Constructing Equilibria in All-Pay Contests with Applications to Competitions with Conditional Investments

Preliminary and incomplete*

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1 Introduction

Many competitions are characterized by asymmetries among competitors. One example is the competition for promotions. Promotion decisions are often based on employees' productivity and effort, as well as on tenure and quality of relationships with superiors. Thus, differences in employees' abilities, social skills, and costs of effort translate to asymmetries in competition. Another example is the competition for rents in a regulated market. Since firms often compete by engaging in lobbying activities, firm-specific attributes such as quality and quantity of political connections, cost of capital, and geographic location frequently affect competition. A third example is research and development (R&D) races. The outcome of such races depends on firms' research technologies, past R&D investments, and access to human capital, all of which may vary across firms.

In addition to the asymmetries among competitors, competitions are also characterized by an "all-pay" feature: irreversible investments are made before the outcome of the competition is known. In some settings, competitors may also commit to investments conditional on winning. In an R&D race, for example, a better prototype not only costs more and improves a firm's chances of winning, but also commits the firm to higher expenditures conditional on winning.

A large literature has modeled competitions using the complete-information all-pay auction (henceforth: all-pay auction) or its variants.[1] The assumption of complete information helps interpret players' payoffs as "economic rents", in contrast to "information rents" that arise in models of competition with private information.[2] Other models of competition postulate a probabilistic relation between competitors' efforts and prize allocation.[3] For a comprehensive treatment of the literature on competitions with sunk investments see Shmuel Nitzan (1994) and Kai Konrad (2007).

This literature has produced important insights. Often, however, tractability required simplifying assumptions that are at odds with important features of real-world competitions. In particular, most existing models accommodate only a limited degree of asymmetry among competitors, and do not allow for a combination of conditional and unconditional investments. Many models are limited to a single prize and two competitors.

The "all-pay contests" of Siegel (2008) overcome many of these limitations. In an all-pay

[1] Examples include rent-seeking and lobbying activities (Hillman & Samet (1987), Hillman & Riley (1989), Baye, Kovenock & de Vries (1993), González-Díaz (2007)), competitions for a monopoly position (Ellingsen (1991)), waiting in line (Clark & Riis (1998)), sales (Varian (1980)), and R&D races (Dasgupta (1986)), competitions for multiple prizes (Clark & Riis (1998) and Barut & Kovenock (1998)), the effect of lobbying caps (Che & Gale (1998, 2006) and Kaplan & Wettstein (2006)), and R&D races with endogenous prizes (Che & Gale (2003)).

[2] For example, Moldovanu & Sela (2001, 2006), Kaplan, Luski, Sela, & Wettstein (2002)), and Parreiras & Rubinchik (2006).

[3] See the seminal papers by Edward Lazear and Sherwin Rosen (1981) and Gordon Tullock (1980) .

contest (henceforth: contest), each player chooses a costly "score", and the players with the highest scores obtain one prize each (relevant ties can be resolved using any tie-breaking rule). Conditional on winning or losing, a player's payoff decreases weakly and continuously with his chosen score. The primitives of the contest are commonly known. This captures players' knowledge of the asymmetries among them.

The generality of players' cost functions allows for a wide degree of heterogeneity among players, including differing production technologies, costs of capital, and prior investments. Different players may also be disadvantaged relative to others in different regions of the competition.[4] In addition, the difference in a player's payoff between winning and losing may depend on his chosen score. This feature accommodates both sunk and conditional investments, player-specific risk attitudes, and player- and score-dependent valuations for a prize.

When all investments are unconditional, each player is characterized by his valuation for a prize, which is the payoff difference between winning and losing, and a weakly increasing, continuous cost function that determines his cost of choosing a score independently of whether he wins or loses. Such contests are *separable*. Single- and multiprize all-pay auctions are separable contests with linear costs.[5]

Siegel (2008) provides a closed-form formula for players' equilibrium payoffs in contests that meet certain Generic Conditions, without solving for an equilibrium. In such "generic" contests, a player's payoff in any equilibrium is the most he can guarantee himself provided no player chooses a score that gives him a negative payoff conditional on winning. This result is useful for understanding the effects of changes in competition structure, such as adding players, adding prizes, and changing prizes' values, and for computing players' aggregate expenditures, or rent dissipation, when all players' valuations are identical (as is the case, for example, when prizes are monetary). The payoff result can also be used to provide a sufficient condition for the participation of precisely one more player than the number of prizes.

The payoff result is, however, silent on the number and structure of equilibria. Many interesting questions regarding equilibrium uniqueness, allocations, scores, individual expenditures, and aggregate expenditures when valuations are different cannot be deduced from the payoff characterization alone. For example, the payoff result implies that a player's expected payoff does not depend on his cost when he loses, so that a player's expected payoff in a modified complete-information first-price auction in which he pays a positive fraction of his bid if he loses does not depend on the size of the fraction. In contrast, equilibrium strategies and, consequently, allocations, scores, and expenditures, do depend on the size of the fraction (as shown in Section 4).

[4] See the example of Section 1.1 in Siegel (2008).

[5] Thus, contests generalize multiprize all-pay auctions by allowing for non-linear, asymmetric costs, and accommodating both sunk and conditional investments.

This paper provides an algorithm that constructs the unique equilibrium for a large class of contests. This class nests single- and multiprize generic all-pay auctions. Theorem 4 shows that when costs are strictly increasing and piecewise analytical, an $(m+1)$-player contest for m prizes has a unique equilibrium. To obtain this result, I first construct the unique equilibrium consisting of "well-behaved" strategies. The construction relies on knowledge of players' payoffs, which is provided by the payoff result. Identifying players' best-response sets (or strategy supports) is the main difficulty in the construction, since a player's best-response set may consist of several disjoint intervals. Once existence and uniqueness of a well-behaved equilibrium are established, Theorem 3 rules out the existence of equilibria that are not well behaved. This step is not straightforward, since little can be assumed about the structure of such equilibria, if they exist. Theorem 4 implies that under the same analyticity assumption an n-player contest has at most one equilibrium in which precisely $m+1$ players participate. The participation result of Siegel (2008) provides a sufficient condition for precisely $m+1$ players to participate in any equilibrium, and therefore for the equilibrium constructed by the algorithm to be the unique equilibrium of an n-player contest.

As an application of the algorithm, I investigate the class of *simple* contests. In a simple contest, a positive fraction $\alpha \leq 1$ of each competitor's costs is sunk, and the remaining $1-\alpha$ is paid only by the winners of the $m \geq 1$ prizes. When $\alpha \neq 1$ the contest is not separable. Competitors have access to the same underlying technology for producing "score", but may differ in how efficiently they employ it. This difference in efficiency is captured by player-specific cost coefficients, which are multiplied by a common cost function representing the common production technology. Competitors may also differ in their valuations for a prize. I show that simple contests have a unique equilibrium, in which the best-response set of every player is an interval. Moreover, after normalizing each player's efficiency by dividing his cost coefficient by his valuation for a prize, I show that the equilibrium strategies of more efficient players first-order stochastically dominate those of less efficient players, and that more efficient players win prizes more often than less efficient players. As α approaches 0, the most efficient players obtain a prize with near certainty. When players differ only in their valuations for a prize, as α approaches 0 the contest becomes efficient and expenditures are maximized, since the prizes are allocated to the players with the highest valuations and equilibrium payoffs are independent of α.

The limit of the equilibria as α approaches 0 is an equilibrium of the corresponding "first-price contest". In this equilibrium, players' payoffs are given by the payoff result and no player chooses weakly dominated strategies with positive probability. The equilibrium is robust to the tie-breaking rule. This provides a selection criterion among the continuum of equilibria of the first-price contest, which are not payoff equivalent.

When $\alpha = 1$, all investments are sunk, so the contest is separable. I provide a closed-form formula for players' equilibrium strategies. If, in addition, the common production technology is linear, we have an all-pay auction. [6]

The rest of the paper is organized as follows. Contests are defined and results from Siegel (2008) are stated in Section 2. Section 3 describes the equilibrium construction algorithm and establishes uniqueness. Section 4 analyzes simple contests. Examples 1 and 2 are detailed in Appendix A. The proofs of the results of Section 3 are in Appendix B. The proofs of the results of Section 4 are in Appendix C.

2 The Model and Previous Results

In a *contest*, n players compete for m homogeneous prizes, $0 < m < n$. The set of players $\{1, \ldots, n\}$ is denoted by N. Players compete by each choosing a score, simultaneously and independently. Player i chooses a score $s_i \in S_i = [0, \infty)$.[7] Each of the m players with the highest scores wins one prize. In case of a relevant tie, any procedure may be used to allocate the tie-related prizes among the tied players.

Given scores $s = (s_1, \ldots, s_n)$, one for each player, *player i's payoff* is

$$u_i(s) = P_i(s)\, v_i(s_i) - (1 - P_i(s))\, c_i(s_i)$$

where $v_i : S_i \to \mathbb{R}$ is *player i's valuation for winning*, $c_i : S_i \to \mathbb{R}$ is *player i's cost of losing*, and $P_i : \times_{j \in N} S_j \to [0, 1]$ is *player i's probability of winning*, which satisfies

$$P_i(s) = \begin{cases} 0 & \text{if } s_j > s_i \text{ for } m \text{ or more players } j \neq i \\ 1 & \text{if } s_j < s_i \text{ for } N - m \text{ or more players } j \neq i \\ \text{any value in } [0,1] & \text{otherwise} \end{cases}$$

such that $\sum_{j=1}^{n} P_j(s) = m$.

Note that a player's probability of winning depends on all players' scores, but his valuation for winning and cost of losing depend only on his chosen score. The primitives of the contest are commonly known.

I make the following assumptions, which are depicted in Figure 1.

A1 *v_i and $-c_i$ are continuous and non-increasing.*

A2 $v_i(0) > 0$ *and* $\lim_{s_i \to \infty} v_i(s_i) < c_i(0) = 0$.

[6] This result generalizes and corrects that of Clark & Riis (1998). They constructed an equilibrium for the multiprize all-pay auction and claimed it was unique. Their proof of uniqueness relied on showing that in any equilibrium the best response set of each player is an interval. Their proof of this latter claim was incorrect.

[7] Siegel (2008) allowed for player-specific, non-negative lower bounds on the strategy space. Since this paper focuses on strictly increasing costs, I have set all lower bounds to 0.

A3 $c_i(s_i) > 0$ *if* $v_i(s_i) = 0$.

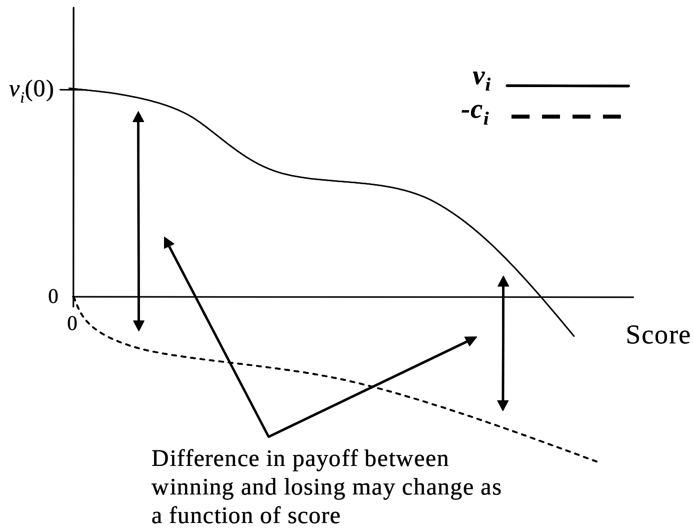

Figure 1: Assumptions A1-A3

Assumption A3 stresses the all-pay nature of contests. It is not satisfied by complete-information first-price auctions, for example, since a player pays nothing if he loses, and is therefore indifferent between losing and winning with a bid that equals his valuation for the prize. But the condition *is* met when an all-pay element is introduced, e.g., when every bidder pays some positive fraction of his bid whether he wins or not, and only the winner pays the balance of his bid.[8] As Figure 1 shows, a player's valuation for a prize, which is the payoff difference between winning and losing, may depend on the player's chosen score.

In a *separable contest*, this difference is constant, so $v_i(s_i) = V_i - c_i(s_i)$ and $u_i(s) = P_i(s) V_i - c_i(s_i)$, where $V_i = v_i(0) > 0$. The value $c_i(s_i)$ can be thought of as player i's cost of choosing score s_i, which does not depend on whether he wins or loses, and V_i could be thought of as player i's valuation for a prize, which does not depend on his chosen score. If a given score can be achieved in different ways, $c_i(s_i)$ corresponds to the least costly way of achieving it. All expenditures are unconditional, and players are risk neutral. Separable contests with linear costs are single- and multi-prize complete-information all-pay auctions (Hillman & Samet (1987), Hillman & Riley (1989), Clark & Riis (1998)).[9]

The following concepts are key in characterizing the payoffs of players in equilibrium.

[8] For a further discussion of the model and the assumptions see Siegel (2008).

[9] Formally, $v_i(s_i) = V_i - s_i, c_i(s_i) = s_i, a_i = 0$, and ties are resolved by randomizing uniformly, where V_i is bidder i's valuation for a prize.

Definition (i) Player i's *reach* r_i is the highest score at which his valuation for winning is 0. That is, $r_i = \max \{s_i \in S_i | v_i(s_i) = 0\}$. Re-index players in (any) decreasing order of their reach, so that $r_1 \geq r_2 \geq \ldots \geq r_n$.

(ii) Player $m + 1$ is the *marginal player.*

(iii) The *threshold* T of the contest is the reach of the marginal player: $T = r_{m+1}$.

(iv) Players i's *power* w_i is his valuation for winning at the threshold. That is, $w_i = v_i(T)$. In particular, the marginal player's power is 0.

In a separable contest, a player's reach is the highest score he can choose by expending no more than his valuation for a prize. For example, in an all-pay auction a player's reach is his valuation for a prize.

Theorem 1 in Siegel (2008), which is stated below, characterizes players' equilibrium payoffs in contests that meet the following two conditions.

Generic Conditions (i) Power Condition - The marginal player is the only player with power 0. (ii) Cost Condition - The marginal player's valuation for winning is strictly decreasing at the threshold, i.e., for every $x \in [a_{m+1}, T)$, $v_{m+1}(x) > v_{m+1}(T) = 0$.[10]

I refer to a contest that meets the Generic Conditions as a *generic contest.* In an m-prize all-pay auction, the Cost Condition is met trivially, because costs are strictly increasing. If the Power Condition is met, i.e., the valuation V_{m+1} of player $m + 1$ is different from those of all other players, the all-pay auction is generic.[11]

Theorem 1 *In any equilibrium of a generic contest, the expected payoff of every player equals the maximum of his power and* 0*.*

Theorem 1 shows that players in $N_W = \{1, \ldots m\}$ ("winning players") have strictly positive expected payoffs, and players in $N_L = \{m + 1, \ldots, n\}$ ("losing players") have expected payoffs of 0. A player *participates* in an equilibrium of a contest if with strictly positive probability he chooses scores associated with strictly positive costs of losing. The following result provides a sufficient condition for players in $N_{L:} \backslash \{m + 1\}$ not to participate in any equilibrium.

Theorem 2 *In a generic contest, if the normalized costs of losing and valuations for winning for the marginal player are, respectively, strictly lower and weakly higher than those of player*

[10] In a separable contest, because $v_{m+1}(x) = V_{m+1} - c_{m+1}(x)$, the cost condition is that for every $x \in [a_{m+1}, T)$, $c_{m+1}(x) < c_{m+1}(T) = V_{m+1}$.

[11] Contests that do not meet the Generic Conditions can be perturbed slightly to meet them. Perturbing the marginal player's valuation for winning around the threshold leads to a contest that meets the Cost Condition. Doing the same for all players with zero power generates a contest that meets the Power Condition.

$i > m+1$, that is

$$\frac{c_{m+1}(x)}{v_{m+1}(0)} < \frac{c_i(x)}{v_i(0)} \text{ for all } x \geq 0 \text{ such that } c_i(x) > 0$$

and

$$\frac{v_{m+1}(x)}{v_{m+1}(0)} \geq \frac{v_i(x)}{v_i(0)} \text{ for all } x \geq 0$$

then player i does not participate in any equilibrium. In particular, if these conditions hold for all players in $N_L \backslash \{m+1\}$, then in any equilibrium only the $m+1$ players in $N_W \cup \{m+1\}$ may participate.

3 Solving for Equilibrium

Recall that a player's strategy is a probability distribution over $[0, \infty)$, and an equilibrium is a profile of strategies, one for each player, such that each player's strategy assigns probability 1 to the player's best-responses.

Identifying players' best-response sets, or strategy supports, is a key step in solving for equilibrium. The difficulty is that with more than two players best-response sets are not necessarily intervals, and may in fact be quite "pathological".[12] It is therefore not immediately obvious how to (1) guarantee the existence of and solve for a "well-behaved" equilibrium and (2) rule out "pathological" equilibria. I do both for a large class of contests, which nests single- and multiprize all-pay auctions. To this end, I consider *regular contests*, defined as follows.

Definition 1 *An n-player, m-prize contest is a* regular contest *if it is generic and meets the following two regularity conditions:*

R1 *The valuation for winning is strictly decreasing and the cost of losing is strictly increasing for all players.*

R2 *The valuation for winning and the cost of losing are piecewise analytical on $[0, T]$ for players $1, \ldots, m+1$.*[13]

Condition R1 and the proof of the Threshold Lemma in Siegel (2008) imply that in every equilibrium of a regular contest players $1, \ldots, m+1$ participate, i.e., choose positive scores with positive probability. In what follows, using the payoff result, I solve for the unique, well-behaved

[12] Example 2 depicts an equilibrium in which a player's best-response set is the Cantor set.

[13] A function f is piecewise analytical on $[0, T]$ if $[0, T]$ can be divided into a finite number of closed intervals such that the restriction of f to each interval is analytical. Analytical functions include polynomials, the exponent function, trigonometric functions, and power functions. Sums, products, compositions, reciprocals, and derivatives of analytical functions are analytical (see, for example, Chapman (2002)).

equilibrium of $(m+1)$-player regular contests. This also shows that n-player regular contests have *at most* one equilibrium in which precisely $m+1$ players participate.

Consider the unique equilibrium of the three-player, two-prize, separable regular contest depicted in Figure 2.[14] Figure 3 depicts players' equilibrium strategies, drawn as cumulative probability distributions (CDFs). In the equilibrium, player 2's best-response set is $(0, x_1] \cup [x_2, 1]$, and that of player 3 is $[0, x_3] \cup [x_4, 1]$. Each player is defined as being "active" on his best-response set, with the possible inclusion of 0.[15]

The algorithm described in Section 3.2 constructs the equilibrium by identifying the players active on each interval (denoted in curly brackets), and the "switching points" above which the

[14] Players' cost functions are given in Appendix B.8.

[15] In the equilibria of Baye et al. (1993), who considered a non-generic single-prize all-pay auction, a player's best-response set may be the union of 0 and a single interval whose lower endpoint is strictly positive. All such equilibria disappear when players' valuations are perturbed slightly to produce unique players with the first- and second-highest valuations (so a single player has power 0). This leaves a single equilibrium, in which the best-response set of each player is an interval (or the singleton 0). A similar perturbation produces a single equilibrium, in which the best-response set of each player is an interval, in González-Díaz (2007). In contrast, the non-interval property that arises here is "fundamental" in nature: it is robust to perturbations in the contest's specification, and, moreover, a player's best-response set may consist of several disjoint intervals of positive length.

set of active players changes ($x_k, 1 \leq k \leq 4$, and 1).[16]

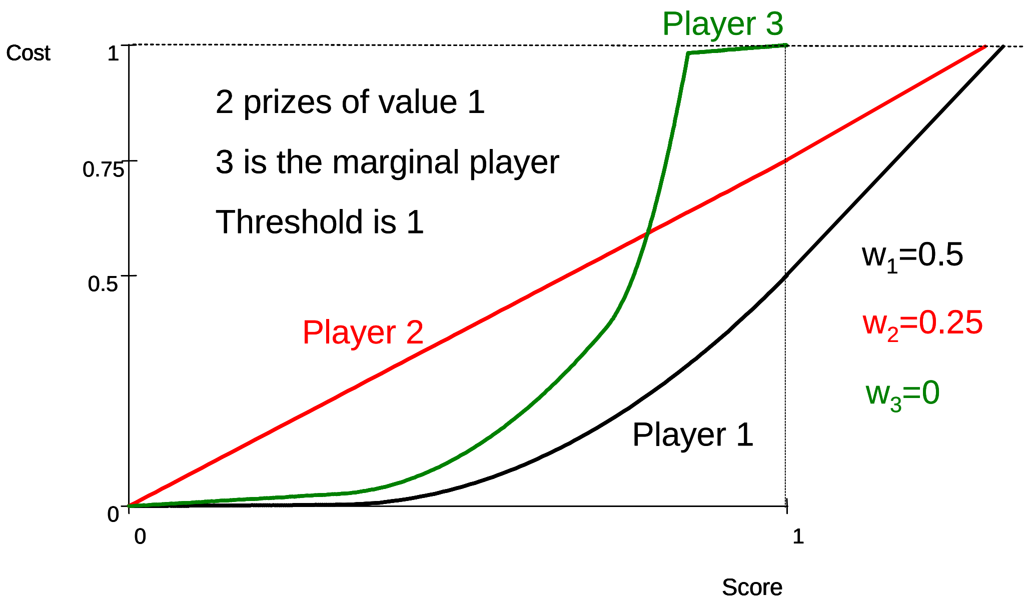

Figure 2: Player's costs, reaches, and powers

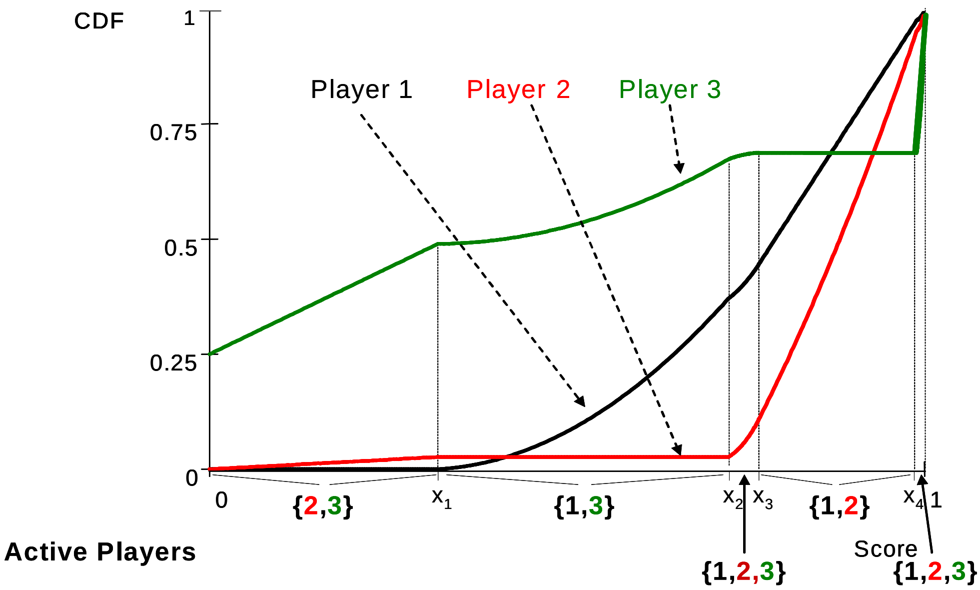

Figure 3: The unique equilibrium

For such a construction to be possible, the equilibrium must be "well-behaved", in that for

[16]The problem of solving for an equilibrium can be formulated as an optimal control problem with linear control, in which the state variables are players' CDFs. The objective is to minimize the sum of players' expected payoffs, subject to the constraints that CDFs are non-decreasing and that no player obtains more than his power. This latter constraint is a state constraint, which precludes the application of Pontryagin's Maximization Principle in its standard form (see Hestenes (1966) and Seierstad & Sydsaeter (1977)). Thus, the standard "pasting conditions" cannot be used, and the same difficulties remain in determining the switching points and sets of active players.

every $x < T$ the set of players active immediately to the right of x must remain constant. This is formalized by the following definition.

Definition 2 *An equilibrium is* ***constructible*** *if for every score $x < T$ there exists some $\bar{x} > x$ such that for each player either every score in $(x, \bar{x})$ is a best response, or no score in $(x, \bar{x})$ is a best response. I refer to equilibria that are not constructible as* ***non-constructible****.*

The algorithm of Section 3.2 solves for a constructible equilibrium $G = (G_1, \ldots, G_{m+1})$ of an $(m+1)$-player regular contest, where $G_i(x)$ is player i's CDF, i.e., the probability that player i chooses a score lower than or equal to x. This is done by using three properties of constructible equilibria to generate a profile of CDFs, and then showing that these CDFs form a constructible equilibrium. The properties are derived in Section 3.1. I begin with an informal overview of these properties and the algorithm.

The first property is that only player $m+1$ has an atom at 0 and the size of the atom is determined by players' payoffs (which are the same in all equilibria). This determines $G(0)$. The second property is that for every $x < T$ the value of G on some right-neighborhood of x is uniquely determined from $G(x)$ and the set of players active immediately to the right of x. This value coincides with the solution to a set of simultaneous equations derived from the condition that active players obtain their equilibrium payoff and inactive players' CDFs do not increase. The third property is that the set of players active to the right of $x < T$ is uniquely determined by $G(x)$, and the first switching point above x can be uniquely determined as well. This is seen by showing that $G(x)$, players' costs in a right-neighborhood of x, and players' payoffs jointly define a "supply function" for hazard rates whose unique positive fixed point identifies the set of players active to the right of x. This uses Condition R2. The first switching point above x is the first point that violates one of the following two equilibrium conditions when G is defined to the right of x as described in the second element above. First, no player should be able to obtain more than his equilibrium payoff. Thus, an inactive player becomes active when other players' CDFs, and therefore his probability of winning, become sufficiently high. Second, players' CDFs must be non-decreasing. Thus, an active player becomes inactive if his CDF would otherwise decrease.

Combining these three properties, the algorithm constructs G by proceeding from 0 to T.[17] Beginning at 0, the set of active players to the right of 0 is determined from $G(0)$, and G is defined up to and including the first switching point above 0. The process is repeated until T is reached. The number of switching points that result is finite, and the resulting G is indeed a constructible equilibrium. In the course of the construction, a player may become active and inactive several times, leading to non-interval supports. This is what happens to players 2 and

[17] Condition R1, the Threshold Lemma of Siegel (2008), and the payoff characterization imply that all players choose scores precisely up to the threshold. It therefore suffices to specify G on $[0, T]$.

3 in the equilibrium of Figure 3 (the construction of this equilibrium is described at the end of Section 3.2 as an application of the algorithm).

Because the local conditions combined with players' payoffs uniquely determine G at every point, the algorithm constructs the unique constructible equilibrium. Theorem 3 in Section 3.3 rules out the existence of non-constructible equilibria using the fact that a constructible equilibrium exists. The remainder of Section 3.3 considers implications for n-player contests. Section 3.4 discusses the much simpler case of two-player contests.

3.1 Properties of a Constructible Equilibrium

Suppose that G is a constructible equilibrium of an $(m+1)$-player regular contest. The value of G at 0 is given by the following lemma, which doesn't rely on constructibility.

Lemma 1 $G_i(0) = 0$ *for* $i < m+1$*, and* $G_{m+1}(0) = \min_{i \leq m} \frac{w_i}{v_i(0)} < 1$.

Now, choose y in $(0,T)$ and suppose y is a best response for player i. Since there are $m+1$ players, payoffs equal powers. Thus, y is a best response for player i if and only if

$$P_i(y)\, v_i(y) - (1 - P_i(y))\, c_i(y) = w_i,$$

where $P_i(y)$ is the probability that i wins a prize when other players choose scores according to G. Equivalently,

$$1 - P_i(y) = 1 - \frac{w_i + c_i(y)}{v_i(y) + c_i(y)}. \tag{1}$$

Since there are m prizes, if G is continuous at y the expression on on left-hand side equals $\Pi_{j \in N \setminus \{i\}} (1 - G_j(y))$, i.e., the probability that all other players choose scores higher than y. In fact, G is continuous on $(0,T)$.[18] Since $v_i(y) + c_i(y)$ is the gain from winning relative to losing, I refer to the right-hand side of Equation (1) as *player* i*'s normalized excess payoff at* y, denoted

$$q_i(y) = 1 - \frac{w_i + c_i(y)}{v_i(y) + c_i(y)} = \frac{v_i(y) - v_i(T)}{v_i(y) + c_i(y)} > 0.$$

Thus,

$$\Pi_{j \in N \setminus \{i\}} (1 - G_j(y)) = q_i(y) \tag{2}$$

if and only if y in $(0,T)$ is a best response for player i.

Let x be a score in $[0,T)$. Considering scores y slightly higher than x, I denote the set of players for whom all such scores are best responses by $A^+(x)$, and refer to it as the set of *players active to the right of* x:

$$A^+(x) = \{i \in N : \text{Equation (2) holds for all } y \in (x,z) \text{ for some } z > x\}.$$

[18] This follows from condition R1 (see Lemma 5 in Appendix B).

That $A^+(x)$ is well defined follows from constructibility of G. Let

$$\bar{x} = \sup\left\{z \in (x,T) : A^+(y) = A^+(x) \text{ for all } y \in [x,z)\right\}.$$

I refer to $\bar{x}$ as the first *switching point* above x, i.e., the first score higher than x at which the set of players active to the right of x changes.

I now show that $G(x)$ and $A^+(x)$ determine the value of G on $[x,\bar{x}]$. By constructibility and continuity of G, if $j \notin A^+(x)$, then G_j does not increase on $[x,\bar{x}]$. Thus, given $G(x)$ and $A^+(x)$, Equation (2) for players j in $A^+(x)$ and score y in $(x,\bar{x}]\backslash T$ leads to a system of $|A^+(x)|$ equations (where $|A|$ denotes the cardinality of a set A) in $|A^+(x)|$ unknowns $(1 - G_j(y))$. The unique solution is given by the following lemma (the proof of this and other results in this section is found in Appendix B):[19]

Lemma 2 *Let $D = \Pi_{j \notin A^+(x)}(1 - G_j(x))$ (if $A^+(x) = N$, then $D = 1$). For every $y \in [x,\bar{x}] \cap [x,T)$,*

$$G_i(y) = \begin{cases} 1 - \dfrac{\Pi_{j\in A^+(x)} q_j(y)^{\frac{1}{|A^+(x)|-1}}}{q_i(y) D^{\frac{1}{|A^+(x)|-1}}} & \text{if } \ i \in A^+(x) \\ G_i(x) & \text{if } \ i \notin A^+(x) \end{cases} \tag{3}$$

and $G_i(T) = 1$.

I now show that $G(x)$ uniquely determines $A^+(x)$, and then show that $\bar{x}$ is uniquely determined as well. For every x in $[0,T)$, let

$$A(x) = \{i \in N : \text{Equation (2) holds with } x \text{ in place of } y\}. \tag{4}$$

I refer to $A(x)$ as the set of *players active at x*. For any x in $(0,T)$, these are the players for whom x is a best response. By right-continuity of q_i and G at every x in $[0,T)$, $A^+(x) \subseteq A(x)$. This inclusion provides an upper bound on $A^+(x)$: *players who are active to the right of x must be active at x*. But this bound is not tight, i.e., $A^+(x)$ may be a strict subset of $A(x)$.[20] Nevertheless, $A(x)$ uniquely determines $A^+(x)$. To see this, I rewrite Equation (2) in terms of marginal percentage changes as follows.

Denote by $\varepsilon_i(y) = -\frac{q_i'(y)}{q_i(y)} > 0$ player i's *semi-elasticity at $y < T$*, and by $h_j(y) = -\frac{(1-G_j(y))'}{1-G_j(y)}$ player j's *hazard rate at y*, where all derivatives denote right-derivatives.[21] For i in $A^+(x)$, by

[19] $|A^+(x)| \geq 2$ is guaranteed by condition R1 (Lemma 5 in Appendix B).

[20] This is the case at x_1 in Figure 3, since $A(x_1) = \{1,2,3\}$ and $A^+(x_1) = \{1,3\}$. The correspondence $x \Rightarrow A(x)$ can be thought of as "right upper-hemi continuous". In general, however, it is not "right lower hemi-continuous".

[21] By condition R2, and since G is given by Equation (3), q_i and G are right-continuously differentiable. $q_i'(y)$ is strictly negative since $q_i'(y) = \left(\frac{v_i(y)-v_i(T)}{v_i(y)+c_i(y)}\right)' = \frac{\overbrace{v_i'(y)}^{\text{Negative}}\overbrace{(c_i(y)+v_i(T))}^{\text{Positive}}+\overbrace{c_i'(y)}^{\text{Positive}}\overbrace{(v_i(T)-v_i(y))}^{\text{Negative}}}{(v_i(y)+c_i(y))^2}$

Equation (2), player i's normalized excess payoff at $y > x$ equals the *product* of the other players' probabilities of choosing scores higher than y, for y sufficiently close to x. Thus, taking natural logs and differentiating Equation (2), $\varepsilon_i(y)$ equals the *sum* $\sum_{j \in N \setminus \{i\}} h_j(y)$ of the other players' hazard rates at y. Since players who are not in $A^+(x)$ have hazard rates of 0 at y,

$$\forall i \in A^+(x) : \varepsilon_i(y) = \sum_{j \in A^+(x) \setminus \{i\}} h_j(y) \tag{5}$$

By right-continuity, Equation (5) holds at x. In addition, since no player can obtain more than his power on a right-neighborhood of x,

$$\forall i \in A(x) : \varepsilon_i(x) \geq \sum_{j \in A^+(x) \setminus \{i\}} h_j(x) \text{ with equality for } i \in A^+(x) \tag{6}$$

Letting $H(x) = \sum_{j \in A^+(x)} h_j(x)$, Equation (5) and Inequality (6) can be combined as

$$\forall i \in A(x) : h_i(x) = \max\{H(x) - \varepsilon_i(x), 0\} \tag{7}$$

Equation (7) pins down players' hazard rates at x. To see this, think of the right-hand side of Equation (7) with H in place of $H(x)$ as player i's "supply curve" of "hazard rate" as a function of "price" H. Then $S_x(H) = \sum_{i \in A(x)} \max\{H - \varepsilon_i(x), 0\}$ is the aggregate supply of hazard rates at x given H. In equilibrium, by adding up Equation (7) for $i \in A(x)$, the aggregate "hazard rates" supplied must equal the actual aggregate hazard rate $H(x)$. Thus, $H(x)$ must satisfy $S_x(H(x)) = H(x)$. To determine $H(x)$ from S_x, note that S_x is a piecewise linear function, whose slope increases by 1 every time H exceeds the semi-elasticity of a player in $A(x)$. Since all semi-elasticities are positive and $|A(x)| \geq 2$,[22] $S'_x(0) = 0$ and $H(x) \neq 0$. So, S_x is a convex function that starts below the diagonal and reaches a slope of at least 2. Therefore, it intersects the diagonal precisely once above 0, at $H(x)$ (see Figure 4 below).

Since players with a positive hazard rate at x are in $A^+(x)$, if $\varepsilon_i(x) < H(x)$ for a player i in $A(x)$, then i is in $A^+(x)$.[23] Since a player l in $A(x)$ must obtain his power immediately to the right of x to be in $A^+(x)$, if $\varepsilon_l(x) > H(x)$, then $l \notin A^+(x)$. This is depicted in Figure 4: $A(x) = \{i, j, l\}$, and $A^+(x) = \{i, j\}$, since $\varepsilon_i(x) < H(x), \varepsilon_j(x) < H(x)$, and $\varepsilon_l(x) > H(x)$. Also, S'_x does not increase at $\varepsilon_k(x)$, since player k is not active at x ($k \notin A(x)$).

[22] Lemma 5 in Appendix B.

[23] By definition of $H(x)$ as the fixed point of $S_x(H)$, there are at least two such players, so $|A^+(x)| \geq 2$.

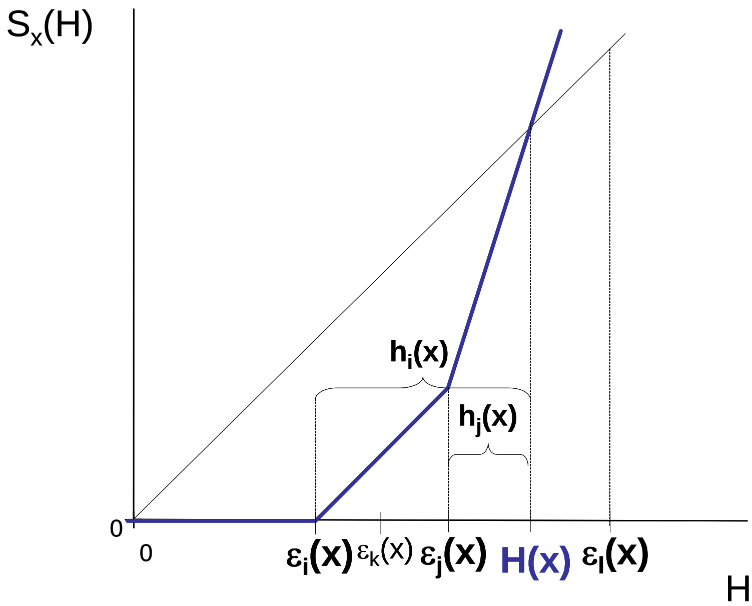

Figure 4: The function S_x and its fixed point $H(x)$

A complication arises when $\varepsilon_i(x) = H(x)$ for a player i in $A(x)$. The correct assignment of such a player is important, since his semi-elasticity slightly above x may differ from the aggregate hazard rate, so he may or may not be active to the right of x, which in turn influences the aggregate hazard rate. That this assignment can be determined unambiguously follows from the assumption of piecewise analytical costs. This is shown in Lemma 6 in Appendix B, which provides the following procedure for deciding whether a player i in $A(x)$ is in $A^+(x)$. Compare $\varepsilon_i(x)$ and $H(x)$; if they are equal, compare their first right-derivatives, etc. (This will "generically" stop at the first derivatives.) The lowest order derivatives of $\varepsilon_i(x)$ and $H(x)$ that differ determine whether player i is in $A^+(x)$ ($<$: in $A^+(x)$, $>$: not in $A^+(x)$). If all derivatives are equal, i is in $A^+(x)$.

Now consider the first switching point $\bar{x} < T$ above x. This is the first score for which $A^+(x) \neq A^+(\bar{x})$. If $j \in A^+(\bar{x}) \setminus A^+(x)$, then $j \in A(\bar{x}) \setminus A^+(x)$, so j obtains his power at $\bar{x}$. If, on the other hand, $j \in A^+(x) \setminus A^+(\bar{x})$, then $h_j(\bar{x}) = 0$. Thus, to identify $\bar{x}$ consider the first point $y > x$ such that Equation (2) holds for a player $j \notin A^+(x)$, or $h_j(y) = 0$ for a player $j \in A^+(x)$, or y is a concatenation point of the cost function of a player in $A^+(x)$ (recall that costs are piecewise-defined functions), or $y = T$. If $y \neq T$, using Equation (4) determine $A(y)$ from $G(y)$, and use $H(y)$ to determine $A^+(y)$ from $A(y)$ as described above. If $A^+(y) \neq A^+(x)$, then $\bar{x} = y$. If $A^+(y) = A^+(x)$, then y is not a true switching point, and the search continues above y for the next candidate switching point. This can only repeat a finite number of times before $\bar{x}$ is identified.[24]

[24]Finiteness can be shown using analyticity, as in the proof of Lemma 6. Players $j \in A^+(x)$ for whom $\varepsilon_i^{(k)}(x) = H^{(k)}(x)$ for all $k \geq 0$ are ignored in the search for the first candidate switching point, etc.

3.2 The Algorithm

The properties derived in the previous subsection suggest the following algorithm for constructing a candidate constructible equilibrium G on $[0, T]$ First, define $G(0)$ as in Lemma 1. Set $x = 0$. Define $A(x)$ from $G(x)$ using Equation (4). Determine $A^{+}(x)$ from $A(x)$ via S_x and its unique fixed point as described above.[25] Identify $\bar{x}$, the first switching point higher than x, as described above. Define G on $[x, \bar{x}]$ using Equation (3). If $\bar{x} \neq T$, set $x = \bar{x}$ and go to Step 2.

For every score x in $(0, T)$ which has been reached in this process, the following points are true.

1. G is continuous and non-decreasing on $(0, x)$ by construction.

2. $\left(1 - \Pi_{j\in N\setminus\{i\}} (1 - G_j(x))\right) v_i(x) - \left(\Pi_{j\in N\setminus\{i\}} (1 - G_j(x))\right) c_i(x) \leq w_i$, with equality if $h_i(x) > 0$, by construction.

3. $G(x) \in (0, 1)$. This follows from the continuity and monotonicity of G up to y, since $G(0) < 1$ (Lemma 1), and if $G_i(x) \geq 1$, then every player $j \neq i$ would obtain strictly more than his power by choosing a score slightly lower than x, violating Point 2.

4. $|A(x)| \geq 2$. This can be seen by induction on the number of switching points up to y, since (i) $|A(0)| \geq 2$ ($A(0) = \left\{i \in N : w_i \leq \min_{j<m+1} \frac{w_j}{v_j(0)}\right\}$), (ii)if $|A(y)| \geq 2$ then $|A^{+}(y)| \geq 2$ for any $y < x$ (see footnote 23), and (iii) $A^{+}(y) \subseteq A(\bar{y})$ for any $y < x$ by construction.

Points 3 and 4 show that the algorithm can proceed from any score $y < T$ that has been reached. To show that the algorithm terminates, it suffices to show that the number of switching points is finite.

Lemma 3 *The number of switching points in $[0, T]$ identified by the algorithm is finite. In addition, $A(x) = N$ for all x sufficiently close to T.*

The construction will therefore reach T by applying the steps above a finite number of times. Thus, the output G is characterized by a partition into a finite number of intervals of positive length, on the interior of which the set of active players remains constant. The value of G on each interval is given by Equation (3). To show that G is an equilibrium, it remains to show that $G_i(T) = 1$.

Lemma 4 *For every player i, $\lim_{x\to T} G_i(x) = G_i(T) = 1$.*

Proposition 1 *G is a constructible equilibrium, which is continuous above* 0.

[25] This requires $|A(x)| \geq 2$, which is shown below. And $|A(x)| \geq 2$ implies $|A^{+}(x)| \geq 2$ (footnote 23).

Proof. G is a profile of probability distribution functions, since it is right-continuous on $[0,T]$, weakly increasing, and $G(T) = 1$ (Point 1 and Lemma 4). It is continuous above 0 (Point 1 and Lemma 4). No player can obtain more than his power and G_i is strictly increasing only where player i obtains precisely his power (Point 2). Thus, best responses are chosen with probability 1, so G is an equilibrium. By the construction procedure, G is constructible (for every $x < T$, every score in $(x, \bar{x})$ is a best response for players in $A^+(x)$ and no score in $(x, \bar{x})$ is a best response for players in $N \backslash A^+(x)$). ■

For an illustration, consider the supply function S_x and its positive fixed point $H(x)$ in the context of Figure 3 above. $A(0) = A^+(0) = \{2, 3\}$. As x increases from 0 to T, the set of active players changes. At the switching point x_1, player 1 becomes active since he obtains his power. This changes S_x and $H(x)$ discontinuously. As a result, $H(x_1)$ falls below player 2's hazard rate, and he becomes inactive immediately above x_1. At x_2, player 2 rejoins the set of active players, and all three players are active up to x_3. Thus, the addition of an active player may or may not cause another to become inactive. At x_3, player 3's hazard rate reaches 0, and he becomes inactive immediately above x_3. Player 3 rejoins the set of active players at x_4, and all three players remain active up to the threshold.[26]

3.3 Equilibrium Uniqueness and Implications

Since the value of G at 0, the switching points, and the corresponding sets of active players are uniquely determined, we have the following.

Corollary 1 *G is the unique constructible equilibrium of an $(m+1)$-player regular contest.*

Proof. Consider a constructible equilibrium $\tilde{G}$, and denote by $\tilde{x}$ the supremum of the scores on which $\tilde{G}$ coincides with G. Since $\tilde{G}(0) = G(0)$ (Lemma 1), both $\tilde{G}$ and G are continuous on $(0,T)$ (Lemma 5 in Appendix B), and $\tilde{G}(T) = G(T) = 1$ (by Condition R1), we have $\tilde{G}(\tilde{x}) = G(\tilde{x})$. By constructibility of $\tilde{G}$ and the construction of G, $\tilde{G}(y) = G(y)$ on a right-neighborhood of $\tilde{x}$, for $\tilde{x} < T$. Thus, $\tilde{x} = T$. ■

Corollary 1 does not apply to non-constructible equilibria, since it assumes that for every $x \in [0,T)$ the set of active players remains constant on (x,y) for some $y \in (x,T)$. To show that the output of the algorithm is the unique equilibrium, the existence of equilibria that are

[26] Bulow & Levin (2006) constructed the equilibrium of a game in which players have linear costs and compete for heterogeneous prizes. Their construction proceeds from the top, without first identifying players' equilibrium payoffs. This is possible because each player's best-response set is an interval and players' marginal costs are identical. Such a procedure does not work here, as the set of players active to the *left* of x cannot be uniquely determined from $G(x)$ and players' powers using local conditions.

not constructible must be ruled out. Theorem 3 does this, using the fact that a constructible equilibrium exists.[27]

Theorem 3 *If a generic $(m+1)$-player contest that meets Condition R1 has a constructible equilibrium G, then G is the unique equilibrium of the contest.*

Since a regular contest has a constructible equilibrium, and the existence of a constructible equilibrium guarantees uniqueness, combining Lemma 3, Proposition 1, and Theorem 3 we have the following.

Theorem 4 *An $(m+1)$-player regular contest has a unique equilibrium, which is constructible. It is characterized by a partition of $[0,T]$ into a finite number of closed intervals with disjoint interiors of positive length, such that the set of active players is constant on the interior of each interval. Thus, each player's best-response set is a finite union of intervals. All players are active on the last interval.*

Proof. Immediate. ■

Example 2 shows that when players' costs are not piecewise analytical, so the contest is not regular, non-constructible equilibria may exist.

An immediate implication of Theorem 4 is that equivalent players play identical strategies.

Corollary 2 *In an $(m+1)$-player regular contest, if $v_i = \gamma v_j$ and $c_i = \gamma c_j$ on $[0,T]$ for some $\gamma > 0$, then players i and j play identical strategies.*

Proof. Multiplying a player's Bernoulli utility function by a positive constant does not affect his strategic behavior, and so does not change the set of equilibria of the contest. Thus, if the strategies of players i and j were different, switching them would lead to a second equilibrium, contradicting uniqueness. ■

I now turn to regular contests with any number of players.

Corollary 3 *Regular contests have at most one equilibrium in which $m+1$ players participate. The candidate for this equilibrium is the unique equilibrium of the reduced contest with players $1,\ldots,m+1$. It is an equilibrium of the original contest if and only if players $m+2,\ldots,n$ cannot obtain strictly positive payoffs by participating. If they can, then in every equilibrium at least $m+2$ players participate.*

[27] Clark & Riis (1998) and, to the best of my reading, Bulow & Levin (2006), who constructed equilibria of similar games with a continuum of pure strategies in which more than two players participate, did not rule out the existence of equilibria that are not constructible. Such equilibria do not arise in Baye et al. (1996), but are not ruled out in the setting of González-Díaz (2007), who extends the analysis of Baye et al. (1996) to more general costs.

Proof. Since valuations for winning are strictly decreasing, players $1, \ldots, m+1$ participate in every equilibrium (see the proof of the Threshold Lemma in Siegel (2008)). Thus, an equilibrium in which precisely $m+1$ players participate must coincide with the unique equilibrium of the reduced contest that involves players $1, \ldots, m+1$. ■

In some regular contests, only players $1, \ldots, m+1$ participate in any equilibrium. A sufficient condition, which is met by generic all-pay auctions, is given in Theorem 2. By Corollary 3, such contests have a unique equilibrium, given by the algorithm.

3.4 Two-Player Contests

Two-player, single-prize contests are relatively simple to solve because equilibrium strategies exist such that every positive score up to the threshold is a best response for both players. This is shown in Theorem 5.

Theorem 5 *In a two-player, single-prize contest that satisfies Condition R1, the unique equilibrium is* $(G_1, G_2)(x) = \left(\frac{c_2(x)}{v_2(x)+c_2(x)}, \frac{w_1+c_1(x)}{v_1(x)+c_1(x)}\right)$ *on* $[0, T]$.

Section 4.1 below provides an application of the result. Theorem 5, applied to separable contests, extends the results of Kaplan & Wettstein (2006) and Che & Gale (2006), who considered two-player separable contests with strictly increasing, ordered cost functions. When costs are not strictly increasing, a generic separable contest may have other equilibria (see Example 1). When the Cost Condition is not met, equilibria may exist in which a player obtains more than his power (see Example 2 in Siegel (2008)). An immediate consequence of Theorem 5 is the following.

Corollary 4 *A generic one-prize contest that meets Condition R1 has at most one equilibrium in which two players participate.*

Proof. Similar to that of Corollary 3 above. ■

4 Simple Contests

As the example of Figures 2 and 3 shows, even when a regular contest has a unique equilibrium, gaps in a player's best-response set may still arise. In this section I identify a class of regular contests with a unique equilibrium, in which every player's best response set is an interval. This class nests generic all-pay auctions.

Consider a situation in which all players share a common underlying technology, captured by a strictly increasing function $c(\cdot)$ with $c(0) = 0$, but may differ in their efficiency of employing this technology. This difference is captured by every player i's idiosyncratic *cost coefficient*

$\gamma_i > 0$. For example, workers at a production plant who compete for promotions based on output use the same equipment to manufacture certain products, but differ in their skill of operating the equipment. A fraction α in $(0, 1]$ of the cost is sunk. The remainder of the cost, $1 - \alpha$, is borne only if the player wins a prize. For every player i, we therefore have $v_i(s_i) = V_i - \gamma_i c(s_i)$ and $c_i = \alpha\gamma_i c(s_i)$, where $V_i > 0$. A generic contest in this family is called a *simple contest* (see Figure 5 below). When $\alpha = 1$ all investments are sunk.[28] If, in addition, $c(x) = x$ and $\gamma_i = 1$ for every player i, we have an all-pay auction.

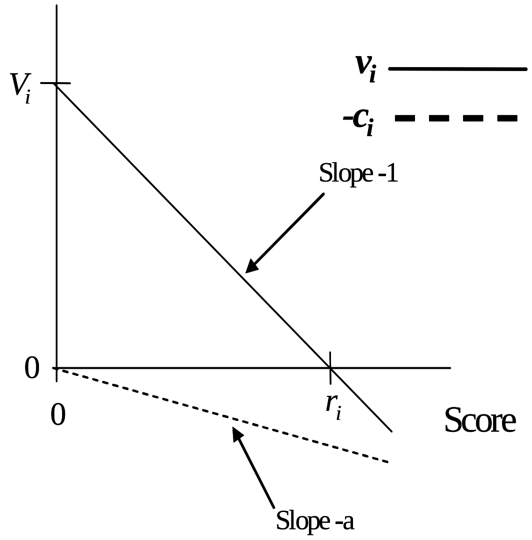

Figure 5: The valuation for winning and cost of losing for player i with $\gamma_i = 1$ in a simple contest with $c(x) = x$

The reach r_i of player i satisfies $v_i(r_i) = 0$, so $r_i = c^{-1}\left(\frac{V_i}{\gamma_i}\right)$. Since c is strictly increasing and players are ordered in decreasing order of reach, $\frac{V_1}{\gamma_1} \geq ... \geq \frac{V_N}{\gamma_N}$. The contest's threshold is $T = r_{m+1} = c^{-1}\left(\frac{V_{m+1}}{\gamma_{m+1}}\right)$, so the range of scores over which players compete, $[0, T]$, is independent of α. The Cost Condition is met since v_i and $-c_i$ are strictly decreasing. For the Power Condition, assume that $\frac{V_{m+1}}{\gamma_{m+1}}$ is distinct. Theorem 1 then shows that the equilibrium payoff of player $i < m + 1$ is $w_i = v_i(T) = V_i - \gamma_i c\left(c^{-1}\left(\frac{V_{m+1}}{\gamma_{m+1}}\right)\right) = V_i - \gamma_i \frac{V_{m+1}}{\gamma_{m+1}}$. We therefore have the following corollary of Theorem 1.

Corollary 5 *In a simple contest, the payoff of every player $i < m + 1$ is $V_i - \gamma_i \frac{V_{m+1}}{\gamma_{m+1}}$. The payoffs of players $m + 1, \ldots, N$ are 0. Payoffs are independent of α and c.*

[28] The contest is then similar to the one in Moldovanu & Sela (2001). The informational assumptions, however, are different. In their model, all players are ex-ante symmetric. The individual coefficients γ_i are privately known and drawn iid from a commonly known distribution. They solve for the symmetric equilibrium, and do not characterize all equilibria. In contrast, the model here is of complete information and has a unique equilibrium.

Corollary 5 shows that a winning player's payoff increases in his valuation for a prize and in the marginal player's cost coefficient, and decreases in the player's cost coefficient and in the marginal player's valuation for a prize. In particular, the payoff of a winning player i is not affected by the characteristics of any player in $N \backslash \{i, m+1\}$. Note that players' powers are ranked according to ratio of their valuation to cost coefficient: $w_i \geq w_j$ if and only if $\frac{V_i}{\gamma_i} \geq \frac{V_j}{\gamma_j}$.

Aggregate expenditures equal the allocation value of the prizes less players' utilities. We therefore obtain the following corollary of Theorem 1.

Corollary 6 *In a simple contest in which* $V_1 = \ldots = V_N = V$*, aggregate expenditures are* $mV - \sum_{i=1}^{m} \left(V - \gamma_i \frac{V}{\gamma_{m+1}} \right) = V \sum_{i=1}^{m} \left(\frac{\gamma_i}{\gamma_{m+1}} \right)$*, and are independent of* α *and* c*.*

Corollary 6 shows that when all valuations are equal, as is the case when prizes are monetary, aggregate expenditures increase in valuations and in each of the winning players' cost coefficients, and decrease in the marginal player's cost coefficient.

4.1 Simple Contests with a Single Prize

Since simple contests satisfy condition R1 (they are regular), Theorems 2 and 5 show that a simple contest with a single prize has a unique equilibrium, described in Theorem 5. We therefore have the following corollary.

Corollary 7 *A simple contest with a single prize has a unique equilibrium. In this equilibrium players* $3, \ldots, n$ *choose* 0 *with probability* 1*. The CDFs of players* 1 *and* 2 *are* $\left(G_1^{\alpha}, G_2^{\alpha}\right)(x) = \left(\frac{\alpha \gamma_2 c(x)}{V_2 - (1-\alpha)\gamma_2 c(x)}, \frac{V_1 - \gamma_1 \frac{V_2}{\gamma_2} + \alpha \gamma_1 c(x)}{V_1 - (1-\alpha)\gamma_1 c(x)} \right)$ *on* $\left[0, c^{-1}\left(\frac{V_2}{\gamma_2}\right)\right]$.

The corollary shows that the unique equilibrium is not independent of α and c. It is straightforward to verify that player 1's CDF first-order stochastically dominates (FOSD) that of player 2. Therefore, player 1 wins the prize with higher probability than player 2 (see Corollary 10 below). Moreover, $\frac{\partial G_1^{\alpha}(x)}{\partial \alpha}, \frac{\partial G_2^{\alpha}(x)}{\partial \alpha} > 0$ for all x in $(0, T)$, so the equilibria for lower values of α first order stochastically dominate those of higher values of α.

In the limit, as α approaches 0, player 1 chooses T with probability 1 and wins the prize with probability 1, and player 2 chooses scores lower or equal to x with probability $\frac{V_1 - \gamma_1 \frac{V_2}{\gamma_2}}{V_1 - \gamma_1 c(x)}$. This is an equilibrium of the limit game in which every player i's payoff is 0 if he loses, and $V_i - \gamma_i c(s_i)$ if he wins when choosing s_i. The equilibrium does not depend on the tie-breaking rule, and gives every player the payoff specified in Theorem 1. Thus, taking the fraction of the all-pay component to 0 can serve as a selection criterion that delivers a unique equilibrium of the limit game. This limit game is not a contest, since Assumption A3 is violated, and has multiple equilibria with differing payoffs.

4.2 Simple Contests with Multiple Prizes

From now on assume that $c(\cdot)$ is piecewise analytical. The following result follows from Theorems 2 and 4.

Corollary 8 *A simple contest with multiple prizes has a unique equilibrium.*

Proof. Let $i = m+2, \ldots, N$. For every $x > 0$,

$$\frac{c_{m+1}(x)}{v_{m+1}(0)} = \frac{\alpha \gamma_{m+1} c(x)}{V_{m+1}} < \frac{\alpha \gamma_i c(x)}{V_i} = \frac{c_i(x)}{v_i(0)},$$

and for every $x \geq 0$,

$$\frac{v_{m+1}(x)}{v_{m+1}(0)} = \frac{V_{m+1} - \gamma_{m+1} c(x)}{V_{m+1}} = 1 - \frac{\gamma_{m+1}}{V_{m+1}} c(x) \geq 1 - \frac{\gamma_i}{V_i} c(x) = \frac{v_i(x)}{v_i(0)}$$

so by Theorems 2, players $m+2, \ldots, N$ do not participate, i.e., choose 0 with probability 1. Applying Theorem 4 to the reduced contest, which includes only players $1, \ldots, m+1$, we obtain the result. ■

Let $a_i = \frac{\gamma_i}{V_i}$, and note that a_i is increasing in i. The following result shows that in the unique equilibrium, the best response set of every player $i \leq m+1$ is an interval whose upper bound is T and whose lower bound increases in the player's power (or, equivalently, decreases in a_i).

Theorem 6 *In the unique equilibrium of a simple contest, every player $i \leq m+1$ is active on the interval $\left[s_i^l, T\right]$ for some $s_i^l \geq 0$, with $s_m^l = s_{m+1}^l = 0$. For $i, j \leq m$, $s_i^l \leq s_j^l$ if and only if $a_j \leq a_i$. Players $m+2, \ldots, N$ do not participate, i.e., they choose 0 with certainty.*

Appendix C contains the proof of Theorem 6 and of other results in this section. To gain some intuition for the result, suppose $\alpha = 1$. In this case, players' semi-elasticities are identical, so the set of players active at x equals the set of players active to the right of x, for every $x \in [0, T)$. Thus, once a player becomes active, he remains active up to the threshold. The higher the normalized power of a player, the later he becomes active.

Having established that each player is active on an interval up to the threshold and that players with higher power become active at higher scores, we proceed to derive an expression for players' equilibrium strategies. Recall that for any $y < T$ the formula for player i's CDF at y is given by Equation (3). Since each player is active on an interval, $D = 1$, and the switching points are the points s_i^l at which players become active. Because players with higher powers become active at higher scores, for every $y < T$ there is a unique $j = 1, \ldots, m$ such that y is in $\left[s_j^l, s_{j-1}^l\right)$ (where $s_0^l = T$) and $A^+\left(s_j^l\right) = j, \ldots, m+1$. Therefore, for y in this $\left[s_j^l, s_{j-1}^l\right)$ we have

$$G_i(y) = \begin{cases} 1 - \frac{\Pi_{k=j}^{m+1} q_k(y)^{\frac{1}{m+1-j}}}{q_i(y)} & \text{if} \quad i \geq j \\ 0 & \text{if} \quad i < j \end{cases}.$$

Substituting

$$q_i(y) = \frac{v_i(y) - v_i(T)}{v_i(y) + c_i(y)} = \frac{a_i\left(\frac{1}{a_{m+1}} - c(x)\right)}{1 - (1-\alpha)\, a_i c(x)}$$

for $i \geq j$, we obtain

$$G_i(y) = 1 - \frac{\Pi_{k=j}^{m+1}\left[\frac{a_k\left(\frac{1}{a_{m+1}} - c(y)\right)}{1-(1-\alpha)a_k c(y)}\right]^{\frac{1}{m+1-j}}}{\frac{a_i\left(\frac{1}{a_{m+1}} - c(y)\right)}{1-(1-\alpha)a_i c(y)}} = 1 - \frac{\left(\frac{1}{a_{m+1}} - c(y)\right)^{\frac{m+2-j}{m+1-j}} \Pi_{k=j}^{m+1}\left[\frac{a_k}{1-(1-\alpha)a_k c(y)}\right]^{\frac{1}{m+1-j}}}{\left(\frac{1}{a_{m+1}} - c(y)\right)\frac{a_i}{1-(1-\alpha)a_i c(y)}}$$

or

$$G_i(y) = 1 - \left(\frac{1}{a_{m+1}} - c(y)\right)^{\frac{1}{m+1-j}} \frac{\Pi_{k=j}^{m+1}\left[\frac{a_k}{1-(1-\alpha)a_k c(y)}\right]^{\frac{1}{m+1-j}}}{\frac{a_i}{1-(1-\alpha)a_i c(y)}} \tag{8}$$

We still have to identify the scores s_i^l at which players become active. For simplicity assume that the a_is are distinct (a similar analysis works when they are not distinct, but the notation becomes more cumbersome). Recall that $s_m = s_{m+1} = 0$. The score s_i^l at which player $i < m$ becomes active is the lowest score x at which he obtains his power. That is, s_i^l is the lowest score x that satisfies

$$\left(1 - \prod_{d=i+1}^{m+1}(1 - G_d(x))\right)(1 - a_i c(x)) - \left(\prod_{d=i+1}^{m+1}(1 - G_d(x))\right)\alpha a_i c(x) = w_i = 1 - \frac{a_i}{a_{m+1}},$$

or

$$\left(\prod_{d=i+1}^{m+1}(1 - G_d(x))\right)(a_i c(x)(1-\alpha) - 1) - a_i c(x) = -\frac{a_i}{a_{m+1}}.$$

After substituting G_d with the right-hand side of Equation (8) (with active players $i+1, \ldots, m+1$) and some algebraic manipulation, it can be shown that s_i^l is the lowest score x that satisfies

$$\frac{\Pi_{k=i+1}^{m} a_k}{a_i^{m-i}} = \frac{1}{(1 - a_{m+1}c(x))} \frac{\Pi_{k=i+1}^{m+1}(1 - a_k c(x)(1-\alpha))}{(1 - a_i c(x)(1-\alpha))^{m-i}}. \tag{9}$$

Equation (9) characterizes s_i^l implicitly, and provides a simple closed-form expression for s_i^l when $\alpha = 1$ (see Section 4.2.1 below). It can also be used to show the following result.

Theorem 7 *For $\alpha \leq 1$ and $i < m$, s_i^l decreases in α. As α approaches 0, s_i^l approaches T.*

The game with $\alpha = 0$ is not a contest, since Assumption A3 is violated. Instead, it is a complete-information multiprize m'th-price auction in which player i's cost of bidding x is $\gamma_i c(x)$. This game has many equilibria, some of which involve players playing weakly-dominated strategies, and some which rely on specific tie-breaking rules. Different equilibria lead to different payoffs. Considering the limit of the equilibria of simple contests as α approaches 0 we obtain the following equilibrium of the game with $\alpha = 0$. Players $1, \ldots, m-1$ bid T with probability 1.

Players $m+2,\ldots,n$ bid 0 with probability 1. Players m and $m+1$ bid the limit of the equilibria of two-player simple contests as α approaches 0. As shown in Section 4.1, this means that player m bids T with probability 1, and player $m+1$ bids according to the CDF $G_{m+1}(x) = \frac{V_m - \gamma_m \frac{V_{m+1}}{\gamma_{m+1}}}{V_m - \gamma_m c(x)}$. Each of players $1,\ldots,m$ wins a prize with probability 1, and players $m+1,\ldots,n$ win a prize with probability 0. Players' payoffs are given by the payoff characterization, and the equilibrium is robust to the tie-breaking rule.

Theorem 7 has the following implication regarding efficiency. Call a simple contest *β-efficient*, for some β in $(0,1)$, if each of the winning players (players with positive power) obtains a prize with probability at least β in the unique equilibrium of the contest.

Corollary 9 *Choose a family of simple contests parameterized by α. For any $\beta < 1$, every simple contest in the family with a small enough $\alpha > 0$ is β-efficient.*

In particular, the corollary shows that when players differ only in their valuations for a prize, efficiency can be approached arbitrarily closely by reducing the unconditional investment component α. The limiting equilibrium corresponding to $\alpha = 0$ is efficient. Since players' payoffs remain the same for all $\alpha > 0$, this immediately implies that as α approaches 0, expenditures approach their maximal value, $\sum_{i=1}^{m} V_i - \sum_{i=1}^{m}\left(V_i - \gamma_i \frac{V_{m+1}}{\gamma_{m+1}}\right) = V_{m+1}\sum_{i=1}^{m}\frac{\gamma_i}{\gamma_{m+1}}$.

That players with higher powers become active at higher scores also implies that their equilibrium CDFs can be ranked in terms of FOSD.

Corollary 10 *For any α in $(0,1]$ and $i < j$, the CDF of player i FOSD that of player j. This implies that player i chooses higher scores than player j, on average, and also that player i wins a prize with higher probability than player j.*

4.2.1 The Case $\alpha = 1$

When $\alpha = 1$ the contest is separable. Equation (8) then simplifies to

$$G_i(y) = 1 - \left(\frac{1}{a_{m+1}} - c(y)\right)^{\frac{1}{m+1-j}} \frac{\Pi_{k=j}^{m+1} a_k^{\frac{1}{m+1-j}}}{a_i} \tag{10}$$

Equation (9) provides the following closed-form expression for s_i^l, $i < m$ (recall that $s_m^l = s_{m+1}^l = 0$)

$$\frac{\Pi_{k=i+1}^{m} a_k}{a_i^{m-i}} = \frac{1}{\left(1 - a_{m+1} c\left(s_i^l\right)\right)} \Rightarrow s_i^l = c^{-1}\left(\frac{1}{a_{m+1}} - \frac{a_i^{m-i}}{\Pi_{k=i+1}^{m+1} a_k}\right). \tag{11}$$

The special case of $c(x) = x$ and $\gamma_i = 1$ is a multi-prize all-pay auction, first analyzed by Clark & Riis (1998). Setting $c(x) = x$ and $\gamma_i = 1$ in Equations (10) and (11) delivers the equilibrium described in their Proposition 1. Theorem 6 above shows that this equilibrium is the unique equilibrium when the marginal player's valuation is distinct.[29] An analysis similar to that of Clark & Riis (1998) shows that individual and aggregate expenditures are independent of the cost function c.

[29] Theorem 6 applied to multiprize all-pay auctions corrects two imprecisions in Clark & Riis (1998). The first is that they claimed uniqueness but provided an incorrect proof of this claim, as discussed in footnote 6 above. The second is that their footnote 6 claims that multiple equilibria arise when players' valuations are not distinct. Theorem 6 shows that the equilibrium is unique even if several players have the same valuation for a prize, as long as the valuation of the marginal player is distinct.

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A Examples 1 and 2

All the examples depict separable contests, so $v_i(\cdot) = V_i - c_i(\cdot)$ for some $V_i > 0$.

Example 1 Multiple equilibria when costs are not strictly increasing.
Let $n = 2, m = 1, V_i = V > 0$,

$$c_1(x) = \begin{cases} 0 & \text{if } \; x < X \\ x - X & \text{if } \; x \geq X \end{cases}$$

for some $X > 0$, and let $c_2(x) = x$. This is a version of an all-pay auction, in which player 1 has an initial advantage of X. For $X \geq V$, the only equilibrium is a pure strategy equilibrium: player 1 chooses X, and player 2 chooses 0. Player 1 wins with certainty, and expenditures are zero.

For $X < V$, (F_1, F_2) is an equilibrium, where

$$F_1(x) = \begin{cases} H_1(x) & \text{if} \quad x < X \\ \frac{X}{V} + \frac{(x-X)}{V} & \text{if} \;\; X \leq x \leq V \\ 1 & \text{if} \quad x > V \end{cases}, F_2(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \frac{X}{V} & \text{if} \;\; 0 \leq x < X \\ \frac{X}{V} + \frac{(x-X)}{V} & \text{if} \;\; X \leq x \leq V \\ 1 & \text{if} \quad x > V \end{cases}$$

and any right-continuous, non-decreasing function H_1, such that $H_1(0) = 0$ and $H_1(x) \leq \frac{x}{V}$.

Example 2 A non-constructible equilibrium.
I construct an equilibrium (C, G, G) of the following contest, in which player 1's best-response set is the Cantor set. Let $n = 3, m = 1, V_i = 1$. To define player 1's cost function, let $c_1(x) = F(x) = x$ and modify $F(x)$ by mimicking the construction of the Cantor set on $[0, 1]$ in the following way. At every removed (a, b) and for every $x \in (a, b)$, let $F(x) = a + \frac{(x-a)^2}{b-a}$. Denote the resulting function by $\tilde{F}$. Then $\tilde{F}(0) = 0, \tilde{F}(1) = 1$, and $\tilde{F}$ is continuous, strictly increasing, equals x precisely on the Cantor set, and is strictly lower than x on its complement. In particular, if player 1's probability of winning when playing x is $\tilde{F}(x)$, then his best-response set is the Cantor set. To achieve this, let C be the Cantor function, and recall that it is continuous and weakly increasing, with $C(0) = 0$ and $C(1) = 1$. Let $G(x)$ satisfy $\tilde{F}(x) = 1 - (1 - G(x))(1 - G(x))$ for all $x \in [0, 1]$. That is,

$$G(x) = 1 - \sqrt{1 - \tilde{F}(x)}$$

Then $G(x)$ is continuous and strictly increasing, with $G(0) = 0$ and $G(1) = 1$. Now, define player 2 and 3's cost functions as

$$c_2(x) = c_3(x) = 1 - (1 - G(x))(1 - C(x)) = 1 - \left(\sqrt{1 - \tilde{F}(x)}\right)(1 - C(x))$$

Then c_2 and c_3 are continuous and strictly increasing, with $c_2(0) = c_3(0) = 0$ and $c_2(1) = c_3(1) = 1$. It is straightforward to verify that (C, G, G) is an equilibrium of the contest, in which player 1's best-response set is the Cantor set.

B Proofs of the Results of Section 3

Lemma 5 *In any equilibrium G of a contest with strictly decreasing valuations for winning and strictly increasing costs of losing, (1) G is continuous on $(0,T)$, and (2) every score in $(0,T)$ is a best response for at least two players.*

Proof. See the Appendix of Siegel (2008). ■

B.1 Proof of Lemma 2

Choose $y \in (x, \bar{x})$, and let $p_i(y) = 1 - G_i(y)$. Since $q_i(y) > 0$ and $p_i(y), D > 0$ (all players choose scores up to the threshold by the Threshold Lemma and strictly decreasing valuations for winning), Equation (2) for $i \in A^+(x)$ can be rewritten as $\Pi_{j\in A^+(x)\setminus\{i\}} p_j(y) = \frac{q_i(y)}{D} > 0$. Taking natural logs,

$$\sum_{j\in A^+(x)\setminus\{i\}} \ln p_j(y) = \ln q_i(y) - \ln D$$

This is a system of $|A^+(x)|$ linear equations in $|A^+(x)|$ unknowns $p_j(y)$. Denote by $I_{M\times M}$ and $1_{M\times M}$ the identity matrix and a matrix of ones, respectively, of dimensions $M \times M$. Then, in vector notation,

$$\left(1_{|A^+(x)|\times|A^+(x)|} - I_{|A^+(x)|\times|A^+(x)|}\right) \ln p(y) = \ln q(y) - \ln D$$

Since $(1_{M\times M} - I_{M\times M})^{-1} = \left(\frac{1}{M-1} \cdot 1_{M\times M} - I_{M\times M}\right)$, we have

$$\ln p_i(y) = \frac{1}{|A^+(x)| - 1} \sum_{j\in A^+(x)} \ln q_j(y) - \ln q_i(y) - \frac{1}{|A^+(x)| - 1} \ln D$$

which gives the result for $y \in (x, \bar{x})$. For $y \in \{x, \bar{x}\}$,$\bar{x} \neq T$, the result follows from left- and right-continuity on $[0,T)$. And $G(T) = 1$ because by Condition R1 no player has best responses above T.

B.2 Proof of Lemma 1

Since positive payoffs imply winning with positive probability at every best response, the Tie Lemma shows that players in $1,\ldots,m$ do not have an atom at 0. Since Condition R1 implies that there are no atoms above 0 and every $x > 0$ is a best response of at least two players (Lemmas 5 and 5), $G_{m+1}(0) \geq \min_{i\leq m} \frac{w_i}{v_i(0)}$. Since no player should be able to obtain more than his power by choosing a score slightly above 0, $G_{m+1}(0) \leq \min_{i\leq m} \frac{w_i}{v_i(0)}$. Strictly decreasing valuations for winning also imply that $w_i = v_i(T) < v_i(0)$, so $G_{m+1}(0) < 1$.

B.3 Statement and Proof of Lemma 6

I show that $A^+(x)$ is exactly all players $i \in A(x)$ with $\varepsilon_i(y) \leq H^{A(x)}(y)$ on some right-neighborhood of x, where $H^{A(x)}(y)$ is the positive fixed point of

$$S_y(H|A(x)) = \sum_{j\in A(x)} \max\{H - \varepsilon_k(y), 0\}$$

Then, by point 1 below, to determine whether $\varepsilon_i(y) \le H^{A(x)}(y)$ on some right-neighborhood of x it suffices to compare the derivatives of $\varepsilon_i(x)$ and $H(x)$, as specified in Section 3, since $S_x(\cdot|A(x)) = S_x(\cdot)$ so $H^{A(x)}(x) = H(x)$.[30]

The phrase **extend G to the right of x with respect to** $\overline{A^+} \subseteq A(x)$ is shorthand for defining G using Equation (3) with $A^+(x) = \overline{A^+}$ on some right-neighborhood of x.

Lemma 6 *A set $\overline{A^+} \subseteq A(x)$ satisfies the following three conditions:*

1. $\overline{A^+} \ge 2$
2. For every player $i \in \overline{A^+}$, $h_i(y) \ge 0$ on some right-neighborhood of x
3. For every player $i \notin \overline{A^+}$, $P_i(y)\, v_i(y) - (1 - P(y))\, c_i(y) < w_i$ on some right-neighborhood of x

when G is extended to the right of x with respect to $\overline{A^+}$, if and only if $\overline{A^+} = \widehat{A^+}(x)$ for

$$\widehat{A^+}(x) = \left\{ i \in A(x) : \varepsilon_i(y) \le H^{A(x)}(y) \text{ on some right-neighborhood of } x \right\}$$

The following points assist in the proof:

1. $H^{A(x)}$ on some right-neighborhood of x can be computed as follows. Order the players in $A(x)$ in any non-decreasing order of semi-elasticity on some right-neighborhood of x (this can be done since, by Condition R2, semi-elasticities are analytical on some right-neighborhood of x, and an analytical function with an accumulation point of roots is identically 0 in the connected component of the accumulation point). Let $L(x)$ be the highest $l \ge 2$ (in this ordering) such that

$$\frac{1}{l-1} \sum_{j \in A(x), j \le l} \varepsilon_j(y) - \varepsilon_l(y) \ge 0$$

on some right-neighborhood of x (that $L(x)$ is well defined follows from Condition R2 as above). Then

$$H^{A(x)}(y) = \frac{1}{L(x) - 1} \sum_{j \in A(x), j \le L(x)} \varepsilon_j(y)$$

on this right-neighborhood of x. This follows from the uniqueness of the fixed point of $S_y(\cdot|A(x))$. Consequently, $H^{A(x)}$ is analytical on this right-neighborhood of x.

[30] $A^+(x)$ can also be constructed as follows. Order the players in $A(x)$ in any non-decreasing order of semi-elasticity on some right-neighborhood of x. $A^+(x)$ is the subset $\{1, \ldots, L(x)\} \subseteq A(x)$, where $L(x)$ is the highest $l \ge 2$ (in this ordering) such that

$$\frac{1}{l-1} \sum_{j \in A(y), j \le l} \varepsilon_j(y) - \varepsilon_l(y) \ge 0$$

on this right-neighborhood of x. This follows from solving the system of Equations (5) and using the arguments in the proof of Lemma 6.

2. $\widehat{A^+}(x) = \left\{ i \in A(x) : \varepsilon_i(y) \leq H^{\widehat{A^+(x)}}(y) \text{ on some right-neighborhood of } x \right\}$, since, by definition of $\widehat{A^+}(x)$ and point 1, $H^{\widehat{A^+(x)}}(y) = H^{A(x)}(y)$ on some right-neighborhood of x.

3. When G is extended to the right of x with respect to $\overline{A^+}, \overline{A^+} \subseteq A(x), \left|\overline{A^+}\right| \geq 2$, condition 2 in the lemma is met for $\overline{A^+}$ if and only if the aggregate hazard rate $\widetilde{H}^{\overline{A^+}}$ equals $H^{\overline{A^+}}$, the fixed point of $S_y\left(\cdot|\overline{A^+}\right)$, on some right-neighborhood of x. Indeed, by taking natural logs and differentiating Equation (2),

$$\widetilde{H}^{\overline{A^+}}(y) = \frac{1}{\left|\overline{A^+}\right| - 1} \sum_{j \in \overline{A^+}} \varepsilon_j(y) \text{ and } \widetilde{H}^{\overline{A^+}}(y) = \sum_{j \in \overline{A^+}} \widetilde{H}^{\overline{A^+}}(y) - \varepsilon_j(y)$$

so if $\widetilde{H}^{\overline{A^+}} = H^{\overline{A^+}}$, condition 2 is met. The other direction follows from the uniqueness of the fixed point of $S_y\left(\cdot|\overline{A^+}\right)$, because $h_i(y) = \widetilde{H}^{\overline{A^+}}(y) - \varepsilon_i(y)$.

Proof. "if": That $\left|\widehat{A^+}\right| \geq 2$ is immediate from point 1. By point 2 and the definition of $H^{\widehat{A^+(x)}}$,

$$H^{\widehat{A^+(x)}}(y) = \sum_{j \in \widehat{A^+}} H^{\widehat{A^+(x)}}(y) - \varepsilon_j(y)$$

so $\widetilde{H}^{\widehat{A^+}} = H^{\widehat{A^+(x)}}$ and by point 2 condition 2 in the lemma is met. Condition 3 is met for players in $A(x) \backslash \widehat{A^+}(x)$ since, by point 2, such players have semi-elasticities strictly higher than the aggregate hazard rate $H^{\widehat{A^+(x)}}$ on some right-neighborhood of x. Condition 3 is trivially met for players in $N \backslash A(x)$ by continuity of G.
"only if": Consider a set $\overline{A^+} \subseteq A(x), \left|\overline{A^+}\right| \geq 2$, that satisfies conditions 2 and 3 of the lemma. Since $\overline{A^+} \subseteq A(x)$, $S_y\left(\cdot|\overline{A^+}\right) \leq S_y(\cdot|A(x))$, so $H^{\widehat{A^+(x)}} = H^{A(x)} \leq H^{\overline{A^+}}$. This implies that $\widehat{A^+}(x) \subseteq \overline{A^+}$. Otherwise, when G is extended to the right of x with respect to $\overline{A^+}$ every player $i \in \widehat{A^+}(x) \backslash \overline{A^+}$ obtains at least his power on some right-neighborhood of x, since by points 2 and 3 above, $\varepsilon_i \leq H^{\widehat{A^+(x)}} \leq H^{\overline{A^+}} = \widetilde{H}^{\overline{A^+}}$, the aggregate hazard rate, which violates condition 3. Since $\widehat{A^+}(x) \subseteq \overline{A^+}$, $H^{\overline{A^+}} \leq H^{\widehat{A^+(x)}}$. By condition 2 and point 3, when G is extended to the right of x with respect to $\overline{A^+}$

$$\forall i \in \overline{A^+} : 0 \leq h_i = \widetilde{H}^{\overline{A^+}} - \varepsilon_i = H^{\overline{A^+}} - \varepsilon_i \leq H^{\widehat{A^+(x)}} - \varepsilon_i$$

on some right-neighborhood of x. So, by point 2, $\overline{A^+} \subseteq \widehat{A^+}(x)$. Therefore, $\overline{A^+} = \widehat{A^+}(x)$. ■

$A^+(x)$ is the set specified by the lemma, since it satisfies conditions 1-3 when G is extended to the right of x with respect to $A^+(x)$.

B.4 Proof of Lemma 3

Lemma 7 *$\forall \tilde{x} < T$, the number of switching points in $[0, \tilde{x}]$ is finite.*

Proof. I assume analytical valuations for winning and costs of losing (the obvious extension applies to piecewise analyticity). Choose $\tilde{x} < T$ and rank players' semi-elasticities at every score in $[0, \tilde{x}]$. Since semi-elasticities are analytical, this ranking can change only finitely many times. Thus, $[0, \tilde{x}]$ can be divided into a finite number of intervals such that the ranking of players' semi-elasticities is constant on each interval. Fix one such interval J. For every subset $B \subseteq N$ of at least two players and every $x \in J$, denote by $t_B(x)$ the highest semi-elasticity of a player who can join the set of active players B and maintain a non-negative hazard rate: $t_B(x) = \frac{1}{|B|-1} \sum_{j \in B} \varepsilon_j(x)$ (the aggregate hazard rates of players in B). Since semi-elasticities are analytical, so is $t_B(\cdot)$. Thus, the interval J can be divided into a finite number of subintervals such on every subinterval each function in $\{\varepsilon_i - t_B : i \in N, B \subseteq N, |B| \geq 2\}$ is either positive, negative, or 0. Clearly, on any such subinterval $L \subseteq J$ an active player can become inactive only if a player with a strictly lower semi-elasticity becomes active (recall that the order of players' semi-elasticities doesn't change on J). Now, suppose in contradiction that the number of switching points in L is infinite. This implies that some player i becomes inactive and active an infinite number of times, which, by the above, implies that some player j with semi-elasticity strictly lower than that of i becomes inactive and active an infinite number of times. Continuing in this way, we obtain a contradiction since the number of players is finite. ■

The following two lemmas show that there are no switching points in some left-neighborhood of T.

Lemma 8 *$\exists \tilde{x} < T$ such that $\forall i \in N : \varepsilon_i(x) < H(x)$ for every $x \in (\tilde{x}, T)$.*

Proof. First, $\forall i, j$:

$$\lim_{x \to T} \frac{\varepsilon_i(x)}{\varepsilon_j(x)} = \lim_{x \to T} \frac{q_i'(x)}{q_i(x)} \frac{q_j(x)}{q_j'(x)} = \frac{q_i'(T)}{q_j'(T)} \lim_{x \to T} \frac{q_j(x)}{q_i(x)} = \frac{q_i'(T)}{q_j'(T)} \frac{q_j'(x)}{q_i'(x)} = 1$$

where the penultimate equality follows from l'Hopital's rule.
Let $\varepsilon_{\min}(x) = \min_{i \in N} \varepsilon_i(x)$ for $x < T$. Then, by the above, $\lim_{x \to T} \frac{\varepsilon_i(x)}{\varepsilon_{\min}(x)} = 1$, so $\frac{\varepsilon_i(x)}{\varepsilon_{\min}(x)} < \frac{n}{n-1}$ for all $x > \tilde{x}$ for some $\tilde{x}$ sufficiently close to T. To conclude, it suffices to show that $\forall x > \tilde{x}$: $H(x) \geq \frac{n}{n-1} \varepsilon_{\min}(x)$. Let $S_x^{\min}(H) = n \max\{H - \varepsilon_{\min}(x), 0\}$. Then $\forall H : S_x(H) \leq S_x^{\min}(H)$ and since $\frac{n}{n-1} \varepsilon_{\min}(x)$ is the unique positive fixed point of $S_x^{\min}$, $H(x) \geq \frac{n}{n-1} \varepsilon_{\min}(x)$. ■

Since active players with semi-elasticities strictly lower than the aggregate hazard rate remain active, in order to complete the proof it suffices to show the following.

Lemma 9 *Every player i has scores x arbitrarily close to T such that $q_i(x) = \Pi_{j \neq i}(1 - G_j(x))$.*

Proof. Suppose, in contradiction, that $\forall x \in (\tilde{x}_i, T) : q_i(x) < \Pi_{j \neq i}(1 - G_j(x))$ for some player i and some $\tilde{x}_i > \tilde{x}$ of the previous lemma. Then, $f(x) = \sum_{j \neq i} \ln(1 - G_j(x)) - \ln q_i(x) > 0$. Since $i \notin A(x)$,

$$\forall x \in (\tilde{x}_i, T) : H(x) = \frac{|A^+(x)|}{|A^+(x)| - 1} \sum_{j \in A^+(x)} \varepsilon_j(x) = \sum_{j \in N} \frac{G_i'(x)}{(1 - G_i(x))} = \sum_{j \neq i} \frac{G_i'(x)}{(1 - G_i(x))}$$

Thus,

$$f'(x) = \varepsilon_i(x) - \frac{|A^+(x)|}{|A^+(x)| - 1} \sum_{j \in A^+(x)} \varepsilon_j(x) \leq \varepsilon_i(x) - \frac{n-1}{n-2} \sum_{j \in A^+(x)} \varepsilon_j(x) =$$

$$-\frac{1}{n-2}\varepsilon_{\min}(x)+o(\varepsilon_{\min}(x))$$

as $x \to T$, by the proof of the previous lemma. Since

$$-\frac{1}{n-2}\int_x^T \varepsilon_{\min}(y)\,dy = \lim_{z\to T}\frac{1}{n-2}\left(\ln q_{\min}(z)-\ln q_{\min}(x)\right) = -\infty$$

f crosses 0 at a score in $(\tilde{x}_i T)$, a contradiction. ■

B.5 Proof of Lemma 4

By Lemma 3, $\exists \tilde{x} < T$ such that $\forall x \in (\tilde{x}, T), A(x) = N$. Equation (3) now implies that

$$\forall x \in (\tilde{x}, T), \forall i \in N : \ln(1-G_i(x)) = \frac{1}{n-1}\sum_{j\in N}\ln q_j(x) - \ln q_i(x)$$

To show that $G_i(x) \underset{x\to T}{\to} 1$, it therefore suffices to show that

$$\frac{1}{n-1}\sum_{j\in N}\ln q_j(x) - \ln q_i(x) \underset{x\to T}{\to} -\infty$$

Since $\ln q_i(x) \underset{x\to T}{\to} -\infty$, it suffices to show that

$$\frac{1}{n-1}\frac{\sum_{j\in N}\ln q_j(x)}{\ln q_i(x)} > 1+\delta \text{ for some } \delta > 0$$

for large enough x. The inequality follows from l'Hopital's rule and the fact that $\lim_{x\to T}\frac{\varepsilon_i(x)}{\varepsilon_j(x)} = 1$, shown in the proof of Lemma 3.

B.6 Proof of Theorem 3

For expositional simplicity, I assume that the number of switching points in G is finite. It is straightforward to accommodate a countably infinite number of switching points by defining the limit of a sequence of switching points to be a switching point and modifying the proof appropriately.

In the following propositions, x_k denotes switching point k in G. The last switching point is T. $A(x)$ and $A^+(x)$ are defined as in Section 3. Choose any equilibrium $\tilde{G}$ of the contest, and recall that $\tilde{G}$ is continuous above 0 because of Condition R1. $\tilde{A}(x)$ denotes the set of players active at x under $\tilde{G}$, i.e., the set of players defined by Equation (4) with $\tilde{G}$ instead of G. Using $A^+(x)$, I show that $A(x) = \tilde{A}(x)$ for all $x \in [0, T]$. The following lemma shows that doing so is sufficient.

Lemma 10 *Let $\tilde{x} \in [0, T]$. If $\forall x \in [0, \tilde{x}] : \tilde{A}(x) = A(x)$, then $\forall x \in [0, \tilde{x}] : \tilde{G}(x) = G(x)$.*

Proof. Similar to that of Proposition 1, since $\tilde{G}(0) = G(0)$ (Lemma 1 does not rely on analyticity), $\tilde{G}$ satisfies the conditions in the definition of constructibility on $[0, \tilde{x})$ (because G does), $\tilde{G}$ is continuous on above 0 (Lemma 5), and $\tilde{G}(T) = G(T) = 1$ (by Condition R1). ■

Now, let x_k be the highest positive switching point such that $\tilde{A}(x) = A(x)$ on $[0, x_k]$, and suppose in contradiction that $x_k < T$. Choose $x \in (x_k, x_{k+1})$ such that $\tilde{A}(x) \neq A(x)$. Since $x_k < T$, such an x exists otherwise Lemma 10 and continuity would imply that $\tilde{A}(x_{k+1}) = A(x_{k+1})$. The following lemmas show that $\tilde{A}(x) \subseteq A(x)$ and $A(x) \subseteq \tilde{A}(x)$, which completes the proof.

Lemma 11 $\tilde{A}(x) \subseteq A(x)$.

Proof. Suppose $\tilde{A}(x) \not\subseteq A(x)$, and let $i_0 \in \tilde{A}(x) \backslash A(x)$. Since $i_0 \notin A(x)$, we have

$$\frac{w_{i_0} + c_{i_0}(y)}{v_{i_0}(y) + c_{i_0}(y)} > P_{i_0}(x) = 1 - \Pi_{j \neq i_0}(1 - G_j(x))$$

or

$$\frac{v_{i_0}(y) - w_{i_0}}{v_{i_0}(y) + c_{i_0}(y)} < \Pi_{j \neq i_0}(1 - G_j(x))$$

and since $i_0 \in \tilde{A}(x)$, we have

$$\frac{v_{i_0}(y) - w_{i_0}}{v_{i_0}(y) + c_{i_0}(y)} = \Pi_{j \neq i_0}\left(1 - \tilde{G}_j(x)\right)$$

so

$$\Pi_{j \neq i_0}\left(1 - \tilde{G}_j(x)\right) < \Pi_{j \neq i_0}(1 - G_j(x))$$

Let $J_1 = N \backslash \{i_0\}$. Then

$$\Pi_{j \in J_1}\left(1 - \tilde{G}_j(x)\right) < \Pi_{j \in J_1}(1 - G_j(x)) \tag{12}$$

By the Threshold Lemma, the expression on each side of Inequality (12) is a product of $n-1$ strictly positive numbers. Therefore, there exists a player $i_1 \in J_1$ such that

$$\Pi_{J_1 \backslash \{i_1\}}\left(1 - \tilde{G}_j(x)\right) < \Pi_{J_1 \backslash \{i_1\}}(1 - G_j(x)) \tag{13}$$

(otherwise multiplying the products of all subsets of size $n-2$ for G and for $\tilde{G}$ would lead to a contradiction).
Now, note that $\forall i \in N : \tilde{G}_i(x_k) = G_i(x_k)$, by Lemma 10, and since $\tilde{G}_i$ is non-decreasing

$$\forall i \notin A^+(x) : \left(1 - \tilde{G}_i(x)\right) \leq (1 - G_i(x)) \tag{14}$$

Let $K_1 = N \backslash J_1 = \{i_0\}$. Since $A^+(x) \subseteq A(x)$ and $i_0 \notin A(x)$, by Inequality (14)

$$\left(1 - \tilde{G}_{j \in K_1}(x)\right) \leq (1 - G_{j \in K_1}(x)) \tag{15}$$

By Inequalities (13) and (15),

$$\Pi_{j \in J_1 \cup K_1 \backslash \{i_1\}}\left(1 - \tilde{G}_j(x)\right) < \Pi_{j \in J_1 \cup K_1 \backslash \{i_1\}}(1 - G_j(x)) \tag{16}$$

Since $N = J_1 \cup K_1$, Inequality (16) shows that $i_1 \notin A^+(x_k)$, otherwise i_1 would obtain under $\tilde{G}$ more than his power by choosing x.

Now repeat the process above, letting $J_{r+1} = J_r \backslash \{i_r\}, K_{r+1} = K_r \cup \{i_r\}$. By induction on r, Inequalities (12),(13),(15), and (16) hold with J_r instead of J_1, K_r instead of K_1, and i_k instead of i_1, so $K_r \cap A^+(x_k) = \phi$. A contradiction is reached at stage $n-1$, since $|K_{n-1}| = N-1$ but $|A^+(x_k)| \geq 2$. ■

Corollary 11 $\forall j \notin A(x), \forall y \in (x_k, x_{k+1}) : \tilde{G}_j(y) = G_j(y) = G(x_k)$.

Proof. Immediate from $\tilde{A}(y) \subseteq A(y)$ applied to all points $y \in (x_k, x_{k+1})$. ■

The next two lemmas establish that $A(x) \subseteq \tilde{A}(x)$.

Lemma 12 *If $A(x) \nsubseteq \tilde{A}(x)$, then $\tilde{G}_i(x) > G_i(x)$ for some $i \in A(x) \backslash \tilde{A}(x)$.*

Proof. Let $B = A(x) \backslash \tilde{A}(x)$, and suppose that $\forall j \in B : \tilde{G}_j(x) \leq G_j(x)$. This implies that $\exists j \in B : \tilde{G}_j(x) < G_j(x)$. Indeed, by Corollary 11 and Equation (3) with $\tilde{A}(x)$ instead of $A^+(x)$ and x instead of y, once G and $\tilde{G}$ agree on $(N/A(x)) \cup B = N/\tilde{A}(x)$ we obtain $\tilde{G}(x) = G(x)$ and therefore $A(x) = \tilde{A}(x)$.
To show that $\exists i \in B$ such that $\tilde{G}_i(x) > G_i(x)$, the following observation is required. Fix some values $\bar{G}_j(x)$ for $j \notin A(x)$ and use Equation (3) to solve for the values $\bar{G}_l(x), l \in A(x)$. Maintaining the value $\bar{G}_l(x)$ for player $l \in A(x)$ and solving for $A(x) \setminus \{l\}$ using Equation (3) gives the same solutions. If we now lower $\bar{G}_l(x)$ and solve for $A(x) \setminus \{l\}$, then the values $\bar{G}_j(x)$ of all players $j \in A(x) \setminus \{l\}$ strictly increase (this is easily seen from Equation (3), since D increases). Observe also that setting $\bar{G}_j(x) = \tilde{G}_j(x)$ for $j \notin \tilde{A}(x)$ and solving for the values $\bar{G}_i(x), i \in \tilde{A}(x)$ using Equation (3) with $\tilde{A}(x)$ instead of $A(x)$ gives $\bar{G}(x) = \tilde{G}(x)$.
Now, consider the following process by which $\tilde{G}(x)$ is "reached" from $G(x)$. Set $\bar{G}_l(x)$ equal to $G_l(x) = \tilde{G}_l(x)$ for $l \notin A(x)$. Take a player $j \in B$ for whom $\tilde{G}_j(x) < G_j(x)$. Then, solving for $A(x) \setminus \{j\}$ using $\bar{G}_j(x) = G_j(x)$ as described above and then lowering $\bar{G}_j(x)$ to $\tilde{G}_j(x)$ and solving again for $A(x) \setminus \{j\}$, raises the solutions above $G_l(x)$ for all $l \in A(x) \setminus \{j\}$. Thus, if $B = \{j\}$ then $A(x) \setminus \{j\} = \tilde{A}(x)$ and $\bar{G}_l(x) = \tilde{G}_l(x), l \notin \tilde{A}(x)$, so the solutions for $\bar{G}_i, i \in \tilde{A}(x)$ coincide with $\tilde{G}$ and player j obtains under $\tilde{G}$ at x more than his power (since under G at x player $j \in A(x)$ obtains precisely his power). If $B \neq \{j\}$, continue this process: take a player $l \in B \backslash \{j\}$ and lower $\bar{G}_l(x)$ obtained in the previous step to $\tilde{G}_l(x)$;[31] solve for $A(x) \setminus \{j, l\}$, and remember that after lowering $\bar{G}_j(x)$ from $G_j(x)$ to $\tilde{G}_j(x)$ all players in $A(x) \setminus \{j\}$ obtained precisely their power, and $l \in A(x) \setminus \{j\}$. Since $\bar{G}_l(x)$ decreases to $\tilde{G}_l(x)$, the solutions for all players in $A(x) \setminus \{j, l\}$ strictly increase and $\tilde{G}_j(x)$ does not change, so player l now obtains more than his payoff. Continuing in this way and recalling that $\bar{G} = \tilde{G}$ once $\bar{G}$ and $\tilde{G}$ agree on $N/\tilde{A}(x)$, we see that the last player in B obtains more than his power under $\tilde{G}$ at x, a contradiction. Therefore, $\tilde{G}_i(x) > G_i(x)$ for some player $i \in B$. ■

Lemma 13 $A(x) \subseteq \tilde{A}(x)$.

[31] Since $\tilde{G}_l(x) \leq G_l(x)$ and lowering $\bar{G}_j(x)$ from $G_j(x)$ to $\tilde{G}_j(x)$ raised the solutions $\bar{G}_l(x)$ for all $l \in A(x) \setminus \{j\}$, we have $\tilde{G}_l(x) < \bar{G}_l(x)$.

Proof. Suppose $A(x) \not\subseteq \tilde{A}(x)$. By the previous lemma, $\tilde{G}_i(x) > G_i(x)$ for some $i \in A(x) \backslash \tilde{A}(x)$. Since $\tilde{G}_i(x_k) = G_i(x_k)$, $\tilde{G}_i(y) > G_i(y)$ for some $y \in (x_k, x)$ such that $i \in \tilde{A}(y)$.[32] This means that $\tilde{A}(y) \neq A(y)$ (otherwise Corollary 11 and Equation (3) would imply $\tilde{G}_i(y) = G_i(y)$). Let $\hat{B} = A(y) \backslash \tilde{A}(y)$.
Now perform a procedure similar to the one described in the previous lemma, reaching $\tilde{G}(y)$ from $G(y)$. Begin with players $l \in \hat{B}$ for whom $\tilde{G}_l(y) > G_l(y)$. Raising $\bar{G}_l(y)$ from $G_l(y)$ to $\tilde{G}_l(y)$ decreases the solutions for all other players, so the order of raising does not matter - the solutions must be raised for all players $l \in \hat{B}$ for whom $\tilde{G}_l(y) > G_l(y)$. If the solutions of any remaining players in $\hat{B}$ now need to be raised to reach their level in $\tilde{G}$, continue the raising process until no more players in $\hat{B}$ need their solutions raised. It cannot be that $\hat{B}$ is exhausted, since $\tilde{G}_i(y) > G_i(y)$ and so far the solutions of all players in $\tilde{A}(y)$ have been repeatedly decreased, starting from their level in G. Thus, there remains a non-empty set $\bar{B} \subseteq \hat{B}$ of players whose solutions must now be decreased to reach their level in $\tilde{G}$. Decreasing these solutions increases the solutions for all other players. By the argument used in the previous lemma, the last player whose solution is decreased receives too high a payoff under $\tilde{G}$ at y. ■

B.7 Proof of Theorem 5

It is straightforward to see that G_1 and G_2 are increasing, continuous, equal 0 at 0 and 1 at T. Thus, they are cumulative distribution functions. Player 1 obtains a payoff of w_1 by choosing any score in $(0, T)$, and at most w_1 by choosing 0. Player 2 obtains a payoff of 0 by choosing any score in $[0, T)$. Since players' valuations for winning are strictly decreasing, no profitable deviation exists in $[T, \infty)$. Thus, (G_1, G_2) is an equilibrium.

Suppose Condition R1 holds. By Lemma 5, both players must be indifferent among all scores in $(0, T)$, so an equilibrium has the form $\left(\frac{u_2+c_2(y)}{v_2(y)+c_2(y)}, \frac{u_i+c_1(y)}{v_1(y)+c_1(y)}\right)$ on $(0, T)$ and neither CDF can reach 1 before T. In fact, the CDF of both players must reach 1 at exactly T: scores above T are strictly dominated for player 2, so neither player has best responses above T. Therefore, we have $u_1 = v_1(T) = w_1$ and $u_2 = v_2(T) = 0$.

B.8 The Example of Section 3

Cost functions are $c_2(x) = \frac{3x}{4}$,

$$c_1(x) = \begin{cases} \frac{x}{100} & \text{if } 0 \leq x \leq 0.31948 \\ \frac{x}{100} + 1.0581\,(x - (0.31948))^2 & \text{if } 0.31948 < x \leq 1 \\ 0.5 + 1.45\,(x-1) & \text{if } 1 < x \end{cases}$$

and

$$c_3(x) = \begin{cases} \frac{x}{12} & \text{if } 0 \leq x \leq 0.31948 \\ \frac{x}{12} + 1.9794\,(x - (0.31948))^2 & \text{if } 0.31948 < x \leq 0.7259 \\ 0.387\,44 + 1.6923\,(x - 0.7259) + 25\,(x - 0.7259)^2 & \text{if } 0.7259 < x \leq 0.85 \\ 0.98247 + \frac{(1-0.98247)}{0.15}\,(x - 0.85) & \text{if } 0.85 < x \end{cases}$$

[32] Let $\bar{z} = \sup_{z \in [x_k, x)} \left\{\tilde{G}_i(z) = G_i(z)\right\}$. By continuity of $\tilde{G}_i$ and G_i, $\bar{z} < x$. If for all $y \in (\bar{z}, x)$ we had $i \notin \tilde{A}(y)$ then $\tilde{G}_i$ would not increase on $(\bar{z}, x)$ so we would have $\tilde{G}_i(x) \leq G_i(x)$.

These cost functions give powers of $0, \frac{1}{4}$ and $\frac{1}{2}$. Perturbing the cost functions slightly does not change the qualitative aspects of the equilibrium.

C Proofs of the Results of Section 4

C.1 Proof of Theorem 6

That players $m+2, \dots, N$ do not participate was shown in Corollary 8. That $s^l_m = s^l_{m+1} = 0$ follows from point 4 in step 4 of the equilibrium construction, since $m, m+1 \in A(0)$. Suppose that for $i, j \le m$, $s^l_i \le s^l_j$. Since i and j have positive powers and are not active below s^l_i, $G_i\left(s^l_i\right) = G_j\left(s^l_i\right) = 0$. Thus, $P_i\left(s^l_i\right) = P_j\left(s^l_i\right)$.[33] Since

$$\frac{u_i\left(s^l_i\right)}{V_i} = P_i\left(s^l_i\right)\left(1 - a_i c\left(s^l_i\right)\right) - \left(1 - P_i\left(s^l_i\right)\right)\alpha a_i c\left(s^l_i\right) = P_i\left(s^l_i\right)\left(1 - (1-\alpha) a_i c\left(s^l_i\right)\right) - \alpha a_i c\left(s^l_i\right)$$

and

$$\frac{u_j\left(s^l_i\right)}{V_j} = P_j\left(s^l_i\right)\left(1 - a_j c\left(s^l_i\right)\right) - \left(1 - P_j\left(s^l_i\right)\right)\alpha a_j c\left(s^l_i\right) = P_j\left(s^l_i\right)\left(1 - (1-\alpha) a_j c\left(s^l_i\right)\right) - \alpha a_j c\left(s^l_i\right),$$

we have

$$\begin{aligned}\frac{u_i\left(s^l_i\right)}{V_i} - \frac{u_j\left(s^l_i\right)}{V_j} &= P_i\left(s^l_i\right)(1-\alpha) c\left(s^l_i\right)(a_j - a_i) + \alpha c\left(s^l_i\right)(a_j - a_i) \\ &= (a_j - a_i) c\left(s^l_i\right)\left(\alpha + P_i\left(s^l_i\right)(1-\alpha)\right).\end{aligned}$$

Also, $\frac{w_i}{V_i} = 1 - a_i c(T)$ and $\frac{w_j}{V_j} = 1 - a_j c(T)$ so $\frac{w_i}{V_i} - \frac{w_j}{V_j} = (a_j - a_i) c(T)$. Since $\frac{w_i}{V_i} = \frac{u_i\left(s^l_i\right)}{V_i}$ and $\frac{w_j}{V_j} \ge \frac{u_j\left(s^l_i\right)}{V_j}$, we have

$$0 \ge \left(\frac{w_i}{V_i} - \frac{w_j}{V_j}\right) - \left(\frac{u_i\left(s^l_i\right)}{V_i} - \frac{u_j\left(s^l_i\right)}{V_j}\right) = (a_j - a_i)\left(c(T) - c\left(s^l_i\right)\left(\alpha + P_i\left(s^l_i\right)(1-\alpha)\right)\right)$$

and since $\alpha > 0$ and $P_i\left(s^l_i\right) \le 1$, $\left(\alpha + P_i\left(s^l_i\right)(1-\alpha)\right) > 0$ so this inequality holds if and only if

$$(a_j - a_i)\left(c(T) - c\left(s^l_i\right)\right) \le 0.$$

Since $c\left(s^l_i\right) < c(T)$, this implies that $a_j \le a_i$, or $w_i \le w_j$. Since $G_{m+1}(0) > 0$ (Lemma 1), $s^l_{m+1} = 0$.

It remains to show that if a player becomes active at some score, then he remains active until the threshold. To do this, let us derive some properties of player's semi-elasticities. We must first normalize players' payoffs so that the prize value is 1 for all players. To this end, note that the contest is strategically equivalent to a contest in which all valuations equal 1 and in which player i's cost is $a_i c$ instead of $\gamma_i c$. We then have

$$q_i(x) = \frac{v_i(x) - v_i(T)}{v_i(x) + c_i(x)} = \frac{a_i\left(\frac{1}{a_{m+1}} - c(x)\right)}{1 - (1-\alpha) a_i c(x)}$$

[33] If $s^l_i = 0$, we consider the limit of the probabilities of winning as the score approaches 0 from above, and similarly for $u_i\left(s^l_i\right)$ and $u_j\left(s^l_i\right)$.

and

$$\varepsilon_i(x) = -\frac{q_i'(x)}{q_i(x)} = \frac{c'(x)(a_i(\alpha-1)+a_{m+1})}{(1-c(x)a_{m+1})(a_ic(x)(\alpha-1)+1)}.$$

Viewed as a function of a_i, we obtain

$$\frac{\partial\varepsilon_i(x)}{\partial a_i} = -\frac{c'(x)(1-\alpha)}{(a_ic(x)(\alpha-1)+1)^2} \le 0$$

so at every score players with higher power have higher semi-elasticities. This means that when a new player becomes active no existing players are "ousted", and that whether an active player remains active depends only on his semi-elasticity and those of players with lower powers. In particular, players m and $m+1$ are always active, since their semi-elasticities are always the lowest. To show that players $1,\ldots,m-1$ are active on an interval, observe that the ratio of semi-elasticities of players $j>i$ is non-decreasing in score:

$$\left(\frac{\varepsilon_j(x)}{\varepsilon_i(x)}\right)' = \left(\frac{(a_j(\alpha-1)+a_{m+1})(a_ic(x)(\alpha-1)+1)}{(a_i(\alpha-1)+a_{m+1})(a_jc(x)(\alpha-1)+1)}\right)'$$

$$= \frac{a_{m+1}-(1-\alpha)a_j}{a_{m+1}-(1-\alpha)a_i}\frac{(1-\alpha)(a_j-a_i)c'(x)}{(a_jc(x)(\alpha-1)+1)^2} \ge 0$$

and also that this ratio is at most 1 (since players with higher power have higher semi-elasticities). Suppose in contradiction that there is a player who is not active on an interval, and let $i\le m-1$ be the player with the highest index (lowest power) among such players. Suppose that player i is active at s_i. Denote by $H^{i+1,\ldots,m+1}(\cdot)$ the fixed point of the "supply function" defined using the semi-elasticities of players $i+1,\ldots,m+1$, and consider a score $s_i'\in[s_i,T]$. By definition of player i and because players with higher powers become active at higher scores, players $m+1,\ldots,i+1$ are active at s_i'. So, because the semi-elasticities of all players $1,\ldots,i-1$ are weakly higher than that of player i, $\varepsilon_i(s_i')\le H(s_i')$ if and only if $\varepsilon_i(s_i')\le H^{i+1,\ldots,m+1}(s_i')$. Let $b=\frac{\varepsilon_i(s_i')}{\varepsilon_i(s_i)}$. For all $j>i$, since $\frac{\varepsilon_j(s_i')}{\varepsilon_i(s_i')}\ge\frac{\varepsilon_j(s_i)}{\varepsilon_i(s_i)}$, we have $\frac{\varepsilon_j(s_i')}{\varepsilon_j(s_i)}\ge\frac{\varepsilon_i(s_i')}{\varepsilon_i(s_i)}=b$. Therefore,

$$H^{i+1,\ldots,m+1}(s_i') = \frac{1}{m-i}\sum_{j=i+1}^{m+1}\varepsilon_j(s_i') \ge \frac{1}{m-i}\sum_{j=i+1}^{m+1}b\varepsilon_j(s_i) = bH^{i+1,\ldots,m+1}(s_i).$$

Because player i is active at s_i, $\varepsilon_i(s_i)\le H(s_i)$ and $\varepsilon_i(s_i)\le H^{i+1,\ldots,m+1}(s_i)$. Therefore, $\varepsilon_i(s_i') = b\varepsilon_i(s_i)\le bH^{i+1,\ldots,m+1}(s_i)\le H^{i+1,\ldots,m+1}(s_i')$ so $\varepsilon_i(s_i')\le H(s_i')$. This shows that once a player becomes active he remains active until the threshold.

C.2 Proof of Theorem 7

Let

$$f(\alpha,x) = \frac{1}{(1-a_{m+1}c(x))}\frac{\Pi_{k=i+1}^{m+1}(1-a_kc(x)(1-\alpha))}{(1-a_ic(x)(1-\alpha))^{m-i}},$$

and note that f is differentiable in $\alpha<1$ and $x<T$. Denote by $s_i^l(\alpha)$ the lowest x such that $f(\alpha,x)=\frac{\Pi_{k=i+1}^{m}a_k}{a_i^{m-i}}$. Note that $\frac{\Pi_{k=i+1}^{m}a_k}{a_i^{m-i}}>1$ (because a_k increases in k) and $f(\alpha,0)=1$ (since

$c(0)=0$). Suppose that when α increases to α' the value of f at $s_i^l(\alpha)$ increases. Then, because $f\left(\alpha', s_i^l(\alpha)\right) > \frac{\Pi_{k=i+1}^{m} a_k}{a_i^{m-i}}$ and $f(\alpha',0) < \frac{\Pi_{k=i+1}^{m} a_k}{a_i^{m-i}}$, the intermediate value theorem shows that $s_i^l(\alpha') < s_i^l(\alpha)$. Therefore, to show that s_i^l decreases in α it suffices to show that $\frac{\partial f(a,x)}{\partial \alpha} > 0$ for $x \in (0,T)$. Since

$$\frac{\partial f}{\partial \alpha} = \frac{\left(\sum_{j=i+1}^{m+1} a_j c(x) \prod_{k\in\{i+1,\dots,m+1\}\setminus j} (1-a_k c(x)(1-\alpha))\right)(1-a_{m+1}c(x))(1-a_i c(x)(1-\alpha))^{m-i}}{\left((1-a_{m+1}c(x))(1-a_i c(x)(1-\alpha))^{m-i}\right)^2}$$

$$-\frac{\Pi_{k=i+1}^{m+1}(1-a_k c(x)(1-\alpha))\left((1-a_{m+1}c(x))(m-i)(1-a_i c(x)(1-\alpha))^{m-i-1} a_i c(x)\right)}{\left((1-a_{m+1}c(x))(1-a_i c(x)(1-\alpha))^{m-i}\right)^2},$$

it suffices to show that

$$\left(\sum_{j=i+1}^{m+1} a_j c(x) \prod_{k\in\{i+1,\dots,m+1\}\setminus j} (1-a_k c(x)(1-\alpha))\right)(1-a_i c(x)(1-\alpha)) >$$

$$\Pi_{k=i+1}^{m+1}(1-a_k c(x)(1-\alpha))((m-i)a_i c(x)).$$

For this inequality to hold, it suffices that for every $j=i+1,\dots m+1$,

$$a_j c(x) > a_i c(x) \text{ and } 1-a_i c(x)(1-\alpha) > 1-a_j c(x)(1-\alpha),$$

and this holds since a_k increases in k. Therefore s_i^l decrease in $\alpha < 1$ for every player $i = 1,\dots,m-1$.

Now consider what happens to s_i^l as α approaches 0. For $x<T$,

$$f(0,x) = \frac{1}{(1-a_{m+1}c(x))}\frac{\Pi_{k=i+1}^{m+1}(1-a_k c(x))}{(1-a_i c(x))^{m-i}} = \frac{\Pi_{k=i+1}^{m}(1-a_k c(x))}{(1-a_i c(x))^{m-i}} \leq 1.$$

Therefore, by uniform continuity of $f(\alpha,x)$ on $[0,\tilde{\alpha}]\times[0,x]$ for any $\tilde{\alpha}\in(0,1)$, s_i^l must approach T as α approaches 0.

D Proof of Corollary 9

Choose $\beta<1$. By Theorem 7 there exist $\tilde{x}<T$ and $\tilde{\alpha}>0$ such that for all $\alpha<\tilde{\alpha}$ and $i<m$, $s_i^l>\tilde{x}$. Choose such $\tilde{x}$ and $\tilde{\alpha}$ that also satisfy (1) $\frac{V_m-\gamma_m\frac{V_{m+1}}{\gamma_{m+1}}+\alpha\gamma_m c(\tilde{x})}{V_m-(1-\alpha)\gamma_m c(\tilde{x})} > \beta$ and (2) $\frac{\alpha\gamma_{m+1}c(\tilde{x})}{V_{m+1}-(1-\alpha)\gamma_{m+1}c(\tilde{x})} < 1-\beta$. Consider the unique equilibrium G of a such a simple contest with $\alpha<\tilde{\alpha}$. Since $G_i(\tilde{x})=0$ for $i=1,\dots,m-1$ and $G_i(0)=1$ for $i=m+2,\dots,n$, Corollary 7 shows that the CDFs of players $m+1$ and m on $[0,\tilde{x}]$ are given by (1) and (2). Since $s_i^l>\tilde{x}$ for $i=1,\dots,m-1$, each of these $m-1$ players beats player $m+1$, and therefore wins a prize, with probability of at least β. Player m chooses scores higher than $\tilde{x}$ with probability of at least β, and therefore wins a prize with probability of at least β^2.

E Proof of Corollary 10

First, $s_j^l \leq s_i^l$, so $G_i(x) \leq G_j(x)$ for any $x \leq s_i^l$. By Theorem 6, both players are active on on $\left[s_i^l, T\right]$. Third, $\varepsilon_i \geq \varepsilon_j$, so because both players are active on $\left[s_i^l, T\right]$, the equilibrium construction algorithm shows us that $h_i \leq h_j$ on $\left[s_i^l, T\right]$. So i starts dropping out later and drops out more slowly than j, which means that G_i FOSD G_j. To see this, recall that $h_i(x) = -\frac{(1-G_i(x))'}{1-G_i(x)}$, so $h_i \leq h_j$ implies that $\frac{(1-G_i)'}{1-G_i} \geq \frac{(1-G_j)'}{1-G_j}$. This implies that for $y \in \left[s_i^l, T\right]$,

$$0 \leq \int_{s_i^l}^{y} \left(\frac{(1-G_i(x))'}{1-G_i(x)} - \frac{(1-G_j(x))'}{1-G_j(x)}\right) dx = \ln\left(\frac{1-G_i(x)}{1-G_j(x)}\right)|_{s_i^l}^{y} = \ln\left(\frac{1-G_i(y)}{1-G_j(y)}\right) - \ln\left(\frac{1-G_i\left(s_i^l\right)}{1-G_j\left(s_i^l\right)}\right)$$

Because $G_i\left(s_i^l\right) \leq G_j\left(s_i^l\right)$, we have $\frac{1-G_i\left(s_i^l\right)}{1-G_j\left(s_i^l\right)} > 1$, so by taking exponents the previous inequality implies $\frac{1-G_i(y)}{1-G_j(y)} \geq 1$, or $G_j(y) \geq G_i(y)$, as required.

This FOSD implies that probability of winning is higher than that of player j for any given score, and hence also in expectation. To see this, note that by choosing $x > 0$ i beats j with probability $G_j(x)$, whereas by choosing x j beats i with probability $G_i(x)$. Therefore, because $G_j(x) \geq G_i(x)$, for any given score i wins with at least as high a probability as j does, i.e., $P_i(x) \geq P_j(x)$. Therefore, $P_i = \int P_i(x)\, dG_i \geq \int P_j(x)\, dG_i$, and because $P_j(\cdot)$ is non-decreasing, by FOSD $\int P_j(x)\, dG_i \geq \int P_j(x)\, dG_j = P_j$.