Other Regarding Preferences in Environments of Risk

Abhinash Borah*†

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Abstract

Other regarding preferences pose a challenge to existing theories of decision making under risk as they may violate the property of stochastic dominance. We introduce a parsimonious decision model (consisting of a set of axioms on behavior and an equivalent utility representation) of such preferences that accommodates violations of stochastic dominace. The key innovation that our theory introduces is a concern for counterfactuals in the decision maker's evaluation of others' outcomes. That is, in evaluating others' outcomes in any given event, the decision maker may care not just about what others actually get in that event (the *ex-post risk* faced by others conditional on that event), but also about what they could have got had other possible events realized instead (the *ex-ante risk* that the others were initially faced with). In our representation, we separate a self regarding component of the decision maker's preferences (her tastes) from an other regarding component (her values). The self-regarding component is represented by a 'vonNeumann-Morgenstern utility function'. The other regarding component is a weighted average of the decision maker's evaluation of the ex-post and ex-ante risk faced by others, where the weights are subjective. We provide two alternative representations corresponding to two different approaches that the decision maker may take in evaluating the risk faced by others. In the first, the decision maker's preferences over lotteries faced by others is linear in probabilities, and represented by a 'vonNeumann-Morgenstern utility function', whereas in the second, her preferences may be non-linear in probabilities and represented by a 'biseparable utility function'.

*abhinash@sas.upenn.edu, Department of Economics, University of Pennsylvania.

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1 Introduction

A decision maker has other regarding preferences if her choices are influenced by a concern not just about her own outcomes, but others' outcomes as well. Over the last couple of decades, economists have collected an impressive body of experimental evidence that strongly suggests that such concerns matter for many decision makers. [1] At the same time they have shown that introducing such concerns into economic models produce novel insights that are of qualitative and quantitative significance. For instance, economists have appealed to other regarding preferences to deepen their understanding of such issues as Ricardian equivalence (Andreoni, 1989), the equity premium puzzle (Abel, 1990), the difference in redistribution policy between United States and Western Europe (Alesina and Angeletos, 2005), amongst others. This body of work suggests that other regarding preferences are an important phenomenon whose theoretical properties deserves thorough study.

In this paper we introduce a decision model, consisting of a set of axioms on behavior and an equivalent 'utility representation', of other regarding preferences in environments of risk. In order to understand the motivation behind proposing a new representation, consider the following example that has been taken from recent experimental work.

Example 1. Consider a decision maker who has to choose between allocating 20 euros either to herself or to some other person. Faced with this decision problem most decision makers, even altruistic ones, would perhaps prefer to keep the money. Let us assume that this is the case. [2] Now consider introducing risk in the environment. In particular, the decision maker is given the option of assigning some probability λ to the other person getting the 20 euros, while retaining the complimentary probability $1-\lambda$ of getting the money herself. Will the decision maker choose λ equal to 0, or will she choose a positive value of λ? Recent experiments conducted by Krawczyk and Le Lec (2008) provide us with an answer to the question. They report that faced with such a choice problem, a non-trivial number of their subjects (about 30%) choose to assign positive probability to the other person getting the 20 euros. On average, these subjects were willing to give a probability of about 0.09 to the other person getting the money.

Models of decision making under risk that economists typically use (for instance, Expected Utility, Rank Dependent Utility, 'Betweenness' based theories, Weighted Utility The-

[1]For a recent survey of the experimental evidence see Cooper and Kagel (2009).

[2]The argument that we make below does not depend on this assumption.

ory, Generalized Expected Utility, Generalized Prospect Theory) cannot explain the choices in the example. The reason they can not is because the choices in the example violate a basic property that all these theories share – *stochastic dominance*. Stochastic dominance requires that if a decision maker prefers one outcome over another, then replacing the inferior outcome with the superior outcome in a lottery (keeping everything else the same) should make her better off. Formally, suppose x and y are two outcomes from some underlying space of outcomes, and l is a lottery over that space. Stochastic dominance requires that if the decision maker prefers outcome x to outcome y, then she must prefer the (compound) lottery that gives outcome x with some positive probability λ and lottery l with complimentary probability $1 - \lambda$ to the (compound) lottery that gives y with probability λ and l with probability $1 - \lambda$. For instance, in Example 1 if a decision maker prefers the outcome that she, rather than the other person, gets the 20 euros then stochastic dominance demands that when she is given the option of assigning some probability λ to the other person getting the money, she should choose λ equal to 0. [3]

Stochastic dominance is central to how economists think about modeling environments featuring risk. In such environments, one of the goals that economists have considered analytically and normatively desirable is the separation of a decision maker's tastes (how desirable does she find an outcome) from her beliefs (how likely does she consider an event). Stochastic dominance, by requiring that in any given event, only realized outcomes in that event matter for the decision maker, is necessary to engender that separation.[4] Conversely, decision makers, like the ones in the example above, whose preferences violate stochastic dominance, may, when assessing their well being in any given event, care about things other than realized outcomes in that event. What are these other things that the decision makers of our example care about?

In order to answer the last question, consider a decision theorist who approaches the decision maker of Example 1, and points out to her the violation of stochastic dominance. How would this decision maker respond? In our opinion, she may defend her choice as follows: "I understand the logic of what you are saying. However, what you fail to see is

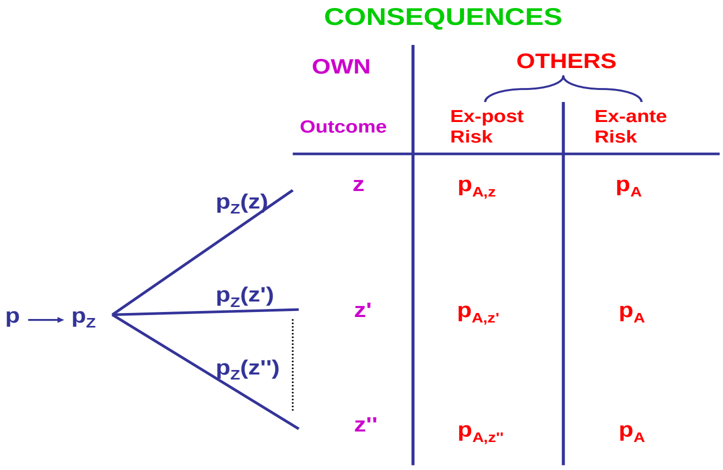
[3] Alternatively, if she prefers that the other person get the 20 euros, then stochastic dominance requires that she should choose $\lambda = 1$.

[4] In that sense, stochastic dominance is a much more fundamental assumption than the famous *Independence* condition of Expected Utility Theory. Violation of stochastic dominance implies a violation of Independence, but not vice versa. For instance, Allais' famous example violates Independence but not stochastic dominance. Models like Machina's Generalized Expected Utility and Quiggin's Rank Dependent Utility, for instance, have given up the Independence condition, but retained stochastic dominance. These models can separate a decision maker's tastes or risk attitudes from her beliefs.

that the impact that the other person's outcomes has on my well being, in any event, is not limited to what he gets in that event only, but extends over to considerations about the *counterfactual*, that is, *what he could have got had other possible events realized instead.* That is the reason why I do not consider my well being in the event that I get the 20 euros in the certain environment to be the same as that in the risky environment. In the latter, given that I care about fairness, I am much better off because, even though the final allocation is unequal, I know that the other person at least had some chance of getting the money, whereas in the former, he had none." In other words, in any given event (in this case, the event that she gets the 20 euros), when this decision maker assesses the impact that the other person's outcomes has on her well being, she cares not just about the *ex-post* prospects that he (other person) is faced with conditional on being in this event, but also about the *ex-ante* prospects that he was faced with originally. Counterfactuals matter for this decision maker precisely because she cares about the ex-ante prospects faced by the other person, even when an event under consideration has ruled out some of the possibilities implied by it.

The experiments of Krawczyk and Le Lec (2008) lend support to the hypothesis that ex-ante and ex-post concerns about the prospects of others may constitute distinct and separate rationales of choice for a decision maker with other regarding preferences. In Example 1 the events of the decision maker getting the money and the other person getting the money are mutually exclusive. Krawczyk and Le Lec consider an alternative treatment in which this is not the case. Under this treatment if the decision maker chose to assign a probability λ to the other person getting the money, then, like before, the former would have a probability λ of getting the money and the latter $1 - \lambda$, but now two independent randomizing devices would be used to determine the outcomes. Krawczyk and Le Lec find that decision makers with other regarding concerns behave differently in the independent treatment than in the correlated one; in particular, in the former they choose higher values of λ than in the latter. It is worth noting that for the independent treatment, in either of the events in which the decision maker gets the 20 euros or nothing, the ex-post and ex-ante prospects facing the other person are the same. Hence, if behavior in this setting differs from the correlated setting, then it naturally follows that both a concern for the ex-ante prospects and ex-post prospects of the other person influence the decision maker's choices. The question then naturally arises: What is the relative importance of each of these concerns for the decision makers, and how do they interact with the concern that she has for her own outcomes? The decision model that we propose in this paper addresses this question from behavioral

Figure 1: DM's relevant set of consequences



primitives. We now provide a brief sketch of this model.

1.1 The Representation and its Interpretation

Assume that there are n individuals, denoted $1, \ldots, n$, about whose outcomes our decision maker (DM) may care. Denote the set of DM's outcomes by the set Z, individual i's outcomes by the set A_i, $i = 1, \ldots, n$, and let $A = \prod_{i=1}^{n} A_i$. Let p be a simple lottery on the allocation space $Z \times A$. Let p_Z and p_A denote the marginal probability measures of p on Z and A respectively. Further, let $p_{A,z}$ denote the conditional probability measure on A with respect to the event that DM gets some outcome $z \in Z$. Under our proposed representation, DM evaluates the lottery p by the function:

$$U(p) = \sum_{z \in Z} p_Z(z)[u(z) + \pi v_z(p_{A,z}) + (1-\pi)v_z(p_A)],$$

where $\pi \in [0, 1]$.

In order to understand the representation, refer to Figure 1 which lists down the relevant set of consequences for DM when she is faced with the lottery p. Consider the event in which DM gets some outcome z in the support of p_Z. In this event three things constitute her relevant set of consequences. First, she cares about her own outcome z. Second, given that she has other regarding concerns, and that her outcomes may be correlated with others' outcomes, she cares about what the others get conditional on her getting outcome z. This concern is specified by the conditional probability measure $p_{A,z}$ over others' outcomes, and we refer to it as the **ex-post risk** facing the others in the event that DM gets z. Finally, given that counterfactuals may matter to her, she cares about the overall prospects that the others were initially faced with. This concern is specified by the marginal probability measure p_A over others' outcomes, and we refer to it as the **ex-ante risk** faced by the others. Under our representation, there is a function u which provides DM's subjective evaluation of her own outcomes. Further, for any outcome z that she gets, there is a function v_z that provides her subjective evaluation of the ex-post and ex-ante risks faced by the others (Note that the domain of the function u is the set of *outcomes* that DM may get, whereas the domain of the function v_z is the set of *lotteries* over others' outcomes). Finally, there is a constant $\pi \in [0, 1]$ that measures the relative importance of the ex-post concern vis-a-vis the ex-ante concern. Accordingly, the term,

$$u(z) + \pi v_z(p_{A,z}) + (1 - \pi)v_z(p_A)$$

gives DM's composite payoffs in the event that she gets outcome z. Once we have accounted for all of her relevant consequences in each such event that is possible under p_Z, we do an 'expected utility' evaluation. That is, the payoffs in each of these events is linearly aggregated using the probability measure p_Z, to arrive at,

$$U(p) = \textstyle\sum_{z \in Z} p_Z(z)[u(z) + \pi v_z(p_{A,z}) + (1 - \pi)v_z(p_A)],$$

We would like to point out here that in one of our representations we allow for the possibility that the weight π may depend on the outcome that DM gets. In that case we have:

$$U(p) = \textstyle\sum_{z \in Z} p_Z(z)[u(z) + \pi_z v_z(p_{A,z}) + (1 - \pi_z)v_z(p_A)],$$

The family of functions v_z, $z \in Z$, are the basic building blocks of our theory. We make a few observations here about their key properties. First consider lotteries p and q in which DM gets some outcome z for sure, while the others are faced with a risky prospect. For such lotteries the ex-post and ex-ante risk faced by the others in the event that DM gets outcome z are the same, and our representation implies that

$$U(p) = u(z) + v_z(p_A), \text{ and } U(q) = u(z) + v_z(q_A).$$

That is,

$$U(p) \geq U(q) \text{ if and only if } v_z(p_A) \geq v_z(q_A).$$

Accordingly, the function v_z captures DM's subjective ranking over lotteries faced by the others when she is guaranteed outcome z. Let us denote this ranking by $\succcurlyeq_z$. We provide two alternative representations corresponding to two different sets of properties that $\succcurlyeq_z$ may satisfy. In the first we require $\succcurlyeq_z$ to satisfy the Independence condition, and the function v_z is linear in probabilities. In the second we allow v_z to be non-linear in probabilities. In this case, we require v_z to satisfy two properties. First, the function v_z is strictly increasing with respect to stochastic dominance; that is, if a lottery p_A over the others' outcomes stochastically dominates another such lottery q_A (with respect to $\succcurlyeq_z$), then v_z assigns a higher value to the former lottery than the latter. Second, v_z satisfies a weak separability property called biseparability, which requires that v_z takes a 'generalized expected utility' form on lotteries whose support have cardinality equal to 2. That is, let $\mathbf{a}$, $\mathbf{a}' \in A$ be outcomes for the others such that $\mathbf{a} \succcurlyeq_z \mathbf{a}'$, and p_A be a lottery that assigns probability λ to $\mathbf{a}$ and $1-\lambda$ to $\mathbf{a}'$. Then, there exists a probability weighting function φ_z (formally, a strictly increasing bijection from $[0,1]$ to $[0,1]$ satisfying $\varphi_z(0) = 0$, $\varphi_z(1) = 1$) such that

$$v_z(p_A) = \varphi_z(\lambda)v_z(\delta_{\mathbf{a}}) + (1 - \varphi_z(\lambda))v_z(\delta_{\mathbf{a}'}),$$

where $\delta_{\mathbf{a}}$ and $\delta_{\mathbf{a}'}$ denotes degenerate lotteries that give $\mathbf{a}$ and $\mathbf{a}'$ respectively with probability 1. The probability weighting function transforms the objective probabilities over others' outcomes into decision weights. These decision weights reflect the decision maker's attitude towards the chance or risk faced by the others.

A few brief comments about the interpretation of the objects that we characterize in the representation are due. We think of the function u which provides DM's subjective ranking of the outcomes that she may get (independent of any consideration of what the others get) as reflecting DM's *tastes*. On the other hand, the family of functions v_z are reflective of DM's *values* or *morals*. [5] Note that DM's value's are of a contingent rather than an absolute nature, that is, how she feels about what others get may depend on what she gets herself. This contingency in values makes our decision model flexible enough to incorporate both 'positive' emotions like altruism, fairness and sympathy and 'negative' ones like spite and envy. Finally, the weight π is a measure of the importance that the decision maker puts

[5] The distinction between tastes and values is motivated by the following quote from Ken Arrow in 'Social Choice and Individual Values': "In general, there will, then be a difference between the ordering of social states according to the direct consumption of the individual and the ordering when the individual adds his general standards of equity. We may refer to the former ordering as reflecting the tastes of the individual and the latter as reflecting his values. The distinction between the two is by no means clear cut...no sharp line can be drawn between tastes and values."

on the actual risk faced by the others relative to the potential risk that they could have faced. The larger π is, the smaller is the concern for counterfactuals in the decision maker's evaluation of other's outcomes.

1.2 A Brief Review of the Literature

The fact that 'rational' economic agents can have other regarding concerns has long been acknowledged by economists, starting with Adam Smith who delved deep into the subject in his *Theory of Moral Sentiments*. Despite this interest in other regarding preferences, the question of whether other regarding preferences pose any peculiar challenges to theories of decision making under risk or uncertainty has not been looked at in any great detail in the literature. For most parts, it has been assumed that theories of decision making under risk or uncertainty that work well in other areas are effective ways of modeling other regarding preferences as well. Harsanyi (1955) is an influential early contribution to the literature. In this paper Harsanyi postulated that an individual's social or moral preference that represents his moral value judgments about allocations in society satisfies the Independence axiom of Expected Utility Theory. Is this a reasonable axiom for social or moral preferences? The question has yielded a spirited debate that has involved many distinguished participants; amongst others, Strotz (1958, 1961), Diamond (1967), Keeney (1980), Broome (1982, 1984), Sen (1985), and Harsanyi (1975, 1978) himself.

One of the most articulate and well known critique of Harsanyi's axiom has been presented by Machina (1989), whose counter example has come to be known as *Machina's Mom* in the literature. The example goes as follows. A mom has a ticket to a movie that she could give either to her son or her daughter. She is indifferent between the daughter getting the ticket or the son getting the ticket, but in a violation of the independence axiom (as indeed stochastic dominance) she strictly prefers a coin flip over each of the sure outcomes. When viewed from the perspective of decision making under uncertainty, Machina's Mom presents a particularly interesting challenge. Mom's choices violate Savage's Event-wise Monotonicity axiom, which is the key axiom in deriving a decision maker's probabilistic beliefs over uncertain events from her choice behavior. It then begs the question: Are such agent's probabilistically sophisticated? Grant (1995) shows that this question can be answered in the affirmative.

Karni and Safra (2002) work in an an environment featuring risk in which the decision maker's preferences may violate stochastic dominance. They look at a problem of dividing one unit of an indivisible good amongst n possible claimants. The objects of choice in their set up are allocation mechanisms which are lotteries over the set of possible ex-post allocations (An ex-post allocation specifies who amongst the n individuals receives the single unit of the indivisible good). The decision maker in their set up has two primitive preference relations over the set of allocation mechanisms – one that represents her choice behavior, and another that represents her notion of fairness. The choice behavior of the decision maker is influenced both by a concern for fairness as well as selfish concerns. Their axiomatic structure allows them to derive the decision makers 'selfish preference relation'. In addition they provide a utility representation for both the fair and selfish preference relations, and the decision maker's choices are represented by a real-valued function defined on the component utilities. The critical axiom in their set up is the one which allows the fair preference relation to be convex in probabilities; that is, if the decision maker finds two allocation mechanisms equally fair, then she finds the allocation mechanism featuring a '$50 - 50$ randomization' of these mechanisms to be strictly fairer.

Maccheroni et al (2008) work in an environment featuring uncertainty, and they focus on decision makers who care about the relative position of their outcomes with respect to their peers' outcomes. They provide representation results in which the decision maker judges her ex-post outcomes based on their intrinsic value to her and how they compare with the distribution of her peers' outcomes. Their representation allows them to establish that behavior may differ significantly based on whether decision makers are more sensitive to social losses (a situation where they do worse then their peers) or social gains (a situation in which they do better then their peers). The preferences of the decision makers in their set up respects Event-wise Monotonicity.

2 The Decision Model

2.1 Preliminaries

We assume that our stylized society comprises of a decision maker (DM) and n other individuals, denoted $1, \ldots, n$. Denote the set of DM's outcomes by the set Z, individual i's outcomes by the set A_i, $i = 1, \ldots, n$, and let $A = \prod_{i=1}^{n} A_i$. We assume that A has at least three elements. We will refer to elements of the sets $Z \times A$ as allocations. We denote the set of simple probability measures (simple lotteries, or just, lotteries, for short) on the sets $Z \times A$, Z and A by Δ, Δ_Z and Δ_A respectively. We will denote elements of Δ by p, q etc., those of Δ_Z by p_Z, q_Z etc., those of Δ_A by p_A, q_A etc. We will often refer to elements of Δ as 'allocation lotteries'. We define convex combination of lotteries in any of these sets in the standard way. [6] For any p in Δ we will denote the marginal probability measures on Z and A by p_Z and p_A respectively; [7] further, we denote the conditional probability measure on A with respect to the event that DM gets $z \in Z$ by $p_{A,z}$. We refer to $p_{A,z}$ as the **ex-post risk** faced by the others when DM gets outcome z, and p_A as the **ex-ante risk** faced by the others. For any p in Δ we will refer to the set

$$\{(z,\, p_{A,z},\, p_A) : z \text{ is in the support of } p_Z\}.$$

as the collection of **risk profiles** under p, and for any particular z in the support of p_Z, we will refer to $(z,\, p_{A,z},\, p_A)$ as the **risk-profile at** z under p. A special class of allocation lotteries are those $p \in \Delta$ in which DM gets some outcome $z \in Z$ for sure, i.e., $p_Z(z) = 1$ for some $z \in Z$. We will represent such a lottery (with a slight abuse of notation) by $p = (z,$ $p_A)$ and refer to it as a **DM-degenerate lottery**.

2.2 Preference and Axioms

DM's preferences are given by a preference relation (weak order) $\succcurlyeq$ on the set,

$$\Delta^+ = \Delta \cup \Delta_Z.$$

[A1] Weak Order
$\succcurlyeq$ is complete and transitive.

We will denote the restriction of the preference relation to the sets Δ and Δ_Z respectively by $\succcurlyeq_\Delta$ and $\succcurlyeq_{\Delta_Z}$. The symmetric and asymmetric components of $\succcurlyeq$ are defined in the usual way and denoted by $\sim$ and $\succ$ respectively. For any $z \in Z$, we use the primitive preference relation $\succcurlyeq$ to define the preference relation $\succcurlyeq_z \subseteq \Delta_A \times \Delta_A$ as follows: for any p_A, $q_A \in \Delta_A$,

$$p_A \succcurlyeq_z q_A \text{ if } (z,\, p_A) \succcurlyeq (z,\, q_A).$$

The preference relation $\succcurlyeq_z$ tells us how DM ranks lotteries over the others' outcomes, when she gets the outcome z for sure. The family of preference relations $(\succcurlyeq_z)_{z\in Z}$ are the basic

[6] For instance, if $p_Z^1, \ldots, p_Z^K \in \Delta_Z$, and $\lambda^1, \ldots, \lambda^K$ are constants in $[0,1]$ that sum to 1, then $\sum_{k=1}^K \lambda^k p_Z^k$ denotes an element in Δ_Z that gives the outcome $z \in Z$ with probability $\sum_{k=1}^K \lambda^k p_Z^k(z)$.

[7] When we reference lotteries like $p_Z \in \Delta_Z$ and $p_A \in \Delta_A$, it will be clear from the context whether we refer to them in the sense of marginal distributions of a lottery $p \in \Delta$, or 'simply' as elements of Δ_Z and Δ_A.

building blocks of our decision model. For any allocation lottery $p \in \Delta$, our decision maker, in the event that she gets the outcome z in the support of p_Z, evaluates the ex-ante and ex-post risk faced by the others according to $\succcurlyeq_z$. Clearly, $\succcurlyeq_z$ is a weak order, and its symmetric and asymmetric components are defined in the usual way and denoted by $\sim_z$ and $\succ_z$ respectively. For any $p_A \in \Delta_A$, the indifference class of p_A under $\succcurlyeq_z$ will be denoted by

$$[p_A]_z = \{q_A \in \Delta_A : q_A \sim_z p_A\}$$

The following binary relations defined on Δ using indifference classes of $\succcurlyeq_z$, $z \in Z$, are a compact way of relating elements in Δ based on properties of their risk profile. Let p, $q \in \Delta$ be such that $p_Z(z)$, $q_Z(z) > 0$, $z \in Z$. Then we will write,

- $p =_z^{RP} q$ if $([p_{A,z}]_z, [p_A]_z) = ([q_{A,z}]_z, [q_A]_z)$
- $p =_{-z}^{RP} q$ if p_Z and q_Z have the same support, and $([p_{A,z'}]_{z'}, [p_A]_{z'}) = ([q_{A,z'}]_{z'}, [q_A]_{z'})$, for all $z' \neq z$ in the common support.

That is, $p =_z^{RP} q$ denotes that the ex-post and ex-ante risk at z under p belong respectively to the same indifference classes of $\succcurlyeq_z$ as the ex-post and ex-ante risk at z under q. From a preference perspective therefore the risk profile under p and q at z may be considered the same.

On the other hand, $p =_{-z}^{RP} q$ indicates that p_Z and q_Z have a common support, and for all outcomes z' other than z in the common support, the ex-post and ex-ante risk at z' under p belong respectively to the same indifference classes of $\succcurlyeq_{z'}$ as the ex-post and ex-ante risk at z under q. Accordingly, from a preference perspective, such p and q may be considered identical in the events that DM gets any outcome z' other than z in the common support. In addition, if it is the case that $p_Z = q_Z$ then, the the only place p and q may 'differ' is in the risk profiles at z. Accordingly, we will say that p, $q \in \Delta$ are **comparable** at z, if $p_Z = q_Z$ and $p =_{-z}^{RP} q$.

Definition 1. *l, $l' \in \Delta^+$ are comparable if either:*

1. *l, $l' \in \Delta_Z$, or*
2. *l, $l' \in \Delta$, and there exists $z \in Z$ such that l and l' are comparable at z.*

We apply the classical axioms of continuity and independence to elements in Δ^+ that are comparable.

[A2] Continuity

Let l, l', $l'' \in \Delta^+$ be such that l, l'' are comparable and $l \succ l' \succ l''$. Then there exists λ, λ' $\in (0,1)$ such that

$$\lambda l + (1-\lambda)l'' \succ l' \succ \lambda' l + (1-\lambda')l''$$

Further, for all p in Δ, there exists q_Z, $q'_Z \in \Delta_Z$, such that $q_Z \succcurlyeq p \succcurlyeq q'_Z$.

[A3] Comparable Independence

Let l_1, l_2, l'_1, l'_2 in Δ^+ be such that l_1, l_2 are comparable, as are l'_1, l'_2. Then, for all $\lambda \in (0,1]$,

$$[l_1 \succ l'_1,\, l_2 \sim l'_2] \Rightarrow \lambda l_1 + (1-\lambda)l_2 \succ \lambda l'_1 + (1-\lambda)l'_2.$$

Comparable Independence identifies the correct domain over which the implication of the classical Independence condition of Expected Utility theory holds. We now briefly highlight the content of this axiom. Comparable Independence implies that $\succcurlyeq_{\Delta_Z}$ (the restriction of $\succcurlyeq$ to Δ_Z) satisfies the classical Independence condition. That is, if p_Z, p'_Z and $p''_Z \in \Delta_Z$, and $p_Z \succ p'_Z$, then for any $\lambda \in (0,1]$,

$$\lambda p_Z + (1-\lambda)p''_Z \succ \lambda p'_Z + (1-\lambda)p''_Z.$$

Further, Comparable Independence also implies that if p, p' and $p'' \in \Delta$ are pairwise comparable at some $z \in Z$, and $p \succ p'$, then for any $\lambda \in (0,1]$,

$$\lambda p + (1-\lambda)p'' \succ \lambda p' + (1-\lambda)p''$$

In particular, since any two DM-degenerate lotteries, (z, q_A) and (z, q'_A), are comparable ay z, it follows that $\succcurlyeq_z$ satisfies the vN-M Independence condition as well. An implication of this is the following.

Lemma 1. *If p, $p' \in \Delta$ are comparable at z, then for any $\lambda \in [0,1]$, p, p' and $\lambda p + (1-\lambda)p'$ are pairwise comparable at z.*

Accordingly, if p, p' and $p'' \in \Delta$ are pairwise comparable at z then so are $\lambda p + (1-\lambda)p''$ and $\lambda p' + (1-\lambda)p''$. That is, the only place $\lambda p + (1-\lambda)p''$ and $\lambda p' + (1-\lambda)p''$ differ is in the event that the decision maker gets the outcome z. Further, the ex-post and ex-ante risks faced by the others at z under $\lambda p + (1-\lambda)p''$ are respectively a $\lambda{:}1-\lambda$ mixture of the ex-post and ex-ante risks at z under p and p''. A corresponding statement holds for the ex-post and ex-ante risks faced by others at z under $\lambda p' + (1-\lambda)p''$. Accordingly, $\lambda p + (1-\lambda)p''$ can be interpreted as a λ:$1-\lambda$ preference mixture of p, p'', and $\lambda p' + (1-\lambda)p''$ can be interpreted as a λ:$1-\lambda$ preference mixture of p', p''. Given this interpretation, Comparable Independence conveys the same message as the classical Independence condition.

[A4] Comparable Monotonicity

Suppose p, $q \in \Delta$ are comparable at $z \in Z$. If $p_{A,z} \succcurlyeq_z q_{A,z}$ and $p_A \succcurlyeq_z q_A$, then $p \succcurlyeq q$.

Comparable Monotonicity says that the decision maker should evaluate the ex-post and ex-ante risk facing the others in the event that she gets outcome $z \in Z$ by the same criterion. In particular, this criterion is given by the preference relation $\succcurlyeq_z$, which tells us how DM ranks lotteries faced by others when she is guaranteed outcome z.

We now formalize a way of deriving DM's ranking over risk profiles. Consider lotteries p, $q \in \Delta$ that are comparable at some $z \in Z$. As indicated above, the 'tie breaker' as far as her preference ranking over p and q goes are the respective risk profiles at z. Thus, if DM, say, prefers p over q, we are naturally lead to the conclusion that the risk profile (z, $p_{A,z}$, p_A) is revealed preferred to the risk profile (z, $q_{A,z}$, q_A). Further, suppose that there exists p', $q' \in \Delta$ that reveal a preference for the risk profile (z, $p'_{A,z}$, p'_A) over the risk profile (z, $q'_{A,z}$, q'_A) (in the above sense), and that the risk profiles at z under q and p' are identical up to indifference class, i.e., $q =_z^{RP} p'$. Then between the pair of choices (i.e., p preferred to q and p' preferred to q'), it seems natural to suggest on grounds of consistency that the risk profile (z, $p_{A,z}$, p_A) is revealed preferred to the risk profile (z, $q'_{A,z}$, q'_A) by our decision maker. There is no reason why the 'choice chain' or 'choice sequence' has to stop at two. More generally, we have the following definition.

Definition 2. *For any p, $q \in \Delta$, z in the support of p_Z, q_Z, the risk profile (z, $p_{A,z}$, p_A) is weakly revealed preferred (resp. strictly revealed preferred) to the risk profile (z, $q_{A,z}$, q_A) if there exists a finite sequence $(p^k, q^k)_{k=1}^K \subseteq \Delta \times \Delta$ satisfying,*

(i) $p^1 = p$, $q^K = q$,

(ii) p^k and q^k are comparable at $z \in Z$, for all $k = 1, \ldots, K$, and

(iii) $q^k =_z^{RP} p^{k+1}$, for all $k = 1, \ldots, K-1$,

such that $p^k \succcurlyeq q^k$ for all $k = 1, \ldots, K$ (resp., $p^k \succcurlyeq q^k$ for all $k = 1, \ldots, K$, with strict preference for some k).

Further, (z, $p_{A,z}$, p_A) is revealed indifferent to (z, $q_{A,z}$, q_A) if $p^k \sim q^k$, for all k.

The following axiom which is in the spirit of the Weak Axiom of Revealed Preference, can be thought of as a consistency axiom on DM's behavior. It says that if a particular sequence of choices reveals a preference for one risk profile over another, then it should not be the case that some other choice sequence reveals a contradictory implication. If this consistency requirement were not to be satisfied by our decision maker, there will be little sense in talking about DM's revealed preference relation over risk profiles.

[A5] Revealed Consistency

For any p, $q \in \Delta$ if the risk profile $(z,\, p_{A,z},\, p_A)$ is weakly revealed preferred to the risk profile $(z,\, q_{A,z},\, q_A)$, then $(z,\, q_{A,z},\, q_A)$ is not strictly revealed preferred to $(z,\, p_{A,z},\, p_A)$.

Our last axiom provides restrictions on DM's preferences between lotteries in Δ and those in Δ_Z. In particular, it requires that preferences across these two sets have to respect an event[8] wise dominance restriction.

[A6] Dominance

Suppose $p \in \Delta$ is such that for each z in the support of p_Z,

(a) the risk profile $(z,\, p_{A,z},\, p_A)$ is revealed indifferent to a risk profile $(z,\, q^z_A,\, q^z_A)$, for some $q^z_A \in \Delta_A$, and

(b) there exists $q^z_Z \in \Delta_Z$ such that $q^z_Z \succcurlyeq$ (resp. $\preccurlyeq$) $(z,\, q^z_A) \in \Delta$, holding with strict preference for some z. Then,

$$\textstyle\sum_{z\in Z} p_Z(z).q^z_Z \succ \text{ (resp. } \prec\text{) } p.$$

The Dominance Axiom incorporates two ideas. First, it says that a risk profile like $(z,\, q^z_A,\, q^z_A)$ which has the same ex-post and ex-ante risk should be evaluated the same as the DM-degenerate lottery $(z,\, q^z_A)$. Second, it requires that once, all the risk profiles have been equated to DM degenerate lotteries, dominance should hold in the usual sense. That is, once it is established (via DM-degenerate lotteries) that for each z in the support of p_Z, the risk profile $(z,\, p_{A,z},\, p_A)$ is dominated by a self lottery $q^z_Z \in \Delta_Z$, then it should be the case that the compound lottery $\sum_{z\in Z} p_Z(z).q^z_Z \in \Delta_Z$ is strictly preferred to p.

The following assumption on the structure of the family of preference relations $(\succcurlyeq_z)_{z\in Z}$ plays an important role in our analysis. In particular, it guarantees that we can identify DM's revealed preferences over risk profiles; if this condition were not to hold, then there does not exist allocation lotteries p, $q \in \Delta$ such that $p =^{RP}_{-z} q$ and $p \neq^{RP}_{z} q$.

[8] Event here refer to the event of DM getting a particular outcome.

[CV.1] Contingent Values

If $\succsim_z$ such that $\succ_z \neq \emptyset$, then there exists $\succsim_{z'} \neq \succsim_z$, with $\succ_{z'} \neq \emptyset$ that satisfies the following condition: For all $p_A \in \Delta_A$, there exists q_A, q'_A, $q''_A \in \Delta_A$ such that $p_A \sim_z q_A$, and
(a) $q_A \sim_{z'} q'_A \sim_{z'} q''_A$,
(b) $q'_A \succsim_z q_A \succsim_z q''_A$.
Further if p_A is not a maximal (resp. minimal) element of $\succsim_z$, then $q'_A \succ_z q_A$ (resp. $q_A \succ_z q''_A$).

3 Representation

3.1 Linear Other Regarding Preferences.

We can now state our first representation theorem.

Theorem 1. *Suppose [CV.1] holds. Then the following statements are equivalent:*

1. *Axioms [A1] - [A6] holds.*
2. *There exists a function $u : Z \to \mathbb{R}$, functions $v_z : \Delta_A \to \mathbb{R}$, $z \in Z$, and constants $\pi_z \in [0, 1]$, $z \in Z$, such that the function $U : \Delta^+ \to \mathbb{R}$ defined by*
   - $U(p_Z) = \sum_{z \in Z} p_Z(z)u(z)$, $\forall\ p_Z \in \Delta_Z$, *and*
   - $U(p) = \sum_{z \in Z} p_Z(z)[u(z) + \pi_z v_z(p_{A,z}) + (1 - \pi_z)v_z(p_A)]$, $\forall\ p \in \Delta$

   represents $\succsim$.

In addition, any triple $(\tilde{u}, (\tilde{v}_z)_{z\in Z}, (\tilde{\pi}_z)_{z\in Z})$ represents $\succsim$ in the above sense if and only if there exists constants $\alpha > 0$ and β such that $\tilde{u} = \alpha u + \beta$, $\tilde{v}_z = \alpha v_z$, for all $z \in Z$, and $\tilde{\pi}_z = \pi_z$ for all $z \in Z$ such that $\succ_z \neq \emptyset$.

Note that each function v_z is a vNM representation of the preference relation $\succsim_z$. The proof of the Theorem is available in the Appendix.

3.2 Linear Other Regarding Preferences with Independent Weights

In Theorem 1 the weights π_z are a function of the outcome z that DM receives. We now provide a representation where there is a unique weight for all z. It should be intuitively clear that to axiomatize this case we need to impose some form of symmetry condition on

DM's preferences. We now make precise this notion of symmetry. In the way of notation, if $p \in \Delta$ is such that the outcomes that DM gets are uncorrelated with the outcomes that others' get, then we shall denote it as a pair: $p = (p_Z,\, p_A)$.

Definition 3. *$p_A,\, p'_A \in \Delta_A$ is **equal gains** for z and z' in Z if for any $q_A \in [p_A]_{z'} \cap [p'_A]_z$, $q'_A \in [p_A]_z \cap [p'_A]_{z'}$,*

$$([z, 1/2; z', 1/2],\, q_A) \sim ([z, 1/2; z', 1/2],\, q'_A), \text{ [9]}$$

*$p,\, q \in \Delta$ are **symmetric** with respect to $z,\, z' \in Z$ if*
(a) $p_Z = q_Z$, with $p_Z(z) = p_Z(z') = \frac{1}{2}$, and $p_A = q_A$,
(b) $p_{A,z} = q_{A,z'}$, $p_{A,z'} = q_{A,z}$, and
(c) $p_{A,z}$, $p_{A,z'}$ is equal gains for z, z'.

The equal gains definition gives us a condition under which the 'subjective difference' between two lotteries, p_A and p'_A in Δ_A, is considered the same by the decision maker under both $\succcurlyeq_z$ and $\succcurlyeq_{z'}$. To see this, consider changing the allocation lottery from $([z, 1/2; z', 1/2]$, $q'_A)$ to $([z, 1/2; z', 1/2],\, q_A)$, and suppose that $p'_A \succ_z p_A$. The only thing that changes for the decision maker is in the other regarding component of her consequences. In particular, in the event that she gets z, the change (under the assumption that $p'_A \succ_z p_A$) makes her better off. Since the change leaves her indifferent overall, it must be that in the event that she gets z', the change makes her worse-off, and this 'negative change' must be of the same magnitude as the positive one. In other words, the improvement under $\succcurlyeq_z$ when the lottery facing the others is changed from q'_A to q_A (or equivalently, from p_A to p'_A) is of the same magnitude as the improvement under $\succcurlyeq_{z'}$ when the lottery facing the others is changed from q_A to q'_A (or equivalently, from p_A to p'_A).

The axiom that we need to ensure that the weights π_z that we derived in the representation result above are all the same requires that if two allocation lotteries $p,\, q \in \Delta$ are symmetric, then the decision maker is indifferent between them.

[A7] Symmetry

If $p,\, q \in \Delta$ are symmetric, then $p \sim q$.

We then have the following representation result:

Theorem 2. *Suppose [CV.1] holds. Then the following statements are equivalent:*

[9]Following standard notation, $[z,\, 1/2;\, z',\, 1/2] \in \Delta_Z$ denotes the lottery that gives DM the outcome z with probability $\frac{1}{2}$, and z' with probability $\frac{1}{2}$.

1. *Axioms [A1] - [A7] holds.*

2. *There exists a function* $u : Z \to \mathbb{R}$, *functions* $v_z : \Delta_A \to \mathbb{R}$, $z \in Z$, *and a constant* $\pi \in [0,\ 1]$ *such that the function* $U : \Delta^+ \to \mathbb{R}$ *defined by*

   - $U(p_Z) = \sum_{z \in Z} p_Z(z)u(z)$, $\forall\ p_Z \in \Delta_Z$, *and*
   - $U(p) = \sum_{z \in Z} p_Z(z)[u(z) + \pi v_z(p_{A,z}) + (1-\pi)v_z(p_A)]$, $\forall\ p \in \Delta$

   represents $\succcurlyeq$.

In addition, any triple $(\tilde{u},\ (\tilde{v}_z)_{z \in Z},\ \tilde{\pi})$ *represents* $\succcurlyeq$ *in the above sense if and only if there exists constants* $\alpha > 0$ *and* β *such that* $\tilde{u} = \alpha u + \beta$, $\tilde{v}_z = \alpha v_z$, *for all* $z \in Z$, *and* $\tilde{\pi} = \pi$, *whenever there exists* $z \in Z$ *such that* $\succ_z \neq \emptyset$.

The proof is available in the appendix.

4 Non-linear other regarding preferences

In this section we introduce our most general representation result. So far in the analysis we have assumed that the preference relations $\succcurlyeq_z$ satisfies the vN-M Independence condition. This condition implies that the decision maker's preferences for the risk faced by others, when she is guaranteed some outcome z, is linear in probabilities. Here we want to allow for the possibility that these preferences may be non-linear in probabilities. Our motivation behind doing this comes from existing models of decision making under risk (for example, Rank Dependent Utility and Generalized Prospect Theory) which emphasize the distinction that decision makers may make between raw probabilities and subjective decision weights. For instance, in this literature it has been highlighted that a decision maker may overweight small chances of receiving a 'good outcome', and underweight larger chances. Such subjective probability weighting has been used to explain phenomenon like the Allais Paradox. We think that subjective weighting of probabilities may play a role when a decision maker with other regarding preferences evaluates the risk faced by others. For instance, it may well be the case that an altruistic decision maker may overweight a small chance that someone she cares about has of getting a good outcome. We want our theory to be rich enough to account for such phenomenon.

Before we make precise the structure we shall impose on $\succcurlyeq_z$, we introduce a topological assumption, and a continuity condition. Note that we will abuse notation here by not distinguishing between an outcome like $\mathbf{a} \in A$, and a lottery in Δ_A that gives the outcome $\mathbf{a}$ with probability 1.

Topological Assumption: *A is a connected topological space. Further, for any $\succcurlyeq_z$, and for any $\mathbf{a} \in A$, $\{\mathbf{a}' \in A : \mathbf{a}' \sim_z \mathbf{a}\}$ is a connected subset of A.*

The topology on A induces the product topology on $A \times A$. As is well known, in this topology a sequence $(\mathbf{a}_k, \mathbf{a}'_k)_{k \in \mathbb{Z}_+} \subseteq A \times A$ converges to $(\mathbf{a}, \mathbf{a}') \in A \times A$ if and only if $(\mathbf{a}_k)_{k \in \mathbb{Z}_+}$ converges to $\mathbf{a}$ and $(\mathbf{a}'_k)_{k \in \mathbb{Z}_+}$ converges to $\mathbf{a}'$. We now propose the following continuity condition for $\succcurlyeq_\Delta$.

[B2.1] Bicontinuity of $\succcurlyeq_\Delta$
Let $z, z' \in Z$, $A' \times A'' \subseteq A \times A$, and $\lambda \in (0, 1]$, and let

$$\widetilde{\Delta} = \{[(z, \mathbf{a}), \lambda; (z', \mathbf{a}'), 1 - \lambda] \in \Delta : (\mathbf{a}, \mathbf{a}') \in A' \times A''\}$$

be such that for any p, $p' \in \widetilde{\Delta}$, p and p' are comparable at z. Then for any $q \in \Delta$, the sets

$$\{(\mathbf{a}, \mathbf{a}') \in A' \times A'' : [(z, \mathbf{a}), \lambda; (z', \mathbf{a}'), 1 - \lambda] \succcurlyeq q\},$$

and,

$$\{(\mathbf{a}, \mathbf{a}') \in A' \times A'' : q \succcurlyeq [(z, \mathbf{a}), \lambda; (z', \mathbf{a}'), 1 - \lambda]\}$$

are closed in $A' \times A''$.

Note that a special case of the above continuity condition is when $\lambda = 1$. This case implies that the preference relation $\succcurlyeq_z$ restricted to A is continuous. That is, for any $p_A \in \Delta_A$, the sets

$$\{\mathbf{a} \in A : \mathbf{a} \succcurlyeq_z p_A\} \text{ and } \{\mathbf{a} \in A : p_A \succcurlyeq_z \mathbf{a}\}$$

are closed in A. This conclusion along with the assumption that $\succcurlyeq_z$ satisfies stochastic dominance, and the topological assumption on A implies that every $p_A \in \Delta_A$ has a certainty equivalent in A with respect to the preference relation $\succcurlyeq_z$; that is for any $p_A \in \Delta_A$, there exists $\mathbf{a}_{p_A} \in A$ such that $p_A \sim_z \mathbf{a}_{p_A}$. We shall continue to use the notation $\mathbf{a}_{p_A}$ to denote the certainty equivalent of $p_A \in \Delta_A$ with respect to $\succcurlyeq_z$. Finally, note that the continuity condition [B2.1] does not preclude the possibility that $z = z'$ and $\lambda \in (0, 1)$.

We now adapt to our environment of risk a definition that Ghirardato and Marinacci (2001) have provided in the context of uncertainty.

Definition 4. *The preference relation $\succsim_z$ is bi-separable if it satisfies stochastic dominance, and admits a representation $V_z : \Delta_A \to \mathbb{R}$ that is unique up to positive affine transformation, for which there exists a strictly increasing bijection φ_z: $[0,1] \to [0,1]$ that satisfies $\varphi_z(0) = 0$, $\varphi_z(1) = 1$, such that, if we let $v_z(\mathbf{a}) = V_z(\mathbf{a})$ for all $\mathbf{a} \in A$, then for all $\mathbf{a}'$, $\mathbf{a}'' \in A$, $\mathbf{a}' \succsim_z \mathbf{a}''$, and all $\lambda \in [0,1]$,*

$$V_z([\mathbf{a}', \lambda; \mathbf{a}'', 1-\lambda]) = \varphi_z(\lambda)v_z(\mathbf{a}') + (1 - \varphi_z(\lambda))v_z(\mathbf{a}'').$$

As the name suggests, biseparable preferences introduce event-wise separability in a very limited sense, viz. lotteries that put positive probability on only two outcomes (for the others) are evaluated by the decision maker in the spirit of generalized expected utility. Other than that the only restriction on $\succsim_z$ is that it respects stochastic dominance. The function φ_z is referred to as a probability weighting function. The probability weighting function has the interpretation that it transforms 'raw' or objective probabilities into decision weights that capture the attitude that DM has towards the chance or risk faced by others.

[B0] Biseparability
For any $\succsim_z$ $(z \in Z)$ such that $\succ_z \neq \emptyset$, $\succsim_z$ is a biseparable preference.

In the subsequent analysis, we shall not distinguish between the functions V_z and v_z, and use the latter to denote both. Further, we will call v_z a biseparable representation of the biseparable preference $\succsim_z$. Some prominent examples of biseparable preferences include Expected Utility, Rank Dependent Utility, and Gul's 'Disappointment Averse' preferences.

Ghirardato et al. (2003) have shown how the structure of biseparable preferences can be used to define 'subjective mixtures' or 'preference averages'. That is, for any two outcomes $\mathbf{a}$, $\mathbf{a}' \in A$, we can identify an outcome $\overline{\mathbf{a}} \in A$ that can be considered as the mid-point on DM's 'preference scale' between $\mathbf{a}$ and $\mathbf{a}'$. Given that the biseparable representation v_z of $\succsim_z$ is unique up to positive affine transformation, such a $\overline{\mathbf{a}}$ is characterized by the equation,

$$v_z(\overline{\mathbf{a}}) = \frac{1}{2}v_z(\mathbf{a}) + \frac{1}{2}v_z(\mathbf{a}').$$

Ghirardato et al. (2003) have shown, in the context of uncertainty, that the notion of a preference average can be equivalently defined from behavioral primitives. We now provide a similar definition in our setting of risk.

Definition 5. *For any $\mathbf{a}$, $\mathbf{a}' \in A$, if $\mathbf{a} \succsim_z \mathbf{a}'$, we say that $\overline{\mathbf{a}} \in A$ is a preference average of*

$\mathbf{a}$ *and* $\mathbf{a}'$ *with respect to* $\succcurlyeq_z$, *denoted* $\frac{1}{2}\mathbf{a} \oplus_z \frac{1}{2}\mathbf{a}'$, *if for all* $\lambda \in [0,1]$,

$$[\mathbf{a},\ \lambda;\ \mathbf{a}',\ 1-\lambda] \sim_z [\mathbf{a}_{[\mathbf{a},\lambda;\overline{\mathbf{a}},1-\lambda]},\ \lambda;\ \mathbf{a}_{[\overline{\mathbf{a}},\lambda;\mathbf{a}',1-\lambda]},\ 1-\lambda].$$

If $\mathbf{a}' \succcurlyeq_z \mathbf{a}$, $\overline{\mathbf{a}}$ *is said to be a preference average of* $\mathbf{a}$ *and* $\mathbf{a}'$ *if it is a preference average of* $\mathbf{a}'$ *and* $\mathbf{a}$.

For a discussion of why the criterion above constitutes a meaningful definition of preference average, the reader is encouraged to refer to the lucid commentary in Ghirardato et al. (2003). Couple of comments are in order. The first is that preference average of any $\mathbf{a}$, $\mathbf{a}' \in A$ need not be unique; there may be multiple elements in A that are a preference average of such $\mathbf{a}$ and $\mathbf{a}'$. All such preference averages form an indifference class of $\succcurlyeq_z$ (see Lemma 2 below), and by $\frac{1}{2}\mathbf{a} \oplus_z \frac{1}{2}\mathbf{a}'$ we shall denote a representative of the indifference class. Second, stochastic dominance implies that if $\overline{\mathbf{a}}$ is a preference average of $\mathbf{a}$ and $\mathbf{a}'$, then $\mathbf{a} \succcurlyeq_z \overline{\mathbf{a}} \succcurlyeq_z \mathbf{a}'$, and this holds with strict preference if $\mathbf{a} \succ_z \mathbf{a}'$. The next Lemma ties down the behavioral and utility approaches of defining preference averages by showing that for biseparable preferences they coincide. (The proof of the Lemma mimics the proof of Proposition 1 in Ghirardato et al. (2003), and the details are omitted).

Lemma 2. *Let* v_z *be a biseparable representation of (the biseparable preference)* $\succcurlyeq_z$. *For any* $\mathbf{a}$, $\mathbf{a}' \in A$, $\overline{\mathbf{a}} \in A$ *is a preference average of* $\mathbf{a}$ *and* $\mathbf{a}'$ *if and only if*

$$v_z(\overline{\mathbf{a}}) = \frac{1}{2}v_z(\mathbf{a}) + \frac{1}{2}v_z(\mathbf{a}').$$

Further, preference average of $\mathbf{a}$ *and* $\mathbf{a}'$ *exist for any* $\mathbf{a}$, $\mathbf{a}'$ *in* A, *and they form an indifference class. That is, if* $\overline{\mathbf{a}}$ *and* $\widehat{\mathbf{a}}$ *are both preference averages of* $\mathbf{a}$ *and* $\mathbf{a}'$, *then* $\overline{\mathbf{a}} \sim_z \widehat{\mathbf{a}}$.

Note that we can now easily define iterated averages like $\frac{1}{2}\mathbf{a} \oplus_z [\frac{1}{2}\mathbf{a} \oplus_z \frac{1}{2}\mathbf{a}']$ which is equivalent to a $\frac{3}{4} : \frac{1}{4}$ mixture of $\mathbf{a}$ and $\mathbf{a}'$, and denoted $\frac{3}{4}\mathbf{a} \oplus_z \frac{1}{4}\mathbf{a}'$. More generally, continuity makes it possible to identify from behavior a $\lambda : 1-\lambda$ mixture of $\mathbf{a}$ and $\mathbf{a}'$ for any $\lambda \in [0,1]$. In addition, we can extend the notion of subjective mixtures to the whole of Δ_A. For any p_A, $p'_A \in \Delta$, if $p_A \succcurlyeq_z p'_A$, we say that $\overline{p}_A$ is a $\lambda : 1-\lambda$ mixture of p_A and p'_A with respect to z, denoted $\lambda p_A \oplus_z (1-\lambda)p'_A$, if

$$\mathbf{a}_{\overline{p}_A} = \lambda \mathbf{a}_{p_A} \oplus_z (1-\lambda)\mathbf{a}_{p'_A}.$$

Further, for any p, $q \in \Delta$ that are comparable at z, and for any $\lambda \in [0,1]$, $\lambda p \oplus_z (1-\lambda)q \in \Delta$ shall denote an element that is comparable with p (and hence with q) at z, and satisfies,

$$(\lambda p \oplus_z (1-\lambda)q)_{A,z} = \lambda p_{A,z} \oplus_z (1-\lambda)q_{A,z}, \quad (\lambda p \oplus_z (1-\lambda)q)_A = \lambda p_A \oplus_z (1-\lambda)q_A.$$

Definition 6. *l, $l' \in \Delta^+$ are* ***mixture comparable*** *if either*

1. *l, $l' \in \Delta_Z$, or*
2. *$l = p$, $l' = q \in \Delta$, are comparable at some $z \in Z$, and, for all $\lambda \in [0,1]$ there exists $\lambda p \oplus_z (1-\lambda)q \in \Delta$.*

In the way of notation, note that for any l, $l' \in X^+$ that are mixture comparable, we shall write $\lambda l \oplus (1-\lambda)l'$ to denote the relevant mixture operation. For instance, for any $p_Z, q_Z \in \Delta_Z$, $\lambda \in [0,1]$, $\lambda p_Z \oplus (1-\lambda)q_Z$ will stand for $\lambda p_Z + (1-\lambda)q_Z$. Having stated this terminology, we shall now reformulate the Continuity and Strong Independence Axiom for the current setting. In particular, we break down the continuity axiom into two parts.

[B2.2] Archimedean Continuity of $\succcurlyeq_{X_0}$
Let p_Z, $q_Z \in \Delta_Z$, $l \in \Delta^+$ be such that $p_Z \succ l \succ q_Z$. Then there exists λ, $\lambda' \in (0,1)$ such that

$$\lambda p_Z \oplus (1-\lambda)q_Z \succ l \succ \lambda' p_Z \oplus (1-\lambda')q_Z$$

Further, for all p in Δ, there exists q_Z, $q'_Z \in \Delta_Z$, such that $q_Z \succcurlyeq p \succcurlyeq q'_Z$.

[B2] Continuity
$\succcurlyeq_\Delta$ is bicontinuous and $\succcurlyeq_{\Delta_Z}$ is Archimedean continuous.

[B3] Comparable Independence
Let l_1, l_2, l'_1, l'_2 in X^+ be such that l_1, l_2 are mixture comparable, as are l'_1 and l'_2. Then, for all $\lambda \in (0,1]$,

$$[l_1 \succ l'_1 \text{ and } l_2 \sim l'_2] \Rightarrow \lambda l_1 \oplus (1-\lambda)l_2 \succ \lambda l'_1 \oplus (1-\lambda)l'_2.$$

In addition, we make the following regularity assumption on the family of preference relations $(\succcurlyeq_z)_{z\in Z}$.

[CV.2] Contingent Values
If $\succcurlyeq_z$ such that $\succ_z \neq \emptyset$, then there exists $\succcurlyeq_{z'} \neq \succcurlyeq_z$, with $\succ_{z'} \neq \emptyset$, such that the following

holds: for all $\widetilde{\mathbf{a}} \in A$*, there exists* $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{a}'' \in A$*, such that* $\widetilde{\mathbf{a}} \sim_z \mathbf{a}$*, and*

(a) $\mathbf{a}' \sim_{z'} \mathbf{a} \sim_{z'} \mathbf{a}''$,

(b) $\mathbf{a}' \succcurlyeq_z \mathbf{a} \succcurlyeq_z \mathbf{a}''$.

Further, if $\mathbf{a}$ *is not a maximal (resp. minimal) element of* $\succcurlyeq_z$*, then* $\mathbf{a}' \succ_z \mathbf{a}$ *(resp.* $\mathbf{a} \succ_z \mathbf{a}''$*).*

Theorem 3. *Suppose [CV.2] holds. Then the following statements are equivalent:*

1. *Axioms [B0], [A1], [B2], [B3], [A4] - [A7] holds.*

2. *There exists a function* $u : Z \to \mathbb{R}$*, functions* $v_z : \Delta_A \to \mathbb{R}$*,* $z \in Z$*, and a constant* $\pi \in [0, 1]$ *such that the function* $U : \Delta^+ \to \mathbb{R}$ *defined by*

    - $U(p_Z) = \sum_{z \in Z} p_Z(z) u(z)$, $\forall\ p_Z \in \Delta_Z$*, and*
    - $U(p) = \sum_{z \in Z} p_Z(z)[u(z) + \pi v_z(p_{A,z}) + (1 - \pi) v_z(p_A)]$, $\forall\ p \in \Delta$

    represents $\succcurlyeq$.

In addition, any triple $(\tilde{u}, (\tilde{v_z})_{z \in Z}, \tilde{\pi})$ *represents* $\succcurlyeq$ *in the above sense if and only if there exists constants* $\alpha > 0$ *and* β *such that* $\tilde{u} = \alpha u + \beta$, $\tilde{v_z} = \alpha v_z$*, for all* $z \in Z$*, and* $\tilde{\pi} = \pi$*, whenever there exists* $z \in Z$ *such that* $\succ_z \neq \emptyset$.

Note that v_z is a biseparable representation of $\succcurlyeq_z$. Further, v_z restricted to A is continuous. Like before Axiom [A7] (Symmetry) plays a role only in establishing that the weights π_z are same for all $z \in Z$ with $\succ_z \neq \emptyset$. The proof of the Theorem is available in the appendix.

5 Application

In this section we use our decision model to address an issue that has been at the center of American electoral politics in recent years, namely, the issue of voters voting against their economic self-interest. Because our decision model makes a separation between tastes and values, it allows us to introduce two distinct rationales that voters may use while making voting decisions. Before introducing the voting model, we would like to highlight some experimental evidence from a recent paper by Feddersen, Gailmard and Sandroni (2009) that supports the hypothesis that voters may indeed vote against their economic interest, but shows that the rationale behind such voting behavior may not be captured by the simple argument that moral issues are more important to voters than economic ones. Their basic hypothesis is that large elections may exhibit a moral bias, viz., alternatives understood by voters to be morally superior are more likely to win in large elections than in small ones. To make this point, they conduct an experimental election with two options – call these A and B. The basic details of their experiment were as follows. First, subjects were broken up into two groups, one consisting of voters and the other of non-voters. Then the voters cast their votes for A or B. Finally, after all voters had cast their vote, one voter was randomly picked, whose choice became the group choice. The number of eligible voters was determined by the experimenters and was varied across different trials of the experiment. Accordingly, the probability of a voter being pivotal (the reciprocal of the number of voters) was directly controlled as a treatment variable in these experiment. As far as payoffs went, option B gave a higher monetary reward to the voters than option A. On the other hand option A was better for the non-voters than option B. One may therefore think of option B as a selfish option for the voters, and option A as a moral option. An interesting pattern of choice that was exhibited by a non-trivial number of voters is the following. When the probability of their vote being pivotal was high, in particular when it was 1, these voters chose option B. On the other hand when the pivot probability was low, they voted for option A. Observe that such voters voted for option A (when pivot probability was low) even though they prefer option B to A. Accordingly, their choices violate stochastic dominance.

We now introduce a theoretical model that addresses the issue. Our primary goal here is to highlight the workings of our decision model and to explain how it differs from standard models. To that end we are going to make the specification of the electoral process extremely simple. For a more detailed treatment of the problem, refer to Borah (2009). Consider an election with two options, 1 and 2, in a society consisting of n voters. We treat n as a parameter of the model. We assume that there are no costs to voting. This will ensure that everyone votes in the election. Further the result of the election will be determined by the following mechanism, which mimic the one used by Feddersen et. al. First, all voters cast their votes. After all voters have reported their choice, one voter is drawn at random, and the choice she reported determines the outcome of the election. Accordingly, pivot probabilities will be a parameter in our model.

We make the extreme assumption that all voters are identical. This greatly simplifies the analysis as it allows us to conduct it in the context of a 'representative voter'. Let us now describe what the problem looks like when viewed from the perspective of one such representative voter (RV). As mentioned above, she can vote for either option 1 or option 2. If option 1 is the group choice, the resulting allocation is $(z^1, \mathbf{a}^1) \in Z \times A$, where z^1 refers to the outcome for RV, and $\mathbf{a}$ the outcomes for everyone else. Similarly, if option 2 is the group choice, the resulting allocation is $(z^2, \mathbf{a}^2) \in Z \times A$, where again z^2 refers to the outcome for RV, and $\mathbf{a}^2$ the outcomes for everyone else.

Note that the probability that RV is pivotal is given by $\lambda = 1/n$. Further, let γ denote the probability that option 1 is the outcome of the election when RV is not pivotal (Of course, γ is an 'endogenous object'). Then the probability distributions over final allocations generated by RV choosing 1 is given by:

$$p^1 = [(z^1, \mathbf{a}^1),\ \lambda + (1-\lambda)\gamma;\ (z^2, \mathbf{a}^2),\ 1 - \lambda - (1-\lambda)\gamma],$$

and that by her choosing 2 is given by:

$$p^2 = [(z^1, \mathbf{a}^1),\ (1-\lambda)\gamma;\ (z^2, \mathbf{a}^2),\ 1 - (1-\lambda)\gamma]$$

Note that if RV's preferences satisfy stochastic dominance, then her vote choice is independent of pivot probabilities, or equivalently, of the number of voters. To understand this claim, suppose, she prefers the allocation $(z^2, \mathbf{a}^2)$ to $(z^1, \mathbf{a}^1)$. Consider any situation in which she is pivotal with probability $\lambda = 1/n$. In this case, taking the other voters' choices as given (that is, taking γ as given), her vote for options 1 and 2 generates respectively the lotteries p^1 and p^2 over final allocations (listed above). Since she prefers the allocation $(z^2, \mathbf{a}^2)$ to $(z^1, \mathbf{a}^1)$, stochastic dominance requires that she must prefer the lottery p^2 to the lottery p^1, and hence must vote for option 2 irrespective of what λ and γ are. Accordingly, we have:

Proposition 1. *If voters preferences satisfy stochastic dominance, then there exists a unique Nash equilibrium (in dominant strategies) that is independent of n in which either everyone votes for option 1 or everyone votes for option 2.*

We now contrast this result with one that is implied by our decision model where voters have other regarding preferences that may violate stochastic dominance. We will consider option 1 to be a 'value choice' or 'moral choice', and 2 to be a 'selfish choice', and further that values matter to our voters. In terms of our decision model, we translate this to mean that

$$u_H = u(z^2) > u(z^1) = u_L,$$

and

$$v_H = v_z(\mathbf{a}^1) > v_z(\mathbf{a}^2) = v_L,$$

for $z = z^1$, z^2. We will further assume that the preference relations $\succcurlyeq_z$, $z = z^1$, z^2, are identical and bi-separable. This means that there there exists a probability weighting function, that is, a strictly increasing bijection $\varphi : [0, 1] \to [0, 1]$ that satisfies $\varphi(0) = 0$, $\varphi(1) = 1$ such that a lottery of the type $[\mathbf{a}^1, r; \mathbf{a}^2, 1 - r]$ is evaluated as,

$$v_z([\mathbf{a}^1, r; \mathbf{a}^2, 1-r]) = \varphi(r)v_z(\mathbf{a}^1) + (1-\varphi(r))v_z(\mathbf{a}^2).$$

As stated above the probability weighting function has the interpretation that it transforms objective probabilities into decision weights. These decision weights capture the attitude that DM has towards the chance or risk faced by others. We will assume that π, the weight that DM puts on the ex-post concern, is equal to $\frac{1}{2}$. Further, define,

$$\nu = \tfrac{u_H - u_L}{v_H - v_L}$$

and assume that:

- $[V1]$ $\nu > 1$.
- $[V2]$ There exists $\underline{\lambda}$, $\overline{\lambda} \in (0, 1)$, such that for all $\widetilde{\lambda} \in (0,\underline{\lambda}) \cup (\overline{\lambda}, 1)$, φ is differentiable, and $\varphi'(\widetilde{\lambda}) > 2\nu - 1$. Further, φ is concave on the interval $[0,\underline{\lambda})$.

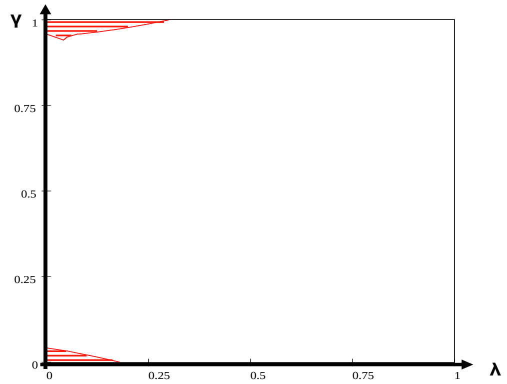
[V1] can be rewritten as

$$u_H + v_L > u_L + v_H$$

The left hand side gives RV's payoffs under our decision model from the allocation $(z^B, \mathbf{a}^B)$, whereas the right hand side gives her payoffs from the allocation $(z^1, \mathbf{a}^1)$. This condition therefore states that RV prefers the allocation $(z^2, \mathbf{a}^2)$ to the allocation $(z^1, \mathbf{a}^1)$.

Assumptions [V1] and [V2] together imply that for all $\widetilde{\lambda} \in (0,\underline{\lambda}) \cup (\overline{\lambda}, 1)$, $\varphi'(\widetilde{\lambda}) > 1$. It follows that there exists a neighborhood of 0 in which $\varphi(\widetilde{\lambda}) > \widetilde{\lambda}$, and there exists a neighborhood of 1 in which $\varphi(\widetilde{\lambda}) < \widetilde{\lambda}$. This implies that the representative voter tends to over-weigh small probabilities and under-weigh large probabilities of her morally superior outcome, $\mathbf{a}^1$, realizing. This phenomenon of over-weighing small probabilities, and under-weighing large ones, which is referred to as regressive probability weighting, has been extensively documented in the literature on decision making under risk starting with the important contribution of Kahneman and Tversky (1979). In that literature regressiveness of probability weighting pertains to a decision maker's attitude to the chances that she is faced with as regards her own outcomes. In our set up this regressiveness pertains to the decision maker's attitude to the chances faced by others.

Figure 2: Payoff difference between voting for options 1 and 2.



Proposition 2. *Under assumptions [V1] and [V2], there exists positive integers $\underline{n}$ and $\overline{n}$, $\underline{n} < \overline{n}$, such that for all $n \leq \underline{n}$, everyone voting for option 2 is the unique symmetric Nash equilibrium (in pure strategies), and for all $n \geq \overline{n}$, everyone voting for option 1 is the unique symmetric Nash equilibrium (in pure strategies).*

The proof is available in the appendix. Here, we briefly go over the reasoning that drives the result. Consider Figure 2 that has been constructed by taking particular values of u_H, u_L, v_H, v_L and functional form for the probability weighting function that is consistent with assumptions [V1] and [V2]. The figure shows the payoff difference for our representative voter from voting for options 1 and 2 as a function of λ, the pivot probability, and γ, the probability that option 1 will be chosen when RV is not pivotal. The shaded area represents those values of λ and γ for which the payoff of voting for 1 exceeds that of voting for 2. The incentives that RV has for voting for option 2 for high values of λ is quite apparent given that she prefers option 2 to option 1. The interesting feature of our model is that for low values of λ, and for suitable values of γ, her vote choice shifts from option 2 to 1. In particular, there are two regions in the λ-γ box of the Figure in which the payoff of voting for option 1 exceeds that of voting for 2. This vote switch is brought about by the role that counterfactuals play in her evaluation of others' outcomes.

Consider first the lower south-west region where both λ and γ are small, which together imply that with high probability option 2 shall be the outcome of the election. In this scenario, RV prefers to add her (small) marginal probability λ of influencing the outcome of the election towards option 1 rather than option 2 –the question is why so? She knows that in all likelihood she will get z^2 the more desirable outcome by here selfish preferences, but this rules out the possibility of the others getting the outcome $\mathbf{a}^1$, the more desirable outcome by her moral preferences. However, given that counterfactuals matter to her, she would still be better off in the event that she get z^2 if the outcome $\mathbf{a}^1$ had a greater chance of realizing in an ex-ante sense. So the relevant tradeoff for her is the following: Does she improve even more the chance of option 2 realizing which shall make her better off via her selfish preferences, or does she improve the chance of option 1 realizing which shall make her better off via the counterfactual consideration (in the event that option 2 is the realized outcome) that $\mathbf{a}^1$ at least had some ex-ante chance of realizing . If she votes for option 1 the ex-ante chance of outcome $\mathbf{a}^1$ realizing is (given by the lottery)

$$[\mathbf{a}^1, \lambda + (1-\lambda)\gamma; \mathbf{a}^2, 1-\lambda + (1-\lambda)\gamma]$$

Her payoffs from this lottery is evaluated as:

$$\varphi(\lambda + (1-\lambda)\gamma)v_H + (1 - \varphi(\lambda + (1-\lambda)\gamma))v_L$$

Given that both λ and γ are small, $\lambda + (1-\lambda)\gamma$ is small. For small probabilities, regressive probability weighting implies that $\varphi(\lambda + (1-\lambda)\gamma) > \lambda + (1-\lambda)\gamma$. This makes increasing the ex-ante probability of $\mathbf{a}^1$ realizing more attractive to her than increasing the probability of z^2 realizing, and determines her choice of option 1. We think of this as a *warm glow effect* – given that she is very likely to get a good personal outcome (z^2), our 'values voter' is willing to forego a small chance of improving her selfish goals for a better ex-ante chance of realizing her moral goals.

Now consider the north-west corner of the λ-γ box, where the payoff of voting for option 1 exceeds that of voting for 2. Given that γ is large, and λ is small in this region, option 1 is very likely to be the outcome of the election. Once again her choice of option 1 in this scenario is driven by her concern for the counterfactual. She knows that in all likelihood the event in which she gets z^1 will realize. Given that this event occurs, the others get $\mathbf{a}^1$. If she were to vote for option 2, it would reduce the ex-ante chance that others have of getting $\mathbf{a}^1$ by λ. Given that the chance of $\mathbf{a}^1$ realizing is close to 1, the regressive nature of probability weighting close to 1, viz., $\varphi(r) < r$, makes the reduction in ex-ante chance unattractive for her. So she prefers adding her marginal probability of λ towards furthering her moral goals than selfish goals. We think of this as a *sacrifice for certainty effect* – given that she is very likely to get the bad personal outcome, our 'values voter' is willing to sacrifice a small chance of improving her personal outcome for the ex-ante certainty of the desirable moral outcome realizing.

Given the structure of payoff differences, it should now be obvious why our result follows. In particular, note that when everyone else is voting for option 1 ($\gamma = 1$), for small pivot probabilities, RV's best response is to vote for option 1.

6 Concluding Remarks

In this paper, we identified a serious challenge that other regarding preferences pose to existing theories of decision making under risk, namely, such preferences may violate the property of stochastic dominance. We argued that such violations are brought about by a concern for counterfactuals that decision makers with other regarding preferences exhibit while evaluating others' outcomes. We provided a parsimonious decision model that accommodates such concerns for counterfactuals. In our representation we separated a self regarding component of preferences from an other regarding component. As far as the other regarding component goes, we showed that this concern can be broken up into a concern for the ex-post risk and the ex-ante risk faced by others. The ex-post risk corresponds to a concern that the decision maker has for the actual outcomes of others whereas the ex-ante risk corresponds to a concern for the process by which others' outcomes are determined.

We applied our decision model to a simple voting problem, and showed that when ex-ante concerns are present the resulting allocation may vary dramatically from when such concerns are absent. At an empirical level our decision model may have implications regarding the distribution of final goods in an economy. For instance, public economists have looked at the question of why there is greater redistribution in some societies than in others, for instance, between Western Europe and United States. One of the explanations that has been advanced for such differences is that individual preferences as it pertains to the outcomes or opportunities of others may vary across societies. A natural question to ask would then be the following: Do differences in attitude amongst decision makers about the ex-ante risk faced by others have an explanatory power in accounting for differences in redistribution policy across societies? Our decision model would provide a simple test of this hypothesis, namely, a test of whether the parameter π statistically differs across such societies.

At a theoretical level, there are two questions that may be explored in future work.

First, it may be a useful exercise to generalize our representation to allow for the decision maker's preferences over own outcomes to be non-linear in probabilities. That is, we may want to explore a representation in which the u function of our representation, is say, Rank Dependent, instead of being Expected Utility. Second, we would like to see our decision model extended to an environment with uncertainty in which no objective probabilities are available.

7 Appendix

7.1 Preliminaries

In this subsection, we provide some preliminary results, as well as introduce some definitions that we shall make use of in proving our representation results. Note that we will abuse notation throughout the Appendix by not distinguishing between an outcome in the set Z or A, and a degenerate lottery that gives that outcome with probability 1.

7.1.1 Preliminary Results

Proof of Lemma 1

Since p and p' are comparable at $z \in Z$, $p_Z = p'_Z$, and so

$$(\lambda p + (1-\lambda)p')_Z = \lambda p_Z + (1-\lambda)p'_Z = p_Z = p'_Z.$$

Further $p_Z = p'_Z$ implies that for all z' in the support of p_Z,

$$(\lambda p + (1-\lambda)p')_{A,z'} = \lambda p_{A,z'} + (1-\lambda)p'_{A,z'}.$$

Further for all z' in the support of p_Z, $z' \neq z$, $p_{A,z'} \sim_{z'} p'_{A,z'}$, and $\succcurlyeq_z$ satisfying Independence implies that

$$\lambda p_{A,z'} + (1-\lambda)p'_{A,z'} \sim_{z'} p_{A,z'} \sim_{z'} p'_{A,z'}.$$

That is,

$$(\lambda p + (1-\lambda)p')_{A,z'} \sim_{z'} p_{A,z'} \sim_{z'} p'_{A,z'}.$$

Similarly, it also follows that

$$(\lambda p + (1-\lambda)p')_A = \lambda p_A + (1-\lambda)p'_A \sim_{z'} p_A \sim_{z'} p'_A. \qquad \square$$

Lemma 3. *For each $p \in \Delta$, there exists $q^*_Z(p) \in \Delta_Z$ such that $p \sim q^*_Z(p)$.*

Proof. By the Continuity axiom, for any $p \in \Delta$, there exists q_Z, $q'_Z \in \Delta_Z$ such that $q_Z \succcurlyeq p \succcurlyeq q'_Z$. If either of those preferences is an indifference, then we are done. So assume that $q_Z \succ p \succ q'_Z$. Then, by Continuity, there exists λ', $\lambda'' \in (0, 1)$, $\lambda' > \lambda''$, such that $\lambda' q_Z + (1 - \lambda')q'_Z \succ p \succ \lambda'' q_Z + (1 - \lambda'')q'_Z$. Let,

$$\Lambda = \{\lambda \in [0, 1] : p \succ \lambda q_Z + (1 - \lambda)q'_Z\}$$

and let $\lambda^* = sup\ \Lambda$. We claim that $p \sim \lambda^* q_Z + (1 - \lambda^*)q'_Z$. To see this, suppose otherwise.

First, suppose that $p \succ \lambda^* q_Z + (1 - \lambda^*)q'_Z$. This implies that $\lambda^* \in \Lambda$. Note that,

$$\lambda' q_Z + (1 - \lambda')q'_Z \succ p \succ \lambda^* q_Z + (1 - \lambda^*)q'_Z$$

This implies that there exists $\lambda \in (0, 1)$, such that letting $\widetilde{\lambda} = \lambda\lambda' + (1 - \lambda)\lambda^*$, we have by Continuity that

$$p \succ \widetilde{\lambda} q_Z + (1 - \widetilde{\lambda})q'_Z.$$

But note that $\lambda' > \lambda^*$, and hence $\widetilde{\lambda} > \lambda^*$. But at the same time $\widetilde{\lambda} \in \Lambda$, which contradicts the fact that $\lambda^* = \sup \Lambda$

Next, suppose that $\lambda^* q_Z + (1 - \lambda^*)q'_Z \succ p$. Then by Continuity, there exists $\lambda \in (0, 1)$ such that letting $\widehat{\lambda} = \lambda\lambda^* + (1 - \lambda)\lambda''$,

$$\widehat{\lambda} q_Z + (1 - \widehat{\lambda})q'_Z \succ p$$

It follows that $\widehat{\lambda}$ is an upper bound of Λ. But at the same time, since $\lambda^* > \lambda''$, $\lambda^* > \widehat{\lambda}$, which contradicts the fact that $\lambda^* = \sup \Lambda$. □

In the subsequent analysis, for any $p \in \Delta$, we shall denote an element of Δ_Z that is indifferent to it by $q^*_Z(p)$.

Lemma 4. *Suppose $\succcurlyeq_z$ satisfies stochastic dominance and vN-M continuity. If $\succ_z \neq \emptyset$, then there does not exist $p \in \Delta$ such that (a) $p_A \succ_z p_{A,z}$ and p_A is the maximal element of $\succcurlyeq_z$, or (b) $p_{A,z} \succ_z p_A$ and p_A is the minimal element of $\succcurlyeq_z$.*

Proof. Consider any $q_A \in \Delta_A$. Note the following mutually exclusive possibilities:
[A] $\mathbf{a} \sim_z \mathbf{a}'$ for all $\mathbf{a}$, $\mathbf{a}'$ in the support of q_A: In this case it must be that $q_A \sim_z \mathbf{a}$ for any $\mathbf{a}$ in the support of q_A. To see this assume otherwise – say $q_A \succ_z \mathbf{a}$. Given that $\succ_z \neq \emptyset$, it follows that there exists $\mathbf{a}''$ satisfying $\mathbf{a}'' \succ_z \mathbf{a}$ or $\mathbf{a} \succ_z \mathbf{a}''$. Assume it is the former.[10] Note

[10] The latter case is dealt analogously

that $\mathbf{a}''$ first order stochastically dominates q_A; hence we have: $\mathbf{a}'' \succ q_A \succ_z \mathbf{a}$. But then by vN-M Continuity, there exists $\lambda \in (0,1)$, such that $q_A \succ_z \lambda\mathbf{a}'' + (1-\lambda)\mathbf{a}'$. But the lottery $\lambda\mathbf{a}'' + (1-\lambda)\mathbf{a}'$ first order stochastically dominates q_A!
[B] $\mathbf{a} \succ_z \mathbf{a}'$ for some $\mathbf{a}$, $\mathbf{a}'$ in the support of q_A: In this case there exists $\overline{\mathbf{a}}$, $\underline{\mathbf{a}}$ in the support of q_A such that $\overline{\mathbf{a}} \succ_z q_A \succ_z \underline{\mathbf{a}}$. This claim is again easily established by appealing to the fact that $\succcurlyeq_z$ satisfies stochastic dominance; so we omit the details here.
Now we proceed to prove the Lemma. Suppose there exists $p \in \Delta$ such that $p_A \succ_z p_{A,z}$ and p_A is the maximal element of $\succcurlyeq_z$. Then we know from above that there exists $\mathbf{a}$ in the support of $p_{A,z}$ such that $p_{A,z} \succcurlyeq_z \mathbf{a}$. Hence, $p_A \succ_z \mathbf{a}$. Clearly, $\mathbf{a}$ is in the support of p_A. This implies (following Case B above) that there exists $\mathbf{a}'$ in the support of p_A such that $\mathbf{a}' \succ_z p_A$. But this contradicts that p_A is the maximal element of $\succcurlyeq_z$. The case of $p_{A,z} \succ_z p_A$ and p_A is the minimal element of $\succcurlyeq_z$ not being possible can be handled analogously. $\square$

It should be obvious that both in the case where $\succcurlyeq_z$ is a vNM preference, as well as when it is biseparable, the assumptions of the last Lemma holds.

7.1.2 A Binary Relation

We define here a binary relation that we shall make use of in proving our representation results. Recall the following notation that we introduced in the text. For any $p_A \in \Delta_A$ and $z \in Z$, the indifference class of p_A under $\succcurlyeq_z$ is denoted by

$$[p_A]_z = \{q_A \in \Delta_A : q_A \sim_z p_A\}$$

Further, $\Delta_A/\sim_z$ shall denote the set of all such indifference classes. We define the binary relations, $\widehat{\succcurlyeq}_z$, $\widehat{\succ}_z$, $\widehat{\sim}_z \subseteq [\Delta_A/{\sim_z}]^2 \times [\Delta_A/{\sim_z}]^2$ as follows:

Definition 7. *$([p_A]_z,\ [q_A]_z)\ \widehat{\succcurlyeq}_z$ (resp. $\widehat{\succ}_z$) $([p'_A]_z,\ [q'_A]_z)$ if there exists a finite sequence $(p^k, q^k)_{k=1}^K \subseteq \Delta \times \Delta$ that reveals a weak (resp. strict) preference for the risk profile $(z,\ p^1_{A,z},\ p^1_A)$ over the risk profile $(z,\ q^K_{A,z},\ q^K_A)$, and $([p^1_{A,z}]_z,\ [p^1_A]_z) = ([p_A]_z,\ [q_A]_z)$, $([q^K_{A,z}]_z,\ [q^K_A]_z) = ([p'_A]_z,\ [q'_A]_z)$.*
Further, $([p_A]_z,\ [q_A]_z)\ \widehat{\sim}_z\ ([p'_A]_z,\ [q'_A]_z)$ if there exists a finite sequence $(p^k, q^k)_{k=1}^K \subseteq \Delta \times \Delta$ that reveals an indifference between the risk profiles $(z,\ p^1_{A,z},\ p^1_A)$ and $(z,\ q^K_{A,z},\ q^K_A)$, and $([p^1_{A,z}]_z,\ [p^1_A]_z) = ([p_A]_z,\ [q_A]_z)$, $([q^K_{A,z}]_z,\ [q^K_A]_z) = ([p'_A]_z,\ [q'_A]_z)$.

The Revealed Consistency axiom guarantees that $\widehat{\sim}_z$ and $\widehat{\succ}_z$ are respectively the symmetric and asymmetric components of $\widehat{\succcurlyeq}_z$. That is,

$([p_A]_z,[q_A]_z) \ \widehat{\sim}_z \ ([p'_A]_z,[q'_A]_z)$ iff $([p_A]_z,[q_A]_z) \ \widehat{\succcurlyeq}_z \ ([p'_A]_z,[q'_A]_z)$ & $([p'_A]_z,[q'_A]_z) \ \widehat{\succcurlyeq}_z \ ([p_A]_z, [q_A]_z)$.

and,

$([p_A]_z,[q_A]_z)\widehat{\succ}_z([p'_A]_z,[q'_A]_z)$ iff $([p_A]_z,[q_A]_z) \ \widehat{\succcurlyeq}_z \ ([p'_A]_z,[q'_A]_z)$ & $\neg \ ([p'_A]_z,[q'_A]_z)\widehat{\succcurlyeq}_z \ ([p_A]_z,[q_A]_z)$.

The next definition uses the binary relation $\widehat{\succcurlyeq}_z$ to formalize a way of associating a risk profile with a 0-degenerate lottery.

Definition 8. *We will say that* $(z, \overline{p}_A) \in \Delta$ *is a **DM-degenerate equivalent** of the risk profile* (z, p_A, q_A) *if* $([p_A], [q_A]) \ \widehat{\sim}_z \ ([\overline{p}_A], [\overline{p}_A])$.

7.1.3 A Topological Structure on $\Delta_A/\sim_z$

We next endow the set $\Delta_A/\sim_z$ with a topology. For any $[p'_A]_z$, $[p''_A]_z \in \Delta_A/\sim_z$, let,

- $][p'_A]_z, [p''_A]_z[= \{[p_A]_z \in \Delta_A/\sim_z : p'_A \succ_z p_A \succ_z p''_A\}$,
- $][p'_A]_z, \rightarrow [= \{[p_A]_z \in \Delta_A/\sim_z : p_A \succ_z p'_A\}$, and
- $] \leftarrow, [p'_A]_z[= \{[p_A]_z \in \Delta_A/\sim_z : p'_A \succ_z p_A\}$.

Since $\succcurlyeq_z$ is a preference relation, it is natural to interpret these sets as preference intervals. Let $[q^{**}_A]_z$ and $[q^*_A]_z$ denote the maximal and minimal indifference classes respectively of $\succcurlyeq_z$ in $\Delta_A/\sim_z$, if such elements exist. That is,

$$[q^{**}_A]_z = \{p_A \in \Delta_A : p_A \succcurlyeq_z p'_A, \text{ for all } p'_A \in \Delta_A\},$$

and

$$[q^{*}_A]_z = \{p_A \in \Delta_A : p'_A \succcurlyeq_z p_A, \text{ for all } p'_A \in \Delta_A\},$$

If $[q^{**}_A]_z$ and/or $[q^*_A]_z$ exist, we shall write,

$$][p'_A]_z, \rightarrow [\ = \ ][p'_A]_z, [q^{**}_A]_z], \text{ and }] \leftarrow, [p'_A]_z[\ = [[q^*_A]_z, [p'_A]_z[$$

We endow the set $\Delta_A/{\sim_z}$ with the order topology of $\succcurlyeq_z$, i.e., the coarsest topology containing all sets of the form $][p'_A]_z, \rightarrow [$ and $] \leftarrow, [p'_A]_z[$, thus all sets of the form $][p'_A]_z, [p''_A]_z[$. We endow $[\Delta_A/{\sim_z}]^2$ with the product topology. A set of the type $C = I \times I' \subseteq [\Delta_A/{\sim_z}]^2$, where I and I' are of the form $][p'_A]_z, [p''_A]_z[$, or $][p'_A]_z, \rightarrow [$, or $] \leftarrow, [p'_A]_z[$ shall be referred to as a **cube** in $[\Delta_A/{\sim_z}]^2$. Our strategy in the proof of the representation results below shall be to first establish that $\widehat{\succcurlyeq}_z$ is a weak order 'locally' on such cubes, and then to extend this 'globally' by tying together these cubes. To that end note that if C, $C' \subseteq [\Delta_A/{\sim_z}]^2$ are cubes, then so is $C \cap C'$, if the intersection happens to be non-empty. Further, if we can establish that $\widehat{\succcurlyeq}_z$ is a weak order on C and C', then Revealed consistency implies that the derived rankings must coincide on $C \cap C'$.

7.2 Representation Results

In this subsection we prove our three representation results.

7.2.1 Proof of Theorem 1

The proof of Theorem 1 proceeds through several Lemmas. Before we move to these we define mixture operations on $\Delta_A/{\sim_z}$ and $[\Delta_A/{\sim_z}]^2$. For any $[p_A]_z$, $[q_A]_z \in \Delta_A/{\sim_z}$, and $\lambda \in [0, 1]$, define

$$\lambda[p_A]_z \;\widehat{\oplus}_z\; (1-\lambda)[q_A]_z = [\lambda p_A + (1-\lambda)q_A]_z$$

Similarly, for any $([p_A]_z, [q_A]_z)$, $([p'_A]_z, [q'_A]_z) \in [\Delta_A/{\sim_z}]^2$, and $\lambda \in [0, 1]$, define

$$\lambda([p_A]_z, [q_A]_z) \;\widehat{\oplus}_z\; (1-\lambda)([p'_A]_z, [q'_A]_z) = (\lambda[p_A]_z \;\widehat{\oplus}_z\; (1-\lambda)[p'_A]_z,\ \lambda[q_A]_z \;\widehat{\oplus}_z\; (1-\lambda)[q'_A]_z)$$

That is,

$$\lambda([p_A]_z, [q_A]_z) \;\widehat{\oplus}_z\; (1-\lambda)([p'_A]_z, [q'_A]_z) = ([\lambda p_A + (1-\lambda)p'_A]_z,\ [\lambda q_A + (1-\lambda)q'_A]_z)$$

Any subset of $[\Delta_A/{\sim_z}]^2$ that is itself a mixture set shall be referred to as a mixture subset of $[\Delta_A/{\sim_z}]^2$. Note that because $\succcurlyeq_z$ satisfies the vN-M Independence condition, any cube $C \subseteq [\Delta_A/{\sim_z}]^2$ is a mixture subset of $[\Delta_A/{\sim_z}]^2$. We state the following Lemma about mixture subsets of $[\Delta_A/{\sim_z}]^2$. The proof is omitted.

Lemma 5. *Every mixture subset of $[\Delta_A/{\sim_z}]^2$, in particular $[\Delta_A/{\sim_z}]^2$ itself, is connected.*

We shall now collect some useful notation to aid the exposition of the next Lemma. We shall denote the restriction of $\widehat{\succcurlyeq}_z$ to any set Λ in $[\Delta_A/{\sim_z}]^2$ by $(\widehat{\succcurlyeq}_z)_\Lambda$. Further, let

$$int(\Delta_A/{\sim_z}) = \{[p_A]_z \in \Delta_A/{\sim_z} : [p_A]_z \neq [q_A^{**}]_z,\, [q_A^*]_z\}$$

$$D^* = \{([q_A]_z,\, [q_A]_z) \in [\Delta_A/{\sim_z}]^2 : [q_A]_z \in \Delta_A/{\sim_z}\}$$

$$D = \{([q_A]_z,\, [q_A]_z) \in [\Delta_A/{\sim_z}]^2 : [q_A]_z \in int(\Delta_A/{\sim_z})\}$$

$$\Omega = \Delta_A/{\sim_z} \times int(\Delta_A/{\sim_z}), \text{ and } \Omega^* = \Omega \cup D^*.$$

Note that if $\widehat{\succcurlyeq}_z$ does not have any extremal elements then, $\Delta_A/{\sim_z} = int(\Delta_A/{\sim_z})$ and $D^* = D$. In that case $D^* \subseteq int(\Delta_A/{\sim_z})$ and it follows that $\Omega^* = \Omega$. The following lemma is the most important one in establishing our representation.

Lemma 6. *$(\widehat{\succcurlyeq}_z)_{\Omega^*}$ is a weak order. Further, there exists*
(i) a function $w_z : \Delta_A \to \mathbb{R}$ that represents $\succcurlyeq_z$ and satisfies: for all $\lambda \in [0,1]$, p_A, $q_A \in \Delta_A$

$$w_z(\lambda p_A + (1-\lambda)q_A) = \lambda w_z(p_A) + (1-\lambda)w_z(q_A), \text{ and}$$

(ii) a constant $\pi_z \in [0,\, 1]$,
such that the function $W_z : \Omega^* \to \mathbb{R}$ given by

$$W_z([p_A]_z,\, [q_A]_z) = \pi_z w_z(p_A) + (1-\pi_z)w_z(q_A)$$

represents $(\widehat{\succcurlyeq}_z)_{\Omega^*}$. Further, another pair $(\widetilde{w_z},\, \widetilde{\pi}_z)$ represents $(\widehat{\succcurlyeq}_z)_{\Omega^*}$ in the above sense iff $\widetilde{w_z}$ is a positive affine transformation of w_z and $\widetilde{\pi}_z = \pi_z$, for all $z \in Z$ such that $\succ_z \neq \emptyset$.

Proof. We first consider those $z \in Z$ such that $\succ_z \neq \emptyset$. We break the argument for such z into three main steps.

Step 1: *For any $([p_A]_z,\, [q_A]_z) \in \Omega$ there exists a cube C containing $([p_A]_z,\, [q_A]_z)$ such that $(\widehat{\succcurlyeq}_z)_C$ satisfies the three vN-M axioms on the mixture space $(C,\, \widehat{\oplus}_z)$.*

Take any $([p_A]_z,\, [q_A]_z) \in \Omega$. Note that by the definition of Ω, $[q_A]_z \neq [q_A^{**}]_z$, $[q_A^*]_z$. We will first do the proof under the assumption that $[p_A]_z \neq [q_A^{**}]_z$, $[q_A^*]_z$. The proof for the case when $[p_A]_z$ is equal to either $[q_A^{**}]_z$ or $[q_A^*]_z$ is similar, and we shall make a few brief comments about this case below. Note that there are two possibilities; first that $p_A \nsim_z q_A$, and second that $p_A \sim_z q_A$. Of course if $p_A \sim_z q_A$, then $[p_A]_z \neq [q_A^{**}]_z$, $[q_A^*]_z$.

In the first case assume without loss of generality that $p_A \succ_z q_A$. Accordingly, there exists $p'_A \in \Delta_A$ such that $p_A \succ_z q_A \succ_z p'_A$. Further, by the Contingent Values assumption (CV.1) there exists $\succcurlyeq_{z'} \neq \succcurlyeq_z$, with $\succ_{z'} \neq \emptyset$, and we can pick appropriate p_A, p'_A for which there exists $\underline{p_A}$, $\overline{p_A} \in [p_A]_{z'}$, $\underline{p'_A}$, $\overline{p'_A} \in [p'_A]_{z'}$ such that $\overline{p_A} \succ_z p_A \succ_z \underline{p_A}$, and $\overline{p'_A} \succ_z p'_A \succ_z \underline{p'_A}$. By Continuity, there exists $\lambda^* \in (0, 1)$, such that

$$q_A \sim_z \lambda^* p_A + (1 - \lambda^*)p'_A.$$

Without loss of generality, let $q_A = \lambda^* p_A + (1 - \lambda^*)p'_A$.

Similarly, in the second case as well pick p_A, $p'_A \in [p_A]_z$ for which there exists $\underline{p_A}$, $\overline{p_A}$ $\in [p_A]_{z'}$, $\underline{p'_A}$, $\overline{p'_A} \in [p'_A]_{z'}$ such that $\overline{p_A} \succ_z p_A \succ_z \underline{p_A}$, and $\overline{p'_A} \succ_z p'_A \succ_z \underline{p'_A}$. Note that it is possible that $p_A = p'_A$. Now for any $\lambda \in (0, 1)$,

$$q_A \sim_z \lambda p_A + (1 - \lambda)p'_A.$$

Pick a positive λ^*, and once again let $q_A = \lambda^* p_A + (1 - \lambda^*)p'_A$.

Continuity of $\succcurlyeq_z$ allows us to pick $\underline{p_A}$, $\overline{p_A}$, $\underline{p'_A}$, $\overline{p'_A}$ such that,

$$\overline{q_A} \equiv \lambda^* \underline{p_A} + (1 - \lambda^*)\overline{p'_A} \succ_z q_A \succ_z \lambda^* \overline{p_A} + (1 - \lambda^*)\underline{p'_A} \equiv \underline{q_A}.$$

We can now define the cube $C \subseteq \Omega$ that the statement of Step 1 requires us to do. Let,

$$C = \,]\, [\underline{p_A}]_z, [\overline{p_A}]_z \,[\, \times \,]\![\underline{q_A}]_z, [\overline{q_A}]_z[.$$

Further, let

$$I_{p_A} = \{p''_A \in [p_A]_{z'} : p''_A = \lambda\overline{p_A} + (1 - \lambda)\underline{p_A}, \, \lambda \in [0, 1]\},$$

and,

$$I_{p'_A} = \{p''_A \in [p'_A]_{z'} : p''_A = \lambda\overline{p_A}' + (1 - \lambda)\underline{p_A}', \, \lambda \in [0, 1]\}.$$

We can now define a subset M of Δ as follows:

$$M = \{q \in \Delta : q_Z(z) = \lambda^*, \, q_Z(z') = 1 - \lambda^*, \, q_{A,z} \in I_{p_A}, \, q_{A,z'} \in I_{p'_A}, \text{ and } ([q_{A,z}]_z, [q_A]_z) \in C\}$$

Since $\succcurlyeq_{z'}$ satisfies the vN-M Independence condition, it follows that for any $\widehat{p_A} \in I_{p_A} \subseteq [p_A]_{z'}$, $\widehat{p_A}' \in I_{p'_A} \subseteq [p'_A]_{z'}$,

$$\lambda^*\widehat{p_A} + (1 - \lambda^*)\widehat{p_A}' \in [q_A]_{z'}.$$

Accordingly, any q, $q' \in M$, are comparable at z. Hence,

$$q \succ q' \Rightarrow ([q_{A,z}]_z, [q_A]_z) \,\widehat{\succ}_z\, ([q'_{A,z}]_z, [q'_A]_z),$$

and,

$$q \sim q' \Rightarrow ([q_{A,z}]_z, [q_A]_z) \widehat{\sim}_z ([q'_{A,z}]_z, [q'_A]_z).$$

Further, it is straightforward to verify that for any $([\widehat{p_A}]_z,[\widehat{q_A}]_z) \in C$, there exists $\widehat{p_A} \in I_{p_A}$, $\widehat{p_A}' \in I_{p'_A}$, such that

$$\widehat{q_A} \sim_z \lambda^* \widehat{p_A} + (1 - \lambda^*)\widehat{p_A}'.$$

That is, for any $([\widehat{p_A}]_z, [\widehat{q_A}]_z) \in C$, there exists $q \in M$ such that $[\widehat{p_A}]_z = [q_{A,z}]_z$, and $[\widehat{q_A}]_z = [q_A]_z$. Accordingly, $\widehat{\succsim}_z$ is a weak order on C.

Next, we show that $(\widehat{\succsim}_z)_C$ satisfies the vN-M Independence axiom. First, note that M is a convex set. That is for any q, $q' \in M$, $\lambda q + (1-\lambda)q' \in M$. Let $([p^1_A]_z, [q^1_A]_z)$, $([p^2_A]_z, [q^2_A]_z)$, $([p^3_A]_z, [q^3_A]_z) \in C$ be such that $([p^1_A]_z, [q^1_A]_z) \widehat{\succ}_z ([p^2_A]_z, [q^2_A]_z)$. Then, there exists q, q', $q'' \in M$ such that $([q_{A,z}]_z ,[q_A]_z) = ([p^1_A]_z, [q^1_A]_z)$, $([q'_{A,z}]_z, [q'_A]_z) = ([p^2_A]_z, [q^2_A]_z)$, $([q''_{A,z}]_z, [q''_A]_z) = ([p^3_A]_z, [q^3_A]_z)$, and $q \succ q'$. By Comparable Independence, it follows that

$$\lambda q + (1-\lambda)q'' \succ \lambda q' + (1-\lambda)q''$$

Accordingly, it follows that
$([(\lambda q + (1-\lambda)q'')_{A,z}]_z, [(\lambda q + (1-\lambda)q'')_A]_z) \widehat{\succ}_z [(\lambda q' + (1-\lambda)q'')_{A,z}]_z, [(\lambda q' + (1-\lambda)q'')_A]_z)$
That is,
$([\lambda[q_{A,z}]_z \widehat{\oplus}_z (1-\lambda)[q''_{A,z}]_z, \lambda[q_A]_z \widehat{\oplus}_z (1-\lambda)[q''_A]_z) \widehat{\succ}_z ([\lambda[q'_{A,z}]_z \widehat{\oplus}_z (1-\lambda)[q''_{A,z}]_z, \lambda[q'_A]_z \widehat{\oplus}_z (1-\lambda)[q''_A]_z)$
or,
$(\lambda[p^1_A]_z \widehat{\oplus}_z (1-\lambda)[p^3_A]_z, \lambda[q^1_A]_z \widehat{\oplus}_z (1-\lambda)[q^3_A]_z) \widehat{\succ}_z (\lambda[p^2_A]_z \widehat{\oplus}_z (1-\lambda)[p^3_A]_z, \lambda[q^2_A]_z \widehat{\oplus}_z (1-\lambda)[q^3_A]_z)$
or,
$\lambda(p^1_A]_z, [q^1_A]_z) \widehat{\oplus}_z (1-\lambda)([p^3_A]_z, [q^3_A]_z) \widehat{\succ}_z \lambda([p^2_A]_z, [q^2_A]_z) \widehat{\oplus}_z (1-\lambda)([p^3_A]_z, [q^3_A]_z)$
Hence, $(\widehat{\succsim}_z)_C$ satisfies the vN-M Independence axiom. It is also straightforward to establish that $(\widehat{\succsim}_z)_C$ satisfies the vN-M Continuity axiom; the details are omitted.

The proof for the case when $[p_A]_z$ is equal to either $[q^{**}_A]_z$, $[q^*_A]_z$ is exactly along similar lines. When $[p_A]_z = [q^{**}_A]_z$, take $\overline{p_A} = p_A$ in the above proof, and define the cube C as follows:

$$C =][\underline{p_A}]_z, [q^{**}_A]_z] \times][\underline{q_A}]_z, [\overline{p_A}]_z[\ .$$

The rest of the details are exactly identical. Similarly, when $[p_A]_z = [q^*_A]_z$, take $\underline{p_A} = p_A$ in the above proof, and define

$$C = [[q^*_A]_z, [\overline{p_A}]_z[\times][\underline{q_A}]_z, [\overline{q_A}]_z[\ .$$

Step 2: *There exists a set $O \subseteq [\Delta_A/{\sim_z}]^2$ containing the set,*

$$D^* = \{([q_A]_z,\, [q_A]_z) \in [\Delta_A/{\sim_z}]^2 : [q_A]_z \in \Delta_A/{\sim_z}\}$$

such that $(\widehat{\succcurlyeq}_z)_O$ is a weak order. Further, there exists
(i) a function $w_z : \Delta_A/{\sim_z} \to \mathbb{R}$ that is unique up to affine transformation and satisfies:

$$w_z(\lambda[q_A]_z \;\widehat{\oplus_z}\; (1-\lambda)[p_A]_z) = \lambda w_z([q_A]_z) + (1-\lambda)w_z([p_A]_z), \text{ for any } \lambda \in [0,1], \text{ and}$$

(ii) a constant $\pi_z \in [0,\, 1]$ that is unique,
such that the function $W_z^0 : O \to \mathbb{R}$ given by

$$W_z^0(([q_A]_z,\, [p_A]_z)) = \pi_z w_z([q_A]_z) + (1-\pi_z)w_z([p_A]_z)$$

represents $(\widehat{\succcurlyeq}_z)_O$.

We know from Step 1 that for any $([q_A]_z,\, [q_A]_z) \in D$, there exists a cube containing $([q_A]_z,\, [q_A]_z)$, which we can take to be $C_{[q_A]_z} = [[\underline{q_A}]_z,\, [\overline{q_A}]_z] \times [[\underline{q_A}]_z,\, [\overline{q_A}]_z] \subseteq [\Delta_A/{\sim_z}]^2$ such that $(\widehat{\succcurlyeq}_z)_{C_{[q_A]_z}}$ satisfies the three vN-M axioms. Further, Comparable Monotonicity implies the following: for any $([p'_A]_z,\, [q'_A]_z), ([p''_A]_z,\, [q''_A]_z) \in C_{[q_A]_z}$,

$$[p'_A]_z \;\widehat{\succcurlyeq}_z\; [p''_A]_z \text{ and } [q'_A]_z \;\widehat{\succcurlyeq}_z\; [q''_A]_z \Rightarrow ([p'_A]_z,\, [q'_A]_z) \;\widehat{\succcurlyeq}_z\; ([p''_A]_z,\, [q''_A]_z).$$

where $[p'_A]_z \;\widehat{\succcurlyeq}_z\; [p''_A]_z$ means $([p'_A]_z,\, [p'_A]_z) \;\widehat{\succcurlyeq}_z\; ([p''_A]_z,\, [p''_A]_z)$ (same for $[q'_A]_z \;\widehat{\succcurlyeq}_z\; [q''_A]_z$). Accordingly, $(\widehat{\succcurlyeq}_z)_{C_{[q_A]}}$ satisfies all the axioms of the Anscombe Aumann Theorem (for finite states) – Weak Order, Archimedean Continuity, vN-M Independence, Monotonicity and Non Degeneracy. It follows that there exists a function $w_z^{q_A} :][\underline{q_A}]_z,\, [\overline{q_A}]_z[\to \mathbb{R}$ that is unique up to positive affine transformation, and a constant $\pi_z^{q_A} \in [0,\, 1]$ that is unique, such that for all $([p'_A]_z,\, [q'_A]_z), ([q''_A]_z,\, [q''_A]_z) \in C_{[q_A]_z}$,

$$([p'_A]_z,\, [q'_A]_z) \;\widehat{\succcurlyeq}_z\; ([p''_A]_z,\, [q''_A]_z) \Leftrightarrow \pi_z^{q_A} w_z^{q_A}([p'_A]_z) + (1-\pi_z^{q_A})w_z^{q_A}([q'_A]_z) \geq \pi_z^{q_A} w_z^{q_A}([p''_A]_z) + (1 - \pi_z^{q_A})w_z^{q_A}([q''_A]_z).$$

Further note that the function w_z satisfies: for all $\lambda \in [0,1]$, $[p_A]_z, [p'_A]_z \in][\underline{q_A}]_z,\, [\overline{q_A}]_z[$.

$$w_z(\lambda[p_A]_z \;\widehat{\oplus_z}\; (1-\lambda)[p'_A]_z) = \lambda w_z([p_A]_z) + (1-\lambda)w_z([p'_A]_z).$$

In addition, for any $([p'_A]_z,\, [q'_A]_z) \in C_{[q_A]_z}$, there exists $[\hat{q_A}]_z \in [[\underline{q_A}]_z,\, [\overline{q_A}]_z]$ such that $([p'_A]_z,\, [q'_A]_z) \;\widehat{\sim}_z\; ([\hat{q_A}]_z,\, [\hat{q_A}]_z)$.

Note that $\widehat{\succcurlyeq}_z$ restricted to D^* is complete. This follows since, any two 0-degenerate lotteries (z, p_A) and (z, p'_A) are comparable at z, and accordingly

$$([p_A]_z,\, [p_A]_z) \;\widehat{\succ}_z\; ([p'_A]_z,\, [p'_A]_z) \text{ if } (z, p_A) \succ_z (z, p'_A),$$

or,

$$([p_A]_z,\, [p_A]_z) \,\widehat{\sim}_z\, ([p'_A]_z,\, [p'_A]_z) \text{ if } (z, p_A) \sim_z (z, p'_A),$$

Now define $O = (\cup_{[q_A]_z \in D} C_{[q_A]_z}) \cup D^*$. We will show that $\widehat{\succsim}_z$ restricted to O is a weak order. Pick any $([p'_A]_z,\, [q'_A]_z) \in C_{[q_A]_z}$, $([p''_A]_z,\, [q''_A]_z) \in C_{[p_A]_z}$. We know that there exists $[\hat{q_A}]_z$, $[\hat{p_A}]_z$ $\in \Delta_A/\sim_z$ such that

$$([p'_A]_z,\, [q'_A]_z) \,\widehat{\sim}_z\, ([\hat{q_A}]_z,\, [\hat{q_A}]_z) \text{ and } ([p''_A]_z,\, [q''_A]_z) \,\widehat{\sim}_z\, ([\hat{p_A}]_z,\, [\hat{p_A}]_z).$$

Accordingly, it follows that

$$([p'_A]_z,\, [q'_A]_z) \,\widehat{\succ}_z\, ([p''_A]_z,\, [q''_A]_z) \text{ if } ([\hat{q_A}]_z,\, [\hat{q_A}]_z) \,\widehat{\succ}_z\, ([\hat{p_A}]_z,\, [\hat{p_A}]_z),$$

or,

$$([p'_A]_z,\, [q'_A]_z) \,\widehat{\sim}_z\, ([p''_A]_z,\, [q''_A]_z) \text{ if } ([\hat{q_A}]_z,\, [\hat{q_A}]_z) \,\widehat{\sim}_z\, ([\hat{p_A}]_z,\, [\hat{p_A}]_z).$$

Hence, $(\widehat{\succsim}_z)_O$ is a weak order.

Now consider any two cubes $C_{[q_A]_z}$ and $C_{[p_A]_z}$ that intersect. Pick $([q'_A]_z,\, [q'_A]_z)$, $([q''_A]_z,$ $[q''_A]_z) \in C_{[q_A]_z} \cap C_{[p_A]_z}$, $[q'_A]_z \neq [q''_A]_z$, and recalibrate the function $w_z^{p_A}$ by setting

$$w_z^{p_A}([q'_A]_z) = w_z^{q_A}([q'_A]_z) \text{ and } w_z^{p_A}([q''_A]_z) = w_z^{q_A}([q''_A]_z)$$

Note that by the uniqueness result of the Anscombe Aumann Theorem, the pair $(w_z^{p_A},\, \pi_z^{p_A})$ continues to represent $(\widehat{\succsim}_z)_{C_{[p_A]}}$. Further, $w_z^{p_A} = w_z^{q_A}$ on $][\underline{p_A}]_z,\, [\overline{p_A}]_z[\,\cap\,][\underline{q_A}]_z,\, [\overline{q_A}]_z[$. Hence it follows that $\pi_z^{p_A} = \pi_z^{q_A}$. Next consider $[q_A]_z$, $[p_A]_z$ such that cubes $C_{[q_A]_z}$ and $C_{[p_A]_z}$ do not intersect. Since the set D is connected, $([q_A]_z,\, [q_A]_z)$ and $([p_A]_z,\, [p_A]_z)$ can be linked by finitely many cubes; that is there are finitely many cubes $C_{[p_A^1]_z}, \ldots, C_{[p_A^m]_z}$, such that $C_{[p_A^1]_z}$ $= C_{[q_A]_z}$, $C_{[p_A^m]_z} = C_{[p_A]_z}$, and each subsequent pairs of $C_{[p_A^j]_z}$'s intersect. Further, we can take $C_{[p_A^j]_z} \cap C_{[p_A^{j-k}]_z} = \emptyset$ for every $k \geq 2$. We can then repeat the above re-calibration exercise over pairs of intersecting cubes in the link. This exercise allows us to define a function w_z on $int(\Delta_A/\sim_z)$, as well as establish $\pi_z^{q_A} = \pi_z^{p_A}$, for all $q_A \neq p_A$, $[q_A]_z$, $[p_A]_z \in int(\Delta_A/\sim_z)$. Finally, for $[\overline{p_A}]_z = [q_A^{**}]_z$, or $[q_A^*]_z$ define

$$w_z([\overline{p_A}]_z) = \lim_{\lambda \to 1} w_z(\lambda[\overline{p_A}]_z \widehat{\oplus}_z (1-\lambda)[p_A]_z),$$

where p_A is any element of $int(\Delta_A/\sim_z)$. This then establishes the claim of Step 2.

Step 3: $(\widehat{\succsim}_z)_{\Omega^*}$ is a weak order, and can be represented as in the statement of the Lemma.

We establish the following claim: for any $([p_A]_z,\ [q_A]_z) \in \Omega^*$ there exists $([p'_A]_z,\ [p'_A]_z)$ in D^* such that $([p_A]_z,\ [q_A]_z)\ \widehat{\sim}_z\ ([p'_A]_z,\ [p'_A]_z)$. To that end, define the function $W_z : \Omega^* \to \mathbb{R}$ by

$$W_z([p_A]_z, [q_A]_z) = \pi_z w_z([p_A]_z) + (1 - \pi_z) w_z([q_A]_z)$$

where w_z and π_z are as in Step 2. For any $[\hat{q_A}]_z \in int(\Delta_A/\sim_z)$, let

$$J_{\hat{q_A}} = \{([p_A]_z, [q_A]_z) \in \Omega : W_z([p_A]_z, [q_A]_z) = W_z([\hat{q_A}]_z, [\hat{q_A}]_z)\}$$

We claim that for all $([p_A]_z,\ [q_A]_z)$, $([p'_A]_z,\ [q'_A]_z) \in J_{\hat{q_A}}$, $([p_A]_z,\ [q_A]_z)\ \widehat{\sim}_z\ ([p'_A]_z,\ [q'_A]_z)$. To see this note that, Step 1 guarantees that for any $([p_A]_z,\ [q_A]_z) \in J_{\hat{q_A}}$, there exists a cube C containing $([p_A]_z,\ [q_A]_z)$ such that $(\widehat{\succsim}_z)_C$ satisfies the three vN-M axioms on the mixture set $(C, \widehat{\oplus}_z)$. Accordingly $(\widehat{\succsim}_z)_C$ can be represented by a vonNeumann-Morgenstern utility function. Consider two such cubes C_1 and C_2 that intersect. Because of the Revealed Consistency Axiom, it follows that for any $([q_A]_z,\ [p_A]_z)$, $([q'_A]_z,\ [p'_A]_z) \in C_1 \cap C_2$, $([q_A]_z,\ [p_A]_z)\ (\widehat{\succsim}_z)_{C_1}\ ([q'_A]_z,\ [p'_A]_z)$ iff $([q_A]_z,\ [p_A]_z)\ (\widehat{\succsim}_z)_{C_2}\ ([q'_A]_z,\ [p'_A]_z)$. Further note that if W_{C_1} and W_{C_2} are two vN-M utility functions that represent $(\widehat{\succsim}_z)_{C_1}$ and $(\widehat{\succsim}_z)_{C_2}$ respectively, these functions can be re-calibrated (in a manner similar to that used in Step 2) and set equal on $C_1 \cap C_2$.

Now, consider the cube $C_{[\hat{q_A}]_z}$ around $([\hat{q_A}]_z,\ [\hat{q_A}]_z)$. We have already established in Step 2 that $(\widehat{\succsim}_z)_{C_{[\hat{q_A}]_z}}$ is represented by the function W_z. Further, $J_{\hat{q_A}}$ is connected. Accordingly, $([\hat{q_A}],\ [\hat{q_A}])$ can be linked to any $([p_A],\ [q_A]) \in J_{\hat{q_A}}$ using a finite number of cubes. On each pair of intersecting cubes $\widehat{\succsim}_z$ must coincide as suggested in the last paragraph. Furthermore the vN-M representations of $\widehat{\succsim}_z$ on these cubes can be re-calibrated and brought in line with W_z. Hence, we may conclude that for all $([q_A]_z,\ [p_A]_z)$, $([q'_A]_z,\ [p'_A]_z) \in J_{\hat{q_A}}$, $([q_A]_z,\ [p_A]_z)\ \widehat{\sim}_z\ ([q'_A]_z,\ [p'_A]_z)$.

Note that if $\pi_z \neq 1$, or if $[q_A^{**}]_z$ and $[q_A^*]_z$ do not exist, then we are done establishing our claim. However, if $\pi_z = 1$, and either $[q_A^{**}]_z$ or $[q_A^*]_z$ exists then members of the set

$$B = \{([p_A]_z,\ [q_A]_z) \in \Omega : [p_A]_z = [q_A^{**}]_z \text{ or } [q_A^*]_z\}$$

are not indifferent to any element of D. In this case it is straightforward to verify that for any $([q_A^{**}]_z,\ [q_A]_z) \in B$, $([q_A^{**}]_z,\ [q_A]_z)\ \widehat{\sim}_z\ ([q_A^{**}]_z,\ [q_A^{**}]_z)$. Similarly, for any $([q_A^*]_z,\ [q_A]_z) \in B$, $([q_A^*]_z,\ [q_A]_z)\ \widehat{\sim}_z\ ([q_A^*]_z,\ [q_A^*]_z)$.

Now consider any $([q_A]_z,\ [p_A]_z)$, $([q'_A]_z,\ [p'_A]_z) \in \Omega^*$. From the argument just made, we know that there exists $([\hat{q_A}]_z, [\hat{q_A}]_z)$, $([\tilde{q_A}]_z, [\tilde{q_A}]_z) \in D^*$, such that $([q_A]_z, [p_A]_z) \mathrel{\widehat{\sim}_z} ([\hat{q_A}]_z, [\hat{q_A}]_z)$ and $([q'_A]_z, [p'_A]_z) \mathrel{\widehat{\sim}_z} ([\tilde{q_A}]_z, [\tilde{q_A}]_z)$. Hence, $([q_A]_z, [p_A]_z) \mathrel{\widehat{\succsim}_z} ([q'_A]_z, [p'_A]_z)$ iff $([\hat{q_A}]_z, [\hat{q_A}]_z) \mathrel{\widehat{\succsim}_z} ([\tilde{q_A}]_z, [\tilde{q_A}]_z)$. Clearly it also follows that,

$$([q_A]_z, [p_A]_z) \widehat{\succsim}_z ([q'_A]_z, [p'_A]_z) \Leftrightarrow W_z(([q_A]_z, [p_A]_z)) \geq W_z(([q'_A]_z, [p'_A]_z))$$

Now note that we may 'extend' the domain of the function w_z to Δ_A by simply giving all elements of an equivalence class, say $[p_A]_z$, the value $w_z([p_A]_z)$. It then follows that for all $\lambda \in [0,1]$, p_A, $q_A \in \Delta_A$,

$$w_z(\lambda p_A + (1 - \lambda)q_A) = \lambda w_z(p_A) + (1 - \lambda)w_z(q_A).$$

The uniqueness statement is simply a re-statement of the essential uniqueness result in Step 2. This then completes the proof for those $z \in Z$ for which $\succ_z \neq \emptyset$.

The proof for those $z \in Z$ for which $\succ_z = \emptyset$ is trivial. Note that for this case $[\Delta_A/{\sim_z} \times \Delta_A/{\sim_z}]$ is a singleton. We can take w_z to be any constant function, and π_z to be any number in [0, 1]. □

Remark 1. Note that the function $w_z : \Delta_A \to \mathbb{R}$ in a 'von-Neumann Morgenstern utility representation' of the preference relation $\succsim_z$. We know that if there is some other function v_z that also happens to be a vN-M utility representation of $\succsim_z$, then w_z and v_z must be positive affine transformations of one another. This is a fact that we shall draw on below.

The last Lemma together with Lemma 4 allow us to conclude:

Corollary 1. For any $p \in \Delta$, and z in the support of p_Z, the risk profile $(z, p_{A,z}, p_A)$ has a DM-degenerate equivalent $(z, \zeta_z(p_{A,z}, p_A)) \in \Delta$ that is unique in the following sense: if (z, q_A) is another DM-degenerate equivalent of $(z, p_{A,z}, p_A)$, then $\zeta_z(p_{A,z}, p_A) \sim_z q_A$. Further, there exists a function $w_z : \Delta_A \to \mathbb{R}$, and a constant $\pi_z \in [0,1]$ such that

$$w_z(\zeta_z(p_{A,z}, p_A)) = \pi_z w_z(p_{A,z}) + (1 - \pi_z)w_z(p_A)$$

The function w_z is unique up to positive affine transformation, and the constant π_z is unique for all z such that $\succ_z \neq \emptyset$.

In the subsequent analysis, for any $p \in \Delta$, and z in the support of p_Z, $(z, \zeta_z(p_{A,z}, p_A)) \in \Delta$ shall denote the 0-degenerate equivalent of the risk profile $(z, p_{A,z}, p_A)$.

Lemma 7. *For all $p \in \Delta$,*

$$p \sim \textstyle\sum_{z \in Z} p_Z(z).q_Z^*(z,\, \zeta_z(p_{A,z},\, p_A)),$$

where $q_Z^*(z,\, \zeta_z(p_{A,z},\, p_A)) \in \Delta_Z$ is such that $(z,\, \zeta_z(p_{A,z},\, p_A)) \sim q_Z^*(z,\, \zeta_z(p_{A,z},\, p_A))$.

Proof. (Note on notation: We shall abuse notation here by not distinguishing between a degenerate lottery in Δ_Z that gives some outcome z for sure, and the outcome itself. That is z shall itself stand for the degenerate lottery)

Note that if the support of p_Z is singleton, that is p is a DM-degenerate lottery, then the conclusion follows immediately from Lemma 3. So assume otherwise. Further, denote

$$q_Z \equiv \textstyle\sum_{z \in Z} p_Z(z).q_Z^*(z,\, \zeta_z(p_{A,z},\, p_A)),$$

and suppose towards a contradiction that $p \nsim q_Z$ – say $p \succ q_Z$ (The case of $q_Z \succ p$ is treated analogously). Suppose first that there exists $z,\, z'$ in the support of q_Z, denoted $S[q_Z]$, such that $z \succ z'$. Let $z'' \in S[q_Z]$ be such that $z'' \succcurlyeq z$ for all $z \in S[q_Z]$. By Dominance, it follows that $z'' \succ p$, and so $z'' \succ p \succ q_Z$. By Continuity it follows that there exists $\lambda \in (0,\, 1)$ such that $p \succ \lambda z'' + (1 - \lambda) q_Z$. Finally, by Dominance $\lambda z'' + (1 - \lambda) q_Z \succ p$, which is absurd. Next consider the case where $z \sim z'$ for all $z,\, z' \in S[q_Z]$. We know (from Continuity) that for any $p \in \Delta$ there exists $z'' \in Z$ such that $z'' \succcurlyeq p \succ q_Z$. It follows that there exists $\lambda \in (0, 1)$ such that $p \succ \lambda z'' + (1 - \lambda) q_Z$. But by Dominance $\lambda z'' + (1 - \lambda) q_Z \succ p$, which is absurd. $\square$

Lemma 8. *Let $(z,\, p_A),\, (z,\, q_A) \in \Delta$. Then, for any $\lambda \in [0, 1]$,*

$$\lambda(z, p_A) + (1 - \lambda)(z, q_A) \sim \lambda q_Z^*(z, p_A) + (1 - \lambda) q_Z^*(z, q_A),$$

where $q_Z^*(z, p_A),\, q_Z^*(z, q_A) \in \Delta_Z$ are such that $q_Z^*(z, p_A) \sim (z, p_A)$ and $q_Z^*(z, q_A) \sim (z, q_A)$.

Proof. Note that $q_Z^*(z, p_A)$ and $q_Z^*(z, q_A)$ are comparable as are (z, p_A) and (z, q_A). The result follows immediately from Comparable Independence. The details are omitted. $\square$

We can now complete the proof of Theorem 1. Since $\succcurlyeq_{\Delta_Z}$ satisfies all the three vN-M axioms it follows from the vN-M Theorem, that there exists a function $u : Z \to \mathbb{R}$ such that the expected utility functional, $Eu \to \mathbb{R}$, defined for any $p_Z \in \Delta_Z$ by

$$Eu(p_Z) = \sum_{z \in Z} p_Z(z) u(z)$$

represents the preference relation $\succcurlyeq_{\Delta_Z}$. Next, following Lemma 3 define a function $U : \Delta \to \mathbb{R}$ as $U(p) = u(q_Z^*(p))$, where $p \sim q_Z^*(p)$. Clearly, the function U represents $\succcurlyeq$. Further, applying Lemma 7 gives us that

$$U(p) = \sum_z p_Z(z).u(q_Z^*((z, \zeta_z(p_{A,z}, p_A)))) = \sum_z p_Z(z)U((z, \zeta_z(p_{A,z}, p_A)))$$

For each $z \in Z$, define the functions $\widehat{v_z} : \Delta_A \to \mathbb{R}$ as $\widehat{v_z}(p_A) = U((z, p_A))$. Note that Lemma 8 establishes that for any p_A, $q_A \in \Delta_A$, $\lambda \in [0, 1]$,

$$(z, \lambda p_A + (1-\lambda)q_A) = \lambda(z, p_A) + (1-\lambda)(z, q_A) \sim \lambda q_Z^*(z, p_A) + (1-\lambda)q_Z^*(z, q_A)$$

That is,

$$q_Z^*(z, \lambda p_A + (1-\lambda)q_A) = \lambda q_Z^*((z, p_A)) + (1-\lambda)q_Z^*((z, q_A))$$

Accordingly, it follows that,

$$\widehat{v_z}(\lambda q_A + (1-\lambda)p_A) = \lambda \widehat{v_z}(q_A) + (1-\lambda)\widehat{v_z}(p_A)$$

Hence, as argued in Remark 1, the function $\widehat{v_z}$ must be a positive affine transformation of the function w_z that we derived in Lemma 6. From the uniqueness result there, it follows that the pair $(\widehat{v_z}, \pi_z)$ represents $\widehat{\succcurlyeq}_z$. So it follows that

$$U(p) = \sum_z p_Z(z)[\pi_z \widehat{v_z}(p_{A,z}) + (1-\pi_z)\widehat{v_z}(p_A)]$$

Finally, define $v_z : \Delta_A \to \mathbb{R}$ to be $v_z(q_A) = \widehat{v_z}(q_A) - u(z)$. It then follows that

$$U(p) = \sum_z p_Z(z)[u(z) + \pi_z v_z(p_{A,z}) + (1-\pi_z)v_z(p_A)]$$

This completes the proof of sufficiency of the axioms. Necessity of the axioms is obvious and we do not provide the details here.

The proof of the uniqueness statement is straightforward as well. First note that if we have a triple $(\tilde{u}, (\tilde{v_z})_{z\in Z}, (\tilde{\pi}_z)_{z\in Z})$ and constants $\alpha > 0$, β such that $\tilde{u} = \alpha u + \beta$, $\tilde{v_z} = \alpha v_z$, for all $z \in Z$ and $\tilde{\pi}_z = \pi_z$, for all z such that $\succ_z \neq \emptyset$, then clearly this triple represents $\succcurlyeq$. Now, consider the converse statement. Suppose the triple $(\tilde{u}, (\tilde{v_z})_{z\in Z}, (\tilde{\pi}_z)_{z\in Z})$ represents $\succcurlyeq$. From the uniqueness result of the vN-M Theorem, it follows that there exists constants $\alpha >$

0 and β such that $\tilde{u} = \alpha u + \beta$. Consider any $(z, q_A) \in \Delta$. We know that there exists $p_Z \in \Delta_Z$ such that $(z, q_A) \sim p_Z$. Accordingly,

$$\tilde{u}(z) + \tilde{v_z}(q_A) = \tilde{u}(p_Z)$$

and hence,

$$\tilde{v_z}(q_A) = \tilde{u}(p_Z) - \tilde{u}(z) = \alpha[u(p_Z) - u(z)] = \alpha v_z(q_A)$$

Finally, note that the uniqueness of π_z follows from the uniqueness result in Lemma 6.

THIS COMPLETES THE PROOF OF THEOREM 1.

7.2.2 Proof of Theorem 2

Note that to simplify on notation for this proof, we use γ, δ etc. to denote elements of the set Δ_A. Begin with a triple $(u, (v_z)_{z\in Z}, (\pi_z)_{z\in Z})$ that represents $\succcurlyeq$ in the sense of Theorem 1. Consider any $z, z' \in Z$ with $\succ_z, \succ_{z'} \neq \emptyset$. There are two cases to consider.

Case I : There exists $\delta \in \Delta_A$ such that $[\delta]_{\sim_z} \neq [\delta]_{\sim_{z'}}$.

In this case, there exists $\gamma \in [\delta]_{\sim_{z'}}$, and $\gamma' \in [\delta]_{\sim_z}$ such that $\gamma \succ_z \delta$ and $\gamma' \succ_{z'} \delta$. In order to establish that $\pi_z = \pi_{z'}$, all we need to do is find $p, q \in \Delta$ such that p and q are symmetric with respect to z, z', for once we do that, the result follows immediately from the Symmetry Axiom. We now proceed to establish that there exists such p and q.

We first show that there exists $\widetilde{\delta} \in \Delta_A$ such that the pair $\delta, \widetilde{\delta}$ is equal gains with respect to z and z'. Consider $p', q' \in \Delta$, where

$$p' = ([z, \tfrac{1}{2}; z', \tfrac{1}{2}], \gamma) \text{ and } q' = ([z, \tfrac{1}{2}; z', \tfrac{1}{2}), \gamma').$$

In case $p' \sim q'$, pick any $\widetilde{\delta} \in [\gamma]_z \cap [\gamma']_{z'}$. Then, the pair $\delta, \widetilde{\delta}$ is equal gains with respect to z and z'. On the other hand suppose $p' \nsim q'$, and without loss of generality, suppose $p' \succ q'$. Then we have that $p' \succ q' \succ ([z, \frac{1}{2}; z', \frac{1}{2}], \delta) \equiv p''$, where the final strict preference follows from Comparable Monotonicity. Note that p' and p'' are comparable; so by Continuity it follows that there exists some $\lambda \in (0, 1)$ such that

$$\lambda p' + (1 \text{ - } \lambda)p'' = ([z, \tfrac{1}{2}; z', \tfrac{1}{2}), \lambda\gamma + (1 \text{ - } \lambda)\delta) \sim q'$$

Now pick any $\widetilde{\delta} \in [\lambda\gamma + (1 - \lambda)\delta]_z \cap [\gamma']_{z'}$. It follows that $\delta, \widetilde{\delta}$ is equal gains with respect to z and z'. Now define p, q as follows: $p_Z = q_Z$ with $p_Z(z) = p_Z(z') = \frac{1}{2}$; $p_{A,z} = \delta$, $p_{A,z'} = \widetilde{\delta}$ and $q_{A,z} = \widetilde{\delta}$, $q_{A,z'} = \delta$. Clearly, p and q are symmetric with respect to z and z'.

Case II : $[\delta]_{\sim_z} = [\delta]_{\sim_{z'}}$ for all $\delta \in \Delta_A$.
In this case, by the Contingent Values assumption (CV.1), we know that there exists $\succcurlyeq_{z''}$, with $\succ_{z''} \neq \emptyset$ for which there exists $\delta \in \Delta_A$ such that $[\delta]_{\sim_z} \neq [\delta]_{\sim_{z''}}$ and $[\delta]_{\sim_{z'}} \neq [\delta]_{\sim_{z''}}$. Based on the analysis in Case I it follows that $\pi_z = \pi_{z''}$, and $\pi_{z'} = \pi_{z''}$, and hence $\pi_z = \pi_{z'}$.

THIS COMPLETES THE PROOF OF THEOREM 2.

7.2.3 Proof of Theorem 3

The proof of Theorem 3 proceeds along similar lines as the proof of Theorem 1. First, we define the appropriate mixture operator on the sets $\Delta_A/\sim_z$ and $[\Delta_A/\sim_z]^2$. For any $[p_A]_z$, $[q_A]_z \in \Delta_A/\sim_z$, and $\lambda \in [0,1]$, define

$$\lambda[p_A]_z \mathbin{\widehat{\oplus}_z} (1-\lambda)[q_A]_z = [\lambda p_A \oplus_z (1-\lambda)q_A]_z$$

Similarly, for any $([p_A]_z, [q_A]_z)$, $([p'_A]_z, [q'_A]_z) \in [\Delta_A/\sim_z]^2$, and $\lambda \in [0,1]$, define

$$\lambda([p_A]_z, [q_A]_z) \mathbin{\widehat{\oplus}_z} (1-\lambda)([p'_A]_z, [q'_A]_z) = (\lambda[p_A]_z \mathbin{\widehat{\oplus}_z} (1-\lambda)[p'_A]_z, \lambda[q_A]_z \mathbin{\widehat{\oplus}_z} (1-\lambda)[q'_A]_z)$$

That is,

$$\lambda([p_A]_z, [q_A]_z) \mathbin{\widehat{\oplus}_z} (1-\lambda)([p'_A]_z, [q'_A]_z) = ([\lambda p_A \oplus_z (1-\lambda)p'_A]_z, [\lambda q_A \oplus_z (1-\lambda)q'_A]_z)$$

Note that for any cube $C \subseteq [\Delta_A/\sim_z]^2$, C endowed with the mixture operator $\widehat{\oplus}_z$, denoted $(C, \widehat{\oplus}_z)$, is a mixture set. The following Lemma is along the lines of Lemma 6, and constitutes the back bone of the proof of the Theorem.

Lemma 9. *$(\widehat{\succcurlyeq}_z)_{\Omega^*}$ is a weak order. Further, there exists*
(i) a function $w_z : \Delta_A \to \mathbb{R}$ that satisfies for all $\lambda \in [0,1]$, p_A, $q_A \in \Delta_A$,

$$w_z(\lambda p_A \oplus_z (1-\lambda)q_A) = \lambda w_z(p_A) + (1-\lambda)w_z(q_A), \text{ and}$$

(ii) a constant $\pi_z \in [0, 1]$,
such that the function $W_z : \Omega^* \to \mathbb{R}$ given by

$$W_z([p_A]_z, [q_A]_z) = \pi_z w_z(p_A) + (1-\pi_z)w_z(q_A)$$

represents $(\widehat{\succcurlyeq}_z)_{\Omega^*}$. Further, another pair $(\widetilde{w_z}, \widetilde{\pi_z})$ represents $(\widehat{\succcurlyeq}_z)_{\Omega^*}$ in the above sense iff $\widetilde{w_z}$ is a positive affine transformation of w_z and $\widetilde{\pi_z} = \pi_z$, for all $z \in Z$ such that $\succ_z \neq \emptyset$.

Proof. Just like in Lemma 6, the case where $\succ_z = \emptyset$ is trivial, and hence ignored here. For those z with $\succ_z \neq \emptyset$, the proof can once again be broken down to three steps. The second and third steps are essentially the same. The first step, though similar in spirit, is different in terms of some of its details, and that is what the proof will focus on.

- **Step 1**: For any $([p_A], [q_A]) \in \Omega$ there exists a cube C containing $([p_A]_z, [q_A]_z)$ such that $(\widehat{\succsim}_z)_C$ satisfies the three vN-M axioms on the mixture set $(C, \widehat{\oplus}_z)$.

Pick any $([p_A]_z, [q_A]_z) \in \Omega$. There may be two possibilities. First, $p_A \nsim_z q_A$, and second $p_A \sim_z q_A$. For the first case assume, without loss of generality that, $p_A \succ_z q_A$, and note that there exists $\mathbf{a}, \mathbf{a}' \in A$ such that $\mathbf{a} \sim_z p_A \succ_z q_A \succ_z \mathbf{a}'$. The fact that we may find $\mathbf{a}$ as specified follows from the fact that any lottery in Δ_A has a certainty equivalent with respect to $\succsim_z$; $\mathbf{a}'$ exists as specified because $q_A \notin [q_A^{**}]_z$ or $[q_A^*]_z$, and that $\succsim_z$ satisfies stochastic dominance. Furthermore, biseparability of $\succsim_z$ implies that there exists $\lambda^* \in (0,1)$ such that,

$$[\mathbf{a},\, \lambda^*;\, \mathbf{a}',\, 1-\lambda^*] \sim_z q_A \text{ [11]}$$

Now consider the case where, $p_A \sim_z q_A$. In this case pick $\mathbf{a}, \mathbf{a}' \in [q_A]_z$. It is possible that $\mathbf{a} = \mathbf{a}'$. Then for any $\lambda \in [0,1]$, since $\succsim_z$ satisfies stochastic dominance, we have that

$$[\mathbf{a}, \lambda; \mathbf{a}', 1-\lambda] \sim_z q_A$$

In this case take any $\lambda^* \in (0,1)$. In either case therefore we can find $\mathbf{a}, \mathbf{a}' \in A$, and some $\lambda^* \in (0,1)$ such that the above preference indifference condition holds. Henceforth, without loss of generality, we shall consider $q_A = [\mathbf{a}, \lambda^*;\, \mathbf{a}', 1-\lambda^*]$. Further, like in the proof of Lemma 6, we will consider the case where $[p_A]_z \neq [q_A^{**}]_z$ or $[q_A^*]_z$. The other cases can be dealt along similar lines.

We know by the assumption of Contingent Values (CV.1) that there exists $\succsim_{z'} \neq \succsim_z$, with $\succ_{z'} \neq \emptyset$, such that for an appropriate choice of $\mathbf{a}$, $\mathbf{a}'$, there exists $\overline{\mathbf{a}}$, $\underline{\mathbf{a}}$ and $\overline{\mathbf{a}}'$, $\underline{\mathbf{a}}'$ that satisfy,

[11] Recall from the definition of a biseparable preference that such preferences are characterized by a utility function $v_z : \Delta_A \to \mathbb{R}$, and a probability weighting function (formally a strictly increasing bijection) $\varphi : [0,1] \to [0,1]$. It follows that $v_z(\mathbf{a}) > v_z(q_A) > v_z(\mathbf{a}')$. Let $r \in [0,1]$ be such that $v_z(q_A) = rv_z(\mathbf{a}) + (1-r)v_z(\mathbf{a}')$. Because φ is surjective, there exists $\lambda^* \in (0,1)$ such that $\varphi(\lambda^*) = r$. Hence

$$v_z(q_A) = \varphi(\lambda^*)v_z(\mathbf{a}) + (1-\varphi(\lambda^*))v_z(\mathbf{a}'),$$

and,

$$[\mathbf{a},\, \lambda^*;\, \mathbf{a}',\, 1-\lambda^*] \sim_z q_A.$$

$$\overline{\mathbf{a}} \sim_{z'} \mathbf{a} \sim_{z'} \underline{\mathbf{a}} \text{ and } \overline{\mathbf{a}} \succ_z \mathbf{a} \succ_z \underline{\mathbf{a}},$$

$$\overline{\mathbf{a}}' \sim_{z'} \mathbf{a}' \sim_{z'} \underline{\mathbf{a}}' \text{ and } \overline{\mathbf{a}}' \succ_z \mathbf{a}' \succ_z \underline{\mathbf{a}}'$$

In particular, Bicontinuity of $\succcurlyeq_\Delta$ allows us to choose $\overline{\mathbf{a}}$, $\underline{\mathbf{a}}$ and $\overline{\mathbf{a}}'$, $\underline{\mathbf{a}}'$ in such a way that:

$$\overline{q_A} = [\underline{\mathbf{a}}, \lambda^*; \overline{\mathbf{a}}', 1-\lambda^*] \succ_z \delta \succ_z [\overline{\mathbf{a}}, \lambda^*; \underline{\mathbf{a}}', 1-\lambda^*] = \underline{q_A}.$$

We can now define the cube $C \subseteq \Omega$ that the statement of Step 1 requires us to do. Define,

$$C =][\underline{\mathbf{a}}]_z, [\overline{\mathbf{a}}]_z[\times][\underline{q_A}]_z, [\overline{q_A}]_z[$$

Further, let,

$$I_{\mathbf{a}} = \{\widehat{\mathbf{a}} \in [\mathbf{a}]_{z'} : \overline{\mathbf{a}} \succcurlyeq_z \widehat{\mathbf{a}} \succcurlyeq_z \underline{\mathbf{a}}\},$$

and

$$I_{\mathbf{a}'} = \{\widehat{\mathbf{a}}' \in [\mathbf{a}']_{z'} : \overline{\mathbf{a}}' \succcurlyeq_z \widehat{\mathbf{a}}' \succcurlyeq_z \underline{\mathbf{a}}'\}.$$

Note that for any $\mathbf{a}_1$, $\mathbf{a}_2 \in I_{\mathbf{a}}$, $\lambda \mathbf{a}_1 \oplus_z (1-\lambda)\mathbf{a}_2 \in I_{\mathbf{a}}$, for all $\lambda \in [0,1]$. [12] Similarly, for any $\mathbf{a}'_1$, $\mathbf{a}'_2 \in I'_{\mathbf{a}}$, $\lambda \mathbf{a}'_1 \oplus_z (1-\lambda)\mathbf{a}'_2 \in I_{\mathbf{a}'}$, for all $\lambda \in [0,1]$. We can now define a subset M of Δ as follows:

$$M = \{p = [(z, \widehat{\mathbf{a}}), \lambda^*; (z', \widehat{\mathbf{a}}'), 1-\lambda^*] \in \Delta : \widehat{\mathbf{a}} \in I_{\mathbf{a}},\ \widehat{\mathbf{a}}' \in I_{\mathbf{a}'}, \text{ and } ([p_{A,z}]_z,\ [p_A]_z) \in C\}.$$

Note that for any $p = [(z, \widehat{\mathbf{a}}), \lambda^*; (z', \widehat{\mathbf{a}}'), 1-\lambda^*] \in M$,

$$[p_{A,z}]_z = [\widehat{\mathbf{a}}]_z \text{ and } [p_A]_z = [[\widehat{\mathbf{a}}, p; \widehat{\mathbf{a}}', 1-p]]_z$$

Since, $\succcurlyeq_{z'}$ is a bi-separable preference, it follows that for any $\widehat{\mathbf{a}} \in I_{\mathbf{a}} \subseteq [\mathbf{a}]_{z'}$, $\widehat{\mathbf{a}}' \in I_{\mathbf{a}'} \subseteq [\mathbf{a}']_{z'}$,

$$[\widehat{\mathbf{a}}, \lambda^*; \widehat{\mathbf{a}}', 1-\lambda^*] \sim_{z'} q_A = [\mathbf{a}, \lambda^*; \mathbf{a}', 1-\lambda^*]$$

Accordingly, any p, $p' \in M$ are comparable at z, and

$$p \succ p' \Rightarrow ([p_{A,z}]_z, [p_A]_z) \widehat{\succ}_z ([p'_{A,z}]_z, [p'_A]_z)$$

[12]This follows since $[\mathbf{a}]_{z'}$ is a connected subset of A. Note that any $\mathbf{a}_1$, $\mathbf{a}_2 \in I_{\mathbf{a}}$, $\lambda \in [0,1]$,

$$W_1 = \{\widehat{\mathbf{a}} \in [\mathbf{a}]_{z'} : \widehat{\mathbf{a}} \succcurlyeq_z \lambda \mathbf{a}_1 \oplus_z (1-\lambda)\mathbf{a}_2\},$$

and,

$$W_2 = \{\widehat{\mathbf{a}} \in [\mathbf{a}]_{z'} : \lambda \mathbf{a}_1 \oplus_z (1-\lambda)\mathbf{a}_2 \succcurlyeq_z \widehat{\mathbf{a}}\}$$

form a separation of $[\mathbf{a}]_{z'}$, and hence their intersection must be nonempty.

$$p \sim p' \Rightarrow ([p_{A,z}]_z, [p_A]_z) \widehat{\sim}_z ([p'_{A,z}]_z, [p'_A]_z)$$

It is straightforward to verify (using Bicontinuity) that for any $([\widehat{p_A}]_z, [\widehat{q_A}]_z) \in C$, there exists $\widehat{\mathbf{a}} \in I_{\mathbf{a}}$, $\widehat{\mathbf{a}}' \in I_{\mathbf{a}'}$, such that $\widehat{\mathbf{a}} \sim_z \widehat{p_A}$, and $\widehat{q_A} \sim_z [\widehat{\mathbf{a}}, \lambda^*; \widehat{\mathbf{a}}', 1-\lambda^*]$. That is for any $([\widehat{p_A}], [\widehat{q_A}])$ $\in C$, there exists $p \in M$ such that $[p_{A,z}]_z = [\widehat{p_A}]_z$, and $[p_A]_z = [\widehat{q_A}]_z$. Accordingly, $\widehat{\succsim}_z$ is a weak order on C.

Next, we show that $(\widehat{\succsim}_z)_C$ satisfies the vN-M Independence axiom. First we establish that for any p, $q \in M$, any $\lambda \in [0,1]$, $\lambda p \oplus_z (1-\lambda)q \in M$. Let

$$p = [(z, \tilde{\mathbf{a}}),\ \lambda^*;\ (z', \tilde{\mathbf{a}}'),\ 1-\lambda^*]$$

$$q = [(z, \hat{\mathbf{a}}),\ \lambda^*;\ (z', \hat{\mathbf{a}}'),\ 1-\lambda^*]$$

Further let $\mathbf{a}_\lambda \in I_{\mathbf{a}}$ be such that,

$$\mathbf{a}_\lambda = \lambda \tilde{\mathbf{a}} \oplus_z (1-\lambda)\hat{\mathbf{a}}.$$

We need to show that there exists, $\mathbf{a}'_\lambda \in I_{\mathbf{a}'}$, such that

$$[\mathbf{a}_\lambda,\ \lambda^*;\ \mathbf{a}'_\lambda,\ 1-\lambda^*] = \lambda[\tilde{\mathbf{a}},\ \lambda^*;\ \tilde{\mathbf{a}}',\ 1-\lambda^*] \oplus_z (1-\lambda)[\hat{\mathbf{a}},\ \lambda^*;\ \hat{\mathbf{a}}',\ 1-\lambda^*]$$

In that case we will have proven that

$$[(z, \mathbf{a}_\lambda),\ \lambda^*;\ (z', \mathbf{a}'_\lambda),\ 1-\lambda^*] = \lambda[(z, \tilde{\mathbf{a}}),\ \lambda^*;\ (z', \tilde{\mathbf{a}}'),\ 1-\lambda^*] \oplus_z (1-\lambda)[(z, \hat{\mathbf{a}}),\ \lambda^*;\ (z', \hat{\mathbf{a}}'),\ 1-\lambda^*],$$

our desired conclusion. By Stochastic Dominance it follows that
$[\mathbf{a}_\lambda,\ \lambda^*;\ \overline{\mathbf{a}}',\ 1-\lambda^*] \succsim_z \overline{q_A} \succsim_z \lambda[\tilde{\mathbf{a}},\ \lambda^*;\ \tilde{\mathbf{a}}',\ 1-\lambda^*] \oplus_z (1-\lambda)[\hat{\mathbf{a}},\ \lambda^*;\ \hat{\mathbf{a}}',\ 1-\lambda^*] \succsim_z \underline{q_A} \succsim_z [\mathbf{a}_\lambda,$ $\lambda^*;\ \underline{\mathbf{a}}',\ 1-\lambda^*]$,
with strict preference holding at least somewhere. Bicontinuity in conjunction with the fact the $[\mathbf{a}']_{z'}$ is a connected subset of A immediately implies the desired conclusion.

Now, let $([p_A^1]_z,\ [q_A^1]_z)$, $([p_A^2]_z,\ [q_A^2]_z)$, $([p_A^3]_z,\ [q_A^3]_z) \in C$ be such that $([p_A^1]_z,\ [q_A^1]_z) \widehat{\succ}_z$ $([p_A^2]_z,\ [q_A^2]_z)$. Then, there exists p, p', $p'' \in M$ such that $([p_{A,z}]_z,\ [p_A]_z) = ([p_A^1]_z,\ [q_A^1]_z)$, $([p'_{A,z}]_z,\ [p'_A]) = ([p_A^2]_z,\ [q_A^2]_z)$, $([p''_{A,z}]_z\ ,[p''_A]_z) = ([p_A^3]_z,\ [q_A^3]_z)$, and $p \succ p'$. By Comparable Independence, it follows that

$$\lambda p \oplus_z (1-\lambda)p'' \succ \lambda p' \oplus_z (1-\lambda)p''$$

Accordingly, it follows that

$$([(\lambda p \oplus_z (1-\lambda)p'')_{A,z}]_z, [(\lambda p \oplus_z (1-\lambda)p'')_A]_z) \;\widehat{\succ}_z$$
$$([(\lambda p' \oplus_z (1-\lambda)p'')_{A,z}]_z, [(\lambda p' \oplus_z (1-\lambda)p'')_A]_z)$$

Or,

$$([\lambda p_{A,z} \oplus_z (1-\lambda)p''_{A,z}]_z, [\lambda p_A \oplus_z (1-\lambda)p''_A]_z) \;\widehat{\succ}_z\; ([\lambda p'_{A,z} \oplus_z (1-\lambda)p''_{A,z}]_z, [\lambda p'_A \oplus_z (1-\lambda)p''_A]_z)$$

That is,

$$(\lambda[p_{A,z}]_z \widehat{\oplus}_z (1-\lambda)[p''_{A,z}]_z, \lambda[p_A]_z \widehat{\oplus}_z (1-\lambda)[p''_A]_z) \;\widehat{\succ}_z$$
$$([\lambda[p'_{A,z}]_z \widehat{\oplus}_z (1-\lambda)[p''_{A,z}]_z, \lambda[p'_A]_z \widehat{\oplus}_z (1-\lambda)[p''_A]_z)$$

or,

$$(\lambda[p^1_A]_z \widehat{\oplus}_z (1-\lambda)[p^3_A]_z, \lambda[q^1_A]_z \widehat{\oplus}_z (1-\lambda)[q^3_A]_z) \;\widehat{\succ}_z\; (\lambda[p^2_A]_z \widehat{\oplus}_z (1-\lambda)[p^3_A]_z, \lambda[q^2_A]_z \widehat{\oplus}_z (1-\lambda)[q^3_A]_z)$$

or,

$$\lambda([p^1_A]_z, [q^1_A]_z) \;\widehat{\oplus}_z\; (1-\lambda)([p^3_A]_z, [q^3_A]_z) \;\widehat{\succ}_z\; \lambda([p^2_A]_z, [q^2_A]_z) \;\widehat{\oplus}_z\; (1-\lambda)([p^3_A]_z, [q^3_A]_z)$$

Hence, $(\widehat{\succcurlyeq}_z)_C$ satisfies the vN-M Independence axiom.

We now establish that $(\widehat{\succcurlyeq}_z)_C$ satisfies the vN-M Continuity axiom. Note that this is equivalent to proving the following: For any p, p', $p'' \in M$ such that $p \succ p' \succ p''$, there exists λ, $\lambda' \in (0,1)$, such that:

$$\lambda\; p \oplus_z (1-\lambda)\; p'' \succ p' \succ \lambda'\; p \oplus_z (1-\lambda')\; p''$$

Suppose otherwise – say that $p' \succcurlyeq \lambda p \oplus_z (1-\lambda)p''$ for all $\lambda \in (0,1)$. We proved above that for all $\lambda \in [0,1]$ there exists $\mathbf{a}_\lambda \in I_{\mathbf{a}}$, $\mathbf{a}'_\lambda \in I_{\mathbf{a}'}$ such that,

$$[(z,\, \mathbf{a}_\lambda),\, \lambda^*;\, (z',\, \mathbf{a}'_\lambda);\, 1-\lambda^*] = \lambda p \oplus_z (1-\lambda)p''.$$

Denote,

$$p = [(z, \tilde{\mathbf{a}}),\, \lambda^*;\, (z', \tilde{\mathbf{a}}'),\, 1-\lambda^*]$$

We may then construct a sequence $(\mathbf{a}_{\lambda_k},\, \mathbf{a}'_{\lambda_k})_{k\in\mathbb{Z}_+} \subseteq I_{\mathbf{a}} \times I_{\mathbf{a}'}$ converging to $(\tilde{\mathbf{a}},\, \tilde{\mathbf{a}'}) \in I_{\mathbf{a}} \times I_{\mathbf{a}'}$, such that for all $k \in \mathbb{Z}_+$,

$$p' \succcurlyeq \lambda_k p \oplus_z (1-\lambda_k)p'' = [(z,\, \mathbf{a}_{\lambda_k}),\, \lambda^*;\, (z',\, \mathbf{a}'_{\lambda_k});\, 1-\lambda^*]$$

Let

$$\Xi = \{(\mathbf{a}_{\lambda_k},\, \mathbf{a}'_{\lambda_k}) \in I_{\mathbf{a}} \times I_{\mathbf{a}'} : p' \succcurlyeq [(z,\, \mathbf{a}_{\lambda_k}),\, \lambda^*;\, (z',\, \mathbf{a}'_{\lambda_k});\, 1-\lambda^*]\}$$

By Bicontinuity the set Ξ is closed in $I_{\mathbf{a}} \times I_{\mathbf{a}'}$. It then follows that $(\tilde{\mathbf{a}}, \tilde{\mathbf{a}'}) \in \Xi$, that is $p' \succcurlyeq p = [(z, \tilde{\mathbf{a}}), \lambda^*; (z', \tilde{\mathbf{a}}'), 1 - \lambda^*]$, which is absurd.

Steps 2 and 3 of Lemma 6 can be replicated here to complete the proof. $\square$

The rest of the steps in proving Theorem 3 replicate those in Theorem 1. Therefore we do not repeat them here. The only other step that needs elaboration is proving that the weights π_z that we get by replicating the steps in Theorem 1 are indeed equal. Consider any $\succcurlyeq_z$, $\succcurlyeq_{z'}$ with $\succ_z$, $\succ_{z'} \neq \emptyset$. Like in the proof of Theorem 2, there are two cases to consider. First suppose that there exists $\widehat{\mathbf{a}} \in A$ such that

$$\{\mathbf{a} \in A : \mathbf{a} \sim_z \widehat{\mathbf{a}}\} \neq \{\mathbf{a} \in A : \mathbf{a} \sim_{z'} \widehat{\mathbf{a}}\}$$

In this case, there exists $\mathbf{a} \in [\widehat{\mathbf{a}}]_{z'}$, $\mathbf{a}' \in [\widehat{\mathbf{a}}]_z$ satisfying $\mathbf{a} \succ_z \widehat{\mathbf{a}}$ and $\mathbf{a}' \succ_{z'} \widehat{\mathbf{a}}$. Like in the proof of Theorem 2 the key is to find a pair p, $q \in \Delta$ such that p and q are symmetric with respect to z and z'.

We will now show that there exists $\widetilde{\mathbf{a}} \in A$ such that the pair $\widehat{\mathbf{a}}$, $\widetilde{\mathbf{a}}$ is equal gains with respect to z and z'. Consider p', $q' \in \Delta$, where

$$p' = ([z, \tfrac{1}{2}; z', \tfrac{1}{2}], \mathbf{a}) \text{ and } q' = ([z, \tfrac{1}{2}; z', \tfrac{1}{2}], \mathbf{a}').$$

In case $p' \sim q'$, pick any $\widetilde{\mathbf{a}} \in [\mathbf{a}]_z \cap [\mathbf{a}']_{z'}$. Then, the pair $\widehat{\mathbf{a}}$, $\widetilde{\mathbf{a}}$ is equal gains with respect to z and z'. On the other hand suppose $p' \nsim q'$, and without loss of generality, suppose $p' \succ q'$. Then we have that $p' \succ q' \succ ([z, \frac{1}{2}; z', \frac{1}{2}], \widehat{\mathbf{a}}) \equiv p''$, where the final strict preference follows from Comparable Monotonicity. Note that p' and p'' are comparable. So by Bicontinuity it follows that there exists some $\lambda \in (0, 1)$, such that

$$\lambda p' \oplus_z (1 \text{ - } \lambda) p'' = ([z, \tfrac{1}{2}; z', \tfrac{1}{2}], \lambda \mathbf{a} \oplus_z (1 - \lambda)\widehat{\mathbf{a}}) \sim q'.$$

Now pick any $\widetilde{\mathbf{a}} \in [\mathbf{a}'']_z \cap [\mathbf{a}']_{z'}$. It follows that $\widehat{\mathbf{a}}$, $\widetilde{\mathbf{a}}$ is equal gains with respect to z and z'. Now define p, q as follows: $p_Z = q_Z$ with $p_Z(z) = p_Z(z') = \frac{1}{2}$; $p_{A,z} = \widehat{\mathbf{a}}$, $p_{A,z'} = \widetilde{\mathbf{a}}$ and $q_{A,z} = \widetilde{\mathbf{a}}$, $q_{A,z'} = \widehat{\mathbf{a}}$. Clearly, p and q are symmetric with respect to z and z'.

On the other hand if

$$\{\mathbf{a} \in A : \mathbf{a} \sim_z \widehat{\mathbf{a}}\} = \{\mathbf{a} \in A : \mathbf{a} \sim_{z'} \widehat{\mathbf{a}}\}$$

for all $\widehat{\mathbf{a}} \in A$, then by the Contingent Values assumption (CV.2), there exists $\succcurlyeq_{z''}$, with $\succ_{z''} \neq \emptyset$ for which there exists $\widehat{\mathbf{a}} \in Z_{-0}$ such that

$$\{\mathbf{a} \in A : \mathbf{a} \sim_z \widehat{\mathbf{a}}\} \neq \{\mathbf{a} \in A : \mathbf{a} \sim_{z''} \widehat{\mathbf{a}}\}$$

and,

$$\{\mathbf{a} \in A : \mathbf{a} \sim_{z'} \widehat{\mathbf{a}}\} \neq \{\mathbf{a} \in A : \mathbf{a} \sim_{z''} \widehat{\mathbf{a}}\}.$$

Based on the argument in the last paragraph, we can then conclude that $\pi_z = \pi_{z''}$, and $\pi_{z'} = \pi_{z''}$, and hence $\pi_z = \pi_{z'}$.
THIS COMPLETES THE PROOF OF THEOREM 3.

7.2.4 Proof of Proposition 2

Recall that the representative voter is pivotal with probability $\lambda = \frac{1}{n}$, and γ denotes the probability that option 1 is the outcome when she is not pivotal. Then the probability distributions over final allocations generated by the representative voter choosing 1 and 2 are respectively,

$$p^1 = [(z^1, \mathbf{a}^1),\ \lambda + (1-\lambda)\gamma;\ (z^2, \mathbf{a}^2),\ 1 - \lambda - (1-\lambda)\gamma],$$

$$p^2 = [(z^1, \mathbf{a}^1),\ (1-\lambda)\gamma;\ (z^2, \mathbf{a}^2),\ 1 - (1-\lambda)\gamma]$$

Under out representation these two lotteries are evaluated as:

$$U(p^1) = u_H + \frac{v_L}{2} - (\lambda + (1-\lambda)\gamma)[u_H - u_L - \frac{1}{2}(v_H - v_L)] + \frac{1}{2}[\varphi(\lambda + (1-\lambda)\gamma)v_H + (1 - \varphi(\lambda + (1-\lambda)\gamma))v_L]$$

and,

$$U(p^2) = u_H + \frac{v_L}{2} - (1-\lambda)\gamma[u_H - u_L - \frac{1}{2}(v_H - v_L)] + \frac{1}{2}[\varphi((1-\lambda)\gamma)v_H + (1 - \varphi((1-\lambda)\gamma))v_L]$$

Subtracting the two gives,

$$U(p^2) - U(p^1) = \lambda[u_H - u_L - \frac{1}{2}(v_H - v_L)] - \frac{1}{2}(v_H - v_L)[\varphi(\lambda + (1-\lambda)\gamma) - \varphi((1-\lambda)\gamma)]$$

Accordingly,

$$U(p^2) - U(p^1) \geq 0 \Leftrightarrow g(\lambda) = \lambda(2\nu - 1) - (\varphi(\lambda + (1-\lambda)\gamma) - \varphi((1-\lambda)\gamma)) \geq 0$$

Now suppose everyone other than RV votes for option 1; i.e., $\gamma = 1$. Then,

$$g(\lambda) = \lambda(2\nu - 1) - (1 - \varphi(1-\lambda))$$

and, for $\lambda \in (0, 1 - \overline{\lambda})$,

$$g'(\lambda) = 2\nu - 1 - \varphi'(1 - \lambda)$$

Let $\lambda' = \min\{1 - \overline{\lambda}, \underline{\lambda}\}$. Then for all $\lambda \in (0, \lambda')$, $g'(\lambda) < 0$. Further, $g(0) = 0$. Hence, $g(\lambda)$ < 0 for all $\lambda \in (0, \lambda')$. Let $\overline{n}$ be any integer greater than $\frac{1}{\lambda'}$. Then,for all $n > \overline{n}$, everyone voting for option 2 is a Nash equilibrium.

Now consider the case when everyone other than RV votes for option 2. That is $\gamma = 0$. Then,

$$g(\lambda) = \lambda(2\nu - 1) - \varphi(\lambda) = \lambda[2\nu - 1 - \frac{\varphi(\lambda)}{\lambda}]$$

Note that, for $\lambda < \lambda'$, $\varphi'(\lambda) > 2\nu - 1$, and since φ is concave over this range, $\frac{\varphi(\lambda)}{\lambda} > \varphi'(\lambda)$. Accordingly, for $\lambda < \lambda'$, $g(\lambda) < 0$, and everyone voting for option 2 can not be a Nash equilibrium. Hence, for all $n \geq \overline{n}$, everyone voting for option 2 is the unique symmetric Nash equilibrium (in pure strategies).

Further, note that when $\gamma = 0$, $g(1) = 2\nu > 0$. By continuity of g, there exists an interval $(\lambda_1, 1]$, such that for all $\lambda \in (\lambda_1, 1]$, $g(\lambda) > 0$, and accordingly everyone voting for option 2 is a Nash equilibrium.

Finally, note that when $\gamma = 1$, $g(1) = 2\nu - 2 > 0$. Once again by the continuity of g, there exists an interval $(\lambda_2, 1]$, such that for all $\lambda \in (\lambda_2, 1]$, $g(\lambda) > 0$, and accordingly everyone voting for option 1 is not a Nash equilibrium. Let, $\lambda'' = \max\{\lambda_1, \lambda_2\}$, and $\underline{n}$ be any integer less than $\frac{1}{n''}$. It follows that for all $n \leq \underline{n}$, everyone voting for option 2 is the unique symmetric Nash equilibrium (in pure strategies). □

8 References

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