Favor-trading with Concave Utility Functions

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Abstract

We study two-player games of favor-trading in a complete information environment standard to the literature, but in contrast to prominent models of favor-trading to date, we assume agents have concave utility functions of the form $u(x) = x^{\alpha}$, $0 < \alpha < 1$, instead of linear utility functions. We characterize equilibria in the concave case and describe qualitative differences to the linear case. We extend several equilibria concepts from previous favor-trading literature, and construct parametric models to numerically analyze and characterize these equilibria in our model.

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1 Introduction

To date the prominent models of favor-trading assume agents have linear preferences and favors are intrinsically of greater benefit than cost. In this paper informal favor-trading is considered to be a form of insurance. We assume agents have concave utility functions and favors derive their value from risk sharing. With sufficiently concave utility functions, agents can beneficially trade favors at some level for any discount factors. This is in contrast to the linear case in which the incentive compatibility of the favors traded is independent of the size of the favors because agents are essentially risk-neutral with respect to favors. Furthermore, if utility functions are linear and agents' discount factors are just large enough to satisfy the incentive compatibility constraint for equality matching, the best the agents can do is to equality match full favors. If the same agents have concave utility functions, we show that the equivalent equality matching equilibria are dominated by equilibria involving a smaller than full first favor, followed by a small second favor if reciprocation has not been received by the time the agent receives the second consecutive favor opportunity. Consequently, the assumption of linear preferences drives some of the results in prominent favor-trading models.

The rest of the paper is organized as follows. We first introduce the relevant literature at the end of this section. Section 2 introduces the model. Section 3 describes equality matching with concave utility functions and generalizes equality matching to multiple states. In section 3.1 we construct two parametric models to numerically analyze multi-state equilibria. The first model simulates a large sample of games, derives payoff functions from the simulations, and uses them to solve for the optimal strategy. The second model solves directly the system of simultaneous payoff equations associated with an equality matching game and uses the results to find the optimal strategy. Section 3.2 presents the results of the numerical tests and a number of conjectures based on them. We describe certain unexpected outcomes in multi-state equality matching including favors above the efficient stage game levels (beyond full-sharing) and optimal multi-state equality matching favor sequences that are not decreasing. We also argue against the efficiency of infinite state equality matching strategies. Section 4 repeats the analysis for strategies we call pseudo-highest symmetric self-generating line equilibria that emulate analogously named strategies from the linear favor-trading literature. Section 5 discusses preliminary work in strategies involving favor-depreciation and other remaining issues. Section 6 concludes. An appendix and reference sections follow.

1.1 Relevant literature

The major contributions to the favor-trading literature by Möbius [12], Hauser and Hopenhayn [8], and Abdulkadiroğlu and Bagwell [1] (AB for short), are all linear models in terms of agents' preferences with intrinsically efficient favors. The model we introduce next retains the fundamental information structure of these models; that is, all information is complete but not always public, but we endogenize the value of favors by assigning agents concave utility functions so that favors may be used to share risk. Furthermore, we assume the absolute cost of doing a favor is equal to the absolute benefit generated. In other words, favors have no intrinsic value in our model. We take a simplified version of the favor-trading model by AB [1] as our benchmark for comparisons.

Outside of the favor-trading literature, our model overlaps with the insurance literature. In particular, Kocherlakota [9] investigates the "Implications of Efficient Risk Sharing without Commitment" as we do, but he uses a macroeconomic model whereas we restrict attention to a two agent game. To our knowledge, this setup has not been covered by any major works in economics or closely related fields.

2 The Model

Since concave functions include linear functions and strictly concave functions can be arbitrarily close to linear functions, we restrict attention to a subclass of concave functions we call α-concave. These functions are "sufficiently concave" for meaningful analysis of the differences concavity can make in favor-trading models. However, we do not claim that concavity alone is sufficient for meaning differences. On the contrary, we believe that the results of linear models could generally be replicated with arbitrary proximity with strictly concave utility functions that are arbitrarily close to their linear counter-parts. However, our goal is to investigate the difference sufficient concavity *can* make relative to linear models. To that end we define α-concave functions below. The domain and range have each been normalized to the unit interval for simplicity. Unless otherwise stated, any future references to concave utility functions imply α-concave utility functions.

Definition 1 *Suppose function $u : [0,1] \to [0,1]$ is such that $u(x) = x^{\alpha}$ for $\alpha \in (0,1)$. Then we call u an α-concave function.*

Consider two identical agents, a and b. Each agent has utility function $u(x) = x^{\alpha}$ for some $\alpha \in (0,1)$. The agents play an infinitely repeated stage game with the following structure: At the beginning of each period nature allocates an opportunity, normalized to size 1, either to agent a or b (but not both), each with probability $p \in (0, 1/2)$, or to neither with probability $1 - 2p$. Opportunities are private information. An agent who receives an opportunity may either use it privately and receive a flow payoff of $u(1) = 1$ or share some or all of it. The amount shared is denoted by x and y for agents a and b, respectively. If agent a receives an opportunity and shares amount x of it, the flow payoffs to (a, b) are $((1-x)^{\alpha}, x^{\alpha})$. Similarly, if agent b does the sharing, payoffs are $(y^{\alpha}, (1-y)^{\alpha})$. Because side payments are not allowed, and reciprocation cannot be explicitly conditioned on future opportunities since they are not publicly observable, shared opportunities are called favors. Favors, including their size, are public information. The stage game is repeated in each subsequent period.

To see how favor-trading works consider the following game called *equality matching (EM)*. In EM of level $z \in (0, 1/2]$, one agent is called *advantaged*, the other *disadvantaged*. The disadvantaged agent is said to owe the advantaged agent a favor of size z. If the disadvantaged agent does a favor of size z, she becomes advantaged and the other disadvantaged. If she does no favor, she remains disadvantaged. Favors of size other than z are not part of equilibrium play and can be deterred by Nash reversion. When $z = 1/2$ (full sharing), the game is called *full equality matching*.

Consider a game of full equality matching between two agents. Suppose agent a is disadvantaged, b advantaged. Let $(\underline{u}_{em}, \bar{u}_{em})$ denote the average discounted payoffs expected by agents

(a, b), or more generally by disadvantaged and advantaged agents, respectively. Let $\sigma_{em}(\underline{u}_{em}, \bar{u}_{em}) = \left(\sigma^a_{em}(\underline{u}_{em}, \bar{u}_{em}), \sigma^b_{em}(\underline{u}_{em}, \bar{u}_{em})\right)$ denote the EM strategy profile that implements the payoff pair $(\underline{u}_{em}, \bar{u}_{em})$. Under σ_{em} the payoffs are

$$\begin{aligned} \underline{u}_{em} &= p\left((1-\delta)u(1-1/2) + \delta\bar{u}_{em}\right) + (1-p)\delta\underline{u}_{em} \\ &= p\left((1-\delta)(1/2)^{\alpha} + \delta\bar{u}_{em}\right) + (1-p)\delta\underline{u}_{em}, \qquad (1) \\ \bar{u}_{em} &= p\left((1-\delta)u(1) + \delta\bar{u}_{em}\right) + p\left((1-\delta)u(1/2) + \delta\underline{u}_{em}\right) + (1-2p)\delta\bar{u}_{em} \\ &= p\left(1-\delta + \delta\bar{u}_{em}\right) + p\left((1-\delta)(1/2)^{\alpha} + \delta\underline{u}_{em}\right) + (1-2p)\delta\bar{u}_{em}. \qquad (2) \end{aligned}$$

The first equation consists of two events: (i) with probability p agent a receives an opportunity, does a full favor $(x = 1/2)$ and becomes the advantaged agent; that is, agent a receives flow payoff $(1/2)^{\alpha}$ and continuation promise $\bar{u}_{em}$, (ii) with probability $(1-p)$ agent a receives no opportunity, so her flow payoff is zero and her continuation promise remains $\underline{u}_{em}$ along with her disadvantaged status. The equation for payoff $\bar{u}_{em}$ consists of three events that occur with probabilities p, p and $1-2p$, respectively: (i) agent b receives an opportunity, does no favor and receives a flow payoff of 1 and her continuation promise remains $\bar{u}_{em}$ as she is still advantaged, (ii) agent a receives an opportunity, shares it $(x = 1/2)$ so agent b receives a flow payoff of $(1/2)^{\alpha}$ but her continuation payoff drops to $\underline{u}_{em}$ because she now owes agent a the next favor, and (iii) neither agent receives a favor opportunity so agent b's flow payoff is zero and her continuation payoff remains $\bar{u}_{em}$.

The two previous equations contain two unknowns, $\underline{u}_{em}$ and $\bar{u}_{em}$, with solutions:

$$\underline{u}_{em} = \frac{p\left(1-\delta+\delta p\left(2+2^{\alpha}\right)\right)}{2^{\alpha}\left(1-\delta(1-2p)\right)}, \qquad (3)$$

$$\bar{u}_{em} = \frac{p\left((1-\delta)\left(1+2^{\alpha}\right)+\delta p\left(2+2^{\alpha}\right)\right)}{2^{\alpha}\left(1-\delta(1-2p)\right)}. \qquad (4)$$

For the EM strategy profile to be a *Nash equilibrium (NE)* in each stage game, neither agent can have a profitable deviation available to her. It is trivial that the advantaged agent has no profitable deviation as she just waits for reciprocation, but does no favors. Public (observable) off-equilibrium path deviations, such as the advantaged agent doing a favor or one of the agents doing the wrong size favor, can easily be deterred by the threat of autarky (no more favors). Therefore, we only need to check that a one-shot deviation for the disadvantaged agent consisting of doing no favor despite having the opportunity to do so followed by σ_{em} play as usual is not profitable. Agent a's discount factor has to be high enough that the *incentive compatibility constraint (ICC)* below is satisfied.

$$\begin{aligned} ICC^a_{em}: \quad & (1-\delta)\tfrac{1}{2^{\alpha}} + \delta\bar{u}_{em} \geq 1-\delta+\delta\underline{u}_{em} \\ \Longleftrightarrow \quad & \bar{u}_{em} - \underline{u}_{em} \geq \tfrac{1-\delta}{\delta}\left(1-\tfrac{1}{2^{\alpha}}\right). \end{aligned}$$

Using equations (3) and (4) we may write ICC^a_{em} as

$$\frac{p((1-\delta)(1+2^{\alpha})+\delta p(2+2^{\alpha}))}{2^{\alpha}(1-\delta(1-2p))} - \frac{p(1-\delta+\delta p(2+2^{\alpha}))}{2^{\alpha}(1-\delta(1-2p))} \geq \frac{1-\delta}{\delta}\left(1-\frac{1}{2^{\alpha}}\right)$$

$$\Longleftrightarrow \quad \frac{\delta p}{1-\delta(1-2p)} \geq 1 - \frac{1}{2^\alpha}$$

$$\Longleftrightarrow \quad \delta \geq \frac{2^\alpha - 1}{2^\alpha - 1 + p(2-2^\alpha)} \equiv \delta_\alpha \tag{5}$$

Figure 1 shows δ_α as p and α vary between $(0, 1/2)$ and $(0, 1)$, respectively.

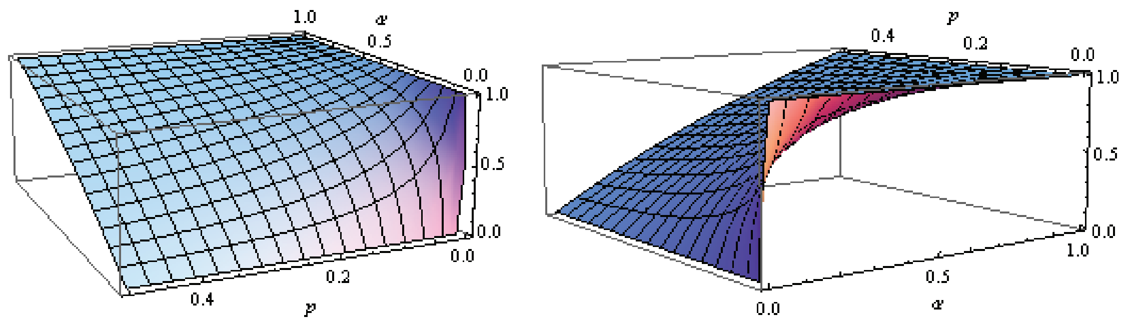

Figure 1: $\delta_\alpha(p, \alpha)$ for full EM with $u(x) = x^\alpha$

We start with equality matching because that is the easiest and most basic way to implement cooperation in the linear model, in particular, in AB [1]. Before proving several differences, we summarize the notation, and introduce the information structure and the equilibrium concept to be used.

2.1 Summary of notation and structure

The summary that follows is meant for reference, but we also need it in the next subsection to formally define several equilibrium concepts. Payoffs are in average discounted values.

Model parameters:	
$i \in \{a, b\}$:	Agents.
$\delta \in (0, 1)$:	Discount factor.
$p \in (0, 1/2)$:	Probability that agent $i \in \{a, b\}$ receives a favor opportunity.
$\alpha \in (0, 1)$	Preference convexity parameter.
Actions:	
$x, y \in [0, 1]$:	Size of favor by agents a, b, respectively.
Payoffs:	
(u, v) :	Current payoffs to agents (a, b).
(u_o, v_o) :	Continuation payoffs to (a, b) when no one does a favor.
(u_i, v_i) :	Continuation payoffs to (a, b) when $i \in \{a, b\}$ does a favor.

Table 1: Summary of notation with concave utility functions

Information structure: Let $t = 1, 2, \ldots$ denote the time index. If agent i receives a t-period favor opportunity, $w_t^i = 1$ and 0 otherwise. Agent i privately observes $W_t^i = \{w_z^i\}_{z=1}^t$. Let

$\tau_t = (x, y)$ denote favors $(x, y) \in (0,1]^2$ agents a and b, respectively, do in period t. If neither agent does a favor, then let $\tau_t = 0$. Both agents observe $T_t = \{\tau_z\}_{z=1}^{t}$. Private history of agent i and public history up to and including period t are denoted by $h_t^i = W_t^i \in \mathcal{H}_t^i$ and $H_t = T_t \in \mathcal{H}_t$, respectively. A strategy for agent i, denoted by σ^i, consists of a favor decision, I_t^i, for each period based on i's private history up to period t, and public history up to period $t-1$. More formally, $I_t^i : \mathcal{H}_t^i \times \mathcal{H}_{t-1} \to [0,1]$ s.t. $I_t^i(\cdot,\cdot) = 0$ when $w_t^i = 0$.

2.2 Strategies and equilibrium concepts

For our solution concept we will use *public perfect equilibrium* (PPE) following Fudenberg, Levine and Maskin [5]. A strategy for agent $i \in \{a, b\}$ is public if it depends only on her current period private information and the public history. In the favor-trading game under study, private information consists of whether or not the agent received a favor opportunity, and public information consists of (public) favors done up to and including the last period. A PPE is a profile of public strategies that form a Nash equilibrium for each period and the corresponding public history. Since the payoff pair $(\underline{u}_{em}, \bar{u}_{em})$ is enforceable (implementable), it follows by symmetry that $(\bar{u}_{em}, \underline{u}_{em})$ is also enforceable, and therefore any utility pair on the line connecting $(\underline{u}_{em}, \bar{u}_{em})$ and $(\bar{u}_{em}, \underline{u}_{em})$ is enforceable with the use of a public randomization device. Off-equilibrium path moves can be deterred by the threat of Nash reversion (autarky). This brings us to the following two definitions.

Definition 2 *Let σ_{aut}^i be such that $I_t^i = 0,\ \forall t$.*

Definition 3 *Let $\mathcal{H}_t^*$ be the set of all public on-equilibrium path histories up to and including period t.*

For example, if two agents are playing a full EM game and agent a is the initial disadvantaged agent, any history such that agent b did the first favor, one of the agents did two consecutive favors or a partial favor, would not be in $\mathcal{H}_t^*$. However, histories that include only private deviations, that is, a disadvantaged agent does not do a favor when she has the opportunity, would still be in $\mathcal{H}_t^*$. Next we define EM formally.

Definition 4 *Given $z \in (0,1]$, $\sigma_{em(z)}^i$ is such that $I_t^i = z$ if agent i is disadvantaged, $w_t^i = 1$ and $h_{t-1} \in \mathcal{H}_{t-1}^*$, otherwise $I_t^i = 0$. Let $\sigma_{em(1)}^i \equiv \sigma_{em}^i$.*

2.3 Basic properties of concave favor-trading games

The lemmas in this section state for the record that the model always has at least the autarky equilibrium and that the first-best outcome is not enforceable.

Claim 5 *In a favor-trading game with preferences $u(x) = x^\alpha$, equilibria always exist.*

Proof. Immediate. Autarky is always an equilibrium. ■

Claim 6 *First-best outcome is not enforceable in equilibrium.*

Proof. To achieve the first-best outcome both agents have to share every favor opportunity equally, that is regardless of whether the other agent has reciprocated. But then each agent has a profitable deviation to not do a favor, which the other agent could not observe as favor opportunities are private information. ■

3 Multi-state equality matching (n-EM)

In this section we generalize equality matching to multiple states.

Lemma 7 (EM is always possible) *Given δ and α, there exists $z \in (0, 1/2]$ such that EM at level z is implementable as a PPE.*

Proof. In appendix. ■

The proof for this lemma follows immediately from the utility function's form $u(x) = x^{\alpha}$. Marginal utility cost of doing an infinitesimal favor goes to a bounded constant:

$$u'(1 - z) = \frac{-\alpha}{z^{1-\alpha}} \rightarrow -\alpha \text{ as } z \rightarrow 0,$$

while the marginal benefit to the recipient goes to infinity:

$$u'(z) = \frac{\alpha}{z^{1-\alpha}} \rightarrow \infty \text{ as } z \rightarrow 0.$$

That is, we can make the cost of doing a favor relative to the benefit arbitrarily small by choosing a small enough size for the favor. This is in contrast to the linear case in which the cost-to-benefit ratio is always $k \in (1, \infty)$ and consequently discount factors need to exceed a certain threshold for EM to be implementable.

In EM games with linear preferences, full equality matching also represents the most efficient incentive compatible form of favor-trading (on symmetric self-generating lines) in the special case of $\delta = \delta^*$. The equivalent form of EM with concave preferences consists of matching half-sized favors as opposed to full-sized ones because maximal efficiency is achieved by sharing the opportunity equally due to α-concave utility functions. Such an EM game is shown in Figure 2 (left). We refer to it as a 2-state EM game because agents alternate between the two states: a advantaged, b disadvantaged, (A, D), and b advantaged, a disadvantaged, (D, A). Similarly, we refer to an EM game with 3-states as a 3-state EM game, such as the one shown in Figure 2 (right). State $(\emptyset, \emptyset)$ refers to a neutral, or even state.

Next we show that 3-state EM strategies dominate 2-state EM strategies when $\delta = \delta^*$. To make the demonstration more concrete, we offer a parametric example of 2-state vs. 3-state EM strategies using values $\alpha = 0.6$ and $p = 0.3$. The threshold discount factor, δ_{α}, is determined by substituting these values into equation (5):

$$\delta_{\alpha} \equiv \frac{2^{\alpha} - 1}{2^{\alpha} - 1 + p\,(2 - 2^{\alpha})} = 0.7802. \qquad (6)$$

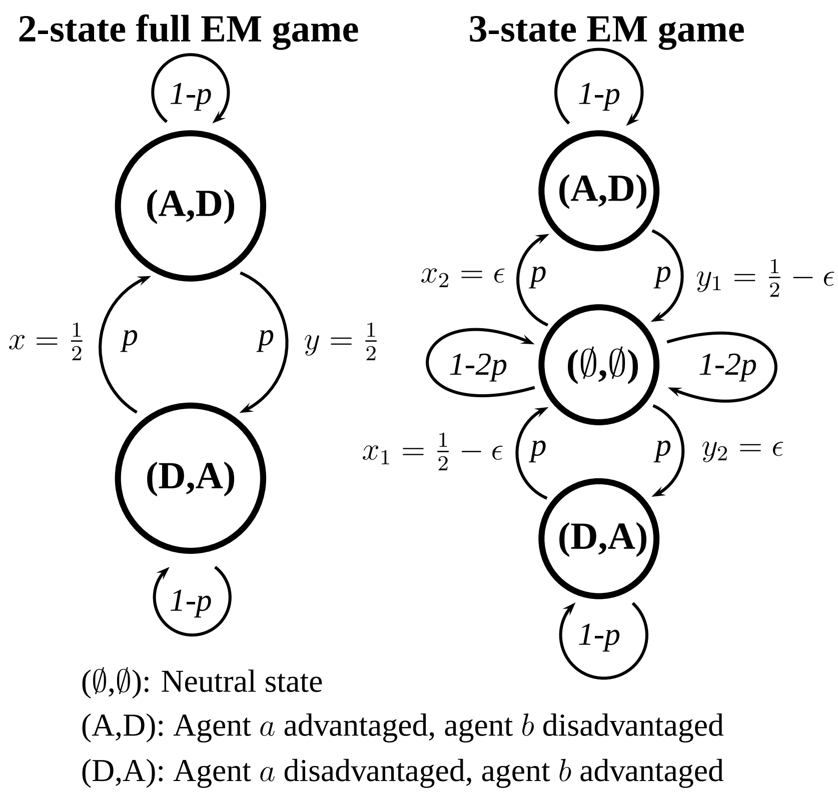

Figure 2: 2-state and 3-state EM automata

We then substitute our values for α, p and $\delta = \delta_\alpha$ into equations (3) and (4) to find our benchmark payoffs:

$$\underline{u}_{em} = \frac{p\left(1-\delta+\delta p\left(2+2^{\alpha}\right)\right)}{2^{\alpha}\left(1-\delta(1-2p)\right)} = 0.3, \tag{7}$$

$$\bar{u}_{em} = \frac{p\left(\left(1-\delta\right)\left(1+2^{\alpha}\right)+\delta p\left(2+2^{\alpha}\right)\right)}{2^{\alpha}\left(1-\delta(1-2p)\right)} = 0.39585, \tag{8}$$

$$\underline{u}_{em} + \bar{u}_{em} = 0.69585. \tag{9}$$

To find the equivalent values for a 3-state EM strategy we need to solve the following system of 3 payoff equations in 3 unknown payoffs,

$$\begin{aligned}
\bar{u} &= p\left(\left(1-\delta\right)+\delta\bar{u}\right)+p\left(\left(1-\delta\right)\left(\tfrac{1}{2}-\varepsilon\right)^{\alpha}+\delta u^{o}\right)+\left(1-2p\right)\delta\bar{u},\\
u^{o} &= p\left(\left(1-\delta\right)\left(1-\varepsilon\right)^{\alpha}+\delta\bar{u}\right)+p\left(\left(1-\delta\right)\varepsilon^{\alpha}+\delta\underline{u}\right)+\left(1-2p\right)\delta u^{o},\\
\underline{u} &= p\left(\left(1-\delta\right)\left(\tfrac{1}{2}+\varepsilon\right)^{\alpha}+\delta u^{o}\right)+\left(1-p\right)\delta\underline{u}.
\end{aligned}$$

To simplify the arithmetic, we first replace the utility function terms as follows:

$$A \equiv \varepsilon^{\alpha},\ B \equiv \left(\tfrac{1}{2}-\varepsilon\right)^{\alpha},\ C \equiv \left(\tfrac{1}{2}+\varepsilon\right)^{\alpha},\ D \equiv \left(1-\varepsilon\right)^{\alpha} \tag{10}$$

For mnemonic reasons, we chose $A < B < C < D$ for ε small. In terms of A, B, C and D, the 3-state EM payoffs are

$$\begin{aligned}\bar{u} &= p\left((1-\delta)+\delta\bar{u}\right)+p\left((1-\delta)B+\delta u^{o}\right)+(1-2p)\,\delta\bar{u},\\ u^{o} &= p\left((1-\delta)D+\delta\bar{u}\right)+p\left((1-\delta)A+\delta\underline{u}\right)+(1-2p)\delta u^{o},\\ \underline{u} &= p\left((1-\delta)C+\delta u^{o}\right)+(1-p)\delta\underline{u}.\end{aligned}$$

Solving and simplifying:

$$\begin{aligned}\bar{u} &= p\tfrac{1+B+((3+A+3B+D)p-2(1+B))\delta+\left(1+B-(3+A+3B+D)p+(1+A+B+C+D)p^{2}\right)\delta^{2}}{(1-(1-p)\delta)(1-(1-3p)\delta)},\\ u^{o} &= p\tfrac{A+D+(-A-D+(1+A+B+C+D)p)\delta}{1-(1-3p)\delta},\\ \underline{u} &= p\tfrac{C+((A+3C+D)p-2C)\delta+\left(C-(A+3C+D)p+(1+A+B+C+D)p^{2}\right)\delta^{2}}{(1-(1-p)\delta)(1-(1-3p)\delta)}.\end{aligned}$$

For later use, the payoff differences between adjacent states are

$$\begin{aligned}\bar{u}-u^{o} &= p\,(1-\delta)\,\tfrac{(1-A+B-D-(1-A+B-D+(-2+A-2B+C+D)p)\delta)}{(1-(1-p)\delta)(1-(1-3p)\delta)},\\ u^{o}-\underline{u} &= p\,(1-\delta)\,\tfrac{(A-C+D+(-A+C-D+(1+A+B-2C+D)p)\delta)}{(1-(1-p)\delta)(1-(1-3p)\delta)}, \qquad (11)\\ \frac{\bar{u}-u^{o}}{u^{o}-\underline{u}} &= \tfrac{1-A+B-D-(1-A+B-D+(-2+A-2B+C+D)p)\delta}{A-C+D-(A-C+D-(1+A+B-2C+D)p)\delta},\\ &= \tfrac{(1-\delta(1-p))(1-A+B-D)+p\delta(1+B-C)}{(1-\delta(1-p))(A-C+D)+p\delta(1+B-C)}. \qquad (12)\end{aligned}$$

With three states, we have two incentive compatibility constraints. To move from the disadvantaged state to the neutral state requires a favor of size $1/2-\varepsilon$ in return for a continuation promise of u^{o}:

$$\begin{aligned}ICC_1: &\quad (1-\delta)\,C+\delta u^{o} \geq 1-\delta+\delta\underline{u}\\ \Longrightarrow &\quad \frac{\delta}{1-\delta} \geq \frac{1-C}{u^{o}-\underline{u}}\\ \Longrightarrow &\quad \delta^{*} = \frac{1-C}{1-C+u^{o}-\underline{u}}. \qquad (13)\end{aligned}$$

To move from the neutral state to the advantaged state requires a favor of size ε in return for a continuation promise of $\bar{u}$:

$$\begin{aligned}ICC_2: &\quad (1-\delta)D+\delta\bar{u} \geq 1-\delta+\delta u^{o}\\ \Longrightarrow &\quad \frac{\delta}{1-\delta} \geq \frac{1-D}{\bar{u}-u^{o}}\\ \Longrightarrow &\quad \delta^{*} = \frac{1-D}{1-D+\bar{u}-u^{*}}. \qquad (14)\end{aligned}$$

Below we show that if ICC_2 holds, then ICC_1 holds, and therefore it is enough to only verify ICC_2.

Claim 8 $ICC_2 \geq 0 \implies ICC_1 \geq 0$.

Proof (by contradiction). Suppose to the contrary that $ICC_2 \geq 0$ but $ICC_1 < 0$. Then $\exists \varepsilon, \delta > 0$ such that $\frac{\delta}{1-\delta} \geq \frac{1-D}{\bar{u}-u^o}$ and $\frac{\delta}{1-\delta} < \frac{1-C}{u^o-\underline{u}}$.

$\implies \delta \geq \frac{1-D}{1-D+\bar{u}-u^o}$ and $\delta < \frac{1-C}{1-C+u^o-\underline{u}}$

$\implies \frac{1-D}{1-D+\bar{u}-u^o} < \frac{1-C}{1-C+u^o-\underline{u}}$

$\implies (1-C)(1-D) + (1-D)(u^o - \underline{u}) < (1-C)(1-D) + (1-C)(\bar{u} - u^o)$

$\implies u^o - \underline{u} - D(u^o - \underline{u}) < \bar{u} - u^o - C(\bar{u} - u^o)$

$\implies C(\bar{u} - u^o) - D(u^o - \underline{u}) < \bar{u} - 2u^o + \underline{u}$

$\implies D(\bar{u} - 2u^o + \underline{u}) < \bar{u} - 2u^o + \underline{u}$

$\implies D > 1$ since $\bar{u} - 2u^o + \underline{u} < 0$ by strict concavity of $u(\cdot)$. The last inequality contradicts the definition of $D \equiv (1-\varepsilon)^\alpha < 1$. ■

Returning to our example, it is sufficient per claim 8 to find an $\varepsilon > 0$ such that ICC_2 binds. Substituting in $\alpha = 0.6,\ p = 0.3$ and $\delta = \delta_\alpha = 0.7802$ from the 2-state problem into our 3-state payoff functions and solving $ICC_2 = 0$ for ε yields $\varepsilon = 0.2176$. Using these values $A \equiv \varepsilon^\alpha = 0.4005$, $B \equiv \left(\frac{1}{2} - \varepsilon\right)^\alpha = 0.4683$, $C \equiv \left(\frac{1}{2} + \varepsilon\right)^\alpha = 0.8195$, $D \equiv (1-\varepsilon)^\alpha = 0.8631$ and $u^o - \underline{u} = \frac{0.498(3.821-\delta)(1-\delta)}{(10-\delta)(1.429-\delta)}$. We can now solve for δ_1 from $ICC_1 : \delta_1 \geq \frac{1-C}{1-C+u^o-\underline{u}}$.

Substituting in values for A, B, C and $D \implies \delta_1 \geq \frac{1-(0.5+\varepsilon)^\alpha}{1-(0.5+\varepsilon)^\alpha + \frac{0.498(3.820-\delta_1)(1-\delta_1)}{(10-\delta_1)(1.429-\delta_1)}}$. Substituting in values for a, ε and solving for $\delta_1 \implies \delta_1 \geq \frac{1-0.819}{1-0.819+\frac{0.498\left(2\delta_1^2-2.402\delta_1+1.903\right)}{\delta_1^2-11.43\delta_1+14.29}} \implies \delta_1 \geq 0.7452$.

That is, if we fix $\delta = \delta^*$ from the two state problem, and increase ε until ICC_2 binds, then ICC_1 will be slack for those ε and δ. In particular, we need $\delta \geq 0.7802$ to satisfy ICC_2 in the example above, whereas ICC_1 holds for $\delta \geq 0.7452$. The payoffs for our example are as follows: $\bar{u} = 0.39941 > 0.39585 = \bar{u}_{em}$, $u^o = 0.36084$ and $\underline{u} = 0.30515 > 0.3 = \underline{u}_{em}$, where $\underline{u}_{em}$ and $\bar{u}_{em}$, from (8) and (7), respectively, are the corresponding payoffs with the 2-state EM strategy. Observe that the total value of the game is higher with the 3-states. Furthermore the total payoff in the 3-state game is higher when the agents are in the middle state: $\underline{u}_{em} + \bar{u}_{em} = 0.69585 < \underline{u} + \bar{u} = 0.70456 < 2u^o = 0.72168$. The example demonstrates a general contrast between favor-trading games with linear and concave utility functions.

We are now ready to generalize equality matching for multiple states. Note that (15) below also implies reversion to autarky if either agent deviates from the equilibrium path ($h_{t-1} \notin \mathcal{H}^*_{t-1}$).

Definition 9 *Suppose that for $i \in \{a, b\}$, $s \in \mathbb{S} \equiv \{1, 2, ..., n\}$ and $t \in \mathbb{N}$, strategy profile σ is such that*

$$y_s \equiv x_{n+1-s} \in (0,1) \text{ and } I^i_s := \begin{cases} x_s & \text{if } s \neq n, w^a_t = 1, h_{t-1} \in \mathcal{H}^*_{t-1}, \\ x_{n+1-s} & \text{if } s \neq 1, w^b_t = 1, h_{t-1} \in \mathcal{H}^*_{t-1}, \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

$$s_{t+1} := s_t + \mathbf{1}_{\left\{\tau_t = \left(x_{s_t}, 0\right)\right\}} - \mathbf{1}_{\left\{\tau_t = \left(0, y_{s_t}\right)\right\}}. \tag{16}$$

*Then we call σ an **n-state EM strategy profile** and denote it by $\boldsymbol{\sigma}_{\mathbf{em^n}}$.*

Definition 10 (Special case of σ_{em^n}) *Suppose that for $i \in \{a, b\}$, $s \in \mathbb{Z}$ and $t \in \mathbb{N}$, strategy profile σ is such that*

$$y_{-s} \equiv x_s \in (0,1) \text{ and } I_s^i := \begin{cases} x_s & \text{if } w_t^a = 1,\ h_{t-1} \in \mathcal{H}_{t-1}^*, \\ y_s & \text{if } w_t^b = 1,\ h_{t-1} \in \mathcal{H}_{t-1}^*, \\ 0 & \text{otherwise}, \end{cases} \tag{17}$$

$$s_0 \equiv 0,\ s_{t+1} := s_t + \mathbf{1}_{\left\{\tau_t = \left(x_{s_t}, 0\right)\right\}} - \mathbf{1}_{\left\{\tau_t = \left(0, y_{s_t}\right)\right\}}. \tag{18}$$

*Then we call σ an **∞-state EM strategy profile** and denote it by $\boldsymbol{\sigma}_{\mathbf{em^\infty}}$.*

For the subsequent definitions and lemmas, we refer to random variables with uppercase letters, realizations of random variables with lowercase letters, and probability distributions with calligraphic (or script uppercase) letters. $\mathcal{U}$ refers to the uniform distribution. $\mathcal{P}$ refers to the distribution of favor opportunities. $\rho_{s,t} \equiv \rho(s,t)$ is the transition probability from state s to t. If a stationary distribution exists, $\pi_s \equiv \pi(s)$ denote the associated probabilities.[1]

Definition 11 *Given an n-state strategy profile σ, let $\pi_s(\sigma)$ denote the stationary probability for state $s = 1, 2, ..., n$, respectively, consistent with σ. If the game has no stationary distribution consistent with σ, let $\pi(\sigma) \equiv \emptyset$. For convenience, let $\pi_s(\sigma) \equiv \pi_s \equiv \pi(s)$ when no ambiguity exists about σ.*

Definition 12 *Given an n-state strategy profile σ, let $s^0 \in \mathbb{S} \equiv \{1, 2, ..., n\}$ denote the starting state, then*

$$u_s(\sigma) := E_0\left[u(\sigma) : s^0 = s\right],\ s = 1, 2, ..., n,$$

defines the expected average discounted payoff to agent a in state s, and

$$u(\sigma) = E\left[(1-\delta)\sum_{t=0}^{\infty} \delta^t\left((1-X^t)^\alpha + (Y^t)^\alpha\right)\right],$$

represents the overall expected average discounted payoff to agent a, where X^t and Y^t are random variables of period t favors by agents a and b, respectively. Let v_s and v be the equivalent payoffs to agent b. If $\pi \neq \emptyset$, $P\left(s^0 = s\right) = \pi_s$, or undefined, and σ is symmetric ($y_s \equiv x_{n+1-s}$ for $s \in \mathbb{N}$ or $y_s \equiv x_{-s}$ for $s \in \mathbb{Z}$), we claim without proof that for finite n^*-EM equilibria

$$u(\sigma) \equiv u(x) = \tfrac{1-\delta}{2}\sum_{s=1}^{n} \pi_s\left((1-x_s)^\alpha + (y_s)^\alpha\right),$$

and we define the value of σ to be $T(\sigma) := u + v \equiv 2u(x)$.

We now return to finish our 2-state vs. 3-state EM comparison for the general case of $\delta > \delta^*$ or for its α-concave equivalent; $\delta > \delta_\alpha$.

Lemma 13 (3-state EM strategy) *For $\delta > \delta_\alpha$ there exists a 3-state EM equilibrium profile, call it σ'_{em^3}, that has strictly higher value than any 2-state EM profile. That is, $T\left(\sigma'_{em^3}\right) > T\left(\sigma_{em^2}\right)$, $\forall \alpha, p, \sigma_{em^2}$.*

---

[1] A stationary probability may also be thought of as the fraction of time spent asymptotically in a given state, or the number of visits to state s in a game with t periods as $t \to \infty$.

Proof. δ_α is defined as the discount factor at which full equality matching becomes incentive compatible in a 2-state EM game (definition 5). Therefore the only σ_{em^2} strategy profile we need to consider consists of full favors by the disadvantaged agent and no favors by the advantaged agent. Call this profile $\sigma^*_{em^2}$.

$$T\left(\sigma^*_{em^2}\right) = \underline{u}_{em}\left(\delta_\alpha\right) + \bar{u}_{em}\left(\delta_\alpha\right) = p\left(1 + (1-x)^\alpha + y^\alpha\right) = p\left(1 + 2^{1-\alpha}\right).$$

For 3-state EM,

$$T\left(\sigma'_{em^3}\right) = 2\left(\pi_1 u_1 + \pi_2 u_2 + \pi_3 u_3\right), \text{ where } \pi = \left(\pi_1, \pi_2, \pi_3\right).$$

π denotes the stationary probabilities. Let m be a matrix of transition probabilities between states induced by a 3-state EM strategy, then π is determined by

$$m^\intercal \pi = \pi \text{ and } \pi_1 + \pi_2 + \pi_3 = 1 \text{ where } m = \begin{bmatrix} 1-p & p & 0 \\ p & 1-2p & p \\ 0 & p & 1-p \end{bmatrix} \implies \pi = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right).$$

It is enough to prove the claim for $\delta = \delta_\alpha$ since for $\delta > \delta_\alpha$ we can use the "$\delta = \delta_\alpha$"-solution because it is incentive compatible for $\delta \geq \delta_\alpha$. The ICC for σ'_{em^3} are:

$$\left(1-\delta_\alpha\right)\left(1-x_1\right)^\alpha + \delta_\alpha u_2 \geq 1 - \delta + \delta_\alpha u_1, \tag{19}$$

$$\left(1-\delta_\alpha\right)\left(1-x_2\right)^\alpha + \delta_\alpha u_3 \geq 1 - \delta + \delta_\alpha u_2. \tag{20}$$

Suppose for our candidate solution we pick x_1 and x_2 such the ICC (19) and (20) bind. Treating the inequality signs in (19) and (20) as equalities and solving yields $x_2 = \left(1 - 2x_1\right)/2$. The arithmetic required for the last step and for an expression for $T\left(\sigma'_{em^3}\right)$ is in the appendix. We just need one point so let $x_1 = 1/3 \implies x_2 = 1/6$. Then

$$\begin{aligned} T\left(\sigma'_{em^3}\right) &= \tfrac{2}{3}p\left(1 + \left(1 - \tfrac{1}{3}\right)^\alpha + \left(\tfrac{1}{3}\right)^\alpha + \left(1 - \tfrac{1}{6}\right)^\alpha + \left(\tfrac{1}{6}\right)^\alpha\right) \\ &= \tfrac{2}{3}p\left(1 + \left(\tfrac{2}{3}\right)^\alpha + \left(\tfrac{5}{6}\right)^\alpha + \left(\tfrac{1}{3}\right)^\alpha + \left(\tfrac{1}{6}\right)^\alpha\right). \end{aligned}$$

A comparison of $T\left(\sigma^*_{em^2}\right)$ to $T\left(\sigma'_{em^3}\right)$ shown in figure 3 concludes the proof. ■

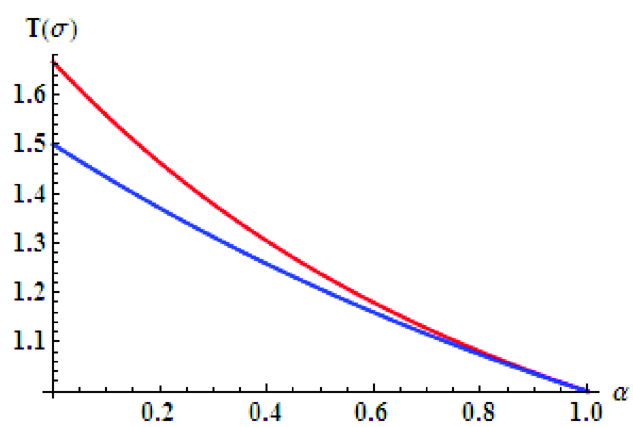

Figure 3: $T(\sigma'_{em^3})$ in red, $T(\sigma^*_{em^2})$ in blue, and $p = \frac{1}{2}$ by normalization

To finish this section we show that the uniform stationary distribution we found in the 3-state EM example holds in general for multi-state EM, except for infinite state strategies. For the latter we show that now stationary distribution exists. Finally, we define locally and globally efficient multi-state equilibria.

Lemma 14 $\pi_s(\sigma_{em^n}) = 1/n$, $s = 1, 2, ..., n$.

Proof. The proof is in the appendix. The proof is a generalized version of the transformation matrix calculation we did in the proof of lemma 13 to compute the fractions of time agents spent in each state in a 3-state EM game. ■

Lemma 15 $\pi(\sigma_{em^\infty}) = \emptyset$.

Proof. By (18) of definition 10 the stochastic process associated with σ_{em^∞} is a simple random walk on $\mathbb{Z}^1$ with transition probabilities $\rho(s, s+1) = \rho(s, s-1) = p$ and $\rho(s, s) = 1 - 2p$. We say that ρ is irreducible since any state $t \in \mathbb{Z}$ is reached with positive probability from any state $s \in \mathbb{Z}$ in a finite number of steps. Furthermore, each $s \in \mathbb{Z}$ is visited infinitely often so each s is recurrent, but there are infinitely many states so the fraction of total visits to s has measure 0. That is, s is null recurrent and therefore no stationary distribution exists over $s \in \mathbb{Z}$ consistent with σ_{em^∞}. A proof of the last statement can be found in Durrett [3] (p. 307, but the details are beyond the scope of this paper). ■

Definition 16 *Suppose $\sigma_{em^{\tilde{n}}}$ is an $\tilde{n}$-state EM strategy profile such that neither agent has a profitable deviation. Then we call $\sigma_{em^{\tilde{n}}}$ an ñ-state EM equilibrium* ***(ñ-EM equilibrium)****. If $T(\sigma_{em^{\tilde{n}}}) \geq T(\sigma'_{em^{\tilde{n}}})$, for all ñ-EM equilibria, $\sigma'_{em^{\tilde{n}}}$, we call $\sigma_{em^{\tilde{n}}}$ a* ***locally efficient*** *ñ-state EM equilibrium, and denote it by $\sigma^*_{em^{\tilde{n}}}$. If $T(\sigma_{em^{\tilde{n}}}) \geq T(\sigma_{em^n})$ for all $n \in \mathbb{N}$ and σ_{em^n}, we call $\sigma_{em^{\tilde{n}}}$ a* ***globally efficient*** *multi-state equality matching equilibrium* ***(n*-EM*** *or* ***globally efficient ñ-EM equilibrium)****, and denote it by $\sigma^*_{em^n}$.*

3.1 Globally efficient n-EM equilibria (n^*-EM): Numerical methods

Imposing concavity on the linear favor-trading model substantially complicates the equilibrium analysis. At this point we are unable to solve the model further in closed-form and therefore in this section we turn to numerical techniques to characterize n^*-EM equilibria. Subsection 3.1.1 introduces our numerical analysis approach and provides an overview of the parametric models we have constructed to carry out the analysis. Details of the two models follow in subsections 3.1.2 and 3.1.3, respectively. We characterize n^*-EM equilibria properties in section 3.2 based on the results of our numerical analysis. However, our solution methods themselves may be of greater interest to some readers than the actual solutions in so far as these techniques may be used to solve other applied game theory problems.

3.1.1 Summary of parametric models

Conjectures in later sections are based on results from two types of parametric computer models; games with *simulated payoffs (SP)* and games with *computed payoffs (CP)*. The computer code for each model is Mathematica 7 based, and consists of a favor solver engine and a favor solver frame. Given numerical values of α, δ, p and n *(number of states)*, the favor solver engine finds a representation for payoffs, $u = \sum_{s=1}^{n} \pi_s u_s$, and incentive compatibility constraints, ICC_s, in terms of x, then solves the constrained nonlinear optimization problem:[2]

$$\max_{\substack{x \geqq 0 \\ \text{subj. to } ICC_s(x) \geq 0,\ \forall s \in \mathbb{S}.}} u(x) \tag{21}$$

The favor solver frame is essentially a series of loops built around the engine code that feeds the engine a set of user-specified values for α, δ and p, and increments n until u stops increasing. The frame also records the results for the optimal n, denoted by n^*, for each triplet (α, δ, p) and constructs user-specified tables and plots out of these results.

The difference between the simulated payoffs model and the computed payoffs model is the part of the engine code that finds $u_s(x)$. The SP-model's engine generates a large set of random favor opportunity sequences, computes the path of the game (sequence of states) consistent with strategy profile σ_{em^n} for each favor opportunity sequence and for each possible starting state, computed the discounted sums agent a's flow payoffs along each path, and takes the averages per starting states to determine $u_s(x),\ \forall s$. The CP-model's engine finds $u_s(x)$ directly by solving the set of n simultaneous payoff equations that characterize σ_{em^n}.

Both models use Mathematica's built-in optimization algorithms to solve problem (21) and the CP-model uses Mathematica's built-in numerical solver to solve the sets of simultaneous payoff equations. The Mathematica code for each model is available in the appendix.

3.1.2 Simulated payoffs (SP) model

The SP-model consists of the following steps:

1. Choose simulation and parameter values: The user specifies the number of *games per state* (I) and *rounds per game* to simulate (J), and either point values or ranges for model parameters α (concavity), δ (discount factor), and p (probability of favor opportunity).

2. Simulate data: The model generates a matrix of random favor opportunities $W \equiv [w_{i,j}]_{I \times J}$ with elements $w_{i,j} \sim \mathcal{P}\{a, b, \emptyset\}$.

3. Process data to estimate payoff functions: We need to map W to payoffs consistent with $\sigma_{em^n}(\alpha, \delta, p)$. To this end, the model defines operators

$$\varrho(s,w) := s + \mathbf{1}_{\{w=a, s \neq n\}} - \mathbf{1}_{\{w=b, s \neq 1\}}, \tag{22}$$

$$\upsilon(s,w)(x,y) := \mathbf{1}_{\{w=a\}}(1-x_s)^\alpha + \mathbf{1}_{\{w=b\}}(y_s)^\alpha$$

$$\iff \upsilon(s,w)(x) := \mathbf{1}_{\{w=a\}}(1-x_s)^\alpha + \mathbf{1}_{\{w=b\}}(x_{n+1-s})^\alpha \text{ by def'n 9} \tag{23}$$

[2] In EM optimization problems $x \equiv \{x_1, ..., x_{n-1}\}$ in (21) because $x_n = 0$ by definition of n-EM and therefore drops out of $u(x)$ and $ICC_s(x)$. For other strategies (with $x_n > 0$), x in (21) should be treated as $x \equiv \{x_1, ..., x_n\}$.

$s_{i,j} = \varrho(s_{i,j-1}, w_{i,j})$ iteratively determines the path of game i consistent with σ_{em^n} given a starting state $s_{i,0} \in \mathbb{S}$. υ maps each state along game i's path to a flow payoff function (of x). We want to find $u_s(x)$ for state s so we apply the following functionals to W given starting state $s_{i,0}$

$$s_{i,0} \times [w_{i,j}]_{I\times J} \mapsto s_{i,0} \times [\varrho(s_{i,j-1}, w_{i,j})]_{I\times J} \equiv [s_{i,j-1}]_{I\times J}, \tag{24}$$

$$[s_{i,j-1}]_{I\times J} \times [w_{i,j}]_{I\times J} \mapsto [\upsilon(s_{i,j-1}, w_{i,j})(x)]_{I\times J} \equiv \left[\upsilon(x)_{i,j-1}\right]_{I\times J}(s). \tag{25}$$

In words, $[\upsilon(x)_{i,j-1}]_{I\times J}(s)$ is an $I \times J$ matrix of flow payoffs in terms of x initiated from state $s_{i,0} = s, \ \forall i$. To find total payoffs the model computes discounted sums along the rows of $[\upsilon(x)_{i,j-1}]_{I\times J}$. We multiply the result by $(1-\delta)$ to convert total payoffs into average discounted payoffs:

$$(1-\delta)\left(\left[\upsilon(x)_{i,j-1}\right]_{I\times J}(s) \cdot \left[\delta^{j-1}\right]_{J\times 1}\right) = [\tilde{u}_s(x)_i]_{I\times 1}, \tag{26}$$
$$\text{where } \tilde{u}_s(x)_i = (1-\delta)\textstyle\sum_{j=1}^{J}\delta^{j-1}\upsilon(x)_{i,j-1}.$$

We use tilde to differentiate estimated payoffs from the true payoffs. However for $J = \infty$,

$$\tilde{u}_s(x) = \tfrac{1}{I}\textstyle\sum_{i=1}^{I}\tilde{u}_s(x)_i \underset{p}{\rightarrow} u_s(x) \text{ as } I \rightarrow \infty \tag{27}$$

by law of large number. $\tilde{u}_s(x)$ contains a small truncation error, $\epsilon_{s_{i,J}} = \delta^J u_{s_{i,J}}(x)$, for each game i because $J < \infty$. We could use $\tilde{u}_s(x) = \frac{1}{1-\delta^J}\frac{1}{I}\sum_{i=1}^{I}\tilde{u}_{s_{0,j}=s}(x)_i$ to compensate, but at the moment the SP-model does not implement any correction scheme for the truncation error. Instead we chose J sufficiently large that errors factored by δ^J are insignificant.

The model produces $\tilde{u}_s(x)$ for all $s \in \{1, 2, ..., n\}$ using the same W. Therefore the simulation generates nI payoff samples in total, and the overall value of the σ_{em^n} is estimated as

$$\begin{aligned} \tilde{u}(x) + \tilde{v}(x) &= 2\tilde{u}(x) \text{ by symmetry, and} \\ \tilde{u}(x) &= \textstyle\sum_{s=1}^{n}\pi_s\tilde{u}_s(x) \text{ by def'n } 12 \\ &= \frac{1}{n}\textstyle\sum_{s=1}^{n}\tilde{u}_s(x) \text{ by def'n } 14. \end{aligned} \tag{28}$$

4. Optimization: Our goal is to find the optimal number of states, n^*, and efficient multi-state EM favors $x^* \equiv \{x_1^*, x_2^*, ..., x_{n^*}^*\}$ for the specified values of α, δ, p and a set of random favor opportunities W. To do so the model performs following steps:

a. Increment n by 1 or start with $n = 2$ if this is the first round.

b. Apply operators ϱ and υ defined by (22) and (23) to W and transformations (24)-(26) to the results to find $\tilde{u}_s(x)$ for $s = 1, 2, ..., n$ as defined in (27).

c. To find $x_1, ..., x_{n-1}$ ($x_n = 0$ by definition of σ_{em^n}), numerically solve the following formulation of nonlinear optimization problem (21):

$$\max_{x_1,...,x_{n-1}>0} \frac{1}{n}\sum_{s=1}^{n} \tilde{u}_s\left(x_1,...,x_{n-1}\right) \quad (29)$$
$$\text{subj. to } x_n{=}0 \text{ and } (1-\delta)((1-x_s)^{\alpha}-1)+\delta(\tilde{u}_{s+1}(x)-\tilde{u}_s(x))\geq 0,\ s{=}1,...,n-1.$$

d. Let $\tilde{u}(n)$ and $\tilde{x}(n)$ denote the solutions to (29). Record these values as well as the corresponding $\tilde{u}_s(n),\ \forall s$, and other desired data.

e. If $n = 2$ or $\tilde{u}(n-1) < \tilde{u}(n)$, repeat from step a. Else $n^* = n - 1$ and $x^* = x(n-1)$.

We defer the technical details concerning Mathematica's optimization algorithms and their scope to subsection 3.1.3 and the appendix.

5. Output: A single point solution is not of particular importance to us, so we run the SP-model on a set of points $(\alpha, \delta, p) \in (0,1)^2 \times (0, 1/2)$ that cover the parameter space. The output is retrieved in tables and plots that describe the general behavior of n^*-EM equilibria. For example, figure 4 depicts globally efficient x_s^* for $s = 1, 2, ..., n^* - 1$ as a functions of p. The exact code for this demonstration is available in the appendix.

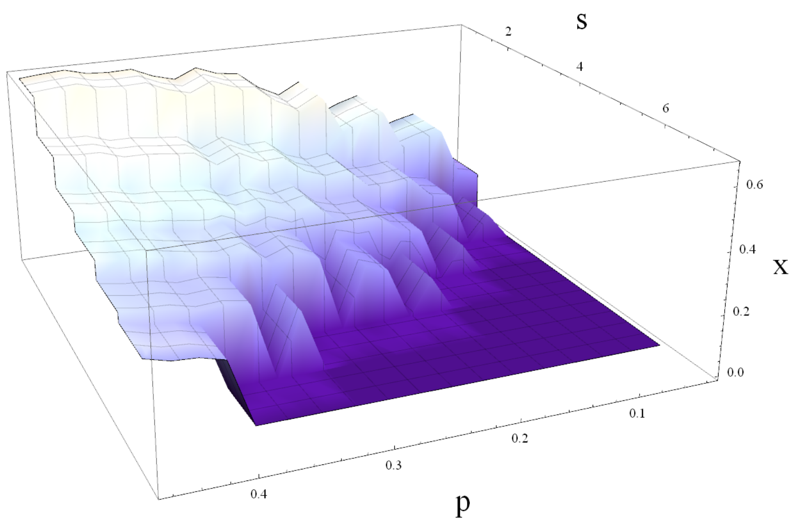

Figure 4: **Simulated payoffs: n^*-EM(p) equilibria**
Algorithm: Interior point
Points: $\alpha = 0.5,\ \delta = 0.8,\ p = 0.05, 0.1, ..., 0.45$

SP-model versus CP-model: The CP-model is more efficient (faster) and more accurate but requires the extra step of solving n simultaneous payoff equations. When that is not possible, the SP-model can fill in the gap. We used the SP-model mainly to double check our results, but we present it here to offer another tool for the continuing research into favor trading equilibria and for other similar problems.

3.1.3 Computed payoffs (CP) model

The CP-model has similar steps to the SP-model:

1. Choose parameter values: The user specifies point values or ranges for model parameters α (concavity), δ (discount factor), and p (probability of favor opportunity).

2. Setup simultaneous payoff equation problem: Given n, we can describe $u_s(\sigma_{em^n})$ with following set of simultaneous equations:

$$\begin{aligned}
u_1 &= p\left((1-\delta)(1-x_1)^\alpha + \delta u_2\right) + (1-p)\delta u_1, \\
u_n &= p(1-\delta+\delta u_n) + p\left((1-\delta)(y_n)^\alpha + \delta u_{n-1}\right) + (1-2p)\delta u_n, \\
u_s &= p\left((1-\delta)(1-x_s)^\alpha + \delta u_{s+1}\right) + p\left((1-\delta)(y_s)^\alpha + \delta u_{s-1}\right) \\
&\quad + (1-2p)\delta u_s \text{ for } s = 2, ..., n-1.
\end{aligned}$$

Substituting $y_s = x_{n+1-s}$ per definition 9 and simplifying

$$u_s := \begin{cases} p\frac{(1-\delta)(1-x_1)^\alpha+\delta u_2}{1-(1-p)\delta}, & \text{s=1,} \\ p\frac{(1-\delta)\left((1-x_s)^\alpha+x_{n+1-s}^\alpha\right)+\delta(u_{s-1}+u_{s+1})}{1-(1-2p)\delta}, & \text{s=2,...,n-1,} \\ p\frac{(1-\delta)(1+x_1^\alpha)+\delta u_{n-1}}{1-(1-p)\delta}, & \text{s=n.} \end{cases} \tag{30}$$

3. Solve payoff equations to find payoff functions: Given parametric values for α, δ and p, and the number of states n, Mathematica can numerically solve (30). Let $\tilde{u}_s(x)$, $s = 1, ..., n$, denote the solution.

4-5. Optimization and output: Same as in the SP-model, except $\tilde{u}_s(x)$, $s = 1, ..., n$, in 4(b) comes from solving (30) instead of simulation based estimates of $u_s(x)$.

Technical details: We used Mathematica's *NSolve* command to solve the payoff equations in the CP-model and the *NMaximize* and FindMaximum commands for optimization. *FindMaximum* looks for a local optimum with an interior point algorithm while *NMaximize uses Differential Evolution* or *Nelder-Mead* simplex algorithm to solve for global optima. *Simulated Annealing* and *Random Search* algorithms were also available, but the other methods performed better. *FindMaximum* was significantly faster and more robust in many cases than the global methods, so we used it to investigate particular aspects of the global equilibria uncovered by the NMaximize. A description of each global algorithm is available in the appendix courtesy of Wolfram Research [21].

Limitations: Limiting factors to arbitrarily high numerical accuracy are time and computing power. In practice this means that characterizing asymptotic behavior of the model is not possible or reliable using numerical techniques. Closed-form solutions would of course be preferable.

3.2 Numerical analysis of n^*-EM equilibria

In this section we present a number of conjectures derived from numerical testing using the models from section 3.1. To support these conjectures we refer to a number of figures and tables interpolated from sets of parametric solutions. The rest of the section illustrates n^*-EM equilibrium behavior as p, δ and α are varied in turn.

3.2.1 Results and resulting conjectures

The following conjectures concern globally efficient multi-state EM equilibria. That is, n-EM equilibria that are optimal across the number of states (n) and favors $(x_1, x_2, ..., x_{n-1})$. In our conjectures we sometimes use the expression "for all" (in quotes) in the context of numerical results to refer to an entire parameter range that we covered at close increments rather than at every actual point. For example, if a numerical result was interpolated from solutions for $p = 0.01, 0.02, ..., 0.49$, we may refer to it as a result "for all" p.

Conjecture 17 *Let $n^*(\alpha, \delta, p)$ be the number of states associated with $\sigma^*_{em^n}$ given α, δ and p. Then the finite difference functions (discrete derivatives) for α, δ and p, respectively, are*

(i) $\Delta\alpha|_{\Delta n^*=1, \Delta\delta=\Delta p=0} < 0$ and increasing in n^* (decreasing in absolute value),
(ii) $\Delta\delta|_{\Delta n^*=1, \Delta\alpha=\Delta p=0} > 0$ and decreasing in n^* (smaller δ-steps per unit Δn^*),
(iii) $\Delta p|_{\Delta n^*=1, \Delta\alpha=\Delta\delta=0} > 0$ and approximately constant.

Support for conjecture. The conjectured relationships were observed in all our numerical tests. Please refer to figures 9, 11, and 13 for plots of $n^*(p), n^*(\delta)$ and $n^*(\alpha)$, respectively. Figure 5 shows the globally efficient number of states, n^*, as a function of δ and p.[3] We suggest further numerical tests to verify these relationships for a larger set of parameter values. Further analysis of the system of n-EM payoff equations may even yield a closed-form solution. ■

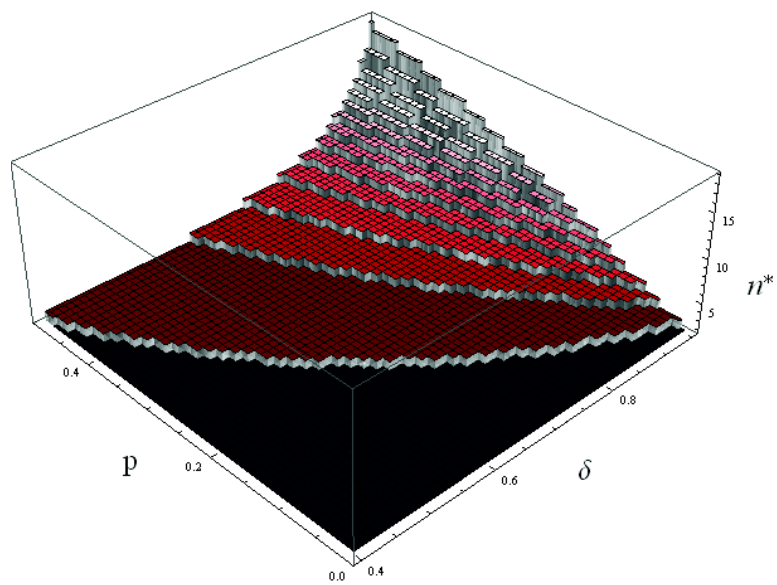

Figure 5: Globally efficient n-EM equilibria: $n^*(\delta, p)$ where $\alpha = 0.5$

[3]Each grid point in figure 5 corresponds to a Mathematica solution for given values of δ and p. We used Mathematica's interior point algorithm to find the optimal favors, $x_1^*, ..., x_n^*$, for each $n = 2, 3, ...$ until $u(x_1^*, ..., x_n^*)$ stopped increasing. n^* was chosen as the n for which $u(x_1^*, ..., x_n^*)$ peaked. The grid points consist of $\alpha = 0.5$, $\delta = 0.4, 0.41, ..., 0.95$ and $p = 0.01, 0.02, ..., 0.49$.

Conjecture 18 *Let* $x^* = \{x_1^*, x_2^*, ..., x_{n^*}^*\} = \left\{y_{n^*}^*, y_{n^*-1}^*, ..., y_1^*\right\}$ *be the set of favors associated with* $\sigma_{em^n}^*$*. Define* $\underline{s}$ *as the first state with* x_s^* *smaller than its successor* x_{s+1}^* *where* $\underline{s} = n^*$ *if none are smaller. That is,* $\underline{s}(\alpha, \delta, p) := \inf\left\{s \in \mathbb{S} : x_s^* < x_{s+1}^* \text{ where } x_{n^*+1}^* \equiv 1\right\}$. *Then given* $\delta, p\ \exists\underline{\alpha} \in (0,1)$ *such that* $\underline{s} < n^*$ *for* $\alpha < \underline{\alpha}$*, otherwise* $\underline{s} = n^*$.

Support for conjecture. The relationship was observed in all our numerical tests. Subsection 3.2.2 covers the conjecture 18 "for all" p with δ fixed when $\alpha \geq \underline{\alpha}$ (see figure 6). Subsection 3.2.3 does the same "for all" δ with p fixed (see figure 10). Subsection 3.2.4 illustrates the behavior of x^* as α varies from values below $\underline{\alpha}$ to values above it (see figure 12) and supports our conjecture that such an $\underline{\alpha}$ exists. Figure 14 illustrates inequality $\underline{s} < n^*$ in more detail. It would be natural to expect, or at least we expected, the optimal favor sequence to be decreasing so this came as a surprise. At the moment we do not attempt to explain this anomaly, but we do confirm it by solving the same model with the additional restriction that favors have to be decreasing. The resulting equilibria favors $\dot{x} = \{\dot{x}_1, \dot{x}_2, ..., \dot{x}_n\}$ are dominated by the original x^* solution; $u(x^*) > u(\dot{x})$. Subsection 3.2.5 offers a numerical analysis with figures and tables of case with $\alpha < \underline{\alpha}$. In particular, figures 15 and 16 illustrate the case "for all" p and "for all" δ, respectively, with α fixed below $\underline{\alpha}$. ■

Claim 19 *Consider* x^* *discussed in conjecture 18. For* $n^* > 2$ n^**-EM equilibria may include (i) favors that are above the socially optimal stage game level of 1/2 (full sharing), and (ii) favor sequences that are non-decreasing.*

Proof. Both findings surprised us which is why we state them above in a claim. We offer the proofs in the form of numerical examples. Please refer to table 2 for results of an n^*-EM equilibrium solution for $\alpha = 0.5$, $\delta = 0.9$ and $p = 0.3, 0.31, ..., 0.49$. Observe that all of the solutions include $x_1^* > 1/2$ and $x_2^* > 1/2$, and toward the higher p-values, $x_3^* > 1/2$. We ran the exact same parametric model was with the same values, except with an additional restriction of $x_s \leq 1/2$ (instead of $x_s \leq 1$). The results are in table 3. Both tables display expected payoff to agent a in the first column under heading u. (Recall that $T = u + v = 2u$, by symmetry). A comparison of the two tables shows that the unrestricted n^*-EM strategies generate slightly higher payoffs. Examples of non-decreasing favor sequences are available in subsection 3.2.5. ■

Conjecture 20 *For* n^* *and* $\underline{s}$ *discussed in conjectures 17 and 18,* $\underline{s}\,(n^*)$ *has approximately the same relationship to* p*,* δ *and* α *as* n^* *has to* p*,* δ *and* α*, respectively.*

Support for conjecture. When $\underline{s}\,(n^*) = n^*$, the result is trivially true. Therefore we only need to consider case (iii) of conjecture 18, that is, when $\underline{s} < n^*$. Figures 14, 15 and 16 in subsection 3.2.5 show the behavior of $\underline{s}$ when $\alpha < \underline{\alpha}$. Recall that $x_{n^*}^* = 0$ by definition of an n-EM strategy. Therefore the outer edge in the figure that forms a geometric ridge in the plots, represents second to last favor by agent a. The referenced plots show this ridge for $x_{n^*-1}^*\,(\alpha)$, $x_{n^*-1}\,(p)$ and $x_{n^*-1}^*\,(\delta)$, respectively. Along each ridge runs a "concave canal" that is formed by the set of smallest nonzero favors $x_{\underline{s}(n^*)}^*\,(\alpha)$, $x_{\underline{s}(n^*)}^*\,(p)$ and $x_{\underline{s}(n^*)}^*\,(\delta)$, respectively. It is apparent from the plots that $x_{\underline{s}(n^*)}^*$ varies in tandem with $x_{n^*-1}^*$ favors in terms of position, and hence it follows that $\frac{\Delta \underline{s}}{\Delta \alpha} \approx \frac{\Delta n^*}{\Delta \alpha}$, $\frac{\Delta \underline{s}}{\Delta p} \approx \frac{\Delta n^*}{\Delta p}$ and $\frac{\Delta \underline{s}}{\Delta \delta} \approx \frac{\Delta n^*}{\Delta \delta}$. ■

3.2.2 Efficient favors as p varies

Figure 6 was interpolated from a set of globally optimal EM favor sequences (x^*) for $\alpha = 0.5$, $\delta = 0.85$ fixed, while p was varied from $p = 0.03, 0.07, ..., 0.47$. The plot shows the globally efficient size of favors (x) across states (s) as the probability of favor opportunities varies from 0 to 0.5. Because the number of states is a discrete variable and favors are optimized across states, this causes the magnitude of each favor to increase as p increases until it becomes optimal to add another state. At this point the size of the favors in the older states tends to fall slightly. The favor levels then climb to and above their previous levels until it becomes optimal to add yet another state, and so forth. This gives the favor surface a jagged outline along the p-axis. The surface would appear even more jagged had we used a larger number of points for $p \in (0, 1/2)$. Along the s-axis favors are decreasing for the given α, however, later we show that this is not necessarily the case for α sufficiently low. The plot also demonstrates lemma 7 in that for any value of p, at least some level of equality matching dominates autarky when utility functions are α-concave.

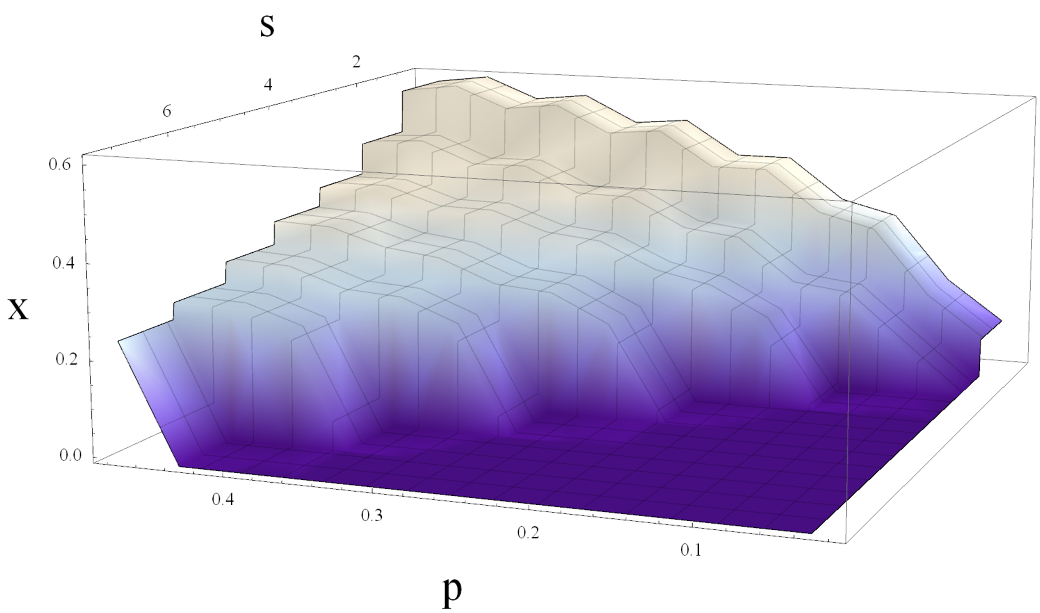

Figure 6: **n^*-EM favor sequence**: $x_1^*(p), x_2^*(p), ..., x_{n^*}^*(p)$
Algorithm: Differential evolution
Points: $\alpha = 0.5,\ \delta = 0.85,\ p = 0.03, 0.07, ..., 0.47$

The corresponding numbers for figure 6 are shown in figure 7 and can found in more detail in the appendix in tables 5 and 6, respectively.

Optimal multi-state EM favors: α = 0.5, δ = 0.85

	u	x_1	x_2	x_3	x_4	x_5	x_6	x_7
p = 0.03	0.034	0.083						
p = 0.07	0.084	0.18	0.1					
p = 0.11	0.14	0.32	0.21					
p = 0.15	0.19	0.45	0.31					
p = 0.19	0.25	0.45	0.32	0.24				
p = 0.23	0.3	0.45	0.33	0.27	0.2			
p = 0.27	0.36	0.52	0.4	0.33	0.26			
p = 0.31	0.41	0.51	0.39	0.35	0.27	0.22		
p = 0.35	0.47	0.58	0.44	0.39	0.31	0.28		
p = 0.39	0.52	0.56	0.45	0.38	0.34	0.28	0.24	
p = 0.43	0.58	0.61	0.5	0.42	0.37	0.32	0.28	
p = 0.47	0.63	0.59	0.49	0.42	0.38	0.33	0.28	0.25

Figure 7: Figure 6 data: Multi-state EM $x^*_s(p)$

The slack in incentive compatibility constraints associated with figure 6 are available in table (figure) 8. The ICC are virtually all tight suggesting the solution may represent a second best strategy profile in at least some situation or for some parameter values.

ICC for optimal multi-state EM equilibria: α = 0.5, δ = 0.85

	IC_1	IC_2	IC_3	IC_4	IC_5	IC_6	IC_7
p = 0.03	0						
p = 0.07	0	0					
p = 0.11	0	0					
p = 0.15	0	0					
p = 0.19	0	0	0.0014				
p = 0.23	0	0	0	0.0025			
p = 0.27	0	0	0	0.0023			
p = 0.31	0	0	0	0.0014	0.0033		
p = 0.35	0	0	0	0.0014	0.0029		
p = 0.39	0	0	0	0	0.0027	0.0037	
p = 0.43	0	0	0	0	0.0025	0.0033	
p = 0.47	0	0	0	0	0.0014	0.0037	0.004

Figure 8: ICC for figure 6: Multi-state EM $x^*_s(p)$

Figure 9 depicts how n^* changes with p. The step function was interpolated from a set of solutions for $\alpha = 0.5$, $\delta = 0.9$ and $p = 0.005, 0.01, ..., 0.495$. The other function is a generic linear approximation. The figure illustrates that steps in p are approximately constant.

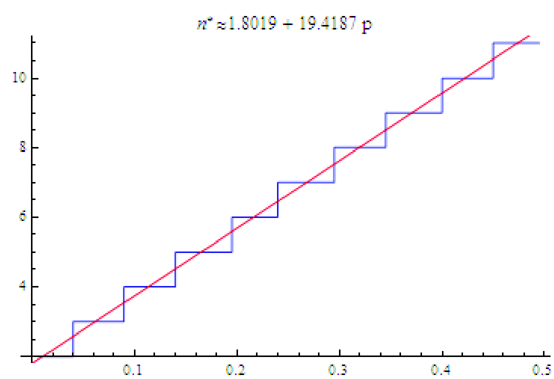

Figure 9: $n^*(p)$ for n-EM equilibria

3.2.3 Efficient favors as δ varies

Figure 10 was interpolated from the multi-state EM solution sequences (favors) for points $(\alpha, \delta, p) = (0.6, 0.37, 0.4)$, $(0.6, 0.41, 0.4)$,..., $(0.6, 0.93, 0.4)$. The regions for $\delta < 0.37$ and $\delta > 0.93$ were excluded from the plots because they would have made the plots visually less informative.[4] $x_s^*(\delta)$ displays many of the same characteristics as the corresponding plot for $x_s^*(p)$ (figure 6); $x_s^*(\delta)$ are jagged along the δ-axis because number of states is a discrete variable, and for δ fixed, $x_s^*(\delta)$ decreases along s for the given α, or for α high enough which we show later. A closer inspection of the plots and the data shows that the dependence of n^* and x_s^*, $s = 1, ..., n^*$ on p is (direct) linear, and (direct) convex on δ for α sufficiently high.

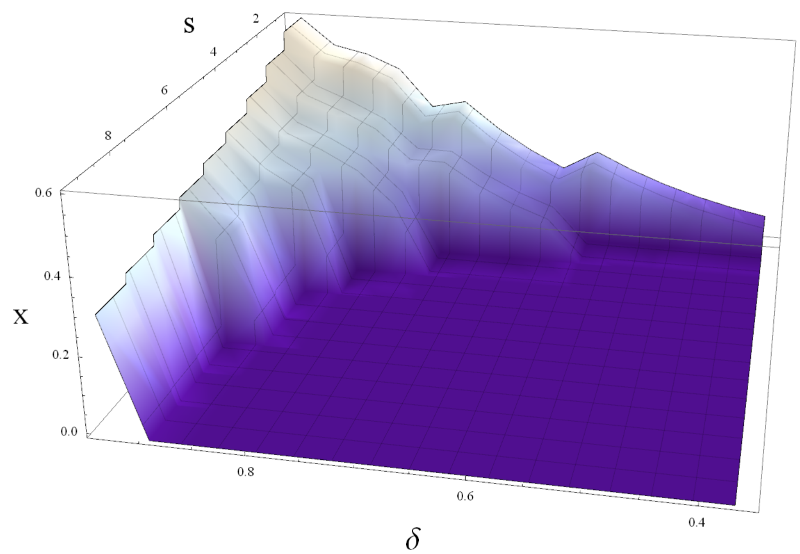

Figure 10: **$\boldsymbol{n^*}$-EM behavior w.r.t. $\boldsymbol{\delta}$:** $x_1^*(\delta), x_2^*(\delta), ..., x_{n^*}^*(\delta)$
Algorithm: Nelder-Mead simplex
Points: $\alpha = 0.6,\ p = 0.4,\ \delta = 0.37, 0.41, ..., 0.93$

The data points and ICC for figure 10 are available in the appendix tables 7 and table 8. As before the incentive compatibility constraints are all tight within round off error. Figure 11 depicts how n^* changes with δ. The step function was interpolated from a set of solutions for $\alpha = 0.5, p = 0.3$ and $\delta = 0.1, 0.11, ..., 0.9$. The other function is a generic approximation. The figure illustrates that steps in δ are decreasing.

3.2.4 Efficient favors as α varies

Figure 12 was interpolated from a set of favor sequences $x^* = \{x_1^*, x_2^*, ..., x_{n^*}^*\}$ obtained by optimizing $\sigma_{em^n}^*(x^*)$ for $\delta = 0.8$ and $p = 0.4$ fixed, while α was varied from $\alpha = 0.2, 0.24, ..., 0.8$.

[4]Solutions for $\delta < 0.37$ each consist of one tiny nonzero favor ($x_1^* < 0.1$) and subsequently would only have added flat space into our plots. The number of nonzero favors increase exponentially in as $\delta \rightarrow 1$ so including the tail end would have dominated the rest of the plot, and obscured other details.

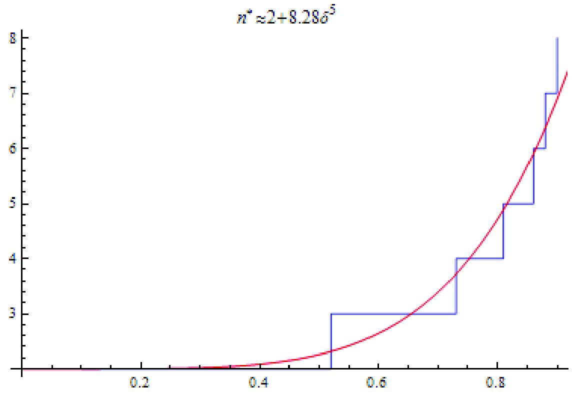

Figure 11: $n^*(\delta)$ for n-EM equilibria

Both the *differential evolution* and *Nelder-Mead* algorithms converged to the same set of global solutions, but performed poorly for smaller α. Later on we use an *interior point* algorithm to find (local) x^* solutions consistent with $\sigma^*_{em^n}$ when $\alpha < 0.2$ (high concavity case). However figure 12 already supports conjecture 18, which states that for α high enough $x^*(\alpha)$ is decreasing, but for α sufficiently low, $x^*(\alpha)$ fails to remain decreasing toward the end of the sequence. As with $x^*(p)$ in p and $x^*_s(\delta)$ in δ, figure 12 shows that $x^*(\alpha)$ is jagged in α because the number of states, s, is treated as a continues variable (for visual effect) even though $s \in \mathbb{N}$ is discrete.

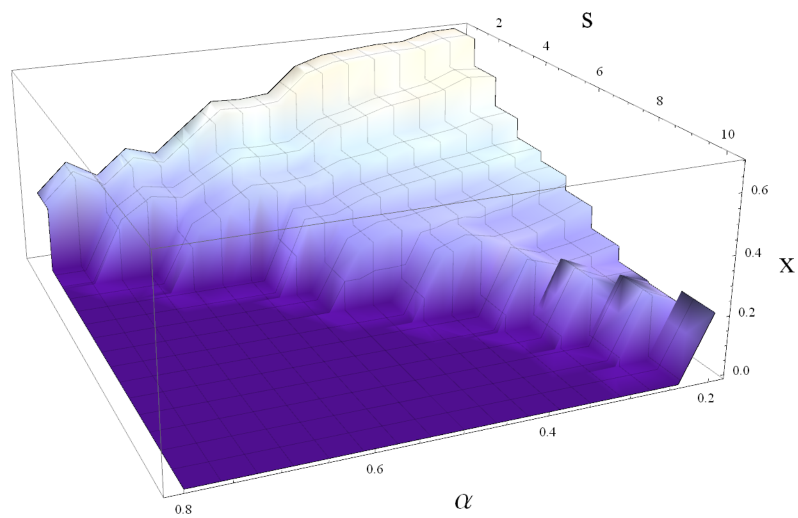

Figure 12: **n^*-EM favor sequence:** $x_1^*(\alpha), x_2^*(\alpha), ..., x_{n^*}^*(\alpha)$
Algorithm: Differential evolution
Points: $\delta = 0.8,\ p = 0.4,\ \alpha = 0.2, 0.24, ..., 0.8$

The Mathematica code used to compute the data points and to plot figure 12 is available in the appendix. The data points and associated payoffs and incentive compatibility constraints may also be found there in tables 9 and 10. The ICC are all close to zero. Figure 13 depicts how n^*

changes with α. The step function was interpolated from a set of solutions for $\delta = 0.7$, $p = 0.3$ and $\alpha = 0.1, 0.11, ..., 0.9$. The other function is a generic approximation. The figure illustrates that steps in α are increasing.

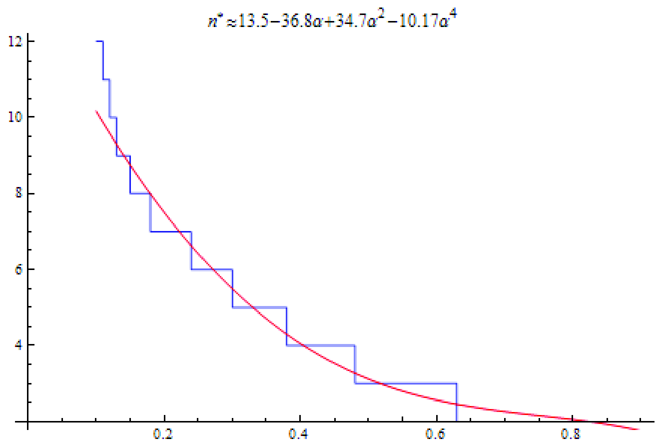

Figure 13: $n^*(\alpha)$ for n-EM equilibria

3.2.5 Analysis of special n^*-EM cases: $\alpha < \underline{\alpha}$

For small α we used Mathematica's interior point algorithm that solves for local optima, but in our case local optima are also global optima. The interior point algorithm is more stable and can handle more variables (longer favor sequences) than Mathematica's global algorithms. Therefore it lets us solve $\sigma^*_{em^n}(x^*)$ for smaller α and gives us a closer look at the behavior of $x^*(\alpha)$ sequences in low α cases. Figure 14 depicts globally efficient $x^*(\alpha)$ sequences for $\alpha < \underline{\alpha}$ that were interpolated from n^*-EM solutions for $\alpha = 0.08$, 0.09, $0.10, ..., 0.35$ (the results for $\alpha = 0.2, 0.3$ are consistent with the results obtained by differential evolution and Nelder-Mead algorithms). The n^*-EM solution data is available in table 11 in the appendix.

Of course the sudden jump in favor sizes toward the very end of the globally efficient series of favors shown in figure 14 could be an error of some sort, perhaps a short-coming in the numerical algorithms employed by Mathematica? However, it seems exceedingly unlikely that three different algorithms (Differential evolution, Nelder-Mead, and interior point) would all produce identical errors. But just to be sure we reran the experiment show in figure 14 with one exception; favors were constrained by $0 \leq x_s \leq x_{s-1}, \forall s$ where $x_0 \equiv 0$, instead of the standard feasibility constraint $0 \leq x_s \leq 1$, $\forall s$. The results are available in table 12 and figure 25 in the appendix. The plot looks identical to its counterpart with unconstrained favor sizes except for the jump in favor values at the end of favor sequences. The payoffs corresponding to the constrained favor sequences were smaller by about 0.02% on average, but more importantly the $x_s \leq x_{s-1}$ constraints bound where $x_s > x_{s-1}$ previously instead of converging to completely different fixed points.

Figure 15 shows the n^* and $x^*(p)$ associated with a globally efficient n^*-EM equilibrium as a function of $p \in (0, 1/2)$ when $\alpha < \underline{\alpha}$. The plot was interpolated from parametric solutions

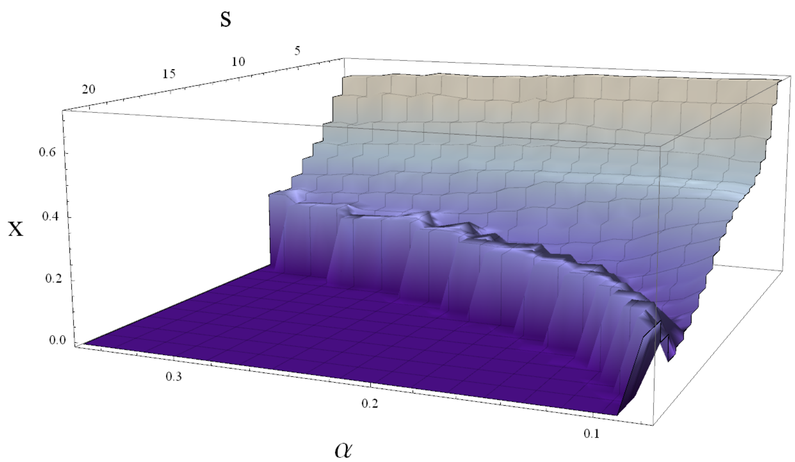

Figure 14: **n^*-EM behavior for $\alpha < \underline{\alpha}$:** $x_1^*(\alpha), x_2^*(\alpha), ..., x_{n^*}^*(\alpha)$
Algorithm: Interior point
Points: $\delta = 0.85,\ p = 0.4,\ \alpha = 0.08, 0.09, ..., 0.35$

for $\alpha = 0.2$, $\delta = 0.85$ and $p = 0.01, 0.02, ..., 0.49$. The corresponding numbers are available in the appendix in table 13. The key take-away from the figure is that while an increase in the probability of favor opportunities (p) increases the number of favors that are optimal as before, it does not eliminate or change the "canal" of minimal nonzero favors defined by $\underline{s}$ in conjecture 18 that appears in the interior of the optimal favor sequence, that is, $\underline{s} < n^*$. Furthermore, $\underline{s}\,(p)$ appears to be directly proportional to $n^*(p)$.

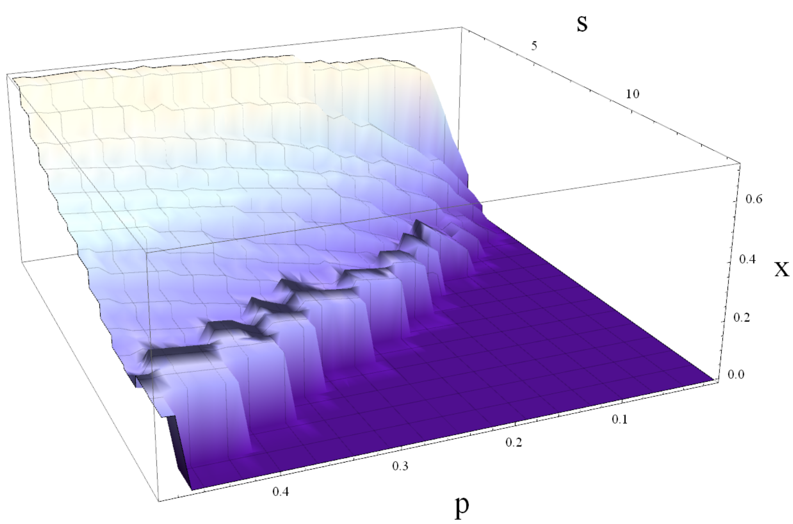

Figure 15: **n^*-EM behavior for $\alpha < \underline{\alpha}$** : $x_1^*(p), x_2^*(p), ..., x_{n^*}^*(p)$
Algorithm: Interior point
Points: $\alpha = 0.2,\ \delta = 0.85,\ p = 0.01, 0.02, ..., 0.49$

Figure 16 shows the same figure with respect to changing δ. The plot was interpolated from parametric solutions for $\alpha = 0.3$, $p = 0.3$ and $\delta = 0.6, 0.61, ..., 0.96$, and the corresponding numbers are available in the appendix in table 14. As before, the "canal" of minimal nonzero favors defined by $\underline{s}$ remains in the interior of the optimal favor sequence; $\underline{s} < n^*$ and $\underline{s}(\delta)$ appears to be directly proportional to $n^*(\delta)$.

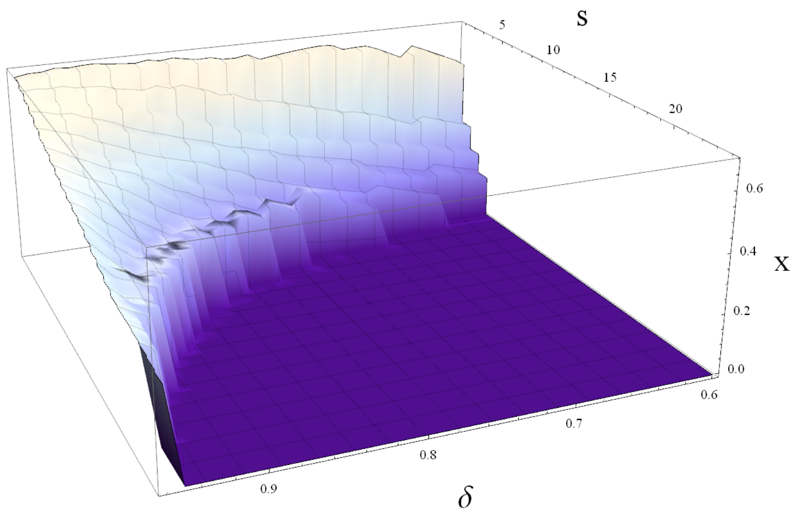

Figure 16: **n^*-EM behavior for $\alpha < \underline{\alpha}$** : $x_1^*(\delta), x_2^*(\delta), ..., x_{n^*}^*(\delta)$
Algorithm: Interior point
Points: $\alpha = 0.3,\ p = 0.4,\ \delta = 0.6, 0.61, ..., 0.96$

3.3 Partial results and conjectures

The conjectures in this subsections are more speculative than the earlier conjectures in this section.

For finite dimensional EM strategies we defined the value of a strategy profile as the sum of average discounted state payoffs for both agents weighted by the stationary probabilities of each state (definition 12). If the starting state is random, this value is also equal to the strategy profile's dynamically defined value. For infinite dimensional EM strategy profiles this is not the case as ∞-EM strategies have no stationary distribution (lemma 15). Therefore the value of σ_{em^∞} has to be computed dynamically and the only sensible starting is state 0. This makes comparisons with finite σ_{em^n} trickier, but our preliminary results suggest that ∞-EM strategies cannot be incentive compatible. If an ∞-EM equilibrium exists, it would be dominated by a truncated (finite) version of that ∞-EM strategy profile. The core intuition is that in EM equilibria the value of the game is the average value of all the states, favors are efficient (up to full sharing) so the bigger the favors, the greater the value of the game. But favors generally have to decrease to provide agents with an incentive to keep doing them. This in turn means that in a game with infinite states there would also be infinitely many low favor (low value) states. Truncating the game at some point ensures that the game is played in high value territory except for two boundary states that nonetheless map the game back into the higher favors region. Eliminating a boundary state may provide a higher return for that point in the game, but it would extend the game into states with lower values until adding another state would be inefficient.

Conjecture 21 *σ_{em^∞} are never globally efficient and may not even exist.*

We define the expected (average discounted) value in an iterative fashion then expand.

$$T\left(\sigma_{em^\infty}\right) := 2\left(1-\delta\right) v_0, \text{ where } v_s := p\frac{(1-x_s)^\alpha + (y_s)^\alpha + \delta(v_{s-1}+v_{s+1})}{1-(1-2p)\delta},\ s\in\mathbb{Z} \tag{31}$$

To make the expansion easier, let $\hat{p} \equiv \frac{p}{1-(1-2p)\delta}$, $\hat{\delta} \equiv \frac{\delta}{1-(1-2p)\delta}$, $\hat{x}_s \equiv (1-x_s)^\alpha + (y_{-s})^\alpha \equiv (1-x_s)^\alpha + (x_{-s})^\alpha$, and in these terms $v_s := \hat{p}\hat{x}_s + \hat{\delta}\left(v_{s-1}+v_{s+1}\right)$ for $s \in \mathbb{Z}$. Expanding v_0 iteratively:

$$\begin{aligned}
v_0 &= \hat{p}\hat{x}_0 + \hat{\delta}\left(v_{-1}+v_{+1}\right)\\
&= \hat{p}\hat{x}_0 + \hat{\delta}\left(\hat{p}\hat{x}_{-1} + \hat{\delta}\left(v_{-2}+v_0\right)\right) + \hat{\delta}\left(\hat{p}\hat{x}_1 + \hat{\delta}\left(v_0+v_2\right)\right)\\
&= \hat{p}\hat{x}_0 + \hat{p}\hat{\delta}\hat{x}_{-1} + \hat{p}\hat{\delta}\hat{x}_1 + \hat{\delta}^2\left(v_{-2}+2v_0+v_2\right)\\
&= \hat{p}\hat{x}_0 + \hat{p}\hat{\delta}\hat{x}_{-1} + \hat{p}\hat{\delta}\hat{x}_1 + \hat{\delta}^2\left(\hat{p}\hat{x}_{-2} + \hat{\delta}\left(v_{-3}+v_{-1}\right)\right)\\
&+ 2\hat{\delta}^2\left(\hat{p}\hat{x}_0 + \hat{\delta}\left(v_{-1}+v_{+1}\right)\right) + \hat{\delta}^2\left(\hat{p}\hat{x}_2 + \hat{\delta}\left(v_1+v_3\right)\right)\\
&= \hat{p}\hat{x}_0\left(1+2\hat{\delta}^2\right) + \hat{p}\hat{\delta}\hat{x}_{-1} + \hat{p}\hat{\delta}\hat{x}_1 + \hat{p}\hat{\delta}^2\hat{x}_{-2} + \hat{p}\hat{\delta}^2\hat{x}_2 + \hat{\delta}^3\left(v_{-3}+3v_{-1}+3v_{+1}+v_3\right)\\
&= \ldots
\end{aligned}$$

To further condense the expansion, let $\check{x}_0 := \hat{x}_0$, $\check{x}_j := \hat{x}_j + \hat{x}_{-j}$ for $t \in \mathbb{N}$. Then

$$\begin{aligned}
v_0 &= \hat{p}\left(1+2\hat{\delta}^2\right)\check{x}_0 + \hat{p}\hat{\delta}\check{x}_1 + \hat{p}\hat{\delta}^2\check{x}_2 + \hat{\delta}^3\left(v_{-3}+3v_{-1}+3v_{+1}+v_3\right)\\
&= \hat{p}\left(1+2\hat{\delta}^2+6\hat{\delta}^4+20\hat{\delta}^6+\ldots\right)\check{x}_0 + \hat{p}\left(\hat{\delta}+3\hat{\delta}^3+10\hat{\delta}^5+35\hat{\delta}^7+\ldots\right)\check{x}_1\\
&+ \hat{p}\left(\hat{\delta}^2+4\hat{\delta}^4+15\hat{\delta}^6+56\hat{\delta}^8+\ldots\right)\check{x}_2 + \hat{p}\left(\hat{\delta}^3+5\hat{\delta}^5+21\hat{\delta}^7+84\hat{\delta}^9+\ldots\right)\check{x}_3 + \ldots
\end{aligned}$$

The $\check{x}_j$ coefficients, say $\check{C}_j$, consist of discounted sums of binomials. Taking their limits shows that $\check{C}_j$ have ${}_2F_1$ Hypergeometric[5] form

$$\begin{aligned}
\check{C}_j &= \sum_{i=0}^{\infty}\hat{p}\binom{2i+j}{i+j}\hat{\delta}^{2i+j} = \hat{p}\hat{\delta}^j{}_2F_1\left(\tfrac{1+j}{2},\tfrac{2+j}{2};1+j;4\hat{\delta}^2\right)\\
\implies \check{C}_0 &= \tfrac{\hat{p}}{\sqrt{1-4\hat{\delta}^2}}, \check{C}_1 = \hat{p}\tfrac{1-\sqrt{1-4\hat{\delta}^2}}{2\hat{\delta}\sqrt{1-4\hat{\delta}^2}}, \check{C}_2 = \hat{p}\tfrac{1-2\hat{\delta}^2-\sqrt{1-4\hat{\delta}^2}}{2\hat{\delta}^2\sqrt{1-4\hat{\delta}^2}}, \ldots\\
\therefore T\left(\sigma_{em^\infty}\right) &= 2\left(1-\delta\right)\hat{p}\sum_{j=0}^{\infty}{}_2F_1\left(\tfrac{1+j}{2},\tfrac{2+j}{2};1+j;4\hat{\delta}^2\right)\check{x}_j
\end{aligned}$$

[5] "The" Hypergeometric function is defined as ${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty}(a)_k(b)_k/(c)_k\ z^k/k!$

$$= \frac{2(1-\delta)p}{1-(1-2p)\delta} \frac{1}{\sqrt{1-\frac{4\delta^2}{(1-(1-2p)\delta)^2}}} (1-x_0)^\alpha + (x_0)^\alpha$$
$$+ \frac{4(1-\delta)p}{1-(1-2p)\delta} \sum_{j=1}^{\infty} {}_2F_1\left(\frac{1+j}{2}, \frac{2+j}{2}; 1+j; \frac{4\delta^2}{(1-(1-2p)\delta)^2}\right) ((1-x_j)^\alpha + (x_{-j})^\alpha)$$

The hypergeometric function is well-suited for numerical and analytical work but we leave further analysis of $T(\sigma_{em^\infty})$ for the future.

Speculation supporting conjecture 21. Suppose an ∞-EM equilibrium exists. We assume the values of the associated favors would have to be decreasing for both theoretical and empirical reasons. In theory we expect favors to decrease to provide agents with an incentive to keep doing them (do a big favor today in return for the promise of bigger reciprocal favors tomorrow). Empirically (numerical testing) we found that favors were decreasing, except in special cases that involved a jump toward the end of the favor sequence. If the sequence is infinite it has no end (to state the obvious), so there would be no reason to expect any jump either. Furthermore, our numerical algorithm worked by optimizing the value of the game for n states, comparing the value to the $n-1$ state value, and adding another state and repeating if the n state value was higher than the $n-1$ state value. The solutions were of course finite, but moreover they were well-behaved in terms of the optimal number of states (see figure 5), and show almost surely that the number of states does not diverge to infinity within the interior of the parameter space. That said, numerical analysis is not well-suited for analysis of asymptotic behavior we so cannot rule out anomalous behavior as $\alpha \to 1$, $\delta \to 1$ or $p \to 1/2$. ■

Conjecture 22 *The globally efficient n^*-EM equilibrium exists and is unique up to a measure zero in parameter space. That is, there may be pairs of points (zero measure) in the parameter space for which the n^* and n^*+1 state values of the game are equal.*

Speculation supporting conjecture 22. Our numerical tests always produced unique n^*-EM outcomes, however, the parameters live on continuous intervals while the optimal number of states is discrete. Let $\tilde{n}$ be the efficient number of states for a triplet of values $\left(\tilde{\alpha}, \tilde{\delta}, \tilde{p}\right)$. Suppose we decrease α, or increase δ or p continuously while the globally efficient multi-state solution is updated until $n^* = \tilde{n}$ turns to $n^* = \tilde{n}+1$. At that point $T\left(\sigma^*_{em^{\tilde{n}}}\right) = T\left(\sigma^*_{em^{\tilde{n}+1}}\right)$, so by definition 16 both are globally efficient multi-state equilibria. For any integer pair $(n, n+1)$ with $n \geq 2$, we should be able to create three distinct pairs of such non-unique EM equilibria (one pair for each parameter), but they would have measure zero mass. It might be possible to use a similar technique to create lines or planes of n-EM intersections by increasing one parameter while lowering another subject to $T\left(\sigma^*_{em^n}\right) = T\left(\sigma^*_{em^{n+1}}\right)$, however there is no reason to suspect that $T\left(\sigma^*_{em^n}\right)$ would remain constant. ■

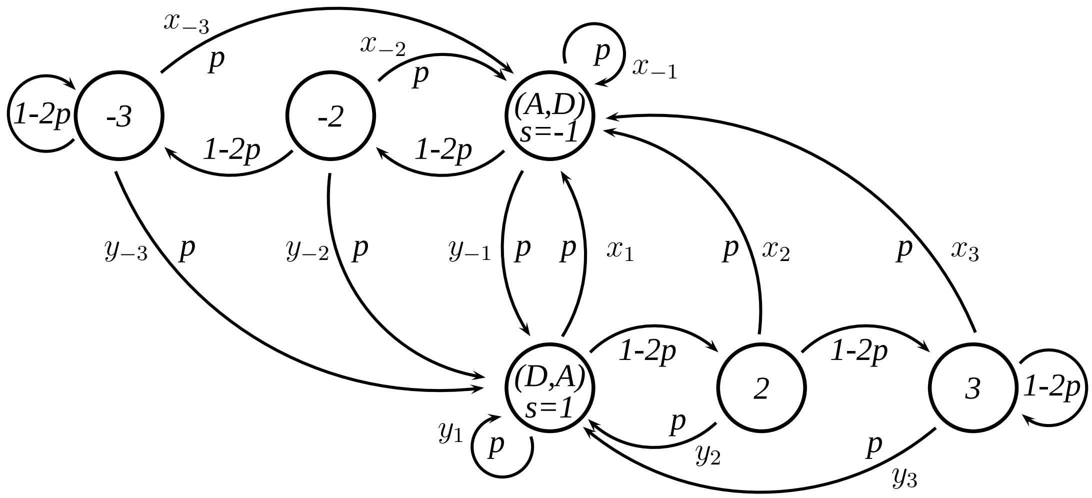

Figure 17: Automata for 6-state pseudo-HSSGL strategy

4 Pseudo-highest symmetric self-generating line (HSSGL) equilibria

In this section we discuss an analog of AB's [1] highest symmetric self-generating line (HSSGL) equilibrium concept for favor-trading with concave utility functions. Our version of HSSGL called *pseudo-HSSGL* involves finite number of states, whereas AB used infinite states, but in the interior both have the same automata representation (see figure 17). Another difference between AB's HSSGL and our *pseudo-HSSGL* is that we do not limit our analysis onto lines (symmetric or otherwise). We call a pseudo-HSSGL strategy with n states an ***n-state pseudo-HSSGL (n-HSSGL)*** strategy, and denote it with σ_{hssgl^n}.

4.1 Globally efficient n-HSSGL equilibria (n^*-HSSGL)

We use the same approach as with n^*-EM strategies to numerically optimize problem (21). As before, it is enough to solve the optimization problem for agent a because we impose symmetry on it by replacing v_s and y_s with u_{-s} and x_{-s}, respectively, where the state space is $\mathbb{S} \equiv \{-n, ..., -1, 1, ..., n\}$. The objective is simply $u(x) = \frac{u_{-1}(x)+u_1(x)}{2}$ because the HSSGL game starts from either state 1 (agent b fully advantaged) or state -1 (agent a fully advantaged). Individual payoffs are described by equation (32) and incentive compatibility constraints by equation (33).

$$u_s := \begin{cases} p\left((1-\delta)(1-x_{-n})^{\alpha}+\delta u_{-1}\right) \\ \quad +p\left((1-\delta)(y_{-n})^{\alpha}+\delta u_1\right)+(1-2p)\delta u_{-n} & \text{s=-n} \\ p\left((1-\delta)(1-x_s)^{\alpha}+\delta u_{-1}\right) \\ \quad +p\left((1-\delta)(y_s)^{\alpha}+\delta u_1\right)+(1-2p)\delta u_{s-1} & \text{s=1-n,...,-1,} \\ p\left((1-\delta)(1-x_s)^{\alpha}+\delta u_{-1}\right) \\ \quad +p\left((1-\delta)(y_s)^{\alpha}+\delta u_1\right)+(1-2p)\delta u_{s+1} & \text{s=1,...,n-1,} \\ p\left((1-\delta)(1-x_n)^{\alpha}+\delta u_{-1}\right) \\ \quad +p\left((1-\delta)(y_n)^{\alpha}+\delta u_1\right)+(1-2p)\delta u_n & \text{s=n.} \end{cases} \tag{32}$$

$$ICC_s(x) := \begin{cases} \mathrm{u}_{-1}-u_{-n}-\frac{1-\delta}{\delta}\left(1-(1-x_{-n})^{\alpha}\right) & \text{s=-n,} \\ \mathrm{u}_{-1}-u_{s-1}-\frac{1-\delta}{\delta}\left(1-(1-x_s)^{\alpha}\right) & \text{s=1-n,...,-1,} \\ \mathrm{u}_{-1}-u_{s+1}-\frac{1-\delta}{\delta}\left(1-(1-x_s)^{\alpha}\right) & \text{s=1,...,n-1,} \\ \mathrm{u}_{-1}-u_n-\frac{1-\delta}{\delta}\left(1-(1-x_n)^{\alpha}\right) & \text{s=n.} \end{cases} \tag{33}$$

The problem is computationally harder than optimizing n-EM strategies, so we used the CP-model with interior point algorithm and applied it to well-behaved regions of the parameter space.

4.2 Numerical analysis of n^*-HSSGL equilibria

For low values of p, our computed n^*-HSSGL results are approximately consistent with AB's original HSSGL equilibria. That is, the size of the "small" favor owed (interest payment) by the advantaged agent grows as periods of no favors pass, while the "large" favor owed by the disadvantaged agent decreases. However, for larger p the optimal n^*-HSSGL strategy appears to converge to four states: large favors in the inner states, and small favors in the outer states.

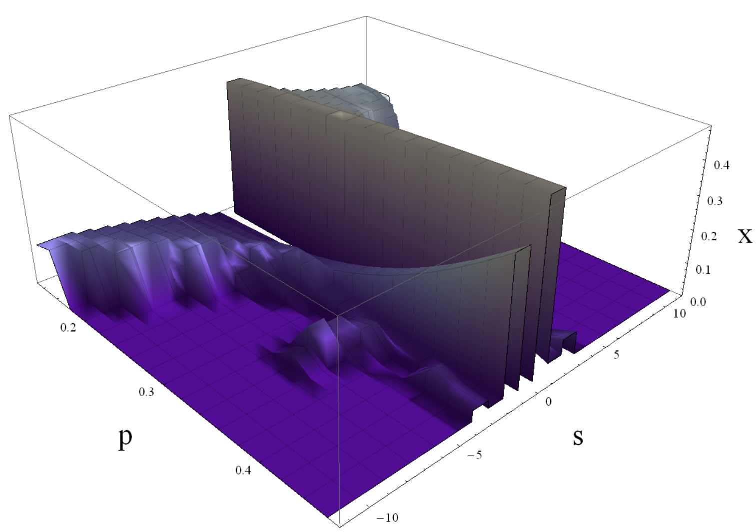

Figure 18: **n^*-HSSGL favors as p varies: View 1**
Points: $\alpha = 0.3,\ \delta = 0.9,\ p = 0.14, 0.16, ..., 0.46$

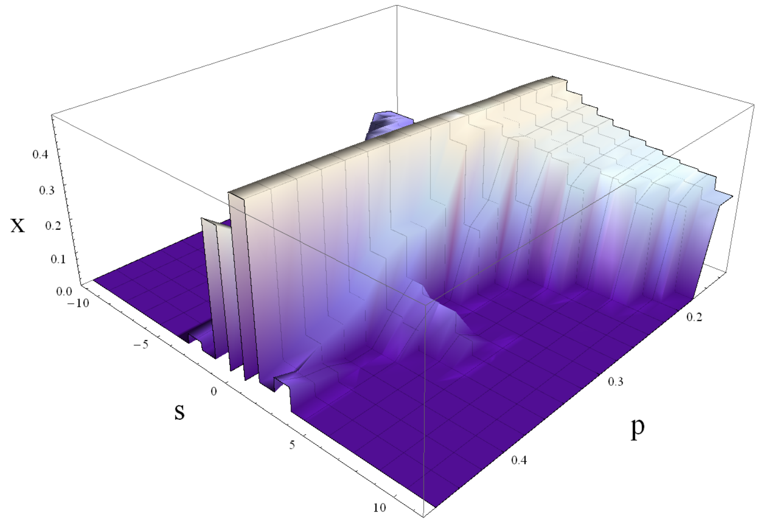

Figure 19: **n^*-HSSGL favors as p varies: View 2**
Points: $\alpha = 0.3,\ \delta = 0.9,\ p = 0.14, 0.16, ..., 0.46$

For the parameter values used to construct figures 18 and 19, the n^*-HSSGL equilibria produce higher payoffs than n^*-EM equilibria for $p = 0.42, 0.44, 0.46$ and lower payoffs otherwise. If we lower δ by 0.1, n^*-HSSGL dominates n^*-EM for all p in our sample. If we lower α by 0.1 n^*-HSSGL dominates n^*-EM for $p = 0.14, 0.16$ and $p \geq 0.34$. However, in each case the n^*-HSSGL dominance increases as p increases, and is insignificant (within margin of error) for low p values.

Figures 26-29 in the appendix illustrate n^*-HSSGL equilibrium behavior as δ and α vary per our CP-model and the (local) interior point algorithm. However, even these numerical solutions were difficult to find in terms of error free parameter space coverage and starting points for the interior point algorithm.

For the δ variation analysis we used $\delta = 0.65$, 0.67,..., 0.95 while $\alpha = p = 0.3$. All n^*-HSSGL equilibria involved 8 or fewer states without strong patterns. However, we also ran the same optimization exercise using n^*-EM strategies, and it turned out that $T\left(\sigma^*_{em^n}\right) > T\left(\sigma^*_{hssgl^n}\right)$ for $\delta < 0.8$ and n^*-HSSGL in turn outperformed n^*-EM increasingly for $\delta > 0.8$.

To investigate n^*-HSSGL responsiveness to changes in concavity, we used $\alpha = 0.1$, $0.12, ...,$ 0.74 while $\delta = 0.9$ and $p = 0.3$. As the game became more linear (α approached 1) the optimal n^*-HSSGL solution included a greater number of states and began to resemble AB's HSSGL strategy. This suggests that pure-form symmetric line equilibria are optimal in the linear model (for some parameter values), but not in the concave model. In comparison to n^*-EM, n^*-HSSGL produced higher payoffs for low α values and lower payoffs for high α.

5 Equilibria with favor-depreciation (FD equilibria)

Consider a strategy that involves a small probability $q \in [0, 1/2]$ that if the disadvantaged agent does not do a favor in the boundary state, the game moves inwards by one state. The idea is to choose q high enough that the agents would do a small favor at the boundary state to avoid the chance of punishment in the form of having to do a positive (bigger) favor next period. We call this mechanism *favor-depreciation.* It is our initial attempt to design equilibria similar to HSSGL equilibria by AB [1] for a multi-state environment, or equilibria in continuous time by Hauser and Hopenhayn [8] for a discrete time environment.

In numerical testing, globally efficient (to use the EM definition 16) *equilibria with favor-depreciation (FD equilibria)* were only marginally better than corresponding n^*-EM equilibria. When the maximum number of states was bound exogenously, FD equilibria performed somewhat better against their locally efficient n-EM counterparts. Figures 20 and 21 show the results of FD equilibria bounded by a maximum of 4 states with $\alpha = 0.1$, $\delta = 0.5$ and $p = 0.07, 0.11, ..., 0.47$. The corresponding results for locally efficient n-EM equilibria are in figures 22 and 23. Column $\%(FD - EM)$ in figure 23 shows the percentage difference between an example set of 4-state FD and EM equilibria payoffs. This was consistent with our expectation since favor-depreciation strategies remove some of the inefficiency involved with boundary states, however we were not expecting as much of a difference for low values of p when the constraint on the number of states is not even binding. The payoff difference starts at 5.4% for $p = 0.07$ and steadily increases to 9% as the probability of favor opportunities is increased to $p = 0.47$. Perhaps the lack of a greater difference is due to a fundamental inefficiency in FD strategies; the (fully) disadvantaged agent is punished with probability q even when she does not do a favor simply because she did not receive a favor opportunity. In our numerical tests, estimates of optimal q were very small for small p (around 0.01) but grew larger (to 0.066) as p increased. The larger p is, the higher the number of efficient states is, and hence the greater the effect of the bound on states.

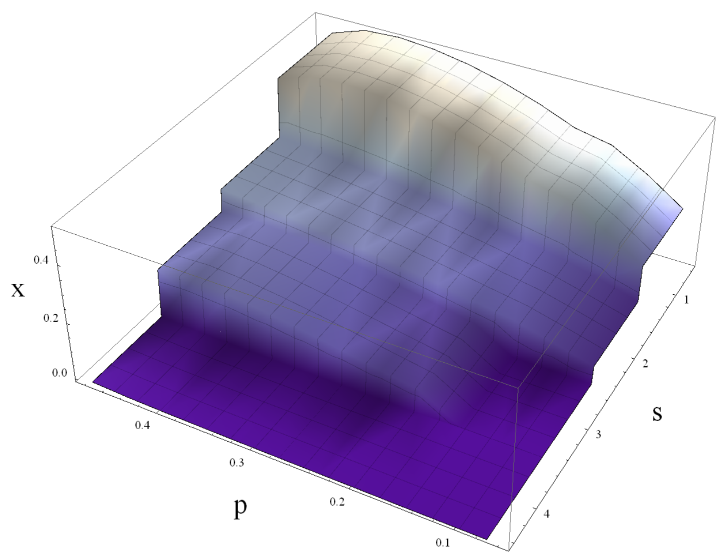

Figure 20: **4-state FD equilibria:** $x_1^*(p), x_2^*(p), ..., x_{n^*}^*(p)$
Algorithm: Interior point
Points: $\alpha = 0.1,\ \delta = 0.5,\ p = 0.07, 0.11, ..., 0.47$

4-state FD equilibria: α = 0.1, δ = 0.5

	u	q	x_1	x_2	x_3	x_4
p = 0.07	0.11675	0.0049581	0.18511	0.06027	0.00030721	
p = 0.11	0.18652	0.00618	0.2703	0.09523	0.0006152	
p = 0.15	0.25705	0.0076243	0.33926	0.12809	0.0010375	
p = 0.19	0.32845	0.0081084	0.36376	0.14902	0.077142	0.00092103
p = 0.23	0.40036	0.0096864	0.41426	0.17136	0.099628	0.0013254
p = 0.27	0.47268	0.011788	0.45508	0.19129	0.12237	0.001866
p = 0.31	0.54537	0.014685	0.48633	0.20905	0.1442	0.0026077
p = 0.35	0.61847	0.018881	0.50755	0.22476	0.16376	0.0036657
p = 0.39	0.6921	0.025466	0.51711	0.23847	0.17925	0.0052718
p = 0.43	0.76651	0.037332	0.51011	0.25028	0.18777	0.0079891
p = 0.47	0.84237	0.066094	0.46958	0.26088	0.18248	0.013757

Figure 21: 4-state FD equilibria: Data for figure 20

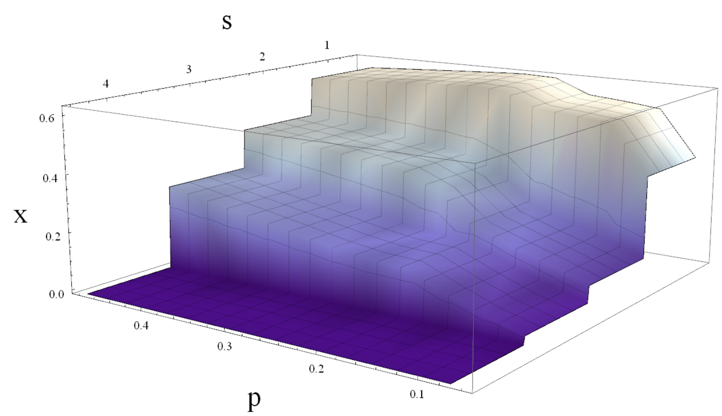

Figure 22: **4-EM equilibria:** $x_1^*(p), x_2^*(p), ..., x_{n^*}^*(p)$
Algorithm: Interior point
Points: $\alpha = 0.1,\ \delta = 0.5,\ p = 0.07, 0.11, ..., 0.47$

Locally efficient 4-EM equilibria: α = 0.1, δ = 0.5

	u	x_1	x_2	x_3	% (FD-EM)
p = 0.07	0.11074	0.37321	0.084913	0.027313	5.428
p = 0.11	0.1765	0.54119	0.12934	0.062631	5.6748
p = 0.15	0.24279	0.55325	0.16898	0.11009	5.875
p = 0.19	0.30944	0.56342	0.2117	0.16449	6.1427
p = 0.23	0.37574	0.61615	0.31745	0.16981	6.5529
p = 0.27	0.44185	0.62012	0.36847	0.18567	6.9763
p = 0.31	0.50801	0.6138	0.38653	0.20814	7.3534
p = 0.35	0.57422	0.60703	0.40253	0.23036	7.7059
p = 0.39	0.64048	0.59996	0.4166	0.25228	8.0591
p = 0.43	0.70676	0.59267	0.42889	0.27385	8.4538
p = 0.47	0.77305	0.58527	0.43959	0.29497	8.9667

Figure 23: 4-EM equilibria: Data for figure 22

5.1 Remaining questions

Other potential equilibria to investigate include a EM-HSSGL hybrids and equilibria involving punishment phases. For example, if no one does a favor, a strategy profile could specify that neither agent do a favor next period either as punishment. The infinite dimensional EM strategies also require more work as we did not have time to apply the expression for $T\left(\sigma_{em^{\infty}}\right)$ that we derived. For finite dimensional strategies, we believe we could generate a full set of data in all dimensions, fit an approximating curve to it, and use the result for comparative statistics and other analysis. It may be possible to analyze the system of simultaneous n-EM payoff equations implicitly, for example, by the use of perturbation analysis.

6 Conclusion

By choosing a concave utility function that is arbitrarily close to a linear utility line, we can obtain equilibria and outcomes that are arbitrarily close to the linear case of favor-trading. Therefore we focused on the family of α-concave functions to emphasize the impact that sufficient concavity can have on favor-trading games. In particular, favor-trading becomes possible for all δ at some level and multi-state strategies become more valuable. For example, when discount factors are just high enough to equality match full favors doing so maximizes expected utility in the linear case within a large class of incentive compatible equilibria, whereas in the concave case we can do better by lowering the favor size and using the generated slack in the incentive compatibility constraints to enforce a second smaller consecutive favor.

In the rest of the paper, we generalized AB's [1] equality matching to multiple states or alternatively, we generalized Möbius' "chips mechanism" to divisible chips. We defined multi-state equality matching for equilibria that were locally efficient for a given number of states, and then for equilibria that were globally efficient across any number of states. We also defined infinite state equality matching strategies, but argued that they either were not incentive compatible or that they would be dominated by finite state equality matching equilibria. We constructed two parametric models to numerically analyze globally efficient multi-state equilibria. The first model simulates a large sample of games, derives payoff functions from the simulations, and finds the number and size of favors that would be optimal for the constructed payoff functions. The second model solves the system of simultaneous payoff equations associated with an equality matching game directly, and uses the results to find the optimal favor sequence given a general strategy profile such as multi-state equality matching or pseudo-highest symmetric self-generating line strategies. We used these models to compute sets of solutions that spanned the parameter space and then interpolated general equilibria characteristics from those results. We further constructed a version of AB's highest symmetric self-generating lines equilibria that followed the same automata except in the border states and was not restricted to lines or infinite number of states. And we also extended the multi-state equality matching model to a class of equilibria that involve favor-depreciation and that dominate globally efficient multi-state equilibria. All strategies were analyzed using our parametric models, but not in as much depth as multi-state equality matching strategies. Further research is needed to find closed-form solutions to these various strategy profiles and to investigate hybrid and other strategies for favor-trading with concave utility functions.

7 Appendix

7.1 Proofs

Proof. (Lemma 7: EM equilibrium always exists)

Consider a simple (2-state) EM strategy profile σ_{em} consisting of favors $x = y = \varepsilon$, and payoffs $\underline{u}_\varepsilon$ and $\bar{u}_\varepsilon$ for the disadvantaged and advantaged agents, respectively. We use a direct proof to show that given any α, δ and p, there exists an ε small enough that the incentive compatibility constraint for simple EM is satisfied.

The payoffs are

$$\begin{aligned}\underline{u}_\varepsilon &= p\left((1-\delta)(1-\varepsilon)^\alpha + \delta\bar{u}_\varepsilon\right) + (1-p)\delta\underline{u}_\varepsilon \\ &= p\frac{(1-\delta)\left(1-\varepsilon\right)^\alpha + \delta\bar{u}_\varepsilon}{1-\delta(1-p)} \\ \bar{u}_\varepsilon &= p\left((1-\delta) + \delta\bar{u}_\varepsilon\right) + p\left((1-\delta)\varepsilon^\alpha + \delta\underline{u}_\varepsilon\right) + (1-2p)\delta\bar{u}_\varepsilon \\ &= p\frac{(1-\delta)\left(1+\varepsilon^\alpha\right) + \delta\underline{u}_\varepsilon}{1-\delta(1-p)}\end{aligned}$$

To obtain explicit equations for $\underline{u}_\varepsilon$ and $\bar{u}_\varepsilon$ the solve the two equations above in two unknowns,

$$\begin{aligned}\underline{u}_\varepsilon &= p\frac{(1-\delta)(1-\varepsilon)^\alpha + p\delta\left(1+(1-\varepsilon)^\alpha+\varepsilon^\alpha\right)}{1-(1-2p)\delta} \\ \bar{u}_\varepsilon &= p\frac{(1-\delta)\left(1+\varepsilon^\alpha\right) + p\delta\left(1+(1-\varepsilon)^\alpha+\varepsilon^\alpha\right)}{1-(1-2p)\delta} \\ \implies \bar{u}_\varepsilon - \underline{u}_\varepsilon &= \frac{p(1-\delta)\left(1-(1-\varepsilon)^\alpha+\varepsilon^\alpha\right)}{1-(1-2p)\delta}\end{aligned} \tag{34}$$

The incentive compatibility constraint is

$$\begin{aligned}(1-\delta)\left(1-\varepsilon\right)^\alpha + \delta\bar{u}_\varepsilon &\geq 1-\delta+\delta\underline{u}_\varepsilon \\ \implies \delta &\geq \frac{1-(1-\varepsilon)^\alpha}{1-(1-\varepsilon)^\alpha+\bar{u}_\varepsilon - \underline{u}_\varepsilon}\end{aligned}$$

We substitute for $\bar{u}_\varepsilon - \underline{u}_\varepsilon$ from equation 34,

$$\delta \geq \frac{1-(1-\varepsilon)^\alpha}{1-(1-\varepsilon)^\alpha + \frac{p(1-\delta)(1-(1-\varepsilon)^\alpha+\varepsilon^\alpha)}{1-(1-2p)\delta}}$$

and solve for δ (here we skip several steps of straightforward simplification),

$$\delta \geq \frac{1}{1-p+p\frac{\varepsilon^\alpha}{1-(1-\varepsilon)^\alpha}} \to 0 \text{ as } \varepsilon \to 0. \tag{35}$$

because $\alpha \in (0,1)$ and $p \in (0,1/2)$ are fixed and $\lim_{\varepsilon \to 0} \frac{\varepsilon^\alpha}{1-(1-\varepsilon)^\alpha} = \infty$. And since $u(\varepsilon) = \varepsilon^\alpha$ is continuous for $\alpha > 0$ and $\varepsilon > 0$, we can always find an ε sufficiently small that inequality 35 is satisfied. ■

Proof. (Lemma 13: $\delta > \delta_\alpha \implies \exists \sigma'_{em^3}$ such that $T\left(\sigma'_{em^3}\right) > T\left(\sigma_{em^2}\right)$)

We first have to solve system of 3-state EM payoff equations when $\delta = \delta_\alpha$:

$$\begin{aligned}
u_3 &= p\left((1-\delta_\alpha) + \delta_\alpha u_3\right) + p\left((1-\delta_\alpha) y_1^\alpha + \delta_\alpha u_2\right) + (1-2p)\delta_\alpha u_3, \\
u_2 &= p\left((1-\delta_\alpha)(1-x_2)^\alpha + \delta_\alpha u_3\right) + p\left((1-\delta_\alpha) y_2^\alpha + \delta_\alpha u_1\right) + (1-2p)\delta_\alpha u_2, \\
u_1 &= p\left((1-\delta_\alpha)(1-x_1)^\alpha + \delta_\alpha u_2\right) + (1-p)\delta_\alpha u_1.
\end{aligned}$$

Recall that $\delta_\alpha = \frac{2^\alpha-1}{2^\alpha-1+p(2-2^\alpha)}$ and by definition of multi-state EM strategies $y_1 = x_2$ and $y_2 = x_1$:

$$\begin{aligned}
u_3 = {} & p\left(\left(1-\tfrac{2^\alpha-1}{2^\alpha-1+p(2-2^\alpha)}\right) + \tfrac{2^\alpha-1}{2^\alpha-1+p(2-2^\alpha)} u_3\right) \\
& + p\left(\left(1-\tfrac{2^\alpha-1}{2^\alpha-1+p(2-2^\alpha)}\right) x_2^\alpha + \tfrac{2^\alpha-1}{2^\alpha-1+p(2-2^\alpha)} u_2\right) \\
& + (1-2p)\tfrac{2^\alpha-1}{2^\alpha-1+p(2-2^\alpha)} u_3, \\
u_2 = {} & p\left(\left(1-\tfrac{2^\alpha-1}{2^\alpha-1+p(2-2^\alpha)}\right)(1-x_2)^\alpha + \tfrac{2^\alpha-1}{2^\alpha-1+p(2-2^\alpha)} u_3\right) \\
& + p\left(\left(1-\tfrac{2^\alpha-1}{2^\alpha-1+p(2-2^\alpha)}\right) x_1^\alpha + \tfrac{2^\alpha-1}{2^\alpha-1+p(2-2^\alpha)} u_1\right) \\
& + (1-2p)\tfrac{2^\alpha-1}{2^\alpha-1+p(2-2^\alpha)} u_2, \\
u_1 = {} & p\left(\left(1-\tfrac{2^\alpha-1}{2^\alpha-1+p(2-2^\alpha)}\right)(1-x_1)^\alpha + \tfrac{2^\alpha-1}{2^\alpha-1+p(2-2^\alpha)} u_2\right) \\
& + (1-p)\tfrac{2^\alpha-1}{2^\alpha-1+p(2-2^\alpha)} u_1.
\end{aligned}$$

Solving

$$\begin{aligned}
u_3 &= p\tfrac{-1+32^\alpha-4^\alpha+(4-4x_1)^\alpha+(1-x_1)^\alpha-2^{1+\alpha}(1-x_1)^\alpha-(1-32^\alpha+4^\alpha)x_1^\alpha+(2-2x_2)^\alpha-(1-x_2)^\alpha+(2^\alpha-1)x_2^\alpha}{2^{1+\alpha}-1}, \\
u_2 &= p\tfrac{-1+2^\alpha+(2-2x_1)^\alpha-(1-x_1)^\alpha+(2^\alpha-1)x_1^\alpha+(1-x_2)^\alpha+x_2^\alpha}{2^{1+\alpha}-1}, \\
u_1 &= p\tfrac{1-2^{1+\alpha}+4^\alpha-(4-4x_1)^\alpha+3(2-2x_1)^\alpha-(1-x_1)^\alpha+(2^\alpha-1)^2x_1^\alpha+(2-2x_2)^\alpha-(1-x_2)^\alpha+(2^\alpha-1)x_2^\alpha}{2^{1+\alpha}-1}.
\end{aligned}$$

And taking the differences and the sum average, and simplifying

$$\begin{aligned}
u_3 - u_2 &= p\left(2-2^\alpha\right)\tfrac{2^\alpha-(2-2x_1)^\alpha+(1-x_1)^\alpha+2^\alpha x_1^\alpha-(1-x_2)^\alpha-x_2^\alpha}{2^{1+\alpha}-1} \\
u_2 - u_1 &= p\left(2-2^\alpha\right)\tfrac{2^\alpha-1-(2-2x_1)^\alpha+(2^\alpha-1)x_1^\alpha+(1-x_2)^\alpha+x_2^\alpha}{2^{1+\alpha}-1} \\
T\left(\sigma'_{em^3}\right) &= \tfrac{2}{3}\left(u_3+u_2+u_1\right) \\
&= \tfrac{2}{3}p\left(1+(1-x_1)^\alpha+x_1^\alpha+(1-x_2)^\alpha+x_2^\alpha\right)
\end{aligned}$$

The incentive compatibility constraints for σ'_{em^3} are

$$\begin{array}{l} \left(1-\delta_\alpha\right)\left(1-x_1\right)^\alpha+\delta_\alpha\, u_2 \geq 1-\delta_\alpha+\delta_\alpha\, u_1 \\ \Longrightarrow \frac{\delta_\alpha}{1-\delta_\alpha}\left(u_2-u_1\right) \geq 1-\left(1-x_1\right)^\alpha \\ \left(1-\delta_\alpha\right)\left(1-x_2\right)^\alpha+\delta_\alpha\, u_3 \geq 1-\delta+\delta_\alpha\, u_2 \\ \Longrightarrow \frac{\delta_\alpha}{1-\delta_\alpha}\left(u_3-u_2\right) \geq 1-\left(1-x_2\right)^\alpha \end{array}$$

Suppose the ICC inequalities bind. Substitute $\frac{\delta_\alpha}{1-\delta_\alpha}=\frac{1}{p}\frac{2^\alpha-1}{2-2^\alpha}$ and the values for u_3-u_2 and u_2-u_1 into the ICC and simplify

$$\begin{array}{l} \frac{(2^\alpha-1)\left(2^\alpha-(2-2x_1)^\alpha+(1-x_1)^\alpha+2^\alpha x_1^\alpha-(1-x_2)^\alpha-x_2^\alpha\right)}{2^{1+\alpha}-1}=1-\left(1-x_1\right)^\alpha \\ \frac{(2^\alpha-1)\left(2^\alpha-(2-2x_1)^\alpha+(1-x_1)^\alpha+2^\alpha x_1^\alpha-(1-x_2)^\alpha-x_2^\alpha\right)}{2^{1+\alpha}-1}=1-\left(1-x_2\right)^\alpha \end{array}$$

Solving yields $x_2=\left(1-2x_1\right)/2$. ■

Proof. (Lemma 14: $\pi_s\left(\sigma_{em^n}\right)=1/n$, $s=1,2,...,n$**)**

Let $\pi^*=(\pi_1^*,\pi_2^*,\ldots,\pi_n^*)$, where $\pi_1^*+\pi_2^*+\ldots+\pi_n^*=1$, denote the fraction of time spent in the corresponding states, and let m denote the transformation matrix of agents between states if they follow σ_{em^n}. In equilibrium, π^* has to satisfy the following equations,

$$m'\pi^*=\pi^* \text{ and } \pi_1^*+\pi_2^*+\ldots+\pi_n^*=1,$$

writing out the m-matrix and π^*-vectors,

$$\begin{pmatrix} 1-p & p & 0 & \cdots & & 0 \\ p & 1-2p & p & 0 & & \vdots \\ 0 & p & 1-2p & p & & \\ \vdots & 0 & p & 1-2p & & \\ & \vdots & 0 & p & & 0 \\ & & \vdots & & \ddots & p \\ 0 & \cdots & & 0 & p & 1-p \end{pmatrix}\begin{pmatrix}\pi_1^*\\ \pi_2^*\\ \vdots\\ \pi_n^*\end{pmatrix}=\begin{pmatrix}\pi_1^*\\ \pi_2^*\\ \vdots\\ \pi_n^*\end{pmatrix}$$

multiplying out the terms,

$$\begin{array}{rl} (1-p)\pi_1^*+p\pi_2^* & =\pi_1^* \\ p\pi_1^*+(1-2p)\pi_2^*+p\pi_3^* & =\pi_2^* \\ p\pi_2^*+(1-2p)\pi_3^*+p\pi_4^* & =\pi_3^* \\ \vdots \qquad\qquad & \vdots \\ p\pi_{n-2}^*+(1-2p)\pi_{n-1}^*+p\pi_n^* & =\pi_{n-1}^* \\ p\pi_{n-1}^*+(1-p)\pi_n^* & =\pi_n^* \end{array}$$

Solving such that $\pi_1^* + \pi_2^* + \ldots + \pi_n^* = 1 \implies \pi^* = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$. ■

7.2 Mathematica code

The following Mathematica code offers a compact example of using simulated payoffs (section 3.1.2) with $\alpha = 0.5$, $\delta = 0.8$ and $p = 0.05, 0.1, ..., .45$. The code constructs payoff functions from the simulated data and solves the n^*-EM optimization problem for those payoff functions and the associated triplet of parameter values (α, δ, p). The solution set is used to interpolate 3D-plot of the optimal favor sequences for the parameter space that was covered (see figure 4).

```
(* Clear old variables and set new values *)
Clear[”Global‘*”];α=.5;δ=.8;II=100;J=10000;
tr1:=p;tr1a=”p”;tr2:=α;tr2a=”α”;tr3:=δ;tr3a=”δ”;Col=1;
v0=1/20;v1=9/20;vInc=1/20;
For[p=v0,p≤v1,p+=vInc,
    PrintTemporary[ToString[tr1a]<>”=”<>ToString[N[tr1]]];
    Clear[avU,xSeq,ICCa];n=1;u[0]=p;
    While[n<3||avU[n-1]<avU[n],n++;
        Clear[x];x[0]=1;x[n+1]=0;x[n]=0;
        (* generate data *)
        W=RandomChoice[{p,p,1-2p}→{1,-1,0},{J,II}];
        Do[S[k]=Drop[FoldList[Max[1,Min[n,Plus[#1,#2]]]&,k,#]&/@W,None,-1],
            {k,1,n}];
        (* derive payoffs and define objective *)
        Do[u[k]=(1-δ)Mean[(Map[(1-x[#])^α&,S[k](W/.-1→0),{2}]+Map[x[n+1-#]^α&,
            -S[k](W/.{1→0}),{2}]).Array[δ^(#-1)&,II]],{k,1,n}];
        objective=Sum[u[k],{k,1,n}]/n;
        (* define incentive and feasibility constraints *)
        ICCa[n]=Table[0≤(1-δ)((1-x[k])^α-1)+δ (u[k+1]-u[k]),{k,1,n-1}];
        constraints=And@@Join[Table[0≤x[k]≤1,{k,1,n-1}],ICCa[n]];
        (* choose starting points and solve for optimal favors *)
        variables=Table[{x[t],Min[2*p,.55]*δ*t^(-α)},{t,1,n-1}];
        {avU[n],xSeq[n]}=FindMaximum[{objective,constraints},variables]
    ];
    nMax=Ordering[Array[avU,n-1,2],-1][[1]]+1;
    Col=Max[Col,Count[xSeq[nMax][[All,2]],_]];
    optimalX0[tr1]=Join[xSeq[nMax][[All,2]],Table[0,{i,30}]]
];
gg={Black,24,”Helvetica”};
data=Flatten[Table[{vv,nn,optimalX0[vv][[nn]]},{vv,v0,v1,vInc},{nn,1,Col}],1];
ListPlot3D[data,AxesLabel→{Text[Style[ToString[tr1a],gg]],Text[Style[”s”,gg]],
    Text[Style[”x”,gg]]},ColorFunction→(ColorData[”LakeColors”][#3*5/3]&)]
```

Our second example applies the computed payoffs model (section 3.1.3) and the differential evolution algorithm to find globally efficient equilibrium favors and associated payoffs for $\alpha = 0.2, .24, ..., .8$ while $\delta = .8$ and $p = .4$. The results are available in figure 12 and table 9.

```
(* Clear old variables and set new values *)
Clear["Global`*"];δ=4/5;p=2/5;Col=1;
tr1:=α;tr1a="α";tr2:=δ;tr2a="δ";tr3:=p;tr3a="p";
(* Define set of system of payoff equations and the ICC *)
equ[n_]:=Join[Table[
    u[s]==(p (δ u[s-1]+δ u[1+s]+(1-δ)(x[1+n-s]^α+(1-x[s])^α)))/(1-(1-2 p)δ),
    {s,2,n-1}],{u[1]==(p (δ u[2]+(1-δ) (1-x[1])^α))/(1-(1-p) δ),
    u[n]==(p (δ u[n-1]+(1-δ) (1+x[1]^α)))/(1-(1-p) δ)}];
ICCa[n_]:=Table[0≤δ (u[s+1]-u[s])-(1-δ)(1-(1-x[s])^α),{s,1,n-1}];
v0=1/5;v1=4/5;vInc=1/25;
For[α=v0, α≤v1, α+=vInc,
    Clear[avU,xSeq];n=1;PrintTemporary[ToString[tr1a]<>"="<>ToString[N[tr1]]];
    While[n<3||avU[n-1]<avU[n],n++;PrintTemporary["n="<>ToString[n]];
        Clear[x];x[0]=1;x[n+1]=0;x[n]=0;
        (* Solve system of payoff equations *)
        uSolv=NSolve[equ[n],Table[u[s],{s,1,n}]][[1]];
        objective=1/n*Sum[u[t],{t,1,n}]/.uSolv;
        constraints=And@@Join[Table[0≤x[k]≤1,{k,1,n-1}],ICCa[n]]/.uSolv;
        (* solve for optimal favors *)
        variables=Table[x[t],{t,1,n-1}];
        {avU[n],xSeq[n]}=NMaximize[{objective,constraints},variables,
            {Method→"DifferentialEvolution",MaxIterations→500}]
    ];
    nMax=Ordering[Array[avU,n-1,2],-1][[1]]+1;
    Col=Max[Col,Count[xSeq[nMax][[All,2]],_]];pay[tr1]=avU[nMax];
    optimalX[tr1]=xSeq[nMax][[All,2]];
    optimalX0[tr1]=Join[xSeq[nMax][[All,2]],Table[0,{i,30}]]
];
gg={Black,24,"Helvetica"};
data=Flatten[Table[{vv,nn,optimalX0[vv][[nn]]},{vv,v0,v1,vInc},{nn,1,Col}],1];
Labeled[TableForm[Table[Prepend[optimalX[vv],pay[vv]],{vv,v0,v1,vInc}],
    TableHeadings→{Table[ToString[tr1a]<>" = "<>ToString[N[vv]],{vv,v0,v1,vInc}],
    Prepend[Table[Subscript[x,t],{t,1,Col}],"u"]}],"n*-EM equilibria: "<>ToString[tr2a]
    <>"="<>ToString[N[tr2]]<>","<>ToString[tr3a]<>"="<>ToString[N[tr3]],Top,
    Frame→True,LabelStyle→Bold]
ListPlot3D[data,AxesLabel→{Text[Style[ToString[tr1a],gg]],Text[Style["s",gg]],
Text[Style["x",gg]]},ColorFunction→(ColorData["LakeColors"][#3*5/3]&)]
```

Our final Mathematica code example was used to generate figures 22 and 23 depicting 4-state FD equilibria. The included code is longer than previous examples only because it includes most of the auxiliary subroutines we use while testing the code. These subroutines provide the user with an option to use either a global or local optimization algorithm, print out intermediate results so that we can see which part of the code or loop is running and to gather debugging information if necessary.

```
(* Clear old variables and set new values *)
```

```
Clear[”Global‘*”];start = SessionTime[];global=0;Col=1;
α=.1;δ=.5;sMax=4;tr1:=p;tr1a=”p”;tr2:=α;tr2a=”α”;tr3:=δ;tr3a=”δ”;
(* Define favor functions and ICC *)
equ[n_]:=Join[Table[
    u[s]==(p(δ u[s-1]+δ u[s+1]+(1-δ) (x[1+n-s]^α+(1-x[s])^α)))/(1-(1-2 p) δ),{s,2,n-1}],
     {u[1]==(p(δ u[2]+(1-δ)((1-x[1])^α+( x[n])^α)))/(1-(1-p)δ),
    u[n]==(δ(p-(1-2 p) x[0])u[n-1]+p(1-δ)((x[1])^α+(1-x[n])^α))/(1-δ(1-p)+(1-2 p)δ x[0])}];
ICCa[n_]:=Append[Table[0≤δ (u[s+1]-u[s])-(1-δ)(1-(1-x[s])^α),
    {s,1,n-1}],0≤(1-δ)((1-x[n])^α-1)+δ*x[0](u[n]-u[n-1])];
v0=7/100;v1=47/100;vInc=4/100;
(* sMax optional limit to number of states *)
For[p=v0,p≤v1,p+=vInc,
    Clear[avU,xSeq];n=1;u[0]=p;
    While[navU[n]),n++;
        Clear[x];x[n+1]=0;uSolv=NSolve[equ[n],Table[u[s],{s,1,n}]][[1]];
        (* Define obj, cons and vars - need to choose starting pts carefully *)
        objective=1/n*Sum[u[t],{t,1,n}]/.uSolv;
        constraints=And@@Join[Table[0≤x[t]≤1,{t,0,n}],ICCa[n]]/.uSolv;
        variables=Prepend[Table[{x[t],Min[2*p,.55]*(sMax+1-n)/sMax},
            {t,1,n}],{x[0],0.01}];varOnly=Table[x[t],{t,0,n}];
        If[global==0,{avU[n],xSeq[n]}=FindMaximum[{objective,constraints},
                variables,MaxIterations→20000],
            {avU[n],xSeq[n]}=NMaximize[{objective,constraints},varOnly,
                {Method→”DifferentialEvolution”,MaxIterations→500}]];
        current = SessionTime[]-start;PrintTemporary[”n = ”<>ToString[n]<>” and ”
            <>ToString[tr1a]<>” = ”<>ToString[N[tr1]]<>” and time = ”
            <>ToString[current]<>” Payoff = ”<>ToString[avU[n]]]
    ];
    nMax=Ordering[Array[avU,n-1,2],-1][[1]]+1;
    Col=Max[Col,Count[xSeq[nMax][[All,2]],_]-1];
    pay[tr1]=avU[nMax];optimalX[tr1]=xSeq[nMax][[All,2]];
    optimalX0[tr1]=Join[xSeq[nMax][[All,2]],Table[0,{i,sMax}]]
];
gg={Black,24,”Helvetica”};
data=Flatten[Table[{vv,nn-1,optimalX0[vv][[nn]]},{vv,v0,v1,vInc},{nn,2,Col+1}],1];
Labeled[TableForm[Table[Prepend[optimalX[vv],pay[vv]],{vv,v0,v1,vInc}],
    TableHeadings→{Table[ToString[tr1a]<>” = ”<>ToString[N[vv]],
    {vv,v0,v1,vInc}],Prepend[Prepend[Table[Subscript[x,t],
    {t,1,Col}],”q”],”u”]}],ToString[sMax]<>”-state FD equilibria: ”
    <>ToString[tr2a]<>” = ”<>ToString[tr2]<>”, ”<>ToString[tr3a]<>” = ”
    <>ToString[N[tr3]],Top,Frame→True,LabelStyle→Bold]
ListPlot3D[data,AxesLabel→{Text[Style[ToString[tr1a],gg]],Text[Style[”s”,gg]],
    Text[Style[”x”,gg]]},ColorFunction→(ColorData[”LakeColors”][#3*5/3]&)]
```

7.3 Numerical algorithms for constrained global optimization

Source: Wolfram Research [21]:

Nelder-Mead

The Nelder-Mead method is a direct search method. For a function of n variables, the algorithm maintains a set of $n+1$ points forming the vertices of a polytope in n-dimensional space. This method is often termed the "simplex" method, which should not be confused with the well-known simplex method for linear programming.

At each iteration, $n+1$ points $x_1, x_2, ..., x_{n+1}$ form a polytope. The points are ordered so that $f(x_1) \leq f(x_2) \leq ... \leq f(x_{n+1})$. A new point is then generated to replace the worst point x_{n+1}.

Let c be the centroid of the polytope consisting of the best n points, $c = \sum_{i=1}^{n} x_i$. A trial point x_t is generated by reflecting the worst point through the centroid, $x_t = c + \alpha(c - x_{n+1})$, where $\alpha > 0$ is a parameter.

If the new point x_t is neither a new worst point nor a new best point, $f(x_1) \leq f(x_t) \leq f(x_n)$, x_t replaces x_{n+1}.

If the new point x_t is better than the best point, $f(x_t) < f(x_1)$, the reflection is very successful and can be carried out further to $x_e = c + \beta(x_t - r)$, where $\beta > 1$ is a parameter to expand the polytope. If the expansion is successful, $f(x_e) < f(x_t)$, x_e replaces x_{n+1}; otherwise the expansion failed, and x_t replaces x_{n+1}.

If the new point x_t is worse than the second worst point, $f(x_t) \geq f(x_n)$, the polytope is assumed to be too large and needs to be contracted. A new trial point is defined as

$$x_c = \begin{cases} c + \gamma\,(x_{n+1} - c), & \text{if } f\,(x_t) \geq f\,(x_{t+1}), \\ c + \gamma\,(x_t - c), & \text{if } f\,(x_t) < f\,(x_{t+1}), \end{cases}$$

where $0 < \gamma < 1$ is a parameter. If $f(x_c) < \min\left(f(x_{n+1}), f(x_t)\right)$, the contraction is successful, and x_c replaces x_{n+1}. Otherwise a further contraction is carried out.

The process is assumed to have converged if the difference between the best function values in the new and old polytope, as well as the distance between the new best point and the old best point, are less than the tolerances provided by `AccuracyGoal` and `PrecisionGoal`.

Strictly speaking, Nelder-Mead is not a true global optimization algorithm; however, in practice it tends to work reasonably well for problems that do not have many local minima.

Differential Evolution

Differential evolution is a simple stochastic function minimizer.

The algorithm maintains a population of m points, $\{x_1, x_2, ..., x_j, ..., x_m\}$, where typically $m \gg n$, with n being the number of variables.

During each iteration of the algorithm, a new population of m points is generated. The j^{th} new point is generated by picking three random points, x_u, x_v and x_w, from the old population, and forming $x_s = x_w + s(x_u - x_v)$, where s is a real scaling factor. Then a new point x_{new} is constructed from x_j and x_s by taking the i^{th} coordinate from x_s with probability ρ and otherwise taking the coordinate from x_j. If $f(x_{new}) < f(x_j)$, then x_{new} replaces x_j in the population. The probability ρ is controlled by the `"CrossProbability"` option.

The process is assumed to have converged if the difference between the best function values in the new and old populations, as well as the distance between the new best point and the old best point, are less than the tolerances provided by `AccuracyGoal` and `PrecisionGoal`.

The differential evolution method is computationally expensive, but is relatively robust and tends to work well for problems that have more local minima.

Simulated Annealing

Simulated annealing is a simple stochastic function minimizer. It is motivated from the physical process of annealing, where a metal object is heated to a high temperature and allowed to cool slowly. The process allows the atomic structure of the metal to settle to a lower energy state, thus becoming a tougher metal. Using optimization terminology, annealing allows the structure to escape from a local minimum, and to explore and settle on a better, hopefully global, minimum.

At each iteration, a new point, x_{new}, is generated in the neighborhood of the current point, x. The radius of the neighborhood decreases with each iteration. The best point found so far, x_{best}, is also tracked.

If $f(x_{new}) \leq f(x_{best})$, x_{new} replaces x_{best} and x. Otherwise, x_{new} replaces x with a probability $e^{b(i,\Delta f,f0)}$. Here b is the function defined by `BoltzmannExponent`, i is the current iteration, Δf is the change in the objective function value, and $f0$ is the value of the objective function from the previous iteration. The default function for b is $\frac{-\Delta f \ \log(i+1)}{10}$.

Like the `RandomSearch` method, `SimulatedAnnealing` uses multiple starting points, and finds an optimum starting from each of them.

The default number of starting points, given by the option `SearchPoints`, is $\min(2d, 50)$, where d is the number of variables.

For each starting point, this is repeated until the maximum number of iterations is reached, the method converges to a point, or the method stays at the same point consecutively for the number of iterations given by `LevelIterations`.

Random Search

The random search algorithm works by generating a population of random starting points and uses a local optimization method from each of the starting points to converge to a local minimum. The best local minimum is chosen to be the solution.

The possible local search methods are Automatic and “`InteriorPoint`”. The default method is Automatic, which uses FindMinimum with unconstrained methods applied to a system with penalty terms added for the constraints. When Method is set to “`InteriorPoint`”, a nonlinear interior-point method is used.

The default number of starting points, given by the option SearchPoints, is $\min\left(10d, 100\right)$, where d is the number of variables.

Convergence for `RandomSearch` is determined by convergence of the local method for each starting point.

`RandomSearch` is fast, but does not scale very well with the dimension of the search space. It also suffers from many of the same limitations as FindMinimum. It is not well suited for discrete problems and others where derivatives or secants give little useful information about the problem.

7.4 Figures

Figure 24 refers to the problem analyzed in lemma 13. We computed the optimal 3-state EM strategies for $\alpha \in \left\{\frac{1}{20}, \frac{2}{20}, ..., \frac{19}{20}\right\}$, $p \in \left\{\frac{1}{20}, \frac{2}{20}, ..., \frac{9}{20}\right\}$ and the corresponding δ_α using Mathematica. The corresponding payoff differences, $u\left(\sigma^*_{em^3}\right) - u\left(\sigma^*_{em^2}\right)$, shown in table 4 in the appendix. Figure 24 shows $u\left(\sigma^*_{em^3}\right)$ and $u\left(\sigma^*_{em^2}\right)$, where $u\left(\sigma^*_{em^3}\right)$ was interpolated from the set of $\sigma^*_{em^3}$ payoffs computed with Mathematica. The point to note is that $u\left(\sigma^*_{em^3}\right) > u\left(\sigma^*_{em^2}\right)$ in numerical testing that spanned the whole feasible region at 5% and 10% increments of α and p, respectively.

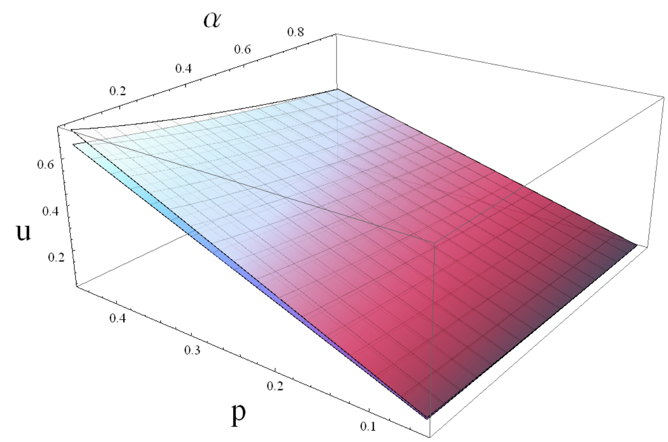

Figure 24: $u\left(\sigma^*_{em^3}\right)$ and $u\left(\sigma^*_{em^2}\right)$ with $\delta = \delta_\alpha$

Figure 25 represents solutions to same problem as figure 14 except with the additional constraint of $x_s \leq x_{s-1}$, $\forall s = 2, 3, ..., n$.

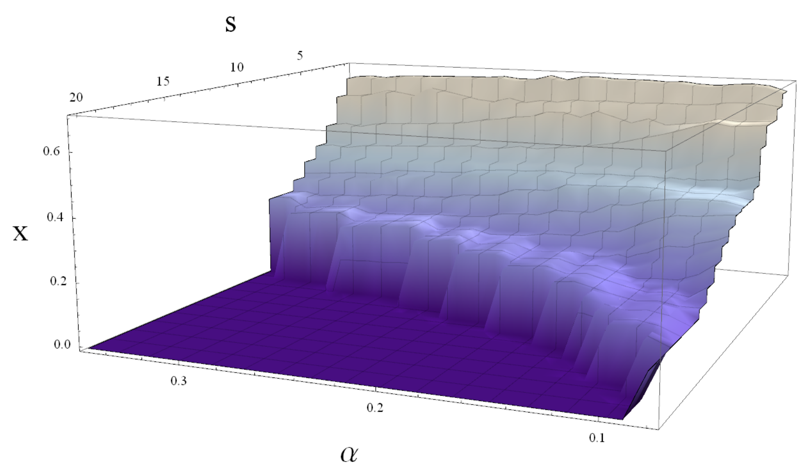

Figure 25: Constrained n^*-EM: $x^*_1(\alpha), ..., x^*_{n^*}(\alpha)$ when $x^*_s \leq x^*_{s-1}$

Figures 26 and 27 illustrate n^*-HSSGL equilibrium behavior as δ changes.

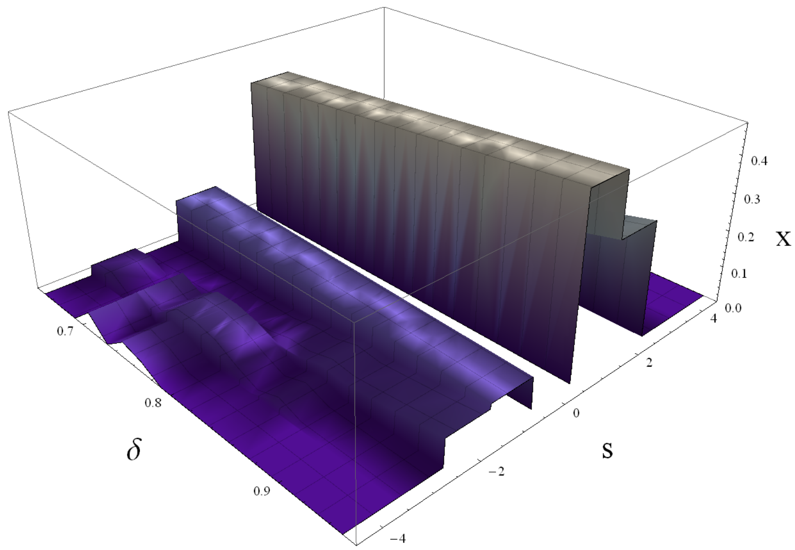

Figure 26: **n^*-HSSGL favors as δ varies: View 1**
Points: $\alpha = 0.3,\ p = 0.3,\ \delta = 0.65, 0.67, ..., 0.95$

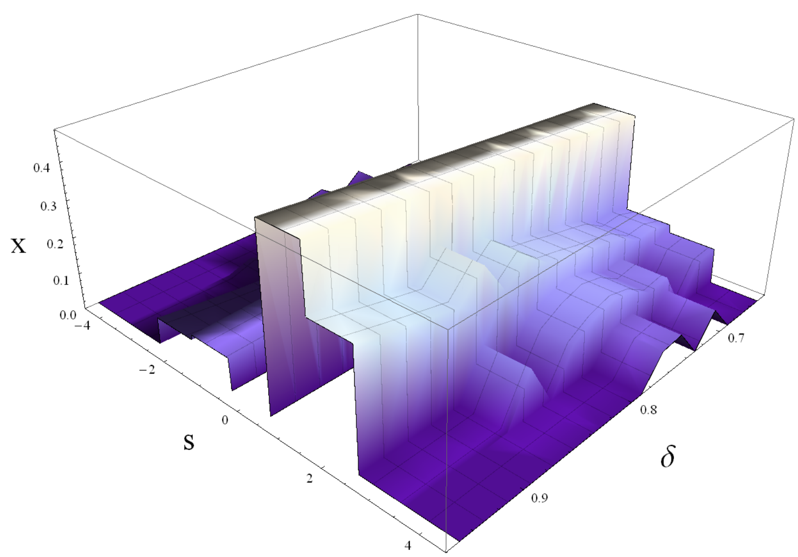

Figure 27: **n^*-HSSGL favors as δ varies: View 2**
Points: $\alpha = 0.3,\ p = 0.3,\ \delta = 0.65, 0.67, ..., 0.95$

Figures 28 and 29 illustrate n^*-HSSGL equilibrium behavior as α changes.

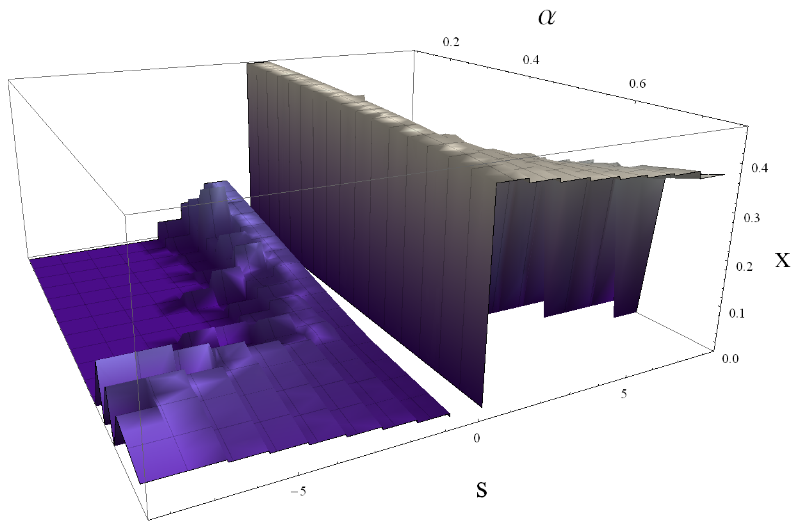

Figure 28: **n^*-HSSGL favors as α varies: View 1**
Points: $\delta = 0.9,\ p = 0.3,\ \alpha = 0.1, 0.12, ..., 0.74$

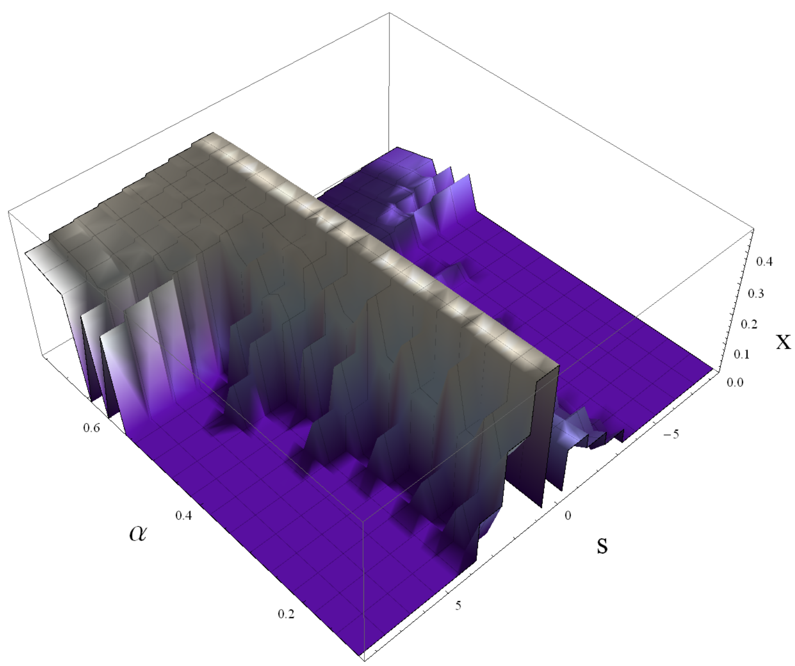

Figure 29: **n^*-HSSGL favors as α varies: View 2**
Points: $\delta = 0.9,\ p = 0.3,\ \alpha = 0.1, 0.12, ..., 0.74$

7.5 Tables

p	u	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
.30	.404	.599	.500	.427	.383	.334	.288	.253			
.31	.418	.615	.516	.442	.395	.347	.302	.266			
.32	.432	.631	.532	.457	.406	.360	.316	.279			
.33	.446	.645	.548	.471	.418	.373	.329	.292			
.34	.460	.647	.555	.479	.426	.381	.338	.303			
.35	.474	.623	.532	.462	.411	.376	.329	.290	.258		
.36	.488	.637	.547	.475	.422	.387	.341	.303	.270		
.37	.502	.650	.560	.488	.433	.397	.353	.315	.282		
.38	.516	.658	.571	.499	.442	.406	.363	.325	.293		
.39	.530	.654	.574	.503	.447	.411	.368	.330	.302		
.40	.544	.641	.557	.489	.437	.404	.364	.324	.290	.262	
.41	.558	.653	.570	.502	.448	.413	.374	.335	.301	.273	
.42	.572	.663	.582	.513	.459	.421	.383	.345	.312	.283	
.43	.586	.664	.587	.519	.465	.426	.390	.352	.319	.292	
.44	.599	.659	.588	.522	.468	.430	.393	.356	.323	.300	
.45	.613	.656	.579	.514	.463	.423	.396	.356	.321	.291	.265
.46	.627	.666	.590	.525	.473	.432	.404	.365	.331	.301	.275
.47	.641	.673	.599	.534	.481	.439	.411	.373	.339	.310	.284
.48	.655	.668	.600	.537	.484	.442	.414	.377	.344	.314	.291
.49	.669	.663	.600	.538	.487	.445	.417	.380	.347	.318	.299

Table 2: n^*-EM and favors above socially efficient size: $x_s^*(p) > 1/2$, $\alpha = 0.5, \delta = 0.9$

p	u	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
.30	.403	.500	.500	.443	.393	.343	.282	.192			
.31	.417	.500	.500	.448	.398	.349	.285	.205			
.32	.431	.500	.500	.453	.403	.354	.289	.217			
.33	.445	.500	.500	.458	.407	.360	.292	.229			
.34	.459	.500	.500	.465	.412	.374	.326	.257	.184		
.35	.473	.500	.500	.470	.416	.378	.332	.261	.195		
.36	.487	.500	.500	.474	.421	.383	.337	.264	.207		
.37	.501	.500	.500	.478	.425	.387	.342	.268	.218		
.38	.515	.500	.500	.483	.430	.392	.348	.270	.228		
.39	.528	.500	.500	.493	.440	.400	.357	.275	.230		
.40	.542	.500	.500	.490	.439	.404	.362	.321	.244	.196	
.41	.556	.500	.500	.494	.443	.407	.366	.327	.248	.206	
.42	.570	.500	.500	.496	.446	.411	.370	.330	.251	.217	
.43	.584	.500	.500	.500	.455	.418	.379	.334	.255	.221	
.44	.598	.500	.500	.500	.458	.421	.382	.336	.263	.225	
.45	.612	.500	.500	.499	.461	.424	.385	.338	.271	.230	
.46	.626	.500	.500	.500	.462	.424	.394	.354	.310	.236	.209
.47	.640	.500	.500	.500	.470	.431	.400	.361	.311	.240	.214
.48	.654	.500	.500	.500	.473	.434	.403	.365	.313	.248	.218
.49	.668	.500	.500	.500	.475	.436	.406	.368	.315	.256	.222

Table 3: n^*-EM with social efficiency constraint on favor size: $x_s^*(p) \leq 1/2$, $\alpha = .5, \delta = .9$

α	$p = .05$	$p = .1$	$p = .15$	$p = .2$	$p = .25$	$p = .3$	$p = .35$	$p = .4$	$p = .45$
.05	.007	.014	.020	.027	.034	.041	.048	.055	.061
.10	.006	.012	.018	.024	.030	.036	.042	.048	.054
.15	.005	.011	.016	.022	.027	.032	.038	.043	.048
.20	.005	.010	.015	.019	.024	.029	.034	.039	.044
.25	.004	.009	.013	.017	.022	.026	.031	.035	.039
.30	.004	.008	.012	.016	.020	.024	.027	.031	.035
.35	.004	.007	.011	.014	.018	.021	.025	.028	.032
.40	.003	.006	.009	.012	.016	.019	.022	.025	.028
.45	.003	.006	.008	.011	.014	.017	.019	.022	.025
.50	.002	.005	.007	.010	.012	.014	.017	.019	.022
.55	.002	.004	.006	.008	.010	.012	.015	.017	.019
.60	.002	.004	.005	.007	.009	.011	.012	.014	.016
.65	.001	.003	.004	.006	.007	.009	.010	.012	.013
.70	.001	.002	.004	.005	.006	.007	.008	.010	.011
.75	.001	.002	.003	.004	.005	.006	.007	.008	.009
.80	.001	.001	.002	.003	.004	.004	.005	.006	.006
.85	.001	.001	.002	.002	.003	.003	.004	.004	.005
.90	.000	.001	.001	.001	.002	.002	.002	.003	.003
.95	.000	.000	.000	.001	.001	.001	.001	.001	.001

Table 4: 3-EM dominates 2-EM: $u(\sigma^*_{em^3}) - u(\sigma^*_{em^2})$ when $\delta = \delta_\alpha$

p	u	x_1	x_2	x_3	x_4	x_5	x_6	x_7
.03	.034	.083						
.07	.084	.175	.104					
.11	.137	.319	.207					
.15	.190	.445	.309					
.19	.245	.446	.318	.243				
.23	.300	.446	.332	.269	.204			
.27	.356	.524	.395	.326	.263			
.31	.410	.508	.395	.352	.269	.219		
.35	.466	.582	.439	.386	.309	.283		
.39	.522	.559	.452	.383	.337	.280	.238	
.43	.577	.608	.500	.422	.374	.321	.278	
.47	.633	.592	.493	.421	.379	.328	.282	.247

Table 5: n^*-EM data for figure 6: u and $x_1^*(p), ..., x_{n^*}^*(p)$, $\alpha = 0.5,\ \delta = 0.85$

p	ICC_1	ICC_2	ICC_3	ICC_4	ICC_5	ICC_6	ICC_7
.03	0						
.07	0	0					
.11	0	0					
.15	0	0					
.19	0	0	.001				
.23	0	0	0	.003			
.27	0	0	0	.002			
.31	0	0	0	.001	.003		
.35	0	0	0	.001	.003		
.39	0	0	0	0	.003	.004	
.43	0	0	0	0	.003	.003	
.47	0	0	0	0	.001	.004	.004

Table 6: n^*-EM: ICC for figure 6, $\alpha = 0.5,\ \delta = 0.85$

δ	u	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
.37	.428	.055								
.41	.433	.076								
.45	.438	.103								
.49	.444	.136								
.53	.449	.176								
.57	.453	.224								
.61	.458	.165	.110							
.65	.465	.216	.149							
.69	.472	.280	.199							
.73	.478	.359	.265							
.77	.484	.336	.266	.200						
.81	.491	.450	.354	.293						
.85	.498	.492	.402	.347	.296					
.89	.505	.515	.443	.392	.356	.311	.275			
.93	.512	.595	.541	.494	.454	.426	.397	.366	.338	.313

Table 7: n^*-EM data for figure 10: u and $x_1^*(\delta), ..., x_{n^*}^*(\delta)$, $\alpha = 0.6,\ p = 0.4$

δ	ICC_1	ICC_2	ICC_3	ICC_4	ICC_5	ICC_6	ICC_7	ICC_8	ICC_9
.37	0								
.41	0								
.45	0								
.49	0								
.53	0								
.57	0								
.61	0	0							
.65	0	0							
.69	0	0							
.73	0	0							
.77	0	0	.002						
.81	0	0	.001						
.85	0	0	0	.002					
.89	0	0	0	0	.001	.002			
.93	0	0	0	0	0	0	.001	.002	.001

Table 8: n^*-EM: ICC for figure 10. $\alpha = 0.6, p = 0.4$

α	u	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
.20	.658	.689	.555	.512	.406	.343	.290	.238	.204	.231
.24	.639	.686	.590	.461	.390	.356	.275	.213	.180	.225
.28	.620	.648	.573	.415	.365	.315	.234	.217	.218	
.32	.602	.653	.535	.419	.346	.291	.238	.229		
.36	.584	.649	.513	.407	.343	.277	.255			
.40	.568	.639	.481	.384	.328	.261	.219			
.44	.551	.604	.458	.371	.305	.251				
.48	.535	.509	.391	.332	.261	.208				
.52	.520	.501	.384	.318	.257					
.56	.504	.506	.388	.323						
.60	.490	.420	.327	.267						
.64	.474	.331	.271	.210						
.68	.462	.364	.290							
.72	.448	.275	.219							
.76	.434	.341								
.80	.424	.237								

Table 9: n^*-EM data for figure 12: u and $x_1^*(\alpha), ..., x_{n^*}^*(\alpha)$, $\delta = 0.8, p = 0.4$

α	ICC_1	ICC_2	ICC_3	ICC_4	ICC_5	ICC_6	ICC_7	ICC_8	ICC_9
.20	.035	.009	0	0	0	.001	.002	0	0
.24	.024	.001	0	0	0	.001	.003	.003	0
.28	.018	0	0	0	0	.002	.004	.001	
.32	.012	0	0	0	.002	.004	.002		
.36	.008	0	0	0	.003	.002			
.40	0	0	0	0	.004	.005			
.44	0	0	0	.002	.004				
.48	0	0	0	.002	.005				
.52	0	0	0	.003					
.56	0	0	0						
.60	0	0	0						
.64	0	0	0						
.68	0	0							
.72	0	0							
.76	0								
.80	0								

Table 10: n^*-EM: ICC for figure 12, $\delta = 0.8, p = 0.4$

α	u	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}	x_{21}
.08	.732	.715	.637	.523	.458	.420	.392	.382	.380	.363	.339	.308	.271	.231	.191	.151	.120	.098	.080	.162	.225	.226
.09	.726	.716	.638	.532	.457	.424	.400	.392	.382	.358	.326	.284	.238	.191	.151	.121	.097	.177	.227	.226		
.10	.720	.714	.635	.519	.468	.426	.400	.391	.379	.351	.312	.265	.213	.169	.136	.111	.163	.227	.227			
.11	.714	.712	.632	.514	.473	.427	.401	.392	.376	.342	.295	.240	.189	.153	.125	.160	.228	.229				
.12	.708	.710	.630	.513	.476	.429	.402	.392	.371	.328	.270	.214	.172	.141	.166	.230	.230					
.13	.702	.708	.629	.517	.477	.430	.404	.392	.361	.305	.243	.195	.160	.179	.234	.232						
.14	.697	.710	.630	.521	.474	.427	.400	.389	.360	.303	.240	.194	.158	.159	.230	.230						
.15	.691	.707	.627	.522	.476	.429	.401	.387	.342	.274	.220	.180	.178	.235	.233							
.16	.686	.709	.629	.530	.472	.426	.399	.385	.340	.272	.219	.178	.165	.232	.231							
.17	.680	.704	.625	.528	.474	.426	.397	.380	.308	.250	.204	.190	.240	.236								
.18	.675	.706	.627	.535	.470	.423	.395	.378	.307	.248	.203	.178	.237	.234								
.19	.670	.708	.628	.541	.468	.421	.393	.377	.305	.246	.201	.168	.234	.231								
.20	.665	.703	.624	.540	.471	.424	.395	.350	.285	.232	.198	.243	.238									
.21	.660	.694	.602	.538	.461	.415	.388	.345	.279	.228	.193	.228	.238									
.22	.655	.696	.618	.538	.467	.418	.388	.322	.265	.226	.255	.245										
.23	.650	.686	.595	.532	.455	.409	.382	.315	.259	.219	.240	.246										
.24	.645	.688	.603	.528	.452	.406	.380	.313	.257	.217	.227	.243										
.25	.641	.683	.600	.530	.456	.407	.361	.299	.252	.263	.255											
.26	.636	.681	.598	.524	.450	.403	.358	.296	.249	.248	.253											
.27	.631	.683	.605	.520	.447	.401	.356	.294	.247	.238	.251											
.28	.627	.684	.612	.517	.444	.398	.355	.292	.245	.228	.248											
.29	.622	.686	.618	.514	.441	.396	.353	.291	.244	.218	.246											
.30	.618	.674	.599	.509	.437	.392	.332	.280	.258	.263												
.31	.613	.675	.605	.506	.434	.391	.330	.278	.249	.260												
.32	.609	.676	.610	.503	.431	.389	.329	.277	.241	.258												
.33	.604	.665	.588	.486	.419	.381	.320	.267	.231	.254												
.34	.600	.664	.594	.496	.425	.375	.317	.279	.276													
.35	.596	.661	.589	.488	.419	.371	.313	.270	.275													

Table 11: n^*-EM data for figure 14: u and $x_1^*(\alpha), ..., x_{n^*}^*(\alpha)$ when $\alpha < \underline{\alpha}$, $\delta = 0.85,\ p = 0.4$

α	u	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}
.08	.731	.671	.627	.589	.589	.523	.399	.367	.367	.360	.332	.297	.256	.208	.166	.140	.140	.140	.140	.140	.140
.09	.725	.690	.636	.585	.584	.500	.379	.379	.379	.353	.314	.266	.210	.165	.165	.165	.165	.165	.165		
.10	.719	.694	.636	.582	.580	.478	.384	.381	.380	.345	.298	.236	.187	.175	.175	.175	.175	.175			
.11	.713	.689	.633	.588	.588	.461	.382	.378	.377	.343	.296	.234	.185	.168	.168	.168	.168	.168			
.12	.708	.683	.623	.571	.571	.439	.389	.382	.371	.328	.266	.210	.180	.180	.180	.180	.180				
.13	.702	.685	.621	.564	.551	.421	.395	.386	.361	.302	.240	.194	.194	.194	.194	.194					
.14	.697	.692	.621	.556	.533	.424	.399	.386	.342	.274	.220	.209	.209	.209	.209						
.15	.691	.689	.620	.576	.518	.420	.395	.384	.340	.271	.218	.202	.202	.202	.202						
.16	.686	.692	.617	.558	.502	.423	.396	.380	.308	.250	.220	.220	.220	.220							
.17	.680	.692	.618	.576	.491	.420	.394	.378	.307	.248	.215	.215	.215	.215							
.18	.675	.691	.620	.593	.481	.416	.391	.376	.305	.246	.210	.210	.210	.210							
.19	.670	.696	.616	.569	.470	.424	.395	.351	.285	.233	.231	.231	.231								
.20	.665	.696	.617	.583	.467	.421	.392	.349	.283	.231	.225	.225	.225								
.21	.660	.680	.606	.581	.459	.413	.387	.345	.279	.228	.217	.217	.217								
.22	.655	.694	.610	.562	.465	.417	.387	.321	.264	.241	.241	.241									
.23	.650	.677	.598	.556	.454	.408	.381	.315	.259	.234	.234	.234									
.24	.645	.676	.613	.545	.450	.405	.379	.313	.257	.228	.228	.228									
.25	.641	.684	.596	.539	.456	.408	.361	.299	.257	.257	.255										
.26	.636	.678	.601	.525	.449	.403	.358	.295	.250	.250	.250										
.27	.631	.677	.614	.519	.446	.400	.356	.294	.247	.244	.244										
.28	.627	.675	.626	.515	.443	.397	.354	.292	.244	.238	.238										
.29	.622	.674	.637	.511	.439	.395	.352	.290	.243	.232	.232										
.30	.618	.671	.603	.508	.436	.392	.331	.280	.260	.260											
.31	.613	.670	.613	.505	.433	.390	.330	.278	.255	.255											
.32	.609	.668	.621	.501	.430	.388	.328	.275	.250	.250											
.33	.604	.659	.603	.487	.419	.381	.320	.268	.241	.241											
.34	.600	.664	.594	.496	.425	.375	.317	.279	.276												
.35	.596	.657	.592	.487	.419	.370	.312	.272	.272												

Table 12: n^*-EM data for figure 25: Favors constrained to decrease. $\delta = 0.85, p = 0.4$

p	u	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
.01	.014	.135	.043												
.02	.029	.225	.095	.028											
.03	.044	.299	.136	.078	.023										
.04	.060	.403	.177	.107	.042										
.05	.076	.498	.218	.137	.068										
.06	.093	.580	.258	.166	.099										
.07	.109	.602	.255	.213	.101	.078									
.08	.125	.612	.289	.233	.119	.102									
.09	.142	.615	.321	.251	.138	.129									
.10	.158	.630	.336	.265	.195	.102	.101								
.11	.175	.631	.363	.282	.210	.117	.125								
.12	.191	.631	.390	.298	.224	.132	.149								
.13	.208	.629	.415	.314	.238	.147	.174								
.14	.225	.625	.439	.329	.252	.162	.199								
.15	.241	.634	.433	.320	.293	.185	.124	.173							
.16	.258	.631	.456	.337	.303	.197	.136	.195							
.17	.275	.648	.490	.361	.316	.213	.152	.210							
.18	.292	.643	.479	.358	.321	.258	.168	.120	.186						
.19	.309	.645	.502	.375	.331	.269	.180	.132	.205						
.20	.326	.680	.548	.407	.348	.285	.197	.149	.208						
.21	.342	.683	.556	.420	.357	.294	.208	.166	.212						
.22	.359	.680	.555	.412	.351	.329	.233	.167	.132	.208					
.23	.376	.686	.567	.428	.362	.336	.243	.177	.147	.211					
.24	.393	.686	.568	.438	.370	.342	.251	.186	.161	.215					
.25	.410	.685	.568	.448	.378	.347	.259	.194	.176	.220					
.26	.427	.683	.567	.458	.386	.353	.266	.202	.191	.225					
.27	.444	.691	.583	.461	.391	.360	.306	.226	.172	.158	.215				
.28	.461	.690	.582	.470	.398	.365	.312	.233	.179	.172	.220				
.29	.478	.689	.581	.479	.406	.370	.318	.240	.187	.186	.224				
.30	.495	.687	.580	.487	.413	.375	.323	.247	.194	.200	.228				
.31	.512	.686	.577	.495	.420	.380	.328	.254	.202	.214	.233				
.32	.529	.687	.582	.503	.428	.386	.334	.261	.209	.231	.236				
.33	.546	.690	.588	.501	.423	.383	.363	.284	.223	.181	.197	.229			
.34	.563	.689	.586	.508	.430	.388	.367	.290	.229	.188	.210	.233			
.35	.580	.689	.588	.516	.438	.394	.371	.296	.236	.194	.225	.236			
.36	.597	.700	.615	.526	.453	.406	.378	.306	.246	.201	.244	.236			
.37	.614	.694	.596	.524	.446	.402	.379	.334	.265	.212	.177	.206	.231		
.38	.631	.693	.596	.531	.453	.407	.383	.339	.270	.219	.183	.219	.235		
.39	.648	.704	.623	.539	.466	.420	.391	.347	.280	.227	.188	.239	.235		
.40	.665	.703	.624	.540	.471	.424	.395	.350	.285	.232	.198	.243	.238		
.41	.682	.701	.624	.540	.476	.428	.398	.354	.289	.238	.209	.247	.241		
.42	.699	.707	.630	.549	.475	.427	.398	.380	.312	.255	.210	.178	.237	.234	
.43	.716	.705	.630	.550	.480	.432	.401	.383	.316	.260	.215	.187	.240	.237	
.44	.733	.704	.630	.550	.485	.436	.405	.385	.320	.264	.220	.197	.244	.240	
.45	.750	.703	.630	.550	.489	.440	.408	.388	.324	.269	.225	.207	.247	.242	
.46	.767	.701	.630	.549	.494	.444	.411	.390	.327	.273	.230	.216	.251	.245	
.47	.785	.699	.630	.549	.498	.448	.414	.393	.331	.278	.235	.226	.254	.248	
.48	.802	.706	.637	.561	.500	.450	.418	.398	.359	.299	.250	.212	.197	.243	.240
.49	.819	.705	.637	.560	.504	.454	.421	.400	.362	.303	.255	.217	.206	.246	.243

Table 13: n^*-EM data for figure 15: u and $x_1^*(p), ..., x_{n^*}^*(p)$ when $\alpha<\underline{\alpha}$. $\alpha = 0.2$, $\delta = 0.85$

δ	u	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}	x_{21}	x_{22}	x_{23}	x_{24}
.60	.584	.548	.314	.224	.150																				
.61	.586	.568	.327	.234	.162																				
.62	.587	.588	.341	.245	.174																				
.63	.588	.608	.356	.257	.187																				
.64	.589	.628	.372	.269	.201																				
.65	.590	.586	.346	.276	.176	.132																			
.66	.592	.608	.364	.285	.187	.145																			
.67	.593	.629	.382	.296	.200	.158																			
.68	.595	.647	.400	.306	.213	.172																			
.69	.596	.645	.412	.315	.223	.185																			
.70	.597	.642	.425	.324	.233	.200																			
.71	.598	.664	.429	.328	.264	.180	.149																		
.72	.600	.666	.443	.339	.274	.191	.164																		
.73	.601	.662	.456	.349	.283	.202	.179																		
.74	.603	.658	.469	.360	.293	.214	.196																		
.75	.604	.652	.482	.371	.303	.226	.214																		
.76	.605	.646	.495	.382	.313	.238	.233																		
.77	.606	.638	.508	.394	.323	.251	.254																		
.78	.608	.661	.513	.399	.347	.266	.208	.213																	
.79	.609	.653	.527	.413	.357	.279	.222	.235																	
.80	.611	.654	.549	.433	.370	.296	.241	.255																	
.81	.612	.665	.552	.440	.378	.326	.255	.208	.228																
.82	.613	.663	.574	.461	.393	.341	.273	.228	.251																
.83	.615	.670	.589	.484	.411	.358	.294	.256	.260																
.84	.616	.679	.607	.496	.423	.382	.317	.263	.229	.250															
.85	.618	.674	.599	.509	.437	.392	.332	.280	.258	.263															
.86	.619	.681	.615	.525	.453	.406	.364	.306	.260	.241	.256														
.87	.621	.674	.602	.538	.467	.419	.377	.322	.279	.275	.271														
.88	.622	.678	.612	.552	.483	.434	.402	.350	.303	.267	.269	.269													
.89	.624	.681	.618	.568	.501	.452	.417	.383	.333	.292	.260	.270	.269												
.90	.625	.688	.634	.589	.525	.475	.438	.412	.367	.325	.289	.262	.279	.271											
.91	.627	.687	.638	.593	.542	.494	.456	.428	.399	.357	.320	.288	.273	.284	.276										
.92	.628	.682	.638	.592	.558	.512	.475	.446	.425	.387	.351	.320	.293	.293	.293	.285									
.93	.630	.687	.650	.611	.579	.536	.501	.471	.447	.429	.396	.363	.333	.307	.285	.300	.294	.285							
.94	.632	.678	.647	.615	.581	.552	.521	.493	.469	.448	.427	.398	.370	.344	.322	.327	.319	.310	.301						
.95	.634	.677	.651	.624	.595	.567	.545	.519	.496	.476	.458	.445	.420	.395	.372	.351	.331	.321	.325	.319	.312	.304			
.96	.636	.669	.648	.626	.604	.581	.556	.543	.523	.504	.487	.472	.459	.444	.423	.404	.385	.367	.351	.345	.343	.338	.332	.325	.317

Table 14: n^*-EM data for figure 16: u and $x_1^*(\delta), ..., x_{n^*}^*(\delta)$ when $\alpha<\underline{\alpha}$, $\alpha = 0.3, p = 0.4$

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