Iterative Weak Dominance and Interval-Dominance Supermodular Games*

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Abstract

This paper extends Milgrom and Robert's treatment of supermodular games in two ways. It points out that their main characterization result holds under a weaker assumption. It refines the arguments to provide bounds on the set of strategies that survive iterative deletion of weakly dominated strategies. I derive the bounds by iterating the best-response correspondence. I give conditions under which they are independent of the order of deletion of dominated strategies. The results have implications for equilibrium selection and dynamic stability in games. The paper generalizes the Interval-Dominance Condition of Quah and Strulovici. *Journal of Economic Literature* Classification Numbers: C72, D81.

Keywords: supermodularity; dominance; equilibrium selection.

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1 Introduction

Milgrom and Roberts [4] and Vives [9] provide useful analyses of the class of supermodular games introduced by Topkis [8]. In a supermodular game, each player's strategy set is partially ordered and there are strategic complementarities that cause a player's best response to be increasing in opponents' strategies. Milgrom and Roberts and Vives describe many applications of the games in the class.

Milgrom and Roberts [4] and Vives [9] demonstrate that supermodular games have a largest and smallest equilibrium. Milgrom and Roberts demonstrate that these extreme equilibria can be obtained by iterating the best-response correspondence and characterize the set of strategies that survive iterative deletion of strictly dominated strategies. This paper extends these insights in two ways. First, it broadens the class of games for which the basic conclusions hold. Second, it provides parallel results for strategies that survive iterative deletion of weakly dominated strategies. The new result forms the basis for equilibrium selection arguments.

I enlarge the class of supermodular games by replacing an increasing-difference condition used by Milgrom and Roberts and Vives with a using a weaker condition, interval dominance. To accomplish this, I generalize the idea of interval dominance introduced by Quah and Strulovici [6]. Section 3 describes some implications of the interval-dominance condition. In particular, it provides an alternative definition of the Quah-Strulovici definition and shows that, when combined with supermodular payoff functions, guarantees that the best-reply correspondence is monotonic when the strategy set is multidimensional.

Section 4 points out a small generalization of the basic result of Milgrom and Roberts characterizing the set of strategies that survive iterative deletion of strictly dominated strategies. Section 5 extends the results to weak dominance. The analysis trades off using a more restrictive solution concept (deleting weakly dominated strategies instead of strongly dominated strategies) with analyzing a broader class of games.

Section 6 discusses applications. This section describes two classes of games that fail to be supermodular and have large sets of equilibria that survive iterative deletion of strongly dominated strategies. I demonstrate that analogs of the methods introduced to study supermodular games partially extend to these games.

2 Preliminaries

There is a finite set of players. I denotes the player set. Each player has a strategy set X_i with typical element x_i. $X = \Pi_{i \in I} X_i$ is the set of strategy profiles. I denote by x_{-i} the strategies of Player i's opponents. Each strategy set is partially ordered by $\geq_i$; $\geq$ denotes the product order derived from the $\geq_i$ (so that $x \geq x'$ if and only if $x_i \geq_i x_i$ for all i). Denote Player i's utility function by $u_i(x_i, x_{-i})$. Denote by $u = (u_i)_{i \in I}$ the set of utility functions. A game in ordered normal form is $\Gamma = (I, X, u, \geq)$.

Consider a set X with a partial order $\geq$ that is transitive, reflexive, and antisymmetric. Following Milgrom and Roberts, I define several basic concepts.

Definition 1. *Given* $T \subset X$, $\bar{b} \in X$ *is called an upper bound for* T *if* $\bar{b} \geq x$ *for all* $x \in T$; *it is the supremum of* T *(denoted* $\sup(T)$*) if for all upper bounds* b *of* T, $b \geq \bar{b}$. *Lower bounds and infimums are defined analogously. A point* x *is a maximal element of* X *if there is no* $y \in X$ *such that* $y > x$ *(that is, no* y *such that* $y \geq x$ *but not* $x \geq y$*); it is the largest element of* X *if* $x \geq y$ *for all* $y \in X$. *Minimal and smallest elements are defined similarly.*

Definition 2. *The set* X *is a lattice if for each two point set* $\{x, y\} \subset X$, *there is a supremum for* $\{x, y\}$ *(denoted* $x \vee y$ *and called the join of* x *and* y*) and an infimum (denoted* $x \wedge y$ *and called the meet of* x *and* y*) in* X. *The lattice is complete if for all nonempty subsets* $T \subset X$, $\inf(T) \in X$ *and* $\sup(T) \in X$. *An interval is a set of the form* $[x, y] \equiv \{z : y \geq z \geq x\}$.

Definition 3. *A sublattice* T *of a lattice* X *is a subset of* X *that is closed under* $\wedge$ *and* $\vee$. *An interval sublattice* T *of a lattice* X *is a sublattice of* X *of the form* $[\underline{x}, \overline{x}]$ *for some* $\underline{x}, \overline{x} \in X$, $\underline{x} \leq \overline{x}$. *A complete sublattice* T *is a sublattice such that the infimum and supremum of every subset of* T *is in* T.

Definition 4. *A chain* $C \subset X$ *is a totally ordered subset of* X, *that is, for any* $x \in C$ *and* $y \in C$, $x \geq y$ *or* $y \geq x$.

Definition 5. *Given a complete lattice* X, *a function* $f : X \to \mathbb{R}$ *is order continuous if it converges along every chain* C *(in both the increasing and decreasing directions), that is, if* $\lim_{x \in C, x \downarrow \inf C} f(x) = f(\inf(C))$ *and* $\lim_{x \in C, x \uparrow \sup C} f(x) = f(\sup(C))$. *It is order upper-semicontinuous if* $\limsup_{x \in C, x \downarrow \inf C} f(x) \leq f(\inf(C))$ *and* $\liminf_{x \in C, x \uparrow \sup C} f(x) \leq f(\sup(C))$.

Definition 6. *A function* $f : X \to \mathbb{R}$ *is supermodular if for all* $x, y \in X$,

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y). \tag{1}$$

Definition 7. *The set* S'' *dominates* S' *in the strong set order (written* $S'' \geq S'$*) if* $x^* \in S'$ *and* $x^{**} \in S''$ *imply that* $x^* \wedge x^{**} \in S'$ *and* $x^* \vee x^{**} \in S''$.

Definition 8. *Given two lattices* X_1 *and* X_2, *a function* $f : X \times Y \to \mathbb{R}$ *has increasing differences in its two arguments* x *and* y *if for all* $x'' \geq x'$, *the difference* $f(x'', y) - f(x', y)$ *is nondecreasing in* y.

There are several ways to weaken the increasing-differences property.

Definition 9. *Given two lattices* X *and* Y, *a function* $f : X \times Y \to \mathbb{R}$ *satisfies the single-crossing property in its two arguments* x *and* y *if for all* $y'' > y'$, $x'' > x'$,

$$f(x'', y') \geq (>) f(x', y') \implies f(x'', y'') \geq (>) f(x', y''). \tag{2}$$

Definition 10. *Let* X *and* Y *be lattices. Assume* $y'' > y'$, $x^* \in \arg\max_{x' \in X} f(x', y')$, *and* $x^{**} \in \arg\max_{x' \in X} f(x', y'')$. *A function* $f : X \times Y \to \mathbb{R}$ *satisfies the interval-dominance property (ID) in its two arguments on* $X \times Y$ *if*

$$f(x \vee x^*, y') \geq (>)\ f(x, y') \implies f(x \vee x^*, y'') \geq (>)\ f(x, y'') \tag{3}$$

and

$$f(x \wedge x^{**}, y'') \geq (>)\ f(x, y'') \implies f(x \wedge x^{**}, y') \geq (>)\ f(x, y'). \tag{4}$$

A function $f : X \times Y \to \mathbb{R}$ satisfies the limited interval-dominance property (ID) in its two arguments on $X \times Y$ if*

$$f(x \vee x^{*}, y') \geq f(x, y') \implies f(x \vee x^{*}, y'') \geq f(x, y'') \tag{5}$$

and

$$f(x \wedge x^{**}, y'') \geq f(x, y'') \implies f(x \wedge x^{**}, y') \geq f(x, y'). \tag{6}$$

Definition 11. *Let X and Y be lattices. Assume $y'' > y'$, $x^* \in \arg\max_{x' \in X} f(x', y')$, and $x^{**} \in \arg\max_{x' \in X} f(x', y'')$. A function $f : X \times Y \to \mathbb{R}$ satisfies the weak interval-dominance property (WID) in its two arguments on the set $X \times Y$ if*

$$f(x \vee x^{*}, y') \geq (>)\ f(x, y') \implies$$

$$\forall y'' \geq y'\ \exists \tilde{x}^{*} \in \arg\max_{x' \in X} f(x', y'), \tilde{x}^{*} \leq x^{*},\ \textit{such that } f(x \vee \tilde{x}^{*}, y'') \geq (>)\ f(x, y'') \tag{7}$$

and

$$f(x \wedge x^{**}, y'') \geq (>)\ f(x, y'') \implies$$

$$\forall y' \leq y''\ \exists \tilde{x}^{**} \in \arg\max_{x' \in X} f(x', y''), \tilde{x}^{**} \geq x^{**},\ \textit{such that } f(x \wedge \tilde{x}^{**}, y') \geq (>)\ f(x, y'). \tag{8}$$

A function $f : X \times Y \to \mathbb{R}$ satisfies the limited weak interval-dominance property (WID) in its two arguments on the set $X \times Y$ if*

$$f(x \vee x^{*}, y') \geq f(x, y') \implies$$

$$\forall y'' \geq y'\ \exists \tilde{x}^{*} \in \arg\max_{x' \in X} f(x', y'), \tilde{x}^{*} \leq x^{*},\ \textit{such that } f(x \vee \tilde{x}^{*}, y'') \geq f(x, y'') \tag{9}$$

and

$$f(x \wedge x^{**}, y'') \geq f(x, y'') \implies$$

$$\forall y' \leq y''\ \exists \tilde{x}^{**} \in \arg\max_{x' \in X} f(x', y''), \tilde{x}^{**} \geq x^{**},\ \textit{such that } f(x \wedge \tilde{x}^{**}, y') \geq f(x, y'). \tag{10}$$

Quah and Strulovici [6] introduce the interval-dominance property under the assumption that X is a subset of the real line. They write the condition differently. I will discuss the (ID) condition in more detail in Section 3 after I define supermodular games. In that section, I present the one-dimensional condition in a way that is transparently equivalent to Quah and Strulovici's formulation. In the one-dimensional case, (ID) is not difficult to understand. Condition (3), for example, reduces to the statement that if x is less than (or equal to) a best response to y' and $y'' > y'$, then x is less than (or equal to) that best response. I demonstrate how a central result in Quah and Strulovici [6], the observation that $\arg\max_{x \in X} f(x, y'') \geq \arg\max_{x \in X} f(x, y')$ whenever $y'' > y'$, extends to higher dimensions.

Any function that satisfies increasing differences must satisfy single crossing. Setting $x'' = x \vee x^{**}$ and $x' = x$, it follows that single crossing implies (3). Single crossing also implies (4). To see this, note that an implication of (the contrapositive of) single crossing is $f(x'', y'') - f(x', y'') \leq 0$ implies that $f(x'', y') - f(x', y') \leq 0$, which implies (4) when $x'' = x$ and $x' = x \wedge x^{**}$. It is straightforward to find examples of functions that satisfy weak interval dominance and not interval dominance, interval dominance but not single crossing, and single crossing but not interval dominance. Milgrom and Shannon [5] is one reference that discusses the relationship between single-crossing and increasing differences. Their paper shows the importance of the single-crossing property for monotone comparative statics. In particular, they show that single-crossing is a necessary condition for $\arg\max_{x \in K} f(x, y'') \geq \arg\max_{x \in K} f(x, y')$ whenever $y'' > y'$ for $K \subset X$.

I did not include analogues to Conditions (4) and (8) in the definitions of single crossing and increasing differences. This is because those implications are implied by the definitions. These conditions follow from (3) and (7) when X is a subset of the real line, but not more generally. The (ID) and (WID) conditions differ because (WID) provides the freedom to replace x^* (x^{**}) with another element of $\arg\max_x f(x, y')$ ($\arg\max_x f(x, y'')$). This difference makes (ID) more difficult to satisfy.

I also introduced variations of (ID) and (WID). The (ID*) and (WID*) conditions weaken (ID) and (WID). Increasing differences, single-crossing, interval dominance, and weak interval dominance all require that if a strict inequality holds at y' it is preserved at y''. For example, if the inequality on the left-hand side of Condition (3) holds strictly, then so does the inequality on the right-hand side. The limited version of these conditions require only that weak inequalities are preserved. The distinction is important for applications.

The conditions play different roles in the analysis. Condition (ID) generalizes the Quah and Strulovici condition. I discuss the connection in the next section. The results on iterative dominance require only the weaker version of (ID). Milgrom and Robert's results about iterative deletion of strictly dominated strategies require only (WID). When (WID) fails, the argument of Milgrom and Roberts no longer works, but a slight modification of the argument does apply to iterative deletion of weakly dominated strategies when (WID*) holds. In Section 6 I present applications in which (WID*) holds but (WID) does not hold.

Definition 12. *The game $\Gamma = (I, X, u, \geq)$ is a supermodular game if, for each $i \in I$:*

(A1) X is a complete lattice;

(A2) $u_i : X \to \mathbb{R}$ is order upper semi-continuous in x_i for fixed x_{-i}; order upper continuous in x_{-i} for fixed x_i; and u_i is bounded above;

(A3) u_i is supermodular in x_i for fixed x_{-i};

(A4) u_i has increasing differences in x_i and x_{-i}.

Definition 13. *The game $\Gamma = (I, X, u, \geq)$ is an interval-dominance supermodular (ID-supermodular) game if, for each $i \in I$:*

(A1) X *is a complete lattice;*

(A2) $u_i : X \to \mathbb{R}$ *is order upper semi-continuous in* x_i *for fixed* x_{-i}*; order upper continuous in* x_{-i} *for fixed* x_i*; and* u_i *is bounded above;*

(A3) u_i *is supermodular in* x_i *for fixed* x_{-i}*;*

(A4′) u_i *satisfies the interval-dominance property in* x_i *and* x_{-i} *on all interval sublattices of* X*.*

The distinction between supermodular and ID-supermodular games is that (A4′) replaces (A4). I point out that small modifications in the arguments of Milgrom and Roberts extends their results to ID-supermodular games. It is also possible to talk of ID*, WID, or WID*-supermodular games in which the appropriate assumption replaces (A4′).

A **pseudo-potential game** is a game for which there exists function $P : X \to \mathbb{R}$ such that $\arg\max_{x_i \in X_i} P(x_i, x_{-i}) \subset \arg\max_{x_i \in X_i} u_i(x_i, x_{-i})$. Dubey, Haimanko, and Zapechelnyuk [1] give conditions under which games with complementarities are pseudo-potential games. Their results imply that finite, two-player (ID) supermodular games are pseudo-potential games. Dubey, Haimanko, and Zapechelnyuk identify several properties of pseudo-potential games.

A useful preliminary observation is Topkis's Monotonicity Theorem.

Fact 1. *Let* S_1 *be a lattice and* S_2 *a partially ordered set. Let* $f(x, y) : S_1 \times S_2 \to \mathbb{R}$*. Suppose that* $f(\cdot)$ *is supermodular in* x *for given* y*. For any sublattice* $A_1 \subset S_1$*, let* $M(A_1) \equiv \arg\max_{z \in A_1} f(z, y)$*.* $M(A_1)$ *is a sublattice of* A_1*.*

In the context of games, the first part of the theorem states that the set of best replies form a sublattice when the payoff function is supermodular in a player's strategy. This result is part of the Topkis Monotonicity Theorem as stated in Milgrom and Roberts [4].

3 Properties of Interval Dominance

Quah and Strulovici [6] define interval dominance as follows.

Definition 14. *Given two lattices* X *and* Y*, assume that* $X \subset \mathbb{R}$*. A function* $f : X \times Y \to \mathbb{R}$ *satisfies the Quah-Strulovici interval-dominance (QSID) property in its two arguments* x *and* y *if for all* $y'' > y'$*,* $x'' > x'$*, (2) holds whenever* $f(x'', y') \geq f(x, y')$ *for all* $x \in [x', x'']$*.*[1]

[1] In fact, Quah and Strulovici frame the definition a bit differently. They assume that f and g are two real-valued functions defined on X and say that g dominates f under the interval dominance order if

$$f(x'') \geq (>)\ f(x') \text{ implies that } g(x'') \geq (>)\ g(x')$$

holds for x' and x'' such that $x'' > x'$ and $f(x'') \geq f(x)$ for all $x \in [x', x'']$. In this definition, f plays the role of $f(\cdot, y')$ in my definition and g plays the role of $f(\cdot, y'')$ in my definition.

Quah and Strulovici [6, Lemma 2] show that if $\arg\max_{x \in [x', x'']} f(x)$ is nonempty for all intervals $[x', x'']$, then (QSID) is equivalent to: If $f(x', y'') \geq f(x, y'')$ for $x \in [x', x'']$, then $f(x', y'') - f(x'', y'') \geq (>)\ 0$ implies $f(x', y') - f(x'', y') \geq (>)\ 0$. This property suggests how Conditions (3) and (4) may be related to (QSID). Loosely, the interval-dominance conditions state that if x' is no greater (less) than the optimizer of $f(\cdot, y')$, then it will also be no greater (less) than the optimizer of $f(\cdot, y'')$ for $y'' > y'$. Imposing the "strict" version of the condition is sufficient to guarantee that reducing the parameter y will lead to a decrease in the set of maximizers of $f(\cdot, y)$. If one does not impose the strict version of the interval-dominance condition, then one must impose both (3) and (4) to guarantee the basic monotonicity property – maximizers of $f(\cdot, y)$ are increasing in y.

Proposition 1. *Given two lattices X and Y, assume that $X \subset \mathbb{R}$. A function $f : X \times Y \to \mathbb{R}$. If f satisfies (QSID) if and only if f satisfies (ID) (or (ID*)) on all intervals $[x', x''] \subset X$.*

Proof. Suppose (ID*) holds on all intervals. If $f(x'', y') \geq f(x, y')$ for all $x \in [x', x'']$, then $x'' \in \arg\max_{x \in [x', x'']} f(x, y')$ and so (3) implies that $f(x'', y'') \geq f(x, y'')$ for $x \in [x', x'']$. If, furthermore, $f(x', y'') \geq f(x'', y'')$, then $x' \in \arg\max_{x \in [x', x'']} f(x, y'')$ so (4) implies that $f(x', y') \geq f(x'', y')$. Consequently, if $f(x'', y') > f(x', y')$ then $f(x'', y'') > f(x', y'')$. It follows that if (ID*) holds on all intervals, then (QSID) holds.

To prove the converse, suppose that (QSID) holds. Fix an interval $[x', x'']$. Let $x^* \in \arg\max_{x \in [x', x'']} f(x, y')$ and $x^{**} \in \arg\max_{x \in [x', x'']} f(x, y'')$. Let $x \in [x', x'']$ be such that $f(x \vee x^*, y') \geq f(x, y')$. If $x \geq x^*$, $x \vee x^* = x$ so certainly $f(x \vee x^*, y'') \geq f(x, y'')$. If $x < x^*$, then $f(x^*, y') \geq f(\hat{x}, y')$ for all $\hat{x} \in [x, x^*]$ by the definition of x^*. Consequently (QSID) implies that $f(x \vee x^*, y'') \geq f(x, y'')$, with strict inequality if $f(x^*, y') > f(x, y')$. Similarly, let $x \in [x', x'']$ be such that $f(x \wedge x^{**}, y'') \geq f(x, y'')$. If $x \leq x^{**}$, $x \wedge x^{**} = x$ so certainly $f(x \wedge x^{**}, y') \geq f(x, y')$. Finally, assume $x > x^{**}$. If $f(x^{**}, y') = f(x^*, y')$, then $f(x \wedge x^{**}, y') = f(x^{**}, y') \geq f(x, y')$. Furthermore, in this case, if $f(x, y') \geq f(x^{**}, y')$, then (QSID) implies $f(x, y'') \geq f(x^{**}, y'')$. It follows that if $f(x \wedge x^{**}, y'') > f(x, y'')$ then $f(x \wedge x^{**}, y') > f(x, y')$, so (8) with strict inequalities must hold. Otherwise, $f(x^*, y') \geq f(\hat{x}, y')$ for all $\hat{x} \in [x^{**}, x^*] \subset [x', x'']$ and $f(x^*, y') > f(x^{**}, y')$. In this case, (QSID) implies that $f(x^*, y'') > f(x^{**}, y'')$, which contradicts the definition of x^{**}. Hence (QSID) implies (ID). ■

Implicit in the statement of Proposition 1 is the assertion that when X is a subset of the real line, (ID*) holds on all intervals if and only if (ID) holds on all intervals. So from the perspective of supermodular games, when X is one dimensional, Conditions (ID) and (ID*) are equivalent. When X is a subset of the real line, if (WIC*) holds on all sublattices, then it holds on all two-point sets. So if (WIC*) holds on all sublattices, (WID) and (WID*) are equivalent to each other and to (ID) and (ID*).

An important property of supermodular games is monotonicity of the best-reply correspondence.

Proposition 2. *Let Γ be an ID*-supermodular game. If $x''_{-i} \geq x'_{-i}$, then*

$$\arg\max_{x_i \in X_i} u_i(x_i, x''_{-i}) \geq \arg\max_{x_i \in X_i} u_i(x_i, x'_{-i}).$$

Proof. Let $x_i^* \in \arg\max_{x_i \in X_i} u_i(x_i, x'_{-i})$ and $x_i^{**} \in \arg\max_{x_i \in X_i} u_i(x_i, x''_{-i})$. Supermodularity implies that $u_i(x_i^* \vee x_i^{**}, x'_{-i}) + u_i(x_i^* \wedge x_i^{**}, x'_{-i}) \geq u_i(x_i^*, x'_{-i}) + u_i(x_i^{**}, x'_{-i})$. It follows from the definition of x_i^* that $u_i(x_i^*, x'_{-i}) \geq u_i(x_i^* \wedge x_i^{**}, x'_{-i})$. Therefore supermodularity implies that $u_i(x_i^* \vee x_i^{**}, x'_{-i}) \geq u_i(x_i^{**}, x'_{-i})$. If $x_i^{**} \geq x_i^*$, then $x_i^* \vee x_i^{**} \in \arg\max_{x_i \in X_i} u_i(x_i, x''_{-i})$ by definition. If $x_i^{**} < x_i^*$, then $x_i^* \vee x_i^{**} \in \arg\max_{x_i \in X_i} u_i(x_i, x''_{-i})$ by (ID*). Similarly (ID*) implies that $x_i^* \wedge x_i^{**} \in \arg\max_{x_i \in X_i} u_i(x_i, x'_{-i})$. ■

Proposition 2 generalizes a result of Milgrom and Shannon [5, Theorem 4] that assumes the single-crossing crossing property rather than (ID) and a result of Quah and Strulovici [6, Theorem 1] that assumes that X is a subset of $\mathbb{R}$. As Milgrom and Shannon note, the lemma holds if one replaces the assumption of super modularity with the weaker assumption of quasi supermodularity.[2]

4 Iterative Deletion of Strictly Dominated Strategies

Theorem 1. *Let Γ be an WID-supermodular game. For each player i, there exist largest and smallest serially undominated strategies, $\underline{x}_i$ and $\overline{x}_i$. Moreover, the strategy profiles $\{\underline{x}_i : i \in I\}$ and $\{\overline{x}_i : i \in I\}$ are pure Nash equilibrium profiles.*

Theorem 1 is Theorem 5 from Milgrom and Roberts [4] under the assumption of weak interval dominance rather than increasing differences. The theorem follows from the next lemma. I include a proof of the lemma to identify precisely where I relax Milgrom and Roberts's condition.

Let

$$U_i(\hat{S}) = \{x_i \in S_i : \text{ for all } x'_i \in S_i \text{ there exists } \hat{x} \in \hat{S} \text{ such that } u_i(x_i, \hat{x}_{-i}) \geq u_i(x'_i, \hat{x}_{-i})\},$$

$U(\hat{S}) = (U_1(\hat{S}), \ldots, U_I(\hat{S}))$, and $\overline{U}(\hat{S})$ denote the interval $[\inf(U(\hat{S})), \sup(U(\hat{S}))]$.

Lemma 1. *Let $\underline{z}, \overline{z} \in X$ be profiles such that $\underline{z} \leq \overline{z}$, let $\underline{B}_i(x)$ and $\overline{B}_i(x)$ denote the smallest and largest best responses for i any $x \in X$, and let $\underline{B}(x)$ and $\overline{B}(x)$ denote the collections $\underline{B}_i(x)$ and $\overline{B}_i(x)$, $i \in I$. Then $\sup U([\underline{z}, \overline{z}]) = \overline{B}(\overline{z})$ and $\inf U([\underline{z}, \overline{z}]) = \underline{B}(\underline{z})$, and $U([\underline{z}, \overline{z}]) = [\underline{B}(\underline{z}), \overline{B}(\overline{z})]$.*

Proof of Lemma 1. The largest and smallest best responses are well defined by Fact 1. By definition, $\underline{B}(\underline{z})$ and $\overline{B}(\overline{z})$ are in $U([\underline{z}, \overline{z}])$, and thus $[\underline{B}(\underline{z}), \overline{B}(\overline{z})] \subset U([\underline{z}, \overline{z}])$. Suppose

[2]A function is quasi-supermodular if $f(x) \geq f(x \wedge y)$ implies $f(x \vee y) \geq f(y)$ and $f(x) > f(x \wedge y)$ implies $f(x \vee y) > f(y)$.

$z \notin [\underline{B}(\underline{z}), \overline{B}(\overline{z})]$ and, in particular, suppose $z_i \not\geq z_i^* \equiv \underline{B}_i(\underline{z})$. I claim that $z_i \notin U([\underline{z}, \overline{z}])$ because z_i is strongly dominated by $z_i \vee z_i^*$. For any $z_i \in [\underline{z}_i, \overline{z}_i]$

$$u_i(z_i \vee z_i^*, \underline{z}_{-i}) - u_i(z_i, \underline{z}_{-i}) \geq u_i(z_i^*, \underline{z}_{-i}) - u_i(z_i \wedge z_i^*, \underline{z}_{-i}) > 0, \tag{11}$$

where the first inequality follows from supermodularity and the second from the definition of z_i^*.

Since z_i^* is the smallest best response to $\underline{z}_{-i}$, (WID) implies that

$$u_i(z_i \vee z_i^*, z_{-i}) > u_i(z_i, z_{-i}) \text{ for all } z_{-i} \in [\underline{z}_{-i}, \overline{z}_{-i}]. \tag{12}$$

An analogous argument applies to show that if $z_i \not\leq \overline{B}_i(\overline{z})$, then z_i is strictly dominated. ■

Milgrom and Roberts [4, Theorem 5] state and prove this result for supermodular games. The proof above follows their proof. They derive inequality (11) and then complete the proof by pointing out that increasing differences implies

$$u_i(z_i \vee \hat{z}_i, z_{-i}) - u_i(z_i, z_{-i}) \geq u_i(z_i \vee \hat{z}_i, \underline{z}_{-i}) - u_i(z_i, \underline{z}_{-i}) \tag{13}$$

provided that $z_{-i} \geq \underline{z}_{-i}$. The lemma follows from (11) and (13). I simply point out that the (WID) condition is sufficient for the result. Notice that (WID*) is not sufficient to guarantee strict inequality in (12).

Milgrom and Roberts use the lemma to prove the theorem. Their proof goes through without modification.

5 Iterative Deletion of Weakly Dominated Strategies

Modifications of the proofs of the Lemma 1 and Theorem 1 allow us to establish descriptions of the set of strategies that survive iterative deletion of weakly dominated strategies.

Definition 15. *Given a game $\Gamma = (I, X, u, \geq)$ and subsets $X_i' \subset X_i$, with $X' = \Pi_{i \in I} X_i'$, player i's strategy $x_i \in X_i'$ is weakly dominated relative to X_i' if there exists $z_i \in X_i'$ such that $u_i(x_i, x_{-i}) \leq u_i(z_i, x_{-i})$ for all $x_{-i} \in X_{-i}'$, with strict inequality for at least one $x_{-i} \in X_{-i}'$.*

Weak dominance will typically delete more strategies than strong dominance. Hence it has the potential to provide more restrictive predictions. I analyze the implications of applying iterative deletion of weakly dominated strategies instead of iterated deletion of strongly dominated strategies. This section studies **iterative interval deletion of weakly dominated strategies**. The procedure iteratively removes weakly dominated strategies beginning with a game $\Gamma^0 = (I, X^0, u, \geq)$ in which $X^0 = [\underline{x}^0, \overline{x}^0]$ and constructs games $\Gamma^k = (I, X^k, u, \geq)$ where $X^k = [\underline{x}^k, \overline{x}^k]$ is the smallest set such that all strategies in $X^{k-1} \setminus X^k$ are weakly dominated with respect to X^{k-1}. I will describe the set of strategies that survive this process, that is, the set of strategies that are in X^k for all

k. It is possible that different ways of deleting weakly dominated strategies will lead to different limit sets. I reference results that identify games in which the order of deletion is essentially unimportant.

The procedure that iteratively deletes dominated strategies works by assuming that existing strategies are in an interval and then finding a (potentially smaller) interval of strategies that are undominated. It is possible that some strategies are weakly dominated but not strictly dominated. If this happens, then the process of iterative deletion of weakly dominated strategies will lead to a small set of surviving strategies. In this section, I point out how to modify Milgrom and Robert's arguments to apply to weak dominance. In the next section, I provide examples in which weak dominance in fact is more selective than strong dominance.

One formal motivation for generalizing Milgrom and Robert's argument is to obtain a characterization result under less restrictive assumptions on the game. Milgrom and Robert's argument depends on (WID). The key to their proof is to show that any strategy z_i that is not greater than or equal to Player i's minimum best response is strictly dominated. The argument uses supermodularity to construct a strategy that is strictly better than z_i against $\underline{z}_{-i}$ and (WID) to conclude that this strategy remains strictly better against $z_{-i} \geq \underline{z}_{-i}$. What if (WID) fails, but (WID*) holds? The identical argument identifies a strategy that is strictly better than z_i against $\underline{z}_{-i}$, but now it is only possible to conclude that this strategy is weakly better against $z_{-i} \geq \underline{z}_{-i}$. Hence if one replaced (WID) by the weaker (WID*), then one must replace strict dominance by more powerful weak dominance to obtain selection results. This discussion provides a first intuition for the results of this section. To provide a close analog to Milgrom and Robert's result, I must invoke (ID*) rather than (WID*). Subsequently, I discuss the implications of iterative deletion of weakly dominated strategies in (WID*)-supermodular games.

Theorem 2. *Let Γ be a finite ID*-supermodular game. For each player i, there exist largest and smallest strategies that survive iterative interval deletion of weakly dominated strategies, $\underline{x}_i$ and $\overline{x}_i$. Moreover, the strategy profiles $\{\underline{x}_i : i \in I\}$ and $\{\overline{x}_i : i \in I\}$ are pure Nash equilibrium profiles.*

Theorem 2 extends Theorem 1 to weak dominance. Notice that I have added the assumption that Γ is finite. I explain the importance of this assumption after the proof.

The theorem requires two preliminary results.

Let $\underline{W}_i(x)$ denote the smallest best response to x and let $\overline{W}_i(x)$ denote the largest best response to x and let $\underline{W}(x)$ and $\overline{W}(x)$ denote the collections $\underline{W}_i(x)$ and $\overline{W}_i(x)$, $i \in I$. Let $E_i(x_i) = \{z_i \in X_i : u_i(x_i, z_{-i}) = u_i(z_i, z_{-i}) \text{ for all } z_{-i} \in X_{-i}\}$.

Lemma 2. *Let Γ be an ID*-supermodular game. Let $\underline{z}, \overline{z} \in X$ be profiles such that $\underline{z} \leq \overline{z}$. There exist largest and smallest strategies that are not weakly dominated. These strategies are, respectively, the largest element in $E_i(\underline{W}_i(\overline{z}))$ and the smallest element in $E_i(\overline{W}_i(\underline{z}))$.*

The way to construct the smallest strategy that is not weakly dominated for Player i is to consider the set of strategies that are best responses to the lowest strategy in $[\underline{z}, \overline{z}]$. If there are multiple best responses, the interval-dominance property suggests that the largest of the best responses performs at least as well than other best responses against higher strategies. This observation makes the largest best response to the smallest strategy a candidate for smallest strategy that is not weakly dominated. In fact, there may be other, smaller, strategies that are equivalent to the largest best response to $\underline{z}_{-i}$ in the sense that these strategies yield identical payoffs against all strategies in $[\underline{z}_{-i}, \overline{z}_{-i}]$. The proof of Lemma 2 shows that there exists a smallest strategy that is equivalent to the largest best response to $\underline{z}_{-i}$ and that this strategy is the smallest strategy that is not weakly dominated.

Proof of Lemma 2. Let $\underline{w}_i = \overline{W}_i(\underline{z})$ be the largest best response to the smallest strategy profile for Player i. It follows from (ID*) that any $x_i \le \underline{w}_i$ is either weakly dominated by $\underline{w}_i$ or equivalent to $\underline{w}_i$ in the sense that $u_i(x_i, x_{-i}) = u_i(\underline{w}_i, x_{-i})$ for all $x_{-i} \in [\underline{z}_{-i}, \overline{z}_{-i}]$.

I claim that $E_i(\underline{w}_i)$ is a lattice. If $x, x' \in E_i(\underline{w}_i)$, then $x_i \vee x_i'$ and $x_i \wedge x_i'$ are best responses to $\underline{z}$. Hence $x_i \vee x_i' \le \underline{w}_i$ by the definition of $\underline{w}_i$. Consequently, by (ID*), $x_i \vee x_i' \in E_i(\underline{w}_i)$. Furthermore, $u_i(x_i \wedge x_i', z_{-i}) \le u_i(x_i \vee x_i', z_{-i})$ for all $z_{-i} \in [\underline{z}_{-i}, \overline{z}_{-i}]$ by (ID*). It follows that

$$2u_i(x_i \vee x_i', z_{-i}) \ge u_i(x_i \vee x_i', z_{-i}) + u_i(x_i \wedge x_i', z_{-i}) \ge u_i(x_i, z_{-i}) + u_i(x_i', z_{-i}) = 2u_i(x_i \vee x_i', z_{-i}), \tag{14}$$

where the second inequality follows from supermodularity and the equation follows because $x_i, x_i', x_i \vee x_i' \in E_i(\underline{w}_i)$. Consequently the first inequality in (14) must be an equation and $x_i \wedge x_i' \in E_i(\underline{w}_i)$ by supermodularity.

Since $E_i(\underline{w}_i)$ is a lattice, it has a smallest element, w_i^*. I claim that w_i^* is the smallest strategy that is not weakly dominated. First observe that if $x_i < w_i^*$, then x_i is weakly dominated. This claim follows because, by definition, x_i is less than $\underline{w}_i$ but not equivalent to $\underline{w}_i$. So it must be weakly dominated by $\underline{w}_i$. Second take any $z_i \not\ge w_i^*$. I claim that $z_i \vee w_i^*$ weakly dominates z_i. The claim follows because

$$u_i(z_i \vee w_i^*, z_{-i}) - u_i(z_i, z_{-i}) \ge u_i(w_i^*, z_{-i}) - u_i(z_i \vee w_i^*, z_{-i})$$

by supermodularity and w_i^* weakly dominates $z_i \vee w_i^*$ by the first observation (since $w_i^* > z_i \vee w_i^*$). Finally, to show that w_i^* is not weakly dominated, assume that w_i^* is weakly dominated by some w_i and argue to a contraction. If w_i weakly dominates w_i^*, then it must also weakly dominate $\underline{w}_i$, since w_i^* and $\underline{w}_i$ are equivalent. But since $\underline{w}_i$ is the largest best response to $\underline{z}$, $w_i \le \underline{w}_i$. Consequently w_i is equivalent to $\underline{w}_i$ or weakly dominated by $\underline{w}_i$. If w_i is equivalent to $\underline{w}_i$ it must also be equivalent to w_i^*, which contradicts the assumption that w_i weakly dominates w_i^*. If w_i is weakly dominated by $\underline{w}_i$, then it must also be weakly dominated by w_i^*, which also contradicts the assumption that w_i weakly dominates w_i^*. It follows that w_i^* is the smallest of $[\underline{z}_i, \overline{z}_i]$ that is not weakly dominated. A similar argument demonstrates that there is a largest element of $[\underline{z}_i, \overline{z}_i]$ that is not weakly dominated. ■

Let $\underline{z}, \overline{z} \in X$ be profiles such that $\underline{z} \le \overline{z}$. Let $\overline{E}_i(x)$ denote the largest element of $E_i(x)$ and $\underline{E}_i(x)$ denote the smallest element of $E_i(x)$. Let $\underline{E}(x)$ and $\overline{E}(x)$ denote the collections $\underline{E}_i(x)$ and $\underline{E}_i(x)$. Define

$$\underline{s}_i = \inf\{x_i \in [\underline{z}_i, \overline{z}_i] : x_i \text{ is not weakly dominated in } [\underline{z}, \overline{z}]\}$$

and

$$\overline{s}_i = \sup\{x_i \in [\underline{z}_i, \overline{z}_i] : x_i \text{ is not weakly dominated in } [\underline{z}, \overline{z}]\}.$$

Now let $V_i([\underline{z}, \overline{z}]) = [\underline{s}_i, \overline{s}_i]$ and $V([\underline{z}, \overline{z}]) = (V_1([\underline{z}, \overline{z}]), \ldots, V_I([\underline{z}, \overline{z}]))$. Finally let $\overline{V}([\underline{z}, \overline{z}])$ denote the interval $[\inf(V([\underline{z}, \overline{z}]), \sup(V([\underline{z}, \overline{z}]))]$.

Lemma 2 implies the following result.

Lemma 3. *Let Γ be an ID*-supermodular game. Let $\underline{z}, \overline{z} \in X$ be profiles such that $\underline{z} \leq \overline{z}$. Then $\sup V([\underline{z}, \overline{z}]) = \overline{E}(\underline{W}(\overline{z}))$ and $\inf V([\underline{z}, \overline{z}]) = \underline{E}(\overline{W}(\underline{z}))$, and $\overline{V}([\underline{z}, \overline{z}]) = [\underline{E}(\overline{W}(\underline{z})), \overline{E}(\underline{W}(\overline{z}))]$.*

Lemma 3 parallels Lemma 1. The first difference is that if $z_i \not\geq z_i^* \equiv \overline{E}(\underline{W}(\overline{z}))$, there is no guarantee that $z_i \vee z_i^*$ strictly dominates z_i. It is possible that $z_i \wedge z_i^*$ is a best response to $\underline{z}_{-i}$. Hence the second inequality in (11) could be weak. The second difference is that one can use weak dominance rather than strict dominance to delete a strategy. So one need only establish that $u_i(z_i \vee z_i^*, z-i) > u_i(z_i, z_{-i})$ for some $z_{-i} \in [\underline{z}, \overline{z}]$. This follows from the definition of z_i^*.

Proof of Theorem 2. The proof of the theorem follows the proof of Theorem 1. One applies Lemma 3 to obtain a decreasing sequence of intervals $[\underline{y}^k, \overline{y}^k]$ such that strategies outside of these intervals are weakly dominated. By monotonicity, $\lim_{k\to\infty} \underline{y}^k$ and $\lim_{k\to\infty} \overline{y}^k$ exist. Denote the limits by $\underline{y}$ and $\overline{y}$ respectively. It is straightforward to show that these limits are Nash Equilibrium profiles. In finite games (where the process of deleting strategies terminates after a finite number of iterations), it follows by construction that $\underline{y}$ and $\overline{y}$ are not weakly dominated by any strategy in $[\underline{y}, \overline{y}]$. From Lemma 3, it follows that anything that survives iterated deletion of weakly dominated strategies must be inside the interval.

The process described only removes strategies outside of the interval $[\underline{y}^k, \overline{y}^k]$. Consequently, it is possible that there are strategies in the interval $[\underline{y}, \overline{y}]$ that are weakly dominated. When the strategy set is finite, it must be the case that $\underline{y}_i$ and $\overline{y}_i$ remain undominated even if additional strategies are deleted. To see this notice that, by construction $\underline{y}_i$ is a best response to $\underline{y}_{-i}$ and the only other best responses to $\underline{y}_{-i}$ in $[\underline{y}_i, \overline{y}_i]$ are equivalent to $\underline{y}_i$. Consequently, $\underline{y}_i$ can only be weakly dominated if $\underline{y}_j$ is deleted for $j \neq i$. Hence no procedure can delete $\underline{y}_i$. Similarly, $\overline{y}_i$ cannot be deleted. This completes the proof of Theorem 2. ■

Theorem 2 uses the assumption that strategy sets are finite. This assumption guarantees that the iterative deletion process terminates in a finite number of steps and, consequently, that $\underline{y}$ and $\overline{y}$ are not weakly dominated. The next example demonstrates that the bounds obtained through the iterative process may be weakly dominated in games in which X_i are infinite.

Example 1. *Consider a three player game in which $X_1 = [0, 1]$ and $X_i = [0, 2]$ for $i = 2, 3$; $u_1(x) = x_1(x_2 - 1)$, $u_i(x) = x_1 x_2 x_3 - x_i^3/3$ for $i = 2, 3$. In this case $\overline{y}^k = (1, 2^{2^{-k}}, 2^{2^{-k}})$ and $\underline{y}^k = (0, 0, 0)$. It follows that $\overline{y} = (1, 1, 1)$ and $\underline{y} = (0, 0, 0)$. Both $\overline{y}$ and $\overline{y}$ are Nash equilibria, but $\overline{y}$ is weakly dominated with respect to strategies in $[\underline{y}, \overline{y}]$.*

Theorem 2 applies to a particular procedure for removal of weakly dominated strategies. Unlike iterative deletion of strictly dominates strategies, the outcome of iterative deletion of weakly dominated strategies may depend on the procedure. On the other hand, for some interesting classes of games, deletion of weakly dominated strategies is essentially independent of the procedure.

Marx and Swinkels [3] show that if a game satisfies the transfer of decision maker indifference (TDI) property, then two "full"[3] procedures for deleting weakly dominated strategies are the same up to the additional or removal of redundant strategies and a renaming of strategies. The TDI property states that if (given the behavior of the other players) Player i is indifferent between two strategies given the behavior of opponents, then all other players are also indifferent between the Player i's choice of strategies. TDI is restrictive, but can be shown to hold in interesting applications including (generically) the examples described in the next section.

Theorem 2 uses the assumption that Γ is an ID*-supermodular game. I am unable to prove an analog to Lemma 2 for WID*-supermodular games. Nevertheless, for the applications are WID*-supermodular games that are not ID*-supermodular games. The following result holds for WID*-supermodular games.

Theorem 3. *Let Γ be a WID*-supermodular game with strategy space $[\underline{z}, \overline{z}]$. For each player i, there exist pure Nash equilibrium profiles $\underline{x}_i$ and $\overline{x}_i$ such that all strategies that survive iterative interval deletion of weakly dominated strategies are contained in $[\underline{x}_i, \overline{x}_i]$. Moreover, there exist an increasing sequences $\{\underline{y}^n\}_{n=1}^{\infty}$ and a decreasing sequence $\{\overline{y}^n\}_{n=1}^{\infty}$ where $\underline{y}^1 = \underline{z}$ and $\overline{y}^1 = \overline{z}$; for $n > 1$, $\underline{y}_i^n = \underline{W}_i(\underline{y}^{n-1})$ and $\overline{y}_i^n = \overline{W}_i(\overline{y}^{n-1})$; and $\underline{x} = \lim_{n\to\infty} \underline{y}^n$, and $\overline{x} = \lim_{n\to\infty} \overline{y}^n$.*

I omit the proof for Theorem 3, which parallels the proof of Theorem 2. The result states that one can bound the set of strategies that survive iterative interval deletion of weakly dominated strategies by taking best responses to the largest and smallest strategies surviving. If the game satisfies (WID*), then the bounds derived in this way may be weak: They are the same bounds associated with iterative deletion of strongly dominated strategies. The result has some value for two reasons. First, the bounds apply when (WID*) replaces the more restrictive (WID) condition. Second, the next section suggests examples under which $\underline{x} = \overline{x}$, so that the result actually selects a unique strategy profile.

The next section presents an example in which the (WID) fails, but the logic of the weak dominance argument still applies. To accommodate this example, I offer an extension.

Definition 16. *Let $\Gamma = (I, X, u, \geq)$ be a game and suppose that there exists a j such that $X_j = X_j' \times X_j''$, where X_j' and X_j'' are complete lattices. The game $\Gamma = (I, X, u, \geq)$ is a interval-dominance supermodular (ID-supermodular) game conditional on X_j' if, for each $i \neq j \in I$:*

(A1) X is a complete lattice;

(A2) $u_i : X \to \mathbb{R}$ is order upper-semicontinuous in x_i for fixed x_{-i};

[3] A "full" procedure stops only if it reaches a stage where there are no weakly dominated strategies.

(A3) *u_i is supermodular in x_i for fixed x_{-i};*

(A4′) *u_i satisfies the interval-dominance property in x_i and x_{-i} on all interval sublattices of X. and*

(A5) *u_j satisfies the interval-dominance property in x''_j and (x_{-j}, x'_j) on all interval sublattices of X.*

I refer to (A5) as conditional interval dominance of u_j (given x'_j).

If Γ is an ID-supermodular game conditional on X'_j, then one can apply dominance arguments component by component. That is, one can fix $x'_j \in X'_j$ and construct a decreasing sequence $[\underline{y}^k, \overline{y}^k]$ such that the X'_j component of $\underline{y}^k$ and $\overline{y}^k$ is equal to x'_j for all k such that strategies outside of these intervals (X'_j fixed) are weakly dominated. That is, even when a game fails to be supermodular it may have enough structure so that the weak dominance arguments that apply can still impose structure on the equilibrium set.

6 Applications

Extending the results about supermodular games from strong to weak dominance is more than a curiosity only if there exist interesting games under which the assumptions of the previous section hold and the arguments reduce the set of predictions. An ideal application would be a ID*-supermodular game that is not supermodular, in which weak dominance arguments have more power to refine the set of equilibria than strong dominance arguments, and in which insights about the structure of equilibria available from the results in this paper have substantive interest.

This section describes two applications. One characteristic that the examples share is a sequential structure. They add a round of strategic behavior to an underlying game. This kind of game is a natural place to expect weak dominance to play a role as weak dominance can place restrictions on off-the-path behavior.

The examples demonstrate that there are games that are not supermodular, but have some of the structure of supermodular games. It will be clear than strong dominance arguments do not restrict the predictions. Detailed analysis of the implications of weak dominance require specialized arguments and appear in separate papers.

6.1 Cheap Talk

Kartik and Sobel [2] apply techniques like those in this paper to provide a selection argument in simple cheap-talk games. In the cheap-talk game, nature selects $t \in T$; one player, the Sender (S), learns t and sends a message $m \in M$; the other player, the Receiver (R), takes an action $a \in A$ in response to m. A strategy for S is a mapping $s : T \to M$. A strategy for R is a mapping $a : M \to A$. Assume that M is a finite, ordered, set and that A and T are equal to the unit interval. Assume that there is a prior distribution on types; for convenience assume that the prior is finitely supported and $p(t)$ is the probability that the type is t.[4] Payoffs depend only on a and t. The payoff

[4] Assuming finite support permits me to write expectations as sums rather than integrals.

to Player i when t is the Sender's type and a is the action of the Receiver is $U^i(a,t)$. Assume that $U^i(\cdot)$ is twice continuously differentiable, with negative second derivative with respect to a and positive cross partial. With this structure, order R strategies in the natural way: $a'' \succ a'$ if $a''(m) \geq a'(m)$ for all m. Order S strategies "backwards" so that $s'' \succ s'$ if and only if $s''(t) \leq s'(t)$ for all t.[5] The payoff functions for the cheap-talk game are $u_S(s,a) = EU^S(a(s(t)),t)$ and $u_R(a,s) = EU^R(a(s(t),t))$, where the expectation is taken using the prior on types. It is straightforward to check that this game satisfies the TDI condition of Marx and Swinkels [3].

Under a monotonicity condition (described below), this class of cheap-talk game is WID*-supermodular but not supermodular. Kartik and Sobel [2] demonstrate that the set of strategies that survive iterative deletion of weakly dominated strategies is strictly smaller than the set of strategies that survive iterative delation of strongly dominated strategies.

In this subsection, I describe several properties of this class of games.

Lemma 4. *Preferences are supermodular in cheap talk games.*

Proof of Lemma 4.

$$\begin{aligned} u_S(s \vee s', a) + u_S(s \wedge s', a) &= E[U^S(a(\min\{s(t), s'(t)\}), t) + U^S(a(\max\{s(t), s'(t)\}), t)] \\ &= E[U^S(a(s(t)), t) + U^S(a(s'(t)), t)] \\ &= u_S(s,a) + u_S(s',a) \end{aligned}$$

$$\begin{aligned} u_R(a \vee a', s) + u_R(a \wedge a', s) &= E[U^R(\max\{a(s(t)), a'(s(t))\}, t) + U^R(\min\{a(s(t)), a'(s(t))\}, t)] \\ &= E[U^R(a(s(t)), t) + U^R(a'(s(t)), t)] \\ &= u_R(a,s) + u_R(a',s) \end{aligned}$$

■

Without further assumptions best responses will not have any monotonicity properties in the basic cheap-talk game. For example, suppose that $U^R(a,t) = -(a-t)^2$, and the prior is uniform on $\{0, 1/N, \dots, k/N, \dots, 1\}$ for some even number N. Assume that M contains messages 0 and 1 with $0 < 1$. If the Sender always sends $m = 0$, then it is a best response for the Receiver to respond to $m = 0$ with .5 and all other messages with 0. Denote this strategy by a^{**}. Let

$$s(t) = \begin{cases} 1 & \text{if } t \in [0, .5] \\ 0 & \text{if } t \in (.5, 1] \end{cases} \text{ and } a(m) = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}.$$

The Receiver prefers $a \wedge a^{**}$ to a when S always sends $m = 0$, but R's preferences reverse when S plays s. Consequently interval dominance does not hold. One can confirm that

[5] This ordering guarantees that R's best response increases when S's strategy increases.

S's preferences violate interval dominance and that the violations do not depend on the choice of order over S's strategies.

Best response correspondences do have some monotonicity properties for a restricted version of the cheap-talk game. Henceforth consider a **monotonic restriction** of the cheap-talk game. In the monotonic restriction, the Sender and Receiver are restricted to monotonic strategies (s is monotonic if $t'' > t'$ implies $s(t'') \geq s(t')$; a is monotonic if $m'' > m'$ implies that $a(m'') \geq a(m')$). See Kartik and Sobel [2] for a justification of the monotonic restriction. I call the monotonic restriction of a cheap-talk game a **monotone cheap-talk game**.

Even with the restriction to monotonic strategies, the cheap-talk game does not satisfy increasing differences.

To see that the Sender's payoff does not satisfy increasing differences, let $s(t) \equiv 0$ and $s'(t) \equiv 1$ so that $u_S(s,a) - u_S(s',a) = E[U^S(a(0),t) - U^S(a(1),t)]$. The right-hand side is not monotonic in $a(0)$ (or in $a(1)$), so the increasing difference condition does not hold.

To see that the Receiver's payoff does not satisfy increasing differences, let $a'(t) \equiv 1$. Hence $u_R(a,s)$ does not depend on s. Fix a message $\tilde{m}$ and let

$$a(m) = \begin{cases} 0 & \text{if } m \leq \tilde{m} \\ 1 & \text{if } m > \tilde{m} \end{cases}$$

so that $u_R(a,s) - u_R(a',s)$. Increasing s will change $u_R(a',s)$ but these changes need not be monotonic.

Lemma 5. *Receiver's preferences in a monotone cheap-talk game satisfy (WID*).*

Proof of Lemma 5. Assume $U^R(a \vee a^*, s') \geq U^R(a, s')$. Let $\tilde{a}^* = \min \arg\max_a U^R(a, s')$ be the smallest best response to s'. I must show that if $s'' \succ s'$, then $U^R(a \vee \tilde{a}^*, s'') \geq U^R(\tilde{a}, s'')$. Let $\mu_s(\cdot \mid m)$ be the posterior distribution over t given $s(t) = m$. The posterior is well defined if there exists t such that $s(t) = m$. It suffices to prove that, for all m in the image of $s''(\cdot)$,

$$\sum_t u_R(a \vee \tilde{a}(m), t)\mu_{s''}(t \mid m) \geq \sum_t u_R(a(m), t)\mu_{s''}(t \mid m). \tag{15}$$

If $s'(t) < m$ for all t, then $s''(t) < m$ for all t (recall that $s'' \succ s'$ implies $s''(t) \leq s'(t)$ for all t). If there exists t such that $s'(t) = m$, then $s'' \succ s'$ implies that $\mu_{s''}$ stochastically dominates $\mu_{s'}$. Since $\tilde{a}^*(m)$ solves $\max \sum_t u_R(a,t)\mu_{s'}(t \mid m)$, it follows from the supermodularity of u_R that the solution to $\max \sum_t u_R(\tilde{a}(m),t)\mu_{s''}(t \mid m)$ is greater than $\tilde{a}^*(m)$ and by concavity of $u_R(\cdot, t)$ that $a(m) \leq \tilde{a}(m)$ implies that inequality (15) holds. If $s'(t) > m$ for all t, then $\tilde{a}(m) = 0$ by definition and inequality (15) holds.

It remains to consider the case in which there does not exist t such that $s'(t) = m$, but $s'(t) < m$ for some t. In this case, define $\underline{m}$ to be

$$\max\{m' < m : \text{ there exists } t \text{ such that } s'(t) = m'\}.$$

It follows that $\tilde{a}^*(m)$ solves $\max \sum_t u_R(a,t)\mu_{s'}(t \mid \underline{m})$. Let $\bar{t} = \max\{t : s'(t) \leq \underline{m}\}$. Since $s'(t) = \underline{m}$ for some t, $\bar{t}$ is well defined. Furthermore, $\tilde{a}(m) \leq \arg\max u_R(a, \bar{t})$. Since

$s'' \succ s'$, $\mu_{s''}(t \mid m) = 0$ if $t < \bar{t}$. It follows that $\tilde{a}^*(m) \leq \arg\max \sum_t u_R(a(m), t)\mu_{s''}(t \mid m)$ and so (15) holds.

A symmetric argument establishes that if $s'' \succ s'$, $U^R(a \wedge a^{**}, s'') \geq U^R(a, s'')$, then $U^R(a \wedge \tilde{a}^{**}, s') \geq U^R(a, s')$ (when $\tilde{a}^{**}$ is the largest best response to s''). ■

In general, the Receiver's preferences satisfy neither (WID) nor (ID*). (WID) and (ID) differ because (WID) allows one to replace a^* with $\tilde{a}$. This replacement only matters for messages not in the image of s' (otherwise the best response is uniquely defined), but it does matter for m not in the image of s'. It is straightforward to construct examples in which (ID) does not hold if a^* is too large outside the range of s'. That is, the Receiver's preferences do not satisfy (ID).

The Receiver's preferences also do not satisfy (WID). To see this, let m_0 denote the lowest message and suppose that $s'(t) > m_0$ for all t, while $s''(t) \equiv m_0$. It follows that $s'' \succ s'$. It is straightforward to construct s', a and a^* such that $U^R(a \vee a^*, s') > U^R(a, s')$ but $U^R(a \vee a^*, s'') < U^R(a, s'')$. For example, let s' be a separating strategy; let a^* be a best response to s'; and let $a(m) = \arg\max \sum_t u_R(a, t)p(t)$ for all m.

Similarly, the Sender's preferences also satisfy (WID*) but not (WID) or (ID*).

Lemma 6. *Sender's preferences in a monotone cheap-talk game satisfy (WID*).*

Proof of Lemma 6. Assume that $U^S(a', s^* \vee s) \geq U^S(a', s)$. Let $\tilde{s}^* = \min \arg\max U^S(a', s)$ be the smallest best response to a'. I must show that if $a'' \succ a'$, then $U^S(a'', s \vee \tilde{s}) \geq U^S(a'', s)$. It suffices to show that, for all t, $s(t) > \tilde{s}^*(t)$ implies that $u_S(a''(\tilde{s}^*(t)), t) \geq u_S(a''(s(t)), t))$. If $s(t) > \tilde{s}^*(t)$, then by definition of $\tilde{s}^*$, $u_S(a'(\tilde{s}^*(t), t) > u_S(a'(s(t)), t)$. The inequality must be strict because $\tilde{s}$ is the smallest best response (so type t sends the highest message that leads to the maximum available payoff) and $\tilde{s}^*(t) < s(t)$. It follows from concavity of $u_S(\cdot, t)$ that $u_S(a''(\tilde{s}^*(t)), t) \geq u_S(a''(s(t)), t))$. Notice that this inequality may be weak (if $a''(\tilde{s}^*(t)) = a''(s(t))$) so that (WID) does not hold.

A symmetric argument establishes that if $a'' \succ a'$, $U^S(a'', s^{**} \wedge s) \geq U^S(a'', s)$ implies that $U^S(a'', \tilde{s}^{**} \wedge s) \geq U^S(a'', s)$, where $\tilde{s}^{**} = \max \arg\max U^S(a'', s)$. ■

(WID) and (ID) differ because (ID) does not require the Sender to use the highest message when indifferent. If s^* does not select the higher of two best responses, then it is possible to have $u_S(a'(s^*(t)), t) \geq u_S(a'(s(t)), t))$ but not $u_S(a''(s^*(t)), t) \geq u_S(a''(s(t)), t)$.

The three preliminary results of this subsection combine to establish the following proposition.

Proposition 3. *Monotone cheap-talk games are (WID*)-supermodular.*

Proposition 3 is not sufficient to guarantee that one can use weak dominance arguments to refine predictions in monotone cheap-talk games. Kartik and Sobel [2] establish that in a widely studied class of monotone cheap-talk games there is a unique equilibrium outcome that survives iterative deletion of weakly dominates strategies (in any order).

6.2 Investment Games

Consider a two-player game in which Player 1 first makes an observable investment ($k \in K$) and then both players simultaneously make decisions ($(x_1, x_2) \in \tilde{X}_1 \times X_2$). I assume that $K, \tilde{X}_1, X_2 \subset [0,1]$ and that $X_1 = K \times \tilde{X}_1$. We can think of the strategy set for Player 1 as pairs $(k, x_1) \in X_1$. Strategies for Player 2 are mappings from K into X_2. As in the previous subsection, I will limit attention to monotonic strategies for Player 2 (if $k'' > k'$, then $x_2(k'') \geq x_2(k')$). Preferences in the game are derived from utility functions $u_i : X_1 \times X_2 \to \mathbb{R}$. Payoff functions for the extended game take the form $U_i((k, x_1), x_2) = u_i(x_1, x_2(k)) + \lambda_i k$. Order Player 1's strategies by the order on X_1 induced by the standard order on $[0,1] \times [0,1]$. Order Player 2's strategies $x_2'' \geq x_2'$ if and only if $x_2''(k) \geq x_2'(k)$ for all K. Provided that the underlying game has generic payoffs (for example, $u_i(x) = u_i(x')$ implies $x = x'$), then the investment games satisfies the TDI condition of Marx and Swinkels [3].

It is possible to interpret this setting as a model of communication about intensions. Under this interpretation, Player 1 first sends a signal to Player 2 and then they play a two-player simultaneous move game. An alternative interpretation is that k is an investment that Player 1 makes prior to the players participating in a two-player game. For the first interpretation, $\lambda_i = 0$ for both i. For the second interpretation, $\lambda_1 < 0 = \lambda_2$.

Definition 17. *An investment game is a* **regular investment game** *if the underlying game determined by* (u_1, u_2) *is a supermodular game (so that* u_i *satisfies increasing differences and is supermodular for* $i = 1, 2$*);* $x_2(\cdot)$ *is monotonic; and* $u_1(\cdot, x_2)$ *is single-peaked in its first argument for all* x_2*.*[6]

Lemma 7. *Preferences are supermodular in regular investment games.*

Proof of Lemma 7. Assume without loss of generality that $k \geq k'$,

$$U_1((k, x_1) \vee (k', x_1''), x_2) + U_1((k, x_1) \wedge (k', x_1'), x_2) =$$
$$u_1(x_1 \vee x_1', x_2(k)) + u_1(x_1 \wedge x_1', x_2(k')) + \lambda_1(k + k').$$

Note that

$$u_1(x_1 \vee x_1', x_2(k)) + u_1(x_1 \wedge x_1', x_2(k')) + \lambda_1(k + k') =$$
$$u_1(x_1 \vee x_1', x_2(k)) + u_1(x_1 \wedge x_1', x_2(k)) + (u_1(x_1 \wedge x_1', x_2(k')) - u_1(x_1 \wedge x_2', x_2(k))) + \lambda_1(k + k') \geq$$
$$u_1(x_1, x_2(k)) + u_1(x_1', x_2(k)) + u_1(x_1 \wedge x_1', x_2(k')) - u_1(x_1 \wedge x_1', x_2(k)) + \lambda_1(k + k') =$$
$$U_1((k, x_1), x_2) + U_1((k', x_1'), x_2) + u_1(x_1', x_2(k)) - u_1(x_1', x_2(k')) + u_1(x_1 \wedge x_1', x_2(k')) - u_1(x_1 \wedge x_1', x_2(k)) \geq$$
$$U_1((k, x_1), x_2) + U_1((k', x_1'), x_2)$$

where the first inequality follows from the supermodularity of $u_1(\cdot, y)$ and the second inequality follows from

$$u_1(x_1', x_2(k)) - u_1(x_1', x_2(k')) + u_1(x_1 \wedge x_1', x_2(k')) - u_1(x_1 \wedge x_1', x_2(k)) \geq 0$$

[6] The function $u_1(\cdot, x_2)$ is single peaked in its first argument if there exists $x_1^*(x_2)$ such that $u_1(x_1, x_2)$ is increasing in x_1 for $x_1 < x_1^*(x_2)$ and decreasing in x_1 for $x_1 > x_1^*(x_2)$.

(because $u_1(\cdot)$ satisfies increasing differences, $x_2(\cdot)$ is monotonic, and $k \geq k'$). This establishes supermodularity of $U_1(\cdot)$.

Supermodularity of U_2 is straightforward:

$$\begin{aligned} U_2((k, x_1), x_2 \vee x_2') + U_2((k, x_1), x_2 \wedge x_2') &= u_2(x_1, \max\{x_2(k), x_2'(k)\}) + u_2(x_1, \min\{x_2(k), x_2'(k)\}) \\ &= U_2((k, x_1), x_2) + U_2((k, x_1), x_2'). \end{aligned}$$

■

Lemma 8. *In a regular investment game, preferences for Player 2 satisfy interval dominance.*

Proof of Lemma 8. Fix a strategy (k', x_1') for Player 1. Let x_2^* be a best response to this strategy. I will show that if $U_2((k', x_1'), x_2 \vee x_2^*) \geq U_2((k', x_1'), x_2)$ and $(k'', x_1'') \geq (k', x_1')$, then there exists $\tilde{x}_2^*$ such that $\tilde{x}_2^* \leq x_2^*$ and $\tilde{x}_2^*$ is a best response to (k', x_1') such that $U_2((k'', x_1''), x_2 \vee \tilde{x}_2^*) \geq U_2((k'', x_1''), x_2)$. To do this, it suffices to show that

$$u_2(x_1', x_2 \vee \tilde{x}_2^*(k')) \geq u_2(x_1', x_2(k')) \implies u_2(x_1'', x_2 \vee \tilde{x}_2^*(k'')) \geq u_2(x_1'', x_2(k'')). \tag{16}$$

If $x_2(k') \geq x_2^*(k')$, (16) clearly holds (with $\tilde{x}_2^* = x_2^*$). If $x_2(k') < x_2^*(k')$, then set

$$\tilde{x}_2^*(k) = \begin{cases} \min\{x_2^*(k), \max\{x_2(k), x_2^*(k')\}\} & \text{if } k \geq k' \\ x_2^*(k) & \text{if } k < k' \end{cases}.$$

To establish (16) it suffices to show

$$u_2(x_1'', \tilde{x}_2^*(k'')) \geq u_2(x_1'', x_2(k'')) \tag{17}$$

whenever $\tilde{x}_2^*(k'') > x_2(k'')$. It follows from the definition of $\tilde{x}_2(\cdot)$ that if $\tilde{x}_2^*(k'') > x_2(k'')$, then $\tilde{x}_2^*(k'') = x_2^*(k')$. By definition, $x_2^*(k')$ is a best response to x_1'. Increasing differences (of $u_2(\cdot)$), implies that the best response(s) to x_1'' is greater than $x_2^*(k')$. Consequently, Inequality (17) follows because $u_2(x_1', \cdot)$ is single peaked and $x_2^*(k') = \tilde{x}_2^*(k'') > x_2(k'')$.

A symmetric argument establishes the other part of the definition of (ID). ■

Lemma 9. *In a regular investment game, Player 1's preferences satisfy conditional ID* given k for all k.*

Proof of Lemma 9. To show that the game satisfies conditional ID* for Player 1, fix a strategy x_2' for Player 2 and let $x_2'' > x_2'$. Let (k^*, x_1^*) satisfy $U_1((k^*, x_1^*), x_2') \geq U_1((k^*, x_1), x_2')$ for all x_1. I want to show that

$$U_1((\tilde{k}^*, x_1) \vee (\tilde{k}^*, \tilde{x}_1^*), x_2') \geq U_1((k^*, x_1), x_2') \implies U_1((k^*, x_1) \vee (\tilde{k}^*, \tilde{x}_1^*), x_2'') \geq U_1((k^*, x), y''). \tag{18}$$

The implication follows directly from the interval dominance property of $u_1(\cdot)$ since in that case

$$U_1((k^*, x) \vee (k^*, x^*), y) = u_1(x \vee x^*, x_2(k^*)) \text{ and } U_1((k^*, x_1), x_2) = u_1(x_1, x_2(k^*))$$

for all x_2. Consequently (18) is equivalent to

$$u_1(x_1 \vee x_1^*, x_2'(k^*)) \geq u_1(x_1, x_2'(k^*)) \implies u_1(x_1 \vee x_1^*, x_2''(k^*)) \geq u_1(x_1, x_2''(k^*)).$$

■

The lemmas of this subsection combine to establish the following proposition.

Proposition 4. *Regular investment games games are (ID*)-supermodular games conditional on K.*

Proposition 4 is not sufficient to guarantee that one can use weak dominance arguments to refine predictions in monotone cheap-talk games. Sobel [7] establishes that a unique equilibrium outcome that survives iterative deletion of weakly dominates strategies (in any order) in a class of games with pre-play communication.

A regular investment game need not satisfy (WID*) for Player one. In order to satisfy (WID*) it must be the case that there exists $(\tilde{k}^*, \tilde{x}_1^*) \leq (k^*, x_1^*)$ such that $U_1((\tilde{k}^*, \tilde{x}_1^*), x_2') \geq U_1((k, x_1), x_2')$ for all (k, x_1) and

$$U_1((k, x_1) \vee (\tilde{k}^*, \tilde{x}_1^*), x_2') \geq U_1((k, x_1), x_2') \implies U_1((k, x_1) \vee (\tilde{k}^*, \tilde{x}_1^*), x_2'') \geq U_1((k, x_1), x_2''). \quad (19)$$

Suppose that $u_1(x_1^*, x_2'(k^*)) > u_1(x_1, x_2)$ unless $(x_1, x_2) = (x_1^*, x_2'(k^*))$. When $x_1 = x_1^*$, $k \geq k^*$, $k < k^*$, and $x_2''(k^*) > x_2''(k) = x_2'(k^*) > x_2'(k)$. Under these assumptions (19) reduces to

$$u_1(x_1^*, x_2'(k^*)) \geq u_1(x_1^*, x_2'(k)) \implies u_1(x_1^*, x_2''(k^*)) \geq u_1(x_1^*, x_2'(k^*)). \quad (20)$$

The implication in (20) fails when $x_2''(k^*) > x_2'(k^*)$.

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