha}$ is the $1-\alpha$ quantile of $\mathcal{T}^p(x) = \min_{\Omega_p^{1/2}(x)\mu \geq 0} \|Z - \mu\|^2$, $\Omega_p(x) = RV_p(x)R'$, $V_p(x)$ is a diagonal matrix with the main diagonal elements $V_p(\underline{N}, x), \ldots, V_p(\bar{N}, x)$, and $Z \sim N(0, I_{\#\mathcal{N}-1})$. Such a test has asymptotic size α and is consistent. As in the case of $T^{GLS}(x)$, the critical values for the $T^p(x)$ test can be simulated following the steps described above.

5 Monte-Carlo experiment

In this section we present a Monte-Carlo study of the small sample properties of the tests. In particular, we are interested how the choice of quantiles τ affects size and power of the tests. In our simulations, we focus on testing the GLS model without covariates x.

We simulate the random signals S and valuations V using the Gaussian copula. Let (Z_1, Z_2) be bivariate normal with zero means, variances equal to one, and the correlation coefficient ρ. Let Φ denote the standard normal CDF. A pair (S, V) is generated as $S = \Phi(Z_1)$, and $V = \Phi(Z_2)$. Nonzero values of ρ correspond to the case of informative signals and selective entry; while $\rho = 0$ corresponds to the case of the LS model.

The details of the computation of the distributions $F(v|S)$ and $F^*(v|N)$ that are needed in order to solve for the equilibrium of the auction are as follows. First, recall that $Z_2|Z_1 \sim N\left(\rho z_1, 1 - \rho^2\right)$, and, consequently, the conditional distribution of V given S is given by

$$\begin{aligned} F(v|S) &= P(V \leq v|S) \\ &= P\left(Z_2 \leq \Phi^{-1}(v) | \Phi^{-1}(S)\right) \\ &= \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(S)}{\sqrt{1-\rho^2}}\right). \end{aligned}$$

Next, note that the marginal distribution of S is uniform on the $[0,1]$ interval, and

$$\begin{aligned} F^{*}(v|N) &= F(v|S \geq \bar{s}(N)) \\ &= \frac{1}{1-\bar{s}} \int_{\bar{s}(N)}^{1} \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(s)}{\sqrt{1-\rho^2}}\right) ds, \end{aligned}$$

where the cutoff signal $\bar{s}(N)$ can be found, given the value of N, as a solution to equation (2.11). Lastly, for $S \geq \bar{s}(N)$, the bids are computed according to the bidding strategy (2.10).

In our simulations, we set $L = 250$, $\mathcal{N} = \{2,3,4,5\}$, $\pi(N) = 1/4$ for all $N \in \mathcal{N}$, and $k = 0.17$. The number of Monte Carlo replications is 1,000; in each replication, the critical values for the T^{GLS} test are obtained using 999 replications. We use the triweight kernel function $K(u) = (35/32)\left(1-u^2\right)^3 1\{|u| \leq 1\}$ for nonparametric kernel estimation. To reflect the fact that the number of active bidders varies from auction to auction depending on the number of potential bidders in the auction N and $p(N)$, we decided to use a bandwidth that depends on N. Specifically, we used $h = (LN\hat{p}(N))^{-2/5}$.

Table 1 reports the results of size simulations for $\rho = 0, 0.5$, and the following sets of quantiles: $\{0.5\}$, $\{0.3, 0.5, 0.7\}$, and $\{0.3, 0.4, 0.5, 0.6, 0.7\}$. While the asymptotic approximation works reasonable well for a small number quantiles, the finite sample size properties of the test deteriorate when the number of quantiles used to construct the test increases. For example, the T^{GLS} test over rejects the null when $\rho = 0$ and $\tau \in \{0.3, 0.4, 0.5, 0.6, 0.7\}$: for the nominal size of 10%, 5%, and 1% the simulated rejection rates are approximately 16%, 10% and 4% respectively. Note also that the rejection rates for $\rho = 0.5$ are smaller than for $\rho = 0.5$ and below the nominal rejection rates. This reflects the fact that the probability of type I error is maximized at $\rho = 0$.

Table 2 reports the *size corrected* power results (the critical values are computed from the simulated distribution of the test statistic under the null). To address the power issue, it is necessary to come up with an alternative. Ideally, this would be achieved by considering a structural model. Absent a structural model, however, we are allowed to consider any configuration of bidding quantiles, in particular we may reverse their order, making *decreasing* as opposed to increasing in N. To do this in the simplest fashion possible, we multiply each quantile by minus one and then add a constant to all quantiles to assure that they are positive.

Table 2 shows that the power increases with the distance from the null. The power also increases when we use quantiles 0.3 and 0.7 in addition to the median. However, we also observe that in some cases the size corrected power decreases with the number of quantiles, when the number of quantiles used to construct the test is large. For example, in the case of $\{0.3, 0.5, 0.7\}$ quantiles, $\rho = 0.9$, and the nominal size of 5%, the simulated rejection rate is about 20%; however, when we use in addition the quantiles 0.4 and 0.6, the rejection rate is only about 18%. In practice, given samples of moderate size, a rule of thumb would be to use 3 fixed quantiles in order to maintain good size and reasonable power.

6 Empirical application

Our dataset consists of 547 auctions for surface paving and grading contracts let by Oklahoma Department of Transportation (ODOT) during the period of January, 2002 to December, 2005.[8] The available data items include all bids, the engineer's estimate, the time length of the contract (in days), the number of items in the proposal and the length of the road. The ODOT implements a policy under which all bids over 7% of the engineer's estimate are typically rejected, so there is a binding reserve price. In reality, we do observe bids above the reserve price (although extremely few winning bids were above the reserve price). We treat these bids as non-serious. They are only used in the estimation of bidding probabilities, but are otherwise eliminated from the sample of bids.

Importantly, we observe the list of eligible bidders (planholders) for each auction. In the vocabulary of this paper, these eligible bidders are the potential bidders. The list of planholders is published on the ODOT website prior to bidding. A firm becomes a planholder through the following process. All projects to be auctioned are advertised by the ODOT 4 to 10 weeks prior to the letting date. These advertisements include the engineer's estimate, a brief summary of the project, location of the work and the type of the work involved. But the advertisements lack detailed schedules of work items which are only revealed in construction plans.

Interested firms can then submit a request for plans and bidding proposals, the documents that contain the specifics of the project (in particular, the items schedule). An important feature of the qualification process is that only eligible firms are allowed access to these documents. A firm is deemed eligible if it satisfies certain qualification requirements. The goal of the qualification process is to ensure that the winning firm will have sufficient expertise and capacity to undertake the project. While the expertise part is typically determined at the pre-qualification stage (in most cases, once per year), the capacity part is project-specific. An important requirement is that the prospective bidder is not qualified for the aggregate amount of work that exceeds $2\frac{1}{2}$ times its current working capital.[9] Given that the bidders know the sizes of all projects to be let but not the project specifics, it is plausible that the decision of a firm to request the plan for a particular project

[8] Our choice of surface paving and grading contracts is motivated by the fact that Hong and Shum (2002), in their study of highway procurement auctions in New Jersey, find little support for common values for this type of contracts. See also De Silva, Dunne, Kankanamge, and Kosmopoulou (2007). This is important because in this paper, we assume independent private values (costs).

[9] This requirement is explicitly stated in the ODOT rule OAC 730:25.

is primarily determined by the project size as well as the sizes of other projects for which it is pre-qualified, in relation to the available capacity of the firm. The capacity may be determined by a number of factors, such as for example the amount of resources committed to other projects, including but not limited to those previously contracted with ODOT.

Figure 1 shows the empirical frequencies (i.e. the histogram) of project sizes. The pattern is highly skewed towards smaller projects: the average project size is about \$3.6 million (from Table 3), but the projects for which there are at least 10 auctions in the dataset have sizes not exceeding \$2.5 million. Since we need at least moderate number of observations to implement our nonparametric tests, we have chosen the set of project sizes $\mathbb{X}$ to be the equal partition of the interval $[0, 2.5]$, i.e. in millions of dollars, $\mathbb{X} = \{0.5, 1, 1.5, 2, 2.5\}$.

The projects of larger size may be, other things equal, more attractive. For example, this would be so if there are economies of scale. There is some evidence of this in the data. Table 5 shows estimated conditional probabilities $\pi(N|x)$, where as before N is the number of potential bidders and x is the size of the project as measured by the engineer's estimate. One can see that the estimated mean $E[N|x]$ increases with x from $E[N|x = 2] = 4.69$ to $E[N|x = 2.5] = 7.97$, and the differences $E[N|x] - E[N|x']$, $x > x'$, are statistically significant for all $x \in \mathbb{X}$.

Table 5 also shows that there still remains a substantial variation in the number of potential bidders even after controlling for project size. This variation will important for our tests. Equally important is that the variation in the number of planholders is likely to be exogenous, uncorrelated with unobservable project characteristics since the latter only become available in the plans.

The variables used in the regressions are described in Table 3.[10] The results of the entry logit regression and OLS bidding regression are presented in Table 4. In the OLS regression, the dependent variable is log(bid), where bid is the amount of bid in millions of dollars. The size of the project has a strongly positive effect on the bids. Clearly, it is the most important variable in the OLS regression. Using it alone produced R^2 of about 0.79, so the impact of the other variables is much smaller. In the order of importance, the next variable is the number of potential bidders N; if it is included in the regression, R^2 increases to about 0.94. We also mention that the project size has a negative (but not statistically significant) effect on the probability of submitting a bid.

The effect of the number of potential bidders is statistically significant in all regressions. Having more potential bidders reduces the bid submission rate: increasing N by 1 reduces the odds of submitting a bid by about 4%. Having more bidders also results in lower bids.

[10] The covariates are basically the same as in other papers on procurement auctions (e.g. Bajari and Ye (2003); Pesendorfer and Jofre-Bonet (2003); Krasnokutskaya (2003); Krasnokutskaya and Seim (2006); Li and Zheng (2005)).

This is of course consistent with the models considered in this paper. The logit regression also shows that project size has a negative effect on the probability of bidding. This effect, however, is not statistically significant.

The complexity of the project is captured by the number of items in the construction plan. This is the variable *Nitems*. Table 3 shows that there is a substantial variation in *Nitems*; the mean is 72 and the standard deviation is 71 item. One might expect that the cost of preparing a bid is an increasing function of *Nitems*, even controlling for the size of the project. The conjectured effect of *Nitems* is therefore to reduce the probability of submitting a bid. The estimate of the *Nitems* coefficient in the logit regression confirms this conjecture. However, the effect is quite small: increasing *Nitems* by one standard deviation, i.e. adding about 70 pay items, reduces the odds by only about 1%.

Included in both regressions are dummy variables for 20 firms that appear on the planholders list most frequently. The other firms are treated as fringe firms. Observe that even though not all firms enter at the same rate and bid similarly, the empirical evidence of asymmetries is strong only for out-of-state firms (the firms with headquarters outside the state of Oklahoma) that enter less frequently and also bid less, and for the following three firms: Broce Construction, Glover Construction and Becco Contractors. Since our model assumes bidder symmetry, we decided to exclude the auctions in which either out-of-state firms or these three firms were on the planholders list.

In the practical implementation of our estimators, we are confronted with the usual bias and variance trade-off. Including more variables will reduce the bias, but at the same time will increase the sample variability of our estimators. Given the preliminary regression results, we only condition on the project size.

Another practical issue is that, as is well known, nonparametric estimators suffer from substantial loss of precision when sample size is very small. When we tried to include all auctions, the estimates of the quantiles $Q(\tau|N,x)$ were highly erratic. The problem is that, because the data is sparse, for some (N,x) pairs the estimated probability $\hat{\pi}(N|x)$ is very close to 0. To make our estimators stable, we decided to exclude those pairs (N,x) where the number of observations that $N_l = N$ and $x_l \in [x-h, x+h]$ is less than 15. The working sample ultimately consisted of 258 auctions, and all the results discussed below were obtained using this smaller sample.

The tests are performed conditional on project size $x \in \mathbb{X}$.[11] We first test the prediction shared by all models considered in this paper, namely that bid submission probabilities $p(N,x)$ are declining in the number of potential bidders N for each x. Refer to Table 6, where the estimated bidding probabilities $p(N,x)$ as well as the results of the tests are reported. The average rate of bid submission is about 62%. Expect for relatively large

[11] The bandwidth was chosen according to the same rule as in Section 5.

projects, $x = \$2.5$ mil., there is a strongly declining pattern over N. For example, for moderately sized projects, $x = \$1.5$ mil., the probability of bidding $p(N,x)$ falls from a relatively large value of 0.56 when there are 3 potential bidders, to about half of that, 0.28 when there are 9 potential bidders. For other values of $x \in \mathbb{X} \setminus \{2.5\}$, the pattern is less pronounced, but the formal tests of the monotonicity restrictions still do not reject the null at the conventional 5% significance level.

We now turn to the tests of the models - LS, S and GLS. First note that, because the procurement auctions are low-bid, the null hypothesis that corresponds to the GLS model must be changed accordingly, i.e. the quantiles must decreasing rather than increasing. Also, the inverse strategy is now

$$\xi(b|N,x) = b - \frac{1 - p(N,x)\, G^*(b|N,x)}{(N-1)\, p(N,x)\, g^*(b|N,x)},$$

and the asymptotic variances of the quantiles are

$$V_Q(N,\tau,x) = \left(\int K(u)^2\, du\right)^{d+1} \frac{(1 - p(N,x)\,\tau)^2}{(N-1)^2 N p^3(N,x)\, g^{*3}(q^*(\tau|N,x)\,|N,x)\pi(N|x)\,\varphi(x)},$$

but all other aspects of estimation are unchanged.

The results of the tests are presented in Table 9. Consider first the results for just one quantile, the median. The GLS model is rejected for relatively small projects, $x = \$0.5$ and $x = \$1$ mil., but it is not rejected for larger projects. Neither LS models are rejected for most project sizes. Since the findings are somewhat counter intuitive; but recall that our Monte-Carlo studies have shown that the power of the tests can be increased substantially if we increase the number of quantiles from one to three. When 0.3, 0.5 and 0.7 quantiles are simultaneously considered, the support for the S model disappears completely at 5% significance for all project sizes. The largest p-value of the test is 0.02. The LS model fairs better, but it too is rejected for all projects but the largest ones, with $x = \$2.5$ mil. The GLS model, on the other hand, is now rejected also for $x = 1$ mil., but similarly to the one quantile case, is not rejected for project sizes $x = \$2.0$ and $x = \$2.5$ mil.

Increasing the number of quantiles from three to five (we have chosen the quantiles 0.3, 0.4, 0.5, 0.6 and 0.7) leads to the same results: the S model is robustly rejected, the LS model is rejected for all but the largest projects. The GLS model is not rejected for the two largest project sizes we considered. The reason that all models are rejected for the small project can be as follows. It is possible that for small projects, firms coordinate their decisions to request plans and become potential bidders, while these decisions are made independently for larger projects. Confirming this hypothesis empirically is likely to be difficult in view of potentially confounding effects of unobserved project heterogeneity.

The fact that the S model is always rejected is probably not very surprising. Recall that the entry cost in the S model is solely the cost of preparing a bid rather than the joint cost that also includes information acquisition. Even for small projects, firms may face uncertainty about the exact level of their construction costs that can only be resolved through costly information acquisition, so it is only natural that this is confirmed empirically. The empirical evidence regarding the LS model can also be explained intuitively. It is plausible that project complexity increases with project size. Our inability to reject the LS model for large projects may be due to the fact that for these projects, the information received by the bidders before the plans are available is relatively less precise.

7 Concluding remarks

In this paper, we have proposed nonparametric tests to discriminate among alternative models of entry in first-price auctions. The models considered are: (a) the Levin and Smith (1994) model with randomized entry strategies, (b) the Samuelson (1985) model that assumes that bidders are perfectly informed about their valuations at the entry stage, and select into the pool of entrants based on this information, and (c) a new model that allows for selective entry but in a less stark form than Samuelson (1985). Specifically, our model assumes that bidders receive signals that are informative about their valuations and make their entry decisions based on these signals.

In the empirical application, we have tested the restrictions of each model against the unrestricted alternatives using a dataset of highway procurement auctions from the Oklahoma Department of Transportation (ODOT). We have found strong evidence for selective entry according to our model. While these findings are encouraging and the testing framework can be used in other applications, this research could be extended in a number of directions.

One important extension would be developing more powerful tests. One could improve power by considering tests based on a continuum of quantiles rather than a finite set. HSS have pursued this approach, developing a Kolmogorov-Smirnov type test. As we have already discussed, their estimation method cannot be directly transferred to our setting. This would be an important extension of our approach left for future work. Similar hypotheses are considered in the recent literature on tests of stochastic dominance and monotonicity (see, for example, Lee, Linton, Whang, Suntory, for Economics, and Disciplines (2006)); however, their approach cannot be applied directly in our case, since private valuations are unobservable, and our statistics are based on kernel density estimators.

Our testing framework is quite general and the empirical findings are by and large intuitive. But there is also an important limitation that the future research should address.

In auction datasets, one typically finds that the variation in bids is only partially explained by their variation within auctions. The between-auction variation is typically present. It is also observed in our dataset. This pattern can be explained within the IPV framework only by unobserved project heterogeneity. Recently, Krasnokutskaya (2003) has developed a structural estimation method that can be applied even in the presence of unobserved heterogeneity. She assumes that the heterogeneity enters into the specification of valuations as a multiplying factor. Her method relies on the fact that the same multiplicative structure carries over to the bids. Unfortunately, this is not true for the models with entry considered in this paper, for reasons largely similar to those described in Section 4.4.

Alternatively, one can explain between-auction variation using a model with affiliated private values (APV), as in Li, Perrigne, and Vuong (2002). Note however that the theoretical models of entry that we build on are all within the IPV setting. It is known that the APV model can lead to qualitatively different predictions (Pinkse and Tan (2005)), so it is an open question if it can lead to testable implications similar to those considered here. We leave this for future research.

Another extension would be to allow bidder asymmetries (e.g., a recent working paper by Krasnokutskaya and Seim (2006)). The obvious difficulty here would the necessity to deal with multiple equilibria. Bajari, Hong, and Ryan (2004) obtain a number of identification results in this direction and estimate a parametric model with multiple equilibria for highway procurement auctions. Finally, incorporating dynamic features as in Pesendorfer and Jofre-Bonet (2003) is also left for future research.

8 Appendix

8.1 Details of the GLS model of entry

Proof of Proposition 3. One can show that a symmetric equilibrium is characterized by a signal cutoff $\bar{s}$ such that only the bidders for whom the news are sufficiently good, $S_i \geq \bar{s}$, enter. Since the bidders who have entered know their true values V_i, the bidding game is essentially a first-price auction with a random number of bidders. It is easy to see that, in a symmetric equilibrium, the probability of winning the auction for a bidder with valuation v is

$$\left(1 - p\left(N\right) + p\left(N\right) F^*\left(v|N\right)\right)^{N-1}. \tag{8.1}$$

To see why, note that because of independence, a bidder wins against a given potential rival either if the rival does not bid, or bids but his valuation conditional on bidding is less than v. This probability is equal to $1 - p\left(N\right) + p\left(N\right) F^*\left(v|N\right)$. By independence, we arrive at (8.1) as the total probability of winning the auction.

A standard envelope formula then applies: the slope of the profit function equals the probability of winning the auction. Since the expected profit obtained by the lowest bidder type is 0 the profit at the bidding stage as a function of v is equal to

$$\Pi^*\left(v\right) = \int_r^v \left(1 - p\left(N\right) + p\left(N\right) F^*\left(\tilde{v}|N\right)\right)^{N-1} d\tilde{v}.$$

(Notice that this profit is gross of entry cost.) Since this profit is also equal to

$$\left(v - B_N\left(v\right)\right)\left(1 - p\left(N\right) + p\left(N\right) F^*\left(v|N\right)\right)^{N-1},$$

we obtain the bidding strategy equation (2.10). Standard single-crossing arguments (e.g., Athey (2002)) imply that that $B_N\left(v\right)$ is strictly increasing in v.

Multiplying $\Pi^*\left(v\right)$ by $f\left(v|s\right)$, the density of valuations conditional on the signal that is obtained at the entry stage, we get the expected equilibrium profit at the bidding stage as a function of the signal:

$$\Pi_N\left(p\left(N\right), s\right) = \int_r^{\bar{v}} f\left(v|s\right) \int_r^v \left(1 - p\left(N\right) + p\left(N\right) F^*\left(\tilde{v}|N\right)\right)^{N-1} d\tilde{v} dv.$$

Changing the order of integration on the right-hand side of the above equation gives equation (2.9) in the statement of the proposition. The rest follows simply by observing that the bidder will make his entry decision by comparing $\Pi_N\left(p\left(N\right), s\right)$ with the entry cost k. Since $\Pi_N\left(p\left(N\right), s\right)$ is increasing in s, there are three possibilities depending on the magnitude of the entry cost k. First, the entry cost can be so small that $\bar{s}\left(N\right) = 0$ and all potential bidders enter (but not necessarily submit a bid). Second, k may be moderate so that $\bar{s}(N) \in (0,1)$. In this case, the bidder with signal $\bar{s}(N)$ is indifferent between entering or not, which gives equation (2.11) in the statement of the proposition.

We now show that for any N, $N' > N$ such that $\bar{s}(N), \bar{s}(N') > 0$, we have $\bar{s}(N') > \bar{s}(N)$. Substituting

$$p(N) = \Pr\{S_i \geq \bar{s}(N), V_i \geq r\}$$

and

$$p(N) F^*(v|N) = \Pr\{S_i \geq s, V_i \in [r, V]\}$$

into (2.9) gives the ex-ante profit from bidding when $s = \bar{s}(N)$:

$$\hat{\Pi}(N,s) = \int_r^{\bar{v}} (1 - F(v|s))(1 - \Pr\{S_i \geq s, V_i \geq r\} + \Pr\{S_i \geq s, V_i \in [r,V]\})^{N-1} dv.$$

Note that by the affiliation assumption, $F(v|s)$ is non-increasing in s (Theorem 23 in Milgrom and Weber (1982)), so $1 - F(v|s)$ is non-decreasing in s. Also, the term within the second parentheses is increasing in s:

$$\begin{aligned} \frac{d}{ds} \Pr\{S_i \geq s, V_i \geq r\} &= \frac{d}{ds} \int_s^1 F(r|\tilde{s})\, d\tilde{s} \\ &= -[1 - F(r|s)], \end{aligned}$$

$$\begin{aligned} \frac{d}{ds} \Pr\{S_i \geq s, V_i \in [r,V]\} &= \frac{d}{ds} \int_s^1 [F(v|\tilde{s}) - F(r|\tilde{s})]\, d\tilde{s} \\ &= F(r|s) - F(v|s), \end{aligned}$$

so that the derivative of

$$1 - \Pr\{S_i \geq s, V_i \geq r\} + \Pr\{S_i \geq s, V_i \in [r,V]\}$$

with respect to s is equal to

$$[1 - F(r|s)] + [F(r|s) - F(v|s)] = 1 - F(v|s) > 0,$$

where the strict inequality follows from the fact that $s = \bar{s}(N) > 0$. It follows that $\hat{\Pi}(N,s)$ is increasing in s and decreasing in N so that the value $s = \bar{s}(N)$ that solves the equation $\hat{\Pi}(N,s) = k$ is increasing in N, i.e. $\bar{s}(N') > \bar{s}(N)$.■

8.2 Details of the estimation method

We begin with the following lemma that shows certain smoothness properties of the conditional bid density g^*.

Lemma 1 *Under Assumption 2(f), for all $N \in \mathcal{N}$ and $x \in \mathcal{X}$, the distribution of bids has the compact support $\left[\underline{b}(N,x), \bar{b}(N,x)\right]$, and $g^*(\cdot|N,\cdot)$ has at least $R+1$ continuos partial derivatives on its interior. Furthermore, $g^*(b|N,x)$ is bounded away from zero.*

Proof of Lemma 1. The proof is similar to that of Proposition 1 of GPV. ■

For kernel estimation, we use kernel functions K satisfying the following standard assumption (see, for example, Newey (1994)).

Assumption 3 *The kernel K has at least $R \geq 2$ continuous and bounded derivatives on $\mathbb{R}$, compactly supported on $[-1,1]$ and is of order R: $\int K(u)\,du = 1$, $\int u^j K(u)\,du = 0$ for $j = 1, ..., R-1$.*

The standard nonparametric regression arguments imply that the estimator of entry probabilities $\hat{p}(N,x)$ is asymptotically normal as well (see, for example, Pagan and Ullah (1999), Theorem 3.5, page 110):

Proposition 4 *Suppose that $x \in \mathcal{X}^\dagger$. Assume that the bandwidth h satisfies as $L \to \infty$: $Lh^d \to \infty$ and $\sqrt{Lh^d}h^R \to 0$. Then, under Assumptions 2 and 3, $\sqrt{Lh^d}\left(\hat{p}(N,x) - p(N,x)\right)$ is asymptotically normal with mean zero and variance*

$$V_p(N,x) = \frac{p(N,x)\left(1 - p(N,x)\right)}{N\pi(N|x)\varphi(x)}\left(\int K(u)^2\,du\right)^d.$$

Moreover, the estimators $\hat{p}(N,x)$ are asymptotically independent for any distinct $N, N' \in \left\{\underline{N}, ...\bar{N}\right\}$ and $x, x' \in \mathcal{X}^\dagger$.

Since the distribution of values and, consequently, the distribution bids have compact supports, the estimator of the density g^* is asymptotically biased near the boundaries. Our quantile approach allows one to avoid the problem by considering only inner intervals of the supports. Specifically, let $[\underline{v}(N,x), \bar{v}(N,x)]$ denote the support of $F^*(v|N,x)$, and let Λ be some compact inner interval, $\Lambda(N,x) = [v_1(N,x), v_2(N,x)] \subset [\underline{v}(N,x), \bar{v}(N,x)]$. The quantile orders corresponding to v_1 and v_2 are given by $\tau_i(N,x) = F^*(v_i(N,x)|N,x)$ for $i = 1, 2$. Hence, we consider quantile orders in $\Upsilon(N,x) = [\tau_1(N,x), \tau_2(N,x)]$. Next, the corresponding inner interval of the support of G^* is given by the values between the τ_1 and τ_2 quantiles: $\Theta(N,x) = [b_1(N,x), b_2(N,x)]$, where $b_i(N,x) = q^*(\tau_i(N,x)|N,x)$, $i = 1, 2$.

Similarly, we define the interval of quantile orders for transformed quantiles: $\Upsilon^{\beta}(N,x) = \left[\tau_1^{\beta}(N,x), \tau_2^{\beta}(N,x)\right]$ such that $\tau_1^{\beta}(N,x) = \{\inf \tau | \beta(\tau,N,x) \geq \tau_1(N,x), \tau \in [0,1]\}$ and $\tau_2^{\beta}(N,x) = \sup\{\tau | \beta(\tau,N,x) \leq \tau_2(N,x), \tau \in [0,1]\}$.

Lemma 2 *Under Assumptions 2 and 3, for all $x \in Interior(\mathcal{X})$ and $N \in \mathcal{N}$,*

(a) $\hat{\varphi}(x) - \varphi(x) = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right)$.

(b) $\hat{\pi}(N|x) - \pi(N|x) = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right)$.

(c) $\hat{p}(N,x) - p(N,x) = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right)$.

(d) $\sup_{b \in \left[\underline{b}(N,x), \bar{b}(N,x)\right]} |\hat{G}^*(b|N,x) - G^*(b|N,x)| = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right)$.

(e) $\sup_{\tau \in \Upsilon(N,x)} |\hat{q}^*(\tau|N,x) - q^*(\tau|N,x)| = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right)$.

(f) $\sup_{b \in \Theta(N,x)} |\hat{g}^*(b|N,x) - g^*(b|N,x)| = O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2} + h^R\right)$.

(g) $\sup_{\tau \in \Upsilon(N,x)} |\hat{Q}^*(\tau|N,x) - Q^*(\tau|N,x)| = O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2} + h^R\right)$.

(h) $\sup_{\tau \in [0,1]} \left|\hat{\beta}(\tau,N,x) - \beta(\tau,N,x)\right| = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right)$.

(i) $\sup_{\tau \in \Upsilon^{\beta}(N,x)} |\hat{Q}^*\left(\hat{\beta}(\tau,N,x)|N,x\right) - Q^*(\beta(\tau,N,x)|N,x)| = O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2} + h^R\right)$.

Proof of Lemma 2. Parts (a)-(c) of the lemma follow from Lemma B.3 of Newey (1994).

For part (d), define a function

$$G_0^*(b,N,x) = Np(N,x)\,\pi(N|x)\,G^*(b|N,x)\,\varphi(x),$$

and its estimator as

$$\hat{G}_0^*(b,N,x) = \frac{1}{h^d L}\sum_{l=1}^{L}\sum_{i=1}^{N_l} y_{il} 1\{N_l = N\}\, 1(b_{il} \leq b)\, K_{*h}(x_l - x),$$

where

$$K_{*h}(x_l - x) = \frac{1}{h^d} K_d\left(\frac{x_l - x}{h}\right), \text{ and}$$

$$K_d\left(\frac{x_l-x}{h}\right) = \prod_{k=1}^{d} K\left(\frac{x_{kl}-x_k}{h}\right). \tag{8.2}$$

Next,

$$\begin{aligned}
E\hat{G}_0^*(b,N,x) &= E\left(1\{N_l=N\}K_{*h}(x_l-x)\sum_{i=1}^{N_l} y_{il}1(b_{il}\le b)\right)\\
&= NE\left(1\{N_l=N\}K_{*h}(x_l-x)y_{il}1(b_{il}\le b)\right)\\
&= NE\left(E\left(1(b_{il}\le b)|N,x_l,y_{il}=1\right)y_{il}1\{N_l=N\}K_{*h}(x_l-x)\right)\\
&= NE\left(G^*(b|N,x_l)p(N,x_l)\pi(N|x_l)K_{*h}(x_l-x)\right)\\
&= N\int G^*(b|N,u)p(N,u)\pi(N|u)K_{*h}(x-u)\varphi(u)\,du\\
&= \int G_0^*(b,N,x+hu)K_d\left(\frac{u}{h}\right)du.
\end{aligned}$$

By Lemma 1, $G^*(b|N,\cdot)$ admits at least $R+1$ continuous derivatives. Then, as in the proof of Lemma B.2 of Newey (1994), Assumptions 2(b), (g) and (h) imply that there exists a constant $c>0$ such that

$$\left|G_0^*(b,N,x)-E\hat{G}_0^*(b,N,x)\right| \le ch^R\left(\int\left|K_d\left(\frac{u}{h}\right)\right|\|u\|^R du\right)\left\|vec\left(D_x^R G_0^*(b,N,x)\right)\right\|,$$

where $\|\cdot\|$ denotes the Euclidean norm, and $D_x^R G_0^*$ denotes the R-th partial derivative of G_0^* with respect to x. It follows then that

$$\sup_{b\in\left[\underline{b}(N,x),\bar{b}(N,x)\right]}\left|G_0^*(b,N,x)-E\hat{G}_0^*(b,N,x)\right| = O\left(h^R\right). \tag{8.3}$$

Now, we show that

$$\sup_{b\in\left[\underline{b}(N,x),\bar{b}(N,x)\right]}|\hat{G}_0^*(b,N,x)-E\hat{G}_0^*(b,N,x)| = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2}\right). \tag{8.4}$$

We follow the approach of Pollard (1984). Consider, for given $N\in\mathcal{N}$ and $x\in Interior(\mathcal{X})$, a class of functions $\mathcal{Z}$ indexed by h and b, with a representative function

$$z_l(b,N,x) = \sum_{i=1}^{N_l} y_{il}1\{N_l=N\}1(b_{il}\le b)h^d K_{*h}(x_l-x).$$

By the result in Pollard (1984) (Problem 28), the class $\mathcal{Z}$ has polynomial discrimination. Theorem 37 in Pollard (1984) (see also Example 38) implies that for any sequences δ_L, α_L

such that $L\delta_L^2\alpha_L^2/\log L \to \infty$, $Ez_l^2(b) \le \delta_L^2$,

$$\alpha_L^{-1}\delta_L^{-2}\sup_{b\in\left[\underline{b}(N,x),\bar{b}(N,x)\right]}|\frac{1}{L}\sum_{l=1}^{L}z_l(b,N,x)-Ez_l(b,N,x)|\to 0 \tag{8.5}$$

almost surely. We claim that this implies

$$\left(\frac{Lh^d}{\log L}\right)^{1/2}\sup_{b\in\left[\underline{b}(N,x),\bar{b}(N,x)\right]}|\hat{G}_0^*(b,N,x)-E\hat{G}_0^*(b,N,x)|.$$

is bounded as $L\to\infty$ almost surely. This implies that

$$\sup_{b\in\left[\underline{b}(N,x),\bar{b}(N,x)\right]}|\hat{G}_0^*(b,N,x)-E\hat{G}_0^*(b,N,x)|=O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2}\right).$$

The proof is by contradiction. Suppose not. Then there exist a sequence $\gamma_L\to\infty$ and a subsequence of L such that along this subsequence

$$\sup_{b\in\left[\underline{b}(N,x),\bar{b}(N,x)\right]}|\hat{G}_0^*(b,N,x)-E\hat{G}_0^*(b,N,x)|\ge\gamma_L\left(\frac{Lh^d}{\log L}\right)^{-1/2}. \tag{8.6}$$

on a set of events $\Omega'\subset\Omega$ with a positive probability measure. Now if we let $\delta_L^2=h^d$ and $\alpha_L=(\frac{Lh^d}{\log L})^{-1/2}\gamma_L^{1/2}$, then the definition of z implies that, along the subsequence, on a set of events Ω',

$$\begin{aligned}
&\alpha_L^{-1}\delta_L^{-2}\sup_{b\in\left[\underline{b}(N,x),\bar{b}(N,x)\right]}|\frac{1}{L}\sum_{l=1}^{L}z_l(b,N,x)-Ez_l(b,N,x)|\\
=\;&\left(\frac{Lh^d}{\log L}\right)^{1/2}\gamma_L^{-1/2}h^{-d}\sup_{b\in\left[\underline{b}(N,x),\bar{b}(N,x)\right]}|\frac{1}{L}\sum_{l=1}^{L}z_l(b,N,x)-Ez_l(b,N,x)|\\
=\;&\left(\frac{Lh^d}{\log L}\right)^{1/2}\gamma_L^{-1/2}\sup_{b\in\left[\underline{b}(N,x),\bar{b}(N,x)\right]}|\hat{G}_0^*(b,N,x)-E\hat{G}_0^*(b,N,x)|\\
\ge\;&\left(\frac{Lh^d}{\log L}\right)^{1/2}\gamma_L^{-1/2}\gamma_L\left(\frac{Lh^d}{\log L}\right)^{-1/2}\\
=\;&\gamma_L^{1/2}\to\infty,
\end{aligned}$$

where the inequality follows by (8.6), a contradiction to (8.5). This establishes (8.4), so that (8.3), (8.4) and the triangle inequality together imply that

$$\sup_{b\in\left[\underline{b}(N,x),\bar{b}(N,x)\right]}|\hat{G}_0^*\left(b,N,x\right)-G_0^*\left(b,N,x\right)|=O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2}+h^R\right). \tag{8.7}$$

To complete the proof, recall that, from the definitions of $G_0^*\left(b,N,x\right)$ and $\hat{G}_0^*\left(b,N,x\right)$,

$$G^*\left(b|N,x\right)=\frac{G_0^*\left(b,N,x\right)}{p\left(N,x\right)\pi\left(N|x\right)\varphi\left(x\right)},\ \text{and}\ \hat{G}^*\left(b|n,x\right)=\frac{\hat{G}_0^*\left(b,N,x\right)}{\hat{p}\left(N,x\right)\hat{\pi}\left(N|x\right)\hat{\varphi}\left(x\right)},$$

so that by the mean-value theorem,

$$\left|\hat{G}^*\left(b|N,x\right)-G^*\left(b|N,x\right)\right|\leq\tilde{C}\left(b,N,x\right)\left\|\begin{pmatrix}\hat{G}_0^*\left(b,N,x\right)-G_0^*\left(b,N,x\right)\\ \hat{p}\left(N,x\right)-p\left(N,x\right)\\ \hat{\pi}\left(N|x\right)-\pi\left(N|x\right)\\ \hat{\varphi}\left(x\right)-\varphi\left(x\right)\end{pmatrix}\right\|, \tag{8.8}$$

where $\tilde{C}\left(b,N,x\right)$ is given by

$$\left|\frac{1}{\tilde{p}\left(N,x\right)\tilde{\pi}\left(N,x\right)\tilde{\varphi}\left(x\right)}\right|\left\|\left(1,\frac{\tilde{G}_0^*\left(b,N,x\right)}{\tilde{p}\left(N,x\right)},\frac{\tilde{G}_0^*\left(b,N,x\right)}{\tilde{\pi}\left(N,x\right)},\frac{\tilde{G}_0^*\left(b,N,x\right)}{\tilde{\varphi}\left(x\right)}\right)\right\|,$$

and $\left\|\left(\tilde{G}^0-G^0,\tilde{p}-p,\tilde{\pi}-\pi,\tilde{\varphi}-\varphi\right)\right\|\leq\left\|\left(\hat{G}^0-G^0,\hat{p}-p,\hat{\pi}-\pi,\hat{\varphi}-\varphi\right)\right\|$ for all (b,N,x). Further, by Assumption 2(b), (c) and (h), and the results in parts (a)-(c) of the lemma, with the probability approaching one $\tilde{p}$, $\tilde{\pi}$ and $\tilde{\varphi}$ are bounded away from zero. The desired result follows from (8.7), (8.8) and parts (a)-(c) of the lemma.

For part (e) of the lemma, since $\hat{G}^*\left(\cdot|N,x\right)$ is monotone by construction,

$$\begin{aligned}P\left(\hat{q}^*\left(\tau_1\left(N,x\right)|N,x\right)<\underline{b}\left(N,x\right)\right) &= P\left(\inf_b\left\{b:\hat{G}^*\left(b|N,x\right)\geq\tau_1\left(N,x\right)\right\}<\underline{b}\left(N,x\right)\right)\\ &= P\left(\hat{G}^*\left(\underline{b}\left(N,x\right)|N,x\right)>\tau_1\left(N,x\right)\right)\\ &= o\left(1\right),\end{aligned}$$

where the last equality is by the result in part (d). Similarly,

$$\begin{aligned}P\left(\hat{q}^*\left(\tau_2\left(N,x\right)|N,x\right)>\overline{b}\left(N,x\right)\right) &= P\left(\hat{G}\left(\overline{b}\left(N,x\right)|N,x\right)<\tau_2\left(N,x\right)\right)\\ &= o\left(1\right).\end{aligned}$$

Hence, for all $x \in \mathit{Interior}(\mathcal{X})$ and $N \in \mathcal{N}$, with the probability approaching one, $\underline{b}(N,x) \leq \hat{q}^*(\tau_1(N,x)|N,x) < \hat{q}^*(\tau_2(N,x)|N,x) \leq \bar{b}(N,x)$. Since the distribution $G^*(b|N,x)$ is continuous in b, $G^*(q^*(\tau|N,x)|N,x) = \tau$, and, for $\tau \in \Upsilon(N,x)$, we can write the identity

$$G^*(\hat{q}^*(\tau|N,x)|N,x) - G^*(q^*(\tau|N,x)|N,x) = G^*(\hat{q}^*(\tau|N,x)|N,x) - \tau. \tag{8.9}$$

Using Lemma 21.1(ii) of van der Vaart (1998), and by the definition of $\hat{G}^*$,

$$0 \leq \hat{G}^*(\hat{q}^*(\tau|N,x)|N,x) - \tau \leq \frac{1}{\hat{p}(N,x)\,\hat{\pi}(N|x)\,\hat{\varphi}(x)\,NLh^d},$$

and by the results in (a)-(c),

$$\hat{G}^*(\hat{q}^*(\tau|N,x)|N,x) = \tau + O_p\left(\left(Lh^d\right)^{-1}\right) \tag{8.10}$$

uniformly over τ. Combining (8.9) and (8.10), and applying the mean-value theorem to the left-hand side of (8.9), we obtain

$$\begin{aligned} & \hat{q}^*(\tau|N,x) - q^*(\tau|N,x) \\ = \quad & \frac{G^*(\hat{q}^*(\tau|N,x)|N,x) - \hat{G}^*(\hat{q}^*(\tau|N,x)|N,x)}{g^*(\widetilde{q}^*(\tau|N,x)|N,x)} + O_p\left(\left(Lh^d\right)^{-1}\right), \end{aligned} \tag{8.11}$$

where $\widetilde{q}^*$ lies between $\hat{q}^*$ and q^* for all (τ, N, x). Now, by Lemma 1, $g^*(b|N,x)$ is bounded away from zero, and the result in part (e) follows from (8.11) and part (d) of the lemma.

To prove part (f), by Lemma 1, $g^*(\cdot|N,\cdot)$ admits at least $R+1$ continuous bounded partial derivatives. Let

$$g_0^*(b,N,x) = p(N,x)\,\pi(N|x)\,\varphi(x)\,g^*(b|N,x), \text{ and} \tag{8.12}$$

$$\hat{g}_0^*(b,N,x) = \hat{p}(N,x)\,\hat{\pi}(N|x)\,\hat{\varphi}(x)\,\hat{g}^*(b|N,x). \tag{8.13}$$

By Lemma B.3 of Newey (1994), $\hat{g}_0^*(b,N,x)$ is uniformly consistent over $b \in \Theta(N,x)$:

$$\sup_{b\in\Theta(N,x)} |\hat{g}_0^*(b,N,x) - g_0^*(b,N,x)| = O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2} + h^R\right). \tag{8.14}$$

By the results in parts (a)-(c), the estimators $\hat{p}(N,x)$, $\hat{\pi}(N|x)$ and $\hat{\varphi}(x)$ converge at the rate faster than that in (8.14). The desired result follows by the same argument as in the proof of part (d), equation (8.8).

Next, we prove part (g). By Lemma 1, $g^*(b|N,x) > c_g > 0$. Then

$$
\begin{aligned}
&\left|\hat{Q}^*(\tau|N,x) - Q^*(\tau|N,x)\right| \\
\leq\ & |\hat{q}^*(\tau|N,x) - q^*(\tau|N,x)| + 2\frac{|\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x) - g^*(q^*(\tau|N,x)|N,x)|}{p(N,x)\,\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x)\,c_g} \\
&+\frac{|\hat{p}(N,x) - p(N,x)|}{\hat{p}(N,x)\,p(N,x)\,\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x)} \\
\leq\ & \left(1 + \frac{2\sup_{b\in\Theta(N,x)}|\partial g^*(b|N,x)/\partial b|}{p(N,x)\,\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x)\,c_g}\right)|\hat{q}(\tau|n,x) - q(\tau|n,x)| \\
&+2\frac{|\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x) - g^*(\hat{q}^*(\tau|N,x)|N,x)|}{p(N,x)\,\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x)\,c_g} \\
&+\frac{|\hat{p}(N,x) - p(N,x)|}{\hat{p}(N,x)\,p(N,x)\,\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x)}. \qquad (8.15)
\end{aligned}
$$

Define an event

$$E_L(N,x) = \{\hat{q}^*(\tau_1(N,x)|N,x) \geq b_1(N,x),\, \hat{q}^*(\tau_2(N,x)|N,x) \leq b_2(N,x)\},$$

and let $\beta_L = \left(\frac{Lh^{d+1+2k}}{\log L}\right)^{1/2} + h^{-R}$. By the result in part (e), $P(E_L^c(N,x)) = o(1)$. Hence, it follows from part (e) of the lemma the estimator $\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x)$ is bounded away from zero with the probability approaching one. Consequently, it follows by Lemma 1 and part (e) of this lemma that the first summand on the right-hand side of (8.15) is $O_p\left(\beta_L^{-1}\right)$ uniformly over $\Upsilon(N,x)$. Next,

$$
\begin{aligned}
&P\left(\sup_{\tau\in\Upsilon(N,x)} \beta_L|\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x) - g^*(\hat{q}^*(\tau|N,x)|N,x)| > M\right) \\
\leq\ & P\left(\sup_{\tau\in\Upsilon(N,x)} \beta_L|\hat{g}^*(\hat{q}^*(\tau|N,x)|N,x) - g^*(\hat{q}^*(\tau|N,x)|N,x)| > M, E_L(x)\right) \\
&+P(E_L^c(x)) \\
\leq\ & P\left(\sup_{b\in\Theta(N,x)} \beta_L|\hat{g}^*(b|N,x) - g^*(b|N,x)| > M\right) + o(1). \qquad (8.16)
\end{aligned}
$$

The result of part (g) follows from parts (c) and (f) of the lemma and (8.16).

For part (h), by Assumption 2(h) and part (c) of the lemma, $\hat{\beta}(\tau,N,x) \rightarrow_p \beta(\tau,N,x)$ for all τ, N, and x. The result of part (h) follows since β is linear in τ (see Andrews (1992); also Theorems 21.9 and 21.10 on pages 337-339 of Davidson (1994)).

Lastly, we prove part (i). We have

$$\sup_{\tau\in\Upsilon^\beta(N,x)}\left|\hat{q}^*\left(\hat{\beta}(\tau,N,x)|N,x\right) - q^*(\beta(\tau,N,x)|N,x)\right|$$

$$
\begin{aligned}
&= \hat{q}^*\left(\hat{\beta}(\tau,N,x)|N,x\right)-q^*\left(\hat{\beta}(\tau,N,x)|N,x\right)\\
&\quad +q^*\left(\hat{\beta}(\tau,N,x)|N,x\right)-q^*\left(\beta(\tau,N,x)|N,x\right)\\
&\leq \sup_{\tau\in\Upsilon(N,x)}\left|\hat{q}^*(\tau|N,x)-q^*(\tau|N,x)\right|+O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2}+h^R\right)\\
&= O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2}+h^R\right),
\end{aligned}
\tag{8.17}
$$

where the inequality follows from part (h) of the lemma and Lemma 1. The result of part (i) follows from the definition of $\hat{Q}^*$ in (4.3) and (8.17). ■

Lemma 3 *Let $\Theta(N,x)$ be as in Lemma 2. Suppose that Assumptions 2 and 3 hold, and that the bandwidth h is such that $Lh^{d+1}\to\infty$, $\sqrt{Lh^{d+1}}h^R\to 0$. Then*

$$
\sqrt{Lh^{d+1}}\left(\hat{g}^*(b|N,x)-g^*(b|N,x)\right)\to_d N\left(0,V_g(b,N,x)\right)
$$

for $b\in\Theta(N,x)$, $x\in Interior(\mathcal{X})$, and $N\in\mathcal{N}$, where $V_g(b,N,x)$ is given by

$$
V_g(N,b,x)=\frac{g^*(b|N,x)}{Np(N,x)\,\pi(N|x)\,\varphi(x)}\left(\int K(u)^2\,du\right)^{d+1}.
$$

Furthermore, $\hat{g}^*(b|N_1,x)$ and $\hat{g}^*(b|N_2,x)$ are asymptotically independent for all $N_1\neq N_2$, $N_1,N_2\in\mathcal{N}$.

Proof of Lemma 3. Consider $g_0^*(b,n,x)$ and $\hat{g}_0^*(b,n,x)$ defined in (8.12) and (8.13) respectively. It follows from parts (a)-(c) of Lemma 2,

$$
\begin{aligned}
&\sqrt{Lh^{d+1}}\left(\hat{g}^*(b|N,x)-g^*(b|N,x)\right)\\
&= \frac{1}{p(N,x)\,\pi(N|x)\,\varphi(x)}\sqrt{Lh^{d+1}}\left(\hat{g}_0^*(b,N,x)-g_0^*(b,N,x)\right)+o_p(1).
\end{aligned}
\tag{8.18}
$$

Furthermore, as in Lemma B2 of Newey (1994), $E\hat{g}_0^*(b,N,x)-g_0^*(b,N,x)=O\left(h^R\right)$ uniformly in $b\in\Theta(N,x)$ for all $x\in Interior(\mathcal{X})$ and $N\in\mathcal{N}$. Thus, it remains to establish asymptotic normality of $\sqrt{Lh^{d+1}}\left(\hat{g}_0^*(b,N,x)-E\hat{g}_0^*(b,N,x)\right)$.

Define

$$
\begin{aligned}
w_{il,N} &= \sqrt{\frac{1}{h^{d+1}}}y_{il}1\{N_l=N\}\,K\left(\frac{b_{il}-b}{h}\right)K_d\left(\frac{x_l-x}{h}\right),\\
\overline{w}_{L,N} &= \frac{1}{NL}\sum_{l=1}^{L}\sum_{i=l}^{N_l}w_{il,N},
\end{aligned}
$$

where K_d is defined in (8.2). With above definitions we have that

$$\sqrt{NLh^{d+1}}\left(\hat{g}_0^*(b,N,x)-E\hat{g}_0^*(b,N,x)\right)=\sqrt{NL}\left(\overline{w}_{L,N}-E\overline{w}_{L,N}\right). \tag{8.19}$$

Then, by the Liapunov CLT (see, for example, Corollary 11.2.1 on page 427 of Lehman and Romano (2005)),

$$\sqrt{NL}\left(\overline{w}_{L,N}-E\overline{w}_{L,N}\right)/\sqrt{NLVar\left(\overline{w}_{L,N}\right)}\to_d N(0,1), \tag{8.20}$$

provided that $Ew_{il,N}^2<\infty$, and for some $\delta>0$,

$$\lim_{L\to\infty}\frac{1}{L^{\delta/2}}E\left|w_{il,N}-Ew_{il,N}\right|^{2+\delta}=0.$$

The last condition follows from the Liapunov's condition (equation (11.12) on page 427 of Lehman and Romano (2005)) and because $w_{il,N}$ are iid Next, $Ew_{il,N}$ is given by

$$\begin{aligned}
&\sqrt{\frac{1}{h^{d+1}}}E\left(p(N,x_l)\pi(N|x_l)\int K\left(\frac{u-b}{h}\right)g^*(u|N,x_l)\,duK_d\left(\frac{x_l-x}{h}\right)\right)\\
=\;&\sqrt{\frac{1}{h^{d+1}}}\int\int p(N,y)\pi(N|y)K\left(\frac{u-b}{h}\right)g^*(u|N,y)K_d\left(\frac{y-x}{h}\right)\varphi(y)\,dudy\\
=\;&\sqrt{h^{d+1}}\\
&\times\int\int p(N,x+hy)\pi(N|x+hy)K(u)g^*(b+hu|N,x+hy)K_d(y)\varphi(x+hy)\,dudy\\
\to\;&0.
\end{aligned}$$

Further, $Ew_{il,N}^2$ is given by

$$\begin{aligned}
&\frac{1}{h^{d+1}}\int\int p(N,y)\pi(N|y)K^2\left(\frac{u-b}{h}\right)g^*(u|N,y)K_d^2\left(\frac{y-x}{h}\right)\varphi(y)\,dudy\\
=\;&\int\int p(N,x+hy)\pi(N|x+hy)K^2(u)g^*(b+hu|N,x+hy)K_d^2(y)\varphi(x+hy)\,dudy\\
<\;&\infty.
\end{aligned}$$

Hence,

$$NLVar\left(\overline{w}_{L,N}\right)\to p(N,x)\pi(N|x)g^*(b|N,x)\varphi(x)\left(\int K^2(u)\,du\right)^{d+1}du. \tag{8.21}$$

Next, $E\left|w_{il,N}\right|^{2+\delta}$ is bounded by

$$
\begin{aligned}
& \frac{1}{h^{(d+1)(1+\delta/2)}}\int\int\left|K\left(\frac{u-b}{h}\right)\right|^{2+\delta} g^{*}\left(u|N,y\right)\left|K_{d}\left(\frac{y-x}{h}\right)\right|^{2+\delta}\varphi\left(y\right)dudy \\
= & \frac{1}{h^{(d+1)\delta/2}}\int\int\left|K\left(u\right)\right|^{2+\delta} g^{*}\left(b+hu|N,x+hy\right)\left|K_{d}\left(y\right)\right|^{2+\delta}\varphi\left(x+hy\right)dudy \\
\leq & \frac{1}{h^{(d+1)\delta/2}}\sup_{u\in[-1,1]}\left|K\left(u\right)\right|^{(d+1)(2+\delta)}\sup_{x\in\mathcal{X}}\varphi\left(x\right)\sup_{b\in\Theta(N,x)} g^{*}\left(b|N,x\right) \\
= & \frac{C}{h^{(d+1)\delta/2}}.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
\frac{1}{L^{\delta/2}}E\left|w_{il,N}-Ew_{il,N}\right|^{2+\delta} & \leq \frac{2^{1+\delta}}{L^{\delta/2}}E\left|w_{il,N}\right|^{2+\delta} \\
& \leq \frac{2^{1+\delta}C}{\left(Lh^{d+1}\right)^{\delta/2}} \\
& \rightarrow 0, \qquad (8.22)
\end{aligned}
$$

since $Lh^{d+1}\rightarrow\infty$ by the assumption. The first result of the lemma follows now from (8.18)-(8.22).

Next, note that the asymptotic covariance of $\overline{w}_{L,N_1}$ and $\overline{w}_{L,N_2}$ involves a product of the two indicator functions, $1\left\{N_l=N_1\right\}1\left\{N_l=N_2\right\}$, which is zero for all $N_1\neq N_2$. The joint asymptotic normality and asymptotic independence of $\hat{g}^{*}\left(b|N_1,x\right)$ and $\hat{g}\left(b|N_2,x\right)$ follows then by the Cramér-Wold device. ■

Proposition 5 *Suppose that $\tau\in(0,1)$ and $x\in\mathcal{X}^{\dagger}$. Assume that the bandwidth h satisfies as $L\rightarrow\infty$: $Lh^{d+1}\rightarrow\infty$ and $\sqrt{Lh^{d+1}}h^{R}\rightarrow 0$. Then, under Assumptions 2 and 3,*

$$
\sqrt{Lh^{d+1}}\left(\hat{Q}^{*}\left(\tau|N,x\right)-Q^{*}\left(\tau|N,x\right)\right)\rightarrow_{d} N\left(0,V_{Q}\left(N,\tau,x\right)\right),
$$

$$
\sqrt{Lh^{d+1}}\left(\hat{Q}^{*}\left(\hat{\beta}\left(\tau,N,x\right)|N,x\right)-Q^{*}\left(\beta\left(\tau,N,x\right)|N,x\right)\right)\rightarrow_{d} N\left(0,V_{Q}\left(N,\beta\left(\tau,N,x\right),x\right)\right),
$$

where

$$
V_{Q}\left(N,\tau,x\right)=\left(\frac{1-p\left(N,x\right)\left(1-\tau\right)}{(N-1)p\left(N,x\right)g^{*2}(q^{*}\left(\tau|N,x\right)|N,x)}\right)^{2}V_{g}\left(N,q^{*}\left(\tau|N,x\right),x\right),
$$

and $V_{g}\left(N,\tau,x\right)$ is defined in Lemma 3. Moreover, for any distinct $N,N'\in\left\{\underline{N},...\bar{N}\right\}$, $\tau,\tau'\in\Upsilon$, and $x,x'\in\mathcal{X}^{\dagger}$, the estimators $\hat{Q}^{*}\left(\tau|N,x\right)$ are asymptotically independent, as well as the estimators $\hat{Q}^{*}\left(\hat{\beta}\left(\tau,N,x\right)|N,x\right)$.

Proof of Proposition 5. First, by Lemma 2 (c), (e) and (f), and the mean-value theorem,

$$\begin{aligned}\hat{Q}^*(\tau|N,x) &= Q^*(\tau|N,x) - \frac{1-p(N,x)(1-\tau)}{(N-1)p(N,x)\widetilde{g}^{*2}(q^*(\tau|N,x)|N,x)} \\ &\quad \times (\hat{g}^*(q^*(\tau|N,x)) - g^*(q^*(\tau|N,x))) + o_p\left(\frac{1}{\sqrt{Lh^{d+1}}}\right), \qquad (8.23)\end{aligned}$$

where $\widetilde{g}^*$ is a mean-value between g^* and $\hat{g}^*$ for $b = q^*(\tau|N,x)$. The result follows then by Lemma 3. ■

Proposition 6 *Let $x \in \mathcal{X}^\dagger$. Assume that $Lh^d \to \infty$ and $\sqrt{Lh^d}h^R \to 0$ as $L \to \infty$, and Assumptions 2 and 3 hold. Then*

$$\sup P_{H_{GLS}}\left(T^{GLS}(x) > c\right) = P_{H_{LS}}\left(T^{GLS}(x) > c\right) \qquad (8.24)$$

$$\to P\left(\mathcal{T}^{GLS}(x) > c\right), \qquad (8.25)$$

where $P_{H_{GLS}}$ and $P_{H_{LS}}$ denotes probabilities under the inequality restrictions of GLS and equality restrictions of LS respectively, and $\mathcal{T}^{GLS}(x)$ is defined in (4.7).

Proof of Proposition 6. The result in (8.24) follows by Lemma 8.2 of Perlman (1969). In order to show (8.25), consider first the case of $k = 1$. By the results in Chapter 21.3.3 of Gourieroux and Monfort (1995), $T^{GLS}(x)$ is asymptotically equivalent to

$$\tilde{T}^{GLS}(x) = \min_{\Omega^{1/2}(\tau,x)\mu \le 0} Lh^{d+1}\|\hat{\gamma} - \mu\|^2,$$

where $\sqrt{Lh^{d+1}}\hat{\gamma} \to_d N(0, I_{\#\mathcal{N}-1})$; however,

$$\begin{aligned}\tilde{T}^{GLS}(x) &= \min_{\Omega^{1/2}(\tau,x)\mu/\sqrt{Lh^{d+1}} \le 0} \left\|\sqrt{Lh^{d+1}}\hat{\gamma} - \mu\right\|^2 \\ &= \min_{\Omega^{1/2}(\tau,x)\mu \le 0} \left\|\sqrt{Lh^{d+1}}\hat{\gamma} - \mu\right\|^2,\end{aligned}$$

and the result follows by the Continuous Mapping Theorem. Extension to the case of $k > 1$ is straightforward since there are no cross τ restrictions in (4.6), and the quantile estimators are asymptotically independent across τ. ■

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Figure 1: Sample frequencies of project sizes

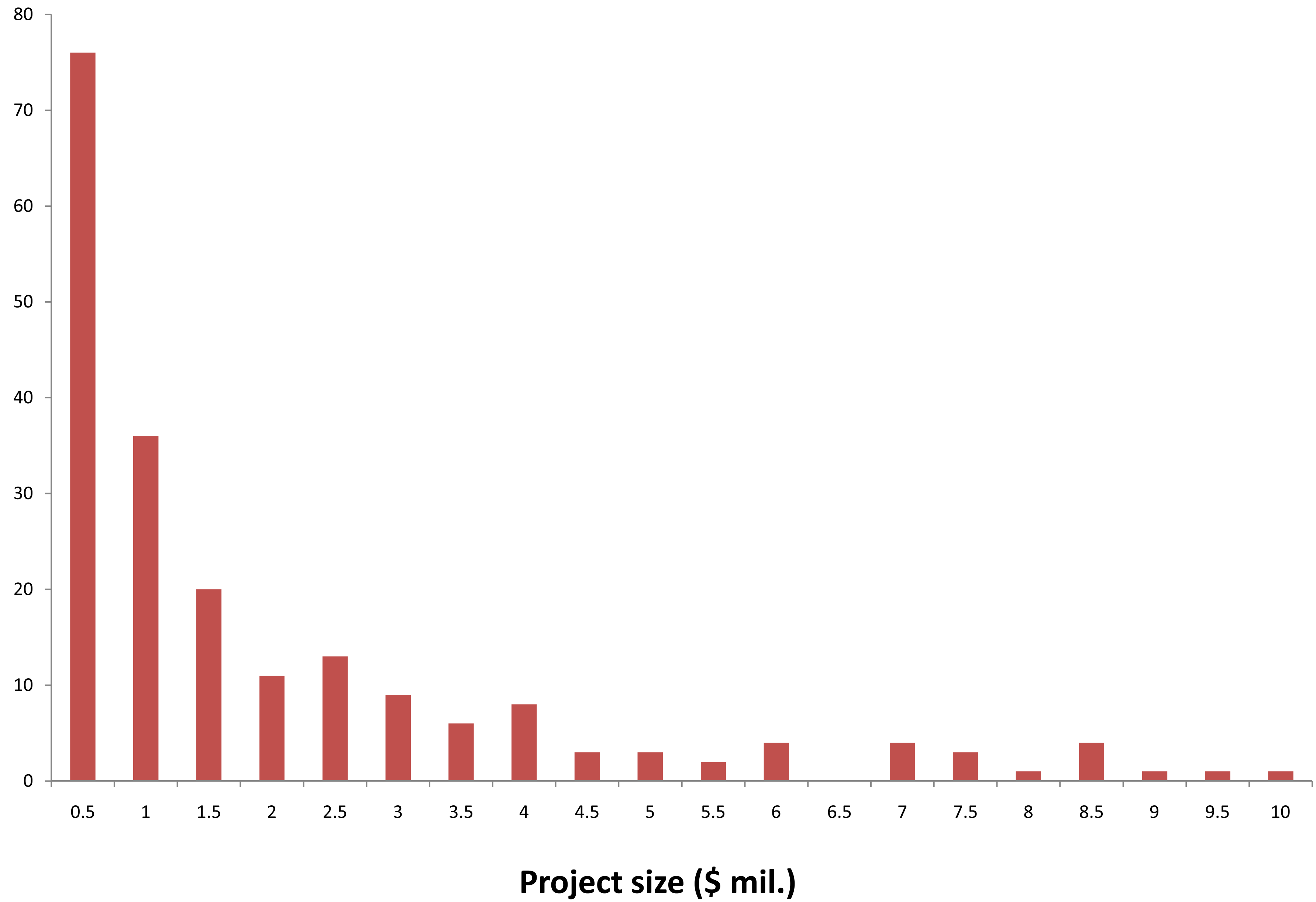

Table 1: Size of the GLS test

	Quantiles		
nominal size	0.5	0.3, 0.5, 0.7	0.3, 0.4, 0.5, 0.6, 0.7
		rho=0	
0.10	0.0760	0.1520	0.1580
0.05	0.0310	0.0730	0.1030
0.01	0.0070	0.0200	0.0360
		rho=0.5	
0.10	0.0420	0.0660	0.0640
0.05	0.0240	0.0350	0.0310
0.01	0.0010	0.0130	0.0100

Table 2: Size-corrected power of the GLS test

	Quantiles		
nominal size	0.5	0.3, 0.5, 0.7	0.3, 0.4, 0.5, 0.6, 0.7
		rho=0.5	
0.10	0.1622	0.2800	0.2683
0.05	0.0941	0.2030	0.1812
0.01	0.0210	0.0630	0.0651
		rho=0.9	
0.10	0.2693	0.3624	0.4124
0.05	0.1842	0.2853	0.2983
0.01	0.0661	0.1231	0.1552

Table 3: Description of Variables

Variable	Description	Mean	Std. Dev	Min	Max
EngEst	The engineer's estimate for the project, in mil. Dollars	3.647	4.488	0.066	24.800
Bid	Bid divided by the engineer's estimate	1.067	0.173	0.385	2.106
Nitems	Number of pay items in the project ad	71.736	70.704	1.000	363.000
Ndays	Number of business days to complete the project	195.995	142.370	10.000	681.000
Length	Length of the road (in miles)	4.699	4.794	0.000	36.63
Distance	Distance in miles from the headquarters of the firm of the bidding firm to the project site	344.237	382.469	0.000	1702.016
Backlog	The total amount of unfinished work on a given day and normalized by the bidder-specific maximum, the value is between 0 and 1	0.219	0.297	0.000	1.000
Npotential	number of planholders	8.299	4.244	2.000	26.000
Nactual	number of actual bidders	3.451	1.428	0.000	7.000
out-of-state	dummy =1 if the firm has headquarters outside the state of Oklahoma	0.136	0.343	0	1

Table 4: Logit and OLS regressions

Variable	Logit Coefficient	Logit s.e.	OLS Coefficient	OLS s.e.
Intercept	**3.565**	1.148	**0.497**	0.123
Log(EngEst)	-0.010	0.034	**0.970**	0.010
Npotential	**-0.043**	0.019	**-0.008**	0.002
Length	0.009	0.012	0.001	0.001
Ndays	0.000	0.001	0.000	0.000
Nitems	**-2.700E-04**	0.000	**4.580E-04**	0.000
Distance	0.000	0.000	0.000	0.000
Backlog	0.137	0.170	0.019	0.019
Out-of-state	**-0.359**	0.166	**-0.034**	0.020
Fringe firm	0.250	0.175	0.027	0.034
Firm				
APAC-OKLAHOMA, INC.	0.303	0.497	0.013	0.025
THE CUMMINS CONST. CO., INC.	0.266	0.385	0.005	0.023
HASKELL LEMON CONST. CO.	-0.206	0.351	0.028	0.023
BROCE CONSTRUCTION CO., INC.	**-1.803**	0.313	0.052	0.029
WESTERN PLAINS CONSTRUCTION COMPANY	-1.338	0.756	-0.011	0.031
BELLCO MATERIALS, INC.	-0.672	0.411	-0.086	0.277
OVERLAND CORPORATION	-0.726	0.399	0.057	0.029
GLOVER CONST. CO., INC.	**-0.919**	0.395	**0.061**	0.003
T & G CONSTRUCTION, INC.	-0.974	0.992	-0.036	0.028
TIGER INDUSTRIAL TRANS. SYS., INC.	-0.061	0.479	-0.004	0.029
HORIZON CONST. CO., INC.	-0.181	0.473	0.009	0.033
CORNELL CONST. CO., INC.	-0.481	0.450	0.007	0.030
SEWELL BROTHERS, INC	-0.075	0.472	-0.018	0.034
BECCO CONTRACTORS, INC.	-1.818	0.876	**-0.068**	0.033
EVANS & ASSOC. CONST. CO., INC.	-0.228	0.582	-0.044	0.036
SHERWOOD CONST. CO., INC.	-0.561	0.416	0.006	0.035
VANTAGE PAVING, INC.	-1.619	0.962	0.030	0.047
ALLEN CONTRACTING, INC.	0.245	0.493	0.014	0.035
DUIT CONSTRUCTION CO., INC.	1.275	0.707	0.027	0.037
MUSKOGEE BRIDGE CO., INC.	-1.122	0.809	0.006	0.037
Observations	4485		1860	
Log-Likelihood	-1543.750			
R2			0.983	

Notes: Significant coefficients (at 5% level) are marked in bold. The dependent variables were: for the logit regression, the indicator variable equal to 1 if the bid is submitted; for the OLS regression, the amount of bid in $ mil.

Table 5: Estimated probability $\pi(N|x)$ of N conditional on project size x

	Project size ($ mil.)									
	x=0.5		x=1		x=1.5		x=2		x=2.5	
N	π(N|x)	s.e.	π(N|x)	s.e.	π(N|x)	s.e.	π(N|x)	s.e.	π(N|x)	s.e.
2	0.07	0.02	0.04	0.02	0.02	0.02	0.01	0.02	0.00	0.00
3	0.26	0.04	0.21	0.04	0.14	0.04	0.07	0.04	0.01	0.02
4	0.22	0.04	0.22	0.04	0.16	0.04	0.10	0.05	0.12	0.06
5	0.15	0.03	0.17	0.04	0.18	0.05	0.13	0.05	0.06	0.04
6	0.17	0.03	0.15	0.03	0.19	0.05	0.26	0.07	0.18	0.07
7	0.02	0.01	0.04	0.02	0.05	0.03	0.11	0.05	0.20	0.07
8	0.05	0.02	0.06	0.02	0.08	0.03	0.11	0.05	0.12	0.06
9	0.02	0.01	0.04	0.02	0.07	0.03	0.08	0.04	0.07	0.05
10	0.01	0.01	0.03	0.02	0.06	0.03	0.06	0.04	0.04	0.03
11	0.01	0.01	0.01	0.01	0.02	0.02	0.04	0.03	0.08	0.05
12	0.01	0.01	0.01	0.01	0.01	0.01	0.00	0.00	0.02	0.03
13					0.00	0.00	0.02	0.02	0.04	0.03
14	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.03	0.03
15					`	0.01	0.02	0.02	0.04	0.03
E[N|x]	4.69	0.02	5.23	0.02	5.95	0.03	6.84	0.04	7.97	0.04

Table 6: Estimated probability of bidding P(N,x)

	Project size ($ mil.)									
	x=0.5		x=1		x=1.5		x=2		x=2.5	
N	P(N,x)	s.e.	P(N,x)	s.e.	P(N,x)	s.e.	P(N,x)	s.e.	P(N,x)	s.e.
2										
3	0.49	0.05	0.53	0.06	0.56	0.09	0.47	0.17		
4	0.45	0.05	0.45	0.05	0.45	0.07	0.56	0.12		
5	0.39	0.05	0.40	0.05	0.38	0.06	0.33	0.09		
6	0.37	0.04	0.36	0.05	0.34	0.05	0.30	0.06	0.27	0.08
7										
8	0.30	0.06	0.31	0.06	0.27	0.07				
9					0.28	0.07	0.35	0.09	0.45	0.11
10										
11									0.21	0.08
Test statistic	0.00		0.00		0.01		4.73		19.74	
P value	0.99		1.00		1.00		0.17		0.00	

Table 7: Estimated medians of costs

	Project size ($ mil.)									
	x=0.5		x=1		x=1.5		x=2		x=2.5	
N	median	s.e.	median	s.e.	median	s.e.	median	s.e.	median	s.e.
2										
3	0.79	0.15	0.78	0.18	0.83	0.22	0.84	0.33		
4	0.89	0.07	0.89	0.08	0.90	0.11	0.94	0.10		
5	0.90	0.08	0.89	0.09	0.84	0.18	0.69	0.45		
6	0.90	0.06	0.90	0.06	0.88	0.07	0.87	0.08	0.84	0.13
7										
8	0.90	0.07	0.92	0.07	0.93	0.09				
9					0.76	0.22	0.82	0.08	0.80	0.03
10										
11									0.81	0.11

Table 8: Estimated transformed medians of costs

	Project size ($ mil.)									
	x=0.5		x=1		x=1.5		x=2		x=2.5	
N	transformed median	s.e.	transformed median	s.e.	transformed median	s.e.	transformed median	s.e.	transformed median	s.e.
2										
3	0.85	0.14	0.85	0.16	0.86	0.20	0.84	0.33		
4	0.92	0.07	0.91	0.08	0.93	0.11	0.95	0.10		
5	0.91	0.07	0.92	0.08	0.89	0.14	0.69	0.45		
6	0.91	0.06	0.91	0.06	0.90	0.07	0.87	0.08	0.85	0.13
7										
8	0.90	0.07	0.92	0.07	0.93	0.09				
9					0.76	0.22	0.82	0.08	0.84	0.07
10										
11									0.81	0.11

Table 9: Test results

Project Size ($ mil.)	GLS statistic	GLS p-value	LS statistic	LS p-value	S statistic	S p-value
Quantiles: 0.5						
0.5	9.56	0.02	9.56	0.05	8.35	0.08
1.0	9.21	0.02	9.21	0.06	5.85	0.21
1.5	6.84	0.11	12.69	0.03	12.24	0.03
2.0	4.68	0.19	17.04	0.00	20.03	0.00
2.5	0.04	0.67	1.26	0.53	3.95	0.14
Quantiles: 0.3, 0.5, 0.7						
0.5	30.45	0.00	30.47	0.00	41.77	0.00
1.0	31.77	0.00	31.77	0.00	41.57	0.00
1.5	20.44	0.02	32.10	0.01	46.46	0.00
2.0	12.89	0.12	44.32	0.00	62.33	0.00
2.5	1.02	0.73	3.82	0.70	15.65	0.02
Quantiles: 0.3, 0.4, 0.5, 0.6, 0.7						
0.5	50.55	0.00	50.57	0.00	65.13	0.00
1.0	54.21	0.00	54.21	0.00	61.78	0.00
1.5	37.96	0.00	57.69	0.00	64.57	0.00
2.0	19.49	0.14	72.63	0.00	104.08	0.00
2.5	1.61	0.85	5.50	0.86	24.11	0.01