Moving the Goalposts: Addressing Limited Overlap in Estimation of Average Treatment Effects by Changing the Estimand[*]

Richard K. Crump[†]   V. Joseph Hotz[‡]   Guido W. Imbens[§]   Oscar Mitnik[¶]

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Abstract

Estimation of average treatment effects under unconfoundedness or selection on observables is often hampered by lack of overlap in the covariate distributions. This lack of overlap can lead to imprecise estimates and can make commonly used estimators sensitive to the choice of specification. In such cases researchers have often used informal methods for trimming the sample or focused on subpopulations of interest. In this paper we develop formal methods for addressing such lack of overlap in which we sacrifice some external validity in exchange for improved internal validity. We characterize optimal subsamples where the average treatment effect can be estimated most precisely, as well optimally weighted average treatment effects. We show the problem of lack of overlap has important connections to the presence of treatment effect heterogeneity: under the assumption of constant conditional average treatment effects the treatment effect can be estimated much more precisely. The efficient estimator for the treatment effect under the assumption of a constant conditional average treatment effect is shown to be identical to the efficient estimator for the optimally weighted average treatment effect. We also develop tests for the null hypotheses of a constant and a zero conditional average treatment effect. The latter is in practice more powerful than the commonly used test for a zero average treatment effect.

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1 Introduction

There is a large literature on estimating average treatment effects under assumptions of unconfoundedness or ignorability following the seminal work by Rubin (1973, 1978) and Rosenbaum and Rubin (1983a). Researchers have developed estimators based on regression methods (e.g., Hahn, 1998, Heckman, Ichimura and Todd, 1998), matching (e.g., Rosenbaum, 1989, Abadie and Imbens, 2004), and methods based on the propensity score (e.g., Rosenbaum and Rubin, 1983a, Hirano, Imbens and Ridder, 2003). Related methods for missing data problems are discussed in Robins, Rotnitzky and Zhao (1995) and Robins and Rotznitzky (1995). See Rosenbaum (2001), Heckman, Lalonde and Smith (1999), Wooldridge (2002), Blundell and Costa-Diaz (2002), Imbens (2004) and Lee (2005a) for surveys of this literature. In practice an important concern in implementing all these methods is that one needs sufficient overlap between covariate distributions in the two subpopulations. Even if there exist areas with sufficient overlap, there may be other parts of the covariate space with few units of one of the treatment levels. Such areas of limited overlap can lead to estimators for average treatment effects with poor finite sample properties. In particular, such estimators can have substantial bias, large variances, as well as considerable sensitivity to the exact specification of the regression functions or propensity score. Heckman, Ichimura and Todd (1997) and Dehejia and Wahba (1999) point out the empirical relevance of this overlap issue.[1]

One strand of the literature has focused on assessing the robustness of existing estimators to a variety of potential problems including lack of overlap. See for example Rosenbaum and Rubin (1983b), Imbens (2003), and Ichino, Mealli, and Nannicini (2005). A second strand of the literature focuses on developing new estimators that are more robust and precise. With this goal in mind researchers have proposed discarding or downweighting observations with covariates in areas with limited overlap. A number of specific methods have been proposed for implementing this. In simplest setting with a discrete covariate Rubin (1977) and Lee (2005b) suggest suggests simply discarding all units with covariate values with either no treated or no control units. Rubin and Cochran (1973) suggest caliper matching where potential matches are dropped if the within-match difference in propensity scores exceeds some threshold level. Dehejia and Wahba (1999) focus on the average treatment effect for the treated and suggest discarding all controls with estimated propensity scores below the smallest value of the propensity score among the treated. Heckman, Ichimura, Smith and Todd (1997) and Heckman, Ichimura and Todd (1998) drop units from the analysis if the estimated density of the covariate distribution conditional on treatment status is below some threshold. Ho, Imai, King and Stuart (2004) propose preprocessing the data by matching units and carrying out parametric inferences using only the matched data. All of these methods have some advantages as well as drawbacks. All of them change the estimand, at least in finite samples. They all do tend to reduce sensitivity of the final estimates to model specification. However, they rely on arbitrary choices regarding

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[1]Dehejia and Wahba (1999) write: "... our methods succeed for a transparent reason: They only use the subset of the comparison group that is comparable to the treatment group, and discard the complement." Heckman, Ichimura and Todd (1997) write "A major finding of this paper is that comparing the incomparable – i.e., violating the common support condition for the matching variables – is a major sources of evaluation bias as conventionally measured."

thresholds for discarding observations, and there are few formal results on their properties.

In this paper we propose a systematic approach to account for subpopulations with limited overlap in the covariates. This approach has asymptotic optimality properties under some conditions and is straightforward to implement. We consider two specific methods. First we focus on average treatment effects within a selected subpopulation defined in terms of covariate values. Conditioning on a subpopulation reduces the effective sample size, thus increasing the variance of the estimated average treatment effect. However, if the subpopulation is chosen appropriately, it may be possibly to estimate the average treatment within this subpopulation more precisely than the average effect for the entire population despite the smaller sample size. It turns out that in general this tradeoff is well defined and leads under some conditions to choosing the subpopulation with the propensity score in an interval $[\alpha, 1 - \alpha]$, with the optimal cutoff value of α solely determined by the distribution of the propensity score. We refer to this as the Optimal Subpopulation Average Treatment Effect (OSATE).

Second, we consider weighted average treatment effects with the weights depending only on the covariates. The first approach of choosing a subpopulation can be viewed as a special case in this framework where the weight function is restricted to be an indicator function. Without imposing this restriction we characterize the weight function that leads to the most precisely estimated average treatment effect. Note that this class of estimands with weights depending on the covariates includes the average treatment effect for the treated where the weight function is proportional to the propensity score. Under the same conditions as before the optimal weight function will again be a function of the propensity score alone, proportional to the product of the propensity score and one minus the propensity score. We refer to this as the Optimally Weighted Average Treatment Effect (OWATE).

The switch to average treatment effect for an optimally selected subpopulation or to a optimally weighted average treatment effect has a second benefit beyond the increase in precision. The subpopulations for treated and control group in this selected or weighted population tend to be more balanced in the distribution of the covariates. This is a consequence of the fact that, under homoskedasticity, the variance of the conditional average treatment effect is proportional to $(e(X) \cdot (1 - e(X)))^{-1}$, and thus lowering the weight on high-variance observations increases the weight on observations with propensity scores close to $1/2$. The increased balance in the selected or weighted sample reduces the sensitivity of any estimators to changes in the specification. In the extreme case where the selected sample is completely balanced in covariates in the two treatment arms one can simply use the average difference in outcomes between treated and control units.

It is important to stress that these methods change the estimand. Instead of focusing on the traditional estimands, the population average treatment effect, the average effect for the subpopulation of the treated or another *a priori* defined subpopulation of interest, we focus on average effects for a statistically defined (weighted) subpopulation.[2] This change of focus is *not* motivated by an intrinsic interest in the subpopulation we ultimately estimate the average causal effect for. Rather, it acknowledges and addresses the difficulties in making inferences about the population of primary interest. Instead of reporting solely the potentially imprecise estimate for

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[2]This is also true for the method proposed by Heckman, Ichimura and Todd, (1998).

the population average treatment effect we advocate reporting both estimates for the population of interest and estimates for subpopulations where we can make more precise inferences. In settings where we cannot ascertain with much confidence the sign of the population average treatment effect such estimates may serve to demonstrate that there are subpopulations that benefit from or are harmed by the program, as well as the extent of this benefit or harm. It is also important to note that the subpopulation for which these estimands are valid are defined in terms of the observed covariate values so that one can determine for each individual whether they are in the relevant subpopulation or not.

This change of estimand is uncommon in econometric analyses.[3] Typically in such analyses the estimand is defined *a priori*, followed by a presentation of estimates that turn out to be more or less precise depending on the actual data. In cases where even large data sets would not permit point identification of the estimand or interest regions of the parameter space consistent with the model may be reported in a bounds analysis of the type developed by Manski (1990, 2003). Here our approach is different and to some extent complementary. Sacrificing some external validity by changing the sample from one that was potentially representative of the population of interest we potentially gain some internal validity by changing it to a sample where we can obtain more precise and credible estimates.[4] Our proposed stress on internal validity at the expense of external validity is similar to that in randomized experiments which are often carried out in populations unrepresentative of the population of interest.[5] More generally, the primacy of internal validity over external validity is advocated in many discussions of causality (see, for example, Shadish, Cook, and Campbell, 2002).

In interpreting our results it is also of interest to consider estimation of the average treatment effect under the assumption that it does not vary with the covariates.[6] This assumption can be quite informative except in the case where the propensity score is constant. Under the assumption that the treatment effect does not vary with covariates the model is a special case of the partial linear model studied by Robinson (1988), Stock (1989) and Robins, Mark and Newey (1992).[7] The efficient estimator for that case turns out to be identical to the efficient estimator in the heterogenous case for the weighted average treatment effect with the weights chosen to obtain the most precisely estimated average treatment effect.

We also develop a set of three new nonparametric tests. We first test the hypothesis that there is no variation in the conditional average treatment effect by covariates. Second, we test the hypothesis that the conditional average treatment effect is zero for all values of the

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[3]One exception is the local average treatment effect introduced by Imbens and Angrist (1994), which is the average effect of the treatment for the subpopulation of compliers.

[4]A separate issue is that in practice in many cases even the original sample is not representative of the population of interest. For example, we are often interested in policies that would extend small pilot versions of job training programs to different locations and times.

[5]Even in those settings this can be controversial and lead to misleading conclusions.

[6]The possible presence of heterogeneity of the treatment effect is an important consideration in much of this literature. See for applications Dehejia and Wahba (1999), Lechner (2002) and others.

[7]Stock (1989) also focuses on estimating the effect of a policy intervention, but he formulates the problem differently. In his approach the intervention does not change the relationship between the covariates and the outcome. Instead it changes the distribution of the covariates in a known manner. The estimand is then the difference between the average value of the regression function given the induced distribution of the covariates and the average value of the regression function given the current distribution of the covariates.

covariates. Third, we test the hypothesis that the optimally weighted average treatment effect is equal to zero.

We illustrate these methods using data from the Greater Avenue for INdependence (GAIN) experiments carried out in California in the nineties. In some of the the improvement in precision from focusing on the restricted sample is substantial. The additional improvement from moving from the optimal subpopulation to the optimally weighted sample is considerably smaller. The increased difficulty in interpretation of the weighted average treatment effect may not be worth this additional increase in precision.

It is important to note that our calculations are not tied to a specific estimator. The results formally refer to differences in the efficiency bound for different subpopulations. As a consequence, they are relevant for all efficient estimators, including the ones proposed by Hahn (1998), Hirano, Imbens and Ridder (2003), Imbens, Newey and Ridder (2004), Robins, Rotnitzky and Zhao (1995). Although not directly applicable to estimators that do not reach the efficiency bound, such as the nearest neighbor matching estimators in Abadie and Imbens (2002) and the local linear estimators in Heckman, Ichimura and Todd (1998), the close relation between those estimators and the efficient ones suggests that with matching the same issues are relevant.

2   A Simple Example

To set the stage for the issues to be discussed in this paper, consider an example with a scalar covariate X taking on two values, 0 and 1. Let N_x be the sample size for the subsample with $X = x$, and let $N = N_0 + N_1$ be the total sample size. Also let $p = N_1/N$ be the population share of $X = 1$ units. Let the average treatment effect conditional on the covariate be equal to τ_x. The population average treatment effect is then $\tau = p \cdot \tau_1 + (1-p) \cdot \tau_0$. Let N_{xw} be the number of observations with covariate $X_i = x$ and treatment indicator $W_i = w$. Also, let $e_x = N_{x1}/N_x$ be the propensity score for $x = 0, 1$. Finally, let $\bar{y}_{xw} = \sum_{i=1}^{N} Y_i \cdot 1\{X_i = x, W_i = w\}/N_{xw}$ be the average within each of the four subpopulations. Assume that the variance of $Y(w)$ given $X_i = x$ is σ^2 for all x.

The natural estimator for the treatment effects for each of the two subpopulations are

$$\hat{\tau}_0 = \bar{y}_{01} - \bar{y}_{00}, \qquad \text{and} \quad \hat{\tau}_1 = \bar{y}_{11} - \bar{y}_{10},$$

with variances

$$V(\hat{\tau}_0) = \sigma^2 \cdot \left(\frac{1}{N_{00}} + \frac{1}{N_{01}} \right) = \frac{\sigma^2}{N \cdot (1-p)} \cdot \frac{1}{e_0 \cdot (1-e_0)},$$

and

$$V(\hat{\tau}_1) = \sigma^2 \cdot \left(\frac{1}{N_{10}} + \frac{1}{N_{11}} \right) = \frac{\sigma^2}{N \cdot p} \cdot \frac{1}{e_1 \cdot (1-e_1)}.$$

The estimator for the population average treatment effect is

$$\hat{\tau} = p \cdot \hat{\tau}_1 + (1-p) \cdot \hat{\tau}_0.$$

[4]

Because the two estimates $\hat{\tau}_0$ and $\hat{\tau}_1$ are independent, the variance of the population average treatment effect is

$$V(\hat{\tau}) = p^2 \cdot V(\hat{\tau}_1) + (1-p)^2 \cdot V(\hat{\tau}_0)$$

$$= \frac{\sigma^2}{N} \cdot \left(\frac{p}{e_1 \cdot (1-e_1)} + \frac{1-p}{e_0 \cdot (1-e_0)} \right) = \frac{\sigma^2}{N} \cdot \mathbb{E} \left[\frac{1}{e_X \cdot (1-e_X)} \right].$$

The first point of the paper concerns the comparison of $V(\hat{\tau})$, $V(\hat{\tau}_0)$, and $V(\hat{\tau}_1)$). Define $V_{\min} = \min(V(\hat{\tau}), V(\hat{\tau}_0), V(\hat{\tau}_1)$. Then

$$V_{\min} = \begin{cases} V(\hat{\tau}_0) & \text{if} & (e_1(1-e_1))/(e_0(1-e_0)) & \leq (1-p)/(2-p), \\ V(\hat{\tau}) & \text{if} & (1-p)/(2-p) \leq & (e_1(1-e_1))/(e_0(1-e_0)) & \leq (1+p)/p, \\ V(\hat{\tau}_1) & \text{if} & (1+p)/p \leq & (e_1(1-e_1))/(e_0(1-e_0)). \end{cases}$$

$$(2.1)$$

The key is the ratio of the product of the propensity score and one minus the propensity score, $e_1(1-e_1)/(e_0(1-e_0))$. If the propensity score for units with $X = 0$ is close to zero or one, we cannot estimate the average treatment effect for this subpopulation precisely. In that case the ratio $e_1(1-e_1)/(e_0(1-e_0))$ will be high and we may be able to estimate the average treatment effect for the $X = x_1$ subpopulation more accurately than for the population as a whole, even though we may lose a substantial number of observations by discarding units with $X_i = 0$. Similarly, if the propensity score for the $X = 1$ subpopulation is close to zero or one, the ratio $e_1(1-e_1)/(e_0(1-e_0))$ is close to zero, and we may be able to estimate the average treatment effect for the $X = x_0$ subpopulation more accurately than for the population as a whole. If the ratio is close to one, we can estimate the average effect for the population as a whole more accurately than for either of the two subpopulations.

The second advantage of focusing on subpopulation average treatment effects is in this case obvious. Within the two subpopulations we can estimate the within-subpopulation average treatment effect without bias by simply differencing average treatment and control outcomes. Thus our results are not sensitive to the choice of estimator, whereas in the population as a whole there is potentially substantial bias from simply differencing average outcomes.

The second point is that one need not limit the choice to the three average treatment effects discussed so far. More generally one may wish to focus on a weighted average treatment effect

$$\tau_\lambda = \lambda \cdot \tau_1 + (1-\lambda) \cdot \tau_0,$$

for fixed λ, which can be estimated as

$$\hat{\tau}_\lambda = \lambda \cdot \hat{\tau}_1 + (1-\lambda) \cdot \hat{\tau}_0,$$

The variance for this weighted average treatment effect is

$$V(\hat{\tau}_\lambda) = \lambda^2 \cdot V(\hat{\tau}_1) + (1-\lambda)^2 \cdot V(\hat{\tau}_0)$$

$$= \lambda^2 \cdot \frac{\sigma^2}{N \cdot p} \cdot \frac{1}{e_1 \cdot (1-e_1)} + (1-\lambda)^2 \cdot \frac{\sigma^2}{N \cdot (1-p)} \cdot \frac{1}{e_0 \cdot (1-e_0)}.$$

[5]

The variance is minimized at

$$\lambda^* = \frac{1/V(\hat{\tau}_1)}{1/V(\hat{\tau}_1) + 1/V(\hat{\tau}_0)} = \frac{p \cdot e_1 \cdot (1 - e_1)}{(1 - p) \cdot e_0 \cdot (1 - e_0) + p \cdot e_1 \cdot (1 - e_1)}. \tag{2.2}$$

with the minimum value for the variance equal to

$$V(\tau_{\lambda^*}) = \frac{\sigma^2}{N} \cdot \frac{1}{((1 - p) \cdot e_0 \cdot (1 - e_0) + p \cdot e_1 \cdot (1 - e_1))} = \frac{\sigma^2}{N} \cdot \frac{1}{\mathbb{E}[e_X \cdot (1 - e_X)]}.$$

The ratio of the variance for the population average to the variance for the optimally weighted average treatment effect is

$$V(\tau_P)/V(\tau_{\lambda^*}) = \mathbb{E}\left[\frac{1}{e_X \cdot (1 - e_X)}\right] \Big/ \frac{1}{\mathbb{E}[e_X \cdot (1 - e_X)]} \tag{2.3}$$

$$= \mathbb{E}\left[\frac{1}{V(e_X)}\right] \Big/ \frac{1}{\mathbb{E}[V(e_X)]}.$$

By Jensen's inequality this is greater than one if $V(e_X) > 0$, that is, if the propensity score varies across the population.

In summary, suppose in this case one is interested in the population average treatment effect τ. One may find that the efficient estimator is imprecise. This is consistent with two different states of the world. In one state the average effect for both of the subpopulations are also imprecisely estimated, and in effect one cannot say much about the effect of the treatment at all. In the other state of the world it is still possible to learn something about the effect of the treatment because one of the subpopulation average treatment effects can be estimated precisely. In that case, which corresponds to the propensity score for one of the two subpopulations being close to zero or one, one may wish to report also the estimator for the precisely estimable average treatment effect to convey the information the data contain about the effect of the treatment. It is important to stress that the message of the paper is not that one should report $\hat{\tau}_m$ or $\hat{\tau}_f$ instead of $\hat{\tau}$. Rather, in cases where $\hat{\tau}_m$ or $\hat{\tau}_f$ are precisely estimable and $\hat{\tau}$ is not, one should report both.

In this paper we generalize this analysis to the case with a vector of potentially continuously distributed covariates. We study the existence and characterization of a partition of the covariates space into two subsets. For one of the subpopulations the average treatment effect is at least as accurately estimable as that for any other subset of the covariate space. This leads to a generalization of (2.1). Under some assumptions this problem has a well-defined solution and these subpopulations have a very simple characterization, namely the set of covariates such that the propensity score is in the closed interval $[a, 1 - a]$. The optimal value of the boundary point a is determined by the distribution of the propensity score and its calculation is straightforward. In addition we characterize the optimally weighted average treatment effect and its variance, the generalization of (2.2) and (2.3).

3   Set Up

The basic framework is standard in this literature (e.g., Rosenbaum and Rubin, 1983; Hahn, 1998; Heckman, Ichimura and Todd, 1998; Hirano, Imbens and Ridder, 2003). We have a

random sample of size N from a large population. For each unit i in the sample, let W_i indicate whether the treatment of interest was received, with $W_i = 1$ if unit i receives the treatment of interest, and $W_i = 0$ if unit i receives the control treatment. Using the potential outcome notation popularized by Rubin (1974), let $Y_i(0)$ denote the outcome for unit i under control and $Y_i(1)$ the outcome under treatment. We observe W_i and Y_i, where

$$Y_i \equiv Y_i(W_i) = W_i \cdot Y_i(1) + (1 - W_i) \cdot Y_i(0).$$

In addition, we observe a vector of pre-treatment variables, or covariates, denoted by X_i. Define the two conditional means, $\mu_w(x) = \mathbb{E}[Y(w)|X = x]$, the two conditional variances, $\sigma_w^2(x) = \text{Var}(Y(w)|X = x)$, the conditional average treatment effect $\tau(x) = \mathbb{E}[Y(1) - Y(0)|X = x] = \mu_1(x) - \mu_0(x)$, and the propensity score, the probability of selection into the $e(x) = \Pr(W = 1|X = x) = \mathbb{E}[W|X = x]$.

Initially we focus on two average treatment effects. The first is the (super-)population average treatment effect

$$\tau_P \equiv \mathbb{E}[Y(1) - Y(0)].$$

We also consider the conditional average treatment effect:

$$\tau_C = \frac{1}{N} \sum_{i=1}^{N} \tau(X_i),$$

where we condition on the observed set of covariates. The reason for focusing on the second one is twofold. First, it is analogous to the common conditioning on covariates in regression analysis. Second, it can be estimated more precisely if there is indeed variation in the treatment effect by covariates.

To solve the identification problem, we maintain throughout the paper the unconfoundedness assumption (Rubin, 1978; Rosenbaum and Rubin, 1983), which asserts that conditional on the pre-treatment variables, the treatment indicator is independent of the potential outcomes. Formally:

Assumption 3.1 (UNCONFOUNDEDNESS)

$$W \perp (Y(0), Y(1)) \ \Big| \ X. \tag{3.4}$$

In addition we assume there is overlap in the covariate distributions:

Assumption 3.2 (OVERLAP)
For some $c > 0$,

$$c \leq e(x) \leq 1 - c.$$

In addition for estimation we often need smoothness conditions on the two regression functions $\mu_w(x)$ and the propensity score $e(x)$.

4 Efficiency Bounds

Next, we review some results for efficient estimation of treatment effects. First we discuss efficient estimators previously developed by Hahn (1998) and Hirano, Imbens and Ridder (2003) for treatment effects allowing for heterogeneity in the treatment effects. Second, we present some results for efficient estimation of treatment effects under a variety of assumptions that restrict the heterogeneity of the treatment effects. This setting is closely related to the partial linear model developed by Robinson (1988).

Hahn (1998) calculates the efficiency bound for τ_P.

Theorem 4.1 (HAHN, 1998) *Suppose Assumptions 3.1 and 3.2 hold. Then the semiparametric efficiency bounds for τ is*

$$V_P^{\text{eff}} = \mathbb{E}\left[(\tau(X) - \tau)^2 + \frac{\sigma_1^2(X)}{e(X)} + \frac{\sigma_0^2(X)}{1 - e(X)}\right]. \tag{4.5}$$

Proof: See Hahn (1998).

Robins, Rotznitzky and Zhao (1995) present a similar result in a missing data setting.

Hahn (1998) also proposes an estimator that achieves the efficiency bound.[8] Hahn's estimator is asymptotically linear,

$$\hat{\tau}_H = \frac{1}{N} \sum_{i=1}^{N} \psi(Y_i, W_i, X_i) + o_p\left(N^{-1/2}\right),$$

where

$$\psi(y, w, x) = w \cdot \frac{y - \mu_1(x)}{e(x)} - (1 - w) \cdot \frac{y - \mu_0(x)}{1 - e(x)} + \mu_1(x) - \mu_0(x) - \tau.$$

One implication of this representation is that we can view Hahn's estimator, as well as the other efficient estimators not only as estimators of the population average treatment effect τ_P but also as estimators of the conditional average treatment effect τ_C. As an estimator of τ_C the efficient estimator $\hat{\tau}_H$ has asymptotic variance

$$\mathbb{E}\left[\frac{\sigma_1^2(X)}{e(X)} + \frac{\sigma_0^2(X)}{1 - e(X)}\right]. \tag{4.6}$$

Furthermore, these estimators are efficient for τ_C:

Theorem 4.2 (EFFICIENCY BOUND FOR τ_C) *Suppose Assumptions 3.1 and 3.2 hold. Then the semiparametric efficiency bounds for τ_C is*

$$V_C^{\text{eff}} = \mathbb{E}\left[\frac{\sigma_1^2(X)}{e(X)} + \frac{\sigma_0^2(X)}{1 - e(X)}\right]. \tag{4.7}$$

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[8] Other efficient estimators have been proposed by Hirano, Imbens and Ridder (2003) and Imbens, Newey and Ridder (2004).

Proof: See Appendix.

Next we consider a larger set of estimands. Instead of looking at the average treatment effect within a subpopulation we consider weighted average treatment effects of the form

$$\tau_{P,g} = \mathbb{E}[\tau(X) \cdot g(X)]/\mathbb{E}[g(X)],$$

for nonnegative functions $g(\cdot)$. For estimands of this type the efficiency bound is given in the following theorem:

Theorem 4.3 (HIRANO, IMBENS AND RIDDER, 2003) *Suppose Assumptions 3.1 and 3.2 hold, and suppose that $g(\cdot)$ is known. Then the semiparametric efficiency bounds for τ_g is*

$$V_{P,g}^{\text{eff}} = \frac{1}{\mathbb{E}[g(X)]^2} \cdot \mathbb{E}\left[g(X)^2 \cdot \left(\frac{\sigma_1^2(X)}{e(X)} + \frac{\sigma_0^2(X)}{1 - e(X)} + (\tau(X) - \tau_g)^2\right)\right]$$

Proof: See Hirano, Imbens and Ridder (2003).

Again there is an asymptotically linear estimator that achieves this efficiency bound. The same argument as above therefore shows that the efficient estimator for $\tau_{P,g}$, as an estimator for the conditional average treatment effect version of this estimand,

$$\tau_{C,g} = \sum_{i=1}^{N} \tau(X_i) \cdot g(X_i) \bigg/ \sum_{i=1}^{N} g(X_i),$$

has asymptotic variance

$$V_{C,g}^{\text{eff}} = \frac{1}{\mathbb{E}[g(X)]^2} \cdot \mathbb{E}\left[\frac{g(X)^2}{e(X)}\sigma_1^2(X) + \frac{g(X)^2}{1 - e(X)}\sigma_0^2(X)\right]. \tag{4.8}$$

Next we consider the case where the weights depend on the propensity score: $g(x) = h(e(x))$. This will be useful later when some estimands of interest have this form. If the propensity score is known this is a special case of the previous result, but if the propensity score is unknown the efficiency bound changes.

Theorem 4.4 (WEIGHTED AVERAGE TREATMENT EFFECTS WITH WEIGHTS DEPENDING ON THE PROPENSITY SCORE) *Suppose Assumptions 3.1 and 3.2 hold, and suppose that the weights are a function of the propensity score: $g(x) = h(e(x))$ with $h(\cdot)$ known. Then the semiparametric efficiency bounds for τ_g is*

$$V_{P,g}^{\text{eff}} = \frac{1}{\mathbb{E}[g(X)]^2} \cdot \mathbb{E}\left[g(X)^2 \cdot \left(\frac{\sigma_1^2(X)}{e(X)} + \frac{\sigma_0^2(X)}{1 - e(X)} + (\tau(X) - \tau_g)^2\right)\right]$$

$$+ \frac{1}{\mathbb{E}[g(X)]^2} \cdot \mathbb{E}\left[e(X)(1 - e(X)) \cdot [h'(e(X))]^2(\tau(X) - \tau_g)^2\right],$$

where $h'(a)$ is the first derivative of $h(a)$.

Proof: See Appendix.

A special case of this arises when $h(a) = a$ so that the weights are proportional to the propensity score. In that case the estimand is the average effect for the treated. The efficiency bound for this case under the unknown propensity score was previously derived by Hahn (1998, Theorem 1). It is equal to

$$V_{P,t}^{\text{eff}} = \frac{1}{\mathbb{E}[e(X)]^2} \cdot \mathbb{E}\left[e(X)^2 \cdot \left(\frac{\sigma_1^2(X)}{e(X)} + \frac{\sigma_0^2(X)}{1 - e(X)} \right) + e(X) \cdot (\tau(X) - \tau_t)^2 \right].$$

Finally, we consider the case where we know that the average treatment effect does not vary by covariates.

Assumption 4.1 (CONSTANT CONDITIONAL AVERAGE TREATMENT EFFECT)
For all x, $\mu_1(x) - \mu_0(x) = \tau$.

This assumption is slightly weaker than assuming a constant treatment effect. Under this assumption the efficiency bound is a generalization of the bound given in Robins, Mark and Newey (1992) to the heteroskedastic case:

Theorem 4.5 (ROBINS, MARK AND NEWEY, 1992) *Suppose Assumptions 3.1, 3.2, and 4.1 hold. Then the semiparametric efficiency bounds for τ is*

$$V_{\text{cons}}^{\text{eff}} = \left(\mathbb{E}\left[\left(\frac{\sigma_1^2(X)}{e(X)} + \frac{\sigma_0^2(X)}{1 - e(X)} \right)^{-1} \right] \right)^{-1}. \tag{4.9}$$

Proof: See Robins, Mark and Newey (1992).

It is interesting to compare the efficiency bound for τ under the constant average treatment effect assumption given in (4.9) with the efficiency bound for the average conditional treatment effect τ_C given in (4.6). By Jensen's inequality the former is smaller, unless $\sigma_1^2(x)/e(x) + \sigma_0^2(x)/(1 - e(x))$ is constant. Under homoskedasticity the ratio of the variances V_C^{eff} and $V_{\text{cons}}^{\text{eff}}$ reduces to

$$\mathbb{E}\left[\frac{1}{V(W|X)} \right] \bigg/ \frac{1}{\mathbb{E}[V(W|X)]},$$

the same expression we obtained in the binary covariate case. This ratio is greater than one unless the propensity score is constant. If the propensity score takes on values close to zero or one this ratio can be large. The implication is that knowledge of the treatment effect being constant as a function of the covariates can be very valuable.

5   Previous Approaches to Dealing with Limited Overlap

In empirical application there is often concern about the overlap assumption (e.g., Dehejia and Wahba, 1999; Heckman, Ichimura, and Todd, 1997). To ensure that there is sufficient overlap researchers have sometimes trimmed their sample by excluding observations with propensity scores close to zero or one. Cochran and Rubin (1977) suggest caliper matching where units

whose match quality is too low according to the distance in terms of the propensity score are left unmatched.

Dehejia and Wahba (1999) focus on the average effect for the treated, They suggest dropping all control units with an estimated propensity score lower than the smallest value, or larger than the largest value, for the estimated propensity score among the treated units. Formally, they first estimate the propensity score. Let the estimated propensity score for unit i be $\hat{e}(X_i)$. Then let $\bar{e}_1$ be the minimum of the $\hat{e}(X_i)$ among treated units and let $\overline{e}_1$ be the maximum of the $\hat{e}(X_i)$ among treated units. DW then drop all control units such that $\hat{e}(X_i) < \bar{e}_1$ or $\hat{e}(X_i) > \overline{e}_1$.

Heckman, Ichimura and Todd (1997) and Heckman, Ichimura, Smith and Todd (1998) also focus on the average effect for the treated. They propose discarding units with covariate values at which the estimated density is below some threshold. The precise method is as follows.[9] First they estimate the propensity score $\hat{e}(x)$. Next, they estimate the density of the estimated propensity score in both treatment arms. Let $\hat{f}_w(e)$ denote the estimated density of the estimated propensity score. The specific estimator they use is a kernel estimator

$$\hat{f}_w(e) = \frac{1}{N_w \cdot h} \sum_{i|W_i=w} K\left(\frac{\hat{e}(X_i) - e}{h}\right),$$

with bandwidth h.[10] First HIT discard observations with $\hat{f}_0(\hat{e}(X_i))$ or $\hat{f}_1(\hat{e}(X_i))$ exactly equal to zero leaving J observations. Observations with the estimated denstiy equal to zero may exist when the kernel has finite support. Smith and Todd for example use a quadratic kernel with $K(u) = (u^2 - 1)^2$ for $|u| \leq 1$ and zero elsewhere. Next, they fix a quantile q (Smith and Todd use $q = 0.02$). Among the J observations with positive densities they rank the $2J$ values of $\hat{f}_0(\hat{e}(X_i))$ and $\hat{f}_1(\hat{e}(X_i))$. They then drop units i with $\hat{f}_0(\hat{e}(X_i))$ or $\hat{f}_1(\hat{e}(X_i))$ less than or equal to c_q, where c_q is the largest real number such that

$$\frac{1}{2J} \sum_{i=1}^{J} \left(1\{\hat{f}_0(\hat{e}(X_i)) < c_q\} + 1\{\hat{f}_1(\hat{e}(X_i)) < c_q\}\right) \leq q.$$

Ho, Imai, King and Stuart (2004) propose combining any specific parametric procedure that the researcher may wish to employ with a nonparametric first stage in which the units are matched to the closest unit of the opposite treatment. This typically leads to a data set that is much more balanced in terms of covariate distributions between treated and control. It therefore thus reduces sensitivity of the parametric model to specific modelling decisions such as the inclusion of covariates or functional form assumptions.

All these methods tend to make the estimators more robust to specification decisions. However, few formal results are available on the properties of these procedures.

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[9] See Heckman, Ichimura and Todd (1997) and Smith and Todd (2005) for details, and Ham, Li and Reagan (2005) for an application of this method.

[10] In their application Smith and Todd (2005) use Silverman's rule of thumb to choose the bandwidth.

6 Alternative Estimands

6.1 The Optimal Subpopulation Average Treatment Effect

First we consider trimming the sample by excluding units with covariates outside of a set $\mathcal{A}$, where $\mathcal{A} \subset \mathbb{X}$, with $\mathbb{X} \subset \mathbb{R}^k$ the covariate space. For a given set $\mathcal{A}$ we define a corresponding average treatment effect $\tau_C(\mathcal{A})$:

$$\tau_C(\mathcal{A}) = \int_{\mathcal{A}} \tau(x) f(x) dx.$$

The efficiency bound for this parameter is

$$V_C^{\text{eff}}(\mathcal{A}) = \mathbb{E}\left[\left.\frac{\sigma_1^2(X)}{e(X)} + \frac{\sigma_0^2(X)}{1 - e(X)}\right| X \in \mathcal{A}\right].$$

Because the relative size of the subpopulation in $\mathcal{A}$ is $q(\mathcal{A}) = \Pr(X \in \mathcal{A})$, the efficiency bound normalized by the original sample size is

$$V_C^{\text{eff}\prime}(\mathcal{A}) = \frac{1}{q(\mathcal{A})} \cdot \mathbb{E}\left[\left.\frac{\sigma_1^2(X)}{e(X)} + \frac{\sigma_0^2(X)}{1 - e(X)}\right| X \in \mathcal{A}\right]. \tag{6.10}$$

We look for an optimal $\mathcal{A}$, denoted by $\mathcal{A}^*$, that minimizes the asymptotic variance (6.10) among all subsets $\mathcal{A}$.

There are two competing effects. First, by excluding units with covariate values outside the set $\mathcal{A}$ one reduces the effective sample size from N to $N \cdot q(\mathcal{A})$. This will increase the asymptotic variance, normalized by the original sample size, by a factor $1/q(\mathcal{A})$. Second, by discarding units with high values for $\sigma_1^2(X)/e(X) + \sigma_0^2(X)/(1 - e(X))$ (that is, units with covariate values such that it is difficult to estimate the average treatment effect) one can lower the conditional expectation $\mathbb{E}[\sigma_1^2(X)/e(X) + \sigma_0^2(X)/(1 - e(X))|X \in \mathcal{A}]$. Optimally choosing $\mathcal{A}$ involves balancing these two effects. The following theorem gives the formal result for the optimal $\mathcal{A}^*$ that minimizes the asymptotic variance.

Theorem 6.1 (OSATE)
Let $\underline{f} \leq f(x) \leq \overline{f}$, and $\sigma^2(x) \leq \overline{\sigma^2}$ for $w = 0, 1$ and all $x \in \mathbb{X}$. We consider sets $\mathcal{A} \subset \mathbb{X}$ that are elements of the sigma algebra of Borel subsets of $\mathbb{R}^k$. Then the Optimal Subpopulation Average Treatment Effect (OSATE) is $\tau_C(\mathcal{A}^*)$, where, if

$$\sup_{x \in \mathbb{X}} \frac{\sigma_1^2(x) \cdot (1 - e(x)) + \sigma_0^2(x) \cdot e(x)}{e(x) \cdot (1 - e(x))} \leq 2 \cdot \mathbb{E}\left[\frac{\sigma_1^2(X) \cdot (1 - e(X)) + \sigma_0^2(X) \cdot e(X)}{e(X) \cdot (1 - e(X))}\right],$$

then $\mathcal{A}^* = \mathbb{X}$ and otherwise,

$$\mathcal{A}^* = \left\{x \in \mathbb{X} \left| \frac{\sigma_1^2(x) \cdot (1 - e(x)) + \sigma_0^2(x) \cdot e(x)}{e(x) \cdot (1 - e(x))} \leq a\right.\right\},$$

where a is a positive solution to

$$a = 2 \cdot \mathbb{E}\left[\frac{\sigma_1^2(X) \cdot (1 - e(X)) + \sigma_0^2(X) \cdot e(X)}{e(X) \cdot (1 - e(X))} \left| \frac{\sigma_1^2(X) \cdot (1 - e(X)) + \sigma_0^2(X) \cdot e(X)}{e(X) \cdot (1 - e(X))} < a\right.\right].$$

[12]

Proof: See Appendix.

The result in this theorem simplifies under homoskedasticity.

Corollary 6.1 OPTIMAL OVERLAP UNDER HOMOSKEDASTICITY *Suppose that $\sigma_w^2(x) = \sigma^2$ for all $w \in \{0, 1\}$ and $x \in \mathbb{X}$. If*

$$\sup_{x \in \mathbb{X}} \frac{1}{e(x) \cdot (1 - e(x))} \leq 2 \cdot \mathbb{E} \left[\frac{1}{e(X) \cdot (1 - e(X))} \right],$$

then $\mathcal{A}^* = \mathbb{X}$. Otherwise,

$$\mathcal{A}^* = \left\{ x \in \mathbb{X} \left| \frac{1}{e(x) \cdot (1 - e(x))} \leq a \right. \right\},$$

where a is a solution to

$$a = 2 \cdot \mathbb{E} \left[\frac{1}{e(X) \cdot (1 - e(X))} \left| \frac{1}{e(X) \cdot (1 - e(X))} < a \right. \right].$$

We can find the smallest value of a that satisfies the first order conditions (and which therefore must correspond to a local minimum for $g(a)$) by iteratively solving equation (**??**). Start with $a_0 = 0$. Calculate

$$\gamma_k = \gamma(a_k) = \mathbb{E}[(e \cdot (1 - e))^{-1} | a_k \leq e \leq 1 - a_k].$$

Note that $\gamma_k > 4$ Then solve a_k by solving for the solution in $(0, 1/2)$ of

$$\frac{1}{a_{k+1} \cdot (1 - a_{k+1})} = 2 \cdot \gamma_k,$$

leading to

$$a_{k+1} = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{2 \cdot \gamma_k}}.$$

In an application we would typically not know the propensity score. In that case we would carry out the calculations with the conditional expectation $\mathbb{E}[(e \cdot (1 - e))^{-1} | a \leq e \leq 1 - a]$ replaced by

$$\sum_{i=1}^{N} \frac{1}{e(X_i) \cdot (1 - e(X_i))} \cdot 1\{a \leq e(X_i) \leq 1 - a\} \left/ \sum_{i=1}^{N} 1\{a \leq e(X_i) \leq 1 - a\}. \right.$$

6.2 The Optimally Weighted Average Treatment Effect

In this section we consider weighted average treatment effects of the form

$$\tau_g = \int_x g(x) \cdot \tau(x) dF(x) \left/ \int_x g(x) dF(x). \right.$$

The following theorem gives the most precisely estimable weighte average treatment effect.

Theorem 6.2 (OWATE)
Suppose Assumptions – hold. Then the Optimal Weighted Average Treatment Effect (OWATE) is τ_{g^*}, where

$$g^*(x) = \left(\frac{\sigma_1^2(x)}{e(x)} + \frac{\sigma_0^2(x)}{1 - e(x)} \right)^{-1},$$

Proof: See Appendix.

Corollary 6.2 *Suppose Assumptions – hold, and that $\sigma_0^2(x) = \sigma_1^2(x) = \sigma^2$ for all x. Then the Optimally Weighted Average Treatment Effect (OWATE) is τ_{g^*}, where*

$$g^*(x) = e(x) \cdot (1 - e(x)).$$

7   Testing

In this section we discuss some statistical tests. Most of the statistical tests discussed in this literature focus on the appropriateness of the various estimators (Heckman and Hotz, 1989). Some attempt to provide assessments of the unconfoundedness assumptions (Imbens, 2004). We focus on three different hypotheses concerning the conditional average treatment effect $\tau(x)$. The first hypothesis we consider is that the conditional average treatment effect is zero for all values of the covariates. The second one is the hypothesis that $\tau(x)$ is constant as a function of x. The third test concerns the hypothesis that the optimally weighted conditional average treatment effect τ_{C,g^*} is equal to zero. The latter test is very simple. The previous results lead to a root-N consistent estimator that is asymptotically normal with zero asymptotic bias so that we can use a simple Wald test. The other tests are more complex, requiring comparisons of nonparametric estimators of regression functions over the entire support of the covariates.

There are some obvious relationships between the null hypotheses. H_0 implies H'_0 and H''_0. The last two hypotheses, H'_0 and H''_0 combined imply H_0. There is no direct relationship between H'_0 and H''_0.

In order to develop these tests we need estimators for the two regression functions. We use the series estimator for the regression function $\mu_w(x)$ developed by Imbens, Newey and Ridder (2004). Let K denote the number of terms in the series. As the basis we use power series. Let $\lambda = (\lambda_1, ..., \lambda_d)$ be a multi-index of dimension d, that is, a d-dimensional vector of non-negative integers, with $|\lambda| = \sum_{k=1}^{d} \lambda_k$, and let $x^\lambda = x_1^{\lambda_1} \ldots x_d^{\lambda_d}$. Consider a series $\{\lambda(r)\}_{r=1}^\infty$ containing all distinct vectors such that $|\lambda(r)|$ is nondecreasing. Let $p_r(x) = x^{\lambda(r)}$, where $P_r(x) = (p_1(x), ..., p_r(x))'$. Given the assumptions below the expectation $\Omega_K = \mathbb{E}[P_K(X)P_K(X)'|W = 1]$ is nonsingular for all K. Hence we can construct a sequence $R_K(x) = \Omega_K^{-1/2} P_K(x)$ with $\mathbb{E}[R_K(X)R_K(X)'|W = 1] = I_K$. Let $R_{kK}(x)$ be the kth element of the vector $R_K(x)$. It will be convenient to work with this sequence of basis function $R_K(x)$. The nonparametric series estimator of the regression function $\mu_w(x)$, given K terms in the series, is given by:

$$\hat{\mu}_w(x) = r^K(x)' \left(\sum_{W_i=w} R_K(X_i)R_K(X_i)' \right)^{-} \sum_{W_i=w} R_K(X_i)Y_i = R_K(x)'\hat{\gamma}_{w,K},$$

[14]

where

$$\hat{\gamma}_{w,K} = \left(\sum_{W_i=w} R_K(X_i) R_K(X_i)' \right)^{-} \sum_{W_i=w} R_K(X_i) Y_i.$$

Define the $N_w \times K$ matrix $R_{w,K}$ with rows equal to $R_K(X_i)$ for units with $W_i = w$, and Y_w to be the N_w vector with elements equal to Y_i for the same units, so that $\hat{\gamma}_{w,K} = (R_{w,K}' R_{w,K})^{-1}(R_{w,K}' Y_w)$. Note that we use A^- here to denote a generalized inverse of A.

Given the estimator $\hat{\mu}_{w,K}(x)$ we estimate the error variance σ_w^2 as

$$\hat{\sigma}_{w,K}^2 = \frac{1}{N_w} \sum_{i|W_i=w} (Y_i - \hat{\mu}_{w,K}(X_i))^2.$$

Let $\Omega_{w,K}$ be the limiting variance of $\sqrt{N}\hat{\gamma}_{w,K}$ as the sample size increases for fixed K. We estimate this variance as

$$\hat{\Omega}_{w,K} = \hat{\sigma}_{w,K}^2 \cdot (R_{w,K}' R_{w,K}/N)^{-1}.$$

We make the following assumptions.

Assumption 7.1 (DISTRIBUTION OF COVARIATES)
$X \in \mathbb{X} \subset \mathbb{R}^d$, where $\mathbb{X}$ is the Cartesian product of intervals $[x_{jL}, x_{jU}]$, $j = 1, \ldots, d$, with $x_{jL} < x_{jU}$. The density of X is bounded away from zero on $\mathbb{X}$.

Assumption 7.2 (PROPENSITY SCORE)
(i) The propensity score is bounded away from zero and one.
(ii) The propensity score is s times continuously differentiable.

Assumption 7.3 (CONDITIONAL OUTCOME DISTRIBUTIONS)
(i) The two regression functions $\mu_w(x)$ are t times continuously differentiable.
(ii) the conditional variance of $Y_i(w)$ given $X_i = x$ is equal to σ_w^2.

Assumption 7.4 (RATES FOR SERIES ESTIMATORS)
$K = N^\nu$, with $< \nu <$.

We assume homoskedasticty, although this assumption is not essential and can be relaxed to allow the conditional variance to depend on x, as long as it is bounded from above and below.

7.1 Testing the Null Hypothesis of Zero Conditional Average Treatment Effects

Here we are interested in the null hypothesis that the conditional average treatment effect $\tau(x)$ is zero for all values of the covariates. Formally,

$$H_0 : \ \forall \ x \in \mathbb{X}, \ \tau(x) = 0.$$

[15]

To test this hypothesis we compare estimators for $\mu_1(x)$ and $\mu_0(x)$. Given our use of series estimators we can compare the estimated parameters $\hat{\gamma}_{0,K}$ and $\hat{\gamma}_{1,K}$. Specifically, we use as the test statistic for the test of the null hypothesis H_0 the normalized quadratic form

$$T = \left((\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K})'(\hat{\Omega}_{1,K} + \hat{\Omega}_{0,K})^{-1}(\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K}) - K \right) / \sqrt{2K}.$$

Theorem 7.1 *Suppose Assumptions ??-?? hold. Then if $\tau(x) = 0$ for all $x \in \mathbb{X}$,*

$$T \xrightarrow{d} \mathcal{N}(0,1).$$

Proof: See Appendix.

7.2 Testing the Null Hypothesis of a Constant Conditional Average Treatment Effect

$$H'_0: \; \exists \; \tau_0, \text{ such that } \forall \; x \in \mathbb{X}, \; \tau(x) = \tau_0.$$

Let $\hat{\tau}_{C,g^*}$ be an estimator for τ_{C,g^*}.

For testing the null hypothesis H'_0 we use the test statistic

$$T' = \sum_{i=1}^{N} \left(\hat{\mu}_1(X_i) - \hat{\mu}_0(X_i) - \hat{\tau}_{C,g^*} \right)^2 \cdot g^*(X_i).$$

7.3 Testing the Null Hypothesis of a Zero Average Treatment Effect

$$H''_0: \; \tau_{C,g^*} = 0.$$

For the testing the third null hypothesis we use the fact that $\hat{\tau}_{g^*}$ has a limiting normal distribution.

8 Some Illustrations Based on Real Data

In this section we apply the methods developed in this paper to data from a set of labor market programs. The data sets differ by the amount of balance between controls and treated, to highlight the effectiveness and importance of ensuring balance in a range of settings. In each case we first calculate the optimal cutoff point α based on an estimate of the propensity score. We report the number of observations discarded by the proposed sample selection. We also report the asymptotic variance for five estimands relative to that for the average treatment effect for the full sample. These five estimands include the average effect for the controls, the average effect for the treated, the OSATE, the OWATE, and the average effect for those with a propensity score between 0.1 and 0.9. The latter is assess the sensitivity to the choice of cutoff values for the propensity score.

8.1 The GAIN Data

In the empirical section of this paper we use data from the Great Avenue to INdependence (GAIN) experiments. These experimental evaluations of job training and job search assistance programs took place in the 1990's in a number of locations in California. As the evaluations are based on formal randomization within the sites, evaluations of the within-site effects are straightforward. Here we focus on comparisons of the control groups in different sites. Such comparisons can be useful to assess whether for the program evaluations it is important to have randomizations. Specifically, to analyze the question whether controls in one location (e.g., Los Angeles) could have provided a comparison group for the program in a second location (e.g., Riverside), one could investigate whether conditional on covariates there is a systematic difference (e.g., a non-zero average difference) between the two groups. This amounts to estimating an average treatment effect on the control group data from the two locations with the location indicator as the treatment dummy.

We compare the control groups in the four locations, Los Angeles, Riverside, Alameda and San Diego with 1400, 1040, 601, and 1154 observations. For each individual we have individual background characteristics, including gender age, ethnicity (hispanic, black, or other), an indicator for high school graduation, an indicator for the presence of exactly one child (all individuals have at least one child), and an indicator for the presence of children under the age of 6, and ten quarters of earnings data. Table 1 presents summary statistics for the covariates by site. All earnings data are in thousands of dollars. The levels of earnings are averages for the subsamples with positive earnings only.

8.2 Optimal Sample Selection and Relative Asymptotic Variances

In Table 2 we report for a number of comparisons the estimated value of the cutoff propensity score α and the number of observations in each of the three groups $e(x) < \alpha$, $\alpha \leq e(x) \leq 1 - \alpha$, and $1 - \alpha < e(x)$ by treatment status. For example, in the first row we compare the control groups of Riverside and Los Angeles. We analyze these data as if the Riverside data are the control group and the Los Angeles data are the treatment group. Using a logistic model for the propensity score with all the covariates from Table 1 plus a quadratic term in age leads to $\hat{\alpha} = 0.095$. Of the Riverside sample 122 have a propensity score below 0.095. 900 have a propensity score between 0.095 and 0.905. 18 have a propensity score above 0.905. For the Los Angeles sample these counts are 6, 1199 and 195 respectively. The last five columns in Table 2 present asymptotic variance for various estimands relative to the asymptotic variance for the average treatment effect. In the column with the heading τ_c the relative variance for the average effect for the controls is reported to be 1.32, implying we cannot estimate this estimand as precisely as the overll average treatment effect. Similarly, we cannot estimate the average effect for the treated as well as the overall average treatment effect. Using the optimal sample we can reduce the variance 0.83 of the original variance. The relative variance of the OWATE is 0.73 times the variance for the overall average treatment effect. Lastly, using a cutoff value for the propensity score of 0.1 leads to a very similar result.

8.3 A Closer Look at the Alameda vs San Diego Comparison

Next we look more closely at the comparison of the control samples of Alameda and San Diego. In Table 3 we report averages for the subsamples by ranges of the propensity score. Using the optimal cutoff for the propensity score leads to a much more balanced sample. In Figure 1 we present histograms of the propensity score for the two subsamples, with the cutoff values indicated. For comparison we present in Figures 2 and 3 the comparisons of the San Diego and Riverside controls and the Los Angeles control and treated samples. In the latter case the true propensity score should be constant.

9 Conclusion

Proof of Theorem 4.2: Suppose we have an estimator $\tilde{\tau}$ for τ_C that asymptotically linear with influence function $\phi(Y, W, X)$, so that

$$\tilde{\tau} - \frac{1}{N}\sum_{i=1}^{N} \tau(X_i) = \frac{1}{N}\sum_{i=1}^{N} \phi(Y_i, W_i, X_i) + o_p\left(N^{-1/2}\right).$$

The Hahn estimator $\hat{\tau}_H$ satisfies

$$\hat{\tau} - \frac{1}{N}\sum_{i=1}^{N} \tau(X_i) = \frac{1}{N}\sum_{i=1}^{N} \psi(Y_i, W_i, X_i) + o_p\left(N^{-1/2}\right),$$

where

$$\psi(y, w, w) = \phi(y, w, x) - \tau(x) = w \cdot \frac{y - \mu_1(x)}{e(x)} - (1 - w) \cdot \frac{y - \mu_0(x)}{1 - e(x)}.$$

Note that $\mathbb{E}[\psi(Y, W, X)|X] = 0$ so that $\mathbb{E}[\psi(Y, W, X) \cdot (\tau(X) - \tau_P)] = 0$. For $\tilde{\tau}$ to be unbiased asymptotically it must be that $\mathbb{E}[\phi(Y, W, X)|X = x] = 0$ for all x, again implying that $\mathbb{E}[\phi(Y, W, X) \cdot (\tau(X) - \tau_P)] = 0$.
For $\tilde{\tau}$ to be more efficient than $\hat{\tau}$ as an estimator for τ_C it must be that $\mathbb{E}[\phi(Y, W, X)^2] < \mathbb{E}[\psi(Y, W, X)^2]$. Because $\hat{\tau}_H$ is efficient for τ_P, it must be that $\mathbb{E}[(\phi(Y, W, X) + \tau(X) - \tau_P)^2] \geq \mathbb{E}[(\psi(Y, W, X) + \tau(X) - \tau_P)^2]$. Because $\mathbb{E}[\phi(Y, W, X) \cdot (\tau(X) - \tau_P)] = 0$ and $\mathbb{E}[\psi(Y, W, X) \cdot (\tau(X) - \tau_P)] = 0$ this cannot both be true.
$\square$

Proof of Theorem 4.4: The derivation of the efficiency bound follows that of Hahn (1998). The density of $(Y(0), Y(1), T, X)$ with respect to some σ-finite measure is

$$\begin{aligned} q(y(0), y(1), t, x) &= f(y(0), y(1)|t, x)f(t|x)f(x) \\ &= f(y(0), y(1)|x)f(t|x)f(x) \\ &= f(y(0), y(1)|x)p(x)^t(1 - p(x))^{1-t}f(x) \end{aligned}$$

The density of the observed data (y, t, x), using the unconfoundedness assumption, is

$$q(y, t, x) = [f_1(y|x)p(x)]^t[f_0(y|x)(1 - p(x))]^{1-t}f(x),$$

where $f_t(y|x) = f_{Y(T)|X}(y(t)|x) = \int f(y(1 - t), y|x)dy(1 - t)$. Consider a regular parametric submodel indexed by θ, with density

$$q(y, t, x) = [f_1(y|x, \theta)p(x, \theta)]^t[f_0(y|x, \theta)(1 - p(x, \theta))]^{1-t}f(x, \theta),$$

which is equal to the true density $q(y, t, x)$ for $\theta = \theta_0$. The score is given by

$$\frac{d}{d\theta}\ln q(y, t, x|\theta) = s(y, t, x|\theta) = t \cdot s_1(y|x, \theta) + (1 - t) \cdot s_0(y|x, \theta) + s_x(x, \theta) + \frac{t - p(x)}{p(x)(1 - p(x))} \cdot p'(x, \theta)$$

where

$$\begin{aligned} s_1(y|x, \theta) &= \frac{d}{d\theta}\ln f_1(y|x, \theta) \\ s_0(y|x, \theta) &= \frac{d}{d\theta}\ln f_0(y|x, \theta) \\ s_x(x|\theta) &= \frac{d}{d\theta}\ln f(x, \theta) \\ p'(x, \theta) &= \frac{d}{d\theta}p(x, \theta) \end{aligned}$$

[19]

The tangent space of the model is the set of functions

$$\mathcal{T} = \{t \cdot s_1(y,x) + (1-t) \cdot s_0(y,x) + s_x(x) + a(x) \cdot (t - p(x))\}$$

where $a(x)$ is any square-integrable measurable function of x and s_1, s_0, and s_x satisfy

$$\int s_1(y,x)f_1(y|x)dy = \mathbb{E}[s_1(Y(1),X)|X=x] = 0, \quad \forall x,$$

$$\int s_0(y,x)f_0(y|x)dy = \mathbb{E}[s_0(Y(0),X)|X=x] = 0, \quad \forall x,$$

$$\int s_x(x)f(x)dx = \mathbb{E}[s_x(X)] = 0.$$

The parameter of interest is

$$\tau_h = \frac{\iint h(p(x))yf_1(y|x)f(x)dydx - \iint h(p(x))yf_0(y|x)f(x)dydx}{\int h(p(x))f(x)dx}.$$

Thus, for the parametric submodel indexed by θ, the parameter of interest is

$$\tau_h(\theta) = \frac{\iint h(p(x,\theta))yf_1(y|x,\theta)f(x,\theta)dydx - \iint h(p(x,\theta))yf_0(y|x,\theta)f(x,\theta)dydx}{\int h(p(x,\theta))f(x,\theta)dx}.$$

We need to find a function $F_{\tau_h}(y,t,x)$ such that for all regular parametric submodels,

$$\frac{\partial \tau_h(\theta_0)}{\partial \theta} = \mathbb{E}[F_{\tau_h}(Y,T,X) \cdot s(Y,T,X|\theta_0)].$$

First, we will calculate $\frac{\partial \tau_h(\theta_0)}{\partial \theta}$. Let $\mu_h = \int h(p(x))f(x)dx$. Then,

$$
\begin{aligned}
\frac{\partial \tau_h(\theta_0)}{\partial \theta} &= \frac{1}{\mu_h}\Bigg[\iint h(p(x,\theta_0))y\left[s_1(y|x,\theta_0)f_1(y|x,\theta_0) - s_0(y|x,\theta_0)f_0(y|x,\theta_0)\right]f(x,\theta_0)dydx \\
&\quad + \int h(p(x,\theta_0))\left[\tau(x) - \tau_h\right]s_x(x|\theta_0)f(x,\theta_0)dx \\
&\quad + \int h'(p(x,\theta_0))p'(x,\theta_0)\left[\tau(x) - \tau_h\right]f(x,\theta_0)dx\Bigg]
\end{aligned}
$$

where $h'(p(x)) = \frac{d}{dp(x)}h(p(x))$. The following choice for F_{τ_h} satisfies the condition:

$$
\begin{aligned}
F_{\tau_h}(Y,T,X) &= \frac{T \cdot h(p(X))}{\mu_h \cdot p(X)}(Y - \mathbb{E}[Y(1)|X]) - \frac{(1-T) \cdot h(p(X))}{\mu_h \cdot (1-p(X))}(Y - \mathbb{E}[Y(0)|X]) \\
&\quad + \frac{h(p(X))}{\mu_h}(\tau(X) - \tau_h) + \frac{(T - p(X)) \cdot h'(p(X))}{\mu_h}(\tau(X) - \tau_h).
\end{aligned}
$$

To see this consider the following product,

$$F_{\tau_h}(Y, T, X) \cdot s(Y, T, X | \theta_0)$$

$$= \frac{T^2 \cdot h(p(X))}{\mu_h \cdot p(X)} s_1(Y, X)(Y - \mathbb{E}[Y(1)|X]) \tag{A.1}$$

$$+ \; 0 \tag{A.2}$$

$$+ \; \frac{T \cdot h(p(X))}{\mu_h \cdot p(X)} s_x(X)(Y - \mathbb{E}[Y(1)|X]) \tag{A.3}$$

$$+ \; \frac{T(T - p(X)) \cdot h(p(X))}{\mu_h \cdot p(X)^2(1 - p(X))} p'(X)(Y - \mathbb{E}[Y(1)|X]) \tag{A.4}$$

$$- \; 0 \tag{A.5}$$

$$- \; \frac{(1 - T)^2 \cdot h(p(X))}{\mu_h \cdot (1 - p(X))} s_0(Y, X)(Y - \mathbb{E}[Y(0)|X]) \tag{A.6}$$

$$- \; \frac{(1 - T) \cdot h(p(X))}{\mu_h \cdot (1 - p(X))} s_x(X)(Y - \mathbb{E}[Y(0)|X]) \tag{A.7}$$

$$- \; \frac{(1 - T) \cdot (T - p(X)) \cdot h(p(X))}{\mu_h \cdot p(X)(1 - p(X))^2} p'(X)(Y - \mathbb{E}[Y(0)|X]) \tag{A.8}$$

$$+ \; \frac{T \cdot h(p(X))}{\mu_h} s_1(Y, X)(\tau(X) - \tau_h) \tag{A.9}$$

$$+ \; \frac{(1 - T) \cdot h(p(X))}{\mu_h} s_0(Y, X)(\tau(X) - \tau_h) \tag{A.10}$$

$$+ \; \frac{h(p(X))}{\mu_h} s_x(X)(\tau(X) - \tau_h) \tag{A.11}$$

$$+ \; \frac{(T - p(X)) \cdot h(p(X))}{\mu_h \cdot p(X)(1 - p(X))} p'(X)(\tau(X) - \tau_h) \tag{A.12}$$

$$+ \; \frac{T(T - p(X)) \cdot h'(p(x))}{\mu_h} s_1(Y, X)(\tau(X) - \tau_h) \tag{A.13}$$

$$+ \; \frac{(1 - T)(T - p(X)) \cdot h'(p(x))}{\mu_h} s_0(Y, X)(\tau(X) - \tau_h) \tag{A.14}$$

$$+ \; \frac{(T - p(X)) \cdot h'(p(x))}{\mu_h} s_x(X)(\tau(X) - \tau_h) \tag{A.15}$$

$$+ \; \frac{(T - p(X))^2 \cdot h'(p(x))}{\mu_h \cdot p(X)(1 - p(X))} p'(X)(\tau(X) - \tau_h). \tag{A.16}$$

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Consider each expectation in turn. Equation (1) yields,

$$\mathbb{E}\left[\frac{T^2 \cdot h(p(X))}{\mu_h \cdot p(X)}s_1(Y,X)(Y - \mathbb{E}[Y(1)|X])\right]$$

$$= \mathbb{E}\left[\frac{T \cdot h(p(X))}{\mu_h \cdot p(X)}s_1(Y(1),X)Y(1)\right] - \mathbb{E}\left[\frac{T \cdot h(p(X))}{\mu_h \cdot p(X)}s_1(Y(1),X)\mathbb{E}[Y(1)|X]\right]$$

$$= \mathbb{E}\left[\frac{h(p(X))}{\mu_h \cdot p(X)}\mathbb{E}[T \cdot s_1(Y(1),X)Y(1)|X]\right]$$

$$\quad - \mathbb{E}\left[\frac{h(p(X))}{\mu_h \cdot p(X)}\mathbb{E}[T \cdot s_1(Y(1),X)|X] \cdot \mathbb{E}[Y(1)|X]\right]$$

$$= \mathbb{E}\left[\frac{h(p(X))}{\mu_h}\mathbb{E}[s_1(Y(1),X)Y(1)|X]\right] - \mathbb{E}\left[\frac{h(p(X))}{\mu_h}\mathbb{E}[s_1(Y(1),X)|X] \cdot \mathbb{E}[Y(1)|X]\right]$$

$$= \mathbb{E}\left[\frac{h(p(X))}{\mu_h}\mathbb{E}[s_1(Y(1),X)Y(1)|X]\right]$$

$$= \frac{1}{\mu_h}\iint h(p(x))ys_1(y,x)f_1(y|x)f(x)dydx.$$

Equation (3) yields,

$$\mathbb{E}\left[\frac{T \cdot h(p(X))}{\mu_h \cdot p(X)}s_x(X)(Y - \mathbb{E}[Y(1)|X])\right]$$

$$= \mathbb{E}\left[\frac{T \cdot h(p(X))}{\mu_h \cdot p(X)}s_x(X)Y(1)\right] - \mathbb{E}\left[\frac{T \cdot h(p(X))}{\mu_h \cdot p(X)}s_x(X)\mathbb{E}[Y(1)|X]\right]$$

$$= \mathbb{E}\left[\frac{h(p(X))}{\mu_h \cdot p(X)}s_x(X)\mathbb{E}[TY(1)|X]\right] - \mathbb{E}\left[\frac{h(p(X))}{\mu_h}s_x(X)\mathbb{E}[Y(1)|X]\right]$$

$$= \mathbb{E}\left[\frac{h(p(X))}{\mu_h}s_x(X)\mathbb{E}[Y(1)|X]\right] - \mathbb{E}\left[\frac{h(p(X))}{\mu_h}s_x(X)\mathbb{E}[Y(1)|X]\right]$$

$$= 0.$$

Equation (4) yields,

$$\mathbb{E}\left[\frac{T(T - p(X)) \cdot h(p(X))}{\mu_h \cdot p(X)^2(1 - p(X))}p'(X)(Y - \mathbb{E}[Y(1)|X])\right]$$

$$= \mathbb{E}\left[\frac{T(T - p(X)) \cdot h(p(X))}{\mu_h \cdot p(X)^2(1 - p(X))}p'(X)(Y(1) - \mathbb{E}[Y(1)|X])\right]$$

$$= \mathbb{E}\left[\frac{h(p(X))}{\mu_h \cdot p(X)^2(1 - p(X))}p'(X)\mathbb{E}[T(T - p(X))Y(1)|X]\right]$$

$$\quad - \mathbb{E}\left[\frac{h(p(X))}{\mu_h \cdot p(X)^2(1 - p(X))}p'(X)\mathbb{E}[T(T - p(X))|X] \cdot \mathbb{E}[Y(1)|X]\right]$$

$$= \mathbb{E}\left[\frac{h(p(X))}{\mu_h \cdot p(X)}p'(X)\mathbb{E}[Y(1)|X]\right] - \mathbb{E}\left[\frac{h(p(X))}{\mu_h \cdot p(X)}p'(X)\mathbb{E}[Y(1)|X]\right]$$

$$= 0.$$

Equation (6) yields,

$$-\mathbb{E}\left[\frac{(1-T)^2 \cdot h(p(X))}{\mu_h \cdot (1-p(X))} s_0(Y,X)(Y-\mathbb{E}[Y(0)|X])\right]$$

$$= -\mathbb{E}\left[\frac{(1-T)\cdot h(p(X))}{\mu_h \cdot (1-p(X))} s_0(Y(0),X)Y(0)\right]$$

$$+\mathbb{E}\left[\frac{(1-T)\cdot h(p(X))}{\mu_h \cdot (1-p(X))} s_0(Y(0),X)\mathbb{E}[Y(0)|X]\right]$$

$$= -\mathbb{E}\left[\frac{h(p(X))}{\mu_h \cdot (1-p(X))}\mathbb{E}[(1-T)s_0(Y(0),X)Y(0)|X]\right]$$

$$+\mathbb{E}\left[\frac{h(p(X))}{\mu_h \cdot (1-p(X))}\mathbb{E}[(1-T)s_0(Y(0),X)|X]\mathbb{E}[Y(0)|X]\right]$$

$$= -\mathbb{E}\left[\frac{h(p(X))}{\mu_h}\mathbb{E}[s_0(Y(0),X)Y(0)|X]\right] + \mathbb{E}\left[\frac{h(p(X))}{\mu_h}\mathbb{E}[s_0(Y(0),X)|X]\mathbb{E}[Y(0)|X]\right]$$

$$= -\mathbb{E}\left[\frac{h(p(X))}{\mu_h}\mathbb{E}[s_0(Y(0),X)Y(0)|X]\right]$$

$$= -\frac{1}{\mu_h}\iint h(p(x))y s_0(y,x) f_0(y|x) f(x) dy dx.$$

Equation (7) yields,

$$-\mathbb{E}\left[\frac{(1-T)\cdot h(p(X))}{\mu_h \cdot (1-p(X))} s_x(X)(Y-\mathbb{E}[Y(0)|X])\right]$$

$$= -\mathbb{E}\left[\frac{(1-T)\cdot h(p(X))}{\mu_h \cdot (1-p(X))} s_x(X)Y(0)\right] + \mathbb{E}\left[\frac{(1-T)\cdot h(p(X))}{\mu_h \cdot (1-p(X))} s_x(X)\mathbb{E}[Y(0)|X]\right]$$

$$= -\mathbb{E}\left[\frac{h(p(X))}{\mu_h \cdot (1-p(X))} s_x(X)\mathbb{E}[(1-T)Y(0)|X]\right] + \mathbb{E}\left[\frac{h(p(X))}{\mu_h} s_x(X)\mathbb{E}[Y(0)|X]\right]$$

$$= -\mathbb{E}\left[\frac{h(p(X))}{\mu_h} s_x(X)\mathbb{E}[Y(0)|X]\right] + \mathbb{E}\left[\frac{h(p(X))}{\mu_h} s_x(X)\mathbb{E}[Y(0)|X]\right]$$

$$= 0$$

Equation (8) yields,

$$-\mathbb{E}\left[\frac{(1-T)\cdot(T-p(X))\cdot h(p(X))}{\mu_h \cdot p(X)(1-p(X))^2} p'(X)(Y-\mathbb{E}[Y(0)|X])\right]$$

$$= -\mathbb{E}\left[\frac{(1-T)\cdot(T-p(X))\cdot h(p(X))}{\mu_h \cdot p(X)(1-p(X))^2} p'(X)(Y(0)-\mathbb{E}[Y(0)|X])\right]$$

$$= -\mathbb{E}\left[\frac{h(p(X))}{\mu_h \cdot p(X)(1-p(X))^2} p'(X)\mathbb{E}[(1-T)(T-p(X))Y(0)|X]\right]$$

$$+\mathbb{E}\left[\frac{h(p(X))}{\mu_h \cdot p(X)(1-p(X))^2} p'(X)\mathbb{E}[(1-T)(T-p(X))|X]\mathbb{E}[Y(0)|X]\right]$$

$$= -\mathbb{E}\left[\frac{h(p(X))}{\mu_h \cdot (1-p(X))} p'(X)\mathbb{E}[Y(0)|X]\right] + \mathbb{E}\left[\frac{h(p(X))}{\mu_h \cdot (1-p(X))} p'(X)\mathbb{E}[Y(0)|X]\right]$$

$$= 0.$$

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Equation (9) yields,

$$\mathbb{E}\left[\frac{T \cdot h(p(X))}{\mu_h}s_1(Y,X)(\tau(X)-\tau_h)\right]$$

$$= \mathbb{E}\left[\frac{T \cdot h(p(X))}{\mu_h}s_1(Y(1),X)(\tau(X)-\tau_h)\right]$$

$$= \mathbb{E}\left[\frac{h(p(X))}{\mu_h}\mathbb{E}[Ts_1(Y(1),X)|X](\tau(X)-\tau_h)\right]$$

$$= \mathbb{E}\left[\frac{p(X) \cdot h(p(X))}{\mu_h}\mathbb{E}[s_1(Y(1),X)|X](\tau(X)-\tau_h)\right]$$

$$= 0.$$

Equation (10) yields,

$$\mathbb{E}\left[\frac{(1-T) \cdot h(p(X))}{\mu_h}s_0(Y,X)(\tau(X)-\tau_h)\right]$$

$$= \mathbb{E}\left[\frac{(1-T) \cdot h(p(X))}{\mu_h}s_0(Y(0),X)(\tau(X)-\tau_h)\right]$$

$$= \mathbb{E}\left[\frac{h(p(X))}{\mu_h}\mathbb{E}[(1-T)s_0(Y(0),X)|X](\tau(X)-\tau_h)\right]$$

$$= \mathbb{E}\left[\frac{(1-p(X)) \cdot h(p(X))}{\mu_h}\mathbb{E}[s_0(Y(0),X)|X](\tau(X)-\tau_h)\right]$$

$$= 0.$$

Equation (11) yields,

$$\mathbb{E}\left[\frac{h(p(X))}{\mu_h}s_x(X)(\tau(X)-\tau_h)\right]$$

$$= \frac{1}{\mu_h}\int h(p(x))(\tau(x)-\tau_h)s_x(x)f(x)dx$$

Equation (12) yields,

$$\mathbb{E}\left[\frac{(T-p(X)) \cdot h(p(X))}{\mu_h \cdot p(X)(1-p(X))}p'(X)(\tau(X)-\tau_h)\right]$$

$$= \mathbb{E}\left[\frac{\mathbb{E}[(T-p(X))|X] \cdot h(p(X))}{\mu_h \cdot p(X)(1-p(X))}p'(X)(\tau(X)-\tau_h)\right]$$

$$= 0.$$

Equation (13) yields,

$$\mathbb{E}\left[\frac{T(T-p(X)) \cdot h'(p(X))}{\mu_h}s_1(Y,X)(\tau(X)-\tau_h)\right]$$

$$= \mathbb{E}\left[\frac{T(T-p(X)) \cdot h'(p(X))}{\mu_h}s_1(Y(1),X)(\tau(X)-\tau_h)\right]$$

$$= \mathbb{E}\left[\frac{h'(p(X))}{\mu_h}\mathbb{E}[T(T-p(X))s_1(Y(1),X)|X](\tau(X)-\tau_h)\right]$$

$$= \mathbb{E}\left[\frac{p(X)(1-p(X)) \cdot h'(p(X))}{\mu_h}\mathbb{E}[s_1(Y(1),X)|X](\tau(X)-\tau_h)\right]$$

$$= 0.$$

Equation (14) yields,

$$\mathbb{E}\left[\frac{(1-T)(T-p(X))\cdot h'(p(X))}{\mu_h}s_0(Y,X)(\tau(X)-\tau_h)\right]$$

$$=\mathbb{E}\left[\frac{(1-T)(T-p(X))\cdot h'(p(X))}{\mu_h}s_0(Y(0),X)(\tau(X)-\tau_h)\right]$$

$$=\mathbb{E}\left[\frac{h'(p(X))}{\mu_h}\mathbb{E}[(1-T)(T-p(X))s_0(Y(0),X)|X](\tau(X)-\tau_h)\right]$$

$$=\mathbb{E}\left[\frac{p(X)(1-p(X))\cdot h'(p(X))}{\mu_h}\mathbb{E}[s_0(Y(0),X)|X](\tau(X)-\tau_h)\right]$$

$$=0.$$

Equation (15) yields,

$$\mathbb{E}\left[\frac{(T-p(X))\cdot h'(p(X))}{\mu_h}s_x(X)(\tau(X)-\tau_h)\right]$$

$$=\mathbb{E}\left[\frac{\mathbb{E}[(T-p(X))|X]\cdot h'(p(X))}{\mu_h}s_x(X)(\tau(X)-\tau_h)\right]$$

$$=0.$$

Equation (16) yields,

$$\mathbb{E}\left[\frac{(T-p(X))^2\cdot h'(p(X))}{\mu_h\cdot p(X)(1-p(X))}p'(X)(\tau(X)-\tau_h)\right]$$

$$=\mathbb{E}\left[\frac{(T^2+p(X)^2-2\cdot Tp(X))\cdot h'(p(X))}{\mu_h\cdot p(X)(1-p(X))}p'(X)(\tau(X)-\tau_h)\right]$$

$$=\mathbb{E}\left[\frac{\mathbb{E}[T+p(X)^2-2\cdot Tp(X)|X]\cdot h'(p(X))}{\mu_h\cdot p(X)(1-p(X))}p'(X)(\tau(X)-\tau_h)\right]$$

$$=\mathbb{E}\left[\frac{p(X)(1-p(X)\cdot h'(p(X))}{\mu_h\cdot p(X)(1-p(X))}p'(X)(\tau(X)-\tau_h)\right]$$

$$=\mathbb{E}\left[\frac{h'(p(X))}{\mu_h}p'(X)(\tau(X)-\tau_h)\right]$$

$$=\frac{1}{\mu_h}\int h'(p(x))p'(x)[\tau(x)-\tau_h]f(x)dx$$

Since $F_{\tau_h}\in\mathcal{T}$, the variance bound is

$$\mathbb{E}[F_{\tau_h}(Y,T,X)^2] = \mathbb{E}\left[\frac{[h(p(X))]^2}{(\mu_h)^2\cdot p(X)}\cdot\mathbb{V}[Y(1)|X]\right]+\mathbb{E}\left[\frac{[h(p(X))]^2}{(\mu_h)^2\cdot(1-p(X))}\cdot\mathbb{V}[Y(0)|X]\right]$$

$$+\mathbb{E}\left[\frac{[h(p(X))+(T-p(X))\cdot h'(p(X))]^2}{(\mu_h)^2}(\tau(X)-\tau_h)^2\right]$$

$$= \mathbb{E}\left[\frac{[h(p(X))]^2}{(\mu_h)^2\cdot p(X)}\cdot\mathbb{V}[Y(1)|X]\right]+\mathbb{E}\left[\frac{[h(p(X))]^2}{(\mu_h)^2\cdot(1-p(X))}\cdot\mathbb{V}[Y(0)|X]\right]$$

$$+\mathbb{E}\left[\frac{[h(p(X))]^2+p(X)(1-p(X))\cdot[h'(p(X))]^2}{(\mu_h)^2}(\tau(X)-\tau_h)^2\right]$$

For the special case of $h(p(x))=p(x)$ the semiparametric efficiency bound is,

$$\mathbb{E}\left[\frac{p(X)}{(\mu_h)^2}\cdot\mathbb{V}[Y(1)|X]\right]+\mathbb{E}\left[\frac{p(X))^2}{(\mu_h)^2\cdot(1-p(X))}\cdot\mathbb{V}[Y(0)|X]\right]+\mathbb{E}\left[\frac{p(X)}{(\mu_h)^2}(\tau(X)-\tau_h)^2\right].$$

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For the special case of $h(p(x)) = p(x)(1 - p(x))$ the semiparametric efficiency bound is,

$$\mathbb{E}\left[\frac{p(X)(1 - p(X))^2}{(\mu_h)^2} \cdot \mathbb{V}[Y(1)|X]\right] + \mathbb{E}\left[\frac{p(X)^2(1 - p(X))}{(\mu_h)^2} \cdot \mathbb{V}[Y(0)|X]\right]$$

$$+ \mathbb{E}\left[\frac{p(X)^2(1 - p(X))^2 + p(X)(1 - p(X)) \cdot (1 - 2 \cdot p(X))^2}{(\mu_h)^2}(\tau(X) - \tau_h)^2\right]$$

which simplifies to

$$\mathbb{E}\left[\frac{p(X)(1 - p(X))^2}{(\mu_h)^2} \cdot \mathbb{V}[Y(1)|X]\right] + \mathbb{E}\left[\frac{p(X)^2(1 - p(X))}{(\mu_h)^2} \cdot \mathbb{V}[Y(0)|X]\right]$$

$$+ \mathbb{E}\left[\frac{p(X)(1 - p(X))(3p(X)^2 - 3p(X) + 1)}{(\mu_h)^2}(\tau(X) - \tau_h)^2\right].$$

Before proving Theorem 7.1 we present a couple of preliminary results.

Lemma A.1 *Suppose Assumptions XX-XX hold. Then* (i)

$$\left\|\hat{\Omega}_{w,K} - \Omega_{w,K}\right\| = O_p\left(\zeta(K)K^{\frac{1}{2}}N^{-\frac{1}{2}}\right),$$

and (ii) *The eigenvalues of* $\Omega_{w,K}$ *are bounded and bounded away from zero and* (iii) *The eigenvalues of* $\hat{\Omega}_{w,K}$ *are bounded and bounded away from zero if* $O_p\left(\zeta(K)K^{\frac{1}{2}}N^{-\frac{1}{2}}\right) = o_p(1)$.

Proof: We will generalize the proof found in Imbens, Newey and Ridder (2004). For (i) we will show

$$\mathbb{E}\left[\left\|\hat{\Omega}_{w,K} - \Omega_{w,K}\right\|^2\right] \leq C \cdot \zeta(K)^2 K/N$$

so that the result follows by Markov's inequality.

$$\mathbb{E}\left[\left\|\hat{\Omega}_{w,K} - \Omega_{w,K}\right\|^2\right]$$

$$= \mathbb{E}\left[\left\|\left(R'_{w,K}R_{w,K}/N_w\right) - \Omega_{w,K}\right\|^2\right]$$

$$= \mathbb{E}\left[\text{tr}\left(\left(R'_{w,K}R_{w,K}/N_w\right) - \Omega_{w,K}\right)'\left(\left(R'_{w,K}R_{w,K}/N_w\right) - \Omega_{w,K}\right)\right]$$

$$= \mathbb{E}\left[\text{tr}\left(R'_{w,K}R_{w,K}R'_{w,K}R_{w,K}/N_w^2 - \Omega_{w,K}\left(R'_{w,K}R_{w,K}/N_w\right) - \left(R'_{w,K}R_{w,K}/N_w\right)\Omega_{w,K} + \Omega_{w,K}^2\right)\right]$$

$$= \text{tr}\left(\mathbb{E}\left[R'_{w,K}R_{w,K}R'_{w,K}R_{w,K}/N_w^2\right] - \Omega_{w,K}\mathbb{E}\left[R'_{w,K}R_{w,K}/N_w\right] - \mathbb{E}\left[R'_{w,K}R_{w,K}/N_w\right]\Omega_{w,K} + \Omega_{w,K}^2\right)$$

$$= \text{tr}\left(\mathbb{E}\left[R'_{w,K}R_{w,K}R'_{w,K}R_{w,K}/N_w^2\right] - 2 \cdot \Omega_{w,K}^2 + \Omega_{w,K}^2\right)$$

$$= \text{tr}\left(\mathbb{E}\left[R'_{w,K}R_{w,K}R'_{w,K}R_{w,K}/N_w^2\right]\right) - \text{tr}\left(\Omega_{w,K}^2\right)$$

The second term is

$$\text{tr}(\Omega_{w,K}^2) = \sum_{k=1}^{K}\sum_{l=1}^{K}\left(\mathbb{E}\left[R_{kK}(X)R_{lK}(X)|W = w\right]\right)^2 \tag{A.17}$$

The first term is

$$\text{tr}\left(\mathbb{E}\left[R'_{w,K}R_{w,K}R'_{w,K}R_{w,K}\right]/N_w^2\right)$$

$$= \mathbb{E}\left[\sum_{k=1}^{K}\sum_{l=1}^{K}\left(\sum_{i|W_i=w}^{N}R_{kK}(X_i)R_{lK}(X_i)\right)^2\right]/N_w^2$$

$$= \sum_{k=1}^{K}\sum_{l=1}^{K}\sum_{i|W_i=w}^{N}\sum_{j|W_j=w}^{N}\mathbb{E}\left[R_{kK}(X_i)R_{lK}(X_i)R_{lK}(X_j)R_{kK}(X_j)|W = w\right]/N_w^2$$

We can then partition this expression into terms with $i = j$,

$$\sum_{k=1}^{K}\sum_{l=1}^{K}\sum_{i|W_i=w}^{N} \mathbb{E}\left[R_{kK}(X_i)^2 R_{lK}(X_i)^2 | W = w\right] / N_w^2 \tag{A.18}$$

and with terms $i \neq j$,

$$N_w(N_w - 1)\sum_{k=1}^{K}\sum_{l=1}^{K} \left(\mathbb{E}\left[R_{kK}(X)R_{lK}(X) | W = w\right]\right)^2 / N_w^2 \tag{A.19}$$

Combining equations (1), (2) and (3) yields,

$$
\begin{aligned}
\mathbb{E}\left[\left\|\hat{\Omega}_{w,K} - \Omega_{w,K}\right\|^2\right] &= \sum_{k=1}^{K}\sum_{l=1}^{K}\sum_{i|W_i=w}^{N} \mathbb{E}\left[R_{kK}(X_i)^2 R_{lK}(X_i)^2 | W = w\right] / N_w^2 \\
&\quad + N_w(N_w - 1)\sum_{k=1}^{K}\sum_{l=1}^{K} \left(\mathbb{E}\left[R_{kK}(X)R_{lK}(X) | W = w\right]\right)^2 / N_w^2 \\
&\quad - \sum_{k=1}^{K}\sum_{l=1}^{K} \left(\mathbb{E}\left[R_{kK}(X)R_{lK}(X) | W = w\right]\right)^2 \\
&= \sum_{k=1}^{K}\sum_{l=1}^{K}\sum_{i|W_i=w}^{N} \mathbb{E}\left[R_{kK}(X_i)^2 R_{lK}(X_i)^2 | W = w\right] / N_w^2 \\
&\quad - \sum_{k=1}^{K}\sum_{l=1}^{K} \left(\mathbb{E}\left[R_{kK}(X)R_{lK}(X) | W = w\right]\right)^2 / N_w \\
&< \sum_{k=1}^{K}\sum_{l=1}^{K}\sum_{i|W_i=w}^{N} \mathbb{E}\left[R_{kK}(X_i)^2 R_{lK}(X_i)^2 | W = w\right] / N_w^2 \\
&= \frac{1}{N_w^2}\sum_{i|W_i=w}^{N} \mathbb{E}\left[\sum_{k=1}^{K}R_{kK}(X_i)^2 \sum_{l=1}^{K}R_{lK}(X_i)^2 | W = w\right] \\
&\leq \frac{1}{N_w^2}\sum_{i|W_i=w}^{N} \zeta(K)^2 \cdot \mathbb{E}\left[\sum_{l=1}^{K}R_{lK}(X_i)^2 | W = w\right] \\
&= \frac{1}{N_w^2}\sum_{i|W_i=w}^{N} \zeta(K)^2 \cdot \sum_{l=1}^{K}\mathbb{E}\left[R_{lK}(X_i)^2 | W = w\right] \\
&= \frac{1}{N_w}\zeta(K)^2 \cdot \operatorname{tr}\left(\Omega_{w,K}\right) \\
&\leq \frac{1}{N_w}\zeta(K)^2 \cdot K \cdot \lambda_{max}\left(\Omega_{w,K}\right) \\
&\leq C \cdot K\zeta(K)^2 / N
\end{aligned}
$$

where the fifth line follows by

$$\zeta(K) = \sup_x \|R_K(x)\| = \sup_x \left(\sum_{k=1}^{K} R_{kK}^2(x)\right)^{\frac{1}{2}}$$

which then implies that

$$\sum_{k=1}^{K} R_{kK}^2(x) \leq \zeta(K)^2.$$

The eighth line follows since the maximum eigenvalue of $\Omega_{w,K}$ is $O(1)$ (see below).

For (ii), let us first show that for any two positive semi-definite matrices A and B, and conformable vectors c and d, if $A \geq B$ in a positive semi-definite sense, then for

$$\lambda_{min}(A) = \min_{c'c=1} c'Ac = c^{*\prime}Ac^*, \quad \lambda_{min}(B) = \min_{d'd=1} d'Bd = d^{*\prime}Bd^*,$$

we have that,

$$\lambda_{min}(A) = c^{*\prime}Ac^* \geq c^{*\prime}Bc^* \geq d^{*\prime}Bd^* = \lambda_{min}(B).$$

Now, let $f_w(x) = f_{X|W}(x|W = w)$ and recall that $\Omega_{w,K} = \mathbb{E}\left[R_K(X)R_K(X)'|W = w\right]$ where $\Omega_{1,K}$ is normalized to equal I_K. Next define

$$c(x) = f_0(x)/f_1(x)$$

and note that by Assumptions 8.1 and 8.2 we have that

$$0 < \underline{c} \leq c(x) \leq \bar{c}.$$

Thus we may define $c(x) \equiv \underline{c} + \tilde{c}(x)$ so that,

$$
\begin{aligned}
\Omega_{0,K} &= \mathbb{E}\left[R_K(x)R_K(x)'|W = 0\right] \\
&= \int R_K(x)R_K(x)' f_0(x)dx \\
&= \int R_K(x)R_K(x)' c(x) f_1(x)dx \\
&= \int R_K(x)R_K(x)' \left(\underline{c} + \tilde{c}(x)\right) f_1(x)dx \\
&= \underline{c} \int R_K(x)R_K(x)' f_1(x)dx + \int R_K(x)R_K(x)'\tilde{c}(x) f_1(x)dx \\
&= \underline{c} \cdot \Omega_{1,K} + \int R_K(x)R_K(x)'\tilde{c}(x) f_1(x)dx \\
&= \underline{c} \cdot \Omega_{1,K} + \tilde{C}
\end{aligned}
$$

Where $\tilde{C}$ is a positive semi-definite matrix. Thus,

$$\Omega_{0,K} \geq \underline{c} \cdot \Omega_{1,K} \Rightarrow \lambda_{min}\left(\Omega_{0,K}\right) \geq \underline{c} \cdot \lambda_{min}\left(\Omega_{1,K}\right) = \underline{c}$$

so that the minimum eigenvalue of $\Omega_{0,K}$ is bounded away from zero. The minimum eigenvalue of $\Omega_{1,K}$ is bounded away from zero by construction. Then note that for a positive definite matrix A, $1/\lambda_{min}(A) = \lambda_{max}(A^{-1})$, so that the eigenvalues of $\Omega_{w,K}$ are also bounded from above.

For (iii) consider the minimum eigenvalue of $\hat{\Omega}_{w,K}$:

$$
\begin{aligned}
\lambda_{min}\left(\hat{\Omega}_{w,K}\right) &= \min_{c'c=1} c'\left(\hat{\Omega}_{w,K}\right)c \\
&= \min_{c'c=1} \left(c'\left(\Omega_{w,K}\right)c + c'\left(\hat{\Omega}_{w,K} - \Omega_{w,K}\right)c\right) \\
&\geq \min_{c'c=1} c'\left(\Omega_{w,K}\right)c + \min_{d'd=1} d'\left(\hat{\Omega}_{w,K} - \Omega_{w,K}\right)d \\
&= \lambda_{min}\left(\Omega_{w,K}\right) + \lambda_{min}\left(\hat{\Omega}_{w,K} - \Omega_{w,K}\right) \\
&\geq \lambda_{min}\left(\Omega_{w,K}\right) - \left\|\hat{\Omega}_{w,K} - \Omega_{w,K}\right\| \\
&= \lambda_{min}\left(\Omega_{w,K}\right) - O_p\left(\zeta(K)K^{\frac{1}{2}}N^{-\frac{1}{2}}\right)
\end{aligned}
$$

[28]

Where the fifth line follows since for a symmetric matrix A

$$\|A\|^2 = \operatorname{tr}\left(A^2\right) \geq \lambda_{min}\left(A^2\right) = \lambda_{min}(A)^2,$$

and since the norm is nonnegative

$$\|A\| \geq -\lambda_{min}(A).$$

The last line follows by part (i). $\square$

Newey (1994) showed that $\zeta(K)$ is $O(K)$, so this lemma implies that if $K^3/N \to 0$ (as implied by Assumption XX), $\|\hat{\Omega}_{w,K} - \Omega_{w,K}\| = o_p(1)$.

Next, define the pseudo true value $\gamma^*_{w,K}$ as

$$\gamma^*_{w,K} \equiv (\mathbb{E}[R_K(X)R_K(X)'|W = w])^{-1}\,\mathbb{E}[R_K(X)Y|W = w] = \Omega^{-1}_{w,K}\mathbb{E}[R_K(X)Y|W = w].$$

and

$$\tilde{\gamma}_{w,K} \equiv \gamma^*_{w,K} + \Omega^{-1}_{w,K}R'_{w,K}\varepsilon_w/N_w$$

where

$$\varepsilon_w \equiv Y_w - \mu_w(\mathbf{X}).$$

Then we can write $\sqrt{N_w}(\tilde{\gamma}_{w,K} - \gamma^*_{w,K})$ as

$$\Omega^{-1}_{w,K}\frac{1}{\sqrt{N_w}}R'_{w,K}\varepsilon_w = \frac{1}{\sqrt{N_w}}\sum_{i|W_i=w}^{N}\Omega^{-1}_{w,K}R_K(X_i)\varepsilon_i$$

with

$$\mathbb{E}[\Omega^{-1}_{w,K}R_K(X_i)\varepsilon_i] = \Omega^{-1}_{w,K}\mathbb{E}\left[R_K(X_i)\mathbb{E}\left[\varepsilon_i|X_i\right]\right] = 0$$

and

$$\mathbb{V}\left[\Omega^{-1}_{w,K}R_K(X_i)\varepsilon_i\right] = \sigma^2_w \cdot \Omega^{-1}_{w,K}$$

Therefore,

$$S_{w,K} \equiv \frac{1}{\sqrt{N_w}}\sum_{i|W_i=w}^{N}\left[\sigma^2_w \cdot \Omega_{w,K}\right]^{-\frac{1}{2}}R_K(X_i)\varepsilon_i \equiv \frac{1}{\sqrt{N_w}}\sum_{i|W_i=w}^{N}Z_i$$

is a normalized summation of N_w independent random vectors distributed with expectation $\mathbf{0}$ and variance-covariance matrix I_K.

Denote the distribution of $S_{w,K}$ by Q_{N_w} and define $\beta_3 \equiv \sum_{i=1}^{N}\mathbb{E}\left\|\frac{Z_i}{\sqrt{N}}\right\|^3$. Then, by Theorem 1.3, Götze (1991), provided that $K \geq 6$,

$$sup_{\mathcal{A} \in A_K}|Q_{N_w}(\mathcal{A}) - \Phi(\mathcal{A})| \leq C_K\beta_3 N^{-\frac{1}{2}}$$

where A_K is the class of all measurable convex sets in K-dimensional Euclidean space, C_K is $O(K)$, and Φ is a multivariate standard Gaussian distribution.

Lemma A.2 *Suppose that $K(N) = N^{\nu}$ where $\nu < \frac{2}{11}$. Then,*

$$\sup_{\mathcal{A} \in A_K}|Q_{N_w}(\mathcal{A}) - \Phi(\mathcal{A})| \to 0$$

[29]

Proof: First we will show that β_3 is $O(K^{\frac{9}{2}}N^{-\frac{1}{2}})$

$$\beta_3 \equiv \sum_{i|W_i=w}^{N} \mathbb{E}\left\|\frac{Z_i}{\sqrt{N_w}}\right\|^3 = N_w^{-\frac{3}{2}} \sum_{i|W_i=w}^{N} \mathbb{E}\left\|\left[\sigma_w^2 \cdot \Omega_{w,K}\right]^{-\frac{1}{2}} R_K(X_i)\varepsilon_i\right\|^3$$

$$= \left(N_w \cdot \sigma_w^2\right)^{-\frac{3}{2}} \sum_{i|W_i=w}^{N} \mathbb{E}\left\|\Omega_{w,K}^{-\frac{1}{2}} R_K(X_i)\varepsilon_i\right\|^3$$

$$\leq \left(N_w \cdot \sigma_w^2\right)^{-\frac{3}{2}} \sum_{i|W_i=w}^{N} \mathbb{E}\left[\|\Omega_{w,K}^{-\frac{1}{2}}\|^3 \|R_K(X_i)\varepsilon_i\|^3\right]$$

First, consider

$$\|\Omega_{w,K}^{-\frac{1}{2}}\|^3 = \left[\text{tr}(\Omega_{w,K}^{-1})\right]^{\frac{3}{2}} \leq \left[K \cdot \lambda_{max}(\Omega_{w,K}^{-1})\right]^{\frac{3}{2}} \leq C \cdot K^{\frac{3}{2}}$$

which is $O(K^{\frac{3}{2}})$ because $\lambda_{min}(\Omega_{w,K})$ is bounded away from zero by Lemma 0.1. Next, consider

$$\mathbb{E}\|R_K(X_i)\varepsilon_i\|^3 \leq \sup_x \|R_K(x)\|^3 \cdot \mathbb{E}|\varepsilon_i|^3 \leq C \cdot K^3$$

where the third moment of ε_i is bounded by Assumption XX and so the factor is $O(K^3)$. Since σ_w^2 is also bounded by Assumption XX, β_3 is $O(K^{\frac{9}{2}}N^{-\frac{1}{2}})$. Thus,

$$C_K \beta_3 N_w^{-\frac{1}{2}} = K \sum_{i|W_i=w}^{N} \mathbb{E}\left\|\frac{Z_i}{\sqrt{N_w}}\right\|^3 N_w^{-\frac{1}{2}} \leq C \cdot K \cdot K^{\frac{9}{2}} N_w^{-\frac{1}{2}} \cdot N_w^{-\frac{1}{2}} = C \cdot K^{\frac{11}{2}} N^{-1}$$

and the result follows. $\square$

We may proceed further to detail conditions under which the quadratic form, $S'_{w,K}S_{w,K}$, properly normalized, converges to a univariate standard Gaussian distribution. The quadratic form $S'_{w,K}S_{w,K}$ can be written as

$$S'_{w,K}S_{w,K} = \sum_{j=1}^{K}\left(\frac{1}{\sqrt{N_w}}\sum_{i|W_i=w}^{N} Z_{ij}\right)^2$$

where Z_{ij} is the jth element of the vector Z_i. Thus, $S'_{w,K}S_{w,K}$ is a sum of K uncorrelated, squared random variables with each random variable converging to a standard Gaussian distribution by the previous result. Intuitively, this sum should converge to a χ^2 random variable with K degrees of freedom.

Lemma A.3 *Under Assumptions XX-XX,*

$$\sup_c \left|P(S'_{w,K}S_{w,K} \leq c) - \chi_K^2(c)\right| \to 0.$$

Proof: Define the set $A(c) \equiv \{S \in \mathbb{R}^K \,|\, S'S \leq c\}$. Note that $A(c)$ is a measurable convex set in $\mathbb{R}^K$. Also note that for $Z_K \sim \mathcal{N}(0, I_K)$, we have that $\chi_K^2(c) = P(Z'_K Z_k \leq c)$. Then,

$$\sup_c \left|P[S'_{w,K}S_{w,K} \leq c] - \chi_K^2(c)\right| = \sup_c \left|P(S'_{w,K}S_{w,K} \leq c) - P(Z'_K Z_K \leq c)\right|$$

$$= \sup_c \left|P(S_{w,K} \in A(c)) - P(Z_K \in A(c))\right|$$

$$\leq \sup_{\mathcal{A}\in A_K} \left|Q_{N_w}(\mathcal{A}) - \Phi(\mathcal{A})\right|$$

$$\leq C_K \beta_3 N^{-\frac{1}{2}}$$

$$= O(K^{\frac{11}{2}} N^{-1})$$

[30]

which is o(1) for $\nu < \frac{2}{11}$ by Lemma 0.2. $\square$

The proper normalization of the quadratic form yields the studentized version, $(S'_{w,K}S_{w,K} - K)/\sqrt{2K}$. This converges to a standard Gaussian distribution by the following lemma.

Lemma A.4 *Under Assumptions XX-XX,*

$$\sup_c \left| P\left(\frac{S'_{w,K}S_{w,K} - K}{\sqrt{2K}} \leq c \right) - \Phi(c) \right| \to 0.$$

Proof:

$$
\begin{aligned}
& \sup_c \left| P\left(\frac{S'_{w,K}S_{w,K} - K}{\sqrt{2K}} \leq c \right) - \Phi(c) \right| \\
= \ & \sup_c \left| P\left(S'_{w,K}S_{w,K} \leq K + c\sqrt{2K} \right) - \Phi(c) \right| \\
\leq \ & \sup_c \left| P\left(S'_{w,K}S_{w,K} \leq K + c\sqrt{2K} \right) - \chi^2(K + c\sqrt{2K}) \right| + \sup_c \left| \chi^2(K + c\sqrt{2K}) - \Phi(c) \right|
\end{aligned}
$$

The first term goes to zero by Lemma 0.3. For the second term we may apply the Berry-Esséen Theorem which yields,

$$\sup_c \left| P\left(\frac{Z'_K Z_K - K}{\sqrt{2K}} \leq c \right) - \Phi(c) \right| \leq C \cdot K^{-\frac{1}{2}}.$$

Thus for $\nu > 0$ the right-hand side converges to zero as well and the result is established. $\square$

In order to proceed we need the following selected results from Imbens, Newey and Ridder (2004). These results establish convergence rates for the estimators of the regression function.

Lemma A.5 (IMBENS, NEWEY AND RIDDER (2004)): *Suppose Assumptions XX-XX hold. Then,*

(i) there is a sequence $\gamma^0_{w,K}$ such that

$$\sup_x \left| \mu_w(x) - R_K(x)'\gamma^0_{w,K} \right| \equiv \sup_x \left| \mu_w(x) - \mu^0_{w,K} \right| = O(K^{-\frac{s}{d}})$$

(ii)

$$\sup_x \left| R_K(x)'\gamma^*_{w,K} - R_K(x)'\gamma^0_{w,K} \right| \equiv \sup_x \left| \mu^*_{w,K} - \mu^0_{w,K} \right| = O_p\left(\zeta(K)^2 K^{-\frac{s}{d}} \right)$$

(iii)

$$\left\| \gamma^*_{w,K} - \gamma^0_{w,k} \right\| = O(\zeta(K)K^{-\frac{s}{d}})$$

(iv)

$$\left\| \hat{\gamma}_{w,K} - \gamma^0_{w,k} \right\| = O_p(K^{\frac{1}{2}}N^{-\frac{1}{2}} + K^{-\frac{s}{d}})$$

The following lemma describes the limiting distribution of the infeasible test statistic.

Lemma A.6 *Under Assumptions XX-XX,*

$$\left(N_w \cdot \left(\hat{\gamma}_{w,K} - \gamma^*_{w,K} \right)' \left(\hat{\sigma}^2_{w,K} \cdot \hat{\Omega}^{-1}_{w,K} \right)^{-1} \left(\hat{\gamma}_{w,K} - \gamma^*_{w,K} \right) - K \right) / \sqrt{2K} \xrightarrow{d} \mathcal{N}(0,1)$$

[31]

Proof: We need only show that,

$$\left\| \left[\hat{\sigma}_{w,K}^2 \cdot \hat{\Omega}_{w,K}^{-1} \right]^{-\frac{1}{2}} \sqrt{N_w} \left(\hat{\gamma}_{w,K} - \gamma_{w,K}^* \right) - S_{w,K} \right\| = o_p(1).$$

then the result follows by Lemmas 0.2, 0.3, and 0.4.

First, notice that we can rewrite $\hat{\gamma}_{w,K}$ as

$$\hat{\gamma}_{w,K} = \gamma_{w,K}^* + \hat{\Omega}_{w,K}^{-1} R_{w,K}' \varepsilon_{w,K} / N_w$$

where

$$\varepsilon_{w,K} \equiv Y_w - R_{w,K} \gamma_{w,K}^*,$$

with ith row equal to

$$\varepsilon_{Ki} = Y_i - R_K(X_i)' \gamma_{w,K}^*.$$

Then,

$$\left\| \left[\hat{\sigma}_{w,K}^2 \cdot \hat{\Omega}_{w,K}^{-1} \right]^{-\frac{1}{2}} \sqrt{N_w} \left(\hat{\gamma}_{w,K} - \gamma_{w,K}^* \right) - S_{w,K} \right\|$$

$$= \left\| \left[\hat{\sigma}_{w,K}^2 \cdot \hat{\Omega}_{w,K}^{-1} \right]^{-\frac{1}{2}} \sqrt{N_w} \cdot \hat{\Omega}_{w,K}^{-1} \cdot R_{w,K}' \varepsilon_{w,K} / N_w - \left[\sigma_w^2 \cdot \Omega_{w,K} \right]^{-\frac{1}{2}} \sqrt{N_w} \cdot R_{w,K}' \varepsilon_w / N_w \right\|$$

$$= \left\| \hat{\sigma}_{w,K}^{-1} \hat{\Omega}_{w,K}^{-\frac{1}{2}} \cdot R_{w,K}' \varepsilon_{w,K} / \sqrt{N_w} - \sigma_w^{-1} \Omega_{w,K}^{-\frac{1}{2}} \cdot R_{w,K}' \varepsilon_w / \sqrt{N_w} \right\|$$

$$= \left\| \hat{\sigma}_{w,K}^{-1} \hat{\Omega}_{w,K}^{-\frac{1}{2}} \cdot R_{w,K}' \varepsilon_{w,K} / \sqrt{N_w} - \hat{\sigma}_{w,K}^{-1} \hat{\Omega}_{w,K}^{-\frac{1}{2}} \cdot R_{w,K}' \varepsilon_w / \sqrt{N_w} \right.$$

$$\left. + \hat{\sigma}_{w,K}^{-1} \hat{\Omega}_{w,K}^{-\frac{1}{2}} \cdot R_{w,K}' \varepsilon_w / \sqrt{N_w} - \sigma_w^{-1} \hat{\Omega}_{w,K}^{-\frac{1}{2}} \cdot R_{w,K}' \varepsilon_w / \sqrt{N_w} \right.$$

$$\left. + \sigma_w^{-1} \hat{\Omega}_{w,K}^{-\frac{1}{2}} \cdot R_{w,K}' \varepsilon_w / \sqrt{N_w} - \sigma_w^{-1} \Omega_{w,K}^{-\frac{1}{2}} \cdot R_{w,K}' \varepsilon_w / \sqrt{N_w} \right\|$$

$$\leq \left\| \hat{\sigma}_{w,K}^{-1} \hat{\Omega}_{w,K}^{-\frac{1}{2}} \cdot R_{w,K}' \varepsilon_{w,K} / \sqrt{N_w} - \hat{\sigma}_{w,K}^{-1} \hat{\Omega}_{w,K}^{-\frac{1}{2}} \cdot R_{w,K}' \varepsilon_w / \sqrt{N_w} \right\|$$

$$+ \left\| \hat{\sigma}_{w,K}^{-1} \hat{\Omega}_{w,K}^{-\frac{1}{2}} \cdot R_{w,K}' \varepsilon_w / \sqrt{N_w} - \sigma_w^{-1} \hat{\Omega}_{w,K}^{-\frac{1}{2}} \cdot R_{w,K}' \varepsilon_w / \sqrt{N_w} \right\|$$

$$+ \left\| \sigma_w^{-1} \hat{\Omega}_{w,K}^{-\frac{1}{2}} \cdot R_{w,K}' \varepsilon_w / \sqrt{N_w} - \sigma_w^{-1} \Omega_{w,K}^{-\frac{1}{2}} \cdot R_{w,K}' \varepsilon_w / \sqrt{N_w} \right\|$$

$$= \left| \hat{\sigma}_{w,K}^{-1} \right| \left\| \hat{\Omega}_{w,K}^{-\frac{1}{2}} R_{w,K}' \left(\varepsilon_{w,K} - \varepsilon_w \right) / \sqrt{N_w} \right\| \tag{A.20}$$

$$+ \left| \hat{\sigma}_{w,K}^{-1} - \sigma_w^{-1} \right| \left\| \hat{\Omega}_{w,K}^{-\frac{1}{2}} \cdot R_{w,K}' \varepsilon_w / \sqrt{N_w} \right\| \tag{A.21}$$

$$+ \left| \sigma_w^{-1} \right| \left\| \left(\hat{\Omega}_{w,K}^{-\frac{1}{2}} - \Omega_{w,K}^{-\frac{1}{2}} \right) R_{w,K}' \varepsilon_w / \sqrt{N_w} \right\| \tag{A.22}$$

First, consider equation (4),

$$\left| \hat{\sigma}_{w,K}^{-1} \right| \left\| \hat{\Omega}_{w,K}^{-\frac{1}{2}} R_{w,K}' \left(\varepsilon_{w,K} - \varepsilon_w \right) / \sqrt{N_w} \right\|$$

$$= \left(\sigma_w^{-1} + o_p \left(N^{-\frac{1}{2}} \right) \right) \cdot \left\| \hat{\Omega}_{w,K}^{-\frac{1}{2}} R_{w,K}' \left(\varepsilon_{w,K} - \varepsilon_w \right) / \sqrt{N_w} \right\|$$

$$= \left(O\left(1 \right) + o_p \left(N^{-\frac{1}{2}} \right) \right) \cdot \left\| \hat{\Omega}_{w,K}^{-\frac{1}{2}} R_{w,K}' \left(\varepsilon_{w,K} - \varepsilon_w \right) / \sqrt{N_w} \right\|$$

[32]

where

$$\mathbb{E}\left\|\hat{\Omega}_{w,K}^{-\frac{1}{2}}R'_{w,K}\left(\varepsilon_{w,K}-\varepsilon_w\right)/\sqrt{N_w}\right\|^2$$

$$=\ \mathbb{E}\left[\frac{1}{N_w}\mathrm{tr}\left(\left(\varepsilon_{w,K}-\varepsilon_w\right)'R_{w,K}\hat{\Omega}_{w,K}^{-1}R'_{w,K}\left(\varepsilon_{w,K}-\varepsilon_w\right)\right)\right]$$

$$=\ \mathbb{E}\left[\left(\left(\varepsilon_{w,K}-\varepsilon_w\right)'R_{w,K}\left(R'_{w,K}R_{w,K}\right)^{-1}R'_{w,K}\left(\varepsilon_{w,K}-\varepsilon_w\right)\right)\right]$$

$$\leq\ \mathbb{E}\left[\left(\varepsilon_{w,K}-\varepsilon_w\right)'\left(\varepsilon_{w,K}-\varepsilon_w\right)\right]$$

$$=\ \mathbb{E}\left[\left(\mu_w(\mathbf{X})-R_{w,K}\gamma^*_{w,K}\right)'\left(\mu_w(\mathbf{X})-R_{w,K}\gamma^*_{w,K}\right)\right]$$

$$\leq\ N_w\cdot\sup_x\left|\mu_w(x)-R_K(x)'\gamma^*_{w,K}\right|^2$$

$$\leq\ N_w\cdot\sup_x\left(\left|\mu_w(x)-R_K(x)'\gamma^0_{w,K}\right|+\left|R_K(x)'\gamma^0_{w,K}-R_K(x)'\gamma^*_{w,K}\right|\right)^2$$

$$=\ N_w\left(O\left(K^{-\frac{s}{d}}\right)+O\left(\zeta(K)^2K^{-\frac{s}{d}}\right)\right)^2$$

$$=\ O\left(N\right)\cdot\left(O\left(\zeta(K)^2K^{-\frac{s}{d}}\right)\right)^2$$

so that equation (4) is $O_p\left(N^{\frac{1}{2}}\zeta(K)^2K^{-\frac{s}{d}}\right)$ by Markov's inequality and consistency of the sample variance. The third line follows since $(I_{N_w}-R_{w,K}(R'_{w,K}R_{w,K})^{-1}R'_{w,K})$ is a projection matrix and so it is positive semi-definite. The seventh line follows from Lemma 0.5 (i) and (ii).

Now consider equation (5),

$$\left|\hat{\sigma}_{w,K}^{-1}-\sigma_w^{-1}\right|\left\|\hat{\Omega}_{w,K}^{-\frac{1}{2}}\cdot R'_{w,K}\varepsilon_w/\sqrt{N_w}\right\|$$

The first factor is $o_p(N^{-\frac{1}{2}})$ and

$$\mathbb{E}\left\|\hat{\Omega}_{w,K}^{-\frac{1}{2}}\cdot R'_{w,K}\varepsilon_w/\sqrt{N_w}\right\|^2$$

$$=\ \mathbb{E}\left[\frac{1}{N_w}\mathrm{tr}\left(\varepsilon'_w R_{w,K}\hat{\Omega}_{w,K}^{-1}R'_{w,K}\varepsilon_w\right)\right]$$

$$=\ \mathbb{E}\left[\mathrm{tr}\left(\varepsilon'_w R_{w,K}\left(R'_{w,K}R_{w,K}\right)^{-1}R'_{w,K}\varepsilon_w\right)\right]$$

$$=\ \mathbb{E}\left[\mathrm{tr}\left(R_{w,K}\left(R'_{w,K}R_{w,K}\right)^{-1}R'_{w,K}\varepsilon_w\varepsilon'_w\right)\right]$$

$$=\ \mathrm{tr}\left(\mathbb{E}\left[R_{w,K}\left(R'_{w,K}R_{w,K}\right)^{-1}R'_{w,K}\mathbb{E}\left[\varepsilon_w\varepsilon'_w|\mathbf{X}\right]\right]\right)$$

$$=\ \sigma_w^2\cdot\mathrm{tr}\left(\mathbb{E}\left[R_{w,K}\left(R'_{w,K}R_{w,K}\right)^{-1}R'_{w,K}\right]\right)$$

$$=\ \sigma_w^2\cdot\mathbb{E}\left[\mathrm{tr}\left(R_{w,K}\left(R'_{w,K}R_{w,K}\right)^{-1}R'_{w,K}\right)\right]$$

$$=\ \sigma_w^2\cdot\mathbb{E}\left[\mathrm{tr}\left(\left(R'_{w,K}R_{w,K}\right)^{-1}R'_{w,K}R_{w,K}\right)\right]$$

$$=\ \sigma_w^2\cdot\mathrm{tr}\left(I_K\right)$$

$$=\ \sigma_w^2\cdot K$$

so that the second factor is $O\left(K^{\frac{1}{2}}\right)$. Thus equation (5) is $o_p\left(K^{\frac{1}{2}}N^{-\frac{1}{2}}\right)$.

Finally, consider equation (6),

$$\left|\sigma_w^{-1}\right|\left\|\left(\hat{\Omega}_{w,K}^{-\frac{1}{2}}-\Omega_{w,K}^{-\frac{1}{2}}\right)R'_{w,K}\varepsilon_w/\sqrt{N_w}\right\|$$

$$\leq\ C\cdot\left\|\hat{\Omega}_{w,K}^{-\frac{1}{2}}-\Omega_{w,K}^{-\frac{1}{2}}\right\|\left\|R'_{w,K}\varepsilon_w/\sqrt{N_w}\right\|$$

The first factor is $O_p\left(\zeta(K)K^{\frac{1}{2}}N^{-\frac{1}{2}}\right)$ by Lemma 0.1 and the continuous mapping theorem, and

$$
\begin{aligned}
& \mathbb{E}\left\|R'_{w,K}\varepsilon_w/\sqrt{N_w}\right\|^2 \\
=\ & \mathbb{E}\left[\frac{1}{N_w}\operatorname{tr}\left(\varepsilon'_w R_{w,K}R'_{w,K}\varepsilon_w\right)\right] \\
=\ & \mathbb{E}\left[\frac{1}{N_w}\operatorname{tr}\left(R'_{w,K}\varepsilon_w\varepsilon'_w R_{w,K}\right)\right] \\
=\ & \operatorname{tr}\left(\frac{1}{N_w}\mathbb{E}\left[R'_{w,K}\mathbb{E}\left[\varepsilon_w\varepsilon'_w|\mathbf{X}\right]R_{w,K}\right]\right) \\
=\ & \sigma_w^2\cdot\operatorname{tr}\left(\mathbb{E}\left[R'_{w,K}R_{w,K}/N_w\right]\right) \\
=\ & \sigma_w^2\cdot\operatorname{tr}\left(\Omega_{w,K}\right) \\
\le\ & \sigma_w^2\cdot K\cdot\lambda_{max}\left(\Omega_{w,K}\right) \\
\le\ & C\cdot K
\end{aligned}
$$

so that the second factor is $O\left(K^{\frac{1}{2}}\right)$ by Assumption XX, Lemma 0.1 (ii) and Markov's inequality. Thus, equation (3) is $O_p\left(\zeta(K)KN^{-\frac{1}{2}}\right)$.

Combining these results yields:

$$
\begin{aligned}
& \left\|\left[\hat{\sigma}_{w,K}^2\cdot\hat{\Omega}_{w,K}^{-1}\right]^{-\frac{1}{2}}\sqrt{N_w}\left(\hat{\gamma}_{w,K}-\gamma^*_{w,K}\right)-S_{w,K}\right\| \\
=\ & O_p\left(N^{\frac{1}{2}}\zeta(K)^2 K^{-\frac{s}{d}}\right)+o_p\left(K^{\frac{1}{2}}N^{-\frac{1}{2}}\right)+O_p\left(\zeta(K)KN^{-\frac{1}{2}}\right) \\
=\ & O_p\left(N^{\frac{1}{2}}K^{(2-\frac{s}{d})}\right)+o_p\left(K^{\frac{1}{2}}N^{-\frac{1}{2}}\right)+O_p\left(K^2 N^{-\frac{1}{2}}\right)
\end{aligned}
$$

All three terms are $o_p(1)$ by Assumption XX and for $\frac{s}{d}>\frac{4\nu+1}{2\nu}$. $\square$

Proof of Theorem 8.1: From the previous lemma we have that [11]

$$
T^*\equiv\left(N_w\cdot\left((\hat{\gamma}_{1,K}-\hat{\gamma}_{0,K})-(\gamma^*_{1,K}-\gamma^*_{0,K})\right)'\cdot\hat{V}^{-1}\cdot\left((\hat{\gamma}_{1,K}-\hat{\gamma}_{0,K})-(\gamma^*_{1,K}-\gamma^*_{0,K})\right)-K\right)/\sqrt{2K}
$$

converges in distribution to a $\mathcal{N}(0,1)$ random variable, where $\hat{V}$ is defined as

$$
\hat{V}\equiv(\hat{\sigma}_{0,K}^2\cdot\hat{\Omega}_{0,K}^{-1}+\hat{\sigma}_{1,K}^2\cdot\hat{\Omega}_{1,K}^{-1}).
$$

To complete the proof we must show that under our assumptions $|T^*-T|=o_p(1)$, where T is defined as

$$
T\equiv\left(N_w\cdot(\hat{\gamma}_{1,K}-\hat{\gamma}_{0,K})'\cdot\hat{V}^{-1}\cdot(\hat{\gamma}_{1,K}-\hat{\gamma}_{0,K})-K\right)/\sqrt{2K}.
$$

Note that under the null hypothesis $\mu_1(x)=\mu_0(x)$ so we may choose the same approximating sequence $\gamma_{1,K}^0=\gamma_{0,K}^0$ for $\mu_{1,K}^0(x)=\mu_{0,K}^0(x)$. Then,

$$
\begin{aligned}
\left\|\gamma^*_{1,K}-\gamma^*_{0,K}\right\| & = \left\|\gamma^*_{1,K}-\gamma_{1,K}^0+\gamma_{0,K}^0-\gamma^*_{0,K}\right\| \\
& \le \left\|\gamma^*_{1,K}-\gamma_{1,K}^0\right\|+\left\|\gamma_{0,K}^0-\gamma^*_{0,K}\right\| \\
& = O(\zeta(K)K^{-\frac{s}{d}}) \quad\quad\quad\quad\quad\quad\quad\quad\quad\text{(A.23)}
\end{aligned}
$$

---
[11] For simplicity of notation we assume $N_1=N_0$

by Lemma 0.5 (iii), and

$$
\begin{aligned}
\|\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K}\| &= \|\hat{\gamma}_{1,K} - \gamma^0_{1,K} + \gamma^0_{0,K} - \hat{\gamma}_{0,K}\| \\
&\leq \|\hat{\gamma}_{1,K} - \gamma^0_{1,K}\| + \|\gamma^0_{0,K} - \hat{\gamma}_{0,K}\| \\
&= O_p(K^{\frac{1}{2}}N^{-\frac{1}{2}} + K^{-\frac{s}{d}})
\end{aligned}
\tag{A.24}
$$

by Lemma 0.5 (iv). So then,

$$
\begin{aligned}
|T^* - T| &= \left| \left(N_w \cdot \left((\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K}) - (\gamma^*_{1,K} - \gamma^*_{0,K}) \right)' \hat{V}^{-1} \left((\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K}) - (\gamma^*_{1,K} - \gamma^*_{0,K}) \right) - K \right) / \sqrt{2K} \right. \\
&\quad \left. - \left(N_w \cdot (\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K})' \hat{V}^{-1} (\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K}) - K \right) / \sqrt{2K} \right| \\
&= \frac{N_w}{\sqrt{2K}} \cdot \left| \left((\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K}) - (\gamma^*_{1,K} - \gamma^*_{0,K}) \right)' \hat{V}^{-1} \left((\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K}) - (\gamma^*_{1,K} - \gamma^*_{0,K}) \right) \right. \\
&\quad \left. - (\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K})' \hat{V}^{-1} (\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K}) \right| \\
&= \frac{N_w}{\sqrt{2K}} \cdot \left| -2 \cdot (\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K})' \hat{V}^{-1} (\gamma^*_{1,K} - \gamma^*_{0,K}) + (\gamma^*_{1,K} - \gamma^*_{0,K})' \hat{V}^{-1} (\gamma^*_{1,K} - \gamma^*_{0,K}) \right| \\
&\leq \frac{N_w}{\sqrt{2K}} \cdot \left(2 \cdot \left| (\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K})' \hat{V}^{-1} (\gamma^*_{1,K} - \gamma^*_{0,K}) \right| + \left| (\gamma^*_{1,K} - \gamma^*_{0,K})' \hat{V}^{-1} (\gamma^*_{1,K} - \gamma^*_{0,K}) \right| \right)
\end{aligned}
$$

Consider the first term,

$$
\begin{aligned}
2 \cdot \left| (\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K})' \hat{V}^{-1} (\gamma^*_{1,K} - \gamma^*_{0,K}) \right| &= 2 \cdot \left| \text{tr} \left((\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K})' \hat{V}^{-1} (\gamma^*_{1,K} - \gamma^*_{0,K}) \right) \right| \\
&\leq 2 \cdot \|\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K}\| \cdot \|\gamma^*_{1,K} - \gamma^*_{0,K}\| \cdot \lambda_{max}(\hat{V}^{-1}) \\
&\leq C \cdot \|\hat{\gamma}_{1,K} - \hat{\gamma}_{0,K}\| \cdot \|\gamma^*_{1,K} - \gamma^*_{0,K}\| + o_p(1) \\
&= \left(O_p(K^{\frac{1}{2}}N^{-\frac{1}{2}} + K^{-\frac{s}{d}}) \cdot O(\zeta(K)K^{-\frac{s}{d}}) \right)
\end{aligned}
$$

Where the third line follows from Lemma 0.1 (iii) and Assumption XX. The last line follows from equations (7) and (8).

Now, consider the second term,

$$
\begin{aligned}
\left| (\gamma^*_{1,K} - \gamma^*_{0,K})' \hat{V}^{-1} (\gamma^*_{1,K} - \gamma^*_{0,K}) \right| &= \left| \text{tr} \left((\gamma^*_{1,K} - \gamma^*_{0,K})' \hat{V}^{-1} (\gamma^*_{1,K} - \gamma^*_{0,K}) \right) \right| \\
&\leq \|\gamma^*_{1,K} - \gamma^*_{0,K}\|^2 \cdot \lambda_{max}(\hat{V}^{-1}) \\
&\leq C \cdot \|\gamma^*_{1,K} - \gamma^*_{0,K}\|^2 + o_p(1) \\
&= O(\zeta(K)^2 K^{-\frac{2s}{d}})
\end{aligned}
$$

Where the third line follows from Lemma 0.1 (iii) and Assumption XX. The last line follows from equation (7).

So then,

$$
\begin{aligned}
|T^* - T| &= \frac{N_w}{\sqrt{2K}} \cdot \left(O_p(K^{\frac{1}{2}}N^{-\frac{1}{2}} + K^{-\frac{s}{d}}) \cdot O(\zeta(K)K^{-\frac{s}{d}}) + O(\zeta(K)^2 K^{-\frac{2s}{d}}) \right) \\
&= O_p\left(N^{\frac{1}{2}}\zeta(K)K^{-\frac{s}{d}} \right) + O_p\left(N\zeta(K)K^{-(\frac{1}{2}+\frac{2s}{d})} \right) + O\left(N\zeta(K)^2 K^{-(\frac{1}{2}+\frac{2s}{d})} \right)
\end{aligned}
$$

For $\frac{s}{d} > \frac{2\nu+1}{2\nu}$ all three terms are $o_p(1)$ and the result follows. $\square$

Before proving Theorem 8.2 we need the following lemma.

[35]

Lemma A.7

$$\lambda_{max}\left(\hat{V}^{-1}\right) \geq \lambda_{max}\left(\left([\hat{V}]_{-1}\right)^{-1}\right)$$

Where $[\quad]_{-1}$ *is defined as the* $(K-1)$ *x* $(K-1)$ *matrix created by eliminating the first row and column of a* $(K \ x \ K)$ *square matrix.*

Proof: The proof follows by the Interlacing Theorem given below.

If A is an n x n positive semi-definite Hermitian matrix with eigenvalues $\lambda_1 \geq ... \geq \lambda_n$, B is a k x k principal submatrix of A with eigenvalues $\tilde{\lambda}_1 \geq ... \geq \tilde{\lambda}_k$, then

$$\lambda_i \geq \tilde{\lambda}_i \geq \lambda_{i+n-k}, \quad i = 1, ..., k.$$

In our case, $\hat{V}$ is positive definite, symmetric and thus positive definite, Hermitian. So then, by the Interlacing Theorem

$$\lambda_{min}\left(\hat{V}\right) \leq \lambda_{min}\left([\hat{V}]_{-1}\right) \Longrightarrow \lambda_{max}\left(\hat{V}^{-1}\right) \geq \lambda_{max}\left(\left([\hat{V}]_{-1}\right)^{-1}\right)$$

$\square$

Proof of Theorem 8.2: When the conditional average treatment effect is constant we may choose the two approximating sequences, $\gamma_{0,K}^0$ and $\gamma_{1,K}^0$, to differ only by way of the first element (the coefficient of the constant term in the approximating sequence). In other words, if $\mu_1(x) - \mu_0(x) = \tau_0$ for all $x \in \mathbb{X}$, then the coefficients of the power series terms involving x^r such that $r > 0$ should be identical for $w = 0, 1$, so that their difference no longer varies with x.

Thus, a natural strategy to test the null hypothesis of a constant conditional average treatment effect is to compare the last $K-1$ elements of $\hat{\gamma}_{1,K}$ and $\hat{\gamma}_{0,K}$ and to reject the null hypothesis when these elements are sufficiently different.

By the proof of Lemma 0.6 we have that,

$$\left\|\left[\hat{\sigma}_{w,K}^2 \cdot \hat{\Omega}_{w,K}^{-1}\right]^{-\frac{1}{2}} \sqrt{N_w}\left(\hat{\gamma}_{w,K} - \gamma_{w,K}^*\right) - S_{w,K}\right\| = o_p(1)$$

so that by Lemma 0.2,

$$\left[\hat{\sigma}_{w,K}^2 \cdot \hat{\Omega}_{w,K}^{-1}\right]^{-\frac{1}{2}} \sqrt{N_w}\left(\hat{\gamma}_{w,K} - \gamma_{w,K}^*\right) \xrightarrow{d} \mathcal{N}\left(0, I_K\right).$$

By the joint convergence in distribution of the elements of $\hat{\gamma}_{w,K}$ we may partition this result as

$$\left[\hat{\sigma}_{w,K}^2 \cdot \left[\hat{\Omega}_{w,K}^{-1}\right]_{1,1}\right]^{-\frac{1}{2}} \sqrt{N_w}\left(\hat{\gamma}_{w,K,1} - \gamma_{w,K,1}^*\right) \xrightarrow{d} \mathcal{N}\left(0, 1\right)$$

and

$$\left[\hat{\sigma}_{w,K}^2 \cdot \left[\hat{\Omega}_{w,K}^{-1}\right]_{-1}\right]^{-\frac{1}{2}} \sqrt{N_w}\left([\hat{\gamma}_{w,K}]_{-1} - [\gamma_{w,K}^*]_{-1}\right) \xrightarrow{d} \mathcal{N}\left(0, I_{K-1}\right) \tag{A.25}$$

where $\hat{\gamma}_{w,K,i}$ is the ith element of $\hat{\gamma}_{w,K}$, $[M]_{i,j}$ is the (i,j) element of a matrix M, and where $[\quad]_{-1}$ also represents the $(K-1)$ vector created by eliminating the first element of a vector of dimension K as well as

the $(K-1)$ x $(K-1)$ matrix created by eliminating the first row and column of a $(K$ x $K)$ square matrix.

By the independence of the control and treated observations, marginal convergence in distribution of (1) for $w = \{0, 1\}$ implies joint convergence in distribution. Then, applying the Cramer-Wold device

$$\left([\hat{V}]_{-1}\right)^{-\frac{1}{2}} \sqrt{N_w} \left([\hat{\gamma}_{1,K}]_{-1} - [\hat{\gamma}_{0,K}]_{-1} - \left([\gamma_{1,K}^*]_{-1} - [\gamma_{0,K}^*]_{-1}\right)\right) \xrightarrow{d} \mathcal{N}(0, I_{K-1})$$

where

$$[\hat{V}]_{-1} = \hat{\sigma}_{0,K}^2 \cdot \left[\hat{\Omega}_{0,K}^{-1}\right]_{-1} + \hat{\sigma}_{1,K}^2 \cdot \left[\hat{\Omega}_{1,K}^{-1}\right]_{-1}$$

so that by the continuous mapping theorem (or following the logic of Lemma 0.3) we have that

$$N_w \left(\hat{\delta} - \delta^*\right)' \left([\hat{V}]_{-1}\right)^{-1} \left(\hat{\delta} - \delta^*\right) \xrightarrow{d} \chi_{K-1}^2$$

where

$$\hat{\delta} = [\hat{\gamma}_{1,K}]_{-1} - [\hat{\gamma}_{0,K}]_{-1}$$

and

$$\delta^* = \left[\gamma_{1,K}^*\right]_{-1} - \left[\gamma_{0,K}^*\right]_{-1}.$$

We need only show that

$$N_w \left\{\left(\hat{\delta} - \delta^*\right)' \left([\hat{V}]_{-1}\right)^{-1} \left(\hat{\delta} - \delta^*\right) - \hat{\delta}' \left([\hat{V}]_{-1}\right)^{-1} \hat{\delta}\right\} = o_p(1) \tag{A.26}$$

to obtain the asymptotic distribution of our feasible test statistic. To continue further we will again use the results summarized in Lemma 0.5. Specifically, note that

$$
\begin{aligned}
\left\|[\gamma_{w,K}^*]_{-1} - [\gamma_{w,K}^0]_{-1}\right\|^2 &= \sum_{i=2}^{K} \left(\gamma_{w,K,i}^* - \gamma_{w,K,i}^0\right)^2 \\
&\leq \sum_{i=1}^{K} \left(\gamma_{w,K,i}^* - \gamma_{w,K,i}^0\right)^2 \\
&= \left\|\gamma_{w,K}^* - \gamma_{w,K}^0\right\|^2 \\
&= O\left(\left[\zeta(K)K^{-\frac{s}{d}}\right]^2\right)
\end{aligned}
\tag{A.27}
$$

by Lemma 0.5 (iii) and

$$
\begin{aligned}
\left\|[\hat{\gamma}_{w,K}]_{-1} - [\gamma_{w,K}^0]_{-1}\right\|^2 &= \sum_{i=2}^{K} \left(\hat{\gamma}_{w,K,i} - \gamma_{w,K,i}^0\right)^2 \\
&\leq \sum_{i=1}^{K} \left(\hat{\gamma}_{w,K,i} - \gamma_{w,K,i}^0\right)^2 \\
&= \left\|\hat{\gamma}_{w,K} - \gamma_{w,K}^0\right\|^2 \\
&= O_p\left(\left[K^{\frac{1}{2}}N^{-\frac{1}{2}} + K^{-\frac{s}{d}}\right]^2\right).
\end{aligned}
\tag{A.28}
$$

[37]

by Lemma 0.5 (iv). We may choose the last $(K-1)$ elements of the approximating sequence to be equal, $[\gamma^0_{1,K}]_{-1} = [\gamma^0_{1,K}]_{-1}$. This allows us to bound $\hat{\delta}$ and δ^* by the following

$$
\begin{aligned}
\left\| \hat{\delta} \right\| &= \left\| [\hat{\gamma}_{1,K}]_{-1} - [\hat{\gamma}_{0,K}]_{-1} \right\| \\
&= \left\| [\hat{\gamma}_{1,K}]_{-1} - [\gamma^0_{1,K}]_{-1} + [\gamma^0_{0,K}]_{-1} - [\hat{\gamma}_{0,K}]_{-1} \right\| \\
&\leq \left\| [\hat{\gamma}_{1,K}]_{-1} - [\gamma^0_{1,K}]_{-1} \right\| + \left\| [\gamma^0_{0,K}]_{-1} - [\hat{\gamma}_{0,K}]_{-1} \right\| \\
&= O_p\left(K^{\frac{1}{2}} N^{-\frac{1}{2}} + K^{-\frac{s}{d}} \right)
\end{aligned}
\tag{A.29}
$$

by equation (12). Also,

$$
\begin{aligned}
\| \delta^* \| &= \left\| [\gamma^*_{1,K}]_{-1} - [\gamma^*_{0,K}]_{-1} \right\| \\
&= \left\| [\gamma^*_{1,K}]_{-1} - [\gamma^0_{1,K}]_{-1} + [\gamma^0_{0,K}]_{-1} - [\gamma^*_{0,K}]_{-1} \right\| \\
&\leq \left\| [\gamma^*_{1,K}]_{-1} - [\gamma^0_{1,K}]_{-1} \right\| + \left\| [\gamma^0_{0,K}]_{-1} - [\gamma^*_{0,K}]_{-1} \right\| \\
&= O\left(\zeta(K) K^{-\frac{s}{d}} \right)
\end{aligned}
\tag{A.30}
$$

by equation (11).

Next we can rewrite the left-hand side of equation (10) as

$$
N_w \left\{ \left(\hat{\delta} - \delta^* \right)' \left([\hat{V}]_{-1} \right)^{-1} \left(\hat{\delta} - \delta^* \right) - \hat{\delta}' \left([\hat{V}]_{-1} \right)^{-1} \hat{\delta} \right\} = N_w \left\{ \delta^{*\prime} \left([\hat{V}]_{-1} \right)^{-1} \delta^* - 2 \cdot \hat{\delta}' \left([\hat{V}]_{-1} \right)^{-1} \delta^* \right\}
$$

Consider the first term,

$$
\begin{aligned}
\left| \delta^{*\prime} \left([\hat{V}]_{-1} \right)^{-1} \delta^* \right| &= \left| \text{tr} \left(\delta^{*\prime} \left([\hat{V}]_{-1} \right)^{-1} \delta^* \right) \right| \\
&\leq \| \delta^* \|^2 \cdot \lambda_{max} \left(\left([\hat{V}]_{-1} \right)^{-1} \right) \\
&\leq \| \delta^* \|^2 \cdot \lambda_{max} (\hat{V}^{-1}) \\
&\leq C \cdot \| \delta^* \|^2 + o_p(1) \\
&= O\left(\zeta(K)^2 K^{-\frac{2s}{d}} \right)
\end{aligned}
$$

The third line follows from Lemma 0.7. The fourth line follows from Lemma 0.1 (iii) and Assumption XX. The last line follows from equation (14). Now consider the second term,

$$
\begin{aligned}
2 \cdot \left| \hat{\delta}' \left([\hat{V}]_{-1} \right)^{-1} \delta^* \right| &= 2 \cdot \left| \text{tr} \left(\hat{\delta}' \left([\hat{V}]_{-1} \right)^{-1} \delta^* \right) \right| \\
&\leq 2 \cdot \left\| \hat{\delta} \right\| \cdot \| \delta^* \| \cdot \lambda_{max} \left(\left([\hat{V}]_{-1} \right)^{-1} \right) \\
&\leq 2 \cdot \left\| \hat{\delta} \right\| \cdot \| \delta^* \| \cdot \lambda_{max} (\hat{V}^{-1}) \\
&\leq C \cdot \left\| \hat{\delta} \right\| \cdot \| \delta^* \| + o_p(1) \\
&= O_p\left(K^{\frac{1}{2}} N^{-\frac{1}{2}} + K^{-\frac{s}{d}} \right) \cdot O\left(\zeta(K) K^{-\frac{s}{d}} \right)
\end{aligned}
$$

The third line follows from Lemma 0.7. The fourth line follows from Lemma 0.1 (iii) and Assumption XX. The last line follows from equations (13) and (14). Thus the left-hand side of equation (10) is of order

$$
O(N) \cdot \left[O\left(\zeta(K)^2 K^{-\frac{2s}{d}} \right) + O_p\left(K^{\frac{1}{2}} N^{-\frac{1}{2}} + K^{-\frac{s}{d}} \right) \cdot O\left(\zeta(K) K^{-\frac{s}{d}} \right) . \right]
$$

[38]

For $\frac{s}{d} > \frac{3\nu+1}{2\nu}$ all three terms are $o_p(1)$ and

$$N_w cdot \hat{\delta}' \left([\hat{V}]_{-1} \right)^{-1} \hat{\delta} \xrightarrow{d} \chi^2_{K-1}.$$

Finally, by Lemma 0.4 replacing, K with $(K-1)$, we have that

$$\frac{1}{\sqrt{2(K-1)}} \left[N_w \cdot \left(\left([\hat{\gamma}_{1,K}]_{-1} - [\hat{\gamma}_{0,K}]_{-1} \right)' \left([\hat{V}]_{-1} \right)^{-1} \left([\hat{\gamma}_{1,K}]_{-1} - [\hat{\gamma}_{0,K}]_{-1} \right) \right) - (K-1) \right] \xrightarrow{d} \mathcal{N}(0,1)$$

$\square$

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Table 1: Summary Statistics Gain Data (Control Samples)

	Los Angeles (LA) N=1400		Riverside (RI) N=1040		Alameda (AL) N=601		San Diego (SD) N=1154	
	mean	(s.d.)	mean	(s.d.)	mean	(s.d.)	mean	(s.d.)
female	0.93	(0.26)	0.88	(0.33)	0.95	(0.22)	0.85	(0.36)
age	38.6	(8.4)	33.8	(8.4)	35.2	(8.7)	33.8	(8.6)
hispanic	0.31	(0.46)	0.27	(0.45)	0.06	(0.24)	0.23	(0.42)
black	0.46	(0.50)	0.15	(0.36)	0.71	(0.45)	0.24	(0.43)
hsdiploma	0.35	(0.48)	0.52	(0.50)	0.61	(0.49)	0.59	(0.49)
1 child	0.32	(0.47)	0.41	(0.49)	0.41	(0.49)	0.43	(0.49)
children under 6	0.11	(0.31)	0.17	(0.37)	0.29	(0.46)	0.13	(0.33)
earnings q-1	1.83	(1.78)	1.87	(1.95)	1.53	(1.46)	2.21	(2.11)
earnings q-2	1.81	(1.73)	2.21	(1.96)	1.57	(1.73)	2.57	(2.43)
earnings q-3	1.81	(1.65)	2.28	(2.08)	1.50	(1.64)	2.62	(2.48)
earnings q-4	1.78	(1.87)	2.11	(1.95)	1.95	(1.95)	2.72	(2.60)
earnings q-5	1.89	(2.17)	2.29	(2.02)	1.92	(2.59)	2.91	(2.49)
earnings q-6	1.86	(1.95)	2.51	(2.16)	1.75	(1.94)	2.76	(2.37)
earnings q-7	1.55	(1.73)	2.44	(2.32)	1.84	(2.02)	3.09	(2.56)
earnings q-8	1.54	(1.92)	2.38	(2.42)	1.59	(2.14)	2.93	(2.62)
earnings q-9	1.55	(1.87)	2.60	(2.43)	1.77	(1.94)	2.91	(2.64)
earnings q-10	1.51	(1.58)	2.60	(2.40)	1.90	(2.44)	2.97	(2.66)
pos earn q-1	0.87	(0.33)	0.77	(0.42)	0.86	(0.35)	0.71	(0.45)
pos earn q-2	0.87	(0.33)	0.76	(0.43)	0.86	(0.35)	0.71	(0.45)
pos earn q-3	0.87	(0.34)	0.76	(0.43)	0.86	(0.35)	0.70	(0.46)
pos earn q-4	0.87	(0.34)	0.75	(0.43)	0.83	(0.38)	0.71	(0.45)
pos earn q-5	0.88	(0.33)	0.75	(0.43)	0.86	(0.35)	0.70	(0.46)
pos earn q-6	0.89	(0.31)	0.76	(0.43)	0.85	(0.35)	0.69	(0.46)
pos earn q-7	0.88	(0.33)	0.75	(0.43)	0.87	(0.34)	0.71	(0.45)
pos earn q-8	0.89	(0.31)	0.76	(0.43)	0.88	(0.33)	0.70	(0.46)
pos earn q-9	0.90	(0.30)	0.76	(0.43)	0.88	(0.33)	0.71	(0.46)
pos earn q-10	0.90	(0.30)	0.75	(0.43)	0.87	(0.34)	0.72	(0.45)

Table 2: OPTIMAL SAMPLE SIZES AND RELATIVE ASYMPTOTIC VARIANCES

| C	T	α	$e(x) \in [0, \alpha)$ C	T	$e(x) \in [\alpha, 1-\alpha]$ C	T	$e(x) \in (1-\alpha, 1]$ C	T	τ_c	τ_t	OSATE	OWATE	$\alpha = 0.1$
RI (c)	LA (c)	0.095	122	6	900	1199	18	195	1.32	1.76	0.83	0.73	0.83
RI (c)	AL (c)	0.076	238	6	800	569	2	26	1.01	1.77	0.66	0.55	0.67
RI (c)	SD (c)	0.139	0	0	1040	1151	0	3	1.06	1.02	1.00	0.99	1.00
AL (c)	LA (c)	0.104	2	0	577	1022	22	378	1.44	0.82	0.75	0.66	0.75
AL (c)	SD (c)	0.088	38	2	547	782	16	370	2.10	0.31	0.22	0.18	0.22
SD (c)	LA (c)	0.099	203	9	939	1237	12	154	0.92	2.59	0.64	0.57	0.64
LA (c)	LA (t)	0.123	0	0	1400	2994	0	1	1.01	1.00	1.00	1.00	1.00
RI (c)	RI (t)	0.083	0	0	1039	4387	1	18	1.01	0.99	1.00	0.99	1.00
AL (c)	AL (t)	0.139	3	0	598	596	0	1	1.04	1.06	1.00	0.99	1.00
SD (c)	SD (t)	0.064	0	0	1154	6973	0	5	1.01	0.99	1.00	0.99	1.01

Table 3: SUMMARY STATISTICS ALAMDEDA AND SAN DIEGO CONTROL SAMPLES

|	All			$e(x) < \alpha$		$\alpha \leq e(x) \leq 1-\alpha$			$1-\alpha < e(x)$	
	AL	SD	[t-stat]	AL	SD	AL	SD	[t-stat]	AL	SD
	601	1154		38	2	547	782		16	370
female	0.95	0.85	[-7.3]	1.00	1.00	0.95	0.92	[-2.0]	0.94	0.70
age	35.2	33.8	[-32.6]	37.6	46.5	35.1	34.9	[-4.0]	34.1	31.4
hispanic	0.06	0.23	[11.0]	0.00	0.00	0.05	0.16	[6.6]	0.38	0.39
black	0.71	0.24	[-21.1]	1.00	1.00	0.71	0.34	[-14.4]	0.13	0.03
hsdiploma	0.61	0.59	[-0.8]	0.55	1.00	0.62	0.57	[-1.6]	0.56	0.62
1 child	0.41	0.43	[0.7]	0.08	0.00	0.44	0.42	[-0.5]	0.31	0.44
children under 6	0.29	0.13	[-7.8]	1.00	1.00	0.25	0.14	[-4.9]	0.06	0.09
earnings q-1	1.53	2.21	[3.5]	—	—	1.51	1.60	[0.4]	1.69	2.73
earnings q-2	1.57	2.57	[4.4]	—	—	1.43	1.82	[1.8]	2.90	3.20
earnings q-3	1.50	2.62	[5.1]	0.25	—	1.34	1.84	[2.5]	2.96	3.18
earnings q-4	1.95	2.72	[3.2]	0.77	—	1.87	2.09	[0.9]	2.91	3.25
earnings q-5	1.92	2.91	[3.2]	—	0.48	1.77	2.23	[1.4]	3.22	3.38
earnings q-6	1.75	2.76	[4.2]	—	1.05	1.64	2.00	[1.4]	2.51	3.37
earnings q-7	1.84	3.09	[4.7]	1.33	—	1.55	2.28	[3.0]	3.67	3.72
earnings q-8	1.59	2.93	[4.7]	—	—	1.28	1.96	[2.7]	3.65	3.63
earnings q-9	1.77	2.91	[4.3]	1.79	—	1.50	2.14	[2.8]	3.49	3.47
earnings q-10	1.90	2.97	[3.5]	1.88	—	1.59	2.29	[2.2]	3.96	3.50
pos earn q-1	0.86	0.71	[-7.3]	1.00	1.00	0.86	0.81	[-2.7]	0.38	0.52
pos earn q-2	0.86	0.71	[-7.5]	1.00	1.00	0.86	0.81	[-2.5]	0.50	0.51
pos earn q-3	0.86	0.70	[-8.1]	0.97	1.00	0.86	0.81	[-2.4]	0.44	0.45
pos earn q-4	0.83	0.71	[-5.7]	0.97	1.00	0.83	0.81	[-1.1]	0.44	0.51
pos earn q-5	0.86	0.70	[-8.1]	1.00	0.50	0.86	0.82	[-2.09]	0.44	0.45
pos earn q-6	0.85	0.69	[-8.2]	1.00	0.50	0.86	0.80	[-2.9]	0.31	0.46
pos earn q-7	0.87	0.71	[-8.2]	0.97	1.00	0.88	0.81	[-3.1]	0.31	0.49
pos earn q-8	0.88	0.70	[-9.3]	1.00	1.00	0.88	0.81	[-3.5]	0.38	0.45
pos earn q-9	0.88	0.71	[-8.8]	0.97	1.00	0.88	0.82	[-3.4]	0.38	0.47
pos earn q-10	0.87	0.72	[-8.0]	0.95	1.00	0.88	0.82	[-3.1]	0.38	0.50

propensity score histogram San Diego, San Diego versus Alameda Comparison
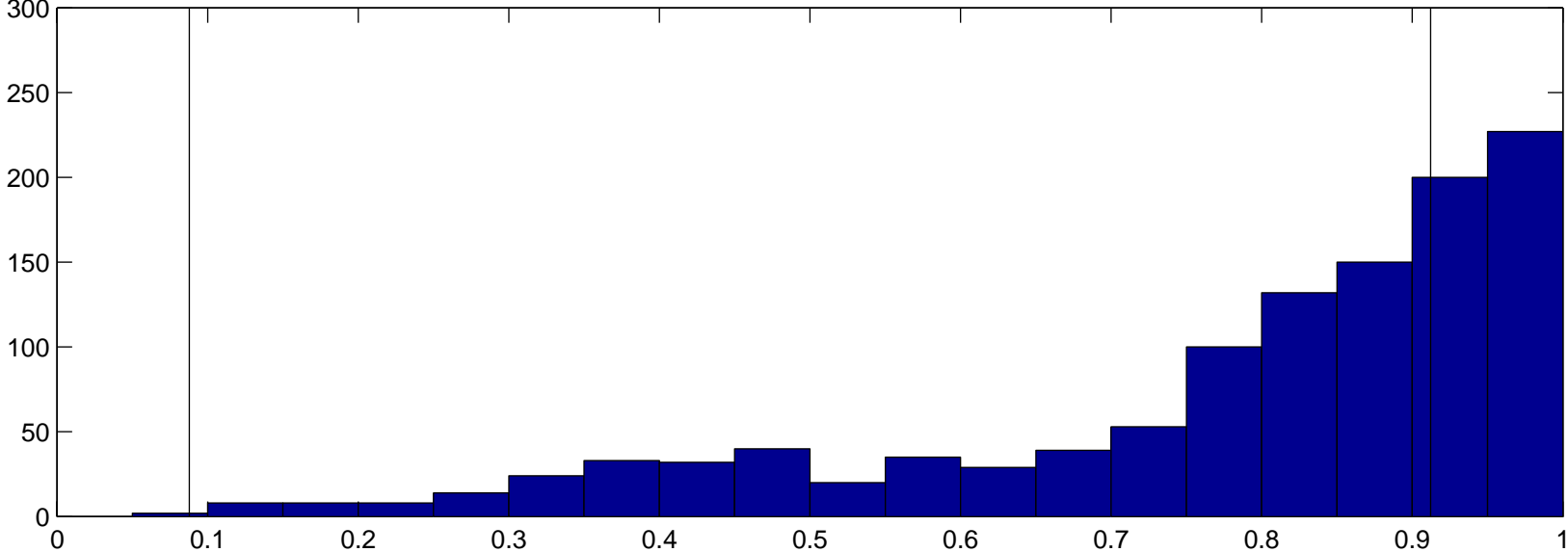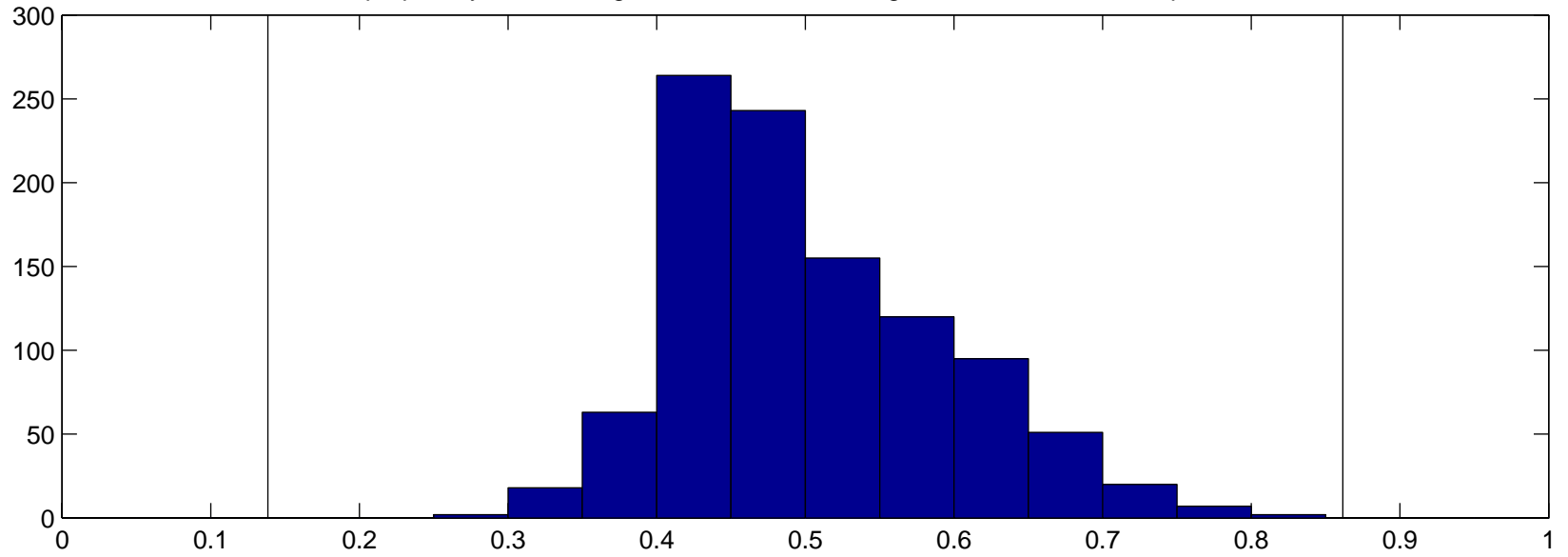
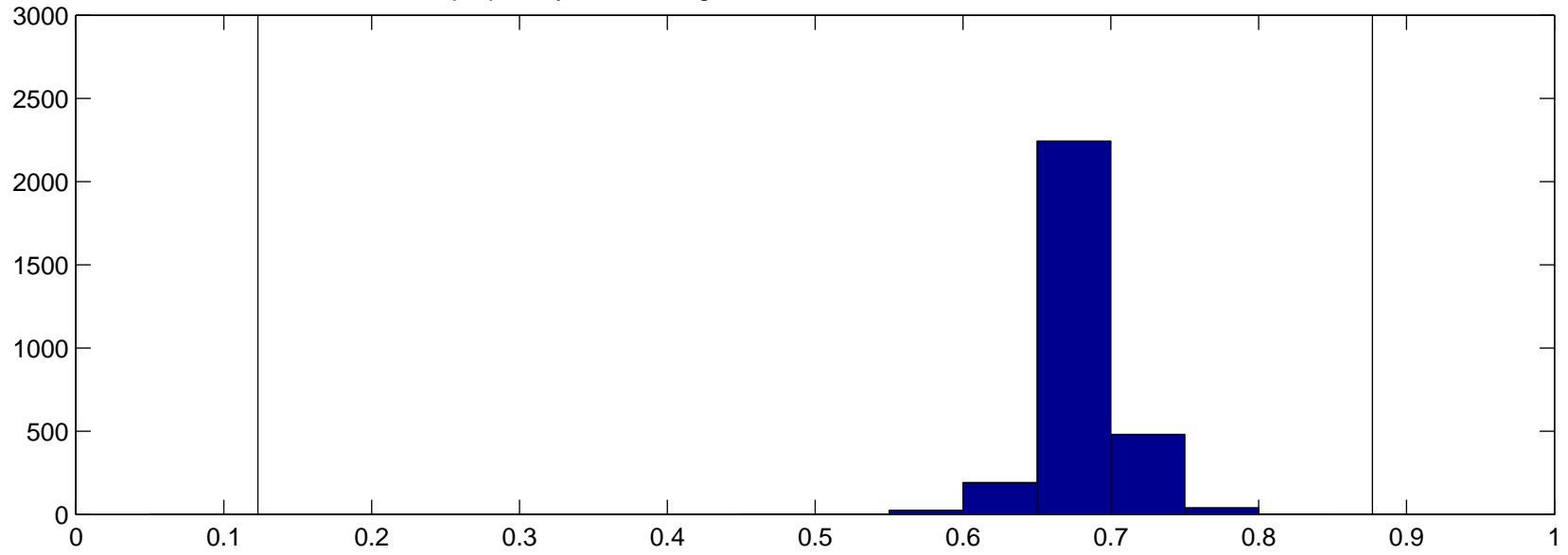propensity score histogram Alameda, San Diego versus Alameda Comparison
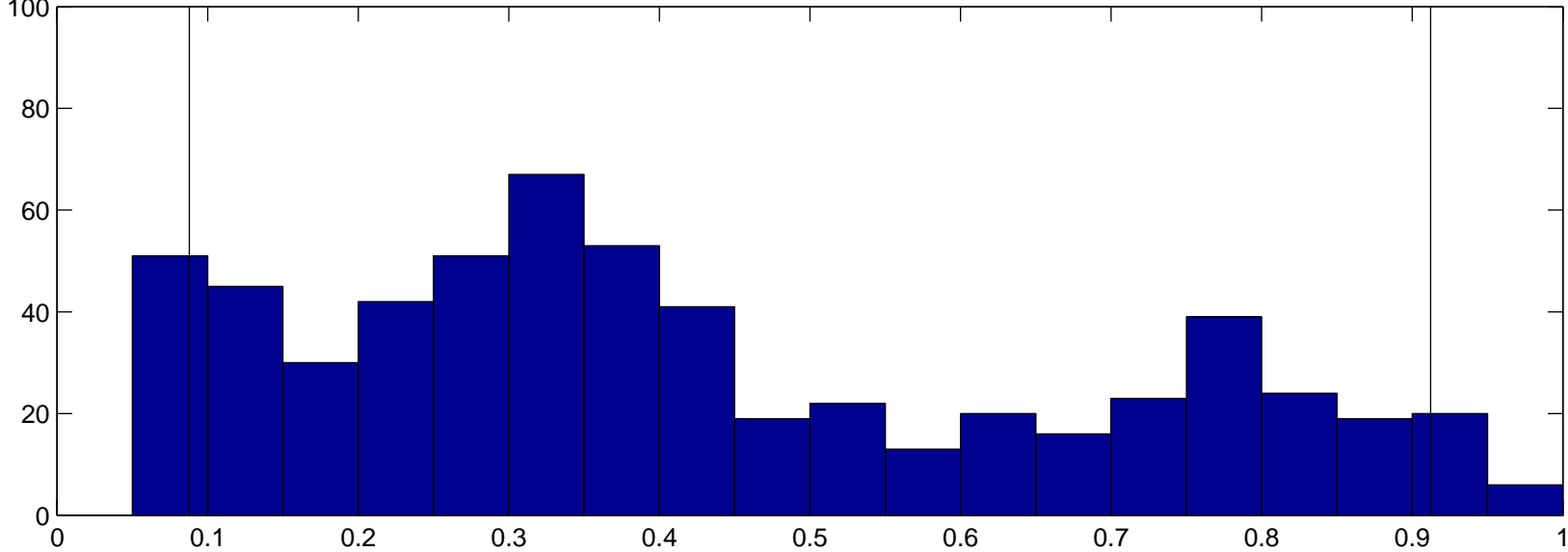
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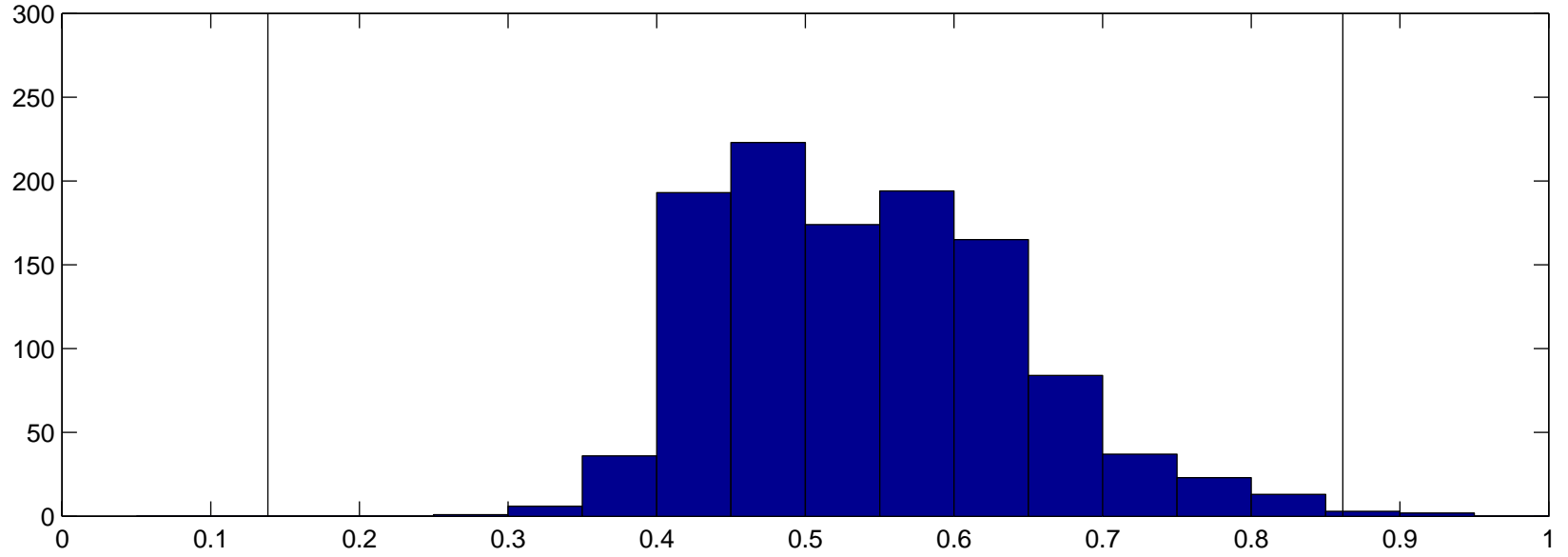
propensity score histogram Riverside, San Diego versus Riverside Comparison

propensity score histogram LA treated, LA controls versus treated

propensity score histogram LA controls, LA controls versus treated