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Generic Existence of Sunspot Equilibria: The Real Asset Case

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Generic Existence of Sunspot Equilibria: the Real Asset Case[*]

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Abstract

This paper establishes the generic existence of sunspot equilibria in a standard two period exchange economy with real assets. We show that for a generic choice of utility functions and endowments, there exists an open set of real asset structures whose payoffs are independent of sunspots such that the economy with this asset structure has a regular sunspot equilibrium. This result also clarifies the relationship between equilibrium multiplicity and existence of sunspot equilibria. Our technique is very general and can be applied to other frameworks as the overlapping generations model with sunspots.

Keywords: sunspots, incomplete markets, endogenous uncertainty.

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1. Introduction

This paper establishes the generic existence of sunspot equilibria in a standard two period exchange economy with real assets; more precisely, we show that for an open dense subset of the set of utility functions and endowments, there exist real assets whose payoffs are independent of sunspots such that the resulting economy with this asset structure has a regular sunspot equilibrium.

The Cass and Shell (1983) Ineffectivity Theorem says that if (i) the market outcome is assured to be Pareto efficient, and (ii) the economy is convex, then sunspots do not matter since a sunspot equilibrium cannot be Pareto efficient under convexity. The so-called "Philadelphia folk theorem" asserts that the converse of the Cass-Shell theorem is also true; that is, in any "class" of market models where the conditions (i) or (ii) are not satisfied, sunspots matter in general. The subsequent literature examined several classes of models to show that the folk theorem is indeed valid in those models, i.e. that they may have sunspot equilibria: models with incomplete markets (Cass (1989,1992), Siconolfi (1989), Balasko (1990), Mas-Colell (1992), Pietra (1992), Suda-Tallon-Villanacci (1992)[1]), with restricted participation/asymmetric information (Cass-Shell (1983), Maskin-Tirole (1987)), with overlapping generations (Shell (1977), Azariadis (1981), Spear (1984), Guesnerie (1986)), with non-convexities (Cass-Polemarchakis (1989), Guesnerie-Laffont (1988), Shell-Wright (1993)); with imperfect competition (Peck-Shell (1988)); with externalities (Spear (1988), Gali (1994)); with public goods (Goenka (1994a,b).

To make our contribution clear, consider the model of incomplete markets with *nominal* assets. The key observation is that the real returns of assets are indeterminate in this model, and thus effectively one can fix the returns exogenously, depending on sunspots.[2] But then if agents trade assets in equilibrium, the level of the agents' future income must depend on sunspots since the endowments are not affected by sunspots, and consequently the equilibrium allocation must depend on sunspots. Thus the existence of sunspot equilibria is automatically ensured by the existence of a competitive equilibria with non-zero asset trades, which is typically the case. Therefore, generic existence of sunspot equilibria is not surprising in models with sunspot dependent assets and/or nominal assets.[3]

[1]This, and the ones which follow, is only a very incomplete lists of references.

[2]This is the essense of the Cass example. See Cass (1989).

[3]The focus of the work on sunspot equilibria with nominal assets (Cass (1992), Pietra (1992),

We consider the case where assets are real and payoffs are independent of sunspots, which is more faithful to the spirit of sunspot analysis. The model does not possess any other immediate source of complication that might help us to generate sunspot equilibria; we consider an economy with finitely many periods, finitely many households, and no restricted participation. There is no intrinsic uncertainty and a single asset whose returns are independent of sunspot is available. Unlike the case of nominal assets, the determinacy of equilibria will be the rule rather than an exception in our model, which raises the following difficulty: we cannot move away from a non-sunspot equilibrium - an equilibrium where households ignore the sunspot signal - by choosing exogenous yield conveniently as in the case of nominal assets. This property is shared in other models listed above where there are no convenient extra variables which virtually make some fundamentals sunspot dependent. Probably due to this technical difficulty, the only available results known to us concern either robust examples, or some constructive argument, which is valid only on a non-generic set of economies. So our generic result is qualitatively different from the previous results.

Our result clarifies the long-standing issue of the relationship between equilibrium multiplicity and existence of sunspot equilibria; the multiplicity of non-sunspot equilibria is not necessary for the existence of sunspot equilibria. To see this, start with an economy which has a unique, regular non-sunspot equilibrium. For instance, consider the case where the initial endowments are Pareto efficient. If the economy is slightly perturbed, then regularity implies that there is a unique non-sunspot equilibrium in the perturbed economy. But if the asset structure is chosen appropriately as in our result, then the economy will also have a regular sunspot equilibrium. That is, the perturbed economy has a robust sunspot equilibrium, as well as a *unique* non-sunspot equilibrium. Of course, since real payoffs of the assets are independent of sunspots, the spot market where households trade the initial endowed goods *and* returns from the assets must have multiple equilibria, but in general none of them are part of the unique non-sunspot equilibrium.

We should emphasize that it is *not* the case that for a given asset structure, the existence of sunspot equilibria is generic. Consider again the case where the initial endowments are Pareto efficient, and fix a non-degenerate asset structure arbitrary. It is then readily verified that the unique equilibrium (no-trade equilibrium) is a regular equilibrium even if sunspot states are taken into account, so

Suda-Tallon-Villanacci (1992)) is therefore the dimension of the set of sunspot equilibria, rather than the existence of a sunspot equilibrium.

by the implicit function theorem, there is an open neighborhood of the economy in which there is a unique equilibrium. Then the unique equilibrium must not depend on sunspots, since there is at least one non-sunspot equilibrium. This observation does not contradict our result; the open neighborhood of the economy in question depends on the asset structure. Our result implies that the neighborhood can be made arbitrarily small by changing the asset structure.

The idea of our proof is simple. Just to exhibit that sunspots matter, it is enough to consider the case where there are two sunspot states, 1 and 2, in the second period. Rather than directly working on the real asset economy which exhibits the technical difficulty described above, consider an auxiliary economy which is identical to the original economy except for the asset. The asset's real payoffs now *depend* on sunspots, say the return is 1 in state 1 and r in state 2. A sunspot equilibrium will exist by the same reason as the case of nominal assets if we set $r \neq 1$. This does not necessarily mean that the equilibrium prices depend on sunspots (consider homothetic preferences), but generically in utility function, it will be the case that the two spot prices, call them p^1 and p^2, are linearly independent, assuming that there is more than one good in each spot. Then we can find a commodity bundle $\mathbf{a}$ which yields the artificial sunspot dependent payoffs by simply solving the simultaneous linear equations $p^1\mathbf{a} = 1$ and $p^2\mathbf{a} = r$. It is easy to check that the artificial equilibrium allocation is in fact an equilibrium allocation of the economy with a real asset that pays the commodity bundle we found as above. Finally, an extra step is needed to establish the generic regularity of the sunspot equilibrium we have obtained.

The trick of using an auxiliary economy to detect sunspot equilibria is quite general. The point is that the auxiliary model has an extra variable which virtually create sunspot dependence, and there is a way to relate an equilibrium of the auxiliary model to the original model. We conjecture that such an auxiliary model can be constructed for the other models considered in the literature. We shall sketch how it can be done for the overlapping generations model in section 4. Note that in the argument above, the sunspot equilibrium generated in the auxiliary equilibrium can be arbitrary close to a non-sunspot equilibrium by choosing r close to 1. This seems to suggest that we can construct a sunspot equilibrium which is close to a stable stationary equilibrium.

We take the structure of sunspot states as given, but this is a little uncomfortable if we wishes to think of sunspots as a way to model endogenous uncertainty, uncertainty that is not described by the primitives of the model. Section 4 also contains some discussions on this.

In the next section we present the model and state the main result. Section 3 contains the proof of the main result. In section 4 we comment on the results and discuss some extensions.

2. The Model

We consider a competitive two-period exchange economy. We assume that there are 2 sunspot states in the second period, but there is no intrinsic uncertainty. To exhibit the existence of sunspot equilibrium, it is enough to consider the case of two sunspots, since a sunspot equilibrium in this setting can be naturally thought as a sunspot equilibrium of a sunspot economy with arbitrary number of sunspots.[4]

At the beginning of the second period, sunspot $s = 1, 2$ occurs with a publicly known probability $\pi^s > 0$. Spot commodity markets open in the first and second period, and there are $C + 1$ commodities in each spot, labelled by $c = 0, 1, 2, ..., C$. $C \geq 1$ is assumed, i.e., there are at least two goods in each spot.

We label each spot by $s = 0, 1, 2$, spot zero corresponding to the first period. There are H households, labelled by $h = 1, 2, ... H$. Household h receives endowments e_h^0 in the first period and e_h^1 in the second period.

There is one asset which yields a commodity bundle $\mathbf{a}$ in the second period. The supply of the asset is zero. In the first period, commodities and the asset are exchanged and consumption takes place. Commodity 0 is a designated numéraire of the economy. Since we shall deal with an economy with a real asset, there is no essential loss of generality in doing so.

Remark 1. *If the distinction among the sunspot states is ignored, we have a deterministic economy with complete markets. We refer to this as the certainty economy.*

The following summarizes the notation:

- $x_h^{c,s}$ is the consumption of commodity c by household h in state s. $x_h^s \equiv (x_h^{c,s})_{c=0}^C$ is the household's consumption plan in state s, $x_h \equiv (x_h^s)_{s=0}^2$, $x \equiv (x_h)_{h=1}^H$

[4]See section 4, however.

- For a vector $z \in I\!R^{C+1}$, we denote by $z^{\setminus}$ the vector obtained from z by dropping the first element, i.e., the element corresponding to the numéraire. For instance, $x_h^{s\setminus}$ denotes the consumption of non numéraire goods of household h in state s.

- $p^s \in I\!R^C$ is the price vector of non-numéraire commodities in spot s, $p \equiv (p^s)_{s=0}^2$, and $\hat{p}^s = (1, p^s) \in I\!R^{C+1}$;

- The price of the asset is denoted by $q \in I\!R$;

- $b_h \in I\!R$ is the demand for the asset by household h, $b \equiv (b_h)_{h=1}^H$.

- λ_h^s will denote the Lagrange multiplier for household h which arises in household's utility maximization problem, $\lambda = (\lambda_h^0, \lambda_h^1, \lambda_h^2)_{h=1}^H$.

Let $\Xi \equiv I\!R_{++}^{3(C+1)H} \times I\!R_{++}^{3H} \times I\!R^H \times I\!R_{++}^{3C} \times I\!R$ be the space of endogenous variables with typical element $\xi = (x, \lambda, b, p, q)$; set $n \equiv 3(C+1)H + 3H + H + 3C + 1$.

Household h's preferences over consumption plans are represented by the utility function $U_h(x_h) = \sum_{s=1}^2 \pi^s u_h(x_h^0, x_h^s)$.

We assume:

Assumption 1. *u_h is C^2, differentiably strictly increasing (i.e. $\forall x_h$, $Du_h(x_h) \gg 0$), differentiably strictly concave (i.e. $\forall x_h$, $D^2u_h(x_h)$ is negative definite) and with the closure of the indifference surfaces contained in $I\!R_{++}^{2(C+1)}$.*

We shall parameterize economies described above by endowment vectors and utility functions. Let $\mathbf{E} = \left\{ e = (e_h^0, e_h^1)_{h=1}^H \in I\!R_{++}^{2(C+1)H} \right\}$ and $\mathbf{U}$ be the set of all utility functions satisfying assumption 1. $\mathbf{U}$ is endowed with the topology of C^2 convergence on compact sets. An economy is completely characterized by $(e, u) \in \mathcal{E}$, where $\mathcal{E} = \mathbf{E} \times \mathbf{U}$ is endowed with the natural topology, and the asset payoffs $\mathbf{a}$. So, we shall call $((e, u), \mathbf{a})$ a real asset economy, or *economy*.

We consider competitive equilibria with self-fulfilling expectations: (x, b, p, q) is an *equilibrium* of economy $((e, u), \mathbf{a})$ if

(H) for each h, (x_h, b_h) solves the following problem given p and q:

$$\begin{aligned}
&max_{x_h, b_h} U_h(x_h) \\
&\text{subject to} \\
&\hat{p}^0(x_h^0 - e_h^0) + qb_h = 0, \\
&\hat{p}^s(x_h^s - e_h^1) - (\hat{p}^s \mathbf{a})b_h = 0 \text{ for } s = 1, 2,
\end{aligned}$$

(M) markets clear, i.e.:

$$\begin{array}{l}\sum_{h=1}^{H}(x_h^0 - e_h^0) = 0, \\ \sum_{h=1}^{H}(x_h^s - e_h^1) = 0 \text{ for } s = 1, 2, \\ \sum_{h=1}^{H} b_h = 0.\end{array}$$

Definition 2.1. *An equilibrium is a sunspot equilibrium if* $x_h^1 \neq x_h^2$ *for some* h.

Remark 2. *There always exists a non-sunspot equilibrium. Moreover, every non-sunspot equilibrium is Pareto efficient and any Pareto efficient equilibrium is a non-sunspot equilibrium.*

Under Assumption 1, the utility maximization condition can be replaced with the corresponding first order condition. So we can equivalently say $\xi = (x, \lambda, b, p, q)$ is an equilibrium if the following holds: define $\Phi\left(\xi; \left((e, u), \mathbf{a}\right)\right)$ by the rule:

$$\Phi\left(\xi; \left((e, u), \mathbf{a}\right)\right) = \left[\begin{array}{cc} \left.\begin{array}{c} \begin{pmatrix} \vdots \\ \frac{\partial}{\partial x_h^s}\pi^s u_h(x_h^0, x_h^s) - \lambda_h^s \hat{p}^s \\ \vdots \end{pmatrix} \\ \begin{pmatrix} \vdots \\ -\hat{p}^s(x_h^s - e_h^s) + (\hat{p}^s \mathbf{a})\, b_h \\ \vdots \end{pmatrix} \end{array}\right\} & \begin{array}{c}\text{for all } h \\ \text{and } s = 1, 2\end{array} \\ \left.\begin{array}{c} \begin{pmatrix} \vdots \\ \sum_s \pi^s \frac{\partial}{\partial x_h^0} u_h(x_h^0, x_h^s) - \lambda_h^0 p^0 \\ \vdots \end{pmatrix} \\ \begin{pmatrix} \vdots \\ -p^0(x_h^0 - e_h^0) - q b_h \\ \vdots \end{pmatrix} \\ -\lambda_h^0 q + \sum_{s=1}^{2} \lambda_h^s (\hat{p}^s \cdot \mathbf{a}) \end{array}\right\} & \text{for all } h \\ \sum_h x_h^0 - e_h^0 & \\ \sum_h x_h^s - e_h^s & \text{for } s = 1, 2 \end{array}\right] \tag{2.1}$$

It can be readily shown that $\Phi\left((x, \lambda, b, p, q); \left((e, u), \mathbf{a}\right)\right) = 0$ if and only if (x, b, p, q) is an equilibrium: note that the remaining markets (for the asset and the numéraire good in state 1 and 2) automatically clear.

Definition 2.2. *ξ is a regular equilibrium of economy $((e,u),\mathbf{a})$ if $\Phi(\xi;((e,u),\mathbf{a}))=0$ and $D_\xi\Phi(\xi;((e,u),\mathbf{a}))$ is an invertible matrix.*

It can be shown that the set of regular equilibria is invariant with respect to the choice of the set of redundant equilibrium equations which are omitted (for instance we could have alternatively eliminated the market clearing conditions for the numéraire commodity in all states), and that the definition above is equivalent to the full rank condition of excess demand function.

If ξ is a regular equilibrium of $((e,u),\mathbf{a})$ then by the implicit function theorem there is a local one-to-one relation between the set of economies and their equilibria. It is straightforward to see that if e is a Pareto efficient allocation of $((e,u),\mathbf{a})$ then $x=e$ constitutes a regular equilibrium allocation of $((e,u),\mathbf{a})$ (see Cass (1992)); moreover the equilibrium is unique. So by Remark 2, using the implicit function theorem and the upper hemicontinuity of the equilibrium correspondence we get:

Lemma 2.3. *If e is a Pareto efficient allocation of $((e,u),\mathbf{a})$ then there exists an open neighborhood $\mathcal{V}$ of $((e,u),\mathbf{a})$ such that $\forall((e',u'),\mathbf{a}')\in\mathcal{V}$ there exists a unique equilibrium, which is a non-sunspot equilibrium.*[5]

Now we are ready to state the main result. Define $\mathcal{E}^*\subset\mathcal{E}$:

$$\mathcal{E}^*=\left\{\begin{array}{c}(e,u)\in\mathcal{E}:\ \text{there is } \mathbf{a}\in I\!R^{C+1} \text{ such that } ((e,u),\mathbf{a}) \text{ has} \\ \text{a regular sunspot equilibrium}\end{array}\right\}$$

Proposition 2.4. *$\mathcal{E}^*$ is an open dense subset of $\mathcal{E}$.*

Thus for a generic choice of (e,u), by the implicit function theorem there exists a neighborhood $\mathcal{V}$ of (e,u) and a non-empty open set $\mathcal{A}\subset I\!R^{C+1}$ such that for any $\mathbf{a}\in\mathcal{A}$, $(e,u)\in\mathcal{V}$, $((e,u),\mathbf{a})$ has a regular sunspot equilibrium, and this can be written as a smooth function of e,u and $\mathbf{a}$ locally. In words, there is a robust way of finding asset structures such that (e,u) has a sunspot equilibrium.

Notice that Proposition 2.4 and Lemma 2.3 are consistent; the choice of the neighborhood $\mathcal{V}$ depends on $\mathbf{a}$ in Lemma 2.3. Also by Lemma 2.3 generic existence cannot be established in the space of $((e,u),\mathbf{a})$, nor of (e,u) for any given $\mathbf{a}$, and so the above result is tight.

[5] A claim along these lines is already in Balasko (1990).

Proposition 2.4 does not show the stronger statement that every sunspot equilibrium of the economy $((e, u), \mathbf{a})$ is regular. However, it can be shown that every sunspot equilibrium is regular generically,[6] so the stronger statement in fact holds.

3. Proof of the Main Result

The problem with an economy with real assets is that the equilibrium is determinate; that is, there is the same number of endogenous variables and equations in (2.1), so there is no degree of freedom, which might help us construct a sunspot equilibrium. Also, a non-sunspot equilibrium always exists. The strategy of our proof is: first, construct an auxiliary model where a parameter of the original economy is made sunspot dependent, so that the auxiliary economy only has sunspot equilibria; and second, show that (at least one of) the sunspot equilibrium we constructed can be obtained as an equilibrium in the asset economy.

We choose here the model similar to Cass (1992) as the auxiliary economy, since we can borrow some important results from it. We use then the asset structure as a parameter, obtaining that for a generic set of endowment and preference distributions there is an open set of asset structures for which sunspot equilibria exist. But certainly this is not the only choice, as we could have used also the endowments, or preferences (e.g. beliefs) to get that for a generic set of asset structures and preferences (endowments), there is an open set of endowments (respectively beliefs) for which there are sunspot equilibria. See section 4.

3.1. Auxiliary Economy

Consider the auxiliary economy, which is an economy identical to $((e, u), \mathbf{a})$ except for the asset structure. There is one asset whose payoff depends on sunspots as follows: the asset pays 1 unit of the numéraire good 0 in state 1 and r unit of good 0 in state 2, where $r \in I\!R_{++}$ is given exogenously. An auxiliary economy is then characterized by $((e, u), r)$.

As before, we can characterize the equilibria of an auxiliary economy by a system of equations as follows: let $\Phi_A(\xi; ((e, u), r))$ be identical to Φ defined

[6] An elaboration of the argument given in Kajii (1991) will do. But note that such a generic regularity result might be vacuous since an economy is also regular if there is no sunspot equilibrium at all.

in (2.1) except for the terms describing the payoff of the asset: $(\hat{p}^1\mathbf{a})$ is replaced with 1 and $(\hat{p}^2\mathbf{a})$ with r. We say that ξ is an *equilibrium* of economy $((e,u),r)$ if $\Phi_A(\xi;((e,u),r)) = 0$, and an equilibrium ξ is said to be regular if $D_\xi\Phi_A(\xi;((e,u),r))$ is invertible.

Remark 3. *If $r = 1$ there exists a non-sunspot equilibrium.*

Applying Cass (1992) , one can show:

Lemma 3.1. *There is an open dense subset $\mathcal{E}_r \subset \mathcal{E}$ such that for any $(\bar{e},\bar{u}) \in \mathcal{E}_r$, whenever ξ is a non-sunspot equilibrium of $((\bar{e},\bar{u}),1)$, $D_\xi\Phi_A(\xi,(\bar{e},\bar{u}),1)$ is non-singular and $b_h \neq 0$ for every h .*

From the above result it follows that all the equilibria of every economy $(\bar{e},\bar{u}) \in \mathcal{E}_r$ are sunspot equilibria whenever $r \neq 1$. Also by the implicit function theorem and the usual boundary argument, again following Cass (1992) one can easily show that:

Lemma 3.2. *For every $(\bar{e},\bar{u}) \in \mathcal{E}_r$ there exists $r^* \in \mathbb{R}$ and an equilibrium ξ^* of $((\bar{e},\bar{u},r^*)$ that is a regular sunspot equilibrium and such that $x_h^{1*} \neq x_h^{2*}$ for every h.*

3.2. Generating a Sunspot Equilibrium

The openness of the set $\mathcal{E}^*$ follows immediately from the implicit function theorem, so we shall concentrate on the density part. Pick $(\bar{e},\bar{u}) \in \mathcal{E}_r$ arbitrarily, and let $\mathcal{V}^*$ be an open neighborhood of $(\bar{e},\bar{u})$. Our goal is to show that there exists $(e,u) \in \mathcal{V}^* \cap \mathcal{E}^*$, which is enough since $\mathcal{E}_r$ is dense in $\mathcal{E}^*$ by Lemma 3.1.

By Lemma 3.2 and an application of the implicit function theorem we can find $r^* \in \mathbb{R}$ and a neighborhood $\mathcal{V}$ of $(\bar{e},\bar{u})$ with $\mathcal{V} \subset \mathcal{V}^*$ such that all economies (e,u,r^*) , $(e,u) \in \mathcal{V}$, have a sunspot equilibrium. This can be, locally, expressed as a smooth function of (e,u). Denote by $\xi(e,u,r^*)$ the function describing this equilibrium around $(\bar{e},\bar{u})$; $\xi(e,u,r^*)$ is a regular equilibrium of $((e,u),r^*)$ and $x_h^1(e,u,r^*) \neq x_h^2(e,u,r^*)$ for every h and for any $(e,u) \in \mathcal{V}$. Moreover, choosing $\mathcal{V}$ small enough, we can find a compact neighborhood G_h^s of $x_h^s(\bar{e},\bar{u},r^*)$ for $s = 1,2$, $h = 1,...H$, such that $G_h^1 \cap G_h^2 = \emptyset$ for every h and $x_h^s(e,u,r^*) \in G_h^s$ for any $(e,u) \in \mathcal{V}$, $s = 1,2$, $h = 1,..,H$. Also, $\Phi_A(\xi;(e,u),r)) = 0$, $x_h \in G_h \forall h \Rightarrow x_h = x_h(e,u,r^*) \forall h$.

Now let

$$\mathcal{D}_1 = \left\{(e,u) \in \mathcal{V}: \ \Phi_A(\xi,(e,u),r^*) = 0 \text{ and } x_h \in G_h \ \forall h \Longrightarrow p^{1,1} \neq p^{1,2}\right\}$$

That is, $\mathcal{D}_1$ is the set of economies in $\mathcal{V}$ such that the price of good 1 differs in the two sunspot states (at every equilibrium under scrutiny). Then the following holds:

Lemma 3.3. *$\mathcal{D}_1$ is open and dense in $\mathcal{V}$.*

A proof can be found in Appendix. It is intuitive that $p^{1,1} \neq p^{1,2}$ will be a knife edge case, but notice that, for instance, if every utility function is homothetic, then no matter how one perturbs endowments e, $p^{1,1} = p^{1,2}$ will hold in equilibrium. This is why we take advantage of the fact that we can freely perturb utility functions.

Let $\mathbf{A} = \left\{ I\!R^{C+1} : \mathbf{a}^c = 0 \text{ for } c > 1\right\}$, that is, $\mathbf{A}$ is the set of real assets that pay in good 0 and 1 only.

Lemma 3.4. *For any $(e,u) \in \mathcal{D}_1$, there is an $\mathbf{a} \in \mathbf{A}$ such that the economy $((e,u),\mathbf{a})$ has a sunspot equilibrium.*

Proof. Pick any $(e,u) \in \mathcal{D}_1$. By construction, there is a regular equilibrium ξ of $((e,u),r^*)$ such that $p^{1,1} \neq p^{1,2}$. Then, since $C > 0$, we can find an $\mathbf{a} \in \mathbf{A}$ which satisfies

$$\begin{aligned} \hat{p}^1 \cdot \mathbf{a} &= 1 \\ \hat{p}^2 \cdot \mathbf{a} &= r^* \end{aligned} \tag{3.1}$$

At p, the real asset $\mathbf{a}$ has payoff structure identical to the numéraire asset r^*, and hence it is easy to see that ξ is also an equilibrium of $((e^*,u^*),\mathbf{a})$. □

It may appear that Proposition 2.4 follows from Lemmas 3.3 and 3.4. But this is not the case. It is true that the equilibrium constructed as in Lemma 3.4 is a regular equilibrium of $((e,u),r^*)$. However it does not necessarily follow that it is a regular equilibrium of $((e,u),\mathbf{a})$.[7] So we need an extra step.

By definition, an equilibrium ξ is a regular equilibrium of $((e,u),\mathbf{a})$ if and only if the determinant $|D_\xi\Phi(\xi,((e,u),\mathbf{a}))|$ is non zero. Fix any $(e,u) \in \mathcal{D}_1$. By construction, the auxiliary economy (e,u,r^*) has a unique equilibrium $\xi(e,u) \in$

[7] See Kajii (1994).

Ξ'; let $\mathbf{a}(e,u)$ be the corresponding real asset that is found in (3.1) (by Lemma 3.4 $\mathbf{a}(e,u)$ is well defined). Define :

$$\mathcal{D}_2 = \{(e,u) \in \mathcal{D}_1 : |D_\xi \Phi(\xi(e,u),((e,u),\mathbf{a}(e,u)))| \neq 0\}$$

So $\mathcal{D}_2$ is the set of economies where the equilibrium we constructed is in fact regular.

Proposition 2.4 is established if we can show that $\mathcal{D}_2$ is non-empty since $\mathcal{V}^*$ can be chosen arbitrary small. In the appendix we will show a stronger result:

Lemma 3.5. *$\mathcal{D}_2$ is open and dense in $\mathcal{D}_1$*

And this completes the proof of Proposition 2.4.

4. Remarks and Extensions

4.1. The Overlapping Generations Model

Our argument which is based on the construction of an auxiliary economy to detect sunspot equilibria is fairly general, and we believe that there are many models where the idea of our argument can be applied. We shall briefly discuss such an instance; the case of an economy with overlapping generations. Consider the standard two-period overlapping generations model with one consumption good available at each date, no intrinsic uncertainty and a single asset, fiat money. There are H consumers in each generations, with endowments of the commodity in the two periods of their life $(e_h^0, e_h^1) \in I\!R^2_{++}$. Extrinsic uncertainty has a Markov structure: there are two sunspot states $s = 1,2$ which may occur at each period and $\pi^{ss'}$ describes the probability of state s' at date $t+1$ given that state s occurred at t. The preferences of an agent born when state s is realized are represented by a utility function $U_h(x_h^s) = \sum_{s'=1}^2 \pi^{ss'} u_h(x_{h,0}^s, x_{h,1}^{ss'})$ satisfying Assumption 1.

Let $x_h^s = (x_{h,0}^s, x_{h,1}^{s1}, x_{h,1}^{s2})$, $x \equiv ((x_h^s)_{s=1}^2)_{h=1}^H$; q^s denotes the price of money in state s, and b_h^s the level of money holding by agent h in state s. Similarly $q = (q^s)_{s=1}^2$, $b = ((b_h^s)_{s=1}^2)_{h=1}^H$.

(x,b,p,q) is a *stationary sunspot equilibrium* of economy (e,u) if $x_h^s \neq x_h^{s'}$ for some h, s and

(H) for each h, s, (x_h^s, b_h^s) solves the following problem:

$$
\begin{aligned}
& max_{x_h^s, b_h^s} U_h^s(x_h^s) \\
& \text{subject to} \\
& x_{h,0}^s - e_h^0 + q^s b_h^s = 0 \\
& x_{h,1}^{ss'} - e_h^1 - q^{s'} b_h^s = 0 \text{ for } s' = 1, 2
\end{aligned}
$$

(M) markets clear, i.e.

$$
\begin{aligned}
& \textstyle\sum_{h=1}^H (x_{h,0}^s - e_h^0) = 0 \text{ for } s = 1, 2 \\
& \textstyle\sum_{h=1}^H (x_{h,1}^{ss'} - e_h^1) = 0 \text{ for } s, s' = 1, 2 \\
& \textstyle\sum_{h=1}^H b_h^s = 0 \text{ for } s = 1, 2
\end{aligned}
$$

For the auxiliary economy, let us consider an economy where endowments of two households can depend on sunspot signals. Formally, define
$\tilde{\mathbf{E}} = \left\{ \tilde{e} = \left((e_1^0 + e_2^0, e_1^1 + e_2^1), (e_h^0, e_h^1)_{h=3}^H \right) \in I\!R_{++}^{2(H-1)} \right\}$, $\mathcal{E} = \tilde{\mathbf{E}} \times \mathbf{U}$.
Let $\mathcal{E}^* \subset \mathcal{E}$ be:

$$
\mathcal{E}^* = \left\{ \begin{array}{l} (\tilde{e}, u) \in \mathcal{E} : \exists (\hat{e}_1^s, \hat{e}_2^s) \in I\!R_{++}^2 \times I\!R_{++}^2, \hat{e}_1^s + \hat{e}_2^s = (e_1^0 + e_2^0, e_1^1 + e_2^1), s = 1, 2, \\ ((\hat{e}_1^s, \hat{e}_2^s)_{s=1}^2, (e_h^0, e_h^1)_{h=3}^H, u) \text{ has a regular stationary sunspot equilibrium} \end{array} \right\}
$$

Then by considering a set of vectors $(\hat{e}_1^s, \hat{e}_2^s)_{s=1}^2$ whose elements vary with the sunspot states, we can generate a sunspot equilibrium; so $(\hat{e}_1, \hat{e}_2)$ plays the role of asset payoff parameter r in the previous section. So we conjecture that stationary a sunspot equilibrium exists, generically; that is,

Conjecture 4.1. *$\mathcal{E}^*$ is open and dense in $\mathcal{E}$.*

4.2. Choice of the Asset

The equilibria of the economies $((e, u), r)$ we considered in constructing $\mathcal{E}^*$ are very close to the efficient equilibria of $((e, u), 1)$. Hence, although p^1 and p^2 are linearly independent, they are very close to each other. Consequently, the vector $\mathbf{a}$ found in (3.1) tends to be very large in norm. Of course, we could normalize the vector $\mathbf{a}$, but then the corresponding equilibrium asset holding b_h would have to be adjusted accordingly.

To see this point more clearly, let us re-examine Lemma 2.3. Consider $(\bar{e}, \bar{u})$ such that $\bar{e}$ is Pareto efficient. Clearly, $(\bar{e}, \bar{u}) \notin \mathcal{E}^*$. Now fix an arbitrary $\bar{\mathbf{a}} \neq 0$, and consider an economy $(\bar{e}, \bar{u}, \bar{\mathbf{a}})$. It is not difficult to see that $((\bar{e}, \bar{u}), \bar{\mathbf{a}})$ has a unique (no trade) equilibrium which is regular. So by the implicit function theorem, there is an open neighborhood $\mathcal{V}(\bar{\mathbf{a}})$ of $(\bar{e}, \bar{u})$ and an open neighborhood $\mathcal{A}(\bar{\mathbf{a}})$ of $\bar{\mathbf{a}}$, such that for any $((e, u), \mathbf{a}) \in \mathcal{V}(\bar{\mathbf{a}}) \times \mathcal{A}(\bar{\mathbf{a}})$, there is a unique equilibrium of $((e, u), \mathbf{a})$. Since a non-sunspot equilibrium always exists, it follows that every $((e, u), \mathbf{a}) \in \mathcal{V}(\bar{\mathbf{a}}) \times \mathcal{A}(\bar{\mathbf{a}})$ has a unique nonsupport equilibrium. Note that even if $(\bar{e}, \bar{u})$ is fixed, the sets $\mathcal{V}(\bar{\mathbf{a}})$ and $\mathcal{A}(\bar{\mathbf{a}})$ depend on the choice of $\bar{\mathbf{a}}$.

On the other hand, by Proposition 2.4, there is a sequence $(e^n, u^n) \in \mathcal{E}^*$ that converges to $(\bar{e}, \bar{u})$ and a sequence $\mathbf{a}^n$ such that any $((e', u'), \mathbf{a}')$ in some neighborhood of $((e^n, u^n), \mathbf{a}^n)$ has a sunspot equilibrium (so, in particular, such $((e', u'), \mathbf{a}')$ must have multiple equilibria). So, it must be the case that $((e^n, u^n), \mathbf{a}^n) \notin \mathcal{V}(\mathbf{a}^n) \times \mathcal{A}(\mathbf{a}^n)$ for all n. Since $\mathbf{a}^n \in \mathcal{A}(\mathbf{a}^n)$ by construction, $(e^n, u^n) \notin \mathcal{V}(\mathbf{a}^n)$ follows. In words, what happens is that as (e^n, u^n) approaches $(\bar{e}, \bar{u})$, our construction must choose $\mathbf{a}^n$ such that $\mathcal{V}(\mathbf{a}^n)$ is so small that $(e^n, u^n) \notin \mathcal{V}(\mathbf{a}^n)$. Also, from the upper hemicontinuity of the equilibrium correspondence and the regularity of the sunspot equilibria we constructed it follows that the sequence $(\mathbf{a}^n)$ does not converge in norm.

4.3. Multiplicity, Complexity, and "Endogenous" Uncertainty

As is mentioned in the Introduction, the existence of sunspot equilibria has little to do with the multiplicity of non-sunspot equilibria, or of spot equilibria (at the non-sunspot equilibria). However, *potential* multiplicity of "spot market equilibria" is necessary for the existence. If the spot market has a unique equilibrium *regardless of the distribution of initial endowments*, then there can be no sunspot equilibrium (when the specification of asset payoffs is sunspot invariant).[8] Our result holds since such a strong uniqueness property is non-generic (in utility function); i.e. we can generically find an asset structure such that at an equilibrium the second period spot economy has multiple equilibria.

Let us conclude with the following very speculative point. If we view sunspots as a modeling device for *endogenous* uncertainty, the specification of the set of relevant sunspot states, and in particular their number, cannot be defined as a part of the primitives of the underlying economy. A step forward in this direction

[8] On the other hand, when asset payoffs are sunspot dependent then we may have a sunspot equilibrium even if the spot market has a unique equilibrium. This applies, for instance to the model considered by Cass and Shell (1983), for which they present an example of an economy that has sunspot equilibria as well as a unique non-sunspot equilibrium.

will be to consider a class of economies whose fundamentals (preferences and endowments) are the same but the structure of sunspot states are different. For an economy (e, u) and an integer n, consider the sunspot economy in which n sunspot states occur with equal chance. Let the *complexity* of an equilibrium ξ be the number of states where the equilibrium allocation non-trivially varies, and call the maximum of those the *complexity of* (e, u). The observation above shows that the complexity of (e, u) is no more than the maximal multiplicity of spot market equilibrium. For instance, if the number of spot market equilibria is no larger than 3 for any distribution of initial endowments, then the maximal complexity of a sunspot equilibrium will be no more than 3.

Of course, one can construct a robust model where there is an arbitrary multiplicity of spot market equilibria by perturbing the "contract curve", but the set of endowments for which such high multiplicity of equilibria obtains will be "small". We conjecture however that the set of economies with high complexity will be "large", since intuitively, what seems to matter is the potential existence of high multiplicity, and so it seems to be generally possible to find some asset structure for which this small area of endowments with high multiplicity of equilibria matters.

Appendix

Proof of Lemma 3.3

Openness. Let $(e^n, u^n) \in \mathcal{V} \setminus \mathcal{D}_1$ for $n = 1, \ldots$, and suppose $(e^n, u^n) \to (e, u) \in \mathcal{V}$. Since $(e^n, u^n) \in \mathcal{V}$, there is a unique equilibrium ξ^n of $((e^n, u^n), r^*)$ such that $x_h^n \in G_h$. $(e^n, u^n) \not\in \mathcal{D}_1$ implies that for each n, $p^{1,1,n} = p^{1,2,n}$. Then by the usual limit argument, we can find a subsequence n_v of $n = 1, .., \xi^{n_v} \to \xi$ where $\Phi_A(\xi, (e, u), r^*) = 0$. By construction, $x_h \in G_h$ in the limit and $p^{1,1} = p^{1,2}$ holds, which implies that $(e, u) \not\in \mathcal{D}$. So, $\mathcal{D}$ is open in $\mathcal{V}$.

Density can be shown by the transversality argument. Notice that because states are sunspots, utility perturbation is tricky. But we can handle it by taking advantage of the fact that $x_h^1 \neq x_h^2$ in the equilibrium in question, according to the technique developed in Kajii (1991).[9] The idea is the following. Let $(\bar{x}_h)_{h=1}^H$ be an equilibrium allocation for the economy $(\bar{e}, \bar{u})$: if we change utility functions $\bar{u}$ to u' such that $\bar{u} = u'$ except for a small neighborhood of $\bar{x}_h^1$, then the equilibrium system Φ_A will be unaffected.

Pick $(e, u) \in \mathcal{V}$ arbitrarily. For each h, we can construct an open convex set $\Theta_h \subset \mathbb{R}^{3(C+1)}$ with $0 \in \Theta_h$, and a C^2 function $\tilde{u}_h : \mathbb{R}_+^{2(C+1)} \times \Theta_h \to \mathbb{R}$ with the property:

$$\tilde{u}_h\left(x^0, x^1; \theta_h^0, \theta_h^1, \theta_h^2\right) = \begin{cases} u_h\left(x^0, x^1\right) + \theta_h^0 x^0 + \theta_h^1 x^1 \text{ if } x^1 \in G_h^1 \\ u_h\left(x^0, x^1\right) + \theta_h^0 x^0 + \theta_h^2 x^1 \text{ if } x^1 \in G_h^2 \end{cases}$$

such that $\tilde{u}_h\left(\cdot; \theta_h^0, \theta_h^1, \theta_h^2\right)$ satisfies Assumption 1 and $(e, \tilde{u}_\theta) \in \mathcal{V}$, where $\tilde{u}_\theta$ is the vector of functions $\left(u_h\left(\cdot; \theta_h\right)\right)_{h=1}^H$. For instance, take a C^∞ function ρ on $\mathbb{R}^{2(C+1)}$ such that $\rho\left(x^0, x^1\right) = 1$ whenever $x^1 \in intG_h$ and $\rho\left(x^0, x^1\right) = 0$ outside some small neighborhood of G_h , and consider $\rho\tilde{u}_h + (1-\rho)u_h$.

Let $\Theta = \prod_h \Theta_h$. Define a C^2 function $\Psi : \Xi' \times \Theta \to \mathbb{R}^n \times \mathbb{R}$:

$$\Psi(\xi, \theta) = \begin{bmatrix} \Phi_A\left(\xi, (e, \tilde{u}_\theta), r^*\right) \\ p^{1,1,} - p^{1,2} \end{bmatrix}$$

For given $(\bar{e}, \theta)$ and r^*, there are one more equations than unknowns, and so if we can show that 0 is a regular value of $\Psi(\cdot, \theta)$ then there is no ξ such that

[9] Suda-Tallon-Villanacci (1992) applied this technique to show the indeterminacy of sunspot equilibria for the case of a nominal asset.

$\Psi(\xi,\theta)=0$, which implies that $(e,\tilde{u}_\theta)\in\mathcal{D}_1$. By the transversality theorem, if 0 is a regular value of Ψ, then there is a dense subset Θ' of Θ such that 0 is a regular value of $\Psi(\cdot,\theta)$ for every $\theta\in\Theta'$. This in particular implies that there is $(e,u)\in\mathcal{D}_1$ arbitrary close to $(\bar{e},\bar{u})$, so the density follows.

So, all we need to show is that 0 is a regular value of Ψ. Pick any ξ,θ such that $\Psi(\xi,\theta)=0$. We need to show that $D\Psi$ evaluated at (ξ,θ) has full row rank. Note that $\Phi_A(\xi,(e,\tilde{u}_\theta),r^*)=0$ and $\xi\in\Xi'$ imply that $\xi=\xi((e,u_\theta),r^*)$ and $x_h^s\in G_h^s$ for all h and $s=1,2$, so at (ξ,θ), $D_x\tilde{u}_\theta$ has the form:

$$\begin{array}{l}\frac{\partial}{\partial x^0}\hat{u}_h\left(x_h^0,x_h^s\right)+\theta_h^0 \text{ for } s=1,2\\ \frac{\partial}{\partial x^1}\hat{u}_h\left(x_h^0,x_h^1\right)+\theta_h^1\\ \frac{\partial}{\partial x^2}\hat{u}_h\left(x_h^0,x_h^2\right)+\theta_h^2\end{array}$$

Therefore, $(D_\xi\Psi, D_\theta\Psi)$ evaluated at (ξ,θ) has the following form:

$$\left[\begin{array}{ccccccc}
\begin{array}{ccc}\ddots & & \\ & D^2U_h(x) & \\ & & \ddots\end{array} & \begin{array}{ccc}\ddots & & 0\\ & -P & \\ 0 & & \ddots\end{array} & 0 & \begin{pmatrix}\vdots\\ \Lambda_h\\ \vdots\end{pmatrix} & 0 & \partial_\theta\\
 & & & & \vdots & \\
\begin{array}{ccc}\ddots & & 0\\ & -P^T & \\ 0 & & \ddots\end{array} & 0 & \begin{array}{ccc}\ddots & & 0\\ & R & \\ 0 & & \ddots\end{array} & \begin{pmatrix}\vdots\\ Z_h\\ \vdots\end{pmatrix} & \begin{pmatrix}b_h\\ 0\\ 0\end{pmatrix} & 0\\
 & & & & \vdots & \\
0 & \begin{array}{ccc}\ddots & & 0\\ & R^T & \\ 0 & & \ddots\end{array} & 0 & 0 & \begin{array}{c}\vdots\\ -\lambda_h^0\\ \vdots\end{array} & 0\\
\cdots\ \tilde{I}\ \cdots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \cdots\ I\ \cdots & 0 & 0 & 0\\
0 & 0 & 0 & \frac{\partial\left(p^{1,1}-p^{1,2}\right)}{\partial p} & 0 & 0
\end{array}\right] \tag{4.1}$$

where P is the $(C+1)3\times 3$ matrix:

$$P\equiv\begin{pmatrix}p^0 & 0 & 0\\ 0 & p^1 & 0\\ 0 & 0 & p^2\end{pmatrix},$$

R denotes the vector:

$$R \equiv \begin{bmatrix} -q \\ 1 \\ r^* \end{bmatrix},$$

$\tilde{I}$ is of dimension $(3C+1) \times 3(C+1)$:

$$\tilde{I} \equiv \begin{pmatrix} \begin{pmatrix} I_C & 0 \end{pmatrix} & & 0 \\ & \begin{pmatrix} I_C & 0 \end{pmatrix} & \\ 0 & & \begin{pmatrix} I_C & 0 \end{pmatrix} \end{pmatrix},$$

Λ_h is the $3(C+1) \times 3C$ matrix:

$$\Lambda_h \equiv \begin{pmatrix} \begin{pmatrix} -\lambda_h^0 I_C \\ 0 \end{pmatrix} & 0 & 0 \\ 0 & \begin{pmatrix} -\lambda_h^1 I_C \\ 0 \end{pmatrix} & 0 \\ 0 & 0 & \begin{pmatrix} -\lambda_h^2 I_C \\ 0 \end{pmatrix} \end{pmatrix},$$

and

$$Z_h^T \equiv \begin{pmatrix} z_h^{\backslash 0} & 0 & .. \\ 0 & z_h^{\backslash 1} & \\ .. & & z_h^{\backslash 2} \end{pmatrix}.$$

The last column of the matrix (4.1) contains the derivative of Ψ with respect to θ, and from the above specification it follows that ∂_θ is a full rank matrix. Thus the top row block is linearly independent of the other blocks and can be eliminated. Since the term $\frac{\partial(p^{1,1}-p^{1,2})}{\partial p}$ is of full row rank, the matrix (4.1) will have full rank iff the submatrix

$$\begin{bmatrix} \ddots & & 0 & & & & \ddots & & 0 \\ & -P^T & & & 0 & & & R & \\ 0 & & \ddots & & & & 0 & & \ddots \\ & & & \ddots & & 0 & & & \\ & 0 & & & R^T & & & 0 & \\ & & & 0 & & \ddots & & & \\ \cdots & \tilde{I} & \cdots & & 0 & & & 0 & \\ & 0 & & & 0 & & \cdots & I & \cdots \end{bmatrix}$$

has full row rank, but this can be readily shown. □

Proof of Lemma 3.5.

The openness is immediate by the implicit function theorem, so we shall show the density part in the following.[10]

Choose any $(\hat{e}, \hat{u}) \in \mathcal{D}_1 \backslash \mathcal{D}_2$, and write $\hat{\xi} = \xi(\hat{e}, \hat{u})$ and $\hat{\mathbf{a}} = \mathbf{a}(\hat{e}, \hat{u})$. We shall fix them through out. We will show that we can always construct an economy $(e', u') \in \mathcal{D}_2$ which is arbitrary close to $(\hat{e}, \hat{u})$ and has a regular equilibrium. To construct such an economy, fix an arbitrary h, say $\hat{h}$. Let $\hat{\lambda}_{\hat{h}}$ be the corresponding element of $\hat{\xi}$, and $\hat{u}_{\hat{h}}$ of $\hat{u}$. Proceeding as in the proof of Lemma 3.3, we can find a small open convex set $\Omega \subset \mathbb{R}$ with $0 \in \Omega$, and a C^2 function $\hat{u}_{\hat{h},\omega} : \mathbb{R}_+^{2(C+1)} \times \Omega \longrightarrow \mathbb{R}$ with the property:

$$\hat{u}_{\hat{h},\omega}(x_0, x_1; \omega) = \left\{ \begin{array}{l} \hat{u}_{\hat{h}}(x_0, x_1) + \omega \hat{\lambda}_{\hat{h}}^0 x_0 + \omega \hat{\lambda}_{\hat{h}}^1 x_1 \text{ if } x_1 \in G_{\hat{h}}^1 \\ \hat{u}_{\hat{h}}(x_0, x_1) + \omega \hat{\lambda}_{\hat{h}}^0 x_0 + \omega \hat{\lambda}_{\hat{h}}^2 x_1 \text{ if } x_1 \in G_{\hat{h}}^2 \end{array} \right\}$$

such that $\hat{u}_{\hat{h},\omega}(.;\omega)$ satisfies Assumption 1 and $(\hat{e}, \hat{u}_\omega) \in \mathcal{V}$, for $\hat{u}_\omega = \{(\hat{u}_h)_{h \neq \hat{h}}, \hat{u}_{\hat{h},\omega}\}$. Let $\hat{\xi}_\omega$ be the vector whose components are equal to $\hat{\xi}$ except that the component $\hat{\lambda}_{\hat{h}}$ is replaced with $\hat{\lambda}_{\hat{h}}(1-\omega)$. Notice that $\frac{\partial}{\partial x_{\hat{h}}^s} \hat{u}_{\hat{h},\omega}(x_0, x_1; \omega) = \frac{\partial}{\partial x_{\hat{h}}^s} \hat{u}_{\hat{h},\omega}(x_0, x_1) + \omega \hat{\lambda}_{\hat{h}}^s$, and so it is readily verified from (2.1) that $\Phi\left(\hat{\xi}_\omega; ((\hat{e}, \hat{u}_\omega), \hat{\mathbf{a}})\right) = 0$, i.e., $\hat{\xi}_\omega$

[10]The idea of the following argument draws on earlier work by Pietra (1992).

is an equilibrium of (e', u'). Recall that $\hat{\xi}$ is a regular equilibrium of the auxiliary economy. By applying the implicit function theorem, choose Ω small enough so that for any $\omega \in \Omega$, $\hat{\xi}_\omega$ is a locally unique equilibrium of $((\hat{e}, \hat{u}_\omega), \hat{\mathbf{a}})$ in G_h.

We want to show that there is arbitrary small ω such that $\hat{\xi}_\omega$ is a regular equilibrium of $((\hat{e}, \hat{u}_\omega), \hat{\mathbf{a}})$. It suffices to show that there is a matrix M which do not depend on ω such that:

$$|D_{\xi}\Phi(\xi, (e, u_\omega), \mathbf{a})| = 0 \Leftrightarrow |(\omega I + M)| = 0, \tag{4.2}$$

since the determinant $|(\omega I + M)|$ is a non-trivial polynomial of ω, so it is non-zero except for a discrete set of ω.

To see that (4.2) holds, first observe that $D_{\xi}\Phi$ can be partitioned into the form $\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{pmatrix}$ where

$$B_{11} = \begin{pmatrix} \ddots & 0 & & \ddots & \\ & D^2U_h(x) & 0 & & -P \\ 0 & & \ddots & & \ddots \\ & & & \ddots & \\ & 0 & 0 & & R^T \\ & & & & \ddots \\ \ddots & & \ddots & & \\ & -P^T & & R & 0 \\ & & \ddots & \ddots & \end{pmatrix}$$

$$B_{12} = \begin{pmatrix} \begin{pmatrix} \vdots \\ \Lambda_h \\ \vdots \end{pmatrix} & \begin{pmatrix} \vdots \\ 0 \\ \vdots \end{pmatrix} \\ \begin{pmatrix} \vdots \\ \tilde{A}_h \\ \vdots \end{pmatrix} & \begin{pmatrix} \vdots \\ -\lambda_0^h I \\ \vdots \end{pmatrix} \\ \begin{pmatrix} \vdots \\ \tilde{Z}_h \\ \vdots \end{pmatrix} & \begin{pmatrix} \vdots \\ \begin{pmatrix} b_h \\ 0 \\ 0 \end{pmatrix} \\ \vdots \end{pmatrix} \end{pmatrix}, \quad B_{21} = \begin{pmatrix} \cdots \tilde{I} \cdots & 0 & 0 \\ 0 & \cdots I \cdots & 0 \end{pmatrix}.$$

And by the standard facts about differentiability of the demand function,[11] B_{11} is invertible, and its inverse has the following diagonal structure (up to appropriate elementary row and column operations): $B_{11}^{-1} = \begin{pmatrix} \ddots & & 0 \\ & \left(B_{11}^h\right)^{-1} & \\ 0 & & \ddots \end{pmatrix}$,

where $\left(B_{11}^h\right)^{-1} = \begin{pmatrix} D^2U_h(x) & 0 & -P \\ 0 & 0 & R^T \\ -P^T & R & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \begin{bmatrix} C^{11,h} & C^{12,h} \end{bmatrix} & C^{2,h} \\ C^{2,h^T} & C^{3,h} \end{pmatrix}$, and $\begin{bmatrix} C^{11,h} & C^{12,h} \end{bmatrix}$ is negative semidefinite, of rank $3C+1$.

Note that $\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{pmatrix} \begin{pmatrix} I & -B_{11}^{-1}B_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} B_{11} & 0 \\ B_{21} & -B_{21}B_{11}^{-1}B_{12} \end{pmatrix}$,so $\left|D_\xi \Phi(\xi, (e, u_{\underset{\omega}{\cdot}}), \mathbf{a})\right| \neq 0$ holds if and only if $\left|B_{21}B_{11}^{-1}B_{12}\right| \neq 0$, where the latter is in fact the determinant of the derivative of excess demand function. A straightforward calculation shows:

$$\left(B_{21}B_{11}^{-1}B_{12}\right)^T = \begin{pmatrix} \sum_h \left(\begin{pmatrix} \bar{C}^{11,h} & C^{12,h} \end{pmatrix} \begin{pmatrix} \Lambda_h \\ \tilde{A}_h \end{pmatrix} + C^{2,h} \tilde{Z}_h \right)^T \\ \sum_h \left(\begin{pmatrix} \bar{C}^{11,h} & C^{12,h} \end{pmatrix} \begin{pmatrix} 0 \\ -\lambda_0^h I \end{pmatrix} + C^{2,h} \begin{pmatrix} b_h \\ 0 \\ 0 \end{pmatrix} \right)^T \end{pmatrix} \quad (4.3)$$

[11] See Balasko-Cass (1991), Theorem 2.

where $\bar{C}^{11,h}$ is obtained from $C^{11,h}$ by deleting rows $C+1$, $2(C+1)$, $3(C+1)$.

Recall that $\Lambda_h = \begin{pmatrix} \begin{pmatrix} -\lambda_0^h I \\ 0 \end{pmatrix} & & 0 \\ & \begin{pmatrix} -\lambda_1^h I \\ 0 \end{pmatrix} & \\ 0 & & \begin{pmatrix} -\lambda_2^h I \\ 0 \end{pmatrix} \end{pmatrix}$, and denoting by $\tilde{C}^{11,h}$ the (square) matrix obtained from $\bar{C}^{11,h}$ by deleting columns $C+1, 2(C+1), 3(C+1)$, we can further simplify $B_{21}B_{11}^{-1}B_{12}$ in (4.3) as follows:

$$= \left[\sum_h \left\{ \left(-\tilde{C}^{11,h} \begin{pmatrix} -\lambda_0^h I & & 0 \\ & -\lambda_1^h I & \\ 0 & & -\lambda_2^h I \end{pmatrix} + C^{12,h}\tilde{A}_h + C^{2,h}\tilde{Z}_h \right), \begin{pmatrix} -C^{12,h}\left(\lambda_0^h I\right) + \\ C^{2,h} \begin{pmatrix} b_h \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \right\} \right]$$

$$= \left[\sum_h \left\{ \begin{pmatrix} \tilde{C}^{11,h} & C^{12,h} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} -\lambda_0^h I & & 0 \\ & -\lambda_1^h I & \\ 0 & & -\lambda_2^h I \end{pmatrix} & 0 \\ -\tilde{A}_h & \lambda_0^h I \end{pmatrix} + C^{2,h} \begin{pmatrix} \tilde{Z}_h & \begin{matrix} b_h \\ 0 \\ 0 \end{matrix} \end{pmatrix} \right\} \right]$$

Note that the matrices $\begin{pmatrix} \tilde{C}^{11,h} & C^{12,h} \end{pmatrix}$ and $\begin{pmatrix} \begin{pmatrix} -\lambda_0^h I & & 0 \\ & -\lambda_1^h I & \\ 0 & & -\lambda_2^h I \end{pmatrix} & 0 \\ -\tilde{A}_h & \lambda_0^h I \end{pmatrix}$ have full rank. Evaluate the last expression above at $\hat{\xi}_\omega$ and pre-multiply it by

$$M_1 \equiv \begin{pmatrix} \hat{\lambda}_0^{\hat{h}} I & & & \\ & \hat{\lambda}_1^{\hat{h}} I & & 0 \\ & & \hat{\lambda}_2^{\hat{h}} I & \\ & -\tilde{A}_h & & \hat{\lambda}_0^{\hat{h}} I \end{pmatrix}^{-1} \begin{pmatrix} \tilde{C}^{11,\hat{h}} & C^{12,\hat{h}} \end{pmatrix}^{-1},$$

we have:

$$\left[-(1-\omega_{\hat{h}})I + M_1 \cdot \left\{ \begin{array}{c} \sum_{h \neq \hat{h}} -\begin{pmatrix} \tilde{C}^{11,h} & \tilde{C}^{12,h} \end{pmatrix} \begin{pmatrix} \hat{\lambda}_0^h I & & & \\ & \hat{\lambda}_1^h I & & 0 \\ & & \hat{\lambda}_2^h I & \\ & -\tilde{A}_h & & \hat{\lambda}_0^h I \end{pmatrix} + \\ \sum \hat{C}^{2,h} \begin{pmatrix} & \hat{b}_h \\ \tilde{Z}_h & 0 \\ & 0 \end{pmatrix} \end{array} \right\} \right]$$

Note that λ_h does not appear in the second term hence ω only appears in the first term of the matrix above. So we obtain the form (4.2) as desired. ■

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