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Dynamic Voluntary Contribution to a Public Project

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Abstract

We consider the dynamic private provision of funds to projects that generate flows of public benefits. Participants have complete information about the environment, but imperfect information about individual actions: each period they observe only the aggregate contribution. Each player may contribute any amount in any period before the contributing horizon is reached. All Nash equilibrium outcomes are characterized. In many cases they are all also perfect Bayesian equilibrium outcomes. If the horizon is long, if the players' preferences are similar, and if they are patient or the period length is short, perfect Bayesian equilibria exist that essentially complete the project. In some of them the completion time shrinks to zero with the period length – efficiency is achieved in the limit.

KEYWORDS: public goods, private provision, voluntary contribution, dynamic games

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1 Introduction

Time is an essential ingredient of many voluntary contribution schemes that finance public projects. Fund drives to support public television or famine relief, or to build church complexes, park systems, or library collections, generally take place over periods of weeks or years. The public is generally kept informed of how much has been contributed to date, but not of each individual's contribution history. Contributions are generally nonrefundable, and individuals are free to contribute more than once and at any time during the drive. The model of this paper is intended to capture these features.

The results are fairly positive. Allowing contributions to be made slowly over time enhances efficiency in some equilibria, even though individual contributions are private information. Nearly efficient perfect Bayesian equilibria exist provided (i) the players evaluate the public good similarly, (ii) the number of periods is large, and (iii) discounting is low or the period length small. The only inefficiency is then a completion delay, which vanishes in some equilibria as the period length shrinks to zero. Dynamics can thus alleviate the well-known inefficiencies of static contribution games.[1]

These positive results are tempered by the fact that our game generally has other equilibria in which no contributions are made. On the other hand, positive results are surprising in view of the following intuition: allowing players to contribute repeatedly can worsen incentives by creating future players upon which current players can free ride. This is the logic behind the negative results of Fershtman and Nitzan (1991) and Admati and Perry (1991). In their models a player can sometimes raise the level of future contributions by lowering his current contribution, which gives him an incentive to free ride on future players. Our game, in contrast, has equilibria in which a player is deterred from contributing too little in one period because doing so causes the other players to contribute nothing in the next.

We consider the model both with and without discounting. In the no-discounting version, a player's payoff is received in the final period of the fund drive. It is then equal to the player's

[1]Static contribution games are studied, e.g., by Andreoni (1988), Bergstrom, et al. (1986), Bernheim (1986), Cornes and Sandler (1996), Moore et. al. (1995), Palfrey and Rosenthal (1984), and Varian (1994).

benefit from the public good, which depends on the total of all past contributions, less the sum of the player's own past contributions. This payoff is appropriate for modeling short fund drives, such as a two-week public radio drive or a two-day telethon to raise money for medical research. In such cases only the final cumulations matter – the time path of contributions is irrelevant for payoffs, and no benefits are received until the drive is over.

The discounting version of the model is more appropriate for long fund-raising drives, such as multi-year campaigns to raise money to build a cathedral, a bridge, a complex of university buildings, a regional road network, or a collection of books for a new library. A player's payoff is then a discounted sum of costs and benefits. His cost in a period is his contribution that period; his benefit is determined by the cumulative prior contributions of all the players. If this benefit is an increasing function of the cumulation, partial benefits are received while funds are still being raised. This is the case if the project is completed and used in stages. For example, the first buildings in the new university complex can be built and put into service before enough has been donated to build the entire complex; the first roads in the new road network can be constructed and used before the towns to be connected have contributed enough to build the final road; the first books for the new library can be purchased and loaned to readers before all the money required for the library collection has been raised. Alternatively, the project may be "binary" in that it generates no benefits until it has been completed, and then yields a constant benefit flow. The textbook example is the building of a bridge – no benefits are generated until the last girder is in place.

The benefit functions we consider range from binary ones to continuous ones that rise linearly with the cumulative contribution up to some "completion point." In intermediate cases benefits rise with the cumulation and jump up at the completion point. The building of a road network is a plausible example: benefits increase with the number of roads that are built and put into service, and they rise discontinuously at the completion point when the linking road is finished. Another example is a charity campaign aimed at famine relief or disease prevention: benefits increase with the number of victims that are fed or treated, but the big payoff comes when the cause of the famine is eliminated or a cure for the disease is discovered. Many charity fund drives exhibit a benefit jump at their conclusion for a different reason: a wealthy donor creates the jump by having committed to contributing a "challenge bonus" when the contribution goal of the drive is reached (Firestone, 1998). Thus, a benefit jump at completion may be due to prior design or to technology.

The static version of our model is a coordination game if each player's benefit jump is large, as in the binary case. Each player is then willing to contribute if the others contribute above a certain level. The static game has an inefficient equilibrium in which no player contributes, and efficient equilibria that complete the project but differ in the distribution of contributions. In this case the interesting question is whether efficient outcomes are achieved by perfect Bayesian equilibria if the players can spread their contributions over time. The answer is negative in the game of Admati and Perry (1991), which has a binary benefit function; each of its two players has too strong an incentive to let the other contribute in the future (see our Section 7). But in our model, allowing even an infinite number of contributing periods does not eliminate the efficient equilibria. We prove this by constructing equilibria that impose the maximal possible punishment on a unilateral deviator – the non-deviators stop contributing forever. Given any equilibrium profile of the static game, the threat of this punishment in the dynamic game induces the players to contribute that profile in the first period. A non-deviator is willing never to contribute again either because he expects the deviator immediately to complete the project alone, or the project is so far from completion that no player wants to contribute given that the others do not contribute. The delicate part of the argument is to show that these strategies can be implemented even though the identity of a deviator need not be common knowledge; beliefs are constructed that allow the players to coordinate their punishment strategies.

The more interesting case is that in which the benefit jumps are sufficiently small that the static game does not have an equilibrium in which contributions are made. (The static game is a prisoners' dilemma if the jumps are nonexistent – contributing nothing is a dominant strategy.) Now the question is whether making the game dynamic creates perfect Bayesian equilibria in which the players contribute. Under certain conditions it does. To prove this we again construct equilibria that punish a deviator by having the non-deviators never contribute again. But now the players contribute only a small amount each period. This increases the incentives to contribute for the following reasons. On the one hand, the cost to a player of deviating in the current period is increased by shifting some of the other players' remaining contributions to the future, to be contributed contingent on no deviation having been detected. This causes a deviation in the current period to reduce the future contributions of the others. On the other hand, the gain to a player from free riding in the current period is decreased by shifting some of that player's remaining contribution to the future since his current contribution in then reduced. The contributions shifted to the future can be adjusted so that the gain from free riding in the current period is always less than its cost. This may require that sizable contributions be postponed. But if the discount factor is high, the rate of contributing can be kept large enough that the cumulative contribution approaches, and in some cases reaches, the completion point.

If the benefit functions are continuous, this logic is the only way to induce contributions – a player will contribute in the current period only if threatened by the withdrawal of future contributions by the others. Thus, in no period can the project be completed. An equilibrium can nonetheless be constructed that completes the project asymptotically in that the cumulation converges to the completion point; approximate completion occurs nearly instantly as the period length converges to zero.

In contrast, if the benefit functions are not continuous, another logic can be used once the project is close to completion. As the cumulative contribution approaches the completion point, the cost of completion becomes small relative to the size of the benefit jumps at completion. Thus, when the benefit functions are not continuous, the continuation game looks much like a static game that has an equilibrium in which the project is completed. Contributions can thus be induced in early periods by the threat of losing future contributions, until the project is close enough to completion that it can be completed in one period. Some equilibria of this type complete the project in a bounded number of periods, and so a fortiori complete it instantaneously as the period length converges to zero.

The equilibria just discussed provide incentives by threatening severe punishments, as a deviation causes the non-deviators never to contribute again. But approximately efficient equilibria that use more forgiving punishments also exist (again, given similar benefit functions, a sufficient number of contributing periods, and high discount factors). We identify specifically a kind of Markov perfect equilibrium that we refer to as a "contribution goal equilibrium." It is characterized by an increasing sequence of goals that the cumulative contribution is to achieve. In each period, the set of players who are to contribute is determined by the smallest goal so far unachieved, and their contributions bring the cumulation up to that goal. (Incentives are strongest if only one player is responsible for attaining each goal – we focus on such an equilibrium.) Thus, in equilibrium the cumulation is raised successively from goal to goal. Punishments are forgiving in that a player who free rides in one period simply delays the achievement of the current and subsequent goals by one period. The outcomes of these equilibria resemble to some extent the successive attainment of announced contribution goals in some fund drives.

The paper starts with a description of the model in Section 2. Nash equilibrium outcomes are characterized in Section 3. Section 4 concerns equilibria that do not complete the project. Section 5 considers equilibria that do complete it and are approximately efficient. In Section 6 we present sufficient conditions for all Nash equilibrium outcomes to be perfect Bayesian equilibrium outcomes. Related literature is discussed in Section 7, and conclusions in Section 8. Longer proofs are in Appendices A and B. The related alternating-contribution model of Admati and Perry (1991) is discussed in Appendix C.

2 The Model

The game has the following elements. The set of players is $N \equiv \{1, \ldots, n\}$, with $n \geq 2$. Each player chooses how much of a private good to contribute to a public project in each period $t \geq 0$. Player i contributes $z_i(t)$ in period t. Contributions are nonrefundable, and they cannot be made after the *contributing horizon,* $\bar{T} \leqslant \infty$, is reached. Budget constraints are assumed to be non-binding. Thus, any $z_i \geq 0$ is feasible in period $t \leq \bar{T}$, and only $z_i = 0$ is feasible in period $t > \bar{T}$.

Denote the contribution vector in period t as $z(t) \equiv (z_1(t), \ldots, z_n(t))$, and the entire contribution sequence as $\{z\} \equiv \{z(t)\}_{t=0}^{\infty}$. Player i's payoff, $U_i(\{z\})$, is specified below.

Each player sees only his own past contributions and the aggregate of the other players' past contributions. Let $Z(t) \equiv \sum_{j \in N} z_j(t)$ be the aggregate contribution in period t, and let $Z_i(t) \equiv Z(t) - z_i(t)$. Player i's *personal history* at the start of period t is then

$$h_i^{t-1} \equiv (z_i(\tau),\, Z_i(\tau))_{\tau=0}^{t-1}.$$

The player's strategy maps each h_i^{t-1} into a contribution that is feasible in period t.

This defines the *contribution game with unobserved contributions,* the subject of most of our results. At times we consider the game in which individual contributions are publicly observed, so that each player's information sets are indexed by the public histories $h^{t-1} \equiv (z(\tau))_{\tau=0}^{t-1}$. To avoid confusion, we always refer to this game explicitly as the *game with observed contributions*. The two games are the same if $n = 2$.

Payoffs depend on cumulative contributions. The individual cumulative contribution of player i at the end of period t is $x_i(t) \equiv \sum_{\tau \leqslant t} z_i(\tau)$. (Set $x_i(-1) \equiv 0$.) The aggregate cumulative contribution, or more simply the *cumulation*, is $X(t) \equiv \sum_{j \in N} x_j(t)$.

In the version of the model without discounting, any benefits and costs borne in a finite number of periods are inconsequential. Thus, if the contributing horizon $\bar{T}$ is finite, payoffs depend only on the final cumulations $X(\bar{T})$ and $x_i(\bar{T})$. The total benefit player i receives from the project in all the periods after $\bar{T}$ is $f_i(X(\bar{T}))$, where f_i is the *benefit function* and is specified below. The cost of contributing enters quasilinearly. Thus, player i's payoff in the no-discounting version of the model, for $\bar{T} < \infty$, is

$$U_i(\{z\}) = f_i(X(\bar{T})) - x_i(\bar{T}). \tag{2.1}$$

This is an appropriate payoff function for modeling short fund drives, such as a two-week public radio campaign. It covers the case in which there is no flow of benefits until the fund drive is over, as when contributions are not used or collected until $\bar{T}$. If the horizon is $\bar{T} = \infty$, the generalization of (2.1) is

$$U_i(\{z\}) = \lim_{t \to \infty} \left[f_i(X(t)) - x_i(t) \right].^2 \tag{2.2}$$

In the discounting version of the model, players discount benefits and costs by a factor $\delta = e^{-r\ell} \in (0,1]$. The discount rate is $r \geq 0$ and the period length is $\ell > 0$. (As we shall see, the no-discounting version of the model is formally the special case $r = 0$.) The contributions in a period are converted into the non-depreciating capital the project uses to generate benefits in that and subsequent periods. Thus, the cost of a contribution is borne when it is made. The total benefit, over all periods, that the project generates for player i if the cumulation is forever fixed at X is still $f_i(X)$. Thus, with discounting and a growing cumulation, player i receives a benefit of $(1-\delta)f_i(X(t))$ in period t. His overall payoff is

$$U_i(\{z\}) \equiv \sum_{t=0}^{\infty} \delta^t \left[(1-\delta) f_i(X(t)) - z_i(t) \right], \tag{2.3}$$

which is the same as (2.2) if $\delta = 1$.[3] If $\delta < 1$ and $f_i(X(t)) > 0$, the project generates a benefit in period t, even if more contributions will be made in subsequent periods. This is true, for example, of long-term fund drives/projects that expand a road network, park system, library, or university; the first of eventually many roads, parks, books, or classrooms are built and put in use even though more donating and building will occur in the future. Alternatively, if $f_i(X) = 0$ until X exceeds some threshold, benefits are not generated while contributions are being collected until the threshold is reached. If the threshold is the completion of the project, benefits are generated only after all contributions have been made. This is the case of a binary project, such as the proverbial building of a bridge.

If $\bar{T} = 0$, so that contributions are allowed only in the first period, (2.3) becomes $U_i(\{z\}) = f_i(Z(0)) - z_i(0)$. This special case is the familiar *static game* in which the players contribute at most once and simultaneously.

The function f_i is the composition of the project's production function with player i's evaluation function for it's services. The binary benefit function commonly studied is zero

[3] Taking $\delta \to 1$ in (2.3) yields (2.2) if $f_i(X(t)) - x_i(t)$ converges, as it does if strategies are undominated.

until the cumulation reaches a certain level, after which it is constant. We study a more general parameterized class of benefit functions. One parameter is a *completion point*, $X^* < \infty$; the project is completed once the cumulation reaches X^*. The project may also generate benefits without being completed, linearly in the cumulation. Thus, for $i \in N$,

$$f_i(X) \equiv \begin{cases} \lambda_i X & \text{for } X < X^* \\ V_i & \text{for } X \geq X^*. \end{cases} \tag{2.4}$$

The player's marginal benefit from a non-completing contribution is λ_i. His benefit from the completed project is V_i. The benefit jump at completion is $b_i \equiv V_i - \lambda_i X^*$.

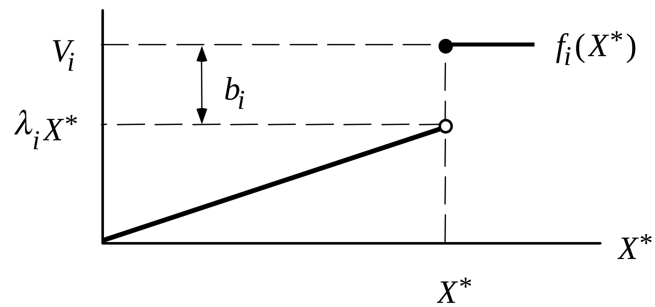

Figure 1

We assume $\lambda_i \geq 0$ and $b_i \geq 0$; these inequalities are equivalent to

$$0 \leqslant \lambda_i \leqslant \frac{V_i}{X^*} \text{ for all } i \in N. \tag{2.5}$$

The polar case $\lambda_i = 0$ yields the *binary benefit function*. The other polar case, $b_i = 0$, yields the *continuous benefit function*. A positive b_i represents strong increasing returns. As discussed in the introduction, it may be due to technology, such as the linking road that completes a network or the girder that completes a bridge. Or it may be due to design, as when some party has publicly committed to contributing a fixed amount if and when the goal of a fund drive is achieved. (If benefits equal λ times the dollars contributed, and B dollars will be contributed if the goal X^* is met, the benefit jump is $b = \lambda B$.)

Payoffs can now be written in a way useful for dynamic programming. Given $\{z\}$, the *completion period* $T(\{z\})$ is the smallest t for which $X(t) \geq X^*$, with $T(\{z\}) = \infty$ if the project is not completed.[4] Refer to $\{z\}$ as *wasteless* if $X(t) \leqslant X^*$ for all t. If it is wasteless, (2.3) and (2.4) imply

$$\begin{aligned} U_i(\{z\}) &= \textstyle\sum_{t=0}^{T} \delta^t \left[\lambda_i Z(t) - z_i(t)\right] + \sum_{t=T}^{\infty} \delta^t \left[(1-\delta) b_i\right] \\ &= \textstyle\sum_{t=0}^{T} \delta^t \left[\lambda_i Z(t) - z_i(t)\right] + \delta^T b_i, \end{aligned} \tag{2.6}$$

where $T = T(\{z\})$. This expresses the payoff as a discounted sum of benefits and costs that are each borne in just one period; it is as though $\lambda_i Z(t) - z_i(t)$ is received in each period prior to completion, and $(1-\delta)b_i$ is received in each subsequent period. If $\lambda_i = 0$, (2.6) becomes $\delta^T V_i - \sum_{t=0}^{T} \delta^t z_i(t)$. If instead $b_i = 0$, it becomes $\sum_{t=0}^{T} \delta^t \left[\lambda_i Z(t) - z_i(t)\right]$, which looks like the payoff of a repeated game with stage game payoffs $\lambda_i Z - z_i$. But the game is not a repeated game: a player's interval of possibly undominated contributions in period t, $[0, X^* - X(t-1)]$, depends on prior contributions, and the completion period is endogenous.

We restrict attention to the case in which free riding is an issue, which is defined by

$$V_i < X^* < \sum_{j=1}^{n} V_j \text{ for all } i \in N. \tag{2.7}$$

The first inequality insures that no player is willing to complete the project alone. The second insures that the project is worthwhile. Assumptions (2.5) and (2.7) together imply $\lambda_i < 1$: the marginal benefit to any player of a non-completing contribution is less than its marginal cost. The players thus have incentives to free ride, but efficiency requires them to complete the project without waste and, if $\delta < 1$, without delay.

3 Nash Equilibria

We restrict attention to pure-strategy equilibria. In this section we characterize the Nash equilibrium outcomes; in Section 6 we show that under a range of parameters, each of them is also a perfect Bayesian equilibrium outcome.

[4] For $T = \infty$, the convention below is that $\delta^T = 0$ for all $\delta \leq 1$.

The static game provides a benchmark. Dropping the time argument, a strategy profile in this game, $(z_1, ..., z_n)$, yields an aggregate contribution Z and payoffs $f_i(Z) - z_i$. Given a contribution $Z_i < X^*$ by the others, player i's best reply contribution is either the completing amount, $X^* - Z_i$, or nothing. (Intermediate amounts are inferior because $\lambda_i \neq 1$.) His marginal benefit from completing the project is $f_i(X^*) - f_i(Z_i) = V_i - \lambda_i Z_i$, and his marginal cost of doing so is $X^* - Z_i$. The former exceeds the latter if and only if the completing amount is less than the player's *critical contribution:*

$$c_i^* \equiv \frac{V_i - \lambda_i X^*}{1 - \lambda_i} = \frac{b_i}{1 - \lambda_i}. \tag{3.1}$$

Thus, the player should complete the project if and only if the amount required does not exceed his critical contribution. The reaction function, for $Z_i < X^*$, is

$$z_i^R(Z_i) \equiv \begin{cases} 0 & \text{if } X^* - Z_i > c_i^* \\ X^* - Z_i & \text{if } X^* - Z_i < c_i^*. \end{cases} \tag{3.2}$$

If f_i is continuous, then $c_i^* = 0$ and contributing nothing is the player's dominant strategy; the static game is a prisoners' dilemma if all the benefit functions are continuous. In general, as (2.7) implies $c_i^* < X^*$, one equilibrium of the static game is always the no-contribution profile $(0, ..., 0)$. A completing equilibrium exists if and only if X^* is not less than the sum of the critical contributions.[5] Completing equilibria thus exist if all the benefit functions are binary, as then $\sum_{i \in N} c_i^* = \sum_{i \in N} V_i > X^*$; the static game is then a coordination game in which each player is willing to contribute if the others contribute. Theorem 0 summarizes.

Theorem 0 *One equilibrium of the static game is always* $(0, ..., 0)$. *Any other* $(z_1, ..., z_n)$ *is an equilibrium if and only if* $\sum_{i \in N} z_i = X^*$ *and* $0 \leq z_i \leqslant c_i^*$ *for all* $i \in N$.

We index the players henceforth by the size of their critical contributions:

$$c_1^* \geq c_2^* \geq ... \geq c_n^*.$$

[5] The profile $\left(c_i^* X^* / \sum_{j \in N} c_j^*\right)_{i \in N}$ is an equilibrium of the static game iff $\sum_{j \in N} c_j^* \geq X^*$.

Because c_i^* is player i's maximum possible equilibrium contribution in the static game, it provides a measure of his incentive to free ride. The smaller is c_i^*, the larger is his incentive to free ride. By this measure, (3.1) implies that the incentive to free ride decreases in b_i, holding the marginal benefit λ_i fixed. But the incentive to free ride increases in λ_i, holding V_i fixed, since the player's marginal benefit from completing the project, $V_i - \lambda_i Z_i$, falls as his benefit $\lambda_i Z_i$ from contributing nothing increases.

We now turn to the dynamic game. Let $g = \{(g_1(t), ..., g_n(t))\}_{t=0}^{\infty}$ be a sequence of nonnegative contributions. The corresponding aggregate contribution in period t, and the aggregate of all contributions but that of player i, are

$$G(t) \equiv \sum_{i=1}^{n} g_i(t) \text{ and } G_i(t) \equiv G(t) - g_i(t).$$

Two necessary conditions for g to be an equilibrium outcome are that it be feasible, so that $G(t) = 0$ for all $t > \bar{T}$, and wasteless, as defined in the previous section. Refer to g as a *candidate outcome* if it is both feasible and wasteless.

A candidate outcome g is a Nash equilibrium outcome if and only if no player wishes unilaterally to deviate from it when doing so is met by a maximal feasible punishment. A strategy profile in which all the other players never contribute imposes the maximal conceivable punishment on a unilateral deviator. This punishment is imposed by the *grim-g strategy profile* in which g is played every period unless an event of the form $Z(t) \neq G(t)$ is witnessed, in which case no player ever contributes again. This strategy profile is feasible, even though individual contributions are unobserved, because it is based only on aggregates. Thus, g is a Nash equilibrium outcome if and only if the grim-g profile is a Nash equilibrium.[6]

This observation leads to an intuition for why more contributions may be obtained with a longer contributing horizon. Consider a contribution vector $z = (z_1, \ldots, z_n)$ that completes the project, but is not an equilibrium of the static game. So, if the others contribute Z_i, some player i prefers to contribute nothing than to contribute z_i:

$$\lambda_i Z_i > V_i - z_i. \tag{3.3}$$

[6] It follows that the game with observed contributions has the same Nash equilibrium outcomes.

So, z is not a first-period equilibrium contribution vector even if $\bar{T} > 0$. If the others contribute Z_i in the first period, the right side of (3.3) is still player i's payoff from contributing z_i, and the left side is a lower bound on his payoff if he deviates to zero – it is his payoff if no player contributes after the deviation. Now, consider an outcome g in which the contributions in z are made in stages over multiple periods. Assuming no discounting, g still gives player i payoff $V_i - z_i$. But his payoff from deviating to zero in the first period, given that it stops future contributions, is $\lambda_i G_i(0)$. So the player will not deviate from the grim-g profile in the first period, given that it is played thereafter, if the contributions of the others in period 0 are so small that

$$\lambda_i G_i(0) < V_i - z_i. \tag{3.4}$$

This shows that a player will contribute in the first period if the others contribute only a small amount then, shifting the bulk of their contributions to the future to be made on a contingent basis.

The generalization of (3.4) to other periods and discount factors is

$$\lambda_i G_i(t) \leqslant \delta^{T(g)-t} b_i + \lambda_i \sum_{\tau=t}^{T(g)} \delta^{\tau-t} G_i(\tau) - (1-\lambda_i) \sum_{\tau=t}^{T(g)} \delta^{\tau-t} g_i(\tau). \tag{3.5}$$

Given that the others play the grim-g strategies and no deviation has occurred, player i prefers to contribute according to g in period t and thereafter, rather than to deviate to zero, if and only if (3.5) holds. The left side is the current value of the player's payoff if he deviates to zero in period t and thereafter, dropping the terms due to previous contributions (see (2.6)). The right side is his payoff if he does not deviate, dropping the same terms from previous periods and again discounting to period t. Note that subtracting an amount from $G_i(t)$ and adding it to $G_i(T(g))$ enlarges the gap between the right and left sides of (3.5). The player's incentive to contribute is thus increased if the others shift some of their current contributions to the future. In addition, if $\delta < 1$ his incentive to contribute is increased if some of his own contribution is shifted to the future: the right side of (3.5) increases if an amount is subtracted from $g_i(t)$ and added to $g_i(T(g))$. That is, since the net benefit of a non-completing contribution, $(\lambda_i - 1)g_i(t)$, is negative, an impatient player prefers to postpone his contributions, ceteris paribus.

Rearrangement of (3.5) yields the *under-contributing constraint:*

$$(1-\lambda_i)g_i(t) \leqslant \sum_{\tau=t+1}^{T(g)} \delta^{\tau-t}\left[\lambda_i G(\tau) - g_i(\tau)\right] + \delta^{T(g)-t} b_i \tag{3.6}$$

$$\text{for } i \in N \text{ and } t \leqslant T(g).$$

As we have seen, (3.6) deters downward deviations (free riding). It puts an upper bound on each player's contribution in each period as a function of the future contributions. Its left side is player i's current net cost of contributing $g_i(t)$. Its right side is his continuation payoff if he does not deviate; given the grim-g strategies, the right side is the payoff he foregoes by not contributing $g_i(t)$.

A second constraint, the *over-contributing constraint,* deters upward deviations. It insures that the completing amounts, $X^* - \sum_{\tau \leqslant t} G(\tau)$, are large enough that no player wants to complete the project prematurely:

$$(\lambda_i - 1)\left(X^* - \sum_{\tau=0}^{t} G(\tau)\right) + b_i \leq \sum_{\tau=t+1}^{T(g)} \delta^{\tau-t}\left(\lambda_i G(\tau) - g_i(\tau)\right) + \delta^{T(g)-t} b_i \tag{3.7}$$

$$\text{for } i \in N \text{ and } t < T(g).$$

The right side of (3.7) is the same as in (3.6), the continuation payoff player i loses by deviating. The left side is the increase in his payoff in period t, over what it is if he contributes $g_i(t)$, if he completes the project then by contributing

$$\bar{z}_i(t) \equiv g_i(t) + X^* - \sum_{\tau=0}^{t} G(\tau).$$

Although (3.6) and (3.7) explicitly deter only two kinds of deviations from the grim-g strategies, they actually deter all deviations. This is shown to prove the following.

Theorem 1 *A candidate outcome g is a Nash equilibrium outcome if and only if it satisfies the under- and over-contributing constraints,* (3.6) *and* (3.7)*.*

Proof. Let $T = T(g)$. The grim-g profile gives player i the payoff

$$U_i^{eq} \equiv \sum_{\tau=0}^{T} \delta^\tau \left(\lambda_i G(\tau) - g_i(\tau)\right) + \delta^T b_i$$

Three kinds of deviation must be considered. The first is for player i to contribute in period $t \leqslant T$ a non-completing $z_i \neq g_i(t)$, and then never to contribute again. His payoff is then

$$U_i^{d1}(z_i, t) \equiv \sum_{\tau=0}^{t-1} \delta^\tau \left(\lambda_i G(\tau) - g_i(\tau)\right) + \delta^t \left[\lambda_i \left(G(t) - g_i(t)\right) - (1 - \lambda_i) z_i\right].$$

As $\lambda_i < 1$ and $z_i \geq 0$, we see that $U_i^{d1}(z_i, t) \leqslant U_i^{d1}(0, t)$.

The second kind of deviation player i could make, at a date $t < T$, is to over-contribute exactly enough to complete the project immediately, i.e., to contribute $\bar{z}_i(t)$. Since no further contributions will be made, player i's payoff from this deviation is

$$U_i^{d2}(t) \equiv U_i^{d1}(0, t) + \delta^t \left[b_i - (1 - \lambda_i)\bar{z}_i(t)\right].$$

The final kind of deviation for player i is to contribute a non-completing amount $z_i \neq g_i(t)$ in a period $t < T$, and also to contribute later. Any such deviation is dominated by a deviation of one of the previous two kinds. If $b_i \leqslant (1-\lambda_i)\bar{z}_i(t)$, the deviation is dominated by contributing zero in all periods $\tau \geq t$; otherwise, the deviation is dominated by contributing the completing amount $\bar{z}_i(t)$ in period t. Thus, $U_i^{eq} \geq U_i^{d1}(0, t)$ for all i and $t \leqslant T$, and $U_i^{eq} \geq U_i^{d2}(t)$ for all i and $t < T$, are jointly necessary and sufficient for g to be an equilibrium outcome. Rearranging these inequalities yields (3.6) and (3.7), respectively. ■

The over-contributing constraint is often not a problem, as the following corollary shows. For example, if the aggregate amount to be contributed in the completing period exceeds each c_i^*, the over-contributing constraint is implied by the under-contributing constraint.

Corollary 1 *Let g be a candidate outcome satisfying* (3.6)*, and let $T = T(g)$. Then g is a Nash equilibrium outcome if any of the following conditions hold:* (*i*) $T = 0$*;* (*ii*) $c_i^* = 0$ *for all i; or* (*iii*) $T < \infty$ *and* $g_i(T-1) + G(T) \geq c_i^*$ *for all i.*

Proof. We show the hypotheses imply (3.7). Obviously (i) does, as (3.7) is vacuous if $T = 0$. So assume $T > 0$, and let $t < T$. Both (ii) and (iii) imply that for $i \in N$,

$$g_i(t) + \sum_{\tau=t+1}^{T} G(\tau) \geq c_i^*.$$

Thus, since $\lambda_i < 1$, $c_i^* = b_i/(1-\lambda_i)$, and $X^* \geq \sum_{\tau=0}^{T} G(\tau)$,

$$(\lambda_i - 1)\left(X^* - \sum_{\tau=0}^{t} G(\tau)\right) + b_i \leq (1-\lambda_i)g_i(t).$$

This and (3.6) imply (3.7). ■

4 Non-Completing Equilibria

Before turning to completing equilibria, we delineate the achievable limits in this section.

Proposition 1 (i) *Nash equilibria exist in which no player contributes.* (ii) *If* $\bar{T} < \infty$ *and* $b_i = 0$ *for all* $i \in N$*, no contributions are made in any equilibrium.* (iii) *If* $\bar{T} = \infty$ *and* $b_i = 0$ *for all* $i \in N$*, the project is not completed in any equilibrium.* (iv) *If* $\bar{T} < \infty$ *or* (v) $\sum_{i \in N} \lambda_i < 1$, *no contributions are made in a non-completing equilibrium.*

Proof. Both (3.6) and (3.7) hold if $g_j(\cdot) \equiv 0$ for all j: (3.6) holds trivially, and (3.7) holds because (2.7) implies $(\lambda_i - 1)X^* + b_i = V_i - X^* < 0$. So Theorem 1 implies (i).

To prove (ii) – (iv), assume one is false. Then an equilibrium outcome g exists such that in some period $T < \infty$, a player i contributes $g_i(T) > 0$ and no contributions are made thereafter. The worst that can happen to player i if he deviates to zero in period T is that no contributions are made thereafter. Let $\hat{g}$ be this outcome. As $b_i = 0$ (cases (ii) and (iii)) or $X(T) < X^*$ (case (iv)), (2.6) implies $U_i(\hat{g}) - U_i(g) = \delta^T(1-\lambda_i)g_i(T) > 0$. This contradicts g's being an equilibrium outcome.

To prove (v), let g be a non-completing outcome with some positive terms. By (2.6),

$$\sum_{i \in N} U_i(g) = \sum_{t=0}^{\infty} \delta^t \left(\sum_{i \in N} \lambda_i - 1\right) G(t) < 0.$$

Thus, since equilibrium payoffs are nonnegative, g is not an equilibrium outcome. ■

Part (i) of Proposition 1 is obvious: if every $j \neq i$ never contributes, player i should either complete the project alone or never contribute. The latter is best because $V_i < X^*$; hence, all players never contributing is an equilibrium. It is not generally a perfect Bayesian equilibrium, but the no-contribution outcome is often a perfect Bayesian equilibrium outcome, as we show in Section 6.

Parts (ii) and (iii) of show that a completing equilibrium does not exist if all benefit functions are continuous. As in the static game, no player then wants to make a completing contribution.

Even if the project cannot be completed, it may be *asymptotically completed* in that $X(t) \to X^*$ as $t \to \infty$, but $X(t) < X^*$ for all t. Contributions must be made infinitely often in such equilibria. Thus, an asymptotically completing equilibrium cannot exist if the contributing horizon $\bar{T}$ is finite, as part (iv) of Proposition 1 indicates.

By part (v), an asymptotically completing equilibrium also does not exist if the sum of the players' marginal benefits from non-completing contributions is less than unity, the marginal cost of contributing. The argument for this starts with the observation that in any equilibrium, the social payoff, i.e., the sum of individual payoffs, must be nonnegative. An outcome g increments the social payoff in each non-completing period t by $\left(\sum_{i \in N} \lambda_i - 1\right) G(t)$, which is negative if $\sum_{i \in N} \lambda_i < 1$ and $G(t) > 0$. Thus, without a positive benefit jump at completion, the social payoff is nonnegative only if no contributions are made.

Part (v) does not apply if all benefit functions are continuous, as then $\sum_{i \in N} \lambda_i = \sum_{i \in N} V_i / X^* > 1$, by (2.7). An asymptotically completing equilibrium may then exist.

5 Completing Equilibria

We now give sufficient conditions for nearly efficient equilibria to exist. Completing and asymptoticaly completing Nash equilibrium outcomes are considered in the first two subsections. In both cases some of the outcomes are also perfect Bayesian equilibrium outcomes, as we show in Section 6. In the third subsection we consider a class of nearly efficient Markov perfect equilibria that use forgiving punishments to deter free riding.

Completing in Finite Time

For an equilibrium that completes the project in finite time to exist, some player must have a discontinuous benefit function, the contributing horizon must be sufficiently long, and the discount factor must be sufficiently large. These conditions are also sufficient, as we now prove. The idea is straightforward. In the equilibrium we construct, the players with positive benefit jumps complete the project immediately once the cumulation is close enough to X^*. Before then, the threat of halting future contributions keeps the players contributing.

The construction is recursive, starting at the completion period T^* to be determined below. In this period the under-contributing constraint (3.6) is $g_i(T^*) \leqslant b_i/(1-\lambda_i) = c_i^*$ for player i. Assuming it binds, the player contributes c_i^* in the completing period. Define

$$c_i(0) \equiv c_i^*. \tag{5.1}$$

Assuming $(c_1(0), ..., c_n(0))$ is contributed in period T^*, constraint (3.6) now defines an upper bound $c_i(1)$ for the penultimate contribution $g_i(T-1)$; it is given by $(1-\lambda_i)c_i(1) = \delta\lambda_i \sum_{j \neq i} c_j(0)$. Continuing in this fashion, making the under-contributing constraints recursively bind yields a sequence $\{c_i(k)\}_{k=0}^{\infty}$ for each $i \in N$:

$$c_i(k) \equiv \delta\left(\frac{\lambda_i}{1-\lambda_i}\right)\sum_{j \neq i} c_j(k-1) \text{ for all } k > 0. \tag{5.2}$$

Quantity $c_i(k)$ is the maximal contribution player i can make in period $T^* - k$ if in each period $\tau > t$, each player $j \in N$ contributes $c_j(T^* - \tau)$. The equilibrium will specify that the players contribute these amounts, except in period 0 when they may contribute less.[7]

To determine the completion period, note that if each player i contributes $c_i(\kappa)$ in every period $T^* - \kappa$, the remaining amount of contributions to be made in periods $T^* - k$ and

[7] We see from (5.1) and (5.2) that any player with $\lambda_i = 0$ does not contribute in this equilibrium until the completing period. Any player with $b_i = 0$ contributes in all but the completing period.

thereafter is

$$R(k) \equiv \sum_{\kappa=0}^{k} \sum_{i \in N} c_i(\kappa). \tag{5.3}$$

(Let $R(-1) \equiv 0$.) In period 0 the remaining amount of contributions, $R(T^*)$, must exceed X^*. Hence, T^* is determined by

$$R(T^* - 1) < X^* \leq R(T^*). \tag{5.4}$$

As is shown in the proof of Proposition 2 below, $R(k)$ does exceed X^* for large k, and hence (5.4) does define a finite T^*, if δ is large and $\sum_{i \in N} b_i$ is positive.

The equilibrium outcome, g^*, is defined in the following way. In period zero each player i contributes a fraction of $c_i(T^*)$:

$$g_i^*(0) \equiv \left(\frac{X^* - R(T^* - 1)}{R(T^*) - R(T^* - 1)} \right) c_i(0). \tag{5.5}$$

In periods $t > 0$ each player $i \in N$ contributes the full amount, $c_i(T^* - t)$:

$$g_i^*(t) \equiv \begin{cases} c_i(T^* - t) & \text{for } 0 < t \leqslant T^* \\ 0 & \text{for } t > T^*. \end{cases} \tag{5.6}$$

Provided T^* is well-defined, Corollary 1 implies g^* is a Nash equilibrium outcome: g^* satisfies the under-contributing constraints by construction, and it satisfies the over-contributing constraints because it specifies the maximal possible contributions, c_i^*, in the completing period. The full proof of the following is in Appendix A.

Proposition 2 *Assume* $\sum_{i \in N} b_i > 0$. *Then* $\delta^* < 1$ *exists such that for each* $\delta \in (\delta^*, 1]$, *a unique* $T^* < \infty$ *is defined by (5.4). If also* $\bar{T} \geq T^*$, *the* g^* *defined by* (5.1) – (5.6) *is a Nash equilibrium outcome that completes the project in period* T^*.

The only source of inefficiency when g^* is played is due to delay. For each $\delta > \delta^*$, let $T^*(\delta)$ be the period in which g^* completes the project. Since $T^*(1) < \infty$, g^* is efficient if $\delta = 1$, and if $\delta \approx 1$ it is nearly efficient.[8] *A fortiori*, the time to completion, $\ell T^*(e^{-r\ell})$, converges to zero as the period length ℓ converges to zero. This proves the following corollary.

[8] $T^*(\cdot)$ is bounded in a neighborhood of $\delta = 1$.

Corollary 2 *Assume* $\sum_{i \in N} b_i > 0$ *and* $\bar{T} > T^*(1)$. *Then* g^* *is a Nash equilibrium outcome if* r *or* ℓ *is sufficiently small;* g^* *finishes the project nearly instantaneously if* ℓ *is nearly zero; and* g^* *is nearly efficient if* r *or* ℓ *is nearly zero.*

Completing Asymptotically

We now construct an asymptotically completing equilibrium outcome for the case in which the benefit functions are continuous. We restrict attention to the case of identical benefit functions: in this subsection we assume $b_i = 0$, $\lambda_i = \lambda$, and $V_i = V = \lambda X^*$ for all $i \in N$. Note that (2.7) implies $\lambda > 1/n$. Thus, if the players contribute at the same rate, each player's utility gain in each period will be positive.

To define the equilibrium, first define the critical discount factor,

$$\hat{\delta} \equiv \frac{1-\lambda}{(n-1)\lambda}. \tag{5.7}$$

Note that $\hat{\delta} < 1$. The equilibrum outcome, which is denoted $\hat{g}$, is symmetric: $\hat{g}_i(t) = \hat{g}_j(t)$ for all $i \neq j$ and $t \geq 0$. It is calculated by converting the under-contributing constraints (3.6) to equalities, imposing symmetry, and solving the resulting system subject to $\sum_{t=0}^{\infty} \hat{g}_i(t) = X^*/n$. This yields

$$\hat{g}_i(t) \equiv \left(\frac{\hat{\delta}}{\delta}\right)^t \left(\frac{\delta - \hat{\delta}}{\delta}\right) \left(\frac{X^*}{n}\right) \text{ for all } t \geq 0 \text{ and } i \in N. \tag{5.8}$$

If $\delta > \hat{\delta}$, then $\sum_{t=0}^{\infty} \hat{g}_i(t) = X^*/n$, and so $\hat{g}$ completes the project asymptotically.

Proposition 3 *Assume the players have identical continuous benefit functions. If* $\bar{T} = \infty$ *and* $\delta \in (\hat{\delta}, 1]$, *then* $\hat{g}$ *is an equilibrium outcome, and it completes the project asymptotically.*

Proof. Observe that $\Sigma_{\tau \geq t} \delta^\tau \hat{g}_i(\tau) = \delta^t \hat{g}_i(t)/(1-\hat{\delta})$. Hence, for $t \geq 0$,

$$\begin{aligned}(1-\lambda)\hat{g}_i(t) &= (1-\lambda)\left(\frac{1-\hat{\delta}}{\hat{\delta}}\right) \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} \hat{g}_i(\tau) \\ &= (n\lambda - 1) \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} \hat{g}_i(\tau).\end{aligned}$$

So (3.6) holds (with equality), since $T(g) = \infty$ and $(n\lambda - 1)\hat{g}_i(\tau) = \lambda G(\tau) - \hat{g}_i(\tau)$. As $c^* = 0$, Corollary 1 (ii) now implies $\hat{g}$ is an equilibrium outcome. ■

Despite the fact that $\hat{g}$ never completes the project, it approximately completes the project arbitrarily quickly as the period length shrinks to zero. In addition, the efficiency loss vanishes as either $r \to 0$ or $\ell \to 0$:

Corollary 3 *In the setting of Proposition* 3, $\hat{g}$ *is an equilibrium outcome for which:* (i) *after any positive amount of time has passed, the contribution needed to complete is nearly zero if* ℓ *is nearly zero; and* (ii) *the efficiency loss is nearly zero if* r *or* ℓ *is nearly zero.*

Proof. $\hat{g}$ generates a cumulation in period t of

$$\hat{X}(t) = n \sum_{\tau=0}^{t} \hat{g}_i(\tau) = \left(1 - \left(\hat{\delta}/\delta\right)^{t+1}\right) X^*.$$

This quantity increases with $\delta = e^{-r\ell}$. Hence, for any $\theta < 1$, the amount of time it takes for the cumulation to exceed θX^* shrinks to zero as $\ell \to 0$. This proves (i). When $\hat{g}$ is played, the sum of the players' payoffs,

$$n(n\lambda - 1) \sum_{\tau=0}^{\infty} \delta^{\tau} \hat{g}_i(\tau) = \frac{(n\lambda - 1)(\delta - \hat{\delta})X^*}{(1 - \hat{\delta})\delta},$$

converges to its upper bound of $nV - X^*$ as $\delta \to 1$. This proves (ii). ■

Forgiving Punishments

Some of the outcomes of the previous subsections may be supported only by equilibria that punish a deviator by halting the contributions of the other players forever – that is the nature of the strategies used to prove the characterizing Theorem 1, and Theorem 2 below. However, completing and asymptotically completing equilibria that have more forgiving punishments do exist. We now present an example of a type we refer to as a *contribution goal equilibrium.*

As the name suggests, an equilibrium of this type is characterized by a sequence of *contribution goals*, $\{X_k\}_{k=0}^{T}$. In equilibrium the cumulation is raised each period to the smallest goal so far unachieved: if $X_{k-1} \leqslant X(t-1) < X_k$, equilibrium play in period t results in $X(t) = X_k$. Punishments are thus forgiving: a player who free rides in just one period simply delays the achievement of the current and subsequent goals by one period. Completion, or asymptotic completion, occurs so long as no player deviates an infinite number of times.

For simplicity we consider only the case of continuous benefit functions, and restrict attention to an an asymptotically completing contribution goal equilibrium in which only one player is responsible for achieving each goal. Thus, a player who is responsible for the current goal cannot gain from the contributions of others until he contributes enough to achieve the goal. If he does not contribute enough, he alone makes up the shortfall in the next period. A player's only gain from free riding is to shift a contribution into the discounted future, as opposed to lowering his total contribution.[9]

We further restrict attention to the case in which the benefit functions are identical. Thus, we again assume $b_i = 0$, $\lambda_i = \lambda$, and $V_i = V = \lambda X^*$ for all $i \in N$. To define the goals, first define $J(p) \equiv p^{n-1} + p^{n-2} + \cdots + p - \frac{1-\lambda}{\lambda}$. This polynomial has a unique positive root, γ, and it is in the interval $(0, 1)$.[10] The goals are given by

$$X_k \equiv \left(1 - \left(\frac{\gamma}{\delta}\right)^k\right) X^* \text{ for all } k \geq 0. \tag{5.9}$$

Thus $X_0 = 0$, and $X_k \to X^*$ monotonically if $\delta \in (\gamma, 1]$. Player i's strategy maps each possible cumulation X into a contribution, $z_i = \sigma_i(X)$. Letting $\iota(k) \equiv k \pmod n$, the strategy is

$$\sigma_i(X) \equiv \begin{cases} X_k - X & \text{if } X \in [X_{k-1}, X_k) \text{ and } i = \iota(k) \\ 0 & \text{otherwise.} \end{cases} \tag{5.10}$$

So each player is responsible for achieving every n^{th} goal. Let $\sigma = (\sigma_1, ..., \sigma_n)$.

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[9] If the players were to contribute equally to achieve each goal, punishments for free riding would be weaker. If the benefit functions are identical, such equilibria exist only if $V \geq (2 - 1/n)(X^*/n)$, and the time they take to raise contributions to θX^*, for any fixed $\theta \in (0, 1)$, is bounded below as $\ell \to 0$; efficiency is not obtained in the limit. See Marx and Matthews (1997).

[10] Since $b = 0$ implies $\lambda \in (\frac{1}{n}, 1)$, $J(0) < 0 < J(1)$. Hence, $\gamma \in (0, 1)$.

These strategies depend only on the publicly observed cumulation. Since a player's payoff depends on the contributions of the others only through their observed aggregate, his beliefs about their individual contributions are irrelevant; given that the others use σ, a player's continuation payoff after any history depends only on the current cumulation and his future actions. The appropriate perfection concept is therefore Markov perfect equilibrium.[11] The proof of Proposition 4 is in Appendix A.

Proposition 4 *If the players have identical continuous benefit functions, $\bar{T} = \infty$, and $\delta \in (\gamma, 1]$, then σ is a Markov perfect equilibrium, and it completes the project asymptotically.*

Observe from (5.9) that X_k increases in δ. This implies that efficiency is obtained in the limit as $r \to 0$ or $\ell \to 0$; the following corollary can be proved as was Corollary 3.

Corollary 4 *In the setting of Proposition 4, the contribution goal equilibrium σ has the following features: (i) after any positive amount of time has passed, the contribution needed to complete the project is nearly zero if ℓ is nearly zero; and (ii) the efficiency loss is nearly zero if r or ℓ is nearly zero.*

The contribution goal equilibrium of Proposition 4 appears most plausible in two cases. The first is if there are only two players: in this case they contribute in alternate periods, with each one refusing to contribute until the other has brought the cumulation up to the appropriate goal. Even though they contribute in alternate periods in equilibrium, this is not so off the equilibrium path – the strategies are not feasible if the players are restricted to contributing only in alternate periods (as they are in Admati and Perry, 1991).

Second, the equilibrium is fairly plausible if the number of players is infinite. The infinite-player game is a device for modeling a situation with a large number of participants; each potential contributor in, say, a New York radio fund drive plausibly views the number of others as infinite. The game is well defined with an infinite number of players because the benefit functions are constant for $X > X^*$. The static game with $n = \infty$ and $b = 0$ has

[11] Fudenberg and Tirole (1991) discuss the Markov perfect concept at length.

only the no-contribution equilibrium. However, one equilibrium of the infinite horizon game with these parameters is the limit of the equilibria defined by (5.9) and (5.10) as $n \to \infty$. In the limiting equilibrium, each player contributes only once, doing so in order to induce all higher-indexed players to contribute later. The goals are given by (5.9) with γ replaced by $1 - \lambda$ (this is the root of $J(p)$ if $n = \infty$). The strategies are given by (5.10) with $i =_n k$ replaced by $i = k$ (renumber the players to start with $i = 0$). Corollary 4 still holds, and so efficiency is obtained in the limit as the period length vanishes. This equilibrium resembles what we see in real short-run fund drives, such as a one-week public radio drive or a one-year university building campaign. Contributors in these situations do tend to contribute only once, and many seem to wait to see how the drive is faring before they contribute. Whether contributions tend to decrease over time, as they do in this equilibrium, is unknown to us.

6 Perfect Bayesian Equilibria

There is obviously no perfection problem with the Markov perfect equilibria of Proposition 4. But the same is not true of the grim equilibria used to prove Theorem 1. Those strategies are not sequentially rational after a history that brings the project so near to completion that some player is willing to complete it alone. Refer to the interval of such cumulations for player i as his *critical set*:[12]

$$C_i \equiv (X^* - c_i^*, X^*). \tag{6.1}$$

If previous contributions yield $X \in C_i$, and the strategies require the others to stop contributing, player i's best reply is to complete the project immediately alone. A grim strategy equilibrium is thus not sequentially rational if any player's critical contribution is positive.

Nonetheless, many Nash equilibrium outcomes are also the outcomes of equilibria that do not have this perfection problem, perfect Bayesian equilibria (PBE). The main theorem of this section is the following.[13]

[12]The critical sets satisfy $C_i \supseteq C_{i+1}$, since $c_i^* \geq c_{i+1}^*$.

[13]Theorem 2 (a) trivially implies that Propositions 3 and 4 identify PBE outcomes. Theorem 2 (b) implies

Theorem 2 *Let g be a Nash equilibrium outcome. Then g is a PBE outcome if*

(a) $c_1^* = c_i^*$ *for all* $i \in N$, *or*

(b) $c_1^* = c_2^*$, *and both* $g_1(t) \geq g_i(t)$ *and* $g_2(t) \geq g_i(t)$ *for all* $t \geq 0$ *and* $i = 3, ..., n$.

Hypothesis (a) requires the players to have the same critical contribution. (The parameters V_i and λ_i may differ.) The important consequence of this is that if the cumulation is in the common critical set, each player can believe that any other player will complete the project on his own, regardless of that player's identity. Under this hypothesis, *every* Nash equilibrium outcome is a PBE outcome.

Hypothesis (b) requires only that the two largest critical contributions be the same. But it also restricts attention to outcomes in which the two players with this largest critical contribution contribute the most every period. A prominent special case is a symmetric outcome, one in which $g_i(t) = g_j(t)$ for all $i \in N$ and $t \geq 0$. Hypothesis (b), like (a), implies that every player's critical set is a subset of at least one other player's critical set; thus, no player can ever believe he is the only one willing to complete the project alone.

The proof of Theorem 2 is in Appendix B. It proceeds by showing that under either hypothesis, there is a sequentially rational way of imposing on any unilateral deviator the maximal punishment – the withholding of all future contributions by the non-deviators. This punishment necessarily deters any deviation from a Nash equilibrium outcome.

In order to explain the proof, it is convenient to consider first the game with observed contributions. A PBE of this game is the same as a subgame perfect equilibrium (SPE). The analog of Theorem 2 (a) holds:

Proposition 5 *If the critical contributions of the players are identical, every Nash equilibrium outcome of the game with observed contributions is a SPE outcome.*[14]

that Proposition 2 identifies a PBE outcome if $b_1 = b_2$ and $\lambda_1 = \lambda_2$, by (5.1) and (5.2).

[14]Thus, the Nash equilibrium outcomes and the PBE outcomes of the games with and without observed contributions are all the same if the critical contributions of the players are identical. (Recall footnote 6.)

We give only an informal argument for Proposition 5. Given a Nash equilibrium outcome of the game with observed contributions, consider the following strategies:

> After any unilateral deviation, all players stop contributing if X is not in the common critical set; otherwise, the non-deviators stop contributing and the deviator immediately completes the project alone. (After a multilateral deviation, let any continuation equilibrium be played.)

Such strategies are sequentially rational after a unilateral deviator puts X in the common critical set: he wants to complete the project alone if the others will not contribute, and given that he does so, they do not want to contribute. Sequential rationality also holds after a deviation that does not put X in the common critical set, since then no player wants to contribute if the others do not. These strategies cause all non-deviators to stop contributing after any unilateral deviation, which deters any deviation from a Nash equilibrium outcome.

This argument relies on the critical contributions being identical. If instead $c_i^* > c_j^*$ for some $i \neq j$, a unilateral deviation by player j could yield an X that is in C_i, but not the closure of C_j. It would then not be a continuation equilibrium for all players to stop contributing, or for player j to complete the project alone. As a result, some Nash equilibrium outcomes may not be SPE outcomes, as is shown by Example 1 below.

The argument for Proposition 5 can be extended to prove Theorem 2 (a). As contributions are unobserved, the players must be given appropriate beliefs. In particular, our proof requires each player to rationalize, if possible, any observed deviation as being unilateral. This makes it rational for a unilateral deviator to believe the others will stop contributing.

To be more specific, consider a vector $\bar{z} \in \Re_+^n$ that is supposed to be contributed in some period. If it is, then i sees that his own contribution is $\bar{z}_i$, and that the aggregate contribution of the others is $\bar{Z}_i = \sum_{k \neq i} \bar{z}_k$. If exactly one player deviates from $\bar{z}$, player i instead observes a pair in the set

$$\begin{aligned} A_i(\bar{z}) \;\equiv\; \{(z_i, Z_i) \in \Re_+^2 \mid\, & (i)\ z_i \neq \bar{z}_i \text{ and } Z_i = \bar{Z}_i, \text{ or} \\ & (ii)\ z_i = \bar{z}_i,\ Z_i \neq \bar{Z}_i, \text{ and } Z_i \geq \textstyle\sum_{k \neq i,j} \bar{z}_k \text{ for some } j \neq i\}. \end{aligned} \tag{6.2}$$

If player i is the deviator, (z_i, Z_i) satisfies (i); if player $j \neq i$ is the deviator, it satisfies (ii) (since $z_j \geq 0$). The reverse is also true: given any $(z_i, Z_i) \in A_i(\bar{z})$, a unilateral deviation from $\bar{z}$ exists that causes player i to observe (z_i, Z_i). Beliefs that rationalize deviations as being unilateral whenever possible thus require player i to believe exactly one player deviated from $\bar{z}$ whenever he observes a pair in $A_i(\bar{z})$.

Given hypothesis (a) in Theorem 2, and given any Nash equilibrium outcome, assume the players have such beliefs when a deviation from it is observed. Consider strategies that satisfy the following:

> Player i contributes his equilibrium amount each period until he sees a deviation. If the (z_i, Z_i) he then sees is in $A_i(\bar{z})$, where $\bar{z}$ was the equilibrium contribution vector that period, he never contributes again if $z_i = \bar{z}_i$ or $X \notin C_i$; otherwise, he immediately contributes enough to complete the project alone.

If all players use such strategies, any unilateral deviation causes the non-deviators to stop contributing, as every player observes a pair in his $A_i(\bar{z})$ set. Each non-deviating i wants to stop contributing because he believes the deviation was unilateral, and hence that each $j \neq i$ observes a pair in $A_j(\bar{z})$. Thus, if i did not deviate, he wants to stop contributing because he believes the deviator will complete the project alone, or because $X \notin C_i$.

While there are generally many ways a player can rationalize a deviation as being unilateral, a particular specification is made in the proof of Theorem 2 in order to make it work for hypothesis (b) as well as (a). Each player, when he knows that one and maybe only one of the others deviated, is required to believe that the only deviator among them was a player who was supposed to have contributed no less than any other.[15] Ties are handled by assuming the deviator was the player with the lowest index who was supposed to have contributed no less than the others.

[15] Every deviation by the others that could be unilateral can be, and perhaps must be, the result of only this player deviating. E.g., suppose $n = 3$, $\bar{z}_1 = 2$, and $\bar{z}_2 = 1$. Then the only way player 3 can rationalize an observation that 1 and 2 together contributed $Z_3 = 1$ as a unilateral deviation is to believe player 1 was the deviator, i.e., that $z_1 = 0$ and $z_2 = 1$.

These beliefs, together with (b), imply that if player 1 (2) sees a possibly unilateral deviation by the others from g, he believes player 2 (1) was the unique deviator among them. Suppose player i unilaterally deviates. Any non-deviator $j \neq 1$ then believes player 1 unilaterally deviated, and hence that player 1 will complete the project alone if $X \in C_1$. If $X \notin C_1$, player j knows 1 will not contribute, but this does not induce him to contribute because $X \notin C_j$ (as $C_j \subseteq C_1$). If player 1 was not the deviator, he believes player 2 was, and hence that player 2 will complete the project alone if $X \in C_2$. If $X \notin C_2$, player 1 knows 2 will not contribute, but this does not induce him to contribute because (b) implies $C_1 = C_2$, and so $X \notin C_1$. Any unilateral deviation from g thus induces the non-deviators to stop contributing, which proves the theorem.[16]

We leave for the future a full treatment of the case in which the critical contributions are different. Heterogeneity creates some interesting phenomena, as we now illustrate with two examples. In Example 1, no contributions are made in the only equilibrium of the static game, but every PBE completes the project if the contributing horizon is longer and finite. Imperfect observability plays no role, as the example has only two players.

Example 1 Let $n = 2$, $X^* = 5$, and $\delta = 1$. The preference parameters are $V_1 = V_2 = 4$, $\lambda_1 = 0$, and $\lambda_2 = 4/5$. Thus, $c_1^* = 4$ and $c_2^* = 0$. As $n = 2$, the game contributions are observed, and so we consider its subgame perfect equilibria.

The no-contribution outcome is the only equilibrium of the static game: player 2 always free rides, since $c_2^* = 0$, and player 1 is unwilling to contribute the entire X^*. It is also a Nash equilibrium outcome for any $\bar{T} \geq 0$: the strategy profile in which no player contributes after any history is a Nash equilibrium, as $c_i^* < X^*$ for both i.

If $0 < \bar{T} < \infty$, the project is completed in every SPE, and the players contribute totals of $x_1 = 4$ and $x_2 = 1$. To prove this, let $\varepsilon > 0$. Consider an arbitrary SPE strategy for player 1, and for 2 a noncontingent strategy according to which he contributes nothing except in period $\bar{T} - 1$, when he contributes $1 + \varepsilon$. This strategy pair leads to a period $\bar{T}$ subgame in which only $R(\bar{T}) \equiv \min(0, X^* - X(\bar{T} - 1))$ is needed to complete the project. Since

[16] The complication in proving Theorem 2 is showing sequential rationality after multilateral deviations.

$R(\bar{T}) \leq 4 - \varepsilon < c_1^*$ and $c_2^* = 0$, the unique subgame equilibrium is $(z_1, z_2) = (R(\bar{T}), 0)$. This gives a payoff of at least $3 - \varepsilon$ to player 2, proving that his payoff in any SPE is not less than 3. Hence, in every SPE the project is completed, player 2 contributes no more than 1, and player 1 contributes no less than 4. Thus, since player 1's payoff must be nonnegative, $x_1 = 4$ and $x_2 = 1$ in every SPE.

In this argument player 2 can induce player 1 to complete the project at $\bar{T}$ because he himself can commit not to contribute at $\bar{T}$ or thereafter. If $\bar{T} = \infty$, commitment is impossible and other SPE outcomes exist. For example, let $\bar{X} \in [1, 4]$, and denote by $s(\bar{X})$ the following strategy profile: at any $t \geq 0$, the players contribute

$$(z_1, z_2) = \begin{cases} (0,\ \bar{X} - X(t-1)) & \textit{if } X(t-1) < \bar{X} \\ (5 - X(t-1),\ 0) & \textit{if } \bar{X} \leq X(t-1) < 5. \end{cases}$$

Thus, player 2 contributes $\bar{X}$ in the first period, and in the next player 1 completes the project. It is easily verified that $s(\bar{X})$ is a SPE if $\bar{T} = \infty$ (and $s(1)$ is a SPE for all $\bar{T} > 0$). Player 1 receives a zero payoff from $s(1)$, and player 2 receives a zero payoff from $s(4)$. These equilibria can be used as punishments to deter contributions. The following is a no-contribution SPE when $\bar{T} = \infty$: In period $t \geq 0$, neither player contributes if $X(t-1) = 0$, and otherwise they play $s(1)$ if $z_1(0) > 0$, and $s(4)$ if $z_1(0) = 0$.

Our final example shows that a PBE outcome when contributions are unobserved need not be a SPE outcome when contributions are observed. Thus, imperfect observability can create new pure-strategy PBE outcomes. This emphasizes that in the proof of Theorem 2 (b), the strategy profile calling for non-deviators to stop contributing may be sequentially rational only because the non-deviators have different beliefs about who deviated.[17]

Example 2 Let $n = 3$, $X^* = 5$, and $\delta = 1$. Players 1 and 2 have the same benefit functions: $V_i = 4$ and $\lambda_i = 0$ for $i = 1, 2$. Player 3 has $V_3 = 4$ and $\lambda_3 = 4/5$. Thus, $c_1^* = c_2^* = 4$ and $c_3^* = 0$. The contributing horizon is $\bar{T} = 1$.

[17] Such beliefs are not pathological: they are consistent with sequential equilibrium in similar games with finite strategies, as Matthews (1998) shows.

Consider outcome $g(0) = (1, 1, 1)$, $g(1) = (1, 1, 0)$. The grim-g strategies are a Nash equilibrium, and so g is a PBE outcome by Theorem 2 (b). The PBE used in the proof has the feature here that if player 3 unilaterally deviates from $g(0)$, player 1 (2) will think player 2 (1) was the deviator, and so each will think the other will complete the project on his own at $t = 1$ (since $5 - 2 < c_i^*$ for $i = 1, 2$). Thus, the deviation by player 3 causes no contributions to be made at $t = 1$, and this deters him from deviating.

In the game with observed contributions, players 1 and 2 are not confused about who deviated. Suppose player 3 deviates from $g(0)$ in this game by contributing nothing. In any equilibrium of the ensuing subgame, players 1 and 2 complete the project without player 3's contributing (as $c_3^* = 0$), so player 3 gains by deviating. Thus, g is not a SPE outcome of this game.[18]

7 Related Literature

We first discuss the closely related literature on dynamic voluntary contribution under complete information.[19] We then turn to recently studied games with similar features.

Dynamic Contribution

Fershtman and Nitzan (1991) (henceforth FN) study a differential game in which contributions become the capital a project uses to generate a flow of public benefits. Its open-loop equilibrium yields low contributions; it is analogous to the no-contribution equilibrium of

[18] The following is a SPE: the players contribute $z(0) = (0, 0, 1)$, $z(1) = (0, 0, 0)$ if $X(0) < 1$, and $z(1) = (R^*/2, R^*/2, 0)$ if $1 \leq X(0) < 5$, where $R^* = 5 - X(0)$ is the amount needed at $t = 1$ for completion. The outcome is $\hat{g}(0) = (0, 0, 1)$ and $\hat{g}(1) = (2, 2, 0)$, which is payoff equivalent to g because it also completes the project and yields the same total contributions for each player (recall that $\delta = 1$).

[19] The following are less related. McMillan (1979) studies a repeated game with a contribution stage game; folk theorems apply. Dynamic incomplete information games with discrete public good and contribution levels are studied in Bliss and Nalebuff (1984), Gradstein (1992), and Vega-Redondo (1995). Delay is caused in these games by the incentive to wait for low-cost types to contribute first.

our static game. Surprisingly, the Markov perfect equilibrium that FN identify yields even lower contributions. In this equilibrium a player's contribution rate is a *decreasing* function of the cumulative contribution. A player can thus induce greater future contributions by decreasing his current contribution, and so he has a strong incentive to free ride. The model differs from ours in many ways: it has continuous time, decaying contributions, and quadratic payoffs. Most importantly, FN restrict attention to a linear Markov perfect equilibrium. As Wirl (1996) shows, the game also has nonlinear Markov perfect equilibria, and some yield greater contributions than does the open-loop equilibrium. The negative conclusion of FN that dynamics aggravates free riding is thus true only of some equilibria. If the model is modified by making time discrete and replacing discounting by the limiting means criterion, Gaitsgory and Nitzan (1994) show that every individually rational payoff vector, including the efficient ones, is attained by some Nash equilibrium.

The model of Bagnoli and Lipman (1989) (henceforth BL), unlike ours or the other models we discuss here, concerns mechanism design rather than the private provision of public goods. A game form is constructed that fully implements, via a refinement of subgame perfect equilibrium, the core of a public goods economy in which the public good is available in discrete levels. The game form resembles our contribution game, but with a central authority committed in each period to halting the process if too little is contributed, and to refunding the excess if too much is contributed. BL's result shows that adding a third party with these relatively small commitment capabilities can overcome free riding. The discreteness of the public good plays a role similar to that of our benefit jump, insuring that a player is willing to contribute once the contribution total is close to that required to raise the public good to its next level. Our asymptotic completion result, Proposition 3, is foreshadowed by an unpublished result in the Appendix of Bagnoli and Lipman (1987), proved for a strictly concave public good production function and two players.

Admati and Perry (1991) (henceforth AP) study a dynamic contribution game that differs from ours in only a few ways, but yields the opposite conclusion that dynamics aggravate free riding. This conclusion is the more surprising because the players have identical binary benefit functions – recall that in this case of our model, an efficient PBE exists for any discount factor and any contributing horizon. We discuss the AP model in some detail.

The *AP game* has two players and an infinite contributing horizon. It differs from ours in that each player is allowed to contribute only in alternate periods, and the cost to a player of contributing z_i is given by a strictly convex function $w(z_i)$ (satisfying $w(0) = 0$).[20] The payoff of player i if completion occurs at T and his contributions are $z_i(0), ..., z_i(T)$ is

$$\delta^T V - \sum_{t=0}^{T} \delta^t w(z_i(t)).$$

AP show that (generically) this game has a unique subgame perfect equilibrium outcome. This *AP outcome* is usually inefficient, most notably if w is linear: "... in the linear case a necessary and sufficient condition for the completion of the project in our equilibrium is that each player would complete the project (immediately) if he was the only player." (p. 268) The AP outcome therefore yields no contributions under our assumptions $w(z) \equiv z$ and (2.7), which is now $V < X^* < 2V$. No contributions are made even if w merely approximates the identity function in the sense that $w'(0) \geq V/X^*$, by AP's Proposition 4.1.

The logic of these results is the following. In the AP game, a player completes the project in some period if that makes him better off than having the other player complete the project in the next period. Thus, in any SPE the player whose turn it is to contribute will complete the project if the required amount is less than the quantity R_1 defined by $V - w(R_1) \equiv \delta V$. Suppose that in some period completion requires a greater contribution, $y + R_1$ for some $y > 0$. If the player whose turn it is contributes just y, the other player will complete the project in the next period by contributing R_1. The player's marginal benefit from contributing $y + R_1$ rather than y is thus $V - \delta V$. This is less than his marginal cost of raising y to $y + R_1$, since the strict convexity of w and $w(0) = 0$ imply

$$w(y + R_1) - w(y) > w(R_1) = V - \delta V. \tag{7.1}$$

[20] AP also study a "subscription game" in which the players alternately pledge amounts to be paid when the project is completed. Its SPE is efficient when w is linear.

So the player does not contribute the completing amount $y+R_1$. This proves that a completing contribution in any SPE cannot exceed R_1. Repeating this argument recursively yields a sequence $\{R_k\}$ such that in any SPE, R_1 is the most that can be contributed in a period to complete the project, R_2 is the most that can be contributed in the penultimate period before it is completed, and so on. The cumulative contribution is thus bounded by $\sum_{k=1}^{\infty} R$. This sum is easily shown to equal V if w is the identity function. Thus, no SPE completes the project when $V < X^*$ and w is sufficiently close to the identity function.

The crux of this argument is that a completing contribution cannot exceed R_1. This is not true for various modifications of the AP game. For example, suppose w is the identity function. Then the inequality in (7.1) is an equality: the player is indifferent between contributing $y + R_1$ to complete the project, and contributing just y to let the other player complete it in the next period. This implies the existence of equilibrium outcomes other than the AP outcome, ones in which a completing contribution is larger than R_1. Completing equilibria can then exist. In Appendix C we show that if δ is sufficiently large and (2.7) holds, the AP game with $w(z) \equiv z$ has a SPE in which player 1 contributes $X^* - V$ in the first period, and player 2 contributes V in the second period to complete the project.[21]

We also show in Appendix C that two other modifications of the AP game have completing equilibria if w just approximates the identity function, but δ is large and (2.7) holds. First, if the infinite contributing horizon is replaced by a finite $\bar{T} > 0$, *every* SPE completes the project. Second, if the players can contribute in any period instead of being required to alternate, some subgame perfect and Markov perfect equilibria complete the project within two periods. These equilibria resemble the simplest of those constructed in Sections 5 and 6. Thus, the completing equilibria of our game are robust to making the contributing cost function slightly convex. The completing equilibria of the AP game with $w(z) \equiv z$ are not robust in this sense: by AP's uniqueness result, they vanish if w is made strictly convex.

[21] This is not made clear in AP. The discussion around Lemma 4.1 gives the incorrect impression that any SPE outcome when w is linear is payoff-equivalent to the AP outcome, and so yields no contributions when $w(z) \equiv z$ and $V < X^*$.

Hence, the difference between AP's negative result and our more positive one should be attributed primarily to the AP players being required to alternate their contributions.

Other Dynamic Models

Some of our results are like those obtained in recent papers on other kinds of dynamic incentive problems. We discuss a sample of them here.

Gale (1995) studies a dynamic investment game in which each player chooses the period in which to make a discrete investment to a project. Each player invests at most once, at which time he begins receiving a flow of benefits from the project. The model can thus be interpreted as one of voluntary contribution to a project that produces an excludable public good each period. The efficient equilibrium consists of all players investing in the first period. Equilibria with delay also exist because each player has an incentive to wait until enough of the others have invested so that the benefit flow has become positive by the time he invests. This is like our model with a positive benefit jump: a player is reluctant to contribute only if the level of previous contributions is low. All subgame perfect equilibria are shown to converge to the efficient one as the period length converges to zero.[22]

Bolton and Harris (1998) study a many-player bandit problem. Each player at each moment can choose to invest in a safe asset, or a risky asset with an unknown expected return. The realized return from an investment in the risky asset provides not only a private return to the player, but also a public return in the form of information that helps all players make better future decisions. The public benefit function is thus stochastic, and determined endogenously by the degree of current uncertainty and the nature of future strategies. Despite such differences, the symmetric Markov perfect equilibrium is somewhat similar to some of the equilibria of our game: e.g., investment in the risky asset is too slow

[22] More recently Gale (1998) studies a related class of games in which players can only increase the level of their actions (e.g. cumulative contributions) over time. For games with positive spillovers, like public good contribution games, he characterizes the limits of all SPE outcomes when there is no discounting and the players can only move one at a time.

because of free-riding, but the inefficiency vanishes as discounting disappears.

Neher (1997) shows that an investor in a start-up firm can, by making her investment piecemeal over time, protect herself from the entrepreneur's inability not to renegotiate down the investor's claim once the investment is sunk. The model differs from ours in its focus on contracts, renegotiation, and transferable assets, but it reaches a similar result: investing over time can mitigate an incentive problem caused by an inability to commit.

8 Conclusions

We have studied a model of dynamic voluntary contribution that reflects features of long fund-raising campaigns. An important assumption is that players only observe the aggregate contribution made each period. This is consistent with real fund-raising campaigns that last weeks or months, and in which the public is periodically encouraged to contribute by informing it of how much is needed to reach the campaign goal. We characterize the pure strategy Nash equilibrium outcomes, and show that they are all perfect Bayesian equilibrium outcomes if the players' preferences are sufficiently similar in the sense of Theorem 2. Our welfare results show that approximately efficient equilibria exist if the players have sufficiently similar benefit functions, if contributions can be made in a sufficiently large number of periods, and if discounting is low or the period length small. Creating future contributors upon which to free ride does increase the incentives to free ride, but they can be countered by the ability of future players to punish past free riders by withholding or postponing their contributions.

If some players realize a positive benefit discontinuity when the project is completed, equilibria can exist that complete the project in a finite number of periods. Such a discontinuity can be due to technology, or to a challange bonus that some party has committed to making when a specified contribution goal is achieved. If the benefit functions are (initially) continuous and the contributing horizon is finite but large, a challenge bonus can create completing equilibria where none existed before. Such completing equilibria generally give players a positive payoff, and so the players themselves have an incentive to commit to making a challenge bonus. (The bonus can be arbitrarily small if δ and $\bar{T}$ are large.) The existence of challenge bonuses is thus consistent with the model.

Also consistent with our results is some experimental evidence. Although almost all public good experiments have been in a static setting (Ledyard, 1995), Dorsey (1992) is an exception. He finds that total contributions are larger if players can contribute to a public good in multiple periods, especially if the contributions are nonrefundable and the benefit function has a completion point.

Our main theoretical result is the identification of two sufficient conditions for any Nash equilibrium outcome to be also the outcome of a perfect Bayesian equilibrium. The proof requires the construction of beliefs and actions off the equilibrium path that make it rational for each non-deviator to impose the maximum possible punishment of never contributing again – despite the fact that the project might be so close to completion that every continuation equilibrium must complete it, and despite the fact that the identity of the deviator might not be common knowledge. One of the sufficient conditions is that the players' critical contributions c_i^* be identical. The second is that the two highest critical contributions be the same and belong to players who contribute the most every period. The latter condition is sufficient only because the identity of a unilateral deviator is not observed.

An interesting extension will be to fully characterize equilibria for the case in which the players' benefit functions are not identical. We only obtain asymptotically completing Nash equilibria in Section 5 for the case in which the benefit functions are identical; and the sufficient conditions in Section 6 for any Nash equilibrium outcome to be a perfect Bayesian equilibrium outcome also require some similarity of the benefit functions. The examples in Section 6 are suggestive, but the heterogeneous case needs more study.

There are other interesting extensions, such as to more general utility functions. More general timing conventions are also of interest, such as that in which benefits are not received until the project is completed or the contributing horizon is reached, whichever comes first. (This timing is covered by the no-discounting version of our model, but not by the discounting version.) It would be fruitful to disentangle the concept of a period as the minimum length of time between contributions, as opposed to the minimum length of time between announcements that update information about received contributions. Finally, it would be interesting to assume contributions generate benefits only after a lag, as when each stage of a project takes time to complete but can be used immediately once it is completed.

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A Appendix: Proofs for Section 5

Proof of Proposition 2

Since each $c_i(\kappa) \geq 0$, each $R(\kappa) \geq 0$. Hence, not more than one $T^* \geq 0$ can satisfy (5.4). Assuming some $T^* < \infty$ does satisfy it, g^* is a well-defined candidate outcome. By construction, it satisfies (3.6) and completes the project in period T^*. As $T^* = 0$ or $g^*(T^*) = (c_1^*, ..., c_n^*)$, Corollary 1 (i) or (iii) implies g^* is a Nash equilibrium outcome.

We now show that (5.4) is satisfied by some $T^* < \infty$ if $\delta = 1$; continuity then establishes that the same is true for all δ in a neighborhood of 1. So assume henceforth that $\delta = 1$. Define $Y(k) \equiv \sum_{i \in N}(1 - \lambda_i)c_i(k)$. Since each $\lambda_i < 1$ and $c_i(k) \geq 0$, $Y(k) \geq 0$. By (5.2),

$$Y(k) = \left(\sum_{i \in N} \lambda_i - 1\right) \sum_{i \in N} c_i(k-1) + Y(k-1). \tag{A1}$$

Iterating, using (5.3) and $(1 - \lambda_i)c_i(0) = b_i$, yields

$$Y(k) = \left(\sum_{i \in N} \lambda_i - 1\right) R(k-1) + \sum_{i \in N} b_i. \tag{A2}$$

Case 1: $\sum_{i \in N} \lambda_i \geq 1$. This implies, since $R(k-1) \geq 0$, that $Y(k) \geq \sum_{i \in N} b_i$. Thus,

$$\begin{aligned}
\sum_{i \in N} c_i(k) &\geq \sum_{i \in N} \left(\frac{1 - \lambda_i}{\max_{j \in N}(1 - \lambda_j)}\right) c_i(k) \\
&= \frac{Y(k)}{\max_{j \in N}(1 - \lambda_j)} \\
&\geq \frac{\sum_{i \in N} b_i}{\max_{j \in N}(1 - \lambda_j)}.
\end{aligned}$$

The last term is independent of k and positive, as $\sum_{i \in N} b_i > 0$. Hence, by (5.3), $R(k) \to \infty$ as $k \to \infty$. This proves that some finite $T^* \geq 0$ satisfies (5.4).

Case 2: $\sum_{i \in N} \lambda_i < 1$. Then $\{Y(k)\}$ is a decreasing nonnegative sequence, and so it converges. Taking limits in (A1) thus shows that $\sum_{i \in N} c_i(k-1) \to 0$ as $k \to \infty$. As each

$c_i(k) \geq 0$, this implies $Y(k) \equiv \sum_{i \in N}(1-\lambda_i)c_i(k) \to 0$. So by (A2),

$$\begin{aligned} \lim_{k\to\infty} R(k-1) &= \frac{\sum_{i\in N} b_i}{1-\sum_{i\in N}\lambda_i} \\ &= \frac{\sum_{i\in N} V_i - X^* \sum_{i\in N}\lambda_i}{1-\sum_{i\in N}\lambda_i} > X^*, \end{aligned}$$

using $\sum_{i\in N} V_i > X^*$. This again proves some finite $T^* \geq 0$ satisfies (5.4). ■

Derivation and Proof of Proposition 4

We first derive the equilibrium. Let $Z_k \equiv X_k - X_{k-1}$. For $k \geq 1$, let H_k be the equilibrium continuation payoff of player $i = \iota(k)$ starting from $X = X_k$, which is the period after this player raises the cumulation to X_k. In the next $n-1$ periods, players $j \neq i$ contribute in turn $Z_{k+1}, Z_{k+2}, \ldots, Z_{k+n-1}$; in the n^{th} subsequent period, player i contributes Z_{k+n}. His continuation payoff in the next period is H_{k+n}. Thus,

$$H_k = \lambda \sum_{\kappa=k+1}^{k+n-1} \delta^{\kappa-k-1} Z_\kappa + (\lambda-1)\delta^{n-1} Z_{k+n} + \delta^n H_{k+n}. \tag{A3}$$

Let $V_i(X)$ be player i's equilibrium value function. Thus $V_{\iota(k)}(X_k) = H_k$, and $V_i(X) = 0$ for $X \geq X^*$. Otherwise, for $X \in [X_{k-1}, X_k)$,

$$V_i(X) \equiv \begin{cases} (\lambda-1)(X_k - X) + \delta V_i(X_k) & \text{if } i = \iota(k) \\ \lambda(X_k - X) + \delta V_i(X_k) & \text{if } i \neq \iota(k). \end{cases} \tag{A4}$$

Since a player has the option of never contributing, $V_i(X_k) \geq 0$ for all $k \geq 0$. Suppose $k \geq 1$ and $i = \iota(k+1)$. If $X = X_{k-1}$, player i is supposed to let another player contribute Z_k in the current period before he contributes Z_{k+1} in the next period. His continuation payoff from this strategy must be no less than what he would get by contributing Z_{k+1} at the same time as Z_k is contributed. Thus, $V_i(X_{k-1}) \geq \lambda Z_k + (\lambda-1)Z_{k+1} + \delta V_i(X_{k+1})$. By (A4), this becomes $\lambda Z_k + \delta V_i(X_k) \geq \lambda Z_k + V_i(X_k)$. Therefore, since $V_i(X_k) \geq 0$,

$$V_{\iota(k+1)}(X_k) = 0 \text{ for all } k \geq 1. \tag{A5}$$

We construct the equilibrium for which (A5) also holds for $k = 0$. This, (A4), and $V_{\iota(k)}(X_k) = H_k$ imply

$$H_k = \delta^{-1}(1-\lambda)Z_k \text{ for all } k \geq 1. \tag{A6}$$

Using this to remove H_k and H_{k+n} from (A3), and letting $\rho_k \equiv \delta^k Z_k$, we obtain a linear homogeneous difference equation:

$$-\left(\frac{1-\lambda}{\lambda}\right)\rho_k + \sum_{\kappa=k+1}^{k+n-1} \rho_\kappa = 0.$$

Since $J(\gamma) = 0$, this equation has a solution of the form $\rho_k = K\gamma^k$, or rather, $Z_k = K\left(\frac{\gamma}{\delta}\right)^k$, where K is a constant. Setting $\sum\limits_{k=1}^{\infty} Z_k = X^*$, we see that $\delta > \gamma$ is required, and $K = X^*(\delta-\gamma)/\gamma$. Thus, $Z_k = \left(\frac{\gamma}{\delta}\right)^{k-1}\left(1-\frac{\gamma}{\delta}\right)X^*$. Formula (5.9) comes from $X_k = \sum_{\kappa=1}^k Z_\kappa$.

Proof of Proposition 4 for $\delta \in (\gamma, 1)$.

We need to show that one-shot deviations are unprofitable. Let the current cumulation be $X \in [X_{k-1}, X_k)$ for some $k \geq 1$, and consider player $i \in N$.

Case $i = \iota(k)$. In this case player i is supposed unilaterally to bring the cumulation up to X_k. He can raise it to any level $Y \geq X$. Any $Y \geq X^*$ is strictly dominated. Choosing a $Y \in [X, X^*)$ and then joining the others in playing σ yields a continuation payoff of

$$W^i(Y, X) \equiv (\lambda - 1)(Y - X) + \delta V_i(Y). \tag{A7}$$

We show that X_k maximizes $W^i(\cdot, X)$ on $[X, X^*)$. We can restrict attention to the points $X, X_k, X_{k+1}, \ldots$, since $W^i(\cdot, X)$ decreases on each interval $[X_{\kappa-1}, X_\kappa)$.[23] Because $W^i(X, X) = \delta V_i(X) \leq V_i(X) = W^i(X_k, X)$, we can restrict attention to X_k, $X_{k+1}, \ldots$.

For $\kappa \geq k$, let $\Delta_\kappa \equiv W(X_{\kappa+1}, X) - W(X_\kappa, X)$. Hence,

$$\Delta_\kappa = (\lambda - 1)Z_{\kappa+1} + \delta[V_i(X_{\kappa+1}) - V_i(X_\kappa)]. \tag{A8}$$

[23] By (A4), $W_Y^i(\cdot, X) = (1-\delta)(\lambda-1) < 0$ if $i =_n \kappa$, and $W_Y^i(\cdot, X) = (1-\delta)\lambda - 1 < 0$ if $i \neq_n \kappa$.

If $i = \iota(\kappa+1)$, then $0 = V_i(X_\kappa) = (\lambda-1)Z_{\kappa+1} + \delta V_i(X_{\kappa+1})$ by (A5) and (A4). Hence, (A8) implies $\Delta_\kappa = 0$. If instead $i \neq \iota(\kappa+1)$, then (A4) yields $V_i(X_\kappa) = \lambda Z_{\kappa+1} + \delta V_i(X_{\kappa+1})$. Using this to eliminate $\delta V_i(X_{\kappa+1})$ from (A8), we obtain

$$\Delta_\kappa = (1-\delta)V_i(X_\kappa) - Z_{\kappa+1}. \tag{A9}$$

By (A3) and (A4), $V_i(X_\kappa)$ is bounded above by $\lambda \sum_{j \geq \kappa+1} \delta^{j-\kappa-1} Z_j$, and this in turn is bounded above by $\lambda Z_{\kappa+1}/(1-\delta)$, since $\{Z_j\}$ is decreasing. Hence, (A9) implies $\Delta_\kappa < 0$. We conclude that $\Delta_\kappa \leq 0$ for all $\kappa \geq k$, and so $Y = X_k$ indeed maximizes $W^i(\cdot, X)$.

Case $i \neq_n k$. In this case another player is supposed to raise the cumulation to X_k in the current period. If player i also contributes, he raises the cumulation to some $Y \geq X_k$. Any $Y \geq X^*$ is strictly dominated. Choosing $Y \in [X_k, X^*)$ and then joining the others in playing σ yields a continuation payoff of

$$\begin{aligned} \hat{W}^i(Y, X) &\equiv \lambda(X_k - X) + (\lambda - 1)(Y - X_k) + \delta V_i(Y) \qquad \text{(A10)} \\ &= \lambda(X_k - X) + W^i(Y, X_k), \end{aligned}$$

where W^i is defined in (A7). We must show that X_k maximizes $\hat{W}^i(\cdot, X_k)$ on $[X_k, X^*)$. We showed above that X_k maximizes $W^i(\cdot, X)$ on $[X, X^*)$ for any $X \in [X_{k-1}, X_k)$. Hence, as $W^i(\cdot, X)$ is continuous in X, X_k maximizes it on $[X_k, X^*)$. So X_k indeed maximizes $\hat{W}^i(\cdot, X_k)$ on $[X_k, X^*)$.

Proof of Proposition 4 for $\delta = 1$.

The goals are now $X_k = (1-\gamma^k)X^*$. Let $i \in N$. Suppose that in some period player i's cumulation has reached x_i, and the cumulation has reached $X < X^*$. We show that conditional on starting from (x_i, X), σ_i is a best reply to σ_{-i}. (We cannot restrict attention to one-shot deviations because $\delta = 1$.)

If σ is played starting from (x_i, X), $X(t) \to X^*$. Let x_i^* be the corresponding limit of player i's cumulative contribution. His conditional payoff is then $U_i^* \equiv \lambda X^* - x_i^*$.

Let $\tilde{\sigma}_i$ be a best reply to σ_{-i}, conditional on reaching (x_i, X). Let the sequences of player i's and the aggregate cumulative contributions when $(\tilde{\sigma}_i, \sigma_{-i})$ is played, starting from (x_i, X), be $\{\tilde{x}_i(\tau)\}$ and $\{\tilde{X}(\tau)\}$. Let the limits of these sequences be $\tilde{x}_i$ and $\tilde{X}$. Player i's conditional payoff is then $\tilde{U}_i = \lambda \min(\tilde{X}, X^*) - \tilde{x}_i$.

The nature of σ_{-i} implies that starting from (x_i, X), no strategy of player i can induce the others to contribute more than $X^* - x_i^*$. Thus, if $\tilde{X} \geq X^*$, he himself must contribute at least x_i^* when he plays $\tilde{\sigma}_i$; that is, $\tilde{x}_i \geq x_i^*$. Therefore, in this case $\tilde{U}_i = \lambda X^* - \tilde{x}_i \leq U_i^*$, which shows that σ_i is a conditional best reply to σ_{-i}.

Now suppose $\tilde{X} < X^*$. The nature of σ_{-i} then implies that $k \geq 1$ and $t < \infty$ exist such that $\tilde{X}(\tau) \in [X_{k-1}, X_k)$ for all $\tau \geq t$. No player except perhaps i contributes after date t. Modify $\tilde{\sigma}_i$ to a strategy $\bar{\sigma}_i$ by replacing it with σ_i at all dates $\tau > t$. According to $\bar{\sigma}_i$, player i at date $t+1$ raises the cumulation from $\tilde{X}(t)$ to X_k, whereupon it is raised successively to $X_{k+1}, X_{k+2}, ...$, and converges to X^*. This yields a continuation payoff of $V_i(\tilde{X}(t))$, and so player i's payoff conditional on (x_i, X) when $(\bar{\sigma}_i, \sigma_{-i})$ is played is

$$\begin{aligned}
\bar{U}_i &= \lambda_i \tilde{X}(t) - \tilde{x}_i(t) + V_i(\tilde{X}(t)) \\
&= \lambda_i \tilde{X} - \tilde{x}_i - \lambda_i \left(\tilde{X} - \tilde{X}(t)\right) + (\tilde{x}_i - \tilde{x}_i(t)) + V_i(\tilde{X}(t)) \\
&= \tilde{U}_i + (1 - \lambda_i)\left(\tilde{X} - \tilde{X}(t)\right) + V_i(\tilde{X}(t)),
\end{aligned}$$

where the last equality uses $\tilde{x}_i - \tilde{x}_i(t) = \tilde{X} - \tilde{X}(t)$. Since $V_i(\tilde{X}(t)) \geq 0$, this proves $\bar{U}_i \geq \tilde{U}_i$. So $\bar{\sigma}_i$ is also a conditional best reply to σ_{-i}. The argument above can now be applied with $\bar{\sigma}_i$ replacing $\tilde{\sigma}_i$, since the cumulation converges to X^* when $(\bar{\sigma}_i, \sigma_{-i})$ is played after (x_i, X) is reached. Thus, σ_i is a conditional best reply to σ_{-i}. ■

B Appendix: Proof of Theorem 2

We construct a PBE that has outcome g, regardless of which hypothesis holds, (a) or (b).

Preliminaries

Let $\bar{z} \in \Re^n_+$. Define $M(\bar{z})$ to be the smallest integer such that $\bar{z}_{M(\bar{z})} \geq \bar{z}_i$ for all $i \in N$. Thus, according to $\bar{z}$, player $M(\bar{z})$ is supposed to contribute the maximum amount. Similarly, let $m(\bar{z})$ be the smallest integer not equal to $M(\bar{z})$ satisfying $\bar{z}_{m(\bar{z})} \geq \bar{z}_i$ for all $i \in N \backslash \{M(\bar{z})\}$. According to $\bar{z}$, player $m(\bar{z})$ contributes the second-largest amount.

Player i can rationalize his observation Z_i as the outcome of a deviation from $\bar{z}$ by at most one of the other players if and only if $Z_i \geq \bar{Z}_i - \bar{z}_k$, where $\bar{z}_k$ is the maximum of the other players' contributions in $\bar{z}$. Recall that $A_i(\bar{z})$ is the set of pairs (z_i, Z_i) that player i can rationalize as a unilateral deviation from $\bar{z}$. Hence, $(z_i, Z_i) \in A_i(\bar{z})$ if and only if either (i) $z_i \neq \bar{z}_i$ and $Z_i = \bar{Z}_i$, or

$$(ii)\ z_i = \bar{z}_i,\ Z_i \neq \bar{Z}_i,\ \text{and } Z_i \geq \begin{cases} \bar{Z}_i - \bar{z}_{M(\bar{z})} & \text{if } i \neq M(\bar{z}) \\ \bar{Z}_i - \bar{z}_{m(\bar{z})} & \text{if } i = M(\bar{z}). \end{cases} \tag{B1}$$

History $h_i^{t-1} = ((z_i(r), Z_i(r))_{r=0}^{t-1}$ determines the aggregates $Z(r) = z_i(r) + Z_i(r)$ and $X(r) = \sum_{r' \leq r} Z(r')$. The corresponding remaining contribution needed to complete the project is $R(t) \equiv \max(0, X^* - X(t-1))$. Refer to h_i^{t-1} as

(i) an *equilibrium history* if $t = 0$ or $Z(r) = G(r)$ for all $r = 0, ..., t-1$;

(ii) a *grim history* if it is not an equilibrium history, and $X(t-1) \notin C_1$ or $t > \bar{T}$; and

(iii) a *completing history* if it is not an equilibrium history, $X(t-1) \in C_1$, and $t \leq \bar{T}$.

These definitions depend only on publicly observed quantities, and every personal history is exactly one of these three types. According to the strategies to be defined, $g(t)$ will be contributed at an equilibrium history; no player will contribute at a grim history; and the project will be (expected to be) completed immediately at a completing history.

Refer to $h^{t-1} = (h_1^{t-1}, ..., h_n^{t-1})$ as a *possible history profile* if it is generated by a play of the game. If one history in it is an equilibrium (grim) (completing) history, then it is common knowledge at h^{t-1} that all personal histories are equilibrium (grim) (completing) histories. The same is true of any truncation, $h^{r-1} = (h_1^{r-1}, ..., h_n^{r-1})$ for $r \leq t$, of h^{t-1}.

Given a completing history h_i^{t-1}, define variables $(\tau, \bar{z}, M, m, z_i, Z_i)$ as follows. Let τ be the smallest integer such that the truncation $h_i^{\tau-1}$ of h_i^{t-1} is also a completing history. The truncation $h_i^{\tau-2}$ is thus an equilibrium or a grim history: some players deviated in period $\tau-1$, and the contributions in period $\tau-1$ caused the cumulative contribution to increase above $X^* - c_1^*$, so that $X(\tau-1) \in C_1$, for the first time. Let

$$\bar{z} \equiv \begin{cases} g(\tau-1) & \text{if } h_i^{\tau-2} \text{ is an equilibrium history} \\ (0, ..., 0) & \text{if } h_i^{\tau-2} \text{ is a grim history.} \end{cases} \tag{B2}$$

According to the strategies to be defined, the contributions in $\bar{z}$ should have been made in period $\tau-1$. Finally, let $M = M(\bar{z})$, $m = m(\bar{z})$, and $(z_i, Z_i) = (z_i(\tau-1), Z_i(\tau-1))$. Since $h_i^{\tau-1}$ is not the same type of history as $h_i^{\tau-2}$, $(z_i, Z_i) \neq (\bar{z}_i, \bar{Z}_i)$ for all $i \in N$: each player knows a deviation occurred in period $\tau-1$.

Let h^{t-1} be a possible profile of completing histories, and define the variables $(\tau, \bar{z}, M, m)$ as above. Under hypothesis (a), all the C_i sets are identical, and so

$$C_M = C_m = C_1. \tag{B3}$$

This also follows from hypothesis (b), as it and (B2) imply $C_1 = C_2$ and $\{M, m\} = \{1, 2\}$. Thus, under either hypothesis, it is common knowledge at h^{t-1} that M and m are both willing to complete the project alone, i.e., that C_M and C_m both contain $X(t-1)$.

Strategies

Player i's strategy, s_i, maps personal histories into contributions. Let

$$s_i(h_i^{t-1}) \equiv \begin{cases} g_i(t) & \text{if } h_i^{t-1} \text{ is an equilibrium history} \\ 0 & \text{if } h_i^{t-1} \text{ is a grim history.} \end{cases} \tag{B4}$$

Now suppose h_i^{t-1} is a completing history, and let $(\tau, \bar{z}, M, m, z_i, Z_i)$ be the variables defined above. There are three cases. First, if $(z_i, Z_i) \in A_i(\bar{z})$, let

$$s_i(h_i^{t-1}) \equiv \begin{cases} R(t) & \text{if } z_i \neq \bar{z}_i \text{ and } X(t-1) \in C_i \\ 0 & \text{otherwise.} \end{cases} \tag{B5}$$

Thus, if player i can rationalize the deviation from $\bar{z}$ as unilateral, he contributes enough to complete the project alone if he himself deviated and $X(t-1) \in C_i$; otherwise he contributes nothing. Since all histories that follow h_i^{t-1} have the same truncation $h_i^{\tau-1}$, player i will never contribute in any period after t if $z_i = \bar{z}_i$, regardless of what happens.

Second, if $(z_i, Z_i) \notin A_i(\bar{z})$ and $i \neq M$, let

$$s_i(h_i^{t-1}) \equiv 0. \tag{B6}$$

All players except M never contribute after period $\tau - 1$ if they cannot rationalize the deviation that period as unilateral.

Third, if $(z_M, Z_M) \notin A_M(\bar{z})$, let

$$s_M(h_M^{t-1}) \equiv \begin{cases} 0 & \text{if } Z_M = 0 \text{ and, for some } j \neq M, \\ & \qquad \bar{z}_j > 0,\ z_M = \bar{Z}_j, \text{ and } X(t-1) \in C_j \\ R(t) & \text{otherwise.} \end{cases} \tag{B7}$$

Player M completes the project immediately if he cannot rationalize the $\tau - 1$ deviation as unilateral, unless the first line of (B7) holds. In that case player M knows j observed $(z_j, Z_j) = (0, \bar{Z}_j)$, and that j will complete the project immediately, by (B5); M then contributes nothing.

This completes the specification of a strategy profile, $s = (s_1, ..., s_n)$.

Beliefs

The beliefs of player i at history h_i^{t-1} are represented by a probability distribution over the past contributions of the other players that is consistent with h_i^{t-1} : it is a distribution $P_i(\cdot \mid h_i^{t-1})$ on the set of nonnegative vectors $\tilde{z}_{-i}^{t-1} = (\tilde{z}_{-i}(0), ..., \tilde{z}_{-i}(t-1))$ that satisfy

$$\sum_{j \neq i} \tilde{z}_j(r) = Z_i(r) \text{ for all } r = 0, ..., t-1. \tag{B8}$$

The beliefs are required to satisfy three properties at completing histories. Let h_i^{t-1} be a completing history, with the variables $(\tau, \bar{z}, M, m, z_i, Z_i)$ defined as above.

First, if possible each player should believe that no more than one of the others deviated at $\tau - 1$. This is done by assuming each player believes when possible that none of the others deviated, except perhaps the one who was to have contributed the most. Thus:

$$P_i(\,\tilde{z}_j = \bar{z}_j \,|\, h_i^{t-1}) = 1 \text{ if } \begin{cases} i \neq M,\ j \notin \{i, M\},\ \text{and } Z_i \geq \bar{Z}_i - \bar{z}_M \\ i = M,\ j \notin \{M, m\},\ \text{and } Z_M \geq \bar{Z}_M - \bar{z}_m. \end{cases} \tag{B9}$$

So, if player $i \neq M$ ($i = M$) knows one of the others deviated, and possibly only one of them deviated, he believes only M (m) deviated. Another implication of (B9) is that player i believes none of the other players deviated if $Z_i = \bar{Z}_i$.

Second, player $i \neq M$ believes M deviated if he knows at least one of the others deviated:

$$P_i(\,\tilde{z}_M = \bar{z}_M \,|\, h_i^{t-1}) = 0 \text{ if } i \neq M \text{ and } Z_i \neq \bar{Z}_i. \tag{B10}$$

Third, if player M knows more than one of the others deviated, and in aggregate the others did contribute, then he believes they all deviated according to an atomless distribution:

$$P_M(\,\tilde{z}_j = a \,|\, h_M^{t-1}) = 0 \text{ if } 0 < Z_M < \bar{Z}_M - \bar{z}_m,\ j \neq M,\ \text{and } a \geq 0. \tag{B11}$$

Let $P = (P_1, ..., P_n)$ be any profile of distributions that satisfy (B9)-(B11) at completing histories, and are consistent with the strategy profile s whenever possible.[24] This yields an assessment, (s, P), which satisfies Bayes' rule when possible (and more).

Sequential Rationality

It remains to show that (s, P) is sequentially rational. Let h_i^{t-1} be a personal history. We show that if player i knows the others use the strategies s_{-i}, and his beliefs $P_i(\cdot \,|\, h_i^{t-1})$ satisfy the above restrictions, then $s_i(h_i^{t-1})$ is his best reply at h_i^{t-1}. There are four cases to consider.

Case 1: h_i^{t-1} is a grim history.

[24] As Fudenberg and Tirole (1991) discuss, other properties should be required of these beliefs. This is not a problem; e.g., player i's beliefs about $\tilde{z}_j(r)$ can be required to depend only on his personal history, h_i^{r-1}, and his observation of the aggregate of the other contributions in that period, $Z_i(r)$.

At this history player i knows the others will not contribute. The same will be true at any future date at which $X \notin C_1$, as the history then will still be grim. If i contributes enough to cause $X \in C_1$, and the date is not greater than $\bar{T}$, the history then will be completing. However, player i will have been the unilateral deviator at the date $\tau - 1$ at which X entered C_1. Each $j \neq i$ at $\tau - 1$ will have observed $(z_j, Z_j) = (0, z_i)$, with $z_i > 0$, when he expected to have observed $(\bar{z}_j, \bar{Z}_j) = (0, 0)$. Thus, $(z_j, Z_j) \in A_j(\bar{z})$ and $z_j = \bar{z}_j$, and so (B5) implies j does not contribute at this history. Player j will also not contribute at any subsequent history, since it will either be grim, if the date exceeds $\bar{T}$, or completing, in which case it yields the same τ and $\bar{z}$.

So, regardless of what player i does at h_i^{t-1} or thereafter, s requires the other players never to contribute again. Player i's best reply is thus to contribute nothing, as (B4) specifies, since $X(t-1) \notin C_1$ implies $X(t-1) \notin C_i$.

Case 2: h_i^{t-1} is an equilibrium history.

As in Case 1, if player i deviates from $g(t)$ at this history, s requires the other players never to contribute thereafter. This is sufficient to deter such a deviation, since it is the same punishing strategy as is used in the grim-g Nash equilibrium.

The remaining cases refer to a completing history h_i^{t-1} and its associated variables $(\tau, \bar{z}, M, m, z_i, Z_i)$. It is common knowledge at any possible history profile containing h_i^{t-1} that each personal history is a completing history, that it gives rise to the same $(\tau, \bar{z}, M, m)$, that s called for $\bar{z}$ to have been played at $\tau - 1$, and that $\bar{z}$ was not played at $\tau - 1$.

Case 3: h_i^{t-1} is a completing history and $(z_i, Z_i) \in A_i(\bar{z})$.

By (B1) and (B9), player i believes the deviation at $\tau - 1$ was unilateral. He thus believes each $j \neq i$ observed some $(z_j, Z_j) \in A_j(\bar{z})$. The proof depends on whether he deviated.

Case 3A: $z_i \neq \bar{z}_i$. In this case player i believes no other player deviated at $\tau - 1$. As he believes the actions of the others are given by (B5), he believes no other player will contribute in period t or thereafter, regardless of what he does. His best reply is to complete the project immediately if $X(t-1) \in C_i$, and otherwise not to contribute, as (B5) specifies.

Case 3B: $z_i = \bar{z}_i$. In this case $Z_i \neq \bar{Z}_i$, and player i believes exactly one of the others deviated at $\tau - 1$, either player M or m, by (B1) and (B9). Let $j \in \{M, m\}$ be who player i believes deviated. By (B3), $C_j = C_1$. The definition of a completing history therefore implies $X(t-1) \in C_j$. Thus, since player i believes j acts according to (B5), he believes j will contribute $R(t)$ in period t. His best reply is to contribute 0, as (B5) specifies.

Case 4: h_i^{t-1} is a completing history, $(z_i, Z_i) \notin A_i(\bar{z})$, and $i \neq M$.

In this case $Z_i \neq \bar{Z}_i$, and player i knows the deviation at $\tau - 1$ was multilateral. By (B10), player i believes M deviated. Thus, given that M contributes according to (B5) or (B7), player i believes that either player M will contribute $R(t)$, or that $Z_M = 0$ and, for some $j \neq M$, $z_M = \bar{Z}_j$, $\bar{z}_j > 0$, and $X(t-1) \in C_j$. In the latter case, player j observed $(z_j, Z_j) = (0, \bar{Z}_j)$ with $z_j \neq \bar{z}_j$, and (B5) implies he will complete the project immediately. So in either case player i believes another player will complete the project immediately. His best reply is to contribute 0, as (B6) specifies.

Case 5: h_M^{t-1} is a completing history and $(z_M, Z_M) \notin A_M(\bar{z})$.

Now $Z_M \neq \bar{Z}_M$, and player M knows the deviation at $\tau - 1$ was multilateral. Since (B3) implies $X(t-1) \in C_M$, player M should complete the project immediately if he believes the others will not contribute in period t or thereafter, regardless of what he does. There are three subcases.

Case 5A: $Z_M = 0$. Then player M knows $j \neq M$ saw $(z_j, Z_j) = (0, z_M)$. As M also knows $\bar{z}$, he knows what j's contribution in period t will be, given (B1), (B5), and (B6). That is, for any possible history h^{t-1} containing the h_M^{t-1} at issue, player M knows

$$s_j(h_j^{t-1}) = \begin{cases} R(t) & \text{if } \bar{z}_j > 0,\ z_M = \bar{Z}_j, \text{ and } X(t-1) \in C_j \\ 0 & \text{otherwise.} \end{cases} \tag{B12}$$

Thus, M should not contribute if the top line of (B12) holds for some $j \neq M$, as (B7) specifies. Otherwise he should complete the project, as (B7) specifies.

Case 5B: $Z_M > 0$ and $Z_M \geq \bar{Z}_M - \bar{z}_m$. Then by (B9) and $Z_M \neq \bar{Z}_M$, player M believes that among the other players, only m deviated from $\bar{z}$. Since the deviation was multilateral, this implies $z_M \neq \bar{z}_M$. Player M thus believes

$$\begin{aligned} Z_m &= z_M + \sum\nolimits_{j \neq M,m} \bar{z}_j \\ &\neq \bar{z}_M + \sum\nolimits_{j \neq M,m} \bar{z}_j = \bar{Z}_M, \end{aligned}$$

and so $(z_m, Z_m) \notin A_m(\bar{z})$. By (B6), player M believes m will never contribute again. Also, as he believes no player $j \notin \{M, m\}$ deviated, he believes they will never contribute again either, by (B5) and (B6). So he should complete the project immediately, as (B7) specifies.

Case 5C: $0 < Z_M < \bar{Z}_M - \bar{z}_m$. Now M knows at least two of the others deviated. By (B11), he believes $z_j \neq \bar{z}_j$ and $z_j \neq z_M + Z_M - \bar{Z}_j$, for any $j \neq M$. (To show this, first set $a = \bar{z}_j$, and then $a = z_M + Z_M - \bar{Z}_j$, in (B11).) As the latter is the same as $Z_j \neq \bar{Z}_j$, player M believes that $(z_j, Z_j) \notin A_j(\bar{z})$. Thus, by (B6), player M believes no other player will ever contribute again. His best reply is to complete the project immediately, as (B7) specifies. ■

C Appendix: Alternating Contributions

In this appendix we show how the negative conclusion of Admati and Perry (1991) (AP) regarding the effect of dynamics on free riding is altered by changing any of three assumptions: that the contributing cost function is strictly convex, that the contributing horizon is infinite, or that players cannot contribute in successive periods.

The AP Game

The AP model has two players, both of whom have the same binary benefit function: $\lambda_i = 0$ and $V_i = V$. We maintain assumption (2.7): $V < X^* < 2V$. The contributing cost function w is increasing and satisfies $w(0) = 0$. Player i's payoff is $\delta^T V - \sum_{t=0}^{T} \delta^t w(z_i(t))$ if completion occurs in period T and he contributes $z_i(0), ..., z_i(T)$. Three more specifications define the *AP game*: (i) w is strictly convex; (ii) the contributing horizon is $\bar{T} = \infty$; and (iii) the players can contribute only in alternate periods, starting with player 1.

Recall from Section 7 that $w(R_1) \equiv (1 - \delta)V$ defines the maximum amount R_1 a player is willing to contribute in order to complete the project this period rather than contribute nothing and have the other player complete the project next period. Starting with R_1, a sequence $\{R_k\}_{k=1}^{\infty}$ is defined recursively so that a player is indifferent between contributing R_k, and contributing zero and having the other player contribute R_k in the next period, given that contributions $R_{k-1}, ..., R_1$ will then be made successively to complete the project. This yields $w(R_k) \equiv \delta^{2k-3}(1-\delta^2)V$ for $k \geq 2$. Let $S_0 \equiv 0$ and $S_q \equiv \sum_{k=1}^{q} R_k$ for $q > 0$.

For any history h^{t-1}, the remaining amount required for completion is

$$R(h^{t-1}) \equiv \max(0, X^* - X(t-1)).$$

We refer to the following strategy as the *AP equilibrium*: for any history h^{t-1}, the player whose turn it is contributes

$$s^A(h^{t-1}) \equiv \begin{cases} R(h^{t-1}) - S_q & \text{if } S_q < R(h^{t-1}) \leq S_{q+1} \text{ for any } q < \infty \\ 0 & \text{if } S_\infty \leq R(h^{t-1}). \end{cases} \tag{C1}$$

Denote the resulting contribution sequence as the *AP outcome*. At the null history h^{-1} that starts the game, $R(h^{-1}) = X^*$. So the AP outcome yields no contributions if $S_\infty \leq X^*$. Otherwise, for some $q < \infty$, the first contribution is $X^* - S_q$, the second is R_q, the third is R_{q-1}, and so on until R_1 is contributed to complete the project. This is generically the only subgame perfect equilibrium (SPE) outcome:

Proposition C1 (Admati and Perry, 1991) *If $X^* \neq S_q$ for any $q < \infty$, the AP outcome is the only SPE outcome of the AP game.*

The Linear AP Game

Modifying the AP game by assuming $w(z) \equiv z$ yields the *linear AP game*. The AP equilibrium is still one of its equilibria. This linear cost function implies $S_\infty = V$. Thus, by the second line of (C1), the AP equilibrium yields no contributions, given that $V < X^*$, even though efficiency requires completion. This is the basis of AP's inefficiency results, Lemma 4.1 and Proposition 4.1.

However, the linear AP game has other equilibria. In particular, the following is an equilibrium that completes the project in two periods if δ is sufficiently large. For any history h^{t-1}, if it is player 1's turn, he contributes

$$s_1^L(h^{t-1}) \equiv \begin{cases} R(h^{t-1}) & \text{if } R(h^{t-1}) < R_1 \\ \max(0, R(h^{t-1}) - V) & \text{if } R(h^{t-1}) \geq R_1; \end{cases} \tag{C2}$$

and if it is player 2's turn, he contributes

$$s_2^L(h^{t-1}) \equiv \begin{cases} R(h^{t-1}) & \text{if } R(h^{t-1}) \leq V \\ 0 & \text{if } R(h^{t-1}) > V. \end{cases} \tag{C3}$$

The outcome of s^L is that player 1 contributes $X^* - V$ in period 0, and player 2 completes the project in the second period by contributing V. Off the equilibrium path, player 1 completes the project if the amount required is less than $R_1 = (1-\delta)V$. Otherwise, he contributes just enough to make sure the next amount required to complete the project is not more than V. Player 2 does not contribute unless the amount required to complete is not more than V, and if it is he completes the project immediately. The equilibrium payoffs are $U_1 = (1+\delta)V - X^*$ and $U_2 = 0$: the first mover receives all the surplus.

Proposition C2 *The profile s^L is a SPE of the linear AP game if $\delta > (X^* - V)/V$.*

Proof. Most of the proof is obvious. Player 1's payoff, $(1+\delta)V - X^*$, is positive by the lower bound hypothesis on δ. Hence, it is optimal for him to contribute $X^* - V$ in order to induce player 2 to complete the project in the next period, rather than not to contribute and so leave the project uncompleted forever. Also in accordance with (C2), the definition of R_1 implies that it is optimal for player 1 to complete the project immediately if the required amount does not exceed R_1.

The interesting part of the proof is showing that if $R(h^{t-1}) \leq V$, player 2 should complete the project immediately, instead of inducing player 1 to complete it next period by contributing just $R(h^{t-1}) - R_1$. In fact, player 2 is indifferent between these actions, as was shown in the text: the inequality in (7.1) is an equality if w is linear. ∎

The equilibrium s^L is not robust to making w strictly convex. As we show in the text near (7.1), if w is strictly convex, player 2 is unwilling to complete the project by contributing more than R_1, contrary to (C3). Indeed, no SPE of the linear game that yields contributions can be robust to making w strictly convex: the AP outcome is the only SPE outcome if w is strictly convex, and by AP's Proposition 4.1, it yields no contributions if w approximates the identity function in the sense that $w'(0) \geq V/X^*$. (Recall that $V/X^* < 1$.)

The Finite Horizon AP Game

An alternative modification of the game, replacing $\bar{T} = \infty$ by $0 < \bar{T} < \infty$, yields the *finite horizon AP game*.[25] Refer to the player who may contribute at $\bar{T}$ (resp. $\bar{T} - 1$) as player b (resp. a). In period $\bar{T}$, player b has no future contributors upon which to free ride. In essence, the finite horizon allows player a to commit to making his last contribution at $\bar{T} - 1$. Player b will complete the project in the last period if it leaves him any nonnegative continuation payoff. The maximum final contribution, $w^{-1}(V)$, is thus greater than it is in the AP game, $R_1 = w^{-1}((1-\delta)V)$, and does not vanish as $\delta \to 0$. Player a's SPE payoff is bounded below by what he gets by contributing just enough at $\bar{T} - 1$ to induce player b to complete the project at $\bar{T}$. Hence, in some cases every SPE completes the project.

To prove this, let δ be sufficiently close to 1, and w to the identity function, that

$$\delta V > w(X^* - w^{-1}(V)). \tag{C4}$$

(Recall that $V > X^* - V$.) This condition implies that player a obtains a positive payoff by contributing $X^* - w^{-1}(V)$ if then player b completes the project in the next period.

Proposition C3 *Given (C4) and any* $0 < \bar{T} < \infty$, *every SPE of the finite horizon AP game with horizon* $\bar{T}$ *completes the project. If also* $w(z) \equiv z$, *the set of all SPE payoffs converges to* $(U_a, U_b) = (2V - X^*, 0)$ *as* $\delta \to 1$.

Proof. Given a SPE, let the equilibrium payoffs be U_a^{eq} and U_b^{eq}. Let R be the contribution needed to complete at $\bar{T} - 1$ when player a does not contribute in any prior period, and player b plays his equilibrium strategy. Let $\varepsilon > 0$. Suppose player a contributes nothing

[25] The $\bar{T} = 0$ case is not interesting, as then only player 1 can contribute.

until $\bar{T}-1$, and he then contributes $\max(0, R - w^{-1}(V) + \varepsilon)$. Player b then completes the project at $\bar{T}$ if it is not completed already, since $w(R(h^{\bar{T}-1})) \leq w(w^{-1}(V) - \varepsilon) < V$. The payoff of player a when he plays this strategy is at least

$$\begin{aligned} \delta^{\bar{T}}V - \delta^{\bar{T}-1}w(\max\left\{0, R - w^{-1}(V) + \varepsilon\right\}) &\geq \delta^{\bar{T}}V - \delta^{\bar{T}-1}w(\max\left\{0, X^* - w^{-1}(V) + \varepsilon\right\}) \\ &= \delta^{\bar{T}-1}\min\left\{\delta V, \delta V - w\left(X^* - w^{-1}(V) + \varepsilon\right)\right\}. \end{aligned}$$

The left side of this inequality is not more than U_a^{eq}. The right side is positive for small ε, by (C4). Hence $U_a^{eq} > 0$, which proves the equilibrium completes the project. If $w(z) \equiv z$, the right side of the inequality is

$$\delta^{\bar{T}-1}\min\left\{\delta V, (1+\delta)V - X^* - \varepsilon\right\} = \delta^{\bar{T}-1}\left[(1+\delta)V - X^* - \varepsilon\right],$$

since $V < X^*$. Thus, $U_a^{eq} \geq 2V - X^*$ as $(\delta, \varepsilon) \to (1, 0)$. As the total surplus $2V - X^*$ is no less than $U_a^{eq} + U_b^{eq}$, and $U_b^{eq} \geq 0$, this proves that $U_a^{eq} = 2V - X^*$ and $U_b^{eq} = 0$ as $\delta \to 1$. ■

So, if discounting is low or the period length short, and w approximates the identity function, making the horizon finite in the AP game insures that every SPE completes the project. This is in contrast to the infinite horizon game in which no contributions are made if w approximates the identity. But Proposition C3 reveals an unpleasant endgame effect: for large δ, nearly all the surplus goes to the player able to contribute at $\bar{T}-1$. This is in contrast to the lack of endgame effects in our game: its set of NE outcomes simply expands as $\bar{T}$ increases (Theorem 1), and the same is true of its set of PBE outcomes if players 1 and 2 (in the general $n \geq 2$ case) have the same critical contributions (Theorem 2 (b)).

The Unrestricted AP Game

Another modification of the AP game is to allow each player to contribute in any period. This *unrestricted AP game* would be a special case of our game if w were linear. We show that it, with finite or infinite contributing horizons, has subgame and Markov perfect equilibria that complete the project if w approximates the identity function. Thus, the completing equilibria of our game are robust to making the contributing cost function strictly convex.

Let w approximate the identity function in so far as $z^* \in \Re^2_+$ exists such that $z_1^*+z_2^* = X^*$ and $w(z_i^*) \leq V$ for $i = 1, 2$. Then define a strategy profile s^0: in period 0 player i contributes $s_i^0(h^{-1}) = z_i^*$, and at any other history h^{t-1} player $i \neq j$ contributes

$$s_i^0(h^{t-1}) = \begin{cases} 0 & \text{if } z_i(0) = z_i^* \\ R(h^{t-1}) & \text{if } z_j(0) = z_j^* \\ z_i^* R(h^{t-1})/X^* & \text{otherwise.} \end{cases} \tag{C5}$$

This profile completes the project in period 0. A unilateral deviator completes it alone; if both players deviate, it is completed by each player i's contributing the fraction z_i^*/X^* of the remaining cost. (Note that s^0 may not be efficient – as w is convex, it may be better to spread the contributions over time.) The proof of the following is straightforward.

Proposition C4 *Suppose $z^* \in \Re^2_+$ satisfies $z_1^* + z_2^* = X^*$ and $w(z_i^*) \leq V$ for $i = 1, 2$. Then s^0 is a SPE of the unrestricted AP game with horizon $\bar{T}$, for any $0 \leq \bar{T} \leq \infty$ and $0 \leq \delta \leq 1$.*

The punishments in s^0 are severe – after a unilateral deviation, the non-deviator refuses ever to contribute again. But completing equilibria with more forgiving punishments also exist. The following is a contribution goal equilibrium (see Section 5) that completes the project in two periods. (A fortiori, it is also a Markov perfect equilibrium, as are s^A and s^L above.) If the player responsible for meeting the first goal does not do so, he is punished only for as many periods it takes him to meet the goal.

Assume δ is so close to 1, and w is so close to the identity function, that (C4) and the following condition hold:[26]

$$w(X^*) - \min_{0 \leq y \leq X^*} (w(y) + w(X^* - y)) < \delta V - \delta^2 V. \tag{C6}$$

Let the first goal be $X_0 \equiv X^* - w^{-1}(V)$. (It can be larger as well.) By (C4), $\delta V - w(X_0) \geq 0$ and $V - w(X^* - X_0) \geq 0$. Thus, if player 1 contributes X_0 in the first period, and player 2 contributes $X^* - X_0$ in the second period to complete the project, both receive a nonnegative

[26] For example, let $w(z) = z + \varepsilon z^2$. Let $\bar{\varepsilon} > 0$ be such that $w(X^*/2) < V$ (recall that $X^*/2 < V$), and assume $\varepsilon \in (0, \bar{\varepsilon})$. Let $\bar{w}(z) = z + \bar{\varepsilon} z^2$. Then (C4) holds if $\delta > \bar{w}(X^*/2)/V$, and (C6) holds if $\varepsilon < 2\delta(1-\delta)V/(X^*)^2$.

payoff. The left side of (C6) increases in X^*, and so (C6) also holds if X^* is replaced by X_0. Thus, (C6) implies that player 1 prefers to contribute X_0 in one period to obtain completion in the next period, rather than to contribute X_0 over two periods to obtain completion in the period after the next. Similarly, player 2 prefers to contribute $X^* - X_0$ in one period to complete the project, rather than to contribute it over two periods to complete the project in the next period. These observations prove the following:

Proposition C5 *If δ and w satisfy (C4) and (C6), the strategy profile s^G defined below is a Markov perfect equilibrium of the unrestricted AP game for any horizon $0 < \bar{T} \leq \infty$:*

$$s_1^G(h^{t-1}) \equiv \max(0, X_0 - X(t-1)), \tag{C7}$$

$$s_2^G(h^{t-1}) \equiv \begin{cases} 0 & \text{if } X(t-1) < X_0 \\ R(h^{t-1}) & \text{if } X(t-1) \geq X_0. \end{cases} \tag{C8}$$

According to s^G, player 1 alone is responsible for achieving the goal X_0. Player 2 does not contribute until it is achieved, which in equilibrium occurs in period 0. Player 2 is then responsible for achieving the final goal, X^*, which he does in period 1. The players contribute in alternate periods, though they are not restricted to doing so. The contributions of player 1 and 2 converge, respectively, to $X^* - V$ and V as w converges to the identity function: in this limit the outcome of s^G converges to that of s^L.