On the Importance of Uniform Sharing Rules for Efficient Matching*

Deniz Dizdar and Benny Moldovanu†

October 28, 2014

Abstract

We study a two-sided matching model with transferable utility where agents are characterized by privately known, multi-dimensional attributes that jointly determine the surplus of each potential partnership. We ask the following question: for what divisions of surplus within matched pairs is it possible to design a mechanism that determines additional payments at the match formation stage and induces information revelation leading to an efficient (surplus-maximizing) matching? Our main result shows that the only robust rules compatible with efficient matching are those that divide realized surplus in a fixed proportion, independently of the attributes of the pair's members: each agent must expect to get the same fixed percentage of surplus in every conceivable match. A more permissive result is obtained for one-dimensional attributes and supermodular surplus functions.

Keywords: Matching, surplus division, premuneration values, interdependent values, multi-dimensional attributes.

1 Introduction

We study a two-sided one-to-one matching (or assignment) market with transferable utility and with a finite number of privately informed agents that need to be matched to form productive relationships. We call the two sides of the market "workers"

*A previous version of this paper was circulated under the title "Surplus Division and Efficient Matching."

†We wish to thank Dennis Gärtner, Philippe Jehiel, Claudio Mezzetti, Georg Nöldeke and Nora Szech for helpful discussions. Both authors are grateful for financial support from the German Science Foundation. Dizdar: Department of Economics, University of Montréal, deniz.dizdar@umontreal.ca. Moldovanu: Department of Economics, University of Bonn, mold@uni-bonn.de.

and "employers." Agents are characterized by multi-dimensional, privately known attributes that jointly determine the value/surplus created by each employer-worker pair. We take as primitives the agents' utilities from a match in the absence of additional payments - these objects were aptly called "premuneration values" by Mailath, Postlewaite and Samuelson (2012, 2013). These authors described how premuneration values are affected by the allocation of property rights: for instance, a standardized contract or "sharing rule" might specify various claims to shares of ex-post realized surplus in every formed partnership. We call the sum of employer and worker premuneration values the match surplus.[1]

We ask the following question: for what forms of premuneration values is it possible to design a mechanism that provides incentives for information revelation leading, for each realization of attributes in the economy, to an efficient (surplus-maximizing) matching? We consider standard mechanisms that include payments between agents or to/from a matchmaker at the match formation stage.

Our main result shows that in settings with multi-dimensional, complementary attributes, the only premuneration values compatible with efficient matching are, in essence, those that correspond to dividing surplus according to the same *fixed proportion* in every match.[2] Thus, to enable efficient match formation it is necessary and sufficient that all workers expect to get the same fixed percentage of surplus in every conceivable match, independently of the attributes of the pair's members, and the same thing must hold for employers! More flexibility is possible when attributes are one-dimensional and match surplus is supermodular. Efficient matching is then compatible with any division that leaves each partner with a fraction of the surplus that is also supermodular.

The equilibrium notion used throughout the paper is the *ex-post equilibrium*. This is a generalization of equilibrium in dominant strategies appropriate for settings with interdependent values, and it embodies a notion of no regret: chosen actions must be considered optimal even after the private information of others is revealed. Ex-post equilibrium is a belief-free notion, and our results do not depend in any way on the distribution of attributes in the population.[3]

[1]Thus, our model is an incomplete information, interdependent values version of the classical assignment models due to Shapley and Shubik (1971) and Crawford and Knoer (1981).

[2]There is some minor additional flexibility, in particular if premuneration values are not required to be independent of whether employers or workers are on the long side of the market. See Condition 1, Remark 1 and Theorem 1 for the precise statements.

[3]See also Bergemann and Morris (2005) for the tight connections between ex-post equilibria and

An interesting illustration for a fixed-proportion rule is offered by the German law governing the sharing of profit among a public sector employer and an employee arising from the employee's invention activity. The law differentiates between universities and all other public institutions.[4] Outside universities - where, presumably, the probability of an employee making a job-related discovery is either nil or very low - the law allows any *ex-ante* negotiated contract governing profit sharing (see §40-1 in *Bundesgesetzblatt III, 422-1*). In marked contrast, independently of circumstances, any university and any researcher working there must divide the profit from the researcher's invention according to *a fixed 30%-70% rule*, with the employee getting the 30% share (see §42-4). The rigidity of this "no-exception" rule is additionally underlined by an explicit mention that all feasible arrangements under §40-1 are not applicable within universities (see §42-5).

The occurrence of inflexible, fixed-proportion rules for sharing ex-post surplus - shares do not vary with attributes and are not the object of negotiation - is a recurrent theme in several interesting literatures that try to explain this somewhat puzzling phenomenon. For example, Newbery and Stiglitz (1979) and Allen (1985), among many others, noted that sharecropping contracts in many rural economies involve shares of around one half for landlord and tenant. This percentage division is observed in widely differing circumstances and has persisted in many places for a considerable length of time.[5]

Our study is at the intersection of several strands of the economic literature. We briefly review below some related papers from each of these strands, emphasizing both the existing relations to our work and the present novel aspects.

1. Matching: An overwhelming majority of studies on two-sided matching has assumed either complete information or *private values*, that is, models in which agents' preferences do not depend on signals privately available to others. In the Gale-Shapley (1962) private values model, one-sided serial dictatorship where women, say, sequentially choose partners according to their preferences leads to a Pareto-optimal matching. Difficulties occur when the stronger *stability* requirement is invoked: a standard result is that no ex-post stable matching can be implemented in dominant strategies if both sides of the market are privately informed (see Roth and Sotomayor, 1990).

"robust design."

[4]Practically all German universities are public.

[5]For example, Chao (1983) noted that a fixed 50-50 ratio was prevalent in China for more than 2000 years. The French and Italian words for "sharecropping" literally mean "50-50 split."

Chakraborty, Citanna and Ostrovsky (2010) showed that there may be no stable matching mechanism even in a one-sided private information model, if preferences on one side of the market (colleges, say) depend on information available to agents on the same side of the market.[6] Liu, Mailath, Postlewaite and Samuelson (2014) developed a notion of incomplete information stability for a matching that is already in place, in a Shapley-Shubik model with private information on one side of the market. They showed that the set of incomplete information stable outcomes is a superset of the set of complete information stable outcomes. They also gave sufficient conditions for incomplete information stable matchings to be efficient.

Hoppe, Moldovanu and Sela (2009) analyzed a two-sided matching model with a finite number of privately informed agents, characterized by complementary one-dimensional attributes.[7] In their model match surplus is divided in a fixed proportion, and they showed that efficient, assortative matching can arise as one of the Bayesian equilibria of a bilateral signaling game. This finding is consistent with the results of the present paper.

Mailath, Postlewaite and Samuelson (2012, 2013) focused on the role of premuneration values in a model where, before matching, agents undertake costly investments in their attributes. Under personalized pricing - that must finely depend on the attributes of the matched pairs - an equilibrium which entails efficient investment and matching always exists in large (continuum) markets, no matter how surplus is shared.[8] In contrast, when personalized pricing is not feasible because sellers can not observe buyers' attributes, premuneration values affect investment incentives, and, typically, equilibrium investments cannot be efficient.

2. Property Rights: A large literature, following Coase (1960), analyzes the effects of the ex-ante allocation of property rights on bargaining outcomes. The interplay between private information and ex-ante property rights in private value settings has been emphasized by Myerson-Satterthwaite (1983) and Cramton-Gibbons-

[6]Che, Kim and Kojima (2012) have shown that efficiency is not compatible with incentive compatibility in a one-sided assignment model in which agents' values over objects are allowed to depend on information of other agents. Inefficiency occurs there because of the assumed lack of monetary transfers.

[7]The complete information version has been popularized by Becker (1973): agents are completely ordered according to their marginal productivity, and efficient matching is assortative. The incomplete information version displays interdependent values.

[8]Thus, as in Cole, Mailath and Postlewaite (2001), market competition eliminates hold-up problems.

Klemperer (1987) in a buyer-seller framework and a partnership dissolution model, respectively.[9] In these papers, agents have one-dimensional types and a value maximizing allocation can be implemented via standard Clarke-Groves-Vickrey schemes. Whenever inefficiencies occur, these stem from the inability to design budget-balanced and individually rational transfers that sustain the value maximizing allocation.[10] Brusco, Lopomo, Robinson and Viswanathan (2007) and Gärtner and Schmutzler (2009) looked at mergers with interdependent values, a setting which is more related to the present study.[11]

In marked contrast to all the above papers, our present analysis completely abstracts from budget-balancedness and individual rationality. The fixed-proportion divisions are dictated here by the mere requirement of value maximization together with incentive compatibility.

3. Multi-dimensional Attributes and Mechanism Design: As mentioned above, we discard the prevalent assumption in most incomplete information studies whereby agents can be described by a single trait such as skill, technology, wealth, or education. This is often not tenable, as workers, say, have many diverse job-relevant characteristics, which are only partially correlated.[12] The present combination of multi-dimensional attributes, private information and interdependent values is usually detrimental to efficient implementation. In fact, Jehiel, Meyer-ter-Vehn, Moldovanu and Zame (2006) have shown that, generically, only trivial social choice functions - where the outcome does not depend on the agents' private information - can be ex-post implemented when values are interdependent and types are multi-dimensional. Jehiel and Moldovanu (2001) have shown that, generically, the efficient allocation

[9]Fieseler, Kittsteiner and Moldovanu (2003) offered a unified treatment that allows for interdependent values and encompasses both the above private values models and Akerlof's (1970) market for lemons.

[10]With several buyers and sellers, the Myerson-Satterthwaite model becomes a one-dimensional, linear incomplete information version of the Shapley-Shubik assignment game. Only in the limit, when the market gets very large, one can reconcile, via almost efficient double-auctions, incentives for information revelation with budget-balancedness and individual rationality.

[11]However, at most one match is formed in these models, and private information consists of, or can be reduced to, one-dimensional types.

[12]Tinbergen (1956) pioneered the analysis of labor markets where jobs and workers are described by several characteristics. The seminal studies of finite and continuum complete information assignment models with traders characterized by multi-dimensional attributes are Shapley and Shubik (1971) and Gretsky, Ostroy and Zame (1992). Dizdar (2012) generalized the matching cum ex-ante investment model due to Cole, Mailath and Postlewaite (2001) along this line. Like other recent, related literature (e.g. Chiappori, McCann and Nesheim 2010) his analysis used tools borrowed from optimal transportation theory. See Villani (2009) for an excellent textbook.

cannot be implemented even if the weaker Bayes-Nash equilibrium concept is used.

Our present insight can be reconciled with those general negative results by noting that the two-sided matching model is not generic. In particular, we assume here that match surplus has the same functional form for all pairs (as a function of the respective attributes), and that the match surplus of any pair depends neither on how agents outside that pair match, nor on what their attributes are. These features are natural for the matching model but are "non-generic."

The sufficiency of fixed-proportion sharing for incentive compatibility is related to the presence of individual utilities that admit a cardinal alignment with social welfare, via appropriate Clark-Groves-Vickrey type transfers. By proving necessity of fixed-proportion divisions, we identify a class of interesting settings for which efficient implementation is possible *only if* such cardinal alignment is possible (see Section 3 for the link with Jehiel, Meyer-ter-Vehn and Moldovanu's (2008) definition of a *cardinal potential*[13]). Our result is also reminiscent of Roberts' (1979) characterization of dominant strategy implementation in private values settings, but both technical assumptions and proof are very different here. The analysis of the special case with one-dimensional types and supermodular match surplus is based on an elegant characterization result due to Bergemann and Välimäki (2002), who generalized previous insights due to Jehiel and Moldovanu (2001) and Dasgupta and Maskin (2000).

The paper is organized as follows: in Section 2 we present the matching model. In Section 3 we state our results. Section 4 concludes. All proofs are in the Appendix.

2 The Matching Model

There are I employers and J workers. All agents have quasi-linear utilities. Each employer e_i ($i \in \mathcal{I} = \{1, ..., I\}$) privately knows his type $x_i \in X$, and each worker w_j ($j \in \mathcal{J} = \{1, ..., J\}$) privately knows his type $y_j \in Y$. X and Y denote the sets of agents' possible types.

For an employer of type x, the utility from a match with a worker of type y is $\gamma(x, y)v(x, y)$. The worker's utility from such a match is $(1 - \gamma(x, y))v(x, y)$. These *premuneration values* describe utilities in the absence of additional payments. Note that we take premuneration values as given and call the sum of employer and worker

[13] They presented several non-generic cases where ex-post implementation is possible. See also Bikchandani (2006) for other such cases, e.g. certain auction settings.

premuneration values the *match surplus* v.[14] We write premuneration values as fractions of their sum to emphasize their dependence on how the gains from partnership formation are divided by pre-specified allocations of property rights and sharing rules.

We assume that the match surplus v satisfies $v : X \times Y \to \mathbb{R}_+$ and that unmatched agents create zero surplus.

Let $\mathcal{M}$ denote the set of all possible one-to-one matchings of employers and workers. If $I \leq J$, these are the injective maps $m : \mathcal{I} \to \mathcal{J}$. A matching $m \in \mathcal{M}$ will be called *efficient* for a type profile $(x_1, ..., x_I, y_1, ..., y_J)$ if and only if it maximizes aggregate surplus

$$u_{m'}(x_1, ..., x_I, y_1, ..., y_J) = \sum_{i=1}^{I} v(x_i, y_{m'(i)})$$

among all $m' \in \mathcal{M}$. Analogous definitions apply for the case $J \leq I$. Efficient matchings are the solutions of a finite linear program (see Shapley and Shubik 1971). We also introduce the notation $v_m^{e_i}$ and $v_m^{w_j}$ for agents' premuneration values in the different matchings $m \in \mathcal{M}$: if e_i and w_j form a match in m, then $v_m^{e_i}(x_1, ..., x_I, y_1, ..., y_J) = \gamma(x_i, y_j)v(x_i, y_j)$ and $v_m^{w_j}(x_1, ..., x_I, y_1, ..., y_J) = (1 - \gamma(x_i, y_j))v(x_i, y_j)$. If e_i is unmatched in m we have $v_m^{e_i}(x_1, ..., x_I, y_1, ..., y_J) = 0$ (similarly, $v_m^{w_j}(x_1, ..., x_I, y_1, ..., y_J) = 0$ if w_j stays unmatched).

This matching model gives rise to a natural social choice setting with interdependent values. Every agent attaches a value to each possible alternative, i.e. matching of employers and workers. This value depends both on the agent's own type and on the type of the partner, but not on the private information of other agents. Moreover, this value does not depend on how other agents match. Thus, there are no allocative externalities, and there are no informational externalities across matched pairs.

2.1 Mechanisms

By the *Revelation Principle*, we may restrict attention to direct revelation mechanisms where truthful reporting by all agents forms an ex-post equilibrium. A direct revelation mechanism (mechanism hereafter) is given by functions $\Psi : X^I \times Y^J \to \mathcal{M}$, $t^{e_i} : X^I \times Y^J \to \mathbb{R}$ and $t^{w_j} : X^I \times Y^J \to \mathbb{R}$, for all $i \in \mathcal{I}$, $j \in \mathcal{J}$. Ψ selects a feasible

[14] This is in the spirit of Shapley and Shubik (1971), Crawford and Knoer (1981), Mailath, Postlewaite and Samuelson (2012, 2013) and Liu, Mailath, Postlewaite and Samuelson (2014).

matching as a function of reports, t^{e_i} is the monetary transfer to employer e_i, and t^{w_j} is the monetary transfer to worker w_j, as functions of reports.

Truth-telling is an ex-post equilibrium if for all employers e_i, for all workers w_j, and for all type profiles $p = (x_1, ..., x_I, y_1, ..., y_J)$, $p' = (x_1, ..., x_i', ..., x_I, y_1, ..., y_J)$ and $p'' = (x_1, ..., x_I, y_1, ..., y_j'', ..., y_J)$ it holds that

$$\begin{aligned} v^{e_i}_{\Psi(p)}(p) + t^{e_i}(p) &\geq v^{e_i}_{\Psi(p')}(p) + t^{e_i}(p') \\ v^{w_j}_{\Psi(p)}(p) + t^{w_j}(p) &\geq v^{w_j}_{\Psi(p'')}(p) + t^{w_j}(p''). \end{aligned}$$

3 The Main Results

For which forms of premuneration values, if any, is it possible to implement the value-maximizing social choice function in ex-post equilibrium? We start with a simple sufficient condition.

Condition 1 *There is a constant $\lambda_0 \in [0,1]$ and functions $g : X \to \mathbb{R}$ and $h : Y \to \mathbb{R}$ such that for all $x \in X$, $y \in Y$ it holds that $(\gamma v)(x,y) = \lambda_0 v(x,y) + g(x) + h(y)$. Moreover, h is constant if $I < J$, and g is constant if $I > J$.*

Lemma 1 *If γv satisfies Condition 1, then the efficient matching is implementable in ex-post equilibrium.*

The proof of Lemma 1 is straightforward. Under Condition 1, it is possible to align all agents' utilities with aggregate surplus, via appropriate Clark-Groves-Vickrey type transfers. When the part of the share that is proportional to match surplus is strictly positive for both sides of the market (i.e. $\lambda_0 \in (0,1)$), then a *strict* cardinal alignment is possible: in this case, aggregate surplus is a *cardinal potential* for the individual utilities (see Jehiel, Meyer-ter-Vehn and Moldovanu 2008).

Remark 1 *If we require that premuneration values be independent of whether employers or workers are on the short side of the market, then Condition 1 implies that γv is of the form $(\gamma v)(x,y) = \lambda_0 v(x,y) + c$, where $\lambda_0 \in [0,1]$ and c is a constant. In this case, premuneration values essentially correspond to dividing surplus in the same fixed proportion in all matches (an additional type- and match-independent transfer c is allowed).*

We now turn to our main results for cases with "complementary" types/attributes. In the remainder of the paper, we assume:

Condition 2 *X and Y are open connected subsets of Euclidean space $\mathbb{R}^n$ for some $n \in \mathbb{N}$, and premuneration values γv and $(1-\gamma)v$ are continuously differentiable.*

We invoke an assumption on v that is known as the *twist condition* in the mathematical literature on optimal transport (see Villani 2009). This is a multi-dimensional generalization of the well-known Spence-Mirrlees condition. While in optimal transport - where measures of agents are matched - the condition is invoked in order to ensure that the optimal transport, corresponding here to the efficient matching, is unique and deterministic, we use it for quite different, technical reasons (see the lemmas in the Appendix).

Condition 3 *i) For all $x \in X$, the continuous mapping from Y to $\mathbb{R}^n$ given by $y \mapsto (\nabla_X v)(x, y)$ is injective.*

ii) For all $y \in Y$, the continuous mapping from X to $\mathbb{R}^n$ given by $x \mapsto (\nabla_Y v)(x, y)$ is injective.

Match surplus functions that fulfill Condition 3 model many interesting complementarities between multi-dimensional types of workers and employers. In particular, v is not additively separable with respect to x and y, so that the precise allocation of match partners really matters for efficiency.[15] As a simple example consider the bilinear match surplus: $v(x, y) = x \cdot y$, where $\cdot$ denotes the standard inner product on $\mathbb{R}^n$. Then $(\nabla_X v)(x, y) = y$ and $(\nabla_Y v)(x, y) = x$, and Condition 3 is satisfied.

We can now state the central results concerning the necessity and sufficiency of fixed-proportion divisions:

Theorem 1 *Let $n \geq 2$, $I, J \geq 2$, and assume that Conditions 2 and 3 are satisfied. Then the following are equivalent:*

i) The efficient matching is implementable in ex-post equilibrium.

ii) Premuneration values satisfy Condition 1.

[15]Note for instance that if v is additively separable and $I = J$, then *all* matchings are efficient, and hence the efficient matching can trivially be implemented, no matter what γ is. This stands in sharp contrast to the result of Theorem 1.

Corollary 1 *The only premuneration values for which the efficient matching can be implemented irrespective of whether employers or workers are on the short side of the market are of the form $(\gamma v)(x, y) = \lambda_0 v(x, y) + c$, where $\lambda_0 \in [0, 1]$ and c is a constant.*

The heart of our proof is concerned with situations with two agents on each side (and hence with two feasible matchings), and it exploits the implications of incentive compatibility on the part of employers for varying worker type profiles. Condition 3 ensures that the subset of types for which both feasible matchings are efficient is a well-behaved manifold. As mentioned earlier, our result is reminiscent of Roberts' (1979) Theorem that shows (under some relatively strong technical conditions) that any dominant-strategy implementable social choice function must maximize a weighted sum of individual utilities plus some alternative-specific constants. Both present assumptions and proof are quite different from Roberts'.[16]

Remark 2 *Mezzetti (2004) has shown that efficiency is always (that is, in our context, for any given γ) attainable with two-stage "generalized Groves" mechanisms where a final allocation is chosen at stage one, and where, subsequently, monetary transfers that depend on the realized ex-post utilities of all agents at that allocation are executed at stage two.*[17] *In particular, such mechanisms would require ex-post transfers across all existing partnerships, contingent on the previously realized surplus in each of these pairs. We think that using ex-post information (whether reported or verifiable) to this extent is somewhat unrealistic in the present environment. For example, group manipulations by partners should be an issue for any mechanism that imposes ex-post transfers across pairs. In our model, there are no contingent payments between pairs or to/from a potential matchmaker after partnerships have formed.*

Our second main result deals with the special case where agents' attributes are one-dimensional. If $n = 1$, then Condition 3 implies that $y \mapsto (\partial_x v)(x, y)$ is either strictly increasing or strictly decreasing. Consequently, v either has strictly

[16] Our main technical result is derived by varying a social choice setting with only two alternatives (Roberts studied a single setting with at least three alternatives), surplus may take here general functional forms, and type spaces are arbitrary connected open sets (Roberts has linear utilities and needs an unbounded type space).

[17] The generalized Groves mechanism has the problem that it does not provide *strict* incentives for truthful reporting of ex-post utilities.

increasing differences or strictly decreasing differences in (x, y).[18] That is, v is either strictly supermodular or strictly submodular. This is the classical one-dimensional assortative/anti-assortative framework à la Becker (1973). We treat here the supermodular case. The submodular one is analogous.

In the one-dimensional supermodular case we find that the class of premuneration values that is compatible with efficient matching is strictly larger than the class defined by Condition 1.

Theorem 2 *Let $n = 1$, $I, J \geq 2$ and assume that Condition 2 holds and that v is strictly supermodular. Then, the efficient matching is implementable in ex-post equilibrium if and only if both γv and $(1-\gamma)v$ are supermodular.*

We derive Theorem 2 by applying a characterization result due to Bergemann and Välimäki (2002). These authors have provided a necessary as well as a set of sufficient conditions for efficient ex-post implementation for one-dimensional types. The logic of our proof is as follows. We first verify that monotonicity in the sense of Definition 4 of Bergemann and Välimäki is satisfied for strictly supermodular match surplus. This is the first part of their set of sufficient conditions (Proposition 3). Then, we show that their necessary condition (Proposition 1) implies that γv and $(1-\gamma)v$ must be supermodular. Finally, we show that the second part of the sufficient conditions is satisfied as well if γv and $(1-\gamma)v$ are supermodular.

4 Conclusion

We have introduced a novel two-sided matching model with a finite number of agents, two-sided incomplete information, interdependent values, and multi-dimensional attributes. We have shown that fixed-proportion sharing rules are the only ones conducive for efficiency in this setting. While our present result is agnostic about the preferred proportion, augmenting our model with, say, a particular ex-ante investment game will introduce new, additional forces that can be used to differentiate between various constant sharing rules.

[18] See also Topkis (1998).

5 Appendix

Proof of Lemma 1. Consider the case $I \leq J$. We make use of the "taxation principle" for ex-post implementation. For employer e_i, and matching $m \in \mathcal{M}$ define $t_m^{e_i}(x_{-i}, y_1, ..., y_J) := \lambda_0 \sum_{l \neq i} v(x_l, y_{m(l)}) - h(y_{m(i)})$. Then, $(\gamma v)(x_i, y_{m(i)}) + t_m^{e_i}(x_{-i}, y_1, ..., y_J) = \lambda_0 \sum_{l=1}^{I} v(x_l, y_{m(l)}) + g(x_i)$, so that it is optimal for e_i to select a matching that maximizes aggregate welfare. Note that strict incentives for truth-telling can be provided only if $\lambda_0 > 0$. For worker w_j, define

$$\begin{aligned} t_m^{w_j}(x_1, ..., x_I, y_{-j}) := \quad & (1-\lambda_0) \sum_{k \in m(\mathcal{I}), k \neq j} v(x_{m^{-1}(k)}, y_k) \\ & + g(x_{m^{-1}(j)}) \mathbf{1}_{j \in m(\mathcal{I})} - h(y_j) \mathbf{1}_{j \notin m(\mathcal{I})}. \end{aligned}$$

Here, $\mathbf{1}_{j \in m(\mathcal{I})} = 1$ if $j \in m(\mathcal{I})$, and $\mathbf{1}_{j \in m(\mathcal{I})} = 0$ otherwise. Note that if $I = J$, then $j \in m(\mathcal{I})$ for all possible matchings m, so that the final (y_j-dependent) term always vanishes. If $I < J$, then h is constant by assumption, and the transfer does not depend on y_j. It follows that if w_j is matched in m, his utility is $((1-\gamma)v)(x_{m^{-1}(j)}, y_j) + t_m^{w_j}(x_1, ..., x_I, y_{-j}) = (1-\lambda_0) \sum_{k \in m(\mathcal{I})} v(x_{m^{-1}(k)}, y_k) - h(y_j)$. Otherwise, his utility is just $t_m^{w_j}(x_1, ..., x_I, y_{-j}) = (1-\lambda_0) \sum_{k \in m(\mathcal{I})} v(x_{m^{-1}(k)}, y_k) - h(y_j)$. Hence, it is optimal for w_j to select a matching that maximizes aggregate welfare (strict incentives for truth-telling can be provided only if $\lambda_0 < 1$). This proves the claim for $I \leq J$. The proof for the case $I \geq J$ is completely analogous. ■

We prepare the proof of Theorem 1 by a sequence of lemmas. The key step is Lemma 5 below. It will be very useful to introduce a cross-difference (two-cycle) linear operator F, which acts on functions $f : X \times Y \to \mathbb{R}$. The operator F_f has arguments $x^1 \in X^1 = X$, $x^2 \in X^2 = X$, $y^1 \in Y^1 = Y$ and $y^2 \in Y^2 = Y$, and it is defined as follows:[19]

$$F_f(x^1, x^2, y^1, y^2) := f(x^1, y^1) + f(x^2, y^2) - f(x^1, y^2) - f(x^2, y^1).$$

We also define the sets

$$A := \{(x^1, x^2, y^1, y^2) \in X \times X \times Y \times Y | F_v(x^1, x^2, y^1, y^2) = 0\},$$

[19] We choose superscripts here because x_1 is already reserved for the type of employer e_1, and so on.

and

$$A_0 := \{(x^1, x^2, y^1, y^2) \in A | \nabla F_v(x^1, x^2, y^1, y^2) \neq 0\},$$

where

$$\nabla F_v(x^1, x^2, y^1, y^2) = (\nabla_{X^1} F_v, \nabla_{X^2} F_v, \nabla_{Y^1} F_v, \nabla_{Y^2} F_v)(x^1, x^2, y^1, y^2).$$

Whenever $x_1 \neq x_2$ or $y_1 \neq y_2$, Condition 3 implies that $\nabla F_v(x_1, x_2, y_1, y_2) \neq 0$. This is repeatedly used below.

Lemma 2 *Let $n \in \mathbb{N}$, $I = J = 2$, and let Conditions 2 and 3 be satisfied. If the efficient matching is ex-post implementable, then the following implications hold for all (x_1, x_2, y_1, y_2):*

$$F_v(x_1, x_2, y_1, y_2) \geq (\leq)\, 0 \Rightarrow F_{\gamma v}(x_1, x_2, y_1, y_2) \geq (\leq)\, 0, \tag{1}$$

$$F_v(x_1, x_2, y_1, y_2) \geq (\leq)\, 0 \Rightarrow F_{(1-\gamma)v}(x_1, x_2, y_1, y_2) \geq (\leq)\, 0. \tag{2}$$

Proof of Lemma 2. There are only two alternative matchings, $m_1 = ((e_1, w_1), (e_2, w_2))$ and $m_2 = ((e_1, w_2), (e_2, w_1))$. Since the efficient matching is ex-post implementable, the taxation principle for ex-post implementation implies that there must be "transfer" functions $t^{e_1}_{m_1}(x_2, y_1, y_2)$ and $t^{e_1}_{m_2}(x_2, y_1, y_2)$ for employer e_1 such that

$$\begin{aligned} F_v(x_1, x_2, y_1, y_2) \quad &> \quad (<)\, 0 \Rightarrow \\ (\gamma v)(x_1, y_1) + t^{e_1}_{m_1}(x_2, y_1, y_2) \quad &\geq \quad (\leq)\, (\gamma v)(x_1, y_2) + t^{e_1}_{m_2}(x_2, y_1, y_2). \end{aligned} \tag{3}$$

For $y_1 \neq y_2$, we have $(\nabla_{X^1} F_v)(x_2, x_2, y_1, y_2) = (\nabla_X v)(x_2, y_1) - (\nabla_X v)(x_2, y_2) \neq 0$ by Condition 3. Hence, in every neighborhood of $x_1 = x_2$, there are x_1' and x_1'' such that $F_v(x_1', x_2, y_1, y_2) > 0$ and $F_v(x_1'', x_2, y_1, y_2) < 0$. Since γv is continuous, relation (3) pins down the difference of transfers as:

$$t^{e_1}_{m_1}(x_2, y_1, y_2) - t^{e_1}_{m_2}(x_2, y_1, y_2) = (\gamma v)(x_2, y_2) - (\gamma v)(x_2, y_1).$$

Plugging this back into (3) yields for all (x_1, x_2, y_1, y_2) with $y_1 \neq y_2$:

$$F_v(x_1, x_2, y_1, y_2) > (<)\, 0 \Rightarrow F_{\gamma v}(x_1, x_2, y_1, y_2) \geq (\leq)\, 0. \tag{4}$$

As $F_v(x_1, x_2, y, y) = F_{\gamma v}(x_1, x_2, y, y) = 0$, relation (4) holds for all (x_1, x_2, y_1, y_2). However, every neighborhood of any $(x_1, x_2, y_1, y_2) \in A$ contains both points at which F_v is strictly positive and points at which F_v is strictly negative. Whenever $x_1 \neq x_2$ or $y_1 \neq y_2$, this follows immediately from $\nabla F_v(x_1, x_2, y_1, y_2) \neq 0$. Otherwise, if $x_1 = x_2$ and $y_1 = y_2$, one may perturb x_2 by an arbitrarily small amount to some x_2' (staying in A since $y_1 = y_2$) and apply the argument to (x_1, x_2', y_1, y_2).

Using continuity of γv, (4) may thus be strengthened to (1). A completely analogous argument applies for worker w_1 and yields (2). ■

To prove Theorem 1, we only need local versions of (1) and (2) at profiles where the efficient matching changes. These are available for general $I, J \geq 2$:

Lemma 3 *Let $n \in \mathbb{N}$, $I, J \geq 2$ and let Conditions 2 and 3 be satisfied. If the efficient matching is ex-post implementable, then for all $(x_1, x_2, y_1, y_2) \in A$, there is an open neighborhood $U_{(x_1,x_2,y_1,y_2)} \subset X \times X \times Y \times Y$ of (x_1, x_2, y_1, y_2) such that for all $(x_1', x_2', y_1', y_2') \in U_{(x_1,x_2,y_1,y_2)}$:*

$$F_v(x_1', x_2', y_1', y_2') \geq (\leq)\, 0 \Rightarrow F_{\gamma v}(x_1', x_2', y_1', y_2') \geq (\leq)\, 0, \tag{5}$$

$$F_v(x_1', x_2', y_1', y_2') \geq (\leq)\, 0 \Rightarrow F_{(1-\gamma)v}(x_1', x_2', y_1', y_2') \geq (\leq)\, 0. \tag{6}$$

Proof of Lemma 3. Given $(x_1, x_2, y_1, y_2) \in A$, fix the types of all other employers and workers (x_i for $i \neq 1, 2$, y_j for $j \neq 1, 2$) such that there is an open neighborhood $U_{(x_1,x_2,y_1,y_2)}$ of (x_1, x_2, y_1, y_2) with the following property: for all $(x_1', x_2', y_1', y_2') \in U_{(x_1,x_2,y_1,y_2)}$, the efficient matching for the profile $(x_1', x_2', x_3, ..., x_I, y_1', y_2', y_3, ..., y_J)$ either matches e_1 to w_1 and e_2 to w_2, or e_1 to w_2 and e_2 to w_1 (depending on the sign of $F_v(x_1', x_2', y_1', y_2')$). From here on, the proof parallels the one of Lemma 2. ■

Lemma 3 has the immediate consequence that *on* A_0, the gradients of F_v, $F_{\gamma v}$ and $F_{(1-\gamma)v}$ must all point in the same direction:

Lemma 4 *Let $n \in \mathbb{N}$, $I, J \geq 2$ and let Conditions 2 and 3 be satisfied. If the efficient matching is ex-post implementable, then there is a unique function $\lambda : A_0 \to [0, 1]$ satisfying*

$$\nabla F_{\gamma v}(x_1, x_2, y_1, y_2) = \lambda(x_1, x_2, y_1, y_2) \nabla F_v(x_1, x_2, y_1, y_2) \tag{7}$$

for all $(x_1, x_2, y_1, y_2) \in A_0$.

Proof of Lemma 4. Since $\nabla F_v(x_1, x_2, y_1, y_2) \neq 0$ for all $(x_1, x_2, y_1, y_2) \in A_0$, (5) yields a unique $\lambda(x_1, x_2, y_1, y_2) \geq 0$ with

$$\nabla F_{\gamma v}(x_1, x_2, y_1, y_2) = \lambda(x_1, x_2, y_1, y_2)\nabla F_v(x_1, x_2, y_1, y_2).$$

Moreover, $\nabla F_{(1-\gamma)v}(x_1, x_2, y_1, y_2) = (1-\lambda(x_1, x_2, y_1, y_2))\nabla F_v(x_1, x_2, y_1, y_2)$ and (6) therefore implies $\lambda(x_1, x_2, y_1, y_2) \in [0, 1]$. ∎

The crucial step in the proof follows now. It shows that for $n \geq 2$ the function λ must be constant. This constant corresponds then to a particular fixed-proportion sharing rule.

Lemma 5 *Let $n \geq 2$, $I, J \geq 2$ and let Conditions 2 and 3 be satisfied. Then the function λ from Lemma 4 must be constant: there is a $\lambda_0 \in [0, 1]$ such that $\lambda \equiv \lambda_0$.*

Proof of Lemma 5. Let us spell out the equalities in (7):

$$\begin{aligned}
(\nabla_X\gamma v)(x_1, y_1) - (\nabla_X\gamma v)(x_1, y_2) &= \lambda(x_1, x_2, y_1, y_2)((\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2)) \\
(\nabla_X\gamma v)(x_2, y_2) - (\nabla_X\gamma v)(x_2, y_1) &= \lambda(x_1, x_2, y_1, y_2)((\nabla_X v)(x_2, y_2) - (\nabla_X v)(x_2, y_1)) \\
(\nabla_Y\gamma v)(x_1, y_1) - (\nabla_Y\gamma v)(x_2, y_1) &= \lambda(x_1, x_2, y_1, y_2)((\nabla_Y v)(x_1, y_1) - (\nabla_Y v)(x_2, y_1)) \\
(\nabla_Y\gamma v)(x_2, y_2) - (\nabla_Y\gamma v)(x_1, y_2) &= \lambda(x_1, x_2, y_1, y_2)((\nabla_Y v)(x_2, y_2) - (\nabla_Y v)(x_1, y_2)).
\end{aligned} \tag{8}$$

Given any $(x_1, x_2, y_1, y_2) \in A_0$, one obtains the same system of equations at $(x_2, x_1, y_1, y_2) \in A_0$, albeit for $\lambda(x_2, x_1, y_1, y_2)$. Thus, the function λ is symmetric with respect to x_1 and x_2. Similarly, it is symmetric with respect to y_1 and y_2. Next, for given $x_1 \in X$ and $y_1 \neq y_2$, the vectors in the first equation of (8) (with $(\nabla_X v)(x_1, y_1) - (\nabla_X v)(x_1, y_2) \neq 0$ on the right hand side) do not depend on how (x_1, y_1, y_2) is completed by x_2 to yield a full profile that lies in A_0. Consequently, $\lambda(x_1, x_2, y_1, y_2) = \lambda(x_1, x_1, y_1, y_2)$ for all these possible choices.

We next show that for a given x_1, λ does in fact not depend on y_1 and y_2 as long as $y_1 \neq y_2$. To this end, start with any $x_1 \in X$ and $y_1 \neq y_2$. We will show that for all $y_2' \neq y_1$ it holds

$$\lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x_1, y_1, y_2'). \tag{9}$$

Then, by symmetry of λ, $\lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x_1, y_2', y_1)$, and repeating the argument will yield that λ is indeed independent of y_1 and y_2 as long as $y_1 \neq y_2$.

So, let us prove (9). Using the first equation of (8), we have:

$$
\begin{aligned}
&\lambda(x_1,x_1,y_1,y_2)((\nabla_X v)(x_1,y_1)-(\nabla_X v)(x_1,y_2))\\
&\quad = \;((\nabla_X \gamma v)(x_1,y_1)-(\nabla_X \gamma v)(x_1,y_2'))+((\nabla_X \gamma v)(x_1,y_2')-(\nabla_X \gamma v)(x_1,y_2))\\
&\quad = \;\lambda(x_1,x_1,y_1,y_2')((\nabla_X v)(x_1,y_1)-(\nabla_X v)(x_1,y_2'))\\
&\quad + \;\lambda(x_1,x_1,y_2',y_2)((\nabla_X v)(x_1,y_2')-(\nabla_X v)(x_1,y_2)).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&(\lambda(x_1,x_1,y_1,y_2')-\lambda(x_1,x_1,y_1,y_2))\,((\nabla_X v)(x_1,y_1)-(\nabla_X v)(x_1,y_2'))\\
&\quad + \;(\lambda(x_1,x_1,y_2',y_2)-\lambda(x_1,x_1,y_1,y_2))\,((\nabla_X v)(x_1,y_2')-(\nabla_X v)(x_1,y_2))\\
&\quad = \;0.
\end{aligned}
\tag{10}
$$

Two cases must now be distinguished.

Case 1: $(\nabla_X v)(x_1,y_1)-(\nabla_X v)(x_1,y_2')$ and $(\nabla_X v)(x_1,y_2')-(\nabla_X v)(x_1,y_2)$ are linearly independent. Then, it follows from (10) that $\lambda(x_1,x_1,y_1,y_2')=\lambda(x_1,x_1,y_1,y_2)$.

Case 2: $(\nabla_X v)(x_1,y_1)-(\nabla_X v)(x_1,y_2')$ and $(\nabla_X v)(x_1,y_2')-(\nabla_X v)(x_1,y_2)$ are linearly dependent. In this case, pick some $y_2''\in Y$ such that $(\nabla_X v)(x_1,y_1)-(\nabla_X v)(x_1,y_2'')$ and $(\nabla_X v)(x_1,y_2'')-(\nabla_X v)(x_1,y_2)$ are linearly independent. This is always possible since $(\nabla_X v)(x_1,\cdot)$ maps open neighborhoods of y_1 one-to-one into $\mathbb{R}^n$, and since for $n\geq 2$, there is no one-to-one continuous mapping from an open set in $\mathbb{R}^n$ to the real line $\mathbb{R}$.[20]

From Case 1, we obtain $\lambda(x_1,x_1,y_1,y_2'')=\lambda(x_1,x_1,y_1,y_2)$. Since $(\nabla_X v)(x_1,y_1)-(\nabla_X v)(x_1,y_2')$ and $(\nabla_X v)(x_1,y_2')-(\nabla_X v)(x_1,y_2'')$ are also linearly independent, we then get $\lambda(x_1,x_1,y_1,y_2')=\lambda(x_1,x_1,y_1,y_2'')$, and hence (9) follows.

The third equation of (8) may be now used in an analogous way to show that for a given y_1, $\lambda(x_1,x_2,y_1,y_1)$ does not depend on x_1 and x_2, as long as $x_1\neq x_2$.

The final ingredient is the following observation: for every $(x_1,x_1,y_1,y_2)\in A_0$, there is a $x_2\neq x_1$ with $(x_1,x_2,y_1,y_2)\in A_0$. Indeed, $(\nabla_{X^2}F_v)(x_1,x_1,y_1,y_2)\neq 0$, so that the set of x_2 for which $(x_1,x_2,y_1,y_2)\in A_0$ is given locally (in a neighborhood of $x_2=x_1$) by a differentiable manifold of dimension $n-1$. Since $n\geq 2$, this manifold must contain points other than x_1. A similar argument applies to $(x_1,x_2,y_1,y_1)\in A_0$.

[20] This is a special case of Brouwer's (1911) classical dimension preservation result: For $k<m$, there is no one-to-one, continuous function from a non-empty open set U of $\mathbb{R}^m$ into $\mathbb{R}^k$.

To conclude the proof, we show that λ is constant on $\{(x_1, x_2, y_1, y_2) \in A_0 | x_1 \neq x_2 \text{ and } y_1 \neq y_2\}$. This set is non-empty by the previous observation (and we have already seen that $\lambda(x_1, x_2, y_1, y_2) = \lambda(x_1, x_1, y_1, y_2)$ and $\lambda(x_1, x_2, y_1, y_2) = \lambda(x_1, x_2, y_1, y_1)$, so that λ is constant on all of A_0 then). Given any $(x_1, x_2, y_1, y_2), (x_1', x_2', y_1', y_2') \in A_0$ with $x_1 \neq x_2$, $y_1 \neq y_2$, $x_1' \neq x_2'$ and $y_1' \neq y_2'$, we have:

$$\begin{aligned}\lambda(x_1, x_2, y_1, y_2) &= \lambda(x_1, x_1, y_1, y_2) = \lambda(x_1, x_1, y_1', y_2') \\ &= \lambda(x_1, x_2'', y_1', y_2') = \lambda(x_1, x_2'', y_1', y_1') \\ &= \lambda(x_1', x_2', y_1', y_1') = \lambda(x_1', x_2', y_1', y_2'),\end{aligned}$$

where $x_2'' \neq x_1$ is any feasible profile completion for (x_1, y_1', y_2'). ■

We are now finally ready to prove Theorem 1.

Proof of Theorem 1. ii)⇒ i): See Lemma 1.

i)⇒ ii): By Lemma 5, there is a $\lambda_0 \in [0, 1]$ such that for all $x \in X$, $y_1, y_2 \in Y$ with $y_1 \neq y_2$ it holds (the profile may be completed to lie in A_0, e.g. by $x' = x$):

$$(\nabla_X \gamma v)(x, y_1) - (\nabla_X \gamma v)(x, y_2) = \lambda_0((\nabla_X v)(x, y_1) - (\nabla_X v)(x, y_2)).$$

Integrating along any path from x_2 to x_1 (X is open and connected in $\mathbb{R}^n$, hence path-connected) yields $F_{\gamma v}(x_1, x_2, y_1, y_2) = \lambda_0 F_v(x_1, x_2, y_1, y_2)$. Hence, by linearity of the operator F, we obtain that $F_{(\gamma-\lambda_0)v} \equiv 0$. A function of two variables has vanishing cross differences if and only if it is additively separable, so that we can write $(\gamma v)(x, y) = \lambda_0 v(x, y) + g(x) + h(y)$. This concludes the proof for the case where $I = J$.

It remains to prove that h must be constant if $I < J$ (the proof that g must be constant when $I > J$ is analogous). Given $y_1 \in Y$, Condition 3 implies that $(\nabla_Y v)(\cdot, y_1)$ vanishes at most in one point. Pick then any $x_1 \in X$ with $(\nabla_Y v)(x_1, y_1) \neq 0$. Set $y_2 = y_1$ and complete the type profile for ($i \neq 1$, $j \neq 1, 2$) such that, for an open neighborhood U of (y_1, y_1), the efficient matching changes only with respect to the partner of e_1 : either w_1 is matched to e_1 and w_2 remains unmatched, or w_2 is matched to e_1 and w_1 remains unmatched. For $(y_1', y_2') \in U$, it follows that $v(x_1, y_1') - v(x_1, y_2') \geq (\leq)\ 0$ implies $((1-\gamma)v)(x_1, y_1') - ((1-\gamma)v)(x_1, y_2') \geq (\leq)\ 0$.

Hence, there is a $\mu(x_1, y_1) \geq 0$ such that

$$(1 - \lambda_0)(\nabla_Y v)(x_1, y_1) - (\nabla_Y h)(y_1) = \mu(x_1, y_1)(\nabla_Y v)(x_1, y_1).$$

In other words, $(\nabla_Y h)(y_1)$ and $(\nabla_Y v)(x_1, y_1)$ are linearly dependent. Finally, let x_1 vary and note that, by Condition 3, the image of $(\nabla_Y v)(\cdot, y_1)$ cannot be concentrated on a line (recall footnote 15). Thus, we obtain that $(\nabla_Y h)(y_1) = 0$. Since y_1 was arbitrary and Y is connected, it follows that the function h must constant. ■

Proof of Theorem 2. Let $I \leq J$ (the proof for $I \geq J$ is analogous). Consider some $i \in \mathcal{I}$ and a given, fixed type profile for all other agents $(x_{-i}, y_1, ..., y_J)$. Given any such type profile, we re-order the workers and employers other than i such that $x^{(1)} \geq ... \geq x^{(I-1)}$ and $y^{(1)} \geq ... \geq y^{(J)}$.

We now verify the monotonicity condition identified by Bergemann and Välimäki.[21] This requires that the set of types of agent i for which a particular social alternative is efficient forms an interval. Let then m_k, $k = 1, ..., I$ denote the matching that matches $x^{(l)}$ to $y^{(l)}$ for $l = 1, ..., k-1$, x_i to $y^{(k)}$ and $x^{(l)}$ to $y^{(l+1)}$ for $l = k, ..., I-1$. Then, for $k = 2, ..., I-1$ it holds that the set

$$\{x_i \in X | u_{m_k}(x_1, ..., x_I, y_1, .., y_J) \geq u_m(x_1, ..., x_I, y_1, .., y_J) \text{ ,} \forall m \in \mathcal{M}\}$$

is simply $[x^{(k)}, x^{(k-1)}]$. For $k = I$ the set is $(\inf X, x^{(I-1)}]$, and for $k = 1$ it is $[x^{(1)}, \sup X)$. Monotonicity for workers j is verified in the same way.

Next, the necessary condition of Bergemann and Välimäki, spelled out for our matching model, requires that at all "switching points" $x_i = x^{(k-1)}$ where the efficient allocation changes, it also holds that

$$\frac{\partial}{\partial x_i}((\gamma v)(x_i, y^{(k-1)}) - (\gamma v)(x_i, y^{(k)})) \geq 0.$$

Given x_i and $y' > y$ we can always complete these to a full type profile such that x_i is a change point at which the efficient match for x_i switches from y to y'. Hence $\frac{\partial}{\partial x}((\gamma v)(x, y') - (\gamma v)(x, y)) \geq 0$ for all x and $y' > y$. So, γv must have increasing differences, i.e. it is supermodular. Since $\frac{\partial}{\partial x_i}((\gamma v)(x_i, y^{(k-1)}) - (\gamma v)(x_i, y^{(k)})) \geq 0$ is

[21] We only verify it for type profiles for which all these inequalities are strict. When some types coincide, it is still straightforward to verify monotonicity but we do not spell out the more cumbersome case distinctions here.

satisfied for all $x_i \in X$ (not just at switching points!), the second part of the sufficient conditions of Bergemann and Välimäki is satisfied. The argument for workers (yielding supermodularity of $(1-\gamma)v$) is analogous. This completes the proof. ■

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