CARESS Working Paper #95-20
Underemployment of Resources and Self-confirming Beliefs[*]

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November 16, 1994
This version: November 1, 1995
comments welcome

Abstract

In a model of exchange with price-taking individuals, the existence of non-trivial underemployment equilibria with walrasian prices is proved for a generic set of economies. The likelihood of the occurrence of these equilibria is higher the farther the economy is from a Pareto optimal initial allocation, and the larger the economy is, when considering log-linear preferences. Moreover, these equilibria can be interpreted as the result of some self-fulfilling beliefs. We show how markets are vulnerable to psychological effects translating aggregate signals into bad expectations, which are nonetheless rational in the sense of being confirmed in equilibrium. The possibility of distortions in market allocations is essentially derived from: 1) myopic individual behavior preventing sufficient experimentation; 2) the timing of "production" decisions; 3) the absence of certain financial contracts; 4) the fear of government restrictions on supply.

1. Introduction

In competitive markets models it is usually assumed that agents believe they can buy and sell any quantity they like at given prices. On the other hand, casual empiricism suggests that individuals are not always bringing all their resources to the markets for trade. This is particularly evident for the labor market, where the effect of the inability to sell is usually called unemployment. Among the many answers provided to explain this particular case of inability to sell, the most widely studied general

[*] We are particularly grateful to Yves Balasko and David Cass for valuable discussions.

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equilibrium explanation is in terms of price rigidities. Prices are assumed to adjust slowly, relative to quantity adjustments, or to be fixed. This literature, on rationing schemes, was especially popular (in Europe, at least) in the 70's and early 80's. We are referring to the work of, among others, Bénassy [2], Drèze [4], Grandmont [6], Malinvaud [13] and Younès [16]. Given their motivation for the existence of rationing equilibria, these authors did not show that nonwalrasian equilibria exist also with walrasian prices. Two exceptions are found in the literature: Hahn [7] and Silvestre [14]. Nevertheless, they only have examples of existence of these equilibria, when the rationing scheme is not specified. We will show that nonwalrasian equilibria with walrasian prices exist generically, in the case of a proportional rationing scheme. Our reason to study these equilibria is more in line with Hahn's [8] interpretation of nonwalrasian equilibria as the result of self-fulfilling beliefs, instead of actual existence of quotas. These beliefs are not fully rational, if one insists that rationality of beliefs implies that they cannot be systematically wrong even off the equilibrium path. In Hahn's language, conjectures are not necessarily rational, but only "reasonable". This is a feature shared by our model. In more modern terminology, this is reminiscent of the concept of "self-confirming" equilibrium of Fudenberg and Levine [5]: beliefs cannot be systematically wrong only on the equilibrium path. Our model differs from Hahn's in that: 1) Hahn's conjectures are monopolistic when agents perceive quantity constraints, which are individualized, and give rise to kinked conjectured demand curves; 2) in Hahn's framework walrasian prices imply full employment; 3) finally, in Hahn's model agents are price takers when unconstrained, then some informational imperfections (unemployment statistics may be unavailable to consumers, for instance) are needed to explain why prices are not adjusted to the walrasian level by those agents who are not constrained (they cannot all be).[1]

We embed the static general equilibrium model into a (particular) dynamic specification, hence gain some insight into the conditions that allow the beliefs of underemployment of resources[2] to be confirmed in equilibrium: limited experimentation, the timing of buy-sell decisions and the absence of certain financial contracts.

We have a pure exchange economy in which consumers think that they will be unable to trade more then a certain percentage –common to everybody, but different across goods– of their initial endowments, and then they end up consuming the

[1] Many macroeconomic models of unemployment can be reinterpreted as models of coordination failures, due to externalities, or strategic complementarities. These arise when there are market imperfections such as, for example, in the market structure (leading to noncompetitive behavior on the product market, as in Hart [9] or Heller [10]), or in the trading process (search models, see Diamond [3]). In this last case, an aggregate variable is also seen as a signal that agents perceive in a totally decentralized way to compute their optimal choices. These choices then aggregate in such a way that this macroeconomic variable r be eventually confirmed in its initial expected value. This process of confirmation is an argument of rational expectations in a strict sense, though. We do not use these kind of market imperfections to explain nonwalrasian equilibria, but we have to give up the more stringent notion of equilibrium.

[2] For labor markets, underemployment sometimes refers to a situation when workers are still hired but provide a less-than-optimal amount of labor. We apologize for using the same term, but with a different meaning.

untraded part.[3] They go to the market valuing their wealth at the current market prices, and formulate their demands. Markets clear. The percentage that consumers anticipated is con¯rmed in the long-run equilibrium: the ratio between e®ective aggregate demand and ideal one {i.e., the one that would appear with full trading of resources{ is eventually the percentage they had anticipated.

The inability to sell depends on the believed possibility that there is a mechanism that restricts the supply. The mechanism which is believed to be possible to occur is a proportional rationing scheme, administratively enforced and giving rise to a loss in the value of resources supplied in excess. The argument in the paper does not crucially depend on this particular form of administrative control, but more generally on the fear that agents have of controls to be applied to them. Note that this is not a monopolistic conjecture, and our agents do not control prices.

We prove that for a generic distribution of initial endowments e and preferences u, given an initially bad conjecture or belief, there exists a one-dimensional (not necessarily smooth) manifold of long-run underemployment equilibria. This manifold contains a path joining the initial endowments e to the walrasian equilibrium allocations. It also contains equilibria where prices are all walrasian. The fact that the \right" prices appear does not rule out unemployment: underemployment equilibria may exist due to the mere fact that agents | who are rational in their behavior as well as in their expectations | don't believe in the ability of the market to allow them to trade to the extent they would like to. Underemployment stems from the fear that a government may interfere with the market clearing process. Market clearing prices do not guarantee full employment as long as the government makes possible to believe that it will intervene.

The model structure resembles that of Balasko and Royer[1]. The authors investigate a dynamic model where the amount of labor unemployment reported the previous day appears in the set of perceived constraints as a quantity limit. Their model allows more general functional forms for the perceived constraints, but yields unemployment only if prices are not walrasian. Moreover, our results somewhat contradict the one obtained by Balasko and Royer, who establish that there is a \corridor" of initial endowments around the contract curve for which unemployment equilibria do not appear. In our model the \size" | i.e., merely the relative Lebesgue measure in the cube $[0;1]^l$, l being the number of goods | of expected percentages of sellable resources, for which the sales constraints are not binding, is a continuous function of the initial endowments. It has full measure one for a Pareto-optimal distribution of initial endowments. On the other hand, the more the economy is specialized | i.e., the consumers are basically endowed in only one good, and then we can expect to be fairly far away from the contract curve | the smaller this size is.

Our results seem to suggest that markets are more vulnerable to underemployment crises exactly when they are more needed, that is when the initial allocation of

[3] It will be clear during the presentation of the model setup that one can read the model as a household production problem with linear production technology. We will not pursue this interpretation explicitly in this paper, and leave it for further development in subsequent papers.

resources is farther away from efficiency and "specialization of labor" requires more dependency on trade.

2. Setup of the model.

We consider a pure exchange economy with m consumers and l goods, with $m, l > 1$. We only allow for positive consumption, that is, the consumption set X is $\Re^l_{++}$. Consumer h is characterized by a standard utility function, $u_h(x_h)$, differentiably strictly increasing, differentiably strictly concave, and whose indifference surfaces have closures contained in the positive orthant. He is endowed with a vector $e_h \in R^l_{++}$. The space of endowments $E = \times_h \Re^l_{++}$ is equipped with the natural topology, and the space of utility functions $U = \times_h U_h$ is endowed with the $C^2 -$ compact open topology. An economy is a pair $(e, u) \in E \times U$.

We imagine that there is a sequence of trading dates $t = 0, 1, 2, \ldots$. At each date t, agents only maximize their utility of consumption at that date. In this sense, they display myopic behavior. Since we are focusing on underemployment not stemming from technological shocks, or other forms of uncertainty regarding fundamentals, we will assume that both preferences and endowments do not change over time.

At the beginning of the trading date and upon having expectations on prices, p^t, each consumer believes that at these prices there is rationing with probability π, or no rationing with probability $1 - \pi$.[4] This situation makes sense especially in economies where supply has been constrained in the past. We assume that the rationing scheme is proportional to the endowment. Also, we assume that consumers only think supply, not demand, rationing is possible. This is because we are interested in explaining situations where there is an excess supply of some sort, i.e. underemployment. Finally, we assume that if rationing occurs, resources supplied in excess will be devalued, i.e. they will fetch a lower price.[5] Before knowing whether there is going to be rationing or not, consumers decide how much of their endowment they are going to bring for exchange, and how much they are going to consume without therefore trading it. Denote this amount by $r^t_h \in [0, 1]^l$ for each consumer h. Then they make plans for exchange given the value of their wealth, and contingent upon whether there is going to be rationing or not. Let $S^t = \{0, 1\}$ represent the space of states of the world at the end of trading date t for each consumer, where $s = 0$ corresponds to "rationing". Then their planned demand at time t will be denoted by $y^{s,t}_h \in \Re^l_+$. Given commodity price expectations, the probability distribution and the level of constraint of the previous

[4] Consumers do not have to assign the same probability, as long as they are close to one another, as it will be clear after Lemma 1.

[5] This is not so unrealistic, if one thinks of some agricultural markets of the European Community, for example. Usually output control and marketing quotas are enforced through a system of cash payments to create the incentives to abide the restrictions. This generates an opportunity cost, since the payments are not received if the quota is not observed. Sometimes, the government has even stopped the payments and forced producers to observe the quotas without monetary payments. In any case, one should take the rationing mechanism described as an abstraction of several more complex schemes of administrative control.

period (i.e., the level of proportional rationing at time $t-1$; if there was any), named r^{t-1}; each consumer chooses $\left(r_h^t, y_h^{s,t}\right)$ to maximize his current utility. After receiving communication of r_h, a central authority decides whether or not to ration the markets. Several reasons determine why this may occur. One may think that the markets may not support too much activity, for example, or that – if really an auctioneer exists and is not just a metaphor for decentralized exchange – , that the auctioneer can be lazy, and does not want to call another price and try to adjust this way demand and supply. Or, simply, it may be part of the government policy to manage the supply of certain commodities. In any case, surpluses will be devalued by an amount $c(p^t)$ if there is rationing. Given the announced state of the world, people submit their contingent demands, and markets clear.

We assume that consumers believe that the probability π is a function of r_h^t and of their expectations of rationing r^e, which we take for simplicity to be equal to r^{t-1}, with an adaptive scheme.

At the end of the day, when trade has taken place, consumers can compute the extent to which the market felt constrained, or was actually constrained. This is r^t, the proportion of unconstrained demand computed at current prices p^t that corresponds to the effective aggregate demand, $\sum_h y_h^{s,t}$.

More formally, each consumer formulates a demand $y_h^t \equiv \left(y_h^{s,t}\right)_s$ and a voluntary constraint r_h^t as solution to the following maximization program:

$$\begin{aligned} &\max_{y_h^t, r_h^t} \pi u_h(y_h^{0,t} + (1-r_h^t)\in e_h) + (1-\pi)u_h(y_h^{1,t} + (1-r_h^t)\in e_h) \\ \text{s.t.}\quad &\pi\left\{p^{0,t}y_h^{0,t} - p^{0,t}\left(r^{t-1}\wedge r_h^t\right)\in e_h - (p^{0,t} - c(p^{0,t}))\left[\left(r_h^t - r^{t-1}\right)\vee 0\right]\in e_h\right\} = 0 \\ &(1-\pi)\left(p^{1,t}y_h^{1,t} - p^{1,t}(r_h^t \in e_h)\right) = 0 \\ &y_h^t \geq 0 \text{ and } r_h^t \in [0,1]^l \end{aligned} \tag{2.1}$$

where the symbol $\wedge$ denotes the coordinate-wise minimum of two vectors, and $\vee$ the coordinate-wise maximum. The consumer does not ask for a negative quantity ($y_h^t \geq 0$), because it would be equivalent to trying to create endowments, which is excluded by assumption. The equilibrium price at time t is a standard market-clearing price.

Definition 1. An equilibrium price at time t associated with the economy (e,u) is a vector $p^t \in \mathbb{R}^{2l}_{++}$ such that

$$\pi\left(\sum_h y_h^{0,t} - r_h^t \in e_h\right) = 0 \tag{2.2}$$

$$(1-\pi)\left(\sum_h y_h^{1,t} - r_h^t \in e_h\right) = 0 \tag{2.3}$$

At each date and state of the world, the amount of rationing or aggregate voluntary constraint $r^{s,t}$ recorded in the economy is computed:

$$\sum_{h=1} y_h^{s,t} = r^{s,t} \times \sum_{h=1} x_h^{s,t}. \tag{2.4}$$

where $x_h^{s,t}$ is calculated as follows. Under the price $p^{s,t}$, $x_h^{s,t}$ is the unconstrained demand coming from the following standard maximization program:

$$\begin{aligned} &\max u_h(x_h^{s,t}) \text{ s.t.} \\ &p^{s,t}x_h^{s,t} = p^{s,t}e_h \\ &x_h^{s,t} \gg 0 \end{aligned} \tag{2.5}$$

The next period this percentage $r^{s,t}$ is used by individuals to gauge the level of supply rationing that may occur, based on the belief that if demand is weak, sales will also be lower.

We are interested in a particular short-run equilibrium of this economy, given by particular beliefs π on the possibility of rationing. We assume that consumers believe that this probability is a function of r_h^t and of r^{t-1}. Moreover, it is easily seen that if

$$\pi(r_h^t, r^{t-1}) = \begin{cases} 0 & r_h^t \leq r^{t-1} \\ 1 & r_h^t \not\leq r^{t-1} \end{cases} \tag{2.6}$$

then the sequence of trades has the following property.

Lemma 1. Given the initial beliefs (2.6), $p^{s,t} = p^t$ for $s = 0, 1$, and $c(p^t)$ close enough to p^t the sequence of trades $\left(r_h^t, y_h^{s,t}\right)$ is equivalently characterized by the following short-run equilibrium:

1) consumers solve

$$\begin{aligned} &\max_{y_h^t} u_h(y_h^{1,t} + [1 - \left(r^{t-1} \wedge 1\right)] \times e_h) \\ \text{s.t.:} \quad &p^t y_h^{1,t} = p^t \left(r^{t-1} \wedge 1\right) \times e_h \\ &y_h^t \geq 0 \end{aligned} \tag{2.7}$$

2) markets clear, i.e.

$$\sum_h y_h^{1,t} - (r^{t-1} \wedge 1) \times e_h = 0 \tag{2.8}$$

3) percentages are computed according to

$$\sum_{h=1} y_h^{1,t} = r^t \times \sum_{h=1} x_h^t. \tag{2.9}$$

Proof. See the Appendix.

Notice that in these equilibria consumers choose not to sell more than r^{t-1} voluntarily, even if they are price-takers. Also note that the equality of p^t across states of the world is trivially consistent with their beliefs, and that given this equality, consumers don't learn from prices whether there is going to be rationing or not. We want to insist on the fact that these beliefs are not irrational, since all along the equilibrium path, no rationing is indeed observed, so π is equal to zero as expected, given the individual behavior. Beliefs need not be consistent off the equilibrium path, but consumers will never learn whether their beliefs about the possibility of rationing are actually incorrect. This occurrence will remain a counterfactual and individuals will never observe it because, given their initial beliefs, they have no incentive for experimentation (people are myopic). This structure is similar to the one in the notion of self-confirming equilibria (see Fudenberg and Levine [5]) in strategic games.

We are interested in equilibria where the expected sellable percentage level is equal to the realized one, motivating the following definition.

Definition 2. The expected sellable percentage r^{t-1} is a self-fulfilling percentage if the ratio between the aggregate constrained demand and the aggregate unconstrained demand, r^t, is equal to r^{t-1}, i.e. $r^t = r^{t-1}$.

As a remark, notice that r^t may not be smaller than 1 at any time t. In terms of expectations, though, Lemma 1 implies that r^{t-1} or $1 \wedge r^{t-1}$ cannot be distinguished as far as rationing is concerned. $r^{j,t-1} \geq 1$ means that agents do not expect any constraint on the sale of commodity j; and no difference on behavior results from assuming that the expected r^j for time t is exactly one. But Definition 2 implies that $r^{t-1} \wedge 1 = r^t \wedge 1$.

We will look at particular steady states of the process, or long-run equilibria, defined as follows.

Definition 3. A pair $(p; r)$ is a $*-$long-run equilibrium for an economy $(e; u)$ if $r^t = r \leq 1$; $p^t = p$ for all t; and

$$\sum_h y_h - r \circ e_h = 0$$

$$\sum_{h=1} y_h - r \circ \sum_{h=1} x_h = 0.$$

We call r an equilibrium sellable percentage.

Notice that there may be other long-run equilibria where $r \not\leq 1$. The reason we are interested in equilibria of Definition 3 appears from the following proposition.

Proposition 1. For any economy $(e; u)$, we have:

1) if p is a $*-$long-run equilibrium price, then it is walrasian;

2) given a walrasian equilibrium price p, there exists a sellable percentage $\hat{r}(e; u; p)$ such that all sellable percentages $r \geq \hat{r}(e; u; p)$ are nonbinding equilibrium sellable percentages, i.e., such that the associated equilibrium allocations are walrasian.

Proof. 1) Considering equations (2.8) and (2.9), one obtains that p is an equilibrium price if and only if (r^j being different from zero, for all $1 \leq j \leq l$)

$$\sum_{h=1} x_h = \sum_{h=1} e_h . \tag{2.10}$$

which means p is walrasian.

2) Denote $(x_h)_{1 \leq h \leq m}$ the walrasian equilibrium allocations associated to $(e; u; p)$. Define for all goods j, $1 \leq j \leq l$,

$$\bar{r}^j(e; u; p) = \operatorname{Max}_{1 \leq h \leq m} \left(1 - \frac{x_h^j}{e_h^j} \right) .$$

Take $r \geq \bar{r}(e; u; p)$. It means that for all j, $1 \leq j \leq l$, and for all h, $1 \leq h \leq m$, $x_h^j \geq (1 - r^j)e_h^j$. In other terms, no consumer is compelled to consume more than what he would like to if he were not constrained at all. So none of the constraints of type (b) is binding since for all h, y_h can be equal to $x_h^j - (1 - r^j)e_h^j$ which is positive then satisfies both (a) and (b), and the constrained demand is the same as the unconstrained one for each agent. ■

It is a property of the model and an easy corollary of Proposition 1 that Pareto efficient endowments do not give rise to underemployment equilibria. If $(e; u)$ is such that e belongs to the set of Pareto optimal allocations for utilities u, then $\bar{r}(e; u; p) = 0$. This is a trivial consequence of the definition of $\bar{r}(e; u; p)$ and of Pareto optimality. The farther away from Pareto efficient initial allocations, the higher the value of $\bar{r}(e; u; p)$.

We will use the fact that the $*$ long-run equilibrium price clears both the constrained and unconstrained markets, by replacing equation (2.9) by equation (2.10) in the system of equilibrium equations. We write the system of equations and inequalities defining an equilibrium for our economy as follows. Set $b_h = y_h + [1 - (r \wedge 1)] 2 e_h$. Then (2.7) and (2.8) become

$$\begin{array}{ll} & \max u_h(b_h) \text{ s.t.} \\ \text{a)} & p b_h = p e_h \\ \text{b)} & b_h \geq [1 - (r \wedge 1)] 2 e_h \end{array} \tag{2.11}$$

and

$$\sum_{h=1} b_h = \sum_{h=1} e_h . \tag{2.12}$$

So the definition of equilibrium is completed by appending the maximization problem (2.5) and the associated condition (2.10). Note that all we are trying to show now is whether it is possible to have walrasian equilibrium prices which clear markets in two economies: one with quantity constraints, and the other without them. In both economies the budget constraints and all but one market clearing equations imply the last one, that is, Walras law applies. We can therefore drop the first equation in both (2.12) and (2.10).

We will need the following useful definition as given in Proposition 1. For any economy $(e; u)$; given a walrasian price p; let

$$S(e; u; p) = \{r \mid r \geq \overline{r}(e; u; p)\}$$

and also let

$$S(e; u) = \cup_p S(e; u; p)$$

Finally, define for all j

$$r^{*j}(e; u) = \min_p \overline{r}^j(e; u; p)$$

Proposition 1 raises an important point: even if walrasian prices appear, this does not guarantee that agents will end up with a Pareto-optimal walrasian allocations through market exchange. The latter depends also on the confidence consumers have on the ability of the market to allow trading of the whole initial resources. When consumers cannot bring to the market their whole initial endowment, they are forced to consume it in a distorted way. We call this situation underemployment of resources, in the sense that endowments cannot be brought to the market for trade and this creates distortions in the consumer's behavior. From now on we will refer to the $*$-long-run equilibrium simply as an equilibrium of our economy.

Definition 4. Given an economy $(e; u)$; an underemployment equilibrium is a pair $(p; r)$ of equilibrium prices and sellable percentages such that $r \notin S(e; u; p)$:

It is clear from Proposition 1 that equilibria exist for our economy with full employment of resources. It is also clear that for all economies $(e; u)$ we have trivial equilibria where no exchange of resources occurs, i.e. where $r = 0$ in equilibrium. In the next section we will address the issue of existence of other, and maybe more plausible, underemployment equilibria.

3. Existence of underemployment equilibria

Our strategy of proof will follow a degree argument, and therefore will be built around the construction of a special, well-behaved economy for which we can show the existence of a unique, regular equilibrium. This strategy essentially follows a framework developed by Smale [15]. Our analysis of continuous functions adapts results from Lloyd [12].

As a first step, we introduce the function describing the first order and market clearing conditions:

$$F(\xi; r^1; e; u) = \left\{ \begin{array}{c} Du_h(x_h) - \lambda_h p \\ -p(x_h - e_h) \\ \left(\sum_h x_h - e_h\right)^{\backslash} \\ Du_h(b_h) - \mu_h p + \alpha_h \\ -p(b_h - e_h) \\ \min\left(\alpha_h^j; b_h^j - [1 - \min(1; r^j)] e_h^j\right) \\ \left(\sum_h b_h - e_h\right)^{\backslash} \end{array} \right.$$

where $\xi = (x; \lambda; p; b; \mu; \alpha; r^{\backslash})$ is the vector of "endogenous" variables, and the superscript backslash on a vector means that the first component of the vector has been dropped. We can also normalize the first commodity price, $p^1 = 1$. This function is defined as $F : \Xi \times R^1 \times E \times U \to \Re^n$, where n is the dimension of the manifold Ξ, and

$$\Xi = \Re^{lm}_{++} \times \Re^{m}_{++} \times \Re^{l-1}_{++} \times \Re^{lm}_{++} \times \Re^{m}_{++} \times \Re^{lm} \times R^{l-1}$$

with $R^{l-1} = (-1; 1+\eta)^{l-1}$, and $\eta > 0$, and where $R^1 = (0; 1]$. Note that the zeros of this function represent the equilibria of our economies $(e; u)$. This is because the first order conditions for the inequality program are known to be representable by the minimum function as we write it, and the only difference between our previous definition of equilibrium and the function F is that its domain is restricted to the R^j's. This is without loss of generality, since if $r^j > 1$, then

$$b_h^j \geq (1 - r^j) e_h^j \text{ and } b_h^j \geq \left[1 - \min(1; r^j)\right] e_h^j$$

are equivalent, provided that $b_h^j > 0$, which is true by assumption on the utility function. By the same token, one can see that if ξ satisfies $F(\xi; r^1; e; u) = 0$ for some $\left(r^1; e; u\right)$, then $r \geq 0$. To prove this, suppose not, and $r^j < 0$. Then $b_h^j > e_h^j$ for all h, which implies $\sum_h b_h^j > \sum_h e_h^j$, a contradiction.

We have excluded the trivial equilibria with $r = 0$. We are going to consider the space of (continuous) functions $\varphi(\xi) = F(\xi; r^1; e; u)$. Since the space $\Theta = R^1 \times E \times U$ is arcwise connected (see Smale [15]), it is possible to construct a continuous homotopy between two functions φ which are associated to different economies, or to possibly different sellable percentages for the same economy. In fact, it is a linear homotopy $H : \Xi \times T \to \Re^n$, with $T = [0; 1]$, of the form

$$H(\xi; t) = \left\{ \begin{array}{c} Dv_h(x_h) - \lambda_h p \\ -p(x_h - \varepsilon_h) \\ \left(\sum_h x_h - \varepsilon_h\right)^{\backslash} \\ Dv_h(b_h) - \mu_h p + \alpha_h \\ -p(b_h - \varepsilon_h) \\ \min\left(\alpha_h^j; b_h^j - [1 - \min(1; \rho^j)] \varepsilon_h^j\right) \\ \left(\sum_h b_h - \varepsilon_h\right)^{\backslash} \end{array} \right.$$

where $v_h = (1-t)u_h + tu'_h$; $\varepsilon_h = (1-t)e_h + te'_h$ and $\rho^1 = (1-t)r^1 + tr^{1'}$; $\rho^j = r^j$; for $j > 1$; for two elements $\left(e; u; r^1\right)$ and $\left(e'; u'; r^{1'}\right)$ in Θ; that is, two functions ϕ and ϕ'. The degree of these functions is well-defined, since their domain and range are smooth boundaryless manifolds of the same dimension, the range $\mathbb{R}^n$ is connected and the projection

$$\pi : H^{-1}(0) \to \Theta$$

is proper, as we establish in the following lemma.

Lemma 2. The projection π is proper.

Proof. Take a converging sequence $\left(e^n; u^n; r^{1n}\right) \to \left(e; u; r^1\right)$. Then $t^{n_k} \to t$; obviously for some n_k. The convergence of $(x^{n_k}; \lambda^{n_k}; p^{n_k})$; for some n_k follows from a standard argument. From

$$\sum e_h^{jn} \geq b_{h'}^{jn} \geq 0$$

for all n and the convergence of $\left\{e^j\right\}^n$; $\left\{b^j_{h'}\right\}^n$ lies in a compact set, hence without loss of generality converges to $b^j_{h'}$; for all h' and all j. Note that $b^j_h > 0$; using the boundary condition. Then $Du_h(b_h) \gg 0$; for all h. Now note that for $j > 1$

$$0 \leq r^{jn} \leq 1 + \delta$$

for all n; and there is a convergent subsequence indexed by n_k (the same only for ease of notation), to r^j. Consider two possible cases: for each h;

1. $b^j_h - (1 - \min(1; r^j))e^j_h > 0$ for some j;
2. $b^j_h = (1 - \min(1; r^j))e^j_h$; for all j.

In case 1), equation $\min(\alpha^j_h; b^j_h - (1 - \min(1; r^j))\varepsilon^j_h) = 0$ implies $\alpha^j_h = 0$ for that j. From the first order conditions of the constrained problem, $\mu^n_h \to \mu_h > 0$; since $\alpha^j_h = 0$. Finally, $\alpha^{jn}_h \to \alpha^j_h$; for all other j's.

Case 2) cannot occur, as we now show. Indeed, 2) implies $r^j = 0$ for all j; otherwise $b_h = (1-r)2e_h < e_h$; and the consumer would be in the interior of the constraint set, violating monotonicity. But $r^1 \in (0; 1]$. ■

Since (modulo 2) degree is a homotopy invariant, all we have to exhibit is an example of a function ϕ; say ϕ_0; which is smooth[6] and with $\deg \phi_0 = 1$. We first consider economies with $m \geq l$. Pick the economy $\theta = (e; u; r^1)$ with

$$e^j_h = \begin{cases} \varepsilon > 0 & \text{if } j \neq h \\ 1 & j = h \end{cases}$$

for $h \leq l$; and $e^j_h = 1$ for all j and all $h > l$;

[6] All we need is that the function be locally continuously differentiable around an equilibrium.

$$u_h(x_h) = \frac{1}{l} \sum_j \log x_h^j$$

and

$$r^1 = \gamma$$

with $\gamma \in \left(0; \frac{l-1}{l}(1-\epsilon)\right)$: It is well-known that any of these economies has a unique walrasian equilibrium price. Moreover, the following is true given the particular symmetry of the endowments.

Lemma 3. The walrasian equilibrium price for the above-described economies is $p = 1$; independently of ϵ:

Proof. See the Appendix.

Once ϵ is chosen appropriately, the optimal constrained choice b_h for consumer $h \leq l$ satisfies two key properties.

Lemma 4. i) There exists a sufficiently small ϵ such that the optimal constrained choice b_h satisfies $b_h^j > (1 - r^j)\epsilon$, for all $j \neq h$; and for all $h \leq l$:

ii) For the above economy, $b_h^j = 1 - r^j$ for $j = h$, all $h \leq l$; when $r^j < \bar{r}^j(e; u; p)$:

Proof. See the Appendix.

We are now ready to show existence of equilibrium for our test economy φ_0; given a fixed ϵ such that Lemma 4 holds and any fixed γ as above (i.e., any fixed r^1).

Lemma 5. For the economy φ_0, there is a unique regular equilibrium.

Proof. See the Appendix.

We now complete the argument on existence for any economy φ.

Theorem 3.1. For all economies $(e; u) \in E \times U$; there is an equilibrium for each r^1: Moreover, for at least a generic set of economies $E^* \times U$; r^{*1} is positive, and there are underemployment equilibria for all $r^1 \in [0; r^{*1})$:

Proof. Take first $m \geq l$: Consider any economy $(e; u)$: Define the function $F : \Xi \times R^1 \times E \times U \to \Re^n$ as above, and the continuous function $\varphi(\xi) = F(\xi; r^1; e; u)$ for this economy and a particular choice of r^1: Construct the homotopy between φ and φ_0. By Lemma 2 and Lemma 5, the degree of φ_0 is well-defined, and is equal to 1. Then the degree of this φ is also well-defined and odd, proving existence of an equilibrium. Note that this can be done for any $r^1 \in (0; 1]$: Consider the space of endowments for which every household trades in each commodity market at the walrasian equilibrium. This set is generic in E. By construction, r^{*1} is well-defined and positive on this set. Since any $r^1 < r^{*1}$ corresponds to an equilibrium $(p; r)$ with $r \notin S(e; u)$; this implies $r \notin S(e; u; p)$; that is, there are nontrivial underemployment equilibria.

Finally, the existence of underemployment equilibria in economies where $m < l$ is established in the following way. First take an economy in the set with $m < l$ individuals, and replicate the economy N times, so that $Nm \geq l$. By the statement of the previous paragraph, there is an equilibrium for this economy for all r^1. Two identical agents will get the same allocation, as they are maximizing the same utility function over the same constraint set, and the argmax is unique. So we can go back to the economy with m agents without changing the equilibrium. To complete the argument, once again one can find a generic subset of E; call it E^* using the argument of the beginning of the proof. ■

4. Nonexistence at a fixed walrasian price: an example

The test economy used to prove Theorem 3.1 may suggest that the walrasian and constrained systems are independent. This conjecture is reinforced by a theorem by Laroque and Polemarchakis [11], showing that for any given rationing mechanism and any given price (not necessarily walrasian) there is an equilibrium with rationing. Notice that, besides the fact that our rationing scheme may not directly belong to their general class, their theorem does not say that for a given p and any r^1 there is an r^{n} yielding an equilibrium. Therefore it doesn't say that the equilibrium is with underemployment, if the price is walrasian. In this sense our result is stronger than previous existence theorems with rationing (not necessarily more general). The question is open regarding whether our theorem can be strengthened, showing that for any given walrasian price p and any r^1 there is an equilibrium (with underemployment). We give a counterexample. Consider an economy with two households and three goods. Assuming $r \leqq 1$, household h maximization problem is (2.11) with the inequality changed into $b_h \geq (1 - r) \cdot e_h$.

For each household, the budget set at a walrasian price p, without the inequality constraints, forms a triangle in $\Re^3_{++}$. At the walrasian allocation, there is an indifference surface tangent to this triangle. Any lower indifference surface cuts the triangle in a (deformed) circular fashion. Taking into account the inequality constraints, observe that the line associated with the constraint $b_1^1 \geq (1 - r^1) e_1^1$ is parallel to the axis of good 2. The lower r^1, the farther away this line from the axis. A similar situation occurs for the constraint on good 2, which is parallel to the axis of good 1. For good 3, the constraint line is parallel to the base of the triangle.

The Edgeworth box in this economy is represented by the intersection of a parallelepiped and a plane (which contains the two triangular budget sets of each consumer). This intersection will in general have the shape of an irregular convex hexagon, with parallel opposite sides, corresponding to the area common to the triangles. Observe that in the Edgeworth box a given r cuts the budget set from opposite

Figure 4.1:

sides for the two households. Graphically, it is therefore convenient to use r_h^j to label the line corresponding to r^j for household h:

We now construct an example of nonexistence of equilibrium which is robust in $e; u; r^1$. Choose a Walrasian allocation x^W (which is inside the hexagon) and an endowment e as in Figure 4.1. At this walrasian equilibrium, household 1 is selling good 1 and buying goods 2 and 3, and viceversa for household 2. Corresponding to x^W; there exist a vector $\overline{r}(e; u; p)$ of nonbinding constraints and related lines $\overline{r}_h^j$: Note that this vector can be computed without completely specifying the degree of convexity of u: Choose $r^1 < \overline{r}^1(e; u; p)$; so x^W is not feasible for household 1: In Figure 4.1, we are now on the line r_1^1: We are forcing household 1 to consume more of good 1. Intuitively, if goods 1 and 2 are complement, this household may want to consume a lot more of good 2 as well, say. This is represented by the shape of household 1's indifference ellipsoids, H1.

The optimal choice for household 1 is then shown at point A. We have to show that there are r^2 and r^3 less than 1 that yield an equilibrium. Graphically, this means that the optimal choice B for household 2 should coincide with A. Choose any r_2^2 and r_2^3: If x_2^W is attainable for household 2, $B = x_2^W$ and trivially there will be no equilibrium. If A is not attainable for 2, than again there is no equilibrium. If x^W is not attainable for household 2, but A is, we can find u_2 that leads to indifference ellipsoids H2: Again, $B \neq A$; and no equilibrium obtains. Small changes in e (in the fiber given by $p); u; r^1$ do not alter the results, and in this sense the example is robust.

5. Some results in specialized economies

The goal of this section is to study some properties of the equilibria when economies are specialized, according to the definition given below. Moreover, we only consider preferences represented by log-linear utility functions.

Consumer h's ($1 \leq h \leq m$) utility function is defined by an element $(a_h^j)_{1 \leq j \leq l}$ taken in the open l-simplex:

$$S_l^o = \left\{ (a^j)_{1 \leq j \leq l} \in \mathbb{R}_{++}^l \;\middle|\; \sum_{j=1}^{j=l} a^j = 1 \right\} .$$

The utility function has the following expression:

$$u_h(x_h) = \sum_{j=1}^{j=l} a_h^j \log(x_h^j) .$$

We focus on economies which are specialized in the sense that, for each good, there are some agents in the economy who are endowed only with this good. Let $L = \{1, 2, \ldots, l\}$ and $H = \{1, 2, \ldots, m\}$ denote the sets of goods and agents respectively. In this section we won't exclude the case where some consumers have no initial endowments in some commodities. To each vector of endowments $e = (e_h)_{1 \leq h \leq m} \in \mathbb{R}_+^{lm} \setminus \{0\}$ we associate the nonempty correspondence $\Gamma_e : H \to L$ where $\Gamma_e(h)$ is the set of goods in which consumer h has non-trivial endowments. In other words:

$$\forall j \in L, j \in \Gamma_e(h) \iff e_h^j > 0 .$$

An economy is fully described by a vector

$$((e_h)_{1 \leq h \leq m}, (a_h)_{1 \leq h \leq m}) \in \mathbb{R}_+^{lm} \setminus \{0\} \times \mathbb{R}_{++}^{lm} .$$

Definition 5. The economy (e, a) is specialized if the associated correspondence Γ_e has the following property: for all $j \in L$, there exists some consumer $h(j) \in H$ such that $\Gamma_e(h(j)) = \{j\}$ (consumer $h(j)$ is only endowed in good j).

In the next two subsections, we will present two properties of equilibria in this setup.

5.1. Underemployment versus employment equilibria

In this part we will try to compute the relative size of underemployment equilibria versus employment equilibria. Since we do not have an explanation of why an equilibrium occurs, we may as well assume that all equilibria are equally likely. This leads to considering the relative (Lebesgue) measure of the two types of equilibria as a notion of likelihood of occurrence of underemployment equilibria. From Theorem 3.1 we know that there is a generic set of economies that display underemployment equilibria when $r^1 < r^{*1}$. Still, we don't know how small this r^{*1} may be. We will find out a relationship between the size of r^{*1} (hence, the likelihood of underemployment equilibria) and that of an economy, in terms of the number of traded commodities. We will content ourselves with computing this likelihood only for a restricted class of economies. We consider economies where the number of agents is bigger than the number of goods: $m \geq l$.

We know that any economy in this class has a unique walrasian equilibrium price p, and we have $r^{*j}(e,a) = \hat{r}^j(e,a,p)$.

Lemma 6. Given an economy (e,a), for all good $j \in L$,

$$r^{*j}(e,a) \geq 1 - a^j_{h(j)} .$$

Proof. See the Appendix.

The question we are interested in is the following. Fix l and m, $m \geq l$, fix a rate $\rho \in (0,1)$, arbitrarily close to 1 and assume that the economy (e,a) is chosen "at random". What are the chances that there exists an underemployment equilibrium (r,p) such that for all $j \in L$, $r^{*j}(e,a) \geq \rho$, i.e., an underemployment equilibrium that involves beliefs r which are arbitrarily close to one?

We must first make clear what we mean by a random economy (e,a). The result we are going to prove does not depend on the choice of e. But as for utility functions, we assume that they are uniformly distributed over the simplex S^o_l: a consumer's utility function is characterized by an a priori equally-likely vector a. Therefore the event $S_j^{\geq\alpha} = \left\{ a \in S^o_l \mid a^j_{h(j)} \geq \alpha \right\}$, for all j, has a probability equal to its relative Lebesgue measure in the simplex, i.e.

$$P\left\{ a \in S^o_l \mid a^j_{h(j)} \geq \alpha \right\} \equiv P\left\{ S_j^{\geq\alpha} \right\} = \mu\left\{ S_j^{\geq\alpha} \right\} = \mu\{S^o_l\}$$

where μ stands for the Lebesgue measure.

Lemma 7. Take any threshold $\alpha \in (0,1)$. Then for all j,

$$P\left\{ S_j^{\geq\alpha} \right\} \leq (1-\alpha)^{l-1} .$$

Proof. See the Appendix.

The main step towards our result is provided by the following lemma.

Lemma 8. Take a $\gamma \in (0,1)$ and a random economy (e,a). The probability that there exists an underemployment equilibrium (p,r) such that for all $j \in L$,

$$r^{*j}(e,a) > 1 - \frac{1}{(l-1)^{\gamma}}$$

is greater than

$$1 - l\left(\frac{1}{e}\right)^{(l-1)^{1-\gamma}}:$$

with e the Neperian number.

Proof. We only need to show that

$$P\left\{S_j^{\leq\alpha}\right\} \leq \left(\frac{1}{e}\right)^{(l-1)^{1-\gamma}}$$

for all j, where $\alpha = 1/(l-1)^\gamma$: Indeed, by Lemma 7 $a^j_{h(j)} < \frac{1}{(l-1)^\gamma}$ implies $r^{*j}(e;a) \geq 1 - a^j_{h(j)} > 1 - \frac{1}{(l-1)^\gamma}$, and

$$P\left\{\cup_j S_j^{\leq\alpha}\right\} \leq l\left(\frac{1}{e}\right)^{(l-1)^{1-\gamma}}$$

in turn, implies the result:

$$P\left\{\cap_j \left(S_j^{\leq\alpha}\right)^c\right\} > 1 - l\left(\frac{1}{e}\right)^{(l-1)^{1-\gamma}}$$

Take the threshold value α in Lemma 7 equal to $\alpha(l) = 1/(l-1)^\gamma$. We have

$$P\left\{S_j^{\leq\alpha}\right\} \leq \left(1 - \frac{1}{(l-1)^\gamma}\right)^{l-1} = \left\{\left(1 - \frac{1}{(l-1)^\gamma}\right)^{(l-1)^\gamma}\right\}^{(l-1)^{1-\gamma}}$$

but we know that the sequence

$$\left(1 - \frac{1}{(l-1)^\gamma}\right)^{(l-1)^\gamma}$$

is increasing and tends towards $1/e$. Hence the lemma. ■

We can conclude this section with the result we were looking for.

Theorem 5.1. *For any ρ arbitrarily close to 1, there is an l large enough such that $\rho < 1 - \frac{1}{(l-1)^\gamma}$ and*

$$P\left\{\cap_j \left\{r^{*j}(e;a) > 1 - \frac{1}{(l-1)^\gamma}\right\}\right\} > 1 - l\left(\frac{1}{e}\right)^{(l-1)^{1-\gamma}}$$

Proof. Obvious from Lemma 8.

In words, the theorem says that for sufficiently large economies, with a number of households not less than the number of traded commodities, and with some specialization, it is likely that an equilibrium will display underemployment. Note that underemployment obtains even if agents are confident they can sell almost all their endowment. We conjecture that this result can be generalized to preferences other than log-linear.

5.2. Dynamic properties of equilibria

We now consider economies where the number of agents is equal to the number of commodities, or $m = l$. We focus on economies which are specialized in the sense of the previous subsection. Let $L = \{1, 2, \dots, l\}$ denote the set of goods. A vector of endowments is $e = (e_h)_{1 \le h \le l} \in R_+^{l^2}$ such that $e_h = (0, \dots, 0, e^h, 0, \dots, 0) \in R_+^l$.

Although so far we have considered only homogeneous expectations of rationing across agents, we could examine heterogenous expectations as well. This is natural in the context of specialized economies. In this setup, at date t agent h has expectations $r^{h,t-1}$ of his ability to sell his endowment of only good h. Sellable percentages associated with other goods don't matter, because the agent is not endowed with those goods. Then the distinction between heterogenous and homogenous expectations becomes irrelevant.

We are going to look at the stability properties of the equilibrium defined by equations (2.8) and (2.9). Note that these two equations define a dynamic process in r, taking r^{t-1} to r^t. Any long-run equilibrium sellable percentage $r < \bar{r}$ can be perturbed and we study whether it will converge back to an equilibrium, even if not the same. The perturbation can be thought as a change in the expectations of rationing each agent has at time t, $r^{h,t-1}$.

As a preliminary step, define the two square matrices $A = (a_h^j)_{1 \le h,j \le l}$ and $A' = (a_h'^j)_{1 \le h,j \le l}$, where $a_h'^j = a_h^j/(1 - a_h^h)$ for $h \neq j$, and $a_h'^h = -1$. It is immediate to check that A is a Markov matrix and 1 is an eigenvalue of A associated to a one-dimensional characteristic subspace. Then A' has rank $l - 1$. To see this, denote by $\tilde{A}$ the diagonal matrix $\mathrm{Diag}(1 - a_1^1, 1 - a_2^2, \dots, 1 - a_l^l)$. We straightforwardly have $A'\tilde{A} = A - I$, where I is the identity matrix. Then $A' = (A - I)\tilde{A}^{-1}$. Let $q = (q^1, \dots, q^l)$ be a vector in the kernel of A' and with first component equal to one and let $v = (v^1, \dots, v^l) \in R_{++}^l$ be an eigenvector of A associated with the unit eigenvalue. Then v and the vector $((1 - a_1^1)^{-1}q^1, \dots, (1 - a_l^l)^{-1}q^l)$ are linearly dependent, and q is in the kernel of A' if and only if $(A - I)(\tilde{A}^{-1}q) = 0$. This is equivalent to saying that the vector $v = \tilde{A}^{-1}q$ is an eigenvector of A associated with the unit eigenvalue.

Theorem 5.2. Consider the dynamic process given by equations (2.8) and (2.9), where the initial expectation of agent h is $r^{h,0}$. Then the sequence $\{r^{h,t}\}_{t=0}^{\infty}$ converges to an equilibrium vector of sellable percentage $(r^1, r^2, \dots, r^l)$ such that:

$$\sum_{j=1} \frac{q^j}{r^j} = \sum_{j=1} \frac{q^j}{r^{j,0}} \tag{5.1}$$

with $q^1 = 1$ and $q \gg 0$.

Proof. See the Appendix.

The point of the preceding proposition is that the hypersurface R_K, in R_{++}^l

defined by the equation

$$\sum_{j=1} \frac{q^j}{x^j} = K$$

is a set of initial beliefs that lead to the same outcome, i.e., the same equilibrium sellable percentage. Choose any equilibrium sellable percentage vector $r < \bar{r}$. R_K being a strictly convex hypersurface, the size of the set of "perturbed" sellable percentages vectors r^0 lying in a neighborhood of r which leads to worse underemployment is greater than the size of those perturbed beliefs leading to more optimistic outcomes.

6. Appendix

Proof of Lemma 1

We will drop the superscript t unless it is not obvious from the context. We will prove the result for $c(p^t) = p^t$. First notice that, if $r \geq 1$, there is nothing to prove. The statement of the Lemma will then follow by continuity. For each consumer h, given that $r_h \leq r$, the maximization problem (2.1) is now

$$\begin{array}{rl} & \max_{y_h^t; r_h^t} u_h(y_h^1 + (1 - r_h) \circ e_h) \\ \text{s.t.} & p y_h^1 = p(r_h \circ e_h) \\ & y_h^1 \geq 0 \end{array}$$

Note that y_h^0 can be fixed arbitrarily. We claim that $r_h^* = r \wedge 1$ is always an optimal choice in this range. Indeed, suppose that $\left(\bar{y}_h^1; \bar{r}_h\right)$ is optimal and $\bar{r}_h < r \wedge 1$. Consider the bundle $z_h = \bar{y}_h^1 + (r \wedge 1 - \bar{r}_h) \circ e_h$. It is feasible with respect to the constraint set

$$p y_h^1 = p(r \wedge 1) \circ e_h \quad \text{and} \quad y_h^1 \geq 0$$

Consider the optimal choice $\hat{y}_h^1$ associated with $r \wedge 1$. Then either $\hat{y}_h^1 = z_h$, or

$$u_h(\hat{y}_h^1 + (1 - r \wedge 1) \circ e_h) > u_h(z_h + (1 - r \wedge 1) \circ e_h) = u_h(\bar{y}_h^1 + (1 - \bar{r}_h) \circ e_h)$$

a contradiction to the optimality of $\left(\bar{y}_h^1; \bar{r}_h\right)$.

Given that $r_h \not\leq r$, the maximization problem (2.1) is now

$$\begin{array}{rl} & \max_{y_h^t; r_h^t} u_h(y_h^1 + (1 - r_h) \circ e_h) \\ \text{s.t.} & p y_h^0 = p(r \wedge r_h) \circ e_h \\ & y_h^0 \geq 0 \end{array}$$

with y_h^1 arbitrary. We claim that the choice of r_h in this range always gives lower utility than $r_h^* = r \wedge 1$. Assume that instead r_h^0 is such that

$$r \wedge r_h^0 = \begin{pmatrix} r_h^{0(1)} \\ r^{(2)} \end{pmatrix}$$

This means that r_h^0 is partly above, and partly below the rationing rule r. We have assumed that the first part of the vector is below, without loss of generality. Let $\hat{y}_h^0$ be the optimal choice associated with this choice of r_h. Let $z_h = \hat{y}_h^0 + (r \wedge 1 - r \wedge r_h^0) \circ e_h$. It is easy to observe that z_h is feasible with respect to the constraint set

$$p y_h = p(r \wedge 1) \circ e_h \quad \text{and} \quad y_h \geq 0$$

and, we observe that

$$u_h(\bar{y}_h + (1 - r \wedge 1) \circ e_h) \geq u_h(z_h + (1 - r \wedge 1) \circ e_h) = u_h(\hat{y}_h^0 + (1 - r \wedge r_h^0) \circ e_h)$$

but then by monotonicity

$$u_h(\hat{y}_h^0 + (1 - r \wedge r_h^0) \circ e_h) > u_h(\hat{y}_h^0 + (1 - r_h^0) \circ e_h)$$

Hence the result stated in the lemma. ■

Proof of Lemma 3.

We compute the walrasian equilibrium associated with this economy. From problem (2.5), we find that the demand function is given by

$$x_h^j = \frac{1}{l}\left(\frac{\epsilon \sum_{j \neq h} p^j + p^h}{p^j}\right) \tag{6.1}$$

and the associated multiplier is

$$\lambda_h = 1/(\epsilon \sum_{j \neq h} p^j + p^h). \tag{6.2}$$

Using condition (2.10), we get

$$\sum_{h=1}^{l} [\epsilon(\sum_{j \neq h} p^j) + p^h] - l(1 + (l-1)\epsilon)p^j + (m-l) \sum_i p^i - (m-l) = 0 \tag{6.3}$$

which can be written as

$$Ap^n = -1 \tag{6.4}$$

with $p^n = \left(p^j\right)_{j \neq 1}$ and

$$A = \begin{bmatrix} 1-l & 1 & \cdots & 1 \\ 1 & 1-l & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & 1-l \end{bmatrix}$$

an $l-1$ square matrix. One can easily check that A has full rank, given $l > 1$. Therefore, the solution to system $Ap^n = -\underline{1}$ is unique. In fact, as it is easy to verify, $p^n = \underline{1}$ is the solution. Observe that the full rankness of A implies that the Walrasian equilibrium is regular in the usual sense. ■

Proof of Lemma 4.

i) Take $\epsilon = 0$ and consider problem (2.11). We will consider the function argmax $b_h^j(\epsilon)$ with domain $\mathbb{R}^l_+$, as opposed to $\mathbb{R}^l_{++}$, but the two coincide on this last domain. By assumption on u, the optimal constrained choice b_h must satisfy $b_h^j(0) > 0 = \epsilon$, for all $j \neq h$. It is easy to check that $b_h^j(\epsilon)$ is continuous in ϵ. Hence there exists an $\epsilon > 0$ small enough so that $b_h^j(\epsilon) - \epsilon > 0$. Then $b_h^j(\epsilon) > (1-r^j)\epsilon$, for all $r^j \leq 1$.

ii) Suppose not. Consider first the case when $l = 2$. Then $b_h^h > 1 - r^h$, and the indifference curve is not tangent to the budget line and there is a feasible point b_h' such that $u_h(b_h') > u_h(b_h)$, a contradiction. More generally, if $l > 2$, and $b_h^h > 1 - r^h$, by point i) b_h lies in the relative interior of the constraint set. Take an open set O, relative to the constraint set, around b_h, and consider the intersection of an indifference surface going through b_h with O. Then we can find another feasible point b_h' such that $u_h(b_h') = u_h(b_h)$. By strict convexity of preferences, there is another feasible point $b_h'' = ab_h + (1-a)b_h'$ in this open set O such that $u_h(b_h'') > u_h(b_h)$, contradicting the optimality of b_h. ■

Proof of Lemma 5.

We are going to show that the following is the unique regular equilibrium

$$x_h^j = \begin{cases} \frac{(l-1)\epsilon^2+1}{l} & \text{for } h \le l \\ 1 & h > l \end{cases}$$

$$\lambda_h = \begin{cases} 1/[(l-1)\epsilon^2+1] & \text{for } h \le l \\ 1/l & h > l \end{cases}$$

$$p = 1$$

$$b_{h\le l}^j = \begin{cases} 1-\gamma & \text{if } j = h \\ \epsilon^2 + \gamma/(l-1) & j \ne h \end{cases}$$

$$b_{h>l}^j = 1$$

$$\mu_h = \begin{cases} (l-1)/l[(l-1)\epsilon^2+\gamma)] & \text{for } h \le l \\ 1/l & h > l \end{cases}$$

$$\alpha_h^j = \begin{cases} \frac{(l-1)(1-\epsilon^2)-l\gamma}{l(1-\gamma)[(l-1)\epsilon^2+\gamma]} & \text{if } j = h \\ 0 & j \ne h; \text{and } h \le l \\ 0 & h > l; \text{ all } j \end{cases}$$

$$r^h = \gamma 1.$$

We need to distinguish between two cases.

Case a. $r^j \le \bar{r}^j(e;u;p) = \frac{(l-1)(1-\epsilon^2)}{l}$; for $j \ne 1$.

Case b. There exists j such that $r^j > \bar{r}^j(e;u;p)$.

We will show that there is only an equilibrium in case a).

Case a. Going back to the demand functions (6.1) of Lemma 3 we can now compute the demand at a walrasian equilibrium of the first l consumers:

$$x_h^j = \frac{\epsilon^2(l-1)+1}{l} \tag{6.5}$$

and the corresponding $\bar{r}^j(e;u;p) = \frac{l-1}{l}(1-\epsilon^2) > 0$ for all j. Then our choice of $r^1 = \gamma < \bar{r}^1(e;u;p)$ makes the equilibrium we are going to find an underemployment equilibrium.

From Lemma 4.ii, we know that $b_h^j = 1 - r^j$ for $j = h$; all $h \le l$ and $r^j < \bar{r}^j(e;u;p)$. Moreover, we know from Lemma 4.i that, assuming our choice of ϵ is sufficiently small, $b_h^j > (1-r^j)\epsilon^2$ for $j \ne h$; again for all $h \le l$ and $r^j < \bar{r}^j(e;u;p)$. If $h > l$; then $b_h^j = x_h^j$, for all j. Assuming that $r < \bar{r}(e;u;p)$; the constrained maximization problem (2.11) can be written as

$$\max \frac{1}{l} \sum_{j\ne h} \log b_h^j + \frac{1}{l}\log(1-r^h); \text{ s.t.}$$
$$\sum_{j\ne h} (b_h^j - \epsilon^2) = r^h$$
$$b_h^j > (1-r^j)\epsilon^2; \text{ for } j \ne h$$

or equivalently

$$\max \frac{1}{l} \sum_{j\ne h} \log b_h^j; \text{ s.t.}$$
$$\sum_{j\ne h} b_h^j = (l-1)\epsilon^2 + r^h$$
$$b_h^j > (1-r^j)\epsilon^2; \text{ for } j \ne h$$

The first order conditions give

$$b_h^j = \frac{r^h + \epsilon^2(l-1)}{l-1}$$

which is greater than $(1-r^j)\epsilon^2$; and multipliers

$$\mu_h = (l-1)/l[(l-1)\epsilon^2 + r^h].$$

One can check that the above values satisfy the Kuhn-Tucker conditions of the true problem for the appropriate choice of the multipliers α for the nonnegativity constraints.

Once again, using the market clearing conditions (2.12) we solve for r^n:

$$\sum_{h \neq j} [r^h + \epsilon(l-1)] + (1-l) r^j = (l-1)^2 \epsilon$$

These can be rewritten as

$$\begin{bmatrix} 1 & A \end{bmatrix} r = 0$$

or as

$$A r^n = -\mathbf{1} r^1 \tag{6.6}$$

Since A has full rank, we get a unique solution for each $r^1 < \bar{r}^1$. It is easy to check that the solution here is $r^n = \gamma \mathbf{1}$ when $r^1 = \gamma$.

Note that this establishes "regularity" of equilibrium. Consider the aggregate excess demand function $\hat{\Phi}(p;r) = 0$ defined by (6.4) and (6.6). We show that $D\hat{\Phi}$ has full rank, since after a simple computation we see that

$$\begin{array}{ccc} & p^n & r^n \\ l-1 & A & 0 \\ l-1 & * & A \end{array}$$

and this clearly has full rank.

Case b.

We are to show that there is no equilibrium corresponding to this case. Without loss of generality, assume that

$$r^j \begin{cases} < \frac{(l-1)(1-\epsilon)}{l} & \text{if} \quad j = 1, \ldots, l_1 \\ \geq \frac{(l-1)(1-\epsilon)}{l} & \text{if} \quad j = l_1 + 1, \ldots, l_1 + l_2 \equiv l \end{cases}$$

Observe that household 1 is one of the first l_1 households, because we chose an economy with $r^1 < \bar{r}^1$. Given the above assumption on r^j, for each $h = l_1 + 1, \ldots, l$

$$b_h^j = \begin{cases} \frac{\epsilon(l-1)+\gamma}{l-1} & \text{if } j \neq h; \\ 1-\gamma & \text{if } j = h \end{cases}$$

is not a solution to the household's maximization problem anymore. Intuitively, the walrasian solution should work for this "unconstrained" household. This is in fact the case.

For $h = l_1 + 1, \ldots, l$,

$$\begin{aligned} \mu_h &= \frac{1}{\epsilon(l-1)+1} \\ b_h^j &= \frac{\epsilon(l-1)+1}{l} \\ \alpha_h^j &= 0; \ \text{for } j = 1, \ldots, l \end{aligned}$$

is the solution to the Kuhn-Tucker conditions.

$$
\begin{array}{lll}
\frac{1}{l b_h^j} - \mu_h p^j + \alpha_h^j = 0 & \text{for } j = 1, \ldots, l & \text{(k1)} \\
-\sum_{h=1}^{l} b_h^j + \varepsilon (l-1) + 1 = 0 & & \text{(2)} \\
\alpha_h^j \geq 0 & \text{for } j \neq h & \text{(k2)} \\
b_h^j - \left(1 - r^j\right) \varepsilon \geq 0 & \text{for } j \neq h & \text{(k3)} \\
\alpha_h^j \left(b_h^j - \left(1 - r^j\right) \varepsilon\right) = 0 & \text{for } j \neq h & \text{(k4)} \\
\alpha_h^h \geq 0 & & \text{(i2)} \\
b_h^h - \left(1 - r^h\right) \geq 0 & & \text{(i3)} \\
\alpha_h^h \left(b_h^h - \left(1 - r^h\right)\right) = 0 & & \text{(i4)}.
\end{array}
$$

In particular, note that equation (k3) holds since we have assumed $\varepsilon \in (0,1)$; and equation (i3) holds since

$$
\begin{aligned}
& \frac{\varepsilon(l-1)+1}{l} - \left(1 - r^i\right) \geq 0 \\
\varepsilon l - \varepsilon + 1 - l + l r^j &= \varepsilon (l-1) - (l-1) + l r^j = \\
= -(1-\varepsilon)(l-1) + l r^k &\geq 0 \iff r^k \geq \frac{(l-1)(1-\varepsilon)}{l}
\end{aligned}
$$

as we assumed.

Summarizing, we have that for $h = 1, \ldots, l_1$,

$$
b_h^j = \begin{cases} \frac{\varepsilon(l-1)+r^h}{l-1} = \varepsilon + \frac{r^h}{l-1} & \text{if } j \neq h; \\ 1 - r^h & \text{if } j = h \end{cases}
$$

and for $h = l_1 + 1, \ldots, l$,

$$
b_h^j = \frac{\varepsilon(l-1)+1}{l} = \varepsilon + \frac{1-\varepsilon}{l}.
$$

We will show that the market clearing conditions do not have a solution.

For goods $j = 1, \ldots, l_1$ we have:

$$
(1-l) r^j + \sum_{i=1,\ldots,l_1; i \neq j} r^i = \frac{(1-l)(1-\varepsilon) l_2}{l}.
$$

And for goods $j = l_1 + 1, \ldots, l$:

$$
\sum_{i=1,\ldots,l_1} \left(\varepsilon + \frac{r^i}{l-1}\right) + l_2 \left(\varepsilon + \frac{1-\varepsilon}{l}\right) - 1 - (l-1)\varepsilon = 0
$$

is

$$
\sum_{i=1,\ldots,l_1} r^i = -\frac{(1-l)(1-\varepsilon) l_1}{l}.
$$

Note that all these equations are equal, and we can keep only one of them. To sum up, we get

$$
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1-l & 1 & \cdots & 1 \\
\cdots & \cdots & \cdots & \cdots \\
1 & 1 & \cdots & 1-l \\
1 & 1 & \cdots & 1
\end{bmatrix}_{(l_1+1)(l_1-1)}
\begin{pmatrix} r^2 \\ r^3 \\ \cdots \\ r^{l_1} \end{pmatrix}
=
\begin{pmatrix} \beta \\ \gamma \\ \cdots \\ \gamma \\ \delta \end{pmatrix}
\tag{6.7}
$$

with

$$
\begin{aligned}
\beta &\equiv \frac{(1-l)(1-\varepsilon)l_2}{l} - (1-l)r^1\\
\gamma &\equiv \frac{(1-l)(1-\varepsilon)l_2}{l} - r^1\\
\delta &\equiv -\frac{(1-l)(1-\varepsilon)l_1}{l} - r^1
\end{aligned}
$$

since r^1 is a parameter. (6:7) is a system of l_1+1 equations in l_1-1 unknowns, that we denote by $Br^n = \eta$. We can show that rank $B \neq$ rank $[B|\eta]$. Indeed, rank $B = l_1 - 1$. On the other hand,

$$
\text{rank } [B|\eta] = \text{rank}\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & \beta\\
1-l & 1 & 1 & \cdots & 1 & \gamma\\
1 & 1-l & 1 & \cdots & 1 & \gamma\\
1 & 1 & 1-l & \cdots & 1 & \gamma\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
1 & 1 & 1 & \cdots & 1-l & \gamma\\
1 & 1 & 1 & \cdots & 1 & \delta
\end{bmatrix}_{(l_1+1)\times l_1} \geq
$$

$$
= \text{rank}\begin{bmatrix}
-l & 0 & 0 & \cdots & 1 & \gamma\\
0 & -l & 0 & \cdots & 1 & \gamma\\
0 & 0 & -l & \cdots & 1 & \gamma\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \cdots & -l_2-1 & (l_1-1)\gamma\\
0 & 0 & 0 & \cdots & 1 & \delta
\end{bmatrix}_{l_1\times l_1} =
$$

$$
= l_1 - 2 + \text{rank}\begin{pmatrix} -l_2-1 & (l_1-1)\gamma\\ 1 & \delta \end{pmatrix}
$$

after some matrix manipulation and using Walras' law to erase the first row. Finally,

$$
\text{rank}\begin{pmatrix} -l_2-1 & (l_1-1)\gamma\\ 1 & \delta \end{pmatrix} = 2 \text{ iff } (l_2+1)\delta + (l_1-1)\gamma \neq 0
$$

but $(l_2+1)\left(-\frac{(1-l)(1-\varepsilon)l_1}{l} - r^1\right) + (l_1-1)\left(\frac{(1-l)(1-\varepsilon)l_2}{l} - r^1\right) < 0$ given our choice of r^1. ■

Proof of Lemma 6.

Fix a good $j \in L$. From the first order conditions of consumer $h(j)$'s maximization program we have, for all $k \in L$:

$$
x^k_{h(j)} = \frac{a^k_{h(j)}}{\lambda_{h(j)} p^k}
$$

where $\lambda_{h(j)}$ is consumer $h(j)$'s Lagrange multiplier. From his budget constraint we have

$$
\lambda_{h(j)} p^j e^j_{h(j)} = 1
$$

We then obtain

$$
\frac{x^j_{h(j)}}{e^j_{h(j)}} = \frac{a^j_{h(j)}}{\lambda_{h(j)} p^j e^j_{h(j)}} = a^j_{h(j)}
$$

By definition of $r^{*j}(e;a) = \hat{r}^j(e;a;p)$,

$$
r^{*j}(e;a) = 1 - \min_{1\leq h\leq m} \frac{x^j_h}{e^j_h} \geq 1 - \frac{x^j_{h(j)}}{e^j_{h(j)}} = 1 - a^j_{h(j)}
$$

as we were supposed to prove. ■

Proof of Lemma 7.

Let $S_j^{\lambda,\alpha} = \left\{ a \in S_l^o \mid a^j_{h(j)} \geq \alpha \right\}$. We consider the orthogonal projection π_l on the hyperplane $x_l = 0$. The probability $P\left(S_j^{\lambda,\alpha}\right)$ is the ratio between the Lebesgue measures of the sets $\pi_l(S_j^{\lambda,\alpha})$ and $\pi_l(S_l^o)$ respectively.

The sets $\pi_l(S_l^o)$ is defined in the (positive orthant of the) hyperplane $x_l = 0$ by the inequality $\sum_{k=1}^{l-1} a^k_{h(j)} \leq 1$. Its subset $\pi_l(S_j^{\lambda,\alpha})$ is defined by the inequality

$$\sum_{k=1;k\neq j}^{l-1} a^k_{h(j)} + \tilde{a}^j_{h(j)} \leq 1 - \alpha$$

where $\tilde{a}^j_{h(j)} = a^j_{h(j)} - \alpha \geq 0$.

It is a simple exercise to check that the Lebesgue measure (in R^l_+) of the subset defined by the inequality

$$\sum_{k=1}^{l-1} x^k \leq y$$

is $y^{l-1}/(l-1)!$. Hence the lemma. ■

Proof of Theorem 5.2

Consider the case where, for all h, $1 \leq h \leq l$, agent h expects at date t a sellable percentage $r^{h,t-1}$ which is actually binding; i.e., $y_h^h = 0$. We know that a necessary and sufficient condition for $r^{h,t-1}$ to be binding is $r^{h,t-1} < 1 - a_h^h$. And if the initial expectations are binding for each agent, then they are binding at all steps of the dynamic process. Then problem (2.7) for each agent becomes

$$\max_{y_h} \sum_{j=1;j\neq h}^{l} a_h^j \log(y_h^j) \quad s.t. \sum_{j=1;j\neq h}^{l} p^{j,t} y_h^j = p^{h,t} r^{h,t-1} e^h$$

Its solution is, for $j \neq h$,

$$y_h^j = \frac{a_h^j p^{h,t} r^{h,t-1} e^h}{p^{j,t}(1 - a_h^h)}$$

and then equation (2.8) for good j is

$$\sum_{h=1} y_h^j = \frac{1}{p^{j,t}} \sum_{h=1;h\neq j} \frac{a_h^j}{1 - a_h^h} p^{h,t} r^{h,t-1} e^h = r^{j,t-1} e^j$$

for all j. Let $q^{h,t} = p^{h,t} r^{h,t-1} e^h$, for all h, $1 \leq h \leq l$. Then the above market clearing equation can be written as

$$A' q^t = 0$$

q^t defines the market clearing price p^t. With the normalization $p^{1,t} = 1/r^{1,t-1}e^1$, such that $p^{j,t} = q^{j,t}/(r^{j,t-1}e^j)$; $q^{1,t} = 1$ and $q^t \gg 0$. Since all the vectors in the kernel of A' are colinear, and we have normalized each q^t, we have $q^t = q$. At the market clearing price p^t, the aggregate walrasian demand of good j is

$$\sum_{h=1} x_h^j = \sum_{h=1} a_h^j \frac{p^{h,t}}{p^{j,t}} e^h = \sum_{h=1} a_h^j \frac{q^h r^{j,t-1} e^j}{q^j r^{h,t-1} e^h} e^h = r^{j,t-1} e^j \sum_{h=1} a_h^j \frac{q^h}{q^{j,t} r^{h,t-1}}$$

Then equation (2.9) is, for all j, $1 \leq j \leq l$,

$$r^{j,t} = \frac{r^{j,t-1} e^j}{r^{j,t-1} e^j \sum_{h=1}^{l} a_h^j \frac{q^h}{q^j r^{h,t-1}}}$$

which is equivalent to

$$\frac{q^j}{r^{j,t}} = \sum_{h=1}^{l} a_h^j \frac{q^h}{r^{h,t-1}}$$

such that, if $v^{j,t}$ denotes $q^j/r^{j,t}$, we obtain for all j, $1 \leq j \leq l$,

$$v^{j,t} = \sum_{h=1}^{l} a_h^j v^{h,t-1}$$

which can be rewritten as $v^t = Av^{t-1}$.

We first have to prove that if $r^{h,t-1}$ is binding then so is $r^{h,t}$; in other words, we have to prove that if $r^{h,t-1} < 1 - a_h^h$, for all h, then $r^{h,t} < 1 - a_h^h$, for all h. Suppose $r^{h,t-1} < 1 - a_h^h$, for all h. Then we have, for all j,

$$\frac{1}{r^{j,t}} = \sum_{h=1}^{l} a_h^j \frac{q^h}{q^j r^{h,t-1}} > \sum_{h=1}^{l} a_h^j \frac{q^h}{q^j (1 - a_h^h)} = \frac{1}{q^j} \sum_{h=1}^{l} \frac{a_h^j}{1 - a_h^h} q^h ;$$

but we know that

$$\sum_{h=1; h \neq j}^{l} \frac{a_h^j}{1 - a_h^h} q^h = q^j$$

so that we obtain, for all j,

$$\frac{1}{r^{j,t}} > \frac{1}{q^j} \left(q^j + \frac{a_j}{1 - a_j} q^j \right) = \frac{1}{1 - a_j}$$

It is a standard result of dynamic systems that the sequence $\{v^t\}$ converges toward the vector $v = (v^j)_{1 \leq j \leq l}$ of the characteristic space associated with the eigenvalue 1 of A whose components satisfy

$$\sum_{j=1}^{l} v^j = \sum_{j=1}^{l} v^{j,t}$$

■

References

[1] Balasko, Y. and Royer, D.: Effective Demand Failures Revisited. Scandinavian Journal of Economics 87: 521-536 (1985).

[2] Bénassy, J.P.: The Economics of Market Disequilibrium. Academic Press, New York 1982.

[3] Diamond, P.: Aggregate Demand Management in Search Equilibrium. Journal of Political Economy 90: 881-894 (1982).

[4] Drèze, J.: Underemployment Equilibria. Cambridge University Press, Cambridge 1991.

[5] Fudenberg, D. and Levine, D.K.: Self-Confirming Equilibrium. Econometrica 61: 523-545 (1993).

[6] Grandmont, J.M.: Temporary General Equilibrium Theory. In: Handbook of Mathematical Economics, K.J. Arrow and M.D. Intriligator eds.. Vol II, Ch.19 North-Holland, Amsterdam 1982.

[7] Hahn, F.: Unsatisfactory Equilibria. IMSSS Technical Report # 247, Stanford University 1977.

[8] Hahn, F.: On Non-Walrasian Equilibria. Review of Economic Studies 45: 1-17 (1978).

[9] Hart, O.: A Model of Imperfect Competition with Keynesian Features. Quarterly Journal of Economics 98: 109-138 (1982).

[10] Heller, W.: Coordinating Failure under Complete Markets with Applications to Effective Demand. In: Equilibrium Analysis: Essays in Honor of Kenneth Arrow, vol 2, W. Heller, R. Starr and D. Starrett eds., 1986.

[11] Laroque, G. and Polemarchakis, H.: On the Structure of the Set of Fixed Price Equilibria. Journal of Mathematical Economics 5: 53-69 (1978).

[12] Lloyd, N.G.: Degree Theory. Cambridge University Press, Cambridge 1978.

[13] Malinvaud, E.: The Theory of Unemployment Reconsidered. Basil Blackwell, Oxford 1977.

[14] Silvestre, J.: Continua of Hahn-Unsatisfactory Equilibria. Economics Letters 5: 201-208 (1980).

[15] Smale, S.: Global Analysis and Economics IIA. Journal of Mathematical Economics 1: 1-14 (1974).

[16] Younès, Y.: On the Role of Money in the Process of Exchange and the Existence of a Non-Walrasian Equilibrium. Review of Economic Studies 42: 489-501 (1975).