CARESS Working Paper #96-10

Asynchronous Choice in Repeated Coordination Games

by

Roger Lagunoff and Akihiko Matsui

LEGES SINE MORIBUS VANAE

UNIVERSITY of PENNSYLVANIA

Center for Analytic Research
in Economics and the Social Sciences
McNEIL BUILDING, 3718 LOCUST WALK
PHILADELPHIA, PA 19104-6297

CARESS Working Paper #96-10

Asynchronous Choice in Repeated Coordination Games

Roger Lagunoff* and Akihiko Matsui†

November 10, 1996‡

Abstract

The standard model of repeated games assumes perfect synchronization in the timing of decisions between the players. In many natural settings, however, choices are made asynchronously so that only one player can move at any given time. This paper studies a family of repeated settings in which choices are asynchronous. Initially, we examine, as a canonical model, a simple two person alternating move game of pure coordination. There, it is shown that for sufficiently patient players, there is a unique perfect equilibrium payoff which Pareto dominates all other payoffs. The result generalizes to any finite number of players and any game in a class of *asynchonrously repeated games* which includes both stochastic and deterministic repetition. The results complement a recent Folk Theorem by Dutta (1995) for stochastic games which can be applied to asynchronously repeated games if a full dimensionality condition holds. A critical feature of the model is the inertia in decisions. We show how the inertia in asynchronous decisions determines the set of equilibrium payoffs.

Journal of Economic Literature Classification Numbers: C72, C73.
Keywords: repeated games, asynchronously repeated games, alternating move games, pure coordination games, stochastic games, inertia.

*Department of Economics, Georgetown University, Washington DC 20057 USA.

†Department of Economics, University of Pennsylvania, Philadelphia, PA 19104, USA, and Institute of Socio-Economic Planning, University of Tsukuba, Ibaraki 305, Japan.

‡We would like to thank Eddie Dekel, Prajit Dutta, Hseuh-ling Huyn, Hideshi Itoh, Mamoru Kaneko, Peter Linhart, George Mailath, Stephen Morris, Roy Radner, Phil Reny, Andrew Postlewaite, Ariel Rubinstein, and numerous seminar participants for many useful discussions and suggestions. We are also grateful for the many useful suggestions by Douglas Gale and three anonymous referees.

1 Introduction

The standard model of repeated strategic play is a discretely repeated, simultaneous move game. This formulation assumes a perfect synchronization in the timing of actions between the players. Alternatively, this assumption may be interpreted as having each player move in ignorance of the other players' current move. The effect of this assumption when the stage game has multiple, Pareto-ranked equilibria is that each player may choose an action consistent with a Pareto inferior stage game equilibrium only because he expects that other players will do the same. Since all players move at once, no player can unilaterally signal his intent to do otherwise.

While the synchronized move is not an unreasonable model of repetition in certain settings, it is not clear why it should *necessarily* be the benchmark setting for repeated play. For one thing, the theory of infinitely repeated games offers little predictive content. It is well known from the Folk Theorem that infinitely repeated games admit a multitude of equilibria.[1] More importantly, it seems natural in many contexts that players move *asynchronously*. For example, financial investment decisions by someone living in New York and someone in Tokyo are asynchronous for the simple reason that they are made in financial markets that operate in different time zones. Investment decisions in an oligopolistic industry which involve some inertia or costly upgrading are also asynchronous. Maskin and Tirole (1988) model precisely this scenario as an alternating move game between duopolists. Birth-death processes common in the models of evolution are examples of asynchronous decision making. Individuals or firms are modelled as exiting at independent random times, often to be replaced by other individuals (newborns) who make irreversible decisions at the time of entry.

The purpose of the present paper is to give some support for the study of a more general class of repeated interactions, those which include asynchronous choices. The analysis is intended to be more suggestive than definitive. While the standard Folk Theorem will continue to be central to our understanding of repeated interactions, the types of games with asynchronous choice which we study will yield, at times strikingly, different results.

To model asynchronous choice we first consider a simple two-player, alternating move game similar to the stucture studied by Maskin and Tirole as the canonical model. Our main result proves an "anti-Folk Theorem" for pure coordination stage games — stage games in which payoffs of all players are identical (up to an affine transformation). The importance of this class of games is emphasized by Marschak and Radner (1972) who try to understand the nature of team problems. We show that if players are sufficiently patient, then there is a unique perfect equilibrium payoff which Pareto dominates all other payoffs. This result starkly contrasts with the Folk Theorem, as it rules out all the inefficient payoffs, particularly the inefficient stage game equilibria.

[1] See, for example, Abreu (1988) or Fudenberg and Maskin (1986). References to earlier results may be found in a survey by Aumann (1981).

As a point of comparison, note that in many evolutionary models efficiency sometimes emerges uniquely when perpetual mutation in behavior occurs in society.[2] The difference between this and the results of standard repeated games is often attributed to the lack of intertemporal strategic interaction in the former. Without such interaction, so the explanation goes, the set of outcomes are reduced to the set of Nash equilibria of one-shot games. From there, the techniques of long run equilibria "select" the Pareto efficient outcome in a two-by-two pure coordination game. This explanation is incomplete since it fails to emphasize the role of inertia. In typical evolutionary games, only a fraction of the entire population may change their actions at any given time. Without some form of inertia the behavior pattern of the population discontinuously changes from one distribution state to another, and the stability arguments used to attain uniqueness and optimality become vacuous. The present paper focuses on inertia in the form of asynchronous choice. Since intertemporal strategic interaction is present in our model, it is precisely the inertia caused by asynchronous timing which is crucial for uniquely attaining the Pareto efficient outcome in certain repeated games.

The paper is organized as follows. Section 2 describes the model and defines the alternating move game and the equilibrium concept. There, the main result is given for games of pure coordination. Section 3 presents generalizations of the canonical alternating move game. In Section 4 we discusses general stage games. We relate a recent result by Dutta (1995) for stochastic games to our model. We also discuss the role of inertia for asynchronous repetition of general stage games. Finally, Section 5 discusses related literature.

2 A Canonical Model of Asynchronous Timing

2.1 Alternating Move Games

Consider an alternating move game which is given by $G = \langle S_1, S_2, u_1, u_2 \rangle$ where S_i $(i = 1, 2)$ is the set of actions of player i, and $u_i : S_1 \times S_2 \rightarrow \Re$ is i's utility function.

After the first decision node, which occurs for all players at time zero, the two players alternately have chances to revise their actions. At the beginning of odd numbered periods, player 1 has a chance to revise his action, whereas at the beginning of even numbered periods player 2 has a chance to revise her own action. In the following, revision nodes refer to the decision nodes other than the first one at time zero. If player 1 chooses s_1 in period t $(t = 1, 3, 5, \ldots)$ and if player 2 chose s_2 in the previous period, then the profile of actions and the realized payoff to player $i = 1, 2$ in period t are $s = (s_1, s_2)$ and $u_i(s)$, respectively. Action profiles and payoffs in even periods are similarly defined except for period 0 in which actions are assumed to be chosen simultaneously.

[2]See, for example, Foster and Young (1990) and Kandori, Mailath and Rob, (1993).

Given a sequence of action profiles $\{s(t)\}_{t=0}^{\infty}$, individuals seek to maximize the (normalized) discounted sum $(1-\delta)\sum_{t=0}^{\infty}\delta^t u_i(s(t))$ where δ is a common discount factor. Let e denote an empty history. Then let H_i $(i=1,2)$ denote the set of all histories after which player i moves. Let $H = H_1 \cup H_2$. Note that $H_1 \cap H_2 = \{e\}$. That is, the players make simultaneous moves only after the empty history. A standard notation denotes the history ending in period t (including the action profile in t) by h^t. Any history h in, say, $H_1\backslash\{e\}$ can be expressed as $h' \circ s_2$ for some $h' \in H_2$. This notation expresses the fact that player 2, the most recent mover, chose action s_2 after history h'.

A *strategy* for player i $(i=1,2)$ is a function $f_i : H_i \to \Delta(S_i)$ mapping histories to mixed actions. Given $f=(f_1,f_2)$ and a history $h^t \in H$, let $\tilde{s}^{t+\tau}(f|h^t)$ $(\tau = 0,1,\ldots)$ be a (stochastic) action profile in the $(t+\tau)th$ period induced by f after history h^t. Given $f=(f_1,f_2)$, player i's payoff after history h^t is given by

$$V_i(f|h^t) \;\equiv\; (1-\delta)\sum_{\tau=0}^{\infty}\delta^{\tau}E\left[u_i(\tilde{s}^{t+\tau}(f|h^t)\;)\right], \tag{1}$$

where $E[\cdot]$ is the expectation operator, and we let $u_i(\tilde{s}(e)) = 0$ to simplify notation. We write $\tilde{s}(h^t) = \tilde{s}^t(f|h^t)$ and use a convenient recursive representation for the payoff $V_i(f|h)$ which is expressed by

$$V_i(f|h) \;=\; (1-\delta)E\left[u_i(\tilde{s}(h\circ f_j(h))\;)\right] \;+\; \delta E\left[V_i(f|h\circ f_j(h))\right], \tag{2}$$

where $j \neq i$ iff $h \in H_j$.

A strategy profile $f^* = (f_1^*, f_2^*)$ is called a *perfect equilibrium (PE)* if for each $i=1,2$, f_i^* is a best response to $f_j^*(j\neq i)$ after every history $h\in H$, i.e.,

$$V_i(f^*|h) \geq V_i(f_i, f_j^*|h)$$

for any of player i's strategies f_i.

The following theorem states that perfect equilibria exist.

Theorem 0 *Given any stage game G, there exists a perfect equilibrium of the alternating move repetition of G.*

Proof Suppose that each player takes a "Markovian" strategy, in which action depends only on the other player's action in the previous period. We can represent such a strategy of player $i=1,2$ by the "Markovian" function $\psi_i \in [\Delta(S_i)]^{\{e\}\cup S_j}$ $(j\neq i)$. The strategy represented by ψ_i is denoted by f_{ψ_i}.

For each $i=1,2$, let $BR^i = (BR^i_\sigma)_{\sigma\in\{e\}\cup S_j})$ satisfy

$$BR^i_\sigma(\psi_i,\psi_j) = \arg\max_{f_{\psi_i'}} V(f_{\psi_i'} f_{\psi_j}|\; h) \tag{3}$$

We write $BR = (BR^1, BR^2)$. Then BR is an upper hemicontinuous correspondence from $[\Delta(S_1)]^{\{e\}\cup S_2} \times [\Delta(S_2)]^{\{e\}\cup S_1}$ into itself where the domain is compact and convex. Therefore, by Kakutani's fixed point theorem, there exists ψ such that $\psi \in BR(\psi)$ holds. Standard arguments show that the corresponding strategy f_{ψ_i} is a best response to $(f_{\psi_j})_{j \neq i}$ within the class of *all* strategies after any history h. Hence, f_ψ is a perfect equilibrium. □

2.2 An "Anti-Folk Theorem" for Pure Coordination Games

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{II} \\
 & & s^*_{II} & \bar{s}_{II} \\
\cline{3-4}
I & s^*_I & a\ ,\ a & b\ ,\ b \\
\cline{3-4}
 & \bar{s}_I & c\ ,\ c & d\ ,\ d \\
\cline{3-4}
\end{array}
$$

A game G is a *pure coordination game* if $u_1 = u_2 = u$. A 2×2 example is given in the figure above. Let s^* denote the profile that gives each player his highest payoff u^*. The first part of our main result states that, independently of the discount factor δ, the unique continuation value after any history in which the current profile is s^* must give the optimal payoff $u^* = u(s^*)$

Theorem 1 *Let f be any perfect equilibrium of an alternating move game of pure coordination. Then, for any history $h \in H$ with $\tilde{s}(h) = s^*$, and for each $i = 1, 2$,*

$$V_i(f|h) = u^*$$

Proof Fix a perfect equilibrium f. Observe, first, that from the construction of payoffs in expression (1) that since $u_1 = u_2$, we can drop the subscript i on continuation values V_i. Define

$$\underline{V} = \inf_{\{h:\, \tilde{s}(h)=s^*\}} V(f|h).$$

This is the infimum value of the game when the current behavior profile is s^*. Since the payoff space is bounded from below, so is this infimum. Fix $\epsilon > 0$. Then there exists $h = h_\epsilon \in H$ such that $\tilde{s}(h) = s^*$ and

$$\underline{V} > V(f|h) - \epsilon. \tag{4}$$

Assume $h \in H_1$. The case of $h \in H_2$ will be proven in the same way. Since f is a perfect equilibrium strategy profile, it must be the case that

$$V(f|h) \geq (1-\delta)u^* \ + \ \delta V(f|h \circ s_1^*), \tag{5}$$

where the right hand side is obtained by taking s_1^* after h. Then $\tilde{s}(h \circ s_1^*) = s^*$ implies

$$V(f|h \circ s_1^*) \geq \underline{V}. \tag{6}$$

Substituting (6) into (5) and using (4), we obtain

$$\underline{V} > u^* - \frac{\epsilon}{1-\delta}. \tag{7}$$

Since ϵ is arbitrary and independent of δ, $\underline{V} \geq u^*$ holds. □

The key to this result is that a one-shot gain from staying at s^* is greater than a possible future loss. Indeed, suppose that Theorem 1 does not hold, i.e., that a player, say, Player 1 switches from s^* at some node. At such a node, if Player 1 deviates to remain at s^*, he gets u^* for that period. Therefore, in order for him to follow the equilibrium strategy, it must be the case that his loss in continuation payoff V' one period after the deviation is greater than the one-shot gain, which is no less than $(1-\delta)[u(s^*) - \max_{s \neq s^*} u(s)]$. In the case of a pure coordination game, this means that the opponent suffers the same loss in the continuation payoff. Now, after this deviation, Player 2 has to switch from s^* at some point in order to punish Player 1. Applying the same argument as above, we see that in equilibrium, their common continuation value after Player 2's deviation is less than V' by at least $(1-\delta)[u(s^*) - \max_{s \neq s^*} u(s)]$. Repeating this argument, we get contradiction since the payoff is bounded.[3]

The second part of the main result is that any continuation value in a perfect equilibrium is arbitrarily close to the Pareto efficient value for sufficiently patient players.

Theorem 2 *For any alternating move game with pure coordination, and for any $\epsilon > 0$, there exists $\bar{\delta} \in (0,1)$ such that if $\delta \in (\bar{\delta}, 1)$, then for all perfect equilibria f of the alternating move game, and for all histories $h \in H$, $V_i(f|\ h) > u_i^* - \epsilon$ for each $i = 1, 2$, and the action profile reaches s^* in a finite number of periods.*

The proof of this theorem is straightforward given Theorem 1. It basically shows that Player 1's optimal action when Player 2 is taking s_2^* is to take s_1^*, and that knowing this, Player 2 has an incentive to take s_2^* provided that she is sufficiently patient.

Proof Assume that s^* is the unique Pareto optimal action profile. The cases of multiple optimal profiles can be proved in an analogous way. Fix $\epsilon > 0$. Let $u_* = \min_s u(s)$. Then let $\bar{\delta}$ satisfy $(1-\bar{\delta})u_* + \bar{\delta}u^* > u^* - \epsilon$. Take any $\delta > \bar{\delta}$ and any perfect equilibrium $f^* = (f_1^*, f_2^*)$.

[3]This logic breaks down if the payoff criterion is either the limit of means or the overtaking criterion since a decrease in payoff can be arbitrarily small since in these cases, the payoff value does not have to strictly decrease.

Observe now that for any $h \in H_1$ with $\tilde{s}(h) = (s_1, s_2^*)$, $f_1^*(h) = s_1^*$. For if not, player 1 has an incentive to deviate to take s_1^* since

$$V(f^*|h) = (1-\delta)E[u(f_1^*(h), s_2^*)] + \delta E[V(f^*|h \circ f_1^*(h))] < u^* = (1-\delta)u^* + \delta V(f^*|h \circ s_1^*)$$

where the last equality comes from Theorem 1. This implies that for any $h' \in H_2$,

$$V(f^*|h') \geq (1-\delta)u_* + \delta u^*.$$

If it were otherwise, Player 2 would have an incentive to switch to s_2^* after h' since he would then suffer at most u_* for one period and obtain u^* forever afterward. The same argument holds for any $h' \in H_1$. Hence, for any $h' \in H$, we have

$$V(f^*|h') \geq u^* - \epsilon.$$

Since this inequality holds for any $\epsilon > 0$, it must be the case that the action profile reaches s^* within a finite number of periods. □

Theorems 1 and 2 together with the existence result jointly establish an optimality result. In every equilibrium, players choose s^* at the beginning of the game and never depart. Note that Theorem 2 seems only to suggest an approximation to s^*. However, this is because we start the process from an arbitrary state. In equilibrium, the initial state h^0 or the state after e is determined by players' simultaneous choice. In determining h^0, they follow a reasoning process similar to the one in the proof of Theorem 2. As a result, they take s^* from the beginning. Note also that while Theorem 1 has no qualification about the discount factor, Theorem 2 requires sufficient patience to compensate for the temporary loss in reaching s^*.

3 Other Asynchronous Timing Structures

Though the optimality results have been stated only for two-person alternating move games with certain stage games, we emphasize that statements corresponding to Theorems 1 and 2 hold for any general asynchronously repeated game of pure coordination and for any (finite) number of players. To clarify what we mean by "general asynchronously repeated game" we sketch a general model of repeated interaction that includes both the standard model and the alternating move game as special cases.

Suppose that there are n players. The stage game is continuously repeated, yielding a flow payoff to each player at each instant of time. A *semi-Markov process* is a stochastic process which makes transitions from state to state in accordance with a Markov chain, but in which the amount of time spent in each state before a transition occurs is random and follows a renewal process. Denote $X_1, X_2, \cdots$ to be an i.i.d. sequence of increments in calendar time which determines calendar dates, $T_1, T_2, \ldots$, with $T_0 = 0$ and $T_k = T_{k-1} + X_k = X_1 + \cdots + X_k$

$(k = 1, 2, \ldots)$ at which decision nodes occur. The T_ks are the dates at which someone may have a chance to revise his/her action. At each decision date T_k, nature draws a state ω from a finite collection $\Omega \equiv \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_n$ according to the Markov transition $p_{\omega'\omega}$ which gives the probability of reaching ω given current state ω'. The set Ω_i identifies the set of states at which player i can change his action. Hence, if $\omega \in \Omega_i$ is drawn at date T_k then i has a move. If $\omega \in \Omega_0$ then an inertial state is reached and no one has a move. The "repeated game feature" is captured by the assumption that the process communicates to every player's decision state.

The model of a repeated game as a semi-Markov process may include some but not necessarily all moves being asynchronous. As special cases we have

Example 1 (The standard repeated game) The calendar dates are deterministic: $T_1 = 1, T_2 = 2, \ldots$, and all decision sets coincide: $\Omega_i = \Omega_j$ for all $i, j = 0, 1, \ldots, n$.

Example 2 (The alternating move game) There are two players, and again the calendar dates are deterministic. The states are the names of the players: $\Omega = \{1, 2\}$, and $p_{ij} = 1$ if $i \neq j$ and $= 0$ otherwise.

Example 3 (Poisson revisions) Let $\{X_k\}$ follow an exponential distribution with parameter $\lambda > 0$ so that $\text{Prob}\{X_k < x\} = 1 - e^{-\lambda x}$. Let $\Omega = \{1, \ldots, n\}$, and $p_{ij} = p_j$ for all i and j. Then player's decision nodes are independent of each others and player i's decision nodes follow a Poisson process almost surely with parameter λp_i.

Examples 2 and 3 may be called *asynchronously repeated games.* They are defined as semi-Markov processes of the kind described above where $\Omega_i \cap \Omega_j = \emptyset$ for all $i, j = 0, 1, \ldots, n$. It is not difficult to extend Theorems 1 and 2 to hold for *any* such asynchronously repeated game.

To see this, consider a representation of individuals' continuation values in the general asynchronous model. To formulate this, define the set of histories H the current behavior $s(h)$ at history h as before. We also let h^{t-} be a *conditioning history* at time t, which includes the same information as history h^t except that it excludes the choice made at time t.

A *strategy* for player i is a history contingent action given by the function $f_i : H^- \to \Delta(S_i)$. Given a history $h^t \in H$ and a strategy profile $f = (f_1, ..., f_n)$, the conditional discounted expected payoff to player i at time t is now given by

$$V_i(f|h^t) = r \int_t^{\infty} e^{-r(\tau - t)} E\left[u_i\left(\tilde{s}(f|h^t)(\tau)\right)\right] d\tau. \tag{8}$$

The analysis made in the previous section applies directly to the present case by replacing the value functions with those obtained above in (8). The same optimality result as before is obtained.[4] Evidently, what matters for such games is the fact that the pure coordination problem is asynchronous rather than the specific structure of asynchronous choice.

4 General Stage Games

4.1 Dutta's Result

Stochastic games are those in which the particular stage game faced by the participants at any given time is represented by a state variable. The evolution of the state is then determined by a transition function which maps (probabilisticially) the current state and action profile into next period's state. It is not difficult to recast an alternating move game as a stochastic game. The state space is the set $S_1 \cup S_2$. If the current state is "s_2" for example then this means that individual 2 moved in the previous period and took action s_2. Since only player 1 has a move in the current period, the stage game give only player 1 a payoff relevent choice between actions. If he then chooses $s_1' \in S_1$ his payoff is $u(s_1', s_2)$, and the transition takes them to the next stage game in which only those payoffs consistent with "s_1'" are reachable.

A recent result of Dutta (1995) proves a Folk Theorem for stochastic games resolves the issue: how general is the "Anti-Folk Theorem" of Theorems 1 and 2? In our setup, Dutta's Theorem is stated after the following notation:

In the two person alternating move game we define

$$m_i(h^t) = \lim_{\delta \to 1} \inf_{f_j} \sup_{f_i} V_i(f|h^0, \delta)$$

where we explicitly express the dependence of V_i on discount factor δ. The payoff $m_i(h^t)$ denotes the long run average minimax payoff for player i given history h^t. Now let W denote the convex hull of the stage game payoff vectors and define

$$F(h^t) = \left\{ w \in W \middle| \; w_i > m_i(h^t), \; i = 1, 2 \right\}$$

which is the set of feasible payoffs given h^t that strictly dominate the long run minimax payoff.

Theorem 3 *(Dutta (1995)) Suppose that for each $i = 1, 2$ and all histories h^t, $m_i(h^t) = m_i$ and $F(h^t) = F$ with $dim(F) = 2$. For any $w \in F$ and any $\epsilon > 0$ there is a $\bar{\delta}$ such that if $\delta \geq \bar{\delta}$, then there is a there is a Perfect equilibrium f such that $V_i(f|h^t, \delta) > w_i - \epsilon$, $\forall h^t \in H$.*

[4] For details, see Lagunoff and Matsui (1995b).

According to this result any payoff in F can be arbitrarily approximated as a Perfect equilibrium payoff for sufficiently patient players if the feasible payoffs and the minimax payoffs do not vary across states and the game has full dimension.[5] Note that the minimax benchmark in a stochastic game need not coincide with the minimax in any of its state contingent stage games. In the alternating move game this means that the lower bound m for equilibrium payoffs need not coincide with the minimax of the stage game.

As for the application to the present model, the sufficient conditions of Dutta's Theorem hold in any alternating move game of full dimension. The perfect coordination games are exceptions to the full dimensionality condition. The conclusion of the Dutta Folk Theorem then applies to all other stage games of full dimension and which are repeated in alternating moves. This is perhaps not too surprising since full dimensionality conditions are typically required for the Folk Theorem.[6] The difference is that in standard repeated games with synchronized choice they are required only for games of more than two players, and that the dimensionality condition is not needed if what should be proved is that any payoff no worse than some one-shot Nash equilibrium payoff is supported in equilibrium. Fudenberg and Maskin's (1986) Folk Theorem, for example, holds perfectly well without full dimensionality when there are two players. By contrast, in our model full dimensionality is required for a Folk Theorem in two person games.

4.2 Inertia and Optimality in General Coordination Games

Dutta's result demonstrates a non genericity of results in Section 2 when player are very patient. In our model, this corresponds to the case when there is very little inertia. It makes sense, then, to re-examine the optimality results for a wider class of stage games when players' discount factors are bounded away from 1.

A more general class of coordination problems is captured by games of *common interest.* A (two-person) normal form game is said to be a *game of common interest* if there is a unique action profile s^* dominating all other profiles, i.e., $u_i(s^*) > u_i(s)$ for all $i = 1,2$ and all $s \neq s^*$. If the players are impatient, synchronous repetition and asynchronous repetition induce qualitatively different results in common interest games. On one hand, in a synchronously repeated game, any Nash equilibrium of the stage game may be taken in equilibrium after any history. On the other hand, with asynchronous revision for a sufficiently small discount factor, any strict Nash equilibrium is absorbing. Therefore, once the action profile reaches s^*, it will stay there forever. However, this also means that Pareto inferior

[5]Full dimensionality here means that the dimension of the set of feasible payoffs coincides with the number of players.

[6]Note that in the literature on standard repeated games, the dimension of the feasible payoffs need be only the number of players *minus* one. In fact, Abreu, Dutta, and Smith (1994) require an even weaker condition: that there is a distinct feasible payoff profile for each player in which that player receives a payoff below that of any other profile.

strict Nash equilibria are also absorbing.

A natural question is: when (for what conditions) is the optimality result obtained? As we have already seen, pure coordination and a large discount factor are sufficient conditions. Though a full characterization for all stage games is beyond the scope of the present paper, we provide a simple sufficient condition for common interest stage games in which the optimality result holds for *some* discount factor.

To see this, let $\bar{u} = \max_{s \neq s^*} \max\{u_1(s), u_2(s)\}$, *i.e.*, the second highest payoff. Let also $u_i* = \min_s u_i(s)$, *i.e.*, the smallest payoff for player i. Assume that by affine transformation, we have $u_1(s^*) = u_2(s^*) = u^*$ and $u_{1*} = u_{2*} = u_*$. Let

$$\alpha = (\bar{u} - u_*)/(u^* - u_*).$$

The larger the α, the closer the second highest payoff is to the highest one relative to the lowest payoff. Then we have the following statement.

Theorem 4 *In an alternating move game with its stage game being a game of common interest, there exists a neighborhood of discount factors $\delta \in (0,1)$ under which the optimal outcome s^* is the unique outcome if*

$$2\alpha + \alpha^2 - \alpha^3 \ < \ 1. \tag{9}$$

For example, the above condition holds if $\alpha < 0.4$. Note that this statement has no restriction on payoff differences except, of course, that it cannot exceed $\bar{u} - u_*$.

Proof Suppose that the current action profile is s^*. Then the worst continuation value for the next mover when he keeps s^* is at least $(1-\delta)u^* \ + \ \delta u_*$. On the other hand, the best continuation value when he switches his action is not more than $(1-\delta)\bar{u} \ + \ \delta(1-\delta)\bar{u} \ + \ \delta^2 u^*$. Therefore, the player always chooses to stay at s^* whenever

$$(1-\delta^2)(1-\alpha) \ > \ \delta. \tag{10}$$

Next, we characterize parameters in which players move toward s^*. Suppose that (10) holds. If player i's has move and the current action profile is (s_i, s_j^*), then he should switch to s_i^* immediately since he then will get u^* forever after. Knowing this, player j will switch to s_j^* when it is his turn to move from any profile $s = (s_1, s_2)$ with $s_1 \neq s_1^*$ and $s_2 \neq s_2^*$, if

$$(1-\delta)u_* \ + \ \delta u^* \ > \ \bar{u}.$$

This condition holds if and only if $\delta > \alpha$. Combining this inequality with (10), we obtain (9) as a sufficient condition under which the unique equilibrium outcome is to play s^* always. □

For any game of common interest which satisfies (9), we therefore have three cases, depending upon the value of the discount factor.[7] (1) If δ is sufficiently large, any individually rational payoff pair can be attained in PE (the Folk Theorem) in a common interest game of full dimension; (2) if δ is sufficiently small, then all strict Nash equilibria are absorbing (a result which is often attained in deterministic dynamics in evolutionary game theory); (3) if δ lies in some intermediate range, then the optimal outcome is uniquely attained in PE.

5 Related Literature

It is clearly desirable to consider a repeated setting in which the timing is endogenous. One reasonable model of endogenous timing is Perry and Reny's bargaining model (1993). They formulate a continuous time bargaining model for a unit surplus. Individuals can make an offer at any time, with a small delay after each offer. They show that if players can react to offers instantaneously then the equilibrium payoffs are bounded by the first and second mover payoffs, respectively, of the Rubinstein bargaining model. Extending such a model of endogenous timing to a repeated setting is an interesting exercise which we leave for future research.

Elsewhere, models of asynchronous choice have been studied in other contexts. Examples include Farrell and Saloner (1985), Lagunoff and Matsui (1995a), Lagunoff (1995), Maskin and Tirole (1988a,b), Matsui and Rob (1992), Morris (1995), and Rubinstein and Wolinsky (1995).

Of these, the results of Rubinstein and Wolinsky (1995) are closest to the present paper. They consider a standard infinite repetition of an extensive form game of pure coordination. In their model the optimal outcome is also uniquely obtained. Strictly speaking, theirs is not an asynchronously repeated game since payoffs only occur at the end of each period. Since the perfect equilibrium of their extensive form stage game is unique, the set of equilibrium payoffs does not shrink with repetition. Nevertheless, the extensive form stage game breaks the "synchronized mistake" in much the same way as in our model.

Farrell and Saloner (1985), Gale (1995), and Morris (1995) are also related to the present paper. They use a backward induction logic similar to our Theorem 2 proof for n-person coordination games with irreversible sequential choices. In Gale, for example, backward induction is used to establish that when, say, the first $k < n$ players cooperate, the $(k+1)$th player also cooperates. By contrast, Morris shows that synchronized coordination unravels when the timing of moves is not common knowledge. Though their logic looks similar to ours at first glance, it is very different. None of these are repeated settings so that there is no fear

[7]The necessity of patience is also seen in Lagunoff and Matsui (1995a), Matsui and Matsuyama (1995) Matsui and Rob (1992). Other mechanisms include constant mutation of Foster and Young (1991) and Kandori, Mailath and Rob (1993), and secret handshake of Robson (1990) and cheap-talk of Matsui (1991).

of "retaliation" in the last period. By contrast, in the repeated coordination game, decisions are never irreversible, and so there is always a chance of "retaliation for bad behavior." It just happens that when the game is one of pure coordination game, this "retaliation," though feasible, does not work since one who "retaliates" also harms himself to the same degree.

The Matsui and Rob (1992), Lagunoff and Matsui (1995a), and Lagunoff (1995) papers use the Poisson revision process of Example 3 in Section 3. New individuals were assumed to enter the game stochastically and asynchronously. In the first two models heterogeneous forecasts across generations provided the impetus for change. In the last model, perfect foresight was assumed and, for this reason, is closest to the present equilibrium model.[8]

References

[1] Abreu, D. (1988), "On the Theory of Infinitely Repeated Games with Discounting," *Econometrica*, 56: 383-398.

[2] Abreu, D., P. Dutta, and L. Smith (1994), "The Folk Theorem for R epeated Games: A NEU Condition," *Econometrica*, 62: 939-48.

[3] Aumann, R. (1981), "Survey of Repeated Games," in *Essays in Game Theory and Mathematical Economics in Honor of Oskar Morgenstern*, Manheim: Bibliographisches Institut.

[4] Blume, L. (1993), "The Statistical Mechanics of Strategic Interaction," *Games and Economic Behavior*, 5: 387-424.

[5] Dutta, P. (1995), "A Folk Theorem for Stochastic Games," *Journal of Economic Theory*, 66: 1-32.

[6] Farrell, J. and G. Saloner (1985), "Standardization, Compatibility, and Innovation," *Rand Journal of Economics*, 16: 70-83.

[7] Foster, D., and H.P. Young (1990), "Stochastic Evolutionary Game Dynamics," *Theoretical Population Biology*, 38: 219-232.

[8] Fudenberg, D. and E. Maskin (1986), "The Folk Theorem in Repeated Games with Discounting or with Incomplete Information," *Econometrica*, 54: 533-56.

[9] Gale, D. (1995), "Dynamic Coordination Games," *Economic Theory*, 5: 1-18.

[8] Related work includes Blume (1993), Matsui (1994), and Matsui and Matsuyama (1995). Individuals' decisions in these papers are asynchronous, however, the players are either assumed to be myopic, or are randomly matched in a large population so that no player accounts for the intertemporal effects of his own behavior.

[10] Kandori, M., G.J. Mailath, and R. Rob (1993), "Learning, Mutation, and Long Run Equilibria in Games," *Econometrica*, 61: 29-56.

[11] Lagunoff, R. (1995), "On the Evolution of Pareto Optimal Behavior in Repeated Coordination Problems," mimeo, University of Pennsylvania.

[12] Lagunoff, R. and A. Matsui (1995a), "Evolution in Mechanisms for Public Projects," *Economic Theory*, 6: 195-223.

[13] Lagunoff, R. and A. Matsui (1995b), "An 'Anti-Folk Theorem' for a Class of Asynchronously Repeated Games," Discussion Paper, University of Pennsylvania.

[14] Marschak, J. and R. Radner (1972), *The Economic Theory of Teams*, Yale University Press, New Haven.

[15] Maskin, E. and J. Tirole (1988a), "A Theory of Dynamic Oligopoly, I: Overview and Quantity Competition with Large Fixed Costs," *Econometrica*, 56: 549-70.

[16] Maskin, E. and J. Tirole (1988b), "A Theory of Dynamic Oligopoly, II: Price Competition, Kinked Demand Curves and Fixed Costs, " *Econometrica*, 56: 571- 600.

[17] Matsui, A. (1991), "Cheap-Talk and Cooperation in a Society," *Journal of Economic Theory*, 54: 245-258.

[18] Matsui, A. (1994), "Evolution and Rationalizabilty," mimeo, University of Pennsylvania, March.

[19] Matsui, A. and K. Matsuyama (1995), "An Approach to Equilibrium Selection," *Journal of Economic Theory*, 65: 415-34.

[20] Matsui, A. and R. Rob (1992), "Evolution with Forward Looking Players," mimeo, University of Pennsylvania.

[21] Morris, S. (1995), "Cooperation and Timing," mimeo, University of Pennsylvania.

[22] Perry, M. and P. Reny (1993), "A Non-cooperative Bargaining Model with Strategically Timed Offers," *Journal of Economic Theory*, 59: 50-77.

[23] Robson, A. (1991), "Efficiency in Evolutionary Games: Darwin, Nash and the Secret Handshake," *Journal of Theoretical Biology*, 144: 379-396.

[24] Rubinstein, A. and A. Wolinsky (1995), "Remarks on Infinitely Repeated Extensive-Form Games," *Games and Economic Behavior*, 9: 110-115.

95-19 "Policy Persistence " by Stephen Coate and Stephen Morris
95-20 "Underemployment of resources and self-confirming beliefs*"by Alessandro Citanna , Herve Cres + and Antonio Villancci
96-01 "Multiplicity of Equilibria" by Christian Ghiglino and Mich Tvede
96-02 "Word-of-Mouth Communication and Community Enforcement" by Illtae Ahn and Matti Suominen
96-03 "Dynamic Daily Returns Among Latin Americans and Other Major World Stock Markets" by Yochanan Shachmurove
96-04 "Class Systems and the Enforcement of Social Norms" by Harold L. Cole, George J. Mailath and Andrew Postlewaite
96-05 "Dynamic Liquidation, Adjustment of Capital Structure, and the Costs of Financial Distress" by Matthias Kahl
96-06 "Approximate Common Knowledge Revisited" by Stephen Morris
96-07 "Approximate Common Knowledge and Co-ordination: Recent Lessons from Game Theory" by Stephen Morris and Hyun Song Shin
96-08 "Affirmative Action in a Competitive Economy" by Andrea Moro and Peter Norman
96-09 "An Alternative Approach to Market Frictions: An Application to the Market for Taxicab Rides" by Ricardo A. Lagos
96-10 "Asynchronous Choice in Repeated Coordination Games" by Roger Lagunoff and Akihiko Matsui