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Co-operation and Timing

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Abstract

If players cannot perfectly synchronize their actions in co-ordination games, the efficient equilibrium is never achieved.

1. Introduction

Consider a simple co-ordination game, where each player can choose either a costly action ("work") or a costless action ("shirk"). Work produces a benefit exceeding its cost only if all other players also work. Such a game has an efficient Nash equilibrium, where all players work. It also has an inefficient Nash equilibrium, where all players shirk.

It is important to understand what conditions facilitate or discourage co-operation. This note shows how an inability to precisely co-ordinate the timing of actions can destroy the possibility of Pareto-improving co-operation. If it is costly to be the first person to start working, and players cannot co-ordinate precisely, then no one is willing to start working. This may be true even though the length of time when that first player must work alone becomes arbitrarily small, and thus the utility cost to him becomes arbitrarily small compared with the future bene¯ts of co-operation. He would like to commit ex ante to working early (when he may be the only one). But without the ability to commit his actions before the play of the game, strategic concerns prevent co-operation.

I consider the following environment. There are N players. Each player observes his own \clock", but observes neither other players' clocks nor their actions. Each player must decide whether or not to work, as a function of the time on his clock. Suppose that the potential individual bene¯ts from co-operation are no more than N times the cost of co-operation. Then as long as clocks are not perfectly synchronized, no player ever works. By contrast, if clocks are perfectly synchronized, any outcome is possible.

This result combines an idea from the computer science literature with recent work on higher order beliefs in economics. Halpern and Moses (1990) showed how asynchronous clocks prevent perfect co-ordination because statements about timing never become common knowledge. In fact, it is straightforward to show that such statements also never become almost common knowledge in the sense of Monderer and Samet (1989) (i.e. \common p-belief" for p close to 1). But Morris, Rob and Shin (1995) showed (in two person games) that if no event is almost common knowledge, then there will be a tendency for the risk dominant equilibrium to be played. A similar argument implies - in this context - that the Pareto-dominated \shirking" equilibrium is always played.

The connection to these papers is discussed in section 3. The model is presented in the next section.

2. Model

A collection of N players are playing a co-ordination game. Action 0 (\shirk") always gives a payo® of 0. Action 1 (\work") always entails a cost c. If all players work, then they all also receive a reward k + c. Thus payo®s for i are given by:

	all other players choose 1	some player chooses 0
i's action:		
1	k	¡c
0	0	0

Each player i must decide what action to take as a function of the time on his clock, ¿i 2 <+. Thus a pure strategy for player i is a measurable function

$s_i : \Re_+ \to \{0,1\}$. Write $s \equiv (s_1, ..., s_N)$ and $s_{-i} \equiv (s_1, ..., s_{i-1}, s_{i+1}, ..., s_N)$.

Players' clocks are not perfectly synchronized. In particular, write $t \in \Re_+$ for the "true time". Each player's clock actually begins x_i seconds after time 0, where each x_i is independently distributed on some interval $[0, \varepsilon]$, where $\varepsilon > 0$, with common continuous density $f(.)$. Write $\pi[.]$ for the probability measure on $[0, \varepsilon]^N$, i.e. an event $E \subseteq [0, \varepsilon]^N$ has probability

$$\pi[E] = \int_{x \in E} \left\{ \prod_{i=1}^{N} f(x_i) \right\} dx$$

In fact, only the following symmetry property of π will be relevant:

$$\pi[\{x : x_i \geq x_j, \text{ for all } j \neq i\}] = \frac{1}{N} \tag{2.1}$$

Thus each player attaches probability $\frac{1}{N}$ to having the slowest clock. Now if player i's clock reads τ_i, he knows that either player j's clock reads $\tau_i + x_i - x_j$ or (if $\tau_i + x_i - x_j < 0$) that j's clock has not started. Write $\zeta_i(\tau_i; s_{-i})$ for the probability i attaches to all other players working, when his clock reads τ_i and their equilibrium strategies are s_{-i}. Thus:

$$\zeta_i(\tau_i; s_{-i}) \equiv \pi\left[\left\{x \in [0, \varepsilon]^N : s_j(\tau_i + x_i - x_j) = 1 \text{ for all } j \neq i\right\}\right]$$

where we adopt the convention that $s_j(x) = 0$ if $x < 0$. Player i discounts the future with discount rate r_i starting from the time his clock starts (i.e. x_i). This is a convention only - we will see that neither the discount rate nor the starting time matter.

Thus player i's (ex ante) utility function is

$$\begin{aligned}
u_i(s) &= \int_{t=x_i}^{\infty} \left\{ \int_{x \in [0,\varepsilon]^N} \left(k \left(\prod_{j=1}^{N} s_j(t - x_j) \right) - c s_i(t - x_i) \right) \left\{ \prod_{j=1}^{N} f(x_j) \right\} dx \right\} e^{-r_i(t - x_i)} dt \\
&= \int_{\tau_i=0}^{\infty} \left\{ \int_{x \in [0,\varepsilon]^N} \left(k \left(\prod_{j=1}^{N} s_j(\tau_i + x_i - x_j) \right) - c s_i(\tau_i) \right) \left\{ \prod_{j=1}^{N} f(x_j) \right\} dx \right\} e^{-r_i \tau_i} d\tau_i \\
&= \int_{\tau_i=0}^{\infty} s_i(\tau_i) \left\{ \int_{x \in [0,\varepsilon]^N} \left(k \prod_{j \neq i} s_j(\tau_i + x_i - x_j) - c \right) \left\{ \prod_{j=1}^{N} f(x_j) \right\} dx \right\} e^{-r_i \tau_i} d\tau_i \\
&= \int_{\tau_i=0}^{\infty} s_i(\tau_i) (k \zeta_i(\tau_i; s_{-i}) - c) e^{-r_i \tau_i} d\tau_i
\end{aligned}$$

Lemma 2.1. s is a Nash equilibrium if and only if, for all i and for almost all τ_i,

$$s_i(\tau_i) = \begin{cases} 1, \text{ if } \gamma_i(\tau_i; s_{-i}) > \frac{c}{k} \\ 0, \text{ if } \gamma_i(\tau_i; s_{-i}) < \frac{c}{k} \end{cases}$$

Proof. Follows immediately from the payoff function.

Notice that, technically, Nash equilibrium allows players to choose actions arbitrarily at a negligible set of τ_i. It would be natural to focus on Nash equilibria where we imposed the additional requirement that actions maximize instantaneous utility at all dates. Then we would require that $s_i(\tau_i) = \begin{cases} 1, \text{ if } \gamma_i(\tau_i; s_{-i}) > \frac{c}{k} \\ 0, \text{ if } \gamma_i(\tau_i; s_{-i}) < \frac{c}{k} \end{cases}$ for all i and all τ_i. If we restricted attention to such refined equilibria, all the analysis and proofs which follow would remain true with "for almost all..." replaced everywhere by "for all...".

Let us first note that if the cost of working is sufficiently low (for given N), work is possible in equilibrium.

Lemma 2.2. If $\frac{c}{k} \leq \frac{1}{N}$, then there is a Nash equilibrium where all players always work ($s_i(\tau_i) = 1$ for all i and $\tau_i \in \Re_+$).

Proof. For all i and $\tau_i \in \Re_+$,

$$\begin{aligned} \gamma_i(\tau_i; s_{-i}) &= \pi[\{x : \tau_i + x_i - x_j \geq 0, \text{ for all } j \neq i\}] \\ &\geq \pi[\{x : x_i - x_j \geq 0, \text{ for all } j \neq i\}] \\ &= \frac{1}{N}, \text{ by (2.1)} \\ &\geq \frac{c}{k} \end{aligned}$$

Thus s is a Nash equilibrium by lemma 2.1.

But if this condition on payoffs is not satisfied, then all players must essentially always shirk.

Lemma 2.3. If $\frac{c}{k} > \frac{1}{N}$, then all Nash equilibria have all players almost always shirking ($s_i(\tau_i) = 0$ for all i and almost all $\tau_i \in \Re_+$).

Thus for any fixed $\frac{c}{k}$, there must always be shirking for sufficiently large N.

Proof. Suppose there exists an equilibrium s where some player i does not almost always shirk. Then we must have $\gamma_i(\tau_i; s_{-i}) > 0$ for some τ_i, which implies that all players do not almost always shirk. So let $m_i(s_i)$ be the first time at which player i following strategy s_i works. Formally, let

$$m_i(s_i) = \sup\{\tau_i : s_i(\theta) = 0 \text{ for almost all } \theta \leq \tau_i\}$$

Let $m = \min_i m_i(s_i)$. Now, for any i,

$$\begin{aligned}
\gamma_i(m; s_{-i}) &= \pi[\{x : s_j(m + x_i - x_j) = 1, \text{ for all } j \neq i\}] \\
&\leq \pi[\{x : m + x_i - x_j \geq m_j(s_j), \text{ for all } j \neq i\}] \\
&\quad \text{since } s_j(\theta) = 0, \text{ for almost all } \theta \leq m_j(s_j) \\
&\leq \pi[\{x : x_i - x_j \geq 0, \text{ for all } j \neq i\}] \\
&\quad \text{since } m_j(s_j) \geq m \text{ for all } j \\
&= \frac{1}{N}, \text{ by (2.1)} \\
&< \frac{c}{k}
\end{aligned}$$

But by continuity of $\gamma_i(\cdot; s_{-i})$, there exists $\delta > 0$ such that $\gamma_i(x; s_{-i}) < \frac{c}{k}$ for all $x \in (m; m + \delta)$. This implies that $s_i(x) = 0$ for almost all $x \in (m; m + \delta)$. Yet this implies $m_i(s_i) \geq m + \delta$ for all i, a contradiction.

Lemma 2.3 relies heavily on the asynchonization. If we allowed $\varepsilon = 0$, so that all clocks were perfectly synchronized, then any symmetric strategy profile would be an equilibrium. In particular, $s_i(\tau_i) = 1$ for all i and τ_i would always be an equilibrium, independent of the values of c and k.

3. Discussion

3.1. Relation to the literature

Why does a lack of synchronization matter so much? Halpern and Moses (1990) made the observation that, with asynchronized clocks, no statement about timing ever becomes common knowledge. In the environment we considered, when does individual i know that all clocks have started, i.e. that $\tau_j \geq 0$ for all j? Only if $\tau_i \geq \varepsilon$, so that he knows that $t \geq \varepsilon$; i knows that everyone knows that all clocks have started only if he knows that all clocks read at least ε, i.e. if $\tau_i \geq 2\varepsilon$. We can verify by induction that there is nth order knowledge that all clocks have started only if each $\tau_i \geq n\varepsilon$. Thus it never becomes common knowledge.

In fact, it not only never becomes common knowledge, it also never becomes common p-belief in the sense of Monderer and Samet (1989), for any p greater than $\frac{1}{N}$. An event is common p-belief if everyone believes it with probability at least p, everyone believes with probability at least p that everyone believes it with probability at least p, and so on. Let us see why statements about timing never become common p-belief, if $p > \frac{1}{N}$. When does everyone believe that all clocks read at least k? Suppose $\tau_i = k$. Then individual i attaches probability exactly $\frac{1}{N}$ to every individual having a time greater than or equal to k. If $p > \frac{1}{N}$, there exists $\delta > 0$, such that individual i attaches probability at least p to every individual having a time greater than or equal to k only if $\tau_i \geq k + \delta$. Thus there is nth order p-belief that all clocks have started only if each $\tau_i \geq n\delta$. Thus it never becomes common p-belief.

Monderer and Samet (1989) showed that common p-belief for p close to 1 is a sufficient condition for outcomes close to common knowledge outcomes. Morris, Rob and Shin (1995) showed how it was a necessary condition in the sense that a lack of common p-belief implied that one (out of many) common knowledge equilibria had to be played. Carlsson and van Damme (1994) earlier showed this in the particular context of noisy signals about payoffs. In this paper, noise about the time plays an analogous role.

3.2. Interpretation

This note examined a particularly simple form of asynchronization. A more realistic scenario might be the following. Up until some (stochastic) "switching time", conditions are not ripe for work and it is a dominant strategy to shirk. After that time, payoffs switch to those in this paper with Pareto-ranked symmetric equilibria. Within ε seconds of the switch, each player is informed of the switch. Each player can choose actions contingent the actual time and on the length of time since he received his message.

The analysis of this note can be extended to this more complex scenario. Depending on the ex ante probability distribution on the switching time, there may exist equilibria where everyone starts working at a certain time (and ignore the arrival of messages). The existence of such an equilibrium will depend on the ex ante distribution of the switching time. But for large N or small $\frac{c}{k}$, it is not possible to co-ordinate decisions to work using the messages. This can be shown by exactly the same kind of argument as in this note: contingent on the real time, each individual will put probability close to $\frac{1}{N}$ on having received the message with the most delay.

How should this failure to use information to co-ordinate be interpreted? Rubinstein (1989) has argued in a related context that, in practise, boundedly rational players with \high order knowledge" will behave as if they had common knowledge. In this context, this argument is unconvincing. If players decided after a certain length of time to behave \as if" there was common knowledge that all clocks had started, any symmetric view of the world would presumably require that they still attach probability $\frac{1}{N}$ to being the last to start working. For large N, this means they will not work.

I interpret this result as suggesting that imperfectly correlated information will typically prevent co-ordination (of either fully rational or boundedly rational players) when players cannot observe others' actions. Since co-ordination does in fact occur, the ability to respond to others' actions is shown to be key to generating co-ordination (see Gale (1993, 1995) and references therein).

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