COMPETITIVE EQUILIBRIA IN DECENTRALIZED MATCHING WITH INCOMPLETE INFORMATION

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Abstract.

This paper shows that all perfect Bayesian equilibria of a dynamic matching game with two-sided incomplete information of independent private values variety converge to competitive equilibria. Buyers purchase a bundle of heterogeneous, indivisible goods and sellers own one unit of an indivisible good. Buyer preferences and endowments as well as seller costs are private information. Agents engage in costly search and meet randomly. The terms of trade are determined through bilateral bargaining between buyers and sellers. The paper considers a market in steady state. It is shown that as frictions disappear, i.e., as discounting and the fixed cost of search become small, all equilibria of the market game converge to perfectly competitive equilibria.

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1. Introduction

This paper shows that all equilibria of a dynamic matching game with two-sided incomplete information of the independent private values variety converge to competitive equilibria. In the model each buyer aims to purchase a bundle of heterogeneous, indivisible objects and each seller owns one unit of a heterogeneous indivisible good (as in Kelso and Crawford (1982) or Gul and Stacchetti (1999)). Buyer preferences and endowments as well as seller costs are private information. Agents engage in costly search and meet randomly. The terms of trade are determined through bilateral bargaining under incomplete information between buyers and sellers. The paper considers a market in steady state and shows that as frictions disappear, that is as

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discounting and the explicit (fixed) cost of search become small for all agents, the market becomes perfectly competitive.

Numerous researchers have explored the non-cooperative foundations for competitive equilibria in the markets for indivisible goods using dynamic matching games. Previous work has focused almost exclusively on markets for an homogeneous good and has mainly assumed complete information. In particular, Gale (1987) and Mortensen and Wright (2002) establish convergence of dynamic matching game equilibria to competitive equilibria as search friction disappear under complete information, while Satterthwaite and Shneyerov (2007) extend the analysis to the two-sided incomplete information case.

Often cited examples of markets, where indivisible goods are exchanged through bilateral negotiations, are the labor and the housing markets. Although cited as motivating examples, neither of these markets fit the mold of a market for an homogeneous good where buyers only differ in their valuations for the good, and sellers only differ in their cost of providing the good. For example, in the labor market potential employees (sellers of labor services) differ in their productivity and their disutility of labor. Firms (buyers of labor services) usually search for multiple employees, that may complement or substitute each other. Also, the vacancies in the firms are rarely exactly alike, and an employees productivity may depend crucially on the type of vacancy that a firm has available. In the housing market the potential homes are far from being homogenous and families who search for homes in the same market, may have diverse needs. Moreover, some home purchases are bundles that include the actual home, nearby parking, architectural services for remodeling the home and brokerage services for the transaction. Also, in neither of these markets is it likely that a seller observes a buyers preferences nor a buyer the sellers outside option. This paper presents a dynamic matching game, with two sided incomplete information, that preserves many of the attributes of markets such as the labor market and the housing market.

A brief description of the model presented here is as follows: In each period a unit measure of each type (of buyers and sellers) from a finite set of types is available for entry and those who expect a non-negative return voluntarily enter the market. The market is in steady-state with the measure of agent types endogenously determined to balance the flow of types through the economy. Once in the market, each agent pays a per period cost, and receives a "draw" from the distribution of active players. Also, finding a bargaining partner takes time and agents discount the future. The probability that any buyer (or seller) is paired with a particular type is proportional to the frequency of that type among all sellers (buyers) active in steady state. After two agents are paired, nature designates a proposer, the proposer offers an (incentive compatible direct) mechanism, and the responder decides whether to participate. During this bargaining stage buyer preferences and endowments, as well as, seller costs are private information. The good that the seller offers, however, is observed by the buyer. If a meeting between a pair results in a trade, then the seller leaves the market, otherwise the agents return to the population of active players. Buyers leave the market voluntarily after they have purchased all the goods that they want.

The analysis here is concerned with convergence of equilibria to a competitive equilibrium as search friction vanish. The competitive equilibrium benchmark under consideration is a "flow" equilibrium as in Gale (1987) or Satterthwaite and Shneyerov (2007), generalized to accommodate heterogenous goods and multi-unit demand. In each period, flow supply is the measure of sellers of a particular good entering the market and flow demand is the measure of agents willing to purchase a particular good entering the market. In a flow equilibrium, the buyer and seller continuation values, which are the implicit prices, equate flow supply to flow demand for each of the goods traded in the market.

The main result in the paper shows that a steady state equilibrium exists for any δ and $c > 0$ and as the discount factor $\delta \to 1$ and the explicit search costs $c \to 0$, all trade takes place at competitive prices. The intuition for the convergence result is as follows: as search becomes increasingly cheap, buyers wait until they have accumulated their most favored bundle, and while accumulating these goods, they reject "high" prices. Also, sellers become more discerning and wait until they receive the best price offer possible. Consequently, at the limit, trade in each good occurs at a unique price and each buyer purchases their most preferred bundle at these prices. Since the market remains in a steady state, the limiting price vector balances the flow supply of goods with the flow demand for the goods and is thus a competitive equilibrium price vector. Incomplete information stops playing a role asymptotically, since all agents anticipate that trade will take place at the Walrasian price.

Although the literature on dynamic matching and search is vast, Satterthwaite and Shneyerov (2007) is the work most closely related to this one. Satterthwaite and Shneyerov (2007) established that equilibria of a dynamic matching game converge to a competitive equilibrium in the case of a single homogeneous good and two sided uncertainty. The analysis provided here differs from Satterthwaite and Shneyerov (2007) in two main respects. First, the homogeneous good, unit demand restrictions are lifted. Second, in Satterthwaite and Shneyerov (2007), the buyers and sellers that meet, are assumed to participate in a double auction where any seller bids her continuation value truthfully. In contrast, here the proposer is allowed to choose any mechanism and so, strategic behavior is allowed for both the buyers and sellers. In related models presented in DeFraja and Sakovics (2001), Serrano (2002) and Wolinsky (1990), convergence to a competitive equilibrium fails. The failure of convergence to competitive equilibrium is caused by the bilateral bargaining protocol in Serrano (2002); results from the inefficiency of aggregating common value information through bilateral meeting in Wolinsky (1990); and is due to a "clones" assumption in DeFraja and Sakovics (2001) (see Lauermann (2006) for a detailed discussion of these issues).

The paper proceeds as follows: Section 2 outlines the dynamic matching and bargaining game as well as the competitive benchmark, Section 3.1 presents the main results that show convergence to a competitive equilibrium, Section 3.2 outlines the equilibrium existence argument, and Section 4 concludes. Proofs that are not included in the main text are in the Appendix.

2. The Model

Buyers and sellers in the economy search for possible trading partners. Each seller owns one indivisible good for sale and each buyer wants to purchase a bundle of the indivisible goods offered for sale. The game progresses in discrete time and agents discount the future with a common discount factor δ. In each period, an agent incurs a positive explicit search cost c and meets pairwise with a potential partner.[1] Either the buyer or the seller is designated as the proposer. The probability that the buyer is designated as the proposer is $\beta \in (0, 1)$. The proposer offers a direct mechanism and the responder chooses whether to participate in the mechanism. If the responder

[1]Although, I assume that all agents share a common discount factor δ and explicit search cost, c, this is for convenience only. All results in the paper go through even if agents have heterogeneous search costs.

participates in the mechanism, then the agents report their types to the mechanism and the mechanism chooses the probability with which a trade occurs and specifies the transfers to be paid by the buyer to the seller. Sellers who trade permanently leave the market. Buyers remain in the economy until they have purchased all the goods that they want, then they leave the market and consume their bundle. Agents who fail to trade return to the searching population. Utility is transferable. In particular, if a buyer of type b consumes bundle G, then she enjoys utility h_{bG}. A seller incurs cost r_s when she sells her good. So trade between b and sellers $s \in G$ creates total transferable utility $f_{bG} = h_{bG} - \sum_{s \in G} r_s$.

2.1. **Population of Types and Private Information.** Let B and S denote the finite sets of initial buyer and seller types and let $I = B \cup S$ denote the set of all initial types. A seller's type specifies the good she owns, x_s, and her reservation value (or cost) r_s. Let X denote the set of goods potentially traded in the market, i.e, if x_s is the good owned by seller s, then $X = \{x_s : s \in S\}$.

A buyer's initial type specifies the buyer's utility function $h_b : \mathcal{P}(S) \to \mathbb{R}$, where $\mathcal{P}(S)$ denotes the set of all subsets of S. The utility function for any buyer b satisfies:

(i) Normalization: $h_{b\emptyset} = 0$,
(ii) Monotonicity: If $G \supset A$, then $h_{bG} \geq h_{bA}$,
(iii) Identity Independence: For any s and s' with $x_s = x_{s'}$ (i.e., for sellers s and s' who own the same good), $h_{bG\cup\{s\}} = h_{bG\cup\{s'\}}$ for all G.

Once in the market, a buyer's type changes after each trade and includes information on all trades that the agent has made, and consequently, the goods that the buyer owns. So, refer to a buyer type by $i = b_G$, where $b \in B$ is the initial type, i.e., her utility function, and G is the set of seller types with whom she has already traded. Consequently, the set of buyer types present in the market is $T = B \times \mathcal{P}(S)$.

In each period, a unit measure of each $b \in B$ and $s \in S$ are available to enter the market. Consequently, in each period a measure $|B|$ of buyers and measure $|S|$ of sellers potentially enter the market. Buyers and sellers, who do not enter the market in a given period, are assumed to have opted for an outside option and are thus not available for entry in any subsequent periods. Let $l = (l_{b1}, ..., l_{b_G}, ..., l_{|B|_{|2^S|}}, l_{s1}, ..., l_{|S|})$ denote the steady state measure of buyers and sellers in the market, i.e., $l \in \mathbb{R}_+^{|B|\times|2^S|+|S|}$. The steady state probability for any seller of meeting buyer b, or any buyer meeting a seller s in a given period is

$$p_b = \frac{l_b}{\max\{L_B, L_S\}} \quad \text{or} \quad p_s = \frac{l_s}{\max\{L_B, L_S\}} \tag{1}$$

where $L_B = \sum_{b \in T} l_b$ and $L_S = \sum_{s \in S} l_s$. The total measure of buyer b and seller s pairs formed in a period is $l_s p_b = l_b p_s$ and the total measure of pairs formed is equal to $\min\{L_B, L_S\}$. The (sub) probability measures p_b and p_s (or type distributions) are commonly known by all agents.

The analysis here assumes *independent private values*. More precisely, if a buyer and seller consummate a trade, then the payoff to each agent depends on the terms of trade and their own private information; but does not depend on their trading partner's private information, i.e., there is no "Lemons" problem. Reference to this assumption, which is stated formally below, is omitted from the statements of the results presented since it is maintained throughout the paper.

Assumption: Independent Private Values. *If buyer b_G and seller s meet, then the buyer observes, x_s, the good that seller s has for sale, while b_G and r_s remain as private information.*

Also, further *assume that the agents know the distribution of types in the economy and that any agent's prior belief about his/her trading partner's type coincides with the distribution of types.* This requirement is stronger than what is needed for showing convergence to a competitive equilibrium. As long as the support of any agent's prior belief coincides with the support of the steady state distribution, the convergence results will continue to hold.

2.2. **Agent Behavior and Strategies.** Let σ_i denote a strategy for type i and $\sigma = (\sigma_i)_{i \in I}$ a strategy profile. Assume that all agents use stationary time-invariant strategies ($\sigma^t = \sigma$ for all t). At the start of each period, the strategy determines whether the agent remains in (or enters) the market and pays the cost c. Denote by $\sigma_i(in)$ the probability that agent i remains in (or enters) the market at the start of any period. If i is paired in the current period and is the proposer, then the strategy σ_i returns a (direct) mechanism choice μ_i. If agent i is the responder, then the strategy specifies whether she accepts to participate in the mechanism. If the responder accepts to participate, then the two agents report types k and j to the mechanism and the mechanism chooses a probability of trade and the transfer paid by the buyer to the seller. The proposer can condition her mechanism offer, and the responder can condition her type report, on the measure of agents in the economy l, other common knowledge parameters of the economy, and the observable characteristics of her partner for the period. The responder can also condition her report to the mechanism on the mechanism chosen by the proposer. Without loss of generality, assume that the proposer offers an individually rational and (interim) incentive compatible mechanism and the responder always participates and reports her type truthfully. Consequently, a strategy for type i is given by $\sigma_i = (\sigma_i(in), \mu_i)$.

The game played in each period can always be modeled as a direct mechanism choice, that satisfies individual rationality and incentive compatibility constraints, by an informed proposer (principal) (see for example Myerson (1983) or Maskin and Tirole (1990)). However, the Lemma below shows that the private information of the proposer does not play a role. In fact, the proposer chooses the same mechanism she would have chosen even if her type was publicly know. In particular, since buyers and sellers are risk neutral, a take-it-or-leave-it offer is an optimal mechanism for the proposer (as in Riley and Zeckhauser (1983)).

Lemma 1. *The equilibrium mechanism choices satisfy the following:*

(i) *If μ_i is the optimal (direct) mechanism choice for type i, when type i is proposing and her type is known to the responder, then the direct mechanism $\mu = (\mu_i)$, i.e., the mechanism that uses μ_i when the proposer reports type i to the mechanism, is an optimal mechanism choice for all i, when the proposer's type is private information.*

(ii) *A take-it-or-leave-it offer is optimal when the proposer's type is know, consequently a type specific take-it-or-leave-it offer is an optimal mechanism for any proposer.*

(iii) *If s and s' own the same good and $r_s \geq r'_s$, then the take-it-or-leave-it offer, $t_s \geq t'_s$ in any equilibrium.*

In the case where each buyer only wants to purchase a single good, then a property analogous to the monotonicity property for seller strategies, given in item (iii), also holds for all buyers. Although Lemma 1 is stated under the assumption that buyers and sellers meet in pairs, the result is more general, and extends to the case where a seller (or buyer) meets with an arbitrary number of buyers (or sellers). If a seller meets with an arbitrary number of buyers, she will propose the optimal auction that she would have proposed had her type been known to the responder.

Let the match probability m_{bs} (or m_{sb}) denote the probability that b and s trade, given that the two are paired in the period and b (or s) is chosen as the proposer. Also, let t_{bs} (or t_{sb}) denote the transfer paid by the buyer to the seller, given that b and s are paired in the period, b (or s) is chosen as the proposer, and they trade. The reward function for a buyer b (or seller s) proposing (responding) to seller s (buyer b) is $\pi_b(\sigma, s) = -c - m_{bs}t_{bs}$ (or $\pi_s(\sigma, b) = -c + m_{bs}(t_{bs} - r_s)$). If an agent has exited the market or has accepted a match in a prior period, then the agent is paired with 0, and $\pi_i(\sigma, 0) = 0$. Also, if the agent does not get paired in a period, then she/he is paired with herself and $\pi_i(\sigma, i) = -c$.

The expected future value at the start of a period for a seller equals the maximum of the value of remaining in the market and the value of leaving the market, i.e., $v_s = \max\{v_s(in), 0\}$. The expected future value at the start of a period for a buyer equals the maximum of the value of remaining in the market and the value of leaving the market and consuming the bundle that she owns, that is, $v_b = \max\{v_b(in), h_{bG(b)}\}$ where for each $b \in T$, $G(b)$ denotes the set of seller types with whom type b has already traded. The value of remaining in the economy, $v_b(in)$, satisfies

$$v_b(in) = -c + \sum\nolimits_s p_s \beta m_{bs}(\delta v_{b\cup s} - t_{bs}) + \sum\nolimits_s p_s(1-\beta)m_{sb}(\delta v_{b\cup s} - t_{sb}) \\ + (1 - \sum\nolimits_s p_s \beta m_{bs} - \sum\nolimits_s p_s(1-\beta)m_{sb})\delta v_b.$$

where the notation $b \cup s$ (or the notation $b \setminus s$) denotes a type b' with $h_b = h_{b'}$ and $G(b') = G(b) \cup \{s\}$ (or $G(b') = G(b) \setminus s$). In words, buyer b pays the search (sampling) cost c, then successfully makes a trade as the responder with seller s with probability $(1-\beta)p_s m_{sb}$; makes a trade when she proposes to buyer s with probability $\beta p_s m_{bs}$; and does not trade in the period and receives her continuation value δv_b with probability $(1 - \sum_s p_s \beta m_{bs} - \sum_s p_s(1-\beta)m_{sb})$. Continuation values are defined similarly for the sellers. Rearranging gives the following for buyer and seller values:

$$v_s(in) = -c + \beta \sum\nolimits_{b\in T} p_b m_{bs}(t_{bs} - r_s - \delta v_s) \\ + (1-\beta) \sum\nolimits_{b\in T} p_b m_{sb}(t_{sb} - r_s - \delta v_s) + \delta v_s, \text{ and}$$

$$v_b(in) = -c + \beta \sum_s p_s m_{bs}(\delta v_{b \cup s} - \delta v_b - t_{bs}) \\ + (1-\beta) \sum_s p_s m_{sb} \delta(v_{b \cup s} - \delta v_b - t_{sb}) + \delta v_b,$$

for any b and s.

2.3. **Steady State.** Assuming that the economy remains in steady state implies that the number of type b buyers (or type s sellers) entering the market in each period must equal the number of that type leaving the market. Consequently, the steady state measure of agents in the economy, l, satisfies the following equations:

$$l_b(\sum_{s \in S} M_{bs} + \sigma_b(out)) = \sigma_b(in)$$
$$l_s(\sum_{b \in T} M_{sb} + \sigma_s(out)) = \sigma_s(in)$$

for all types $b \in T$ with $b(G) = \emptyset$ and all sellers $s \in S$, where $M_{bs} = p_s(\beta m_{bs} + (1-\beta)m_{sb})$ denotes the fraction of type b buyers who successfully trade with type s sellers in a period; $\sigma_b(out) = (1 - \sum_{s \in S} M_{bs})(1 - \sigma_b(in))$ denotes the fraction of type b buyers, who failed to trade in the previous period, that choose to leave at start of the current period; and $\sigma_b(in) \leq 1$ is the flow of new buyers into the market at the start of the period. Also,

$$l_b(\sum_{s \in S} M_{bs} + \sigma_b(out)) = \sigma_b(in) \sum_{s \in G(b)} l_{b \setminus s} M_{b \setminus ss}$$

for $b \in T$ with $b(G) \neq \emptyset$, where $\sigma_b(in) \sum_{s \in G(b)} l_{b \setminus s} M_{b \setminus ss}$ is the measure of newly created type b buyers who remain in the market, that is buyer types who were an "s" away from type b who traded with type s in the previous period.

2.4. **Equilibrium.** A steady state search equilibrium is comprised of a mutually compatible strategy profile σ and steady state measure l. That is to say, the measure l satisfies the steady state equations, given that agents use strategy profile σ and, the strategy profile σ comprises a Perfect Bayesian Equilibrium for the market game, given that the steady state measure of agent is l.

2.5. **The Competitive Benchmark.** The competitive equilibrium benchmark considered here is a "flow" equilibrium as in Gale (1987) or Satterthwaite and Shneyerov (2007), generalized to accommodate heterogenous goods and multi-unit demand. In each period, flow supply is the measure of sellers of a particular good entering the market and flow demand is the measure of agents willing to purchase a particular good entering the market. In a flow equilibrium, the buyer and seller continuation values, which are the implicit prices, equate flow supply to flow demand for each good that is traded in the market. The competitive equilibrium allocations for economy $I = B \cup S$ is described by the following linear program (and its dual) which is the classical Assignment Problem where fractional assignments are permitted. This formulation is a generalization of Shapley and Shubik (1972) to a setting where buyers can purchase multiple commodities as in Kelso and Crawford (1982) or Gul and Stacchetti (1999, 2000).

Primal

$$P = \max_{q \geq 0} \sum_{b \in B} \sum_{G \subset S} q_{bG} (h_{bG} - \sum_{s \in G} r_s)$$

Subject to

$$\sum_{b \in B} \sum_{s \ni G} q_{bG} \leq 1 \text{ for all } s, \tag{2}$$

$$\sum_{G \subset S} q_{bG} \leq 1 \text{ for all } b. \tag{3}$$

Dual

$$D = \min_{v \geq 0} \sum_{B} v_b + \sum_{S} v_s$$

Subject to

$$v_b + \sum_{s \in G} v_s \geq h_{bG} - \sum_{s \in G} r_s \ \forall b, G.$$

The vector q that solves the program is a competitive allocation and denotes the measure of matches between buyer b and sellers in the set G that are created in each period of time. Any vector v that solves the dual program is a competitive equilibrium utility vector and the competitive price of a traded good is $p_{x_s} = v_s + r_s$. The constraint given by equation (2) states that the flow demand for seller of type s, i.e., $\sum_{b \in B} \sum_{s \ni G} q_{bG}$, must be less than the flow supply of that type, which is at most one. This constraint will bind, if the good's price is positive, or more precisely, if $v_s > 0$ and thus $p_{x_s} = v_s + r_s > r_s$. The constraint given by equation (3) states that the flow supply to buyers of type b, must be less than the flow demand by type b, which is at most one. Again, this constraint will bind if $v_b > 0$. Together inequalities (2) and (3) ensure market clearing. Observe that, if q solves the primal and v the dual, then each buyer consumes her most preferred bundle, sellers offer their good only if $p_{x_s} \geq r_s$, and all markets clear. Conversely, if q is a competitive allocation and p a competitive price, then q solves the primal Assignment Problem by the first welfare theorem; and buyer values $v_b = \max_{G \subset S} h_{bG} - \sum_{s \in G} p_{x_s}$ and seller values $v_s = \max\{0, p_{x_s} - r_s\}$ solve the dual Assignment Problem.

3. Existence of Equilibrium and Convergence to Competitive Equilibria

The development in this section shows that as search becomes costless, i.e., $\delta \to 1$ and $c \to 0$, any sequence of steady state equilibria converges to a competitive equilibrium (Theorem 1, Corollary 1 and Corollary 2). Also, for any configuration of search frictions, that is, for any $\delta \in [0,1]$ and $c > 0$, a steady state search equilibrium exists (Theorem 2 and Corollary 3).

3.1. **Convergence to Competitive Equilibria.**

The analysis focuses on sequences of equilibria (l^n, σ^n), and the associated sequences of equilibrium match probabilities m^n, type distributions p^n and values v^n, as search costs disappear, i.e., as $(c^n, \delta^n) \to (0, 1)$. Let

$$q^n_{b(i)G(i)} = l^n_i \sigma^n_i(out) + (1 - \sigma^n_i(in)) \sum\nolimits_{s \in G(i)} l^n_{i\backslash s} M^n_{i\backslash ss}$$

denote measure of buyers with initial type $b(i) \in B$ leaving the market with bundle $G(i)$. Since the market is in steady state $0 \leq q^n_{bG} \leq 1$ for all $b \in B$, $G \subset S$ and n. Also, let

$$e^n_{bG} = h_{bG} - \sum\nolimits_{s \in G} r_s - \delta^n v^n_b - \sum\nolimits_{s \in G} \delta^n v^n_s$$

denote the *Excess* between any initial buyer type $b \in B$ and sellers in the set G; and similarly

$$e^n_{is} = \delta^n v^n_{i \cup s} - \delta^n v^n_i - \delta^n v^n_s - r_s$$

denote the excess between buyer $i \in T$ and seller s. Note that, $0 \leq v^n_i \leq \bar{h}$; and $-(|S| + 1)(\bar{h} + \bar{r}) \leq e^n_{ij} \leq \bar{h}$, where $\bar{h} = \max_{b,G} h_{bs}$ and $\bar{r} = \max_s r_s$. Consequently, the sequence (q^n, e^n, v^n, p^n) is included in a compact set and has a convergent subsequence. From hereon restrict attention to convergent subsequences $(q^n, e^n, v^n, p^n) \to (\hat{q}, \hat{e}, \hat{v}, \hat{p})$.

The main result of the paper, Theorem 1, is formulated under two assumptions: Uniform Rate of Convergence (URC) and Free First Draw (FD). The (URC) assumption requires that the explicit search costs c^n do not converge to zero faster than the implicit time costs $1 - \delta^n$. This assumption, stated more precisely below, ensures that the market is not clogged up by agents, that have no hope of trading and no incentive to leave, accumulating in the economy. Since remaining in the economy remains costly for any n, agents who do not have positive continuation values, voluntarily exit the market.

Assumption: Uniform Rate of Convergence (URC)**.** $\limsup_n \frac{1-\delta^n}{c^n} \leq r < \infty$.

It should be pointed out that (URC) is satisfied in Satterthwaite and Shneyerov (2007) implicitly. Suppose, as in Satterthwaite and Shneyerov (2007), that the per-period explicit search cost $c = \Delta t \kappa$, where Δt is the period length and $\kappa > 0$ is the explicit cost, and the time cost $\delta = e^{-\rho \Delta t}$, where $\rho > 0$ is the discount rate. Then, as search frictions become small, or formally, as the period length Δt, shrinks to zero, $\frac{1-e^{-\rho\Delta t}}{\Delta t \kappa} \to \frac{\rho}{\kappa} > 0$.

Assumption (FD), stated below, requires, in the first period for any agent, that the agent not pay the search cost c, that is, one draw (the first draw) from the distribution agents in the market is free for all agents. This assumption ensures that all agents enter the market and there are no coordination problems in entry that could result in a missing market. If the first draw was not for free, then no agent entering the economy is an equilibrium. Also, Example 1 at the end of the section, outlines a more robust demonstration of a coordination failure. At the end of this subsection, this assumption that the first draw comes for free is relaxed.

Assumption: Free First Draw (FD)**.** *The search cost c is not paid in the first period for an agent and a unit measure of each type enters the market in each period; thus all agents sample the distribution at least once.*

Formally, Assumption (FD) requires that the choice of not-entering the market and opting for an outside option is not available to agents at the start of their first period in the market. This choice becomes available only after one period in the market. Consequently, (FD) implies that, $1 \leq l_b$, for all b and $1 \leq l_s$ for all s. Observe, even if not entering was an available action in the first period, all agents weakly prefer to sample the market since they get this first draw at no cost. Consequently, the game where the action of not entering is available to agents in their first period, always admits an equilibrium where each agent samples the market at least once (see Theorem 2).

The following theorem shows that, under the (URC) and (FD) assumptions, any sequence of steady state search equilibria converges to a competitive equilibrium. The proof first shows that the per-period exit rate of buyers with goods in the set G (i.e., $\hat{q}_{bG}$) is a feasible choice for the Assignment Problem, and so, the flow creation of value in the economy is at most as large as the maximized value of the Assignment

Problem. The argument proceeds to show that the *Excess* $\hat{e}_{ij}$ between any buyer i and seller j as well as the *Excess* between any initial type b and sellers in the set G, i.e., $\hat{e}_{bG}$ is non-positive. *No Excess* then implies that the vector of equilibrium values v is a feasible choice for the dual of the Assignment Problem, and consequently, that the flow creation of value in the economy is at least as large as the maximized value of the Assignment Problem.[2] The (URC) and (FD) assumptions in conjunction allows one to show that, if a buyer waits long enough, then she can meet any seller and make this seller a take-it-or-leave-it offer. This drives the *Excess* between any two agents to zero as search frictions vanish.

Theorem 1. *Assume (URC) and (FD). If $(q^n, v^n) \to (\hat{q}, \hat{v})$, then $\hat{q}$ solves the primal Assignment Problem and is a competitive equilibrium allocation; $\hat{v}$ solves the dual Assignment Problem and is a competitive equilibrium utility vector; and $\hat{v}_s + r_s$ is a competitive equilibrium price for good x_s.*

Proof. Note that $\sum_{G \subset S} l^n_{b_G}$ is the measure of buyers, whose initial type was b, present in the market. $\sum_{G \subset S} l^n_{b_G}$ is in steady state since it is the sum of the steady state measures $l^n_{b_G}$. The number of buyers, whose initial type was b, permanently leaving the market in each period is $\sum_{G \subset S} q^n_{bG}$ and the number entering is 1. Consequently, steady state implies $\sum_G q^n_{bG} \le \sigma^n_b(in) = 1$.

Note that $\sum_b \sum_{s \ni G} l^n_{b_G} + l^n_s$ denotes the measure of agents who own the good that initially belonged to a seller of type s and this measure is also in steady state since it is a sum of steady state variables. In each period, the measure of agents leaving with a good that initially belonged to a seller of type s is

$$\sum_b \sum_{s \ni G} q^n_{bG} + l^n_s \sigma^n_s(out)$$

and the number of type s agents entering the market is $\sigma_s(in)$. Consequently,

$$\sum_b \sum_{s \ni G} q^n_{bG} + l^n_s \sigma^n_s(out) \le 1.$$

Taking limits shows

$$\sum\nolimits_b \sum\nolimits_{s \ni G} \hat{q}_{bG} + \hat{l}_s \hat{\sigma}_s(out) \le 1 \text{ for all } s \text{ and}$$
$$\sum\nolimits_G \hat{q}_{bG} \le 1 \text{ for all } b.$$

[2]Observe that the constraint of the dual Assignment Problem only requires *No Excess*.

This implies that the vector $\hat{q}$ satisfies equation (2) and equation (3) and is feasible for the primal Assignment Problem. Consequently,

$$\sum\nolimits_{b\in B}\sum\nolimits_{G\subset S}\hat{q}_{bG}(h_{bG}-\sum\nolimits_{s\in G}r_s)\leq P.$$

By Lemma 4, given in the Appendix, $\hat{e}_{bG}\leq 0$ for all b and G , this implies that $\hat{v}$ is feasible for the dual and consequently, $\sum_B \hat{v}_b+\sum_S \hat{v}_s\geq D$. But,

$$\sum_B \hat{v}_b+\sum_S \hat{v}_s\leq \sum\nolimits_{b\in B}\sum\nolimits_{G\subset S}\hat{q}_{bG}(h_{bG}-\sum\nolimits_{s\in G}r_s)$$

by Lemma 5, given in the Appendix. Consequently,

$$D\leq \sum_B \hat{v}_b+\sum_S \hat{v}_s=\sum\nolimits_{b\in B}\sum\nolimits_{G\subset S}\hat{q}_{bG}(h_{bG}-\sum\nolimits_{s\in G}r_s)\leq P=D$$

and so $\sum_{B\times S}\hat{q}_{bs}f_{bs}=P$ proving that $\hat{q}$ is a competitive allocation and $\hat{v}$ is a competitive equilibrium utility vector. □

In the development below Assumption (FD) is dropped and replaced by a "tighter" version. In particular, the following assumption provides an (almost) necessary and sufficient condition for every sequence of search equilibria to convergence to a competitive equilibrium. The condition requires that only an arbitrarily small, but positive, measure of the lowest cost seller of each good receive their first draw for free (or enter the market by accident).

Assumption: FD for Low Cost Sellers (FDL)**.** *In each period, there is $\varepsilon_x>0$ entry by the lowest cost seller of each good x.*

Again, formally this assumption requires that the choice of not-entering the market and opting for an outside option is not available for a fraction $\varepsilon_x>0$ of sellers at the start of their first period in the market. This choice becomes available only after one period in the market.

The following corollary of Theorem 1 establishes convergence to competitive equilibrium under Assumption (FDL). The argument shows that (FDL) and (URC) together are sufficient to show *No Excess* between any two agents.

Corollary 1. *Assume (URC) and (FDL). If $(q^n,v^n)\rightarrow(\hat{q},\hat{v})$, then $\hat{q}$ solves the primal Assignment Problem and is a competitive equilibrium allocation; $\hat{v}$ solves the* *dual Assignment Problem and is a competitive equilibrium utility vector; and $\hat{v}_s + r_s$ is a competitive equilibrium price for good x_s.*

Proof. To show convergence, $\hat{e}_{bG} \leq 0$ (no excess) is established. Once, $\hat{e}_{bG} \leq 0$, then the corollary follows from the argument in Theorem 1. Lemma 2 and Lemma 3, given in the Appendix, are valid under the assumption of this corollary. Also, Lemma 4 can also be applied as follows: Let i_x denote the lowest cost seller of good x. By Lemma 4, $\hat{e}_{Gb} \leq 0$ for all b and $G \subset \{i_x\}$.

For any two sellers of good x, $v_s - v_{s'} \leq v_s(in) - v_{s'}(in)$ and so,

$$(v_s(in) - v_{s'}(in))(1-\delta) \leq \beta \sum\nolimits_{b\in T} p_b m_{bs}(r_{s'} + \delta v_{s'} - r_s - \delta v_s) \\ + (1-\beta) \sum\nolimits_{b\in T} p_b m_{sb}(r_{s'} + \delta v_{s'} - r_s - \delta v_s)$$

Also, suppose, without loss of generality, that $r_{s'} \geq r_s$.

$$(v_s - v_{s'})(1-\delta) \leq ((r_{s'} - r_s) - \delta(v_s - v_{s'})) \sum\nolimits_{b\in T} M_{sb}$$

$$v_s - v_{s'} \leq (r_{s'} - r_s) \frac{\sum_{b\in T} M_{sb}}{(1-\delta) + \delta \sum_{b\in T} M_{sb}} \leq r_{s'} - r_s$$

Consequently, $\delta v_s + r_s \leq \delta v_{s'} + r_{s'}$. For any set G of sellers, let H denote the set of sellers where each $s \in G$ is replaced by i_{x_s}, i.e., the lowest cost seller who owns the same good as seller s. So, $h_{bG} = h_{bH}$, also, $\delta v_s + r_s \leq \delta v_{s'} + r_{s'}$ for any $s' \in G$ and $s \in H$ with $x_{s'} = x_s$. Consequently,

$$e_{bG} = h_{bG} - \sum\nolimits_{s\in G} (\delta v_s + r_s) - \delta v_b \leq h_{bH} - \sum\nolimits_{s\in H} (\delta v_s + r_s) - \delta v_b = e_{bH}$$

However, $e^n_{bH} \to \hat{e}_{bH} \leq 0$ since $H \subset i_x$. So, $\hat{e}_{bG} = \lim e^n_{bG} \leq \lim e^n_{bH} \leq 0$ proving that $\hat{e}_{bG} \leq 0$. □

As pointed out the condition outlined in Assumption (FDL), or a similar condition imposed on the buyer side of the market, is also necessary in the following limited sense: if Assumption (FDL) does not hold, then there exists a sequence of steady state equilibria for an economy that fails to converge to a competitive equilibrium of that economy. The following is such an example.

Example: Necessity of FDL. *Consider an economy with two buyer types and two seller types, where each buyer wants to purchase only one good and the two seller types own two different goods. Let $h_{12} = h_{21} = 0$ and $h_{11} = h_{22} = 1$, that is h is* *super-modular; buyer 1 likes seller 1's good and buyer 2 likes seller 2's good. Suppose* $r_1 = r_2 = 0$*. Let* $\delta = 1$*. For any* $c \leq 1/2$*, a unit measure of type 1 buyers and a unit measure of type 1 sellers entering, no type 2 buyers or sellers entering and all meetings resulting in a trade at a price of* $1/2$ *is an equilibrium. Clearly such a sequence does not converge to the competitive equilibrium of the economy. However, if a tiny fraction* ε_2 *of type 2 sellers where to enter in each period, then for* $c \leq \frac{\varepsilon_2}{1+\varepsilon_2}$ *the buyers of type 2 would also find it profitable to enter. This results in the markets for both goods operating and leads to convergence to a competitive equilibrium.*

The above example showed that without an assumption along the lines of (FDL), which can be imposed either on the buyer side, or on the seller side of the market, convergence to a competitive equilibrium may fail due to coordination problems. However, even if convergence to a competitive equilibrium fails, the limit will nevertheless have a competitive flavor. In particular, the following corollary to Theorem 1 shows that the limiting allocation and values will comprise a competitive equilibrium for the sub-markets that are open. More precisely, pick any convergent sequence (q^n, v^n). Let s denote a seller for whom $\lim_k l_s^{n_k} > 0$ for some subsequence $\{n_k\}$ of $\{n\}$ where the subsequence may depend on the seller s, and let $\hat{S}$ denote the set of all such sellers, i.e., $\hat{S}$ is the set of sellers who are present in the economy at some limit. Then, $(\hat{q}, \hat{v})$ comprises a search equilibrium for the economy where the set of buyers is B and the set of sellers is $\hat{S} \subset S$. The proof argues that the *Excess* between any buyer b and seller and s in $B \cup \hat{S}$ disappears along a subsequence where $\lim_k l_s^{n_k} > 0$. This in turn implies that the *Excess* vanishes along the sequence $\{n\}$ since $\{e^n\}$ has a limit by assumption.

Corollary 2. *Assume (URC). Suppose* $(q^n, v^n) \to (\hat{q}, \hat{v})$*, let* $\hat{S} = \{s \in S : \limsup_n l_s^n > 0\}$*, then* $\hat{q}$ *solves the primal Assignment Problem and is a competitive equilibrium allocation for the economy with agents in* $\hat{I} = B \cup \hat{S}$*;* $\hat{v}$ *solves the dual Assignment Problem and is a competitive equilibrium utility vector for the economy with agents in* $\hat{I} = B \cup \hat{S}$*; and* $\hat{v}_s + r_s$ *is a competitive equilibrium price for good* x_s*.*

Proof. If $s \notin \hat{S}$ then $lim_n l_s^n = 0$ which implies that $q_{bG} = 0$ for any $G \ni s$. Consequently, q is feasible for the economy $\hat{I}$. By assumption, for each $s \in \hat{S}$, there exists a sequence $\{n_k\} \subset \{n\}$ and an integer K such that $l_s^{n_k} > 0$ for all $n_k > K$ and $\lim_k l_s^{n_k} > 0$. Along this sequence the excess between any buyer and the particular seller disappears by Lemma 4. However, since v_n converges along $\{n\}$, $\lim v^{n_k} = \hat{v}$, and $\lim_n e^n = \lim_k e^{n_k} = 0$, proving *No Excess* for any b and $s \in \hat{S}$, and completing the proof. □

3.2. **Existence of a Steady State Search Equilibrium.** The main theorem, proved in this subsection, establishes that an equilibrium exists, for any $\delta \in [0, 1]$ and $c > 0$. In the model presented here, without an assumption along the lines of (FD) or (FDL), a trivial no-trade equilibrium always exists. Consequently, for a meaningful existence result, the following theorem below assumes (FD) (or (FDL)) and establishes the existence of an equilibrium with trade, that is an equilibrium where the markets for all the goods are open. The proof of the theorem involves a straight forward application of Kakutani's fixed point theorem on a mapping defined from the set of feasible measures l, strategy profiles σ and values v, into itself.

Theorem 2. *Assume (FD) or (FDL). For any (c, δ) a search equilibrium (l, σ) exists.*

Example 1 demonstrated that without (FDL) there may exist sequences of equilibria that fail to converge to competitive equilibria. However, the following corollary maintains two additional assumptions and shows that, even without (FDL), there exists a sequence of equilibria that converges to a competitive equilibrium for the economy.

The first additional assumption (UNQ), requires that the set of goods traded in any competitive is unique, that is, the same goods are traded in any competitive equilibrium. This assumption is trivially satisfied in the models with an homogeneous good such as Gale (1987) and Satterthwaite and Shneyerov (2007). Also, the assumption is satisfied generically in economies where buyers have unit demand and sellers have unit supply.

Assumption: Uniqueness (UNQ)**.** *The set of goods traded in any competitive equilibrium is the same. That is if good x is not traded in one competitive equilibrium, then it is not traded in any other CE.*

Assumption (DR) requires the goods in the economy are substitutes for each other from the point of view of all buyers. This assumption is always trivially satisfied in economies where buyers have unit demand.

Assumption: Decreasing Returns (DR)**.** *If* $G \subset H$*, then* $h_{bH\cup\{s\}} - h_{bH} \leq h_{bG\cup\{s\}} - h_{bG}$ *for all* b *and* s*.*

The argument for the corollary is as follows: First fix the set of goods traded in any competitive equilibrium. Assume at a small measure of the lowest cost seller of each of these goods enters in each period, i.e., (FDL) holds for the traded goods. Given this assumption a sequence of equilibria, that converges to a competitive equilibrium exists by Corollary 1 and Theorem 2. However, if the measure of sellers with (FDL) is picked sufficiently small, then for sufficiently small c^n and $1-\delta^n$, the measure of sellers of the traded goods entering the economy must exceed the measure of sellers entering due to the (FDL) assumption. Consequently, the (FDL) assumption is non-binding and can be dropped thus proving the existence of the desired sequence of convergent equilibria. The convergent sequence, however, converges to an competitive equilibrium for the economy where the set of traded goods is a subset of the set of all goods. Assumption (DR) is then used to show that this also a Competitive Equilibrium for the set of all goods.

Corollary 3. *Assume (URC), (UNQ) and (DR). There exists a sequence* $(q^n, v^n) \to (\hat{q}, \hat{v})$*, such that* $\hat{q}$ *solves the primal Assignment Problem and is a competitive equilibrium allocation;* $\hat{v}$ *solves the dual Assignment Problem and is a competitive equilibrium utility vector; and* $\hat{v}_s + r_s$ *is a competitive equilibrium price for good* x_s*.*

Proof. By (UNQ), the set of goods can be partitioned into two sets $H \subset X$ and $X \setminus H$ where H denotes the set of goods that are traded in any competitive equilibrium. Let q_x denote the measure of good x traded by the lowest cost sellers of good x, i.e., by sellers $S_{\underline{\mathrm{X}}} = \{s : x_s = x \text{ and } r_s \leq r'_s \text{ for all } s' \text{ with } x_{s'} = x\}$, traded in a competitive equilibrium. More precisely

$$q_x = \sum\nolimits_{s \in S_{\underline{\mathrm{X}}}} \sum\nolimits_b \sum\nolimits_{s \ni G} q_{bG}$$

Also, let $\underline{\mathrm{q}}_x = min_{q \in Q} q_x$ where Q denotes the set of competitive allocations. Note that Q is a compact and convex set and $\underline{\mathrm{q}}_x > 0$ for any $x \in H$. Assume (FDL) for all $x \in H$ and let the measure of low cost sellers of good $x \in H$ receiving the first draw free be $0 < \varepsilon_x < \underline{\mathrm{q}}_x$. Observe that given this set-up, the sequence of equilibria will converge to $\hat{q}$, which is competitive equilibrium for the economy comprised of sellers such that $x_s \in H$ and $b \in B$. Also, observe that since only goods in H are traded, q is also an efficient allocation for the original economy I. For any buyer b with $\hat{l}_b > 0$, $\hat{e}_{bG} \leq 0$ for any $G \subset S$. For any buyer with $\hat{l}_b = 0$, $\hat{e}_{bG} \leq 0$ for any $G \subset \{s : x_s \in X \setminus H\}$. This is because otherwise, i.e., is $\hat{e}_{bG} > 0$, then allocating to b, who is not trading, the goods in G, which are not being traded, would improve the efficiency of the matching which would contradict that the matching $\hat{q}$ is efficient. So $\hat{e}_{bG} \leq 0$ for $G \subset \{s : x_s \in X \setminus H\}$. Also, for $\hat{l}_b = 0$, $\hat{e}_{bG} \leq 0$ for any $G \subset \{s : x_s \in H\}$ by Corollary 1. But, $\hat{e}_{bG} \leq 0$ for $G \subset \{s : x_s \in H\}$ and $G \subset \{s : x_s \in X \setminus H\}$ in conjunction with (DR) implies that $\hat{e}_{bG} \leq 0$ for all b and $G \subset S$. This, in turn, shows that the allocation $\hat{q}$ is a competitive equilibrium allocation for I and $\hat{v}$ is a competitive utility vector.

Now observe that for sufficiently large n, $\sigma^n_{s_x}(in) > \varepsilon_x$ since the measure of lowest cost sellers leaving the market must converge to competitive competitive equilibrium which exceeds $\underline{q}_x$. This implies that for n sufficiently large $v_s(in) \geq 0$. This shows that we can drop the (FDL) assumption which is not binding for sufficiently large n and just take entry by type s_x to equal $\sigma^n_{s_x}(in)$. □

4. Discussion and Conclusion

This paper presented a model where buyers purchase a bundle of indivisible, heterogeneous goods from sellers who are each endowed with one unit of a good. Trade takes place in a decentralized market under two sided incomplete information. A small measure of the lowest cost seller of each good is assumed to sample the market at least once. Under this assumption an equilibrium is shown to exist (Theorem 2) and any sequence of equilibria is shown to converge to a competitive equilibrium.

The model presented here considered the case where agents bargain pairwise, where as other studies in the literature, such as Satterthwaite and Shneyerov (2007), analyze bargaining in larger coalitions. The convergence result is not sensitive to this assumption. In particular, the results presented here are robust to any random matching technology as long as any buyer and seller whose exist with positive measure in the economy meet with positive probability. Also, the analysis proceeded under the assumption of two sided incomplete information. However, all the results presented also go through without alteration under complete information. Finally, a central assumption in the model maintained throughout the paper was that the economy remains in steady state. An immediate way to extend this model is to drop the steady state assumption and consider a non-stationary market with finitely many, instead of a continuum, of agents entering in each period. Under such a formulation, the goal would be to show that trade always occurs at competitive prices and that the market clears on average.

Appendix A. Omitted Proofs

A.1. **Proof of Lemma 1.** Proof of item (i) is bellow. Item (ii) follows from Riley and Zeckhauser (1983). Item (iii) follows since Corollary 1 showed that $\delta v_s + r_s \leq \delta v_{s'} + r_{s'}$ for two sellers of the same good with $r_s \leq r_{s'}$. The take-it-or-leave-it offer can be viewed as the choice of an optimal monopoly price where marginal cost is equal to $\delta v_s + r_s$. Consequently, if $\delta v_s + r_s \leq \delta v_{s'} + r_{s'}$, then the optimal monopoly price $t_s \leq t_{s'}$.

I show, if $\mu_i = (m_i, t_i)$ is the optimal mechanism for type i, when type i is proposing and her type is known to the responder, then the mechanism $\mu = (\mu_i)$ is an optimal mechanism choice for all i, when the proposer's type is private information. The proof follows the line of reasoning in Yilankaya (1999). By the Inscrutability Principal of Myerson (1983), we can assume, without loss of generality, that all proposers choose the same mechanism. This mechanism choice by the proposer cannot do better in expectation than the ex-ante optimal mechanism. Below it is shown that the ex-ante optimal mechanism is just $\mu = (\mu_i)$. However, the mechanism choice $\mu = (\mu_i)$ is also available for the proposer and will therefore be chosen. Let $\hat{r}_s = r_s + \delta v_s \geq 0$ and $\hat{h}_{b_G} = \delta(v_{b_{G \cup s}} - v_{b_G})$. The ex-ante problem for the proposer, if the proposer is a seller of good x, is as follows:

$$\max_{m,t \geq 0} \sum\nolimits_{B_{\mathcal{P}(G)} \times S} p_b p_s (t_{bs} - m_{bs} \hat{r}_s)$$

Subject to the incentive compatibility constraint $\sum_S p_s(m_{js}\hat{h}_b - t_{js}) \leq \sum_S p_s(m_{bs}\hat{h}_b - t_{bs})$, the individual rationality constraint $\sum_S p_s(m_{bs}\hat{h}_b - t_{bs}) \geq 0$ and the resource constraint $m_{bs} \leq 1$. Alter the problem as follows: let $h_b = \max\{\hat{h}_b, 0\}$ for all $b \in B_{\mathcal{P}(G)}$ and add a buyer 0 with $h_0 < 0$ and $p_0 = 0$. Also, drop the individual rationality constraint for all buyers except for buyer 0. The modified ex-ante problem is then

$$V^x = \max_{m,t\geq 0} \sum\nolimits_{B_{\mathcal{P}(G)}\times S} p_b p_s (t_{bs} - m_{b,s}\hat{r}_s)$$

$$\sum\nolimits_S p_s(m_{is}h_b - t_{is}) \leq \sum\nolimits_S p_s(m_{b,s}h_b - t_{b_G,s}) \ \forall b \text{ and } i \qquad (\alpha^x_{bj})$$

$$0 \leq \sum\nolimits_S p_s(m_{0s}h_b - t_{0s}) \qquad (\psi^x)$$

$$m_{bs} \leq 1 \qquad (\gamma^x_{bs})$$

where the set B now includes buyer 0 and the Lagrange multipliers are given to the right. The modified problem and the original problem have the same payoff for the proposer. To see this pick any solution to the original problem and take $m_{0s} = t_{0s} = 0$. To show that this is feasible for the modified problem we only need to check constraints for agents with $\hat{h}_b \leq 0$ and for h_0. However, for any such agent, their utility in both the original and modified problem is zero which is their maximum attainable utility. Consequently, the solution to the original problem is feasible for the modified problem and the payoff for the modified problem is at least as large as the original problem. Pick any solution to the modified problem and set $m_{bs} = t_{bs} = 0$ for any $h_b \leq 0$ which must also a solution to the modified problem. To show that this solution is also feasible for the original problem note that for any $h_b \leq 0$, $m_{bs} = 0$ and $t_{bs} = 0$ and so these agents receive their highest possible utility and both IR and IC constraints for these agents hold. Also, dropping the individual rationality constraints causes no change in value since any agent can still guarantee non-negative payoff by pretending to be agent 0. Consequently, the value for the original problem must be at least as large as the value for the modified problem showing that the two payoffs are equal. The dual of the ex-ante problem is

$$D^x = \min_{\gamma^x\geq 0, \alpha\geq 0, \psi\geq 0} \sum\nolimits_{B\times S} \gamma^x_{bs}$$

$$\sum\nolimits_{j\in B}(h_b\alpha^x_{bj} - \hat{h}_j\alpha^x_{jb}) \leq \frac{\gamma^x_{bs}}{p_s} + p_b\hat{r}_s \ \forall\ b \neq 0 \text{ and } s,$$

$$\sum\nolimits_{j\in B}(\alpha^x_{jb} - \alpha^x_{bj}) + p_b \leq 0 \ \forall\ b \neq 0,$$

$$\sum\nolimits_{j\in B}(\alpha^x_{j0} - \alpha^x_{0j}) \leq \psi^x,$$

$$\sum\nolimits_{j\in B}(h_0\alpha^x_{0j} - h_j\alpha^x_{j0}) + \hat{h}_0\psi^x \leq \frac{\gamma^x_{0s}}{p_s} \ \forall\ s.$$

Observe that the last constraint for the dual problem is satisfied automatically and can be ignored since, for any choice of $\gamma^x \geq 0, \alpha \geq 0, \psi \geq 0$, the left hand side is always non-positive and the right hand side is always non-negative.

Similarly, the problem when $\hat{r}_s$ is know and it's program dual are formulated as follows:

$$V_s = \max_{m,t\geq 0} \sum_B p_b(t_{bs} - m_{bs}\hat{r}_s)$$

subject to, incentive compatibility, $h_b m_{js} - t_{js} \leq h_b m_{bs} - t_{bs}$ for all b and j, individual rationality, $0 \leq h_0 m_{0s} - t_{0s}$ and $m_{bs} \leq 1$. The dual is as follows

$$D_s = \min_{\gamma\geq 0} \sum_b \gamma_{bs}$$

subject to:

$$\sum_{j\in B}(h_b\alpha^s_{bj} - \hat{h}_j\alpha^s_{jb}) \leq p_b\hat{r}_s + \gamma_{bs} \ \forall\ b \neq 0,$$
$$\sum_{j\in B}(\alpha^s_{jb} - \alpha^s_{bj}) + p_b \leq 0 \ \forall\ b \neq 0,$$
$$\sum_{j\in B}(\alpha^s_{j0} - \alpha^s_{0j}) \leq \psi^s,$$
$$\sum_{j\in B}(h_0\alpha^s_{0j} - h_j\alpha^s_{j0}) + h_0\psi^s \leq \gamma_{0s}.$$

Again, the last constraint for the dual problem can be ignored.

Let m_s and t_s solve the mechanism choice problem when the cost is r_s is known, and let α_s γ_s and ψ_s denote a dual solution. Observe that $m = (m_s)$ and $t = (t_s)$ is a feasible choice for the ex-ante problem since if each choice satisfies IC and IR_0 separately, they satisfy IC and IR_0 on average. This implies that $V^x \geq \sum_s p_s V_s$. Also, observe that $\alpha = (\alpha^s)$, $\gamma = (p_s\gamma^s)$ and $\psi = \max_s\{\psi^s\}$ is feasible for the dual of the ex-ante problem. Consequently, $D^x \leq \sum_{B\times S} p_s\gamma_{sb}$. However, $D^x \leq \sum_{B\times S} p_s\gamma_{sb} = \sum_s p_s V_s \leq V^x$ and so $V^x = \sum_s p_s V_s$ completing the proof.

A.2. **Proof of Theorem 1.**

Lemma 2 (No Excess 1)**.** *If* $\max\{\hat{p}_b, \hat{p}_s\} > 0$, *then* $e_{bs} \leq 0$.

Proof. For any c^n and δ^n a seller (or buyer) can offer to sell her good for $\delta^n v_{b\cup s} - \delta^n v_b - \varepsilon$ and ensure that buyer b purchases if they meet, since the payoff that buyer b gets from purchasing the good strictly exceeds her continuation payoff $\delta^n v_b$. Also, any buyer can offer to buy a good for $r_s + \delta^n v^n_s + \varepsilon$, and ensure that she makes a purchase if she meets seller s. Consequently,

$$\begin{aligned} v_s^n &\geq -c^n + (1-\beta)p_b^n(\delta^n v_{b\cup s} - \delta^n v_b - r_s) + \delta^n(1-(1-\beta)p_b^n)v_s^n \\ (1-\delta^n)v_s^n &\geq -c^n + (1-\beta)p_b^n(\delta^n v_{b\cup s} - \delta^n v_b - \delta^n v_s^n - r_s) \\ (1-\delta^n)v_s^n &\geq -c^n + (1-\beta)p_b^n e_{bs}^n \end{aligned}$$

and

$$\begin{aligned} (1-\delta^n)v_b^n &\geq -c^n + \beta p_s^n(\delta^n v_{b\cup s} - \delta^n v_b - \delta^n v_s^n - r_s), \\ (1-\delta^n)v_b^n &\geq -c^n + \beta p_s^n e_{bs}^n \end{aligned}$$

Taking limits shows that $\hat{p}_b \hat{e}_{bs} \leq 0$ and $\hat{p}_s \hat{e}_{bs} \leq 0$. However, since $\max\{\hat{p}_b, \hat{p}_s\} > 0$, $\hat{e}_{bs} \leq 0$. □

Lemma 3. *Let* $L^n = \max\{L_B^n, L_S^n\}$, $\lim_n c^n L^n = 0$ *and* $\lim_n (1-\delta^n)L^n = 0$.

Proof. If $\limsup L^n < \infty$, then since $0 \leq L^n$, $\lim_n c^n L^n = 0$ and $\lim_n (1-\delta^n)L^n = 0$. If $\limsup L^n = \infty$, then there must exist an agent type i with $\hat{p}_i > 0$ for whom $l_i^n \to \infty$ and so the value from staying in the market for this type agent must be non-negative for all n large. Consequently, if $\limsup L^n = \infty$, then there exists a buyer b with $\hat{p}_b > 0$ and $v_b^n(in) \geq 0$, or a seller s with $\hat{p}_s > 0$ and $v_s^n(in) \geq 0$, for all n large. Assume, without loss of generality, that there exists a buyer b with $\hat{p}_b > 0$ and $v_b^n(in) \geq 0$. This implies for sufficiently large n,

$$\begin{aligned} (1-\delta^n)l_b^n v_b^n + l_b^n c^n = \beta \sum_S l_b^n p_s^n m_{bs}^n (\delta^n v_{b\cup s} - t_{bs}^n - \delta^n v_b^n) \\ + (1-\beta) \sum_S l_b^n p_s^n m_{bs}^n (\delta^n v_{b\cup s} - t_{sb}^n - \delta^n v_b^n) \end{aligned}$$

However, for $m_{bs}^n > 0$, $t_{bs}^n \geq \delta^n v_s^n + r_s$, so $\delta^n v_{b\cup s} - t_{bs}^n - \delta^n v_b^n \leq \delta^n v_{b\cup s} - \delta^n v_b - \delta^n v_s^n - r_s$ and $\delta^n v_{b\cup s} - t_{sb}^n - \delta^n v_b^n \leq \delta^n v_{b\cup s} - \delta^n v_b - \delta^n v_s^n - r_s$. Consequently,

$$\begin{aligned} (1-\delta^n)l_b^n v_b^n + l_b^n c^n &\leq \beta \sum_S l_b^n p_s^n m_{bs}^n e_{bs}^n + (1-\beta) \sum_S l_b^n p_s^n m_{bs}^n e_{bs}^n \\ (1-\delta^n)l_b^n v_b^n + l_b^n c^n &\leq \bar{e}_{bs}^n \sum_S l_b^n M_{bs}^n \leq \bar{e}_{bs}^n \\ \frac{1}{p_b^n}((1-\delta^n)l_b^n v_b^n + l_b^n c^n) &\leq \frac{\bar{e}_{bs}^n}{p_b^n} \end{aligned}$$

However, since $\hat{p}_b > 0$, by Lemma 2, $\lim_n e^n_{bj} = 0$ for any j. This implies that

$$\lim_n \frac{1}{p^n_b}((1-\delta^n)l^n_b v^n_b + l^n_b c^n) \leq \lim_n \frac{\overline{e}^n_{bs}}{p^n_b}$$
$$\lim_n (1-\delta^n)L^n v^n_b + L^n c^n = 0$$

Observe since $\lim L^n c^n = 0$, by Assumption 1, $\lim_n (1-\delta^n)L^n = 0$. □

Lemma 4. *If $\hat{p}_b = 0$, then also $\hat{e}_{bG} \leq 0$, consequently, $\hat{e}_{bG} \leq 0$ for all b and G.*

Proof. By the argument provided in Lemma 2,

$$(1-\delta^n)v^n_b \geq -c^n + \beta p^n_{s1}(\delta^n v_{b_{\{s1\}}} - \delta^n v^n_b - \delta^n v^n_{s1} - r^n_{s1}).$$

Multiply both sides by $L^n = \max\{L^n_B, L^n_S\}$ which gives

$$((1-\delta^n)v^n_b + c^n)L^n \geq \beta L^n p^n_{s1}(\delta^n v^n_{b_{\{s1\}}} - \delta^n v^n_b - \delta^n v^n_{s1} - r^n_{s1}).$$

Note that $L^n p^n_s = l^n_s \geq 1$ for all n. However, by Lemma 3

$$\lim_n((1-\delta^n)v^n_b + c^n)L^n = 0.$$

Consequently,

$$\hat{v}_{b_{\{s1\}}} - \hat{v}_b - \hat{v}_{s1} - r_{s1} \leq 0.$$

Also, again by the argument provided in Lemma 2,

$$((1-\delta^n)v^n_{b_{\{s2\}}} + c^n)L^n \geq \beta L^n p^n_{s2}(\delta^n v_{b_{\{s_1,s2\}}} - \delta^n v^n_{b\{s_1\}} - \delta^n v^n_{s2} - r^n_{s2}).$$

So, $\hat{v}_{b_{\{s_1,s2\}}} - \hat{v}_{b_{\{s_1\}}} - \hat{v}_{s2} - r_{s2} \leq 0$. Substituting gives

$$\hat{v}_{b_{\{s_1,s2\}}} - \hat{v}_b - \hat{v}_{s2} - \hat{v}_{s1} - r_{s1} - r_{s2} \leq 0$$

Repeating $|G|$ times shows that

$$\hat{v}_{b_G} - \hat{v}_b - \sum\nolimits_{s\in G}(\hat{v}_s + r_s) \leq 0.$$

However, $v^n_{b_G} \geq h_{bG}$ for all n and so $\hat{v}_{b_G} \geq h_{bG}$. Thus

$$h_{bG} - \hat{v}_b - \sum\nolimits_{s\in G}(\hat{v}_s + r_s) \leq 0$$

proving the result. □

Lemma 5. $\sum_B \hat{v}_b + \sum_S \hat{v}_s \leq \sum_b \sum_G \hat{q}_{bG}(h_{bG} - \sum_{s\in G} r_s)$.

Proof. The value equations for the buyers implies

$$l_b v_b(1-\delta) \leq \beta \sum_s l_b p_s m_{bs}(\sigma_{b\cup s}(in)\delta(v_{b\cup s} - h_{i_b G(b\cup s)}) + \delta h_{i_b G(b\cup s)} - \delta v_b - t_{bs})$$
$$+ (1-\beta)\sum_s l_b p_s m_{sb}(\sigma_{b\cup s}(in)\delta(v_{b\cup s} - h_{i_b G(b\cup s)}) + \delta h_{i_b G(b\cup s)} - \delta v_b - t_{sb})$$

Summing up over all buyers and taking the limit as $\delta \to 1$ and observing that t_{bs} goes to $v_s + r_s$ for any b and s with $m_{bs} > 0$ gives

$$0 \leq \sum_{b_G \in T} \sum_{s\in S} (1-\hat{\sigma}_{b_{G\cup s}}(in))(\beta \hat{l}_{b_G}\hat{p}_s \hat{m}_{b_G s} + (1-\beta)\hat{l}_{b_G}\hat{p}_s\hat{m}_{sb_G})(h_{bG\cup s} - \hat{v}_b - \sum_{j\in G\cup s}(\hat{v}_j + r_j))$$

rearranging shows that

$$0 \leq \sum_{b_G \in T} \sum_{s\in S} (1-\hat{\sigma}_{b_G}(in))(\beta \hat{l}_{b_{G\setminus s}}\hat{p}_s \hat{m}_{b_{G\setminus s} s} + (1-\beta)\hat{l}_{b_{G\setminus s}}\hat{p}_s\hat{m}_{sb_{G\setminus s}})(h_{bG} - \hat{v}_b - \sum_{j\in G}(\hat{v}_j + r_j))$$

However

$$\sum_{s\in S}(1-\hat{\sigma}_{b_G}(in))(\beta \hat{l}_{b_{G\setminus s}}\hat{p}_s \hat{m}_{b_{G\setminus s} s} + (1-\beta)\hat{l}_{b_{G\setminus s}}\hat{p}_s\hat{m}_{sb_{G\setminus s}}) = \hat{q}_{bG}$$

which implies that

$$0 \leq \sum_{b_G \in T} \hat{q}_{bG}(h_{bG} - \hat{v}_b - \sum_{j\in G}(\hat{v}_j + r_j))$$
$$0 \leq \sum_b \sum_G \hat{q}_{bG}(h_{bG} - \hat{v}_b - \sum_{j\in G}(\hat{v}_j + r_j))$$

Observe that for b with $\hat{v}_b > 0$ $\sum_b \sum_G \hat{q}_{bG} = 1$ and for s with $\hat{v}_s > 0$, $\sum_b \sum_{s \ni G} \hat{q}_{bG} = 1$ so

$$\sum_B \hat{v}_b + \sum_S \hat{v}_s \leq \sum_b \sum_G \hat{q}_{bG}(h_{bG} - \sum_{s\in G} r_s)$$

proving the result.

□

A.3. **Proof of Theorem 2.** For any c and δ, $0 \leq v_i \leq \bar{h}$. Let $V = \{v \in R^{|B|\times 2^{|S|}+|S|} : 0 \leq v_i \leq \bar{h}\}$ denote the set of possible values. The steady state measure for any agent i is bounded. For all b and s, $v_s(in) < 0$. then $l_s = 1$ and $v_b(in) < 0$, then $l_b = 1$, by Assumption (FD). If $v_s(in) \geq 0$, then $l_s = 1/(\sum_{b\in T} M_{sb} + \sigma_s(out))$. If $v_b(in) \geq 0$, then $l_b = 1/(\sum_{s\in S} M_{bs} + \sigma_b(out))$. Also, if $\sigma_b(in)\sum_{s\in G(b)} l_{b\setminus s}M_{b\setminus ss}$, then

$$l_b = \frac{\sigma_b(in)\sum_{s\in G(b)} l_{b\setminus s}M_{b\setminus ss}}{\sum_{s\in S} M_{bs} + \sigma_b(out)}.$$

Observe that if $v_i(in) \geq 0$, then $v_i(in) \leq -c + \sum_j M_{ij}\bar{h}$ and so $c/\bar{h} \leq \sum_j M_{ij}$. Consequently, $1 \leq l_i \leq \frac{\bar{h}}{c}$. Let $\Lambda = \{l : 1 \leq l_i \leq \frac{\bar{h}}{c}\}$ denote the set of possible steady state measures. Let m_{bs} and transfer t_{bs} be the mechanism choice by the buyers and m_{sb} and t_{sb} the mechanism choice by the sellers, and $\sigma_i = (\sigma_{i0}, \mu_i)$. Start with any $l \in \Lambda$, $\sigma \in \Sigma$, $v \in V$ and let

$$\begin{aligned} l'_s(l,\sigma,v) =& \frac{1}{\max\{c/\bar{h}, \sum_{j\in T} M_{js}p_j\}}, \\ l'_b(l,\sigma,v) =& \frac{1}{\max\{c/\bar{h}, \sum_{s\in S} M_{bs}\}} \text{ and,} \\ l'_j(l,\sigma,v) =& \frac{\sigma_j(in)\sum_{s\in G(j)} l_{j\setminus s}M_{j\setminus ss}}{\max\{c/\bar{h}, \sum_{s\in S} M_{js}\}} \end{aligned}$$

where the M's are calculated according to σ, p and l. This defines a continuous function from $\Lambda \times \Sigma \times V$ into Λ, where $(l,\sigma,v) \mapsto l'_i$ for each i.

Let

$$\begin{aligned} v'_b(in|l,\sigma,v) = \max\nolimits_{(m'_b,t'_b)\geq 0} -c + \beta \sum p_s(m'_{bs}(\delta v_{b\cup s} - \delta v_b) - t'_{bs}) \\ + (1-\beta)\sum_s p_s(m_{sb}(\delta v_{b\cup s} - \delta v_b) - t_{sb}) + \delta v_b \end{aligned}$$

subject to

$$\begin{aligned} t'_{bs} - m'_{bs}(r_s + \delta v_s) &\geq t'_{bj} - m'_{bj}(r_s - \delta v_s) \text{ for all } s \text{ and } j \in S \\ t'_{bs} - m'_{bs}(r_s + \delta v_s) &\geq 0 \text{ for all } s \\ m'_{bs} &\leq 1 \text{ for all } s. \end{aligned}$$

Also, let $S'_{b,1}(l,\sigma,v)$ denote the set of maximizers for the above program and

$$S'_{b,0}(l,\sigma,v) = \arg\max_{\sigma_0\in\Delta\{in,out\}} \sigma_0 v'_b(in,l,\sigma,v) + (1-\sigma_0)h_{bG(b)}.$$

Similarly, for a seller, let

$$\begin{aligned} v'_s(in|l,\sigma,v) = \max\nolimits_{m'_s,t'_s} -c + (1-\beta)\sum\nolimits_{b\in T} p_b(t'_{sb} - m'_{sb}(r_s + \delta v_s)) \\ + \beta \sum\nolimits_{b\in T} p_b(t_{bs} + m_{bs}(r_s + \delta v_s)) + \delta v_s \end{aligned}$$

subject to

$$\begin{aligned} m'_{sb}(\delta v_{b\cup s} - \delta v_b) - t'_{sb} &\geq m'_{sj}(\delta v_{b\cup s} - \delta v_b) - t'_{sj} \text{ for all } b \text{ and } j \in T \\ m'_{sb}(\delta v_{b\cup s} - \delta v_b) - t'_{sb} &\geq 0 \text{ for all } b \\ m'_{sb} &\leq 1 \text{ for all } b \end{aligned}$$

Also, let $S'_{s,1}(l,\sigma,v)$ denote the set of maximizers for the above program and

$$S'_{s,0}(l,\sigma,v) = \arg \max_{\sigma_0 \in \Delta\{in,out\}} \sigma_0 v'_s(in,l,\sigma,v).$$

Finally let $S'_i(l,\sigma,v) = S'_{i,0}(l,\sigma,v) \times S'_{s,1}(l,\sigma,v)$ and $S'(l,\sigma,v) = \prod_i S'_i(l,\sigma,v)$. This process defines a continuous function from $\Lambda \times \Sigma \times V$ into V where $(l,\sigma,v) \mapsto v'_i$ for each i and defines an upper-hemi-continuous, convex compact valued correspondence from $\Lambda \times \Sigma \times V$ into V where $(l,\sigma,v) \mapsto S'$, by Berge's Theorem of the Maximum.

However, we have defined an UHC correspondence $(l,\sigma,v) \mapsto (l',S',v')$. This correspondence maps $\Lambda\times\Sigma\times V$ into $\Lambda\times\Sigma\times V$, it is upper-hemi-continuous, compact, and convex valued; thus by Kakutani's theorem has a fixed point. This fixed point is an equilibrium for the economy.

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