CARESS Working Paper 98-09

Consensus and Co-existence in an Interactive Process of Opinion Formation

Valentina Corradi(+) and Antonella Ianni (++,+)*

(+) University of Pennsylvania
Department of Economics
3718 Locust Walk
Philadelphia, PA 19104-6297, USA
corradi@econ.sas.upenn.edu

(++) University of Southampton
Department of Economics
Highfield
Southampton, SO17 1BJ, UK
a.ianni@soton.ac.uk

April, 27 1998

Abstract

This paper studies a simple dynamic model of pre-electoral opinion formation, where individuals repeatedly form their opinions as to which (out of two) candidate to support. The behavorial rules we analyze allow for various forms of discontinuities, that we characterize in terms of threshold, and are defined in a locally interactive setting. We focus on both the long-run and short-run behavior of the process. It is shown that the asymptotics of the process depend crucially on the particular form of discontinuity that we postulate. In particular the process may show a tendency towards consensus, in the sense that all individuals conform in their opinion, or the system may be absorbed in configurations in which different opinions co-exist. We finally analyze various forms of asymmetries in the threshold.

Keywords : Consensus, Fixation, Local Interaction, Threshold Voter Model

Jel Classification : D72, D82, C62

*We gratefully acknowledge helpful conversations with Luca Anderlini, David Cass, Paul Labys and Larry Samuelson

1 Introduction

This paper studies a simple dynamic model of pre-electoral public opinion formation and aims at explaining the stylized fact that the support for one candidate is often observed within large geographical regions. Indeed from a casual look at the U.S. presidential election results, we can see that there are large geographical regions characterized by a rather homogeneous voting behavior, only partially explained by generic socio-economic variables.

An intuitive explanation of this stylized fact is that the private opinion of each voter affects and is affected by the opinion of other voters. However, because of communication costs, each individual tends to communicate and exchange opinions only with her/his neighbors. Furthermore each voter tends to hold her/his current opinion if most of her/his neighbors agree, while tends to change opinion if most of her/his neighbors disagree. We shall see that the joint effect of local interaction and tendency to agree with the opinion held by the majority, contribute to explain the creation of large homogeneous areas. This suggests to model the process of public opinion formation as a dynamic process characterized by the emergence of interactive patterns of behavior. Hereafter with the term private opinion we mean the position held by a single individual, with the term public opinion we mean the configuration representing the position held by the collection of individuals at a given time. We shall consider a countable population of individuals that repeatedly choose to support one of the two candidates.[1] Each individual has a well defined preference structure over the final electoral outcome, that depends on the distribution of votes across the population. At a random Poisson time each agent revises her/his opinion, by observing the opinions held by her/his neighbors. At any instant of time the probability that more than one voter revises her/his opinion is zero. After observing the neighbors' behavior each agent revises her/his own (i.e. when deciding the candidate for whom (s)he would vote if the election were in that moment) in such a way as to maximize her/his expected utility. The basic assumption of our model is that each individual receives a higher utility if (s)he votes for the winner. This can be justified, for example, on the basis of the existence of some side payment.

Another explanation of why voters prefer to vote for the winner, is the so called bandwagon effect in the public opinion formation. The bandwagon effect has been extensively studied, from

[1]Some of our results straightforwardly hold also the case of a finite number of individuals, others do not; we should be more precise about this in Sections 4 and 5.

both a theoretical and an empirical/experimental point of view, in the political science literature (e.g. Henshel and Johnston 1987, for a survey and a new explanation). According to the bandwagon theory, there is a strong effect from opinion polls to voters decisions. Basically the fact that polls give an advantage to a candidate induces voters to switch their preference in favor of that candidate. This direct effect from polls to voters' opinion is mainly explained as a desire of conformity, both in the sense that people conform to what they believe is the proper norm and in the sense that people feel reassured in adopting the opinion held by the majority. Henshel and Johnston (1987) also point out an indirect effect: if polls give advantage to a candidate, more funds, endorsements will be directed to that candidate, by this increasing her/his share of support. Broadly speaking opinion polls can be interpreted as an indicator of the opinion held by the entire population. Via lab experiments, Nadeau, Cloutier and Guay (1987) provide evidence in favor of a bandwagon effect in opinion formation: in particular they find that individuals tend to switch their opinion towards the one expressed by the majority of the participants in the experiment. Bandwagon effects are also relevant in economics, for example they play a role in the explanation of technologies diffusions (see e.g. Kandori and Rob, 1998). As the bandwagon theory explains why each individual tends to conform to the majority, we should expect to observe growing consensus toward the same candidate over time. However such theory does not allow for a spatial characterization of the process. Hence it fails to explain how this happens along the dynamics. As we previously mentioned, stylized facts support that consensus is reached in larger and larger areas within which voters hold the same opinion. To capture this spatial element of the dynamics, we shall postulate that an individual's opinion is more strongly influenced by the opinion held by the other individuals with whom (s)he interacts more often, like relatives, friends, neighbors, and /or colleagues. As we shall see below, in order to explain the stylized fact of interest, we need to take into consideration local interaction, in addition to the tendency to follow the majority.

In the paper, we propose a model in which, in order to maximize her/his expected utility each individual chooses the opinion held by at least a given fraction of the population. However as we said above, each individual cannot observe the population, but only her/his neighbors. Each voter assumes that the distribution of opinions across the neighbors is a good representation of the distribution of opinions across the population. Thus (s)he changes opinion when a certain proportion of neighbors disagree with her/his opinion. This provides discontinuous behavioral rules, that we formalize in terms of thresholds. A different approach is followed in a companion paper (Ianni and Corradi, 1998), where such discontinuities are smoothed in different fashions.

We shall show that the process of public opinion formation can be formulated in terms of a particular type of interactive particle process, known as the threshold voter. Given that, we can rely on several results available in the probabilistic literature in order to study the dynamics and the asymptotics of the public opinion formation process. We shall see that as time goes to infinity, either we reach consensus, that is asymptotically everyone will hold the same position, or we have coexistence, in the sense that both opinions may survive. In the consensus case we observe the formation of larger and larger connected areas of individuals holding the same opinion (so called clusters), until one region will invade all the others. Because of the attractivity property of the voter process (i.e. each individual tends to conform to her/his neighbors) also in the coexistence case we shall observe large homogeneous regions, simply there won't be a single region invading the others. In this sense the existence of large homogeneous regions can be seen either as a picture of the dynamics (consensus case) or as a picture of the limiting behavior (in the co-existence case). We shall finally consider the case of global interaction, where each voter can observe the behaviour of the whole population. In this case each voter changes opinion at most one time and we shall have consensus on either of the candidates, depending on the initial distribution of opinions across the population.

Local interaction models have received increasing attention in the last few years, especially in the learning/bounded rationality theory. Blume (1993) analyzes stochastic strategy revision processes via a class of (local) interactive particle models, known as Ising models. One of the main differences between Ising models and the voter models we shall treat below, is that the former are ergodic while the latter are nonergodic. Durlauf (1993) uses local interaction among industries in order to explain the possibility of nonergodic growth, Anderlini and Ianni (1996) use local interactive systems in order to explain price dispersion; Ellison (1993) and Morris (1997) analyze the conditions under which a strategy played by a minimum threshold of neighboring individuals will spread out to the entire population. Finally nonergodic, discrete time locally interactive systems are studied by Eshel, Samuelson and Shaked (1998) in order to explain the long run survival of certain strictly dominated strategies, such as altruism. Nevertheless, none of the papers cited above employ the class of voter models we shall use below. One of the nice feature of voter models is that they enable us to study cluster formation, that is the formation of larger and larger areas of homogeneous opinions.[2]

This paper is organized as follows. Section 2 presents a basic model of private opinion formation and derives the rule followed by each voter when revising her/his opinion. In Section 3 by considering the joint, interactive behavior of all voters, we obtain the dynamics for the public opinion formation process. We shall see that the public opinion process can be treated either as a threshold voter process or as a linear voter process. Section 4 first analyzes the cases in which consensus occurs and then analyzes the case of fixation, in which each individual can change opinion only a finite number of times. In general fixation is characterized by the coexistence of both opinions. Section 5 provides a slight modification of the basic model that implies a new behavioral rule, according to which the rate at which each individual changes opinion depends on his/her current opinion. We shall show that consensus (and so clustering) occurs. Finally Section 6 treats the case in which each individual can observe the entire population. Some concluding remarks are given in Section 7.

2 The Individual Process of Opinion Formation

We now formalize the process of pre-electoral opinion formation on the part of each single voter. Two candidates, A and B run the elections, to be held at some future date, and the winner will be decided by simple majority voting. There are countably many voters, that form their own opinion as to which candidate to support, when the election are held. Our purpose is to analyze how any single voter changes opinion over time, starting from the moment in which the two candidates are announced, until the elections date. As there are countably many agents, the chance for a single voter of being pivotal is negligible. Hence we take the two electoral outcomes, A *wins* (resp. B *wins*) as unknown states of nature for each voter. We call them A and B respectively. Also let a, (*resp.* b) be an opinion in favor of candidate A, (*resp.* B). For the sake of simplicity we shall assume voters have identical preferences, formalized by an utility function depending on the opinion and on the unknown state of nature. All agents behave in an identical manner, though asymmetries may arise because of differences in the information set they possess. This can be summarized by

[2] The relationship between cluster formation and the so called herd behavior (e.g. Banerjee 1992, Bikchandani, Hirshleifer and Welch 1992) is studied in Corradi and Ianni (1998).

the following table.

	A wins	B wins
vote a	$U(A, a)$	$U(B, a)$
vote b	$U(A, b)$	$U(B, b)$

We can assume that each voter gets higher utility if (s)he had voted the for candidate who won. This can easily be formalized by the introduction of a side payment, say $\varepsilon > 0$ that a voter gets if (s)he had voted for the winner. Assuming an utility function separable in its arguments, we have

	A wins	B wins
vote a	$U(A) + \varepsilon$	$U(B)$
vote b	$U(A)$	$U(B) + \varepsilon$

Given any probability distribution, P, over the state space $\{A, B\}$, it is easy to notice that $E_P U(a) \geq E_P U(b)$ if and only if $P(A) \geq \frac{1}{2}$. Hence a voter would maximize her/his expected utility by voting for the candidate who is supported by the majority of the electorate. As already mentioned, this type of rules can be justified on the basis of a bandwagon effect. It should be noted that the side payment ε is given exogeneously. In this way we are disregarding any strategic element on the part of the two candidates, just to focus on the opinion formation process on the part of the individual voter. Endogeneous determination of the side payments (for example in terms of distribution of some public goods) via interaction between a strategic behavior on the part of the candidate and the opinion process of the voters, will be object of future research. The case of asymmetries in the side payments is considered in Section 5; we shall see that the dynamics is much less articulate than in the symmetric case. Suppose that ε_A (*resp.* ε_B) is the side payment obtained by an individual who voted for A (*resp.* B) and A, (*resp.* B) wins. We shall see that under mild conditions if $\varepsilon_A > \varepsilon_B$, then we have consensus on A, and viceversa. Thus it is natural to think that if the candidates act strategically, then $\varepsilon_A = \varepsilon_B = \varepsilon$.

If, whenever forming an opinion, the voter knew exactly the proportion with which, say, A is supported in the population, then (s)he would form an opinion consistently with that of the majority. However each individual cannot observe the distribution of opinions across the entire population, or simply it may be costly to collect this information. We assume that each agent can just observe the opinion held by her/his neighbors. (S)he assumes that the distribution of opinions across the neighbors is a good representation of the distribution of opinions across the population. Then each voter flips opinion if the number of neighbors that disagree is above a certain threshold.

This delivers a discontinuos (threshold) rule.[3]

We shall analyze different specifications of the model. In all specifications a voter conforms to the majority of observed opinions, whenever there is a strict majority. However, when both candidates are supported by an equal (finite) number of neighbors, then (s)he may behave differently accordingly to the specification of the model. Formally, if ν_A denotes the proportions of neighbors favoring A, the behavioral rules that each voter follows are as stated below:

if $\nu_A > \frac{1}{2}$ choose A

if $\nu_A < \frac{1}{2}$ choose B

if $\nu_A = \frac{1}{2}$ then:

(1) choose A (resp. B) with probability $\frac{1}{2}$

(2) keep the opinion currently held

(3) change opinion

where (1),(2),(3) refer to the different specifications of the model that we study. In this sense $\frac{1}{2}$ can be seen as a critical value or threshold of the model. As we shall see, the dynamics as well the asymptotic properties of the model, depend on specification of this threshold. Note that, as the number of neighbors is finite, the concept of majority among neighbors is well defined.

A possible objection to the argument given above is that opinion polls are available to the public and can be interpreted as an unbiased estimator of the distribution of opinions across the population; such a possibility will be explored in Section 6.

So far we have analyzed the process of individual opinion formation on the part of a single agent at a given point in time. In order to study the process of aggregate opinion formation, we need to analyze the joint evolution of the opinions held by each voter. This is accomplished in the next section.

3 The Aggregate Process of Opinion Formation

We have seen in the previous section that each individual revises her/his opinion according to the opinion held by the majority of her/his neighbors. In this section we shall show that, after

[3] In a companion paper (Ianni and Corradi, 1998) we analyze both the case in which each individual has an uninformative prior about the distribution of opinions across the population, and updates her/his prior by sampling from the neighbors, and the case of a linear rule, according to which if $\alpha\%$ of neighbors disagree, that individual changes opinion with probability equal to α.

imposing some further structure on the process of individual opinion formation, the process of aggregate opinion formation can be formally treated either as a linear or as a nonlinear (threshold) voter model. We first need to define what we mean by *neighbors*. In order to do this we need to define more precisely the local structure. Each individual lives at a site in a one-dimensional lattice, denoted by Z. We shall denote by $x \in Z$, the individual on site x. The neighbors of individual x are those individuals living at the sites to the immediate right and left of x. Hereafter we shall assume that each site looks at $2N$ neighbors, N on the right and N on the left, where N is some natural, finite number. Thus, for example, the neighbors of the individual at site 0 are $\{-N, -N+1 \ldots -1, 1, 2, \ldots N\}$. Each site revises her/his opinion at a random Poisson time with intensity parameter equal to one, so that the expected length of time between two consecutive opinion reassessments at the same site is equal to one. The Poisson random times are independent across sites. The Poisson assumption ensures that, at any instant of time t, each site has a positive probability of reassessing her/his opinion, but the probability that two sites reassess their opinions simultaneously is zero. With the term configuration we mean the state of the system, at a given time, that is the collection of opinions held by the individuals at different sites at that given time. The public opinion formation process is then the evolution over time of these configurations.

We now need to introduce some notation. We denote by $\eta_t(x)$ the opinion held at time t by the individual at site x, $x \in Z$. For the sake of simplicity, we say that $\eta_t(x) = 1$ if the individual at site x favors candidate A and $\eta_t(x) = 0$ if the individual at site x favors candidate B. Thus $\eta_t(x)$ represent the private opinion of the individual at site x, while $\eta_t = (\eta_t(x), x \in Z)$ represents the public (aggregate) opinion at time t. It is immediate to see that η_t evolves on the space $\{0,1\}^Z$; in fact the state of the system, or configuration, at time t is the collection of the opinions $\{0,1\}$ held by the individuals at the different sites on the one-dimensional lattice Z. In order to analyze the dynamics and the asymptotic behavior of η_t we have to be more precise about the rule followed by each site when revising her/his opinion. As each voter's neighborhood consists of $2N$ other voters, the rule described in (1)-(3), can be reformulated in terms of N, instead of in terms of proportions. Hereafter $|\cdot|$ denotes the Euclidean norm, # denotes the cardinality of a set, and

$$\eta^L = (\eta_t^L(x), x \in Z, t \geq 0) \tag{1}$$

denotes the public opinion formation process, when $N = 1$ and if both neighbors disagree (agree) then each individual change (does not change) opinion, while instead when one neighbor disagrees and the other agrees, the site flips opinion with probability 1/2.

$$\eta^{\tau_c} = (\eta_t^{\tau_c}(x), x \in Z, t \geq 0), \tag{2}$$

denotes the public opinion process when each individual changes opinion if at least N neighbors disagree with the opinion (s)he currently holds. Finally,

$$\eta^{\tau_f} = (\eta_t^{\tau_f}(x), x \in Z, t \geq 0), \tag{3}$$

denotes the public opinion formation process when each individual changes opinion whenever at least $N+1$ disagree with the opinion currently held.

In the sequel we shall use the following facts.

FACT 3.1

$$\eta^L = (\eta_t^L(x), x \in Z, t \geq 0)$$

is a (nearest neighbor) linear voter model.
PROOF: η^L is an interactive particle Markov process evolving on $\{0,1\}^Z$. The value of the configuration η^L at site x changes at rate $c^L(x, \eta)$, where

$$c^L(x, \eta) = \frac{1}{2} \#\{y: \ |y - x| = 1, \ \eta(y) \neq \eta(x)\} \tag{4}$$

Thus from Bramson and Griffeath (1980, p.183) we know that η^L is a (nearest neighbor) linear voter model on a one-dimensional lattice. ♣

FACT 3.2

$$\eta^{\tau_c} = (\eta_t^{\tau_c}(x), x \in Z, \ t \geq 0)$$

is a threshold voter model with threshold $\tau_c = N$
PROOF: η^{τ_c} is an interactive particle Markov process evolving on $\{0,1\}^Z$. The value of the configuration η^{τ_c} at site x flips at rate $c^{\tau_c}(x, \eta)$, where

$$c^{\tau_c}(x, \eta) = 1 \ if \ \#\{y : |y - x| \leq N, \eta(x) \neq \eta(y)\} \geq N \quad and \ 0 \ otherwise \tag{5}$$

The statement then follows from Andjel, Liggett and Mountford (1992, p.74) ♣

FACT 3.3

$$\eta^{\tau_f} = (\eta_t^{\tau_f}(x), x \in Z, \ t \geq 0)$$

is a threshold voter model with threshold $\tau_f = N + 1$

PROOF: by the same argument used in the proof of Fact 3.2, by noting that

$$c^{\tau_f}(x,\eta) = 1 \; if \; \#\{y : |y-x| \leq N, \eta(x) \neq \eta(y)\} \geq N+1 \quad and \; 0 \; otherwise \tag{6}$$

The statement then follows from Durrett and Steiff (1993, p.232) ♣

It should be noted that in the linear voter model, each site changes opinion at rate proportional to the number of neighbors that disagree, while in the threshold voter the site changes opinion if there is at least a given number (threshold) of neighbors that disagree.

Given the facts above, we can now analyze the dynamics and the asymptotic behavior of $\eta^L, \eta^{\tau_c}, \eta^{\tau_f}$, by borrowing some results already available in the probabilistic literature.

4 The Dynamics and the Asymptotic Behavior of the Aggregate Opinion Formation Process

As each site will never change opinion when all neighbors agree, a configuration of either $all\ favoring\ A$ or $all\ favoring\ B$ is an absorbing state for the system. A first question that we wish to address is whether such configurations will be reached as t gets large. More precisely we address the problem of whether consensus will eventually occur or not, where by consensus we mean that, as t gets large, the probability that two generic sites have a different opinion, approaches zero. Thus consensus is a situation in which all sites will eventually agree on the same position. We shall see below that consensus occurs when the flip rate are as in (4)-(5), while consensus does not occur in general when the flip rates are as in (6).

Needless to say, in reality elections will be held at some given date in the future. Thus even if the private opinion formation process follows rules (1) or (2) (and so the flip rates are as in (4)-(5)), we shall never observe all individuals voting for the same candidate. This is because the process is stopped at the election time. However we shall see that in the consensus case, clustering occurs, that is there is a formation over time of larger and larger connected groups of individuals favoring the same candidate. Thus when stopping the process at the election time, we have a picture showing large connected groups favoring A followed by large connected groups favoring B, and so on. In these cases what we observe is the dynamics towards consensus, stopped at the election date. In this sense candidates (or more plausibly the incumbent) can affect the electoral outcome, by choosing the election date.

Linear and nonlinear (threshold) voter models are non-ergodic, in the sense that the limiting distribution depends on the initial distribution. Without loss of generality, we can assume that time 0 is the time at which the names of the two candidates running the elections are publicly announced. Several of the propositions stated below, require the initial distribution to be a Bernoulli product measure with density $\rho \in (0,1)$, denoted as μ^ρ, that is $\mu^\rho(x : \eta_0(x) = A, x \in Z) = \rho, \rho \in (0,1)$. This means that if the elections were held at time 0 (i.e. the announcement of the two candidates and the elections occur at the same time), then each site would vote for A with probability ρ and would vote for B with probability $1-\rho$, independently of the choices of the other sites. We may think that ρ is affected by the candidate platform and/or by a speech made by the candidate at the nomination time. Needless to say candidate A has a strong incentive to raise the value of ρ (and viceversa candidate B). We shall see that the voter process is nonergodic and a high value of ρ increases the chances that candidate A will win. If the initial distribution is μ^ρ, we start from a state of individual independence. The interaction between sites will then start at $t > 0$, when sites, at Poisson random times, will start revising their opinion by observing their neighbors. However the first of our propositions, requires only that the initial distribution is translation invariant, that is it does not depend on the specific site. In fact

PROPOSITION 4.1

For any translation invariant initial distribution and for any $x, y \in Z$,

$$(i) \quad \lim_{t\to\infty} P(\eta_t^{\tau_c}(x) = \eta_t^{\tau_c}(y)) = 1$$

and

$$(ii) \quad \lim_{t\to\infty} P(\eta_t^{L}(x) = \eta_t^{L}(y)) = 1$$

PROOF: (i) Given the flip rate characterizing η^{τ_c}, as in equation (5), the statement follows from Theorem 1a in Andjel, Liggett and Mountford (1992).

(ii) See equation (2.1) in Cox and Griffeath (1986). ♣

Obviously the proposition above would be rather trivial if we started from a configuration of either *all favor A* or *all favor B*. We shall show below that consensus implies cluster formation.

For a cluster in Z we mean a connected line of individuals agreeing on the same opinion. For cluster size we mean the number of connected sites favoring the same opinion. Hereafter the same argument applies to η^{τ_c} and to η^L, so for the sake of simplicity we may drop the superscript. On the other hand the initial distribution plays a crucial role, so that as a superscript we add the initial distribution. We can now define the mean cluster size around the origin at time t as

$$C(\eta_t^\mu) = \lim_{n\to\infty} \frac{2n}{\#\ clusters\ of\ \eta_t^\mu\ on\ [-n,n]}$$

where η^μ is either the linear or the threshold voter, with initial distribution μ. We shall see below that, when consensus occurs, the mean cluster size grows with t and approaches infinity as t approaches infinity.

PROPOSITION 4.2

Suppose that the initial distribution of η^{τ_c} (*resp.* η^L) is a Bernoulli-product measure with density $\rho \in (0,1)$. Then, as $t \to \infty$, $C(\eta_t^{\mu_\rho,\tau_c}) \to \infty$ (*resp.* $C(\eta_t^{\mu_\rho,L}) \to \infty$).

PROOF: As the proof is the same for both η^{μ_ρ,τ_c} and for $\eta^{\mu_\rho,L}$ we shall drop the superscrits τ_c, L. We say that x is a border for η^{μ_ρ} at time t, if $\eta_t^{\mu_\rho}(x-1) \neq \eta_t^{\mu_\rho}(x)$. A border measure for $\eta_t^{\mu_\rho}$ is then defined as

$$\mu^\rho(\eta_t^{\mu_\rho} : \eta_t^{\mu_\rho}(x-1) \neq \eta_t^{\mu_\rho}(x), x \in Z)$$

The Bernoulli product measure with density $\rho \in (0,1)$ is translation invariant (i.e. it does not depend on the specific site) and it is also mixing (for a definition of mixing border measure, see Bramson and Griffeath 1980, p.187-188). Thus by the Birkoff ergodic theorem a strong law of large number follows, ensuring that with probability one,

$$\lim_{n\to\infty} \frac{\#\ of\ borders\ of\ \eta_t^{\mu_\rho}\ in\ [-n,n]}{2n} = P(\eta_t^{\mu_\rho}\ has\ a\ border\ at\ 1)$$

As the number of borders and the number of clusters in $[-n,n]$ can differ by at most one, given the definition of mean cluster size, we have that

$$C(\eta_t^{\mu_\rho}) = \frac{1}{P(\eta_t^{\mu_\rho}\ has\ a\ border\ at\ 1)} = \frac{1}{P(\eta_t^{\mu_\rho}(0) \neq \eta_t^{\mu_\rho}(1))}$$

where the last equality follows straightforwardly from the definition of border at site 1. Now from Proposition 4.1, we know that $\forall x, y \in Z$,

$$\lim_{t\to\infty} P(\eta_t^{\mu_\rho}(x) = \eta_t^{\mu_\rho}(y)) = 1$$

so that by setting $x = 0, y = 1$, we have that $\lim_{t\to\infty} P(\eta_t^{\mu_\rho}(0) \neq \eta_t^{\mu_\rho}(1)) = 0$, and the desired result then follows. ♣

A more informative result is available for η^L, that is for the case in which when half of the neighbors disagree, the site flips opinion with probability 1/2. In fact

PROPOSITION 4.3

If the initial distribution is a Bernoulli product measure with density $\rho \in (0, 1)$, then as t gets large

$$\frac{1}{\sqrt{t}} C(\eta_t^{\mu_\rho, L}) \xrightarrow{p} \frac{\sqrt{\pi}}{2\rho(1-\rho)}$$

PROOF: From Theorem 1c in Bramson and Griffeath (1980, p.191), since any Bernoulli product measure is mixing. ♣

Proposition 4.3 shows that for the case of η^L, the size of clusters increases over time at rate $\sqrt{t}$.

We have stated Propositions 4.1-4.3 in terms of a countable number of individuals. It is worthwhile to see whether the same results hold for the case of a finite population. Basically this is true for Proposition 4.1, in fact Part (ii) follows from Cox (1989, p.1333), and Part (i) follows from the fact that no configurations, but *all* A and *all* B, regardless of the number of agents, can be stable over time. On the other hand, Propostions 4.2 and 4.3 hold only for the countable case. An intuitive reason is that, over a segment of finite lenght n, we cannot have a cluster growing over time, as time approaches infinity. On a more formal level, we notice that in the proof of Proposition 4.2 we make use of the Birkoff ergodic theorem, that holds only for $n \to \infty$.

Thus for η^{τ_c} and η^L, as t gets large we get consensus on only one candidate. In general we may have convergence on a configuration of either $all\ favor\ A$ or of $all\ favor\ B$. Furthermore, as η^{τ_c}, η^L are non-ergodic processes, their limiting distribution will depend on the initial distribution. In particular, as we shall see below, the limiting distribution is a mixture of pointmass distributions δ_A, δ_B, where δ_A (*resp.* δ_B) denotes a configuration in which all individuals favor A (*resp.* B). The weights of this mixture depend on the initial state of the system. Herafter with $\mu_t^{\tau_c}, \mu_t^L$ we denote the distribution of the configuration $\eta_t^{\tau_c}, \eta_t^L$, respectively, at time t. We have

PROPOSITION 4.4

If the initial distribution is a Bernoulli product measure with density $\rho \in (0, 1)$, then

$$(i) \quad \lim_{t\to\infty} \mu_t^L = \rho\delta_A + (1-\rho)\delta_B$$

$$(ii) \quad \lim_{t \to \infty} \mu_t^{\tau_c} = D(\rho)\delta_A + (1 - D(\rho))\delta_B$$

for some constant, depending on ρ, $0 < D(\rho) < 1$.

PROOF: (i) It is shown in Corradi and Ianni (1998, Theorem 2.2) that the assumptions of Corollary 1.13(a) of Liggett (1985, p.231) are satisfied. The statement then follows from that Corollary.

(ii) From Theorem 1b in Andjel, Liggett and Mountford (1992). ♣

Thus from Propostion 4.4 we know that with probability ρ, $(resp.\ D(\rho))$, μ_t^L, $(resp.\ \mu_t^{\tau_c})$ converges to a configuration of $all\ favoring\ A$, and with probability $1 - \rho$, $(resp.\ 1 - D(\rho))$, μ_t^L, $(resp.\ \mu_t^{\tau_c})$ converges to a configuaration of $all\ favor\ B$. So the only difference in the limiting distribution of μ_t^L and $\mu_t^{\tau_c}$ lies in the weights of the mixture of pointmass distributions.

As we shall see below, in the case where each site changes opinion only if more than half of her/his neighbors disagree with the position (s)he currently holds, in general we do not have consensus on only one candidate, and so we do not observe cluster formation along the dynamics. Intuitively the main difference between the dynamic behavior of $\eta_t^{\tau_f}$ and that of $\eta_t^L, \eta_t^{\tau_c}$ is the following: as a site changes opinion only if more than half of her/his neighbors disagree (i.e. if at least $N + 1$ neighbors disagree), any time the site changes opinion, the number of sites in her/his neighborhood that disagree with the position held by that site, decreases (by one). For this reason we expect that over time, each site changes opinion less frequently than in the other previous two cases. In fact, as we shall see below, any site can change opinion only a finite number of times. Furthermore intuition suggest that both the opinions will in general survive in the long-run, so that we may have co-existence. For example if $N = 1$, all configurations characterized by two or more connected sites favoring A, followed by two or more connected sites favoring B, and so on, are stable over time and can be seen as stationary limiting configurations. In fact we shall have an entire family of stationary configurations, characterized by the co-existence of both opinions. Nevertheless also a configuration of *all* A and a configuration of *all* B are stationary limiting distributions. We shall see below that, when the number of neighbors is finite, but large, configurations characterized by large relatively homogeneous areas are likely to be observed. We have

PROPOSITION 4.5

If the flip rate is as in (6), then each site switches opinions only a finite number of times and the system fixates, that is $\forall x \in Z$, with probability one

$$\lim_{t\to\infty} \eta_t^{\tau_f}(x) = \eta_\infty^{\tau_f}(x)$$

PROOF: Let $\delta_{x,y}(t) = 1$ if $\eta_t^{\tau_f}(x) \neq \eta_t^{\tau_f}(y)$, and 0 otherwise. Following the lines of Durrett and Steiff (1993, Theorem 1), we can define an energy function at time t, as

$$\Upsilon_t = \sum_{x,y:|x-y|\leq N} e^{-|x+y|}\delta_{x,y}(t)$$

with $\Upsilon_0 < \infty$. Let $\Upsilon_t(x)$ be the energy at site x, i.e.

$$\Upsilon_t(x) = \sum_{y:|x-y|\leq N} e^{-|x+y|}\delta_{x,y}(t)$$

We shall show that $\forall x \in Z$, whenever the site flips, the energy at that site drops by a strictly positive amount. For the sake of simplicity, take $x = 0$. Thus we have:

$$\Upsilon_t(0) = \sum_{y=-N}^{N} e^{-|y|}\delta_{0,y}(t)$$

Note that whenever $\sum_{y=-N}^{N} \delta_{0,y}(t) < N$ site 0 will not flip anymore, and when this occurs for all sites in Z the system fixates. Now let $\alpha = \#\{y \in \{-N,\ldots,-1,0,1,\ldots N\} : \eta_t^{\tau_f}(0) \neq \eta_t^{\tau_f}(y)\}$ (i.e. α is the number of sites disagreeing with the origin at time t). Suppose that at time t site 0 flips, and recall that this can only happen if $\alpha > N$. Then the drop in the energy at site 0, after the flip occurred, is bounded from below by

$$e^{-2N}(\alpha - (2N - \alpha))$$

which is strictly positive as (i) N is a finite integer (ii) $\alpha \geq \tau > N$. Thus we have a strictly positive drop of energy at site 0. The same argument applies to any generic site. Thus as any site drops energy whenever it flips, and the energy at time 0 is finite, it follows that each site can flip only a finite number of times, so that fixation occurs. ♣

Thus when the flip rate is as in (6), as t gets large, support for both candidates may survive, in other words the two different opinions can co-exist as time gets large. In particular we have infinitely many stationary configurations $\eta_\infty^{\tau_f}$; as we mentioned above for $N = 1$, configurations as $\ldots AABBAABBAABBAA\ldots$ or $\ldots AAABBBAAABBBAAA\ldots$ are stationary. For N finite but sufficiently large, even if, in general, we may not have consensus (and hence we may not have clustering), we shall tend to observe large regions characterized by the prevalence of one opinion alternated by large regions characterized by the prevalence of the opposite opinion. In fact any configuration characterized by segments of less than N consecutive sites favoring the same position, cannot be stable over time, and so cannot be a stationary limiting configuration.

Finally it is worthwhile to point out that Proposition 4.5 holds also for the case of a finite population. In fact the proof of that proposition is valid, regardless the number of sites we consider. In particular, as each site can switch opinion only a finite number of times, if we have finitely many sites, fixation will occur in finite time.

5 Allowing for Differences in the Side Payments

The model of private opinion formation described in Section 2 was characterized by the fact that the side payment obtained by each voter if (s)he had voted for the winner, is the same regardless of which candidate wins the elections. A slight modification of this basic model allows for differences in the side payments. For example we may think at the following preference structure, homogeneous across all voters,

	A wins	B wins
vote a	$U(A)+\varepsilon_A$	$U(B)$
vote b	$U(A)$	$U(B)+\varepsilon_B$

with $\varepsilon_A \neq \varepsilon_B$. As we mentioned in Section 3, if candidates act strategically, the most natural equilibrium side payment is $\varepsilon_A = \varepsilon_B$. However it is interesting to see whether, and under which conditions, in the case of $\varepsilon_A > \varepsilon_B$ we have consensus on A, and viceversa. Straightforward calculation shows that $E_pU(a) \geq E_pU(b)$ if and only if $P(A) \geq \frac{\epsilon_B}{\epsilon_A+\epsilon_B}$. As before, individuals can observe only their neighbors' opinions, and they assume that the distribution of opinions across the neighboring sites, is a good approximation of the distribution of opinions across the entire population. Hereafter, ν_A denotes the proportion of neighbors favoring A, and we let $\theta = \frac{\varepsilon_A}{\varepsilon_B+\varepsilon_A}$ Hence, we can reformulate the rule as:

if $\nu_A > \theta$ choose A

if $\nu_A < \theta$ choose B

if $\nu_A = \theta$ then choose either of the following

(i) choose A

(ii) choose B

(iii) choose A with probability 1/2.

Without loss of generality we shall assume that $\theta < 0.5$. Furthermore, in order to avoid integer problems, we shall assume that θ is rational and such that $2\theta N$ is an integer.[4]

We find it convenient to reformulate the rules in terms of number of individuals, instead of proportions. Hence the rules become:

if $2N\nu_A > 2N\theta$, choose A

if $2N\nu_A < 2N\theta$, choose B

if $2N\nu_A = 2N\theta$ choose either of the following

(i) choose A

(ii) choose B

(iii) choose A with probability 1/2.

It is easy to notice that, whenever $\theta < 0.5$, as assumed, the flip rates generated by the above rule are asymmetric, in the sense that to flip from B to A it suffices to have $2N\theta$ (or $2N\theta + 1$, depending on the rule) neighbors disagreeing. Let us define $\eta_t^{(i)}, \eta_t^{(ii)}, \eta_t^{(iii)}$, the processes defined by rule (i), (ii) and (iii) respectively.

PROPOSITION 5.1

Suppose that individuals follow rule (i).

Define E_t the following event:

E_t= (at time t there is a string of at least $2N\theta + 1$ consecutive sites favoring A). Also define

$$\tau_E = \{\inf \ t : E_t \ occurs\}$$

If $P(\tau_E < \infty) = 1$, then

$$\lim_{t\to\infty} P(\eta_t^{(i)}(x) = A, \forall x \in Z) = 1$$

PROOF:

For the sake of simplicity, and without loss of generality, assume that $\theta = 1/3$, $2N = 6$, so that $2\theta N + 1 = 3$, thus a site favoring B switches to A if at least two neighbors choose A. Consider the most unfavorable situation (in terms of prevalence of individuals favoring A), in which E_t occurs at some finite t. This is

$$\ldots BBBBBBBBAAABBBBBBBB\ldots$$

[4] Otherwise we can just assume θ to be rational, and set $\theta N = [N\theta]$ *or* $[N\theta] + 1$, where $[\cdot]$ denotes the integer part.

At anytime a site favoring A is called to revise his/her position, (s)he does not flip; the same holds for all sites favoring B, but for the two on the immediate left and right of the string of A. When any of them is called to revise her/his position (s)he will switch to A. As in any period each site has a positive probability of being called to revise her/his position, we shall next have a configuration

$$\dots BBBBBBAAAAAAABBBBBB\dots$$

by the same argument above when any of the two B-sites on the immediate left and right of the cluster of A will be called to revise her/his postion (s)he will switch to A, while no site will ever switch to B. Thus as $t \to \infty$ we shall have consensus on A. Note that a similar argument holds for any arbitrary $\theta < 0.5$. ♣

Thus if at any given finite time we have a string of at least $2N\theta + 1$ consecutive sites favoring A, we shall have consensus on A, as $t \to \infty$. This property of the asymmetric threshold voter is related to the concept of contagion proposed by Morris (1997), although he considers a strategic setting, while we do not. According to Morris, we have contagion when a strategy played by only a finite number of individuals will eventually spread out to the whole population. In a strategic setting, also Ellison (1993) shows that certain types of dominant strategy (1/2 dominant in his terminology), if played by at least one couple of players on a line, will spread out to the entire population. While Ellison employs a lattice structure, Morris does not require any particular spatial structure.

We shall see now that when at time 0, each site in Z favors A with probability ρ and favors B with probability $1-\rho$, $\rho \in (0,1)$, then the statement of Proposition 5.1 necessarily holds.

PROPOSITION 5.2

Let the initial distribution be a Bernoulli product measure with density $\rho \in (0,1)$, then

$$\lim_{t\to\infty} P(\eta_t^{(i)}(x) = A, \forall x \in Z) = 1$$

PROOF:

It suffices to show that the event E_t, occurs at $t = 0$, with probability one, then the statement follows from Proposition 5.1. We shall consider the case of M individuals, lying on $-M/2, -M/2+1, \dots -1, 0, 1, 2, \dots M/2$ and then let M go to infinity. From a very large urn with a proportion ρ of A-balls (i.e. individuals favoring A) and $1-\rho$ of B-balls we draw with replacement $(M/(2N\theta+1))$ $2N\theta+1$-tuple of balls (e.g. if $\theta = 1/3$, and $2N = 6$ we draw 3 balls at once, put them back in to the urn, and repeat the process $M/3$ times). As the urn is very large, if we draw $2N\theta+1$ balls consecutively, the chance that the first is say A is the same that the second or the third is A. We proceed as follows: we first draw $2N\theta+1$ balls and place them on sites $-M/2, (-M/2)+1, \dots (-M/2)+2N\theta+1$, put the $2N\theta+1$ balls back in to the urn, pick another $2N\theta+1$-tuple, and place them on sites $(-M/2)+2N\theta+2, \dots (-M/2)+2(2N\theta+1)$, and so on until we have covered all the M sites. The probability of getting a $2N\theta+1$-tuple of As is given by $\rho^{2N\theta+1}$, while $1-\rho^{2N\theta+1}$ is the probability that at least a ball in the $2N\theta+1$-tuple is a B. Hence

$$P(E_0) = 1-(1-\rho^{2N\theta+1})^{M/(2N\theta+1)} \to 1 \;\; as\ M \to \infty$$

The desidered result then follows from Proposition 5.1. ♣

It is immediate to see that Proposition 5.1 (as well as Proposition 5.3 below) holds also for a finite population, while this is not the case for Proposition 5.2.

We now consider the two nonlinear rules (ii) and (iii).

PROPOSITION 5.3

Suppose that individuals follow rule (ii) (resp. (iii)).
Define E_t the following event:
$E_t =$ (at time t there is a string of at least $2N\theta+2$ consecutive sites favoring A). Also define

$$\tau_E = \{\inf\ t : E_t\ occurs\}$$

If $P(\tau_E < \infty) = 1$, then

$$\lim_{t\to\infty} P(\eta_t^{(ii)}(x) = A, \forall x \in Z) = 1$$

$$(resp.\ \lim_{t\to\infty} P(\eta_t^{(iii)}(x) = A, \forall x \in Z) = 1)$$

PROOF:

The proof is the same for both rules. Without loss of generality set $\theta = 1/3$, $2N = 6$, so that $2N\theta+2 = 4$. Suppose the most unfavorable situations for A in which E_t may occur, i.e.

$$\dots BBBBBBBBAAAABBBBBBBB\dots$$

Note that under both rule (ii) and (iii), any site A will never flip to B when called to revise her/his position. Now in both cases the two site B-sites on the immediate left and right of the A-string, would flip to A, while the second two immediate B on the left and right of the A group will not flip in case (ii) and would flip to A with probability 1/2, in case (iii). The result then follows from the same argument given in the prooof of Proposition 5.1. ♣

6 Global Interaction

For the sake of completeness, we now wish to analyze the case of global interaction, that is the case in which each voter can observe the entire population. For example we may assume that opinion polls are an unbiased and very precise estimator of the distribution of opinions across the population that is available to each individual. In all other respects, we shall keep the same setting used in the previous sections. To avoid some technicalities related to the problem of defining proportions when we have a countable population, in this section we assume that the population is large, but finite. Hence ν_A^t denotes the proportion of individuals favoring A at time t.[5]

Hereafter let $\theta = \frac{\varepsilon_B}{\varepsilon_A + \varepsilon_B}$. If $\varepsilon_A = \varepsilon_B$, $\theta = 1/2$, then the flip rate from B to A is the same as that from A to B. In what follows θ is any number strictly between 0 and 1.

We re-state the rules followed in the previous section as:

if $\nu_A^t > \theta$ choose A

if $\nu_A^t < \theta$ choose B

if $\nu_A^t = \theta$, choose either of the following:

(i) choose A

(ii) choose B

(iii) choose A with probability 1/2.

We shall denote the processes $\eta_t^{g,(i)}, \eta_t^{g,(ii)}, \eta_t^{g,(iii)}$ respectively.

PROPOSITION 6.1

(a) If $\nu_A^0 > \theta$, then for all the cases, (i),(ii) and (iii), we shall have consensus on A, if $\nu_A^0 < \theta$, then, for all the cases, (i),(ii) and (iii) we shall have consensus on B. If instead $\theta = \nu_A^0$, then

(i) $\lim_{t\to\infty} P(\eta_t^{g,(i)}(x) = A) = 1, \forall x \in Z$

(ii) $\lim_{t\to\infty} P(\eta_t^{g,(ii)}(x) = B) = 1, \forall x \in Z$

[5] In the case of a countable population, we can define $\nu_A^t = plim_{n\to\infty} \frac{\#\{x:\eta_t(x)=A, x\in Z\}}{2n}$, provided it exists, where $plim$ denots the probability limit. We conjecture that, if the initial distribution is μ^ρ, the Bernoulli product measure with density $\rho \in (0,1)$, then such a limit exists, because of the Birkoff ergodic theorem.

(iii) We may have consensus on either A or B, for $\eta_t^{g,(iii)}$.

(b) In all the cases, each site will flip at most once, so that we shall have fixation.

PROOF:

(a) Suppose that $\nu_A^0 > \theta$. The first site that is called to revise her/his position at $t > 0$ will not flip if it already favors A, while (s)he will flip if is favoring B. The same will hold when a second site will be called to revise her/his position. Thus ν_A^t would monotonically approach a configuration of all A as $t \to \infty$. By the same argument, ν_A^t will monotonically approach a configuration of all B if we start from $\nu_A^0 < \theta$.

If $\nu_A^0 = \theta$, then :

(i) (ii) Any site favoring A (*resp.* B) will not flip, when called to revise her/his position. Any site favoring B (*resp.* A) will flip to A (*resp.* B).

(iii) The first site called to revise her/his position will flip with probability 1/2, then the statement follows by the argument above.

(b) Suppose individuals follow rule (i). If $\nu_A^0 \geq \theta$ the first time a site is called to revise her/his position either will not flip, if it was already favoring A, or it will flip to A, if it was favoring B. The second time that site will be called, (s)he will not flip anymore. Thus any site will flip at most once. A similar argument applies to rule (ii) and (iii) ♣.

7 Conclusions

In this paper we have analyzed a simple model of individual and aggregate opinion formation. Each voter (out of countably many) gets a higher utility if (s)he votes for the candidate who wins the elections. This can be explained either in terms of a side payment, or in terms of some bandwagon effect, according to which individuals wish to conform to the opinion held by the majority. Local interaction and conformity desire are the basic ingredients of our model. More precisely, each individual in forming her/his own opinion as to which (out of two) candidate to vote, follows a simple nonlinear (threshold) rule. Broadly speaking if the proportion of neighbors favoring A is above (resp. below) a certain threshold, called θ, (s)he chooses (resp. does not choses) A. When instead the proportion is equal to θ (s)he follows three different rules: (i) choose A (resp. B) with probability 1/2, (ii) choose A (resp. "B") if currently favoring B (resp. A) (iii) choose A (resp. B) if currently favoring A (resp. B). Of particular interest is the case of $\theta = 0.5$. We show that, when $\theta = 0.5$, in case (i)and (ii) consensus arises, that is in the long run everyone conforms on the same opinion. The dynamics towards consensus is characterized by cluster formation, that is by the formation of larger and larger homogeneous areas. On the other hand, in case (iii) each site can change opinion only a finite number of times and so the system fixates. Furthermore there is an entire family of stationary configurations characterized by the co-existence of the two opinion, although configurations of *all* A or *all* B are also stationary limiting configurations. If the number of neighbors is sufficiently large, also in the co-existence case we observe large relatively homogeneous regions. The case of $\theta \neq 0.5$ is much less articulate. If $\theta < 0.5$, under mild conditions, we shall have consensus on A, and viceversa when $\theta > 0.5$.

In this paper we have concentrated on the process of individual and aggregate opinion formation, on the part of the voters, disregarding any strategic behavior on the part of the candidates. It will be interesting to see how a candidate can increase her/his share of the votes, given that the aggregate opinion process evolves over time and space as a threshold voter model. Intuition suggests that each candidate, in an attempt to increase her/his share of votes, should try to *bribe* individuals living on the borders, where for borders we simply mean two individuals of different opinions living on connected sites. Another question of interest is the endogeneous determination of the election date. Suppose that the incumbent can choose the election date, then for any initial configuration, the choice of the election date can be seen as an optimal stopping time problem. We leave these and other issues for future research.

References

[1] Anderlini L. and A. Ianni (1996), Path dependence and learning from neighbors, Games and Economic Behavior , 13, 141-177

[2] Andjel E.D., T.M. Liggett and T. Mountford (1992), Clustering in one-dimensional threshold voter models, Stochastic Processes and their Applications , 42, 73-90

[3] Banerjee A.V. (1992) A simple model of herd behavior, Quarterly Economic Journal , 107, 797-817

[4] Bikhchandani S., D. Hirshleifer and I. Welch (1992), A theory of fads, fashions, customs and cultural change as information cascades, Journal of Political Economy , 100, 992-1026

[5] Bramson M. and D. Griffeath (1980), Clustering and dispersion rates for some interactive particle systems on Z, Annals of Probability , 7, 418-432

[6] Blume L. E. (1993), The statistical mechanics of strategic interaction, Games and Economic Behavior , 5, 387-424.

[7] Corradi V. and A. Ianni (1998), A note on the dynamic properties of imitative behavior, Mimeo, University of Southampton.

[8] Cox J.T. and D. Griffeath (1986), Diffusive clustering in the two dimensional voter model, Annals of Probability , 14, 347-370.

[9] Cox J.T. (1989), Coalescing random walks and voter model consensus times on the torus in Z^d, Annals of Probability , 17, 1333-1366.

[10] Durlauf S.N. (1993), Nonergodic growth, Review of Economic Studies , 60, 349-366

[11] Durrett R. and J.E. Steiff (1993), Fixation results for threshold voter, Annals of Probability , 21, 232-247

[12] Ellison G. (1993), Learning, local interaction, and coordination, Econometrica , 61, 1047-1071

[13] Eshel I., L. Samuelson and A. Shaked (1998), Altruists, egoists, and hooligans in a local interaction model, American Economic Review , 88, 157-179

[14] Henschel R.L. and W. Johnston (1987), The emergency of bandwagon effects: a theory, The Sociological Quarterly , 28, 493-511

[15] Ianni A. and V. Corradi (1998), A simple model of private and public opinion formation, Mimeo, University of Pennsylvania

[16] Kandori M. and R. Rob (1998), Bandwagon effect and long-run technology choice, Games and Economic Behavior , 22, 30-60.

[17] Liggett T.M. (1985) Interactive Particle Systems , Springer and Verlag, Berlin

[18] Morris S. (1997), Contagion, Caress W.P. 9701, University of Pennsylvania, Economics

[19] Nadeau R., E. Cloutier and J.H. Guay (1993), New evidence about the existence of a bandwagon effect in the opinion formation process, International Political Science Review , 14, 203-213.