Penn Institute for Economic Research
Department of Economics
University of Pennsylvania
3718 Locust Walk
Philadelphia, PA 19104-6297
pier@econ.upenn.edu
http://economics.sas.upenn.edu/pier

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"Network Formation and Systemic Risk, Second Version"

by

Selman Erol and Rakesh Vohra

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Network Formation and Systemic Risk*

Selman Erol† Rakesh Vohra‡

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Abstract

This paper introduces a model of endogenous network formation and systemic risk. In it, strategic agents form networks that efficiently trade-off the possibility of systemic risk with the benefits of trade. Efficiency is a consequence of the high risk of contagion which forces agents to endogenize their externalities. Second, fundamentally 'safer' economies generate much higher interconnectedness, which in turn leads to higher systemic risk. Third, the structure of the network formed depends crucially on whether the shocks to the system are believed to be correlated or independent of each other. This underlines the importance of specifying the shock structure before investigating a given network as a particular network and shock structure could be incompatible.

JEL classification: D85, G01.

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†Department of Economics, University of Pennsylvania (email: erols@sas.upenn.edu).

‡Department of Economics and Department of Electrical & Systems Engineering, University of Pennsylvania (email: rvohra@seas.upenn.edu).

1 Introduction

The awkward chain of events that upset the bankers, began with the collapse of Lehmann Brothers in 2008. Panic spread, the dollar wavered and world markets fell. Interconnectedness of the financial system, some suggested, allowed Lehmann's fall to threaten the stability of the entire system. Thus prompted, scholars have sought to characterize the networks that would allow shocks to one part of the financial network to be spread and amplified. Blume et al. (2013) as well as Vivier-Lirimonty (2006), for example, argue that dense interconnections pave the way to systemic failures. In contrast, Allen and Gale (2000) as well as Freixas et al. (2000), argue that a more interconnected architecture protects the system against contagion because the losses of a distressed institution can be divided among many creditors. Other than very few of them, a common feature of these and other papers (Acemoglu et al. (2013), Eboli (2013), Elliott et al. (2014), Gai et al. (2011), Glasserman and Young (2014)) is an exogenously given network. A node (or subset of them) is subjected to a shock and its propagation studied as the size of the shock varies. Absent are reasons for the presence of links between agents.[1] After all, one could eliminate the possibility of a system-wide failure by never forming links. This is, of course, silly. The presence of a link between two agents represents a potentially lucrative joint opportunity. Yet, every link increases the possibility of contagion. If agents form links anticipating the possibility of system-wide failure, what kinds of networks would they form? In particular, do they form networks that are susceptible to contagion?

In the model we use to answer these questions, agents form links. The payoff to each party that shares a link is uncertain and depends upon the future realization of a random variable (which we will call a shock) and actions taken contingent on the shock. Specifically, there are three stages. In stage one, agents form links which can be interpreted as partnerships or joint ventures. In stage two, each link formed is subjected to a shock. In stage three, with full knowledge of the realized shocks, each agent decides whether to 'default' or not. The payoff of an agent depends on the action she takes in the third stage as well the actions of her neighbors and the realized shocks. The default decision corresponds to exiting from every partnership formed in stage one. If the only Nash equilibrium of the game in stage three is that everyone defaults, we call that a system wide failure. In our model, default is the result of a 'loss of confidence' rather than simple 'spillover' effects.[2]

In the benchmark version of this model we show that the network formed in stage one,

[1]Blume et al. (2013) and Farboodi (2014) are exceptions.

[2]Glasserman and Young (2014) argue that spillover effects have only a limited impact. They suggest that the "mere possibility (rather than the actuality) of a default can lead to a general and widespread decline in valuations....."

efficiently trades-off the risk of system wide failure with the benefits of trade. Efficiency is a consequence of the high risk of contagion which forces agents to endogenize their externalities. When the risk of contagion is *decreased*, the networks formed are no longer efficient. Another factor detrimental for the level of efficiency is the extent to which parties can cooperate during the network formation stage.

A second contribution is to examine how the probability of system wide failure varies with a change in the distribution of shocks. In a setting where shocks are independent and binary (good or bad), the probability of system wide failure *increases* with an increase in the probability of a good shock, up to the point at which the formed network becomes a complete graph, i.e. every pair of agents is linked. After this point, systemic risk declines. Intuitively, as partnerships become less risky, agents are encouraged to form more partnerships increasing interconnectedness which increases the probability of system wide failure.

Our third contribution is to show that the structure of the network formed in stage one depends on whether the shocks to the system are believed to be correlated or independent of each other. When shocks are perfectly correlated, the network formed in stage one is a complete graph, i.e., a link between every pair of agents. We think this finding relevant to the debate between two theories of financial destruction advanced to explain the 2008 financial crisis. The first, mentioned above, is dubbed the 'domino theory'. The alternative, advocated most prominently by Edward Lazear [3], is dubbed 'popcorn'. Lazear describes it thusly in a 2011 opinion piece in the Wall Street Journal:

> "The popcorn theory emphasizes a different mechanism. When popcorn is made (the old fashioned way), oil and corn kernels are placed in the bottom of a pan, heat is applied and the kernels pop. Were the first kernel to pop removed from the pan, there would be no noticeable difference. The other kernels would pop anyway because of the heat. The fundamental structural cause is the heat, not the fact that one kernel popped, triggering others to follow.
>
> Many who believe that bailouts will solve Europe's problems cite the Sept. 15, 2008 bankruptcy of Lehman Brothers as evidence of what allowing one domino to fall can do to an economy. This is a misreading of the historical record. Our financial crisis was mostly a popcorn phenomenon. At the risk of sounding defensive (I was in the government at the time), I believe that Lehman's downfall was more a result of the factors that weakened our economic structure than the cause of the crisis."

[3] Chair of the US President's Council of Economic Advisers during the 2007-2008 financial crisis.

Our model suggests that underlying structural weaknesses (as modeled by strong correlations between shocks) and greater interconnectedness can coexist. Therefore, it would be incorrect to highlight the interconnectedness of the system and suggest it *alone* as the cause of instability.

Our model differs from the prior literature in three important ways.

1. The networks we study are formed endogenously.
   Babus (2013) also has a model of network formation, but one in which agents share the goal of minimizing the probability of system wide default. In our model agents are concerned with their own expected payoffs and only indirectly with the possibility of system wide failure. Acemoglu et al. (2013) also discusses network formation, but the links permitted to form are constrained by an exogenously given 'opportunity' network is exogenously given. If the opportunity network is complete then all links are permitted. However, the networks that are compared in their paper arise from different opportunity networks; one complete and the other a ring. When the opportunity network is complete, agents form a complete network. When the opportunity network is a ring, agents form a ring. In our model, agents are free to partner with anyone and we compare networks that arise within the same world of possibilities. In this sense our network formation is genuinely endogenous. Zawadowski (2013) models the decision of agents to purchase default insurance on their counter-parties. This can be interpreted as a model of network formation, but it is not a model of an agent choosing a particular counter-party. Rather, the counter-parties are fixed, and default insurance serves to change the terms of trade with an existing counter-party. The model in Farboodi (2014) includes network formation with the same solution concept we employ. However, the models are different. In particular, default in her model is not strategic. Blume et al. (2013) has networks that form endogenously. However, the risk of a node defaulting is non-strategic and independent of the network formed. In our model, the likelihood of a node defaulting depends on the structure of the network formed.

2. We examine the effects of a distribution that generates the shocks rather than the effects of fixed shocks applied to particular nodes.
   Glasserman and Young (2014) is the only exception we are aware of, but the networks they consider are exogenously given.

3. The decision to default in our model is strategic.
   We are unaware of prior work that incorporates this. In Blume et al. (2013), Acemoglu et al. (2013), Eboli (2013), Elliott et al. (2014), Gai et al. (2011), Glasserman and

Young (2014) and Farboodi (2014) for example, default is triggered when a parameter beyond the control of an agent falls below some threshold.

In section 2, we give a formal description of the model. Section 3 characterizes the set of agents that choose to default in stage three for a given realized network and realization of shocks. Section 4 uses these results to characterize the structure of the realized networks. Section 5 investigates efficiency and systemic risk of the networks formed. Section 6 discusses correlated shocks and section 7 describes some extensions to the basic model. We propose some future work in Section 8.

2 The Model

Denote by N a finite set of agents.[4] Each pair of agents in N can form a joint venture. We frequently refer to agents as **nodes** and each potential partnership as a **potential edge**.

A potential edge e, a subset of N with two elements, represents a bilateral contract whose payoff to each party is contingent on some future realized state θ^e and actions that each incident[5] node can take upon realization of θ^e. The set of possible values of θ^e is Θ, a finite set of real numbers.

The model has three stages. In stage one, the stage of network formation, agents, by mutual consent, decide which potential edges to pick. The edges picked are called realized. The set of **realized edges** is denoted E. The corresponding network denoted (N, E), is called a **realized network**.

In stage two, for each realized edge e, θ^e is chosen by nature identically and independently across edges via a distribution ϕ over Θ. We relax the independence assumption in Section 6. We denote by $(N, E, \boldsymbol{\theta})$ the realized network and vector of realized θ's.

In stage three, with full knowledge of $(N, E, \boldsymbol{\theta})$ each agent n chooses one of two possible actions called B (business as usual) or D (default), denoted by a_n. Agent n enjoys the sum of payoffs $u_n(a_n, a_m; \theta^{\{n,m\}})$ over all of his neighbors[6] m in (N, E).

We make two assumptions about payoff functions. The first is that if an agent n in (N, E) has degree one and the counter-party defaults, it is the unique best response for agent n to default as well. Formally:

[4]We abuse notation by using N to denote the cardinality of the set when appropriate.

[5]A node v is incident to an edge e if $v \in e$.

[6]Two distinct nodes v and v' are neighbors if $\{v, v'\} \subset E$. In this case v and v' are also said to be adjacent.

Assumption 1. *$u_n(D, D; \theta) > u_n(B, D; \theta)$ for all n and θ.*

The second assumption is a supermodularity which can be interpreted as a form increasing returns in fulfilling the terms of the partnership.

Assumption 2. *$u_n(D, D; \theta) + u_n(B, B; \theta) > u_n(B, D; \theta) + u_n(D, B; \theta)$ for all n and θ.*

If we focus on a pair of agents (n, m) and denote by e the realized edge between them, the payoff matrix of the game they are engaged in stage three is the following (player n is the row player and m the column player):

	B	D
B	$u_n(B, B; \theta^e)$, $u_m(B, B; \theta^e)$	$u_n(B, D; \theta^e)$, $u_m(D, B; \theta^e)$
D	$u_n(D, B; \theta^e)$, $u_m(B, D; \theta^e)$	$u_n(D, D; \theta^e)$, $u_m(D, D; \theta^e)$

A special case of this game is the coordination game of Carlsson and van Damme (1993) reproduced below that will be considered in section 4:

	B	D
B	θ^e, θ^e	$\theta^e - 1$, 0
D	0, $\theta^e - 1$	0, 0

It is clear from this last table that a pair of agents that share a realized edge play a coordination game whose payoffs depend upon the realized state variable θ^e. Following Carlsson and van Damme (1993), the game has a natural interpretation. In stage one the agents get together to pursue a joint investment. In stage two, θ^e is realized, i.e. new information arrives about the profitability of the project. In stage three, agents are allowed to reassess their decision to continue with the project or not. For other examples of games of this kind and their applications in finance see Morris and Shin (2003).

Two features of the model deserve discussion. First, in contrast to prior literature, shocks, in the form of realized states, apply to edges rather than nodes. In section 7 we extend our model to allow for shocks to nodes as well as edges. However, we believe shocks to edges to be of independent interest. An agent's solvency depends on the outcomes of the many investments she has chosen to make. The interesting case is when these investments required coordination with at least one other agent, a joint venture if you will. It is the outcome of this joint venture that will determine whether the participants decide to continue or walk away.

The second is that an agent must default on all partnerships or none. While extreme, it is not, we argue, unreasonable. Were an agent free to default on any subset of its partnerships, we could model this by splitting each node in (N, E) into as many copies of itself as its degree.[7] Each copy would be incident to exactly one of the edges that were previously incident to the original node. Thus, our model could easily accommodate this possibility. However, this has the effect of treating a single entity as a collection of independent smaller entities which we think inaccurate. Institutions considering default face liquidity constraints, which restrict, at best, the number of parties they can repay. When a company fails to pay sufficiently many of its creditors, the creditors will force the company into bankruptcy. While entities like countries can indeed selectively default, there is a knock-on effect. Countries that selectively default, have their credit ratings downgraded which raise their borrowing costs for the other activities they are engaged in. Thus, it is entirely reasonable to suppose that the default decisions associated with the edges a node is incident to must be linked. Ours is an extreme, but simple, version of such a linkage.

2.1 Solution concepts

Here we describe the solution concepts to be employed in stages one and three. We begin with stage three as the outcomes in this stage will determine the choices made by agents in stage one.

Agents enter stage three knowing $(N, E, \boldsymbol{\theta})$. With this knowledge, each simultaneously chooses action B or D. We do not allow actions chosen in stage three to be conditioned on what happens in earlier stages. The outcome in stage three is assumed to be a Nash equilibrium. While 'everybody plays D' is a Nash equilibrium, by Assumption 1, it need not be the only one. We focus on the Nash equilibrium in which largest (with respect to set inclusion) set of agents, among all Nash equilibria, play B. Call this the **cooperating equilibrium**. The proposition below shows that the cooperating equilibrium is well-defined and unique, by using rationalizable strategies.

A realized network along with realized states, $(N, E, \boldsymbol{\theta})$, exhibits **system wide failure** if in the cooperating equilibrium of the game all agents in N choose D.[8] In this case, agents can coordinate on nothing but action D. The probability of system wide failure of a realized network is called its **systemic risk**.

[7] The degree of a node in a graph is the number of edges incident to it.

[8] This is equivalent to saying that 'everybody plays D' is the only Nash equilibrium.

Proposition 1. *A cooperating equilibrium is well-defined and unique.*

Proof. Fix $(N, E, \boldsymbol{\theta})$. The profile where all agents in N play D is a Nash equilibrium by Assumption 1. Hence, D is rationalizable for everyone. Let M be the set of agents who have the unique rationalizable action D. For agents in $N \setminus M$, both B and D are rationalizable.

Consider an agent $n \notin M$. B is rationalizable, i.e., B is a best response to some strategy profile, say a_{-n}, of agents $-n$ in which agents in M play D. Let $\Delta(s_{-n})$ be the difference in payoffs for n between playing B and D against strategy profile s_{-n} of $-n$. $\Delta(a_{-n}) \geq 0$ since B is a best reply to a_{-n}.

Now consider the strategy profile b_{-n} of agents $-n$ such that agents in M play D and the rest play B. We will prove that $\Delta(b_{-n}) \geq \Delta(a_{-n})$. In a_{-n}, players in $N \setminus M$ could be playing B or D. Let $K \subseteq N \setminus M$ be those agents who play D in a_{-n} and let Γ_n be the set of neighbors of n in the realized network (N, E). Then $\Delta(b_{-n}) - \Delta(a_{-n}) =$

$$\sum_{k \in K \cap \Gamma_n} \left(u_n\left(B, B; \theta^{\{n,k\}}\right) - u_n\left(D, B; \theta^{\{n,k\}}\right)\right) - \left(u_n\left(B, D; \theta^{\{n,k\}}\right) - u_n\left(D, D; \theta^{\{n,k\}}\right)\right)$$

which is positive by Assumption 2.

As $\Delta(b_{-n}) \geq \Delta(a_{-n}) \geq 0$ it follows that B is a best reply by n to b_{-n}. This argument works for every agent in $N \setminus M$, not just n. Also, recall that D is the unique rationalizable action for agents in M so that D is the unique best reply to any strategy profile in which all agents in M play D. Therefore, a profile where all agents in M play D and all agents in $N \setminus M$ choose B is a Nash equilibrium.

Note that in any Nash equilibrium, everyone in M must play D since it is their unique rationalizable action. Therefore, "M plays D, M^c plays B" is the unique cooperating equilibrium. $\square$

The proof suggests an *equivalent definition* of a cooperating equilibrium: the rationalizable strategy profile in which those who have the unique rationalizable action D play D, while the remainder play B.

Recall that rationalizable actions are those which remain after the iterated elimination of strictly dominated actions. The iteration is as follows. Those agents who have a strictly dominant action D play D. Then, knowing that these agents play D, it becomes strictly dominant to play D for other agents to do so. This iteration stops in a finite number of steps as N is finite. The remaining action profiles are the rationalizable ones, and the cooperating equilibrium is given by the profile in which whoever is not reached in the iteration plays B.

One conceptual contribution is to establish a natural analogy between contagion of sequential defaults and rationalizable strategies. First, agents whose incident edges have realized states that cause them to default in any best response, no matter what other players do, default. Then, some agents, knowing that some of their counter-parties will default in any best response, choose to default in any best response. Then some more agents and so on. This resembles the fictitious sequential default idea put forth by Eisenberg and Noe (2001) and used as the basis of many balance-sheet contagion models of systemic risk. But, in our model rational agents foresee the sequential default and act strategically, whereas in their model, and the papers that build on it, it is an iterative process on auto-pilot that leads to default.

In stage one, agents know the distribution by which nature assigns states and the equilibrium selection in stage three. Therefore, they are in a position to evaluate their expected payoff in each possible realized network. Using this knowledge they decide which links to form. Here we describe how the realized network is formed.

Consider a candidate network (N, E) and a coalition of agents $V \subset N$. A **feasible deviation** by V allows agents in V

1. to add any absent edges within V, and
2. to delete any edges incident to at least one vertex in V.

A **profitable deviation** by V is a feasible deviation in which all members of V receive strictly higher expected payoff.[9]

A realized network (N, E) is called **pairwise stable** if there are no profitable deviations by any $V \subset N$ with $|V| \leq 2$ (see Jackson (2010)). G is in the **core**[10] if there are no profitable deviations for any $V \subset N$. We assume that the network formed in stage one is in the core. In the sequel we discuss how our main results change under weaker notions of stability.

Our use of the core can be justified as the *strong* Nash equilibrium of a non-cooperative network formation game played between the members of N. Each agent simultaneously proposes to a subset of agents to form an edge. The cost of each proposal is $b > 0$. If

[9] The requirement that all agents in a profitable deviation receive strictly higher payoff prevents ‘cycling’. To illustrate, consider three nodes $N = \{v_1, v_2, v_3\}$ and $E = \{\{v_1, v_2\}, \{v_2, v_3\}\}$. Suppose v_1 and v_3 deviating to $E' = \{\{v_1, v_3\}, \{v_2, v_3\}\}$, leaves v_1 indifferent and v_3 strictly better off. However, E' is just isomorphic to E and there is no good sense in which v_1 would bother deviating to E'. v_1 and v_2 could very well want to deviate back to E from E'. The same argument applies for E with one element as well. As one can see in this example, precluding 'weak' deviations would be overly restrictive, in particular almost trivially imposing very strong forms of symmetry on any candidate network to be formed.

[10] Farboodi (2014) calls this solution concept group stable as she uses the word to describe something else in that paper. We believe core is an appropriate name for this solution concept, illuminating the resemblance to the cooperative game notion of the core.

a proposal is reciprocated, the corresponding edge is formed. The owners of the edge are refunded b. If a proposal is not reciprocated, b is not refunded and the edge is not formed.

Notice that in any Nash equilibrium of this game, all proposals must be mutual. Consider a *strong* Nash equilibrium of the proposal game. A coalition V can make mutual proposals between themselves to form a missing edge, or undo a proposal by any member which would delete the corresponding edge. Therefore, *strong* Nash equilibria of this game correspond to core networks in the way we have defined it.

3 Structure of the Cooperating Equilibrium

For a given $(N, E, \boldsymbol{\theta})$ we characterize the structure of a cooperating equilibrium. In what follows, the following notation will be useful.

Let $\Theta = \{\theta_0, \theta_1, ..., \theta_k\}$ be the set of possible states. For each $v \in N$, let $\Delta_v(\theta) = u_v(B, D; \theta) - u_v(D, D; \theta)$, $\Delta'_v(\theta) = u_v(B, B; \theta) - u_v(D, B; \theta)$; be the gains to v from deviating to B from from D. Denote the vector of these gains by

$$\Delta_v = (\Delta_v(\theta_0), \Delta_v(\theta_1), ..., \Delta_v(\theta_k), \Delta'_v(\theta_0), \Delta'_v(\theta_1), ...\Delta'_v(\theta_k)) \in \mathbb{R}^{2k+2}.$$

Let $V^c = N \setminus V$ denote the complement of V in N for $V \subset N$. For a given $(N, E, \boldsymbol{\theta})$ let $d(v)$ be the degree of $v \in N$ and $d(v, V, \theta_s)$ be the number of edges in state θ_s which are incident to v and V. Let

$$\pi_s(V|v) = \frac{d(v, V, \theta_s)}{d(v)}$$

be the portion of v's edges that are incident to V and has state θ_s. Denote the vector of these ratios respectively for V^c and V by

$$\pi^v(V) = (\pi_0(V^c|v), \pi_1(V^c|v), ..., \pi_k(V^c|v), \pi_0(V|v), \pi_1(V|v), ..., \pi_k(V|v)) \in \mathbb{R}^{2k+2}.$$

Strictly speaking our notation should depend upon $(N, E, \boldsymbol{\theta})$. However, as these are all fixed in stage three we omit doing so.

Notice that $\Delta_v(\theta) < 0$ and $\Delta_v(\theta) \leq \Delta'_v(\theta)$ for all θ and v (by Assumptions 1 and 2). The following lemma characterizes an agent's best response to the actions of other agents.

Lemma 1. *Consider a $V \subset N$ and $v \in N$. Suppose that agents in $V \setminus \{v\}$ play B, and agents in $(N \setminus V) \setminus \{v\}$ play D. Then D (B) is the unique best reply of v if and only if $\Delta_v \cdot \pi^v(V) < 0$ ($\Delta_v \cdot \pi^v(V) > 0$).*

Proof. Straightforward. □

Call a $V \subset N$ **strategically cohesive** if for all $v \in V$

$$\Delta_v \cdot \pi^v(V) \geq 0.$$

Proposition 2. *In the cooperating equilibrium, an agent v plays B if and only if there exists a strategically cohesive set V with $v \in V$.*

Proof. (If part) By the 1, $\Delta_v \cdot \pi^v(V) \geq 0$ implies that B is a best reply by v if all players in V play B and others play D. D is rationalizable for every player, therefore, B can never be eliminated for players in V. For all players in V playing B is rationalizable. Hence in the cooperating equilibrium, all of V, in particular v, play B.

(Only if part) Suppose not. Then N is not strategically cohesive (since $v \in N$) and there exists $v_1 \in N$ such that $\Delta_{v_1} \cdot \pi^{v_1}(N) < 0$. Notice that $\pi^{v'}(V') = \pi^{v'}(V'/\{v'\}) = \pi^{v'}(V' \cup \{v'\})$ for any V' and v' since nodes are not adjacent to themselves. Then, $\Delta_{v_1} \cdot \pi^{v_1}(N/\{v_1\}) < 0$. By Lemma 1, v_1's best response to $N/\{v_1\}$ playing B is D. By Assumption 2, v_1's best response to any strategy profile, in particular any strategy profile not eliminated, is D. Thus, v_1 plays D in a cooperating equilibrium. Hence $v_1 \neq v$. Let $N_1 = N/\{v_1\}$. $v \in N_1$. Therefore, by supposition, N_1 is not strategically cohesive. Hence, there exists $v_2 \in N$ such that $\Delta_{v_2} \cdot \pi^{v_2}(N_1) < 0$. Similarly, by Lemma 1 and Assumption 2, v_2's best response to any profile in which $N_1/\{v_2\}$ plays B, in particular any strategy profile not eliminated, is D. Thus v_2 plays D in the cooperating equilibrium, and $v_2 \neq v$. Let $N_2 = N_1/\{v_2\}$. Since N is finite and v plays D in the cooperating equilibrium, we reach a contradiction in a finite number of steps. □

Lemma 2. *If V and V' are both strategically cohesive, then $V \cup V'$ is also strategically cohesive.*

Proof. Consider a $v \in V$. We show that $\Delta_v \cdot [\pi^v(V \cup V') - \pi^v(V)] \geq 0$. In this summation the t'th component is $\Delta_v(\theta_t) \times [\pi_t((V \cup V')^c\,|v) - \pi_t(V^c|v)]$ and $k+t$'th component is $\Delta'_v(\theta_t) \times [\pi_t((V \cup V')\,|v) - \pi_t(V|v)]$. The terms in the brackets add up to 0. Hence the sum of these two terms is equal to $[\Delta'_v(\theta_t) - \Delta_v(\theta_t)] \times [\pi_t((V \cup V')\,|v) - \pi_t(V|v)] \geq 0$ by Assumption 2.

Therefore, $\Delta_v \cdot \pi^v(V \cup V') = \Delta_v \cdot \pi^v(V) + \Delta_v \cdot [\pi^v(V \cup V') - \pi^v(V)] \geq \Delta_v \cdot \pi^v(V) \geq 0.$ □

Call a set $V \subset N$ **maximally cohesive** if it is the largest strategically cohesive set. This is well-defined by Lemma 2.

Proposition 3. *In the cooperating equilibrium, all members of the maximally cohesive set play B, all the others play D.*

Resilience to system wide failure at stage three is determined by the existence of a strategically cohesive set.[11] Strategic cohesiveness is determined by both Δ_v and $\pi^v(V)$. The first captures the effect of payoffs, while the second captures the effect of the structure of the realized network with states. This suggests that the correct ex-post notion of fragility cannot rely on purely network centric measures. Even if one were to look for an appropriate network centric component of a good measure, it would not be measures like too-interconnected-to-fail (which is silent about the neighbors of the neighbors of the too-interconnected node), or degree sequences (which is silent about local structures), but rather cohesiveness which incorporates the idea of a group of nodes reinforcing each other and resisting contagion that began elsewhere.

To separate the effects of network and payoff structure we make some simplifying assumptions and examine their consequences below.

3.1 Separating network and payoff effects

We suppose that $\theta < 1$ for all $\theta \in \Theta$ and that payoff functions are the same across agents: $u_v \equiv u$. In particular:

Assumption 3. $u_v(B, B; \theta) = \theta$, $u_v(B, D; \theta) = \theta - 1$, $u_v(D, B; \theta) = u_v(D, D; \theta) = 0$ *(in line with Carlsson and van Damme (1993)).*

For each $V \subset N$ and $v \in N$ let $d(v, V)$ be the number of v's neighbors that are in V. Let

$$\pi(V|v) = \frac{d(v, V)}{d(v)}.$$

Let

$$\pi(v) = (\pi_0(N|v), \pi_1(N|v), ..., \pi_k(N|v)).$$

Given $(N, E, \boldsymbol{\theta})$, a set $V \subset N$ is said to be **ex-post cohesive** if $\pi(V|v) + \theta \cdot \pi(v) \geq 1$ for all $v \in V$. The term $\theta \cdot \pi(v)$ captures v's individual resilience from his payoffs, $\pi(V|v)$ captures the collective resilience of V as a function of network structure. If V is sufficiently resilient individually and collectively, then it is ex-post cohesive. Notice that under Assumption 3, strategic cohesiveness reduces to ex-post cohesiveness.

[11] One can think of strategic cohesive sets as 'firebreaks'.

For a given $(N, E, \boldsymbol{\theta})$, vertices within an ex-post cohesive set all play B. Thus, they resist default through mutual 'support'. To illustrate, suppose a vertex v's incident edges all have states that are negative valued. Then, $1 - \theta \cdot \pi(v) > 1$ so that v cannot be part of any ex-post cohesive set. Thus, v defaults for sure. As another example, suppose all elements of Θ are positive. Then, the maximally cohesive set would be N itself for *any* possible case of $(N, E, \boldsymbol{\theta})$. Thus, in any realization, all agents play B. Similarly, if all states in Θ were negative, the maximally cohesive set would be the empty set. In every realization all agents would choose D, i.e., there would be certainty of system wide failure.

3.1.1 p-cohesiveness

Ex-post cohesiveness is closely related to p-cohesiveness introduced in Morris (2000). The significance and relevance of p-cohesiveness is further illuminated in Glasserman and Young (2014). Given $p \in \mathbb{R}$, a set V is **p-cohesive** if for all $v \in V$, $\pi(V|v) \geq p$. p-cohesiveness imposes a uniform bound on the number of neighbors each vertex in V has in V. Ex-post cohesiveness imposes heterogeneous bounds on the same quantity that depend solely on the realized characteristics of v, particularly how v's edge states are distributed.[12] Notice that if Θ was a singleton, say $\{\theta_0\}$, ex-post cohesiveness would be equivalent to $(1-\theta_0)$-cohesiveness.

p-cohesiveness is an ex-ante concept relying only on the structure of (V, E). Ex-post cohesiveness, as its name suggests, is an ex-post concept that depends on $(N, E, \boldsymbol{\theta})$. To illustrate, consider a realized edge with a "bad state" $\theta < 0$ in which $\Delta_v(\theta)$ and $\Delta'_v(\theta)$ are very small. The presence of such an edge would help a set 'containing' the edge become "more" p-cohesive, however it makes it "less" ex-post cohesive. In this sense, lack of strategic cohesiveness is the appropriate ex-post notion of fragility taking into account the variety in states, while lack of p-cohesiveness is possibly an appropriate ex-ante notion of fragility when the states of edges are not yet realized.

3.2 Two states

We introduce a further simplification, $|\Theta| = 2$, with one state being positive and the other negative. This will be convenient for the analysis of the network formation stage and is sufficient to capture most of the essential intuition.

Assumption 4. *$\Theta = \{\theta_0, \theta_1\}$, $\theta_1 < 0 < \theta_0$.*

In addition:

[12] Ex-post cohesiveness can trivially be applied to situation in which edges have heterogeneous volumes.

Assumption 5. $0 < \theta_0 < min\{\frac{1}{N-1}, \frac{-\theta_1}{N-2}\}$.

Assumption 5 ensures that the maximum possible sum of gains from trade scale linearly with N. Another way to interpret this is that the system as a whole cannot withstand bad shocks that make up a fraction of more than $1/N$ of all edges. This assumption simplifies contagion dynamics and buys us great technical convenience in the benchmark model as we will see in the next proposition. We relax this assumption later on.

A **path** between two nodes v_0 and v_{k+1} is a sequence of nodes $v_0, v_1, ..., v_k, v_{k+1}$ such that $\{v_i, v_{i+1}\} \subset E$ for all $i = 0, 1, ..., k$. Two nodes are connected nodes if there is path between them. A subset V of nodes is a connected set if any two elements of V are connected by a path that is resides entirely in V. $V \subset N$ **maximally** connected if V is connected and there is no strict superset of V that is connected.

Proposition 4. *Fix $(N, E, \boldsymbol{\theta})$. A set $V \subset N$ of nodes is ex-post cohesive if and only if it is (ex-ante) maximally connected and (ex-post) all edges with endpoints in V have state θ_0.*

Proof. Choose any $V \subset N$ and any $v \in V$. Observe that $\pi(V|v) = 1$ if and only if all of v's neighbors are in V. Otherwise $\pi(V|v) \leq 1 - \frac{1}{N-1}$. Also, $\theta \cdot \pi(v) = \theta_0$ if and only if all edges of v are θ_0. Otherwise $\theta \cdot \pi(v) \leq \frac{N-1}{N}\theta_0 + \frac{1}{N}\theta_1 < 0$. Note that $1 - \frac{1}{N-1} + \theta_0 < 1$ and $1 + \frac{N-1}{N}\theta_0 + \frac{1}{N}\theta_1 < 1$. Therefore, $\pi(V|v) + \theta \cdot \pi(v) \geq 1$ if only if both $\pi(V|v) = 1$ and $\theta \cdot \pi(v) = \theta_0$ hold. Equivalently, V is ex-post cohesive if and only if for any $v \in V$ all of v's neighbors are in V and all edges incident to v are in state θ_0. □

In the cooperating equilibrium, an agent defaults even if only one edge in the agent's maximally connected component is in the bad state. This is a consequence of the strong contagion embedded in Assumption 5. The condition $0 < \theta_0 < \frac{-\theta_1}{N-2}$ ensures that anyone incident to at least one bad edge defaults. The condition $0 < \theta_0 < \frac{1}{N-1}$ ensures that anyone who has at least one defaulting neighbor also defaults. In a later section, we relax this assumption and discuss its consequences.

4 Network Formation

In this section we characterize the set of core networks under the assumptions stated previously. We show that a core network consists of a collection of node disjoint complete subgraphs[13]. By forming into complete subgraphs agents increase the benefits they enjoy from partnerships. However, the complete subgraphs formed are limited in size[14] and or-

[13] A graph (N', E') is a subgraph of (N, E) if $N' \subset N$ and $E' \subset E$.

[14] The size of a subgraph or a subset of edges is the number of edges in it.

der[15], and are disjoint. In this way agents ensure that a default in one portion of the realized network does not spread to the entire network. This extreme structure is a consequence of the spareness of our model. However, it suggests that more generally we should expect to see collections of densely connected clusters that are themselves sparsely connected to each other. Blume et al. (2013) have a similar finding in their paper.

We first need to determine an agent's expected payoff in various realized networks. Recall that nature determines states identically and independently across edges. Let α be the probability that an edge has state θ_0 and $1-\alpha$ be the probability that it has state θ_1. Consider $v \in N$ and suppose that in a realized network, v has degree d and the maximally connected component that contains v has e edges. By virtue of Proposition 4 we need only consider the case where everyone in the maximally connected component defaults or no one does. The probability that every node in the relevant component defaults is $1-\alpha^e$. In this case, v gets 0. The probability that no one in the relevant component defaults is α^e. In this case, v gets $d\theta_0$. So v's expected payoff in stage two is $d\alpha^e\theta_0$. Using this, we can find what happens in stage one.

Being pairwise stable, henceforth stable, is a necessary condition for being a core network. We first identify conditions on stable networks, then move onto core networks.

4.1 Stable networks

Lemma 3. *Any stable network consists of disjoint complete subgraphs.*

Proof. Suppose, for a contradiction, a stable network with two non-adjacent nodes v' and v'' in the same connected component. Take a path $v' = v_1$, v_2,... $v_t = v''$ between v' and v''. Insert the edge $\{v', v''\}$ and delete $\{v', v_2\}$, as well as $\{v_{t-1}, v''\}$. The degrees of v' and v are unchanged but the number of edges in the component that contains them strictly decreases. Hence, this is a profitable pairwise deviation by v' and v'' which contradicts stability. Therefore, in any stable network all nodes within the same connected component are adjacent, which completes the proof. □

The orders of these complete subgraphs are not arbitrary. Let $U(d) := d\alpha^{(0.5)d(d+1)}$, and $d^* = \arg\max_{d\in\mathbb{N}} U(d)$. For generic α, d^* is well defined. Note that $U(d)$ is strictly increasing in $d \in \mathbb{N}$ up to d^*, and strictly decreasing after d^*. Further, d^* is an increasing step function of α. Let $h^* \geq d^*$ be the largest integer h such that $U(1) \leq U(h)$. Let $h^{**} \leq d^*$ be the largest integer such that $\frac{1}{\alpha} \leq \frac{h+1}{h}\alpha^{(0.5)h(h+1)} = \frac{U(h+1)}{h\alpha^h}$.

[15] The order of a subgraph or a subset of nodes is the number of nodes in it.

Proposition 5. *Any network that consists of disjoint complete subgraphs, each with order between $h^{**}+1$ and h^*+1, is stable. Call these* ***uniform stable networks.***[16]

Proof. Consider a uniform-stable network and suppose that there is a profitable bilateral deviation by two nodes. Take one of them, let her have degree d, and let her have $e = d(d+1)/2$ edges in her complete subgraph. Suppose that in the bilateral profitable deviation she deletes x of her incident edges in her complete subgraph, and adds $t \in \{0, 1\}$ new edges.

If $x = d$, her payoff is at most $\alpha = U(1) \leq U(d)$ (since $1 \leq d \leq h^*$) which cannot be a profitable deviation. So $x < d$, which means she is still incident to $e - x$ edges in her old component. Then her payoff is at most $(d-x+t)\alpha^{e-x+t}$. If $t - x \leq 0$, this is less than $d\alpha^e$ since $y\alpha^y$ is strictly increasing up to k^* in $y \in \mathbb{N}$ and $h \leq k^*$. Then $t - x > 0$, which is possible only when $t = 1$ and $x = 0$. This is true for the other deviator as well. Therefore, these two deviators keep all their previous edges and connect to each other with a new edge.

Let the other deviator have degree d'. Without loss of generality, let $d \leq d'$. Then, the deviator with the smaller degree has her payoff moved from $d\alpha^{(0.5)d(d+1)}$ to $(d+1)\alpha^{1+(0.5)d(d+1)+(0.5)d'(d'+1)}$ which is less than or equal to $(d+1)\alpha^{1+d(d+1)}$. This being a profitable deviation immediately implies $d < h^{**}$, which is a contradiction. □

4.2 Core networks

Lemma 4. *If a network is in the core, it consists of a collection of disjoint complete subgraphs, all but one of order (d^*+1). The remaining complete subgraph is of order at most d^*+1.*

Proof. By Lemma 3 a core network (if it exists) is composed of disjoint complete subgraphs. The payoff to an agent in a $(d+1)$-complete subgraph is $U(d) = d\alpha^{(0.5)d(d+1)}\theta_0$. This is strictly increasing up to d^*.

First, no complete subgraph can have order $d+1 > d^*+1$ in the realized network. Otherwise, d^*+1 members could deviate by forming a (d^*+1)-complete subgraph and cutting all other edges. This would be a strict improvement since d^* is the unique maximizer of $U(d)$.

Second, there cannot be two complete subgraphs of order $d+1 < d^*+1$. Suppose not. Let there be $d'+1$ nodes all together in these two complete subgraphs. Then $min\{d'+1, d^*+1\}$

[16] This is close to a complete characterization of all stable networks in the following sense. Any complete subgraph in any stable network has to be of order at most h^*+1. Moreover, there can be at most one complete subgraph with order less than $h^{**}+1$. The bound on the smallest order depends on what the second smallest order is, and is more involved to characterize.

nodes would have a profitable deviation by forming an isolated complete subgraphs since $U(d)$ is increasing in d up to d^*. $\square$

A realized network in the core necessarily consists of a collection of complete subgraphs of order $d^* + 1$ and one 'left-over' complete subgraph with order different from $(d^* + 1)$. To avoid having to deal with the 'left-over' we make a parity assumption about N. For the remainder of the analysis we assume $N \equiv 0\,(mod\, d^* + 1)$. In fact, without this assumption, the core may be empty. To see why, assume that the 'left-over' complete subgraph is of order 1. This single left-over agent would like to have any edge rather than having none, and any other agent would be happy to form that edge since that extra edge does not carry excess risk. We would expect a pairwise deviation which would contradict the stability. However, even in this case, $N - 1$ agents don't have a deviation among themselves without using the single left over node. In section 7 we consider solution concepts other than core or stability as well.

Theorem 1. *For $N \equiv 0\,(mod\, d^* + 1)$, the core is non-empty, unique (up to permutations) and consists of disjoint $(d^* + 1)$-complete subgraphs.*

Proof. Assuming non-emptiness of the core and the parity assumption, Lemma 4 suffices to yield uniqueness once we have existence. It remains to show that a realized network $G = (N, E)$ consisting of disjoint complete subgraphs $C_1, C_2, ..., C_k$ all of order $(d^* + 1)$ (for k such that $N = k\,(d^* + 1)$) is a core network.

For any profitable deviation by V' from G to G', define $\phi(V', G')$ as the number of edges between V' and N/V' in G'. Let the minimum of ϕ be attained at (V^*, G^*).

Consider G^*. Take a node $v' \in V^*$ that is adjacent to N/V^*. Suppose that there exists $v'' \in V^*$ such that v' is connected but not adjacent to. Cut one edge connecting v' to N/V^* and join the missing edge between v' and v''. This new graph, say G'', is also a profitable deviation by V^* from G. This is because when we move from G^* to G'', the degrees of all nodes in V^* weakly increase, and their component sizes weakly decreases. However, $\phi(V^*, G'') < \phi(V^*, G^*)$, which is a contradiction. Therefore, any node in V^* that is connected to v' is adjacent to it. The same holds for any node that is adjacent to N/V^*.

Take a node in V^* with minimal degree, say v with degree d. Let $d' \geq 0$ be the number of v's neighbors in N/V^*. Suppose $d' \geq 1$. By the last paragraph, a node in V^* that is connected to a neighbor of v can only be a neighbor of v. Therefore, any neighbor of v in V^* has at most $d - d'$ neighbors in V^*, hence at least $d' \geq 1$ neighbors in N/V^*. So by the last paragraph, v and his $d - d'$ neighbors in V^* are all adjacent to each other, forming $(0.5)(d - d' + 1)(d - d')$

edges. Each of them have at least d' edges to N/V^*, so that makes $d'(d-d'+1)$ edges. Finally, since nodes in N/V^* have not deviated from G and are connected to each other, they are all adjacent to each other, forming $(0.5)d'(d'-1)$ edges. Therefore, in v's maximally connected component, there are at least $(0.5)d(d+1)$ edges, so that his payoff is at most $U(d)$. Now suppose $d'=0$. Then all v's d neighbors are in V^*, hence all have degree at least d. Then again, v's component has at least $d(d+1)/2$ edges, so that his payoff is at most $U(d)$. In both cases, v's payoff in G^* is at most $U(d) \leq U(d^*)$; contradiction with profitable deviation from G. □

Theorem 2. *For $N < d^*+1$, the unique core network is the N-complete subgraph.*

Proof. Recall that $d\alpha^{(0.5)d(d+1)}$ is increasing in d up to $d^* > N$. The remainder of the proof follows the proof of Theorem 1 by replacing d^*+1 with N. We omit the details. □

5 Efficiency and Systemic Risk

In this section we define what it means for a network to be efficient and show that a network is efficient if and only if it is in the core. The other stable networks are inefficient, which suggests that some inefficiencies in observed networks stem from the inability of large groups to coordinate.

We also identify another source of inefficiency by relaxing the assumptions governing the strength of contagion. When bad shocks are highly contagious, any expected externality that a node imposes on others turns back on itself, and is naturally internalized. On the other hand, when bad shocks are weakly contagious, agents don't need to consider anyone other than their immediate neighbors. As a consequence, they don't internalize their externalities which leads to excess connectivity and inefficiency.

We further show that systemic risk in the efficient/core network increases as the probability α of a good shock increases. This follows the safety belt argument: as the economy gets safer, agents form networks with higher systemic risk. This intuition, however, may change with different notions of systemic risk.

5.1 Efficiency

5.1.1 The efficient network

Call a realized network (N, E) **efficient** if it maximizes the sum of expected payoffs of agents among all realized networks. Consider a connected subgraph with e edges. A node in the subgraph with degree d enjoys an expected payoff of $d\alpha^e\theta_0$. Therefore, the sum of payoffs of nodes within the graph is $2e\alpha^e\theta_0$. Here we use the well known fact that the sum of degrees is twice the number of edges. It follows then, that the problem of finding an efficient network devolves into two parts: how to partition nodes into maximally connected components, and how many edges to put into each component.

Let $k^* = \arg\max_{y\in\mathbb{N}} y\alpha^y$. For generic α this is well defined.[17] Note that $y\alpha^y$ is strictly increasing in $y \in \mathbb{N}$ up to k^* and strictly decreasing after k^*. Note also that when maximizing $y\alpha^y$ over the non-negative reals, the maximum occurs at a number $y^* = -\frac{1}{log(\alpha)}$ where $\alpha^{y^*} = \mathbf{e}^{-1}$. Here $\mathbf{e}$ is Euler's constant and y^* lies in the interval $(\frac{\alpha}{1-\alpha}, \frac{1}{1-\alpha})$.

Theorem 3. *If $N \equiv 0\,(mod\, d^* + 1)$, a network is efficient if and only if it is in the core.*

Proof. Recall that $U(x) = x\alpha^{(0.5)x(x+1)}$. Let $\mathcal{U} = \{u \in \mathbb{R} \,|\, u = U(x) \text{ for some } x \in \mathbb{N}\}$. The maximum of $\mathcal{U}$ is achieved, uniquely, at $x = d^*$. Let $\bar{u} = U(d^*)$. Notice that this is the average payoff at the core network. We will prove that the average is strictly less in any other network.

Consider an efficient network G and suppose it to be made up of a collection of disjoint connected components: $C^1, C^2, C^3, \ldots$. Consider component C^i and suppose it has q_i edges. The total payoff of C^i scales with $2q_i\alpha^{q_i}$. If $q_i \neq k^*$ we can improve total payoff by deleting or adding (if not complete graph) edges to C^i. Therefore, we can assume that $q_i = k^*$, or that C^i is complete.

Let r_i be the largest integer such that $r_i(r_i-1)/2 < q_i \leq r_i(r_i+1)/2$. Let w_i be such that $q_i = r_i(r_i-1)/2 + w_i$, where $1 \leq w_i \leq r_i$. Note that there must be at least $r_i + 1$ nodes in C^i.

Case 1: $1 \leq w_i \leq \frac{r_i-1}{2}$.

The average degree of nodes in C^i is at most $\frac{2k^*}{r_i+1} = \frac{r_i(r_i-1)+2w_i}{r_i+1} \leq r_i - 1$. Note that $k^* = q_i \geq (r_i-1)r_i/2+1$. Hence the average payoff per node is at most $(r_i-1)\alpha^{k^*} < (r_i-1)\alpha^{\frac{(r_i-1)r_i}{2}} \leq \bar{u}$. So the average payoff is strictly less than $\bar{u}$.

Case 2: $r_i - 1 \geq w_i \geq \frac{r_i}{2}$.

Since $w_i < r_i$, $k^* = q_i \leq r_i(r_i+1)/2 - 1$. The average degree of nodes in C^i is at most $\frac{2k^*}{r_i+1} \leq \frac{r_i(r_i+1)-2}{r_i+1} \leq r_i - \frac{2}{r_i+1}$. Note that $k^* = q_i = (r_i-1)r_i/2 + w_i \geq r_i^2/2$. Hence the average payoff per node is at most $\left(r_i - \frac{2}{r_i+1}\right)\alpha^{r_i^2/2}$. Now we show that this is strictly less than $(r_i-1)\alpha^{(r_i^2-r_i)/2} = U(r_i-1)$. That is equivalent to showing that $\alpha < \left(\frac{r_i+1}{r_i+2}\right)^{2/r_i}$. Recall that k^*

[17] For α such that $(1-\alpha)^{-1}$ is integral, there are two integers in the arg max: $\frac{\alpha}{1-\alpha}$ and $\frac{1}{1-\alpha}$. In other cases, the arg max is unique: it is the unique integer in the open interval $(\frac{\alpha}{1-\alpha}, \frac{1}{1-\alpha})$, i.e. $\lfloor\frac{1}{1-\alpha}\rfloor$.

is the unique integer between $\alpha/(1-\alpha)$ and $1/(1-\alpha)$. Therefore, $\alpha \leq 1-\frac{1}{k^*+1} \leq 1-\frac{2}{r_i(r_i+1)}$. Hence, it suffices to verify that

$$1-\frac{2}{r_i(r_i+1)} < \left(\frac{r_i+1}{r_i+2}\right)^{\frac{2}{r_i}} \iff \left(\frac{r_i+1}{r_i+2}\right)^{\frac{2}{r_i}} > \frac{(r_i+2)(r_i-1)}{(r_i)(r_i+1)}$$

$$\iff (r_i+2)\log\left(1-\frac{1}{r_i+2}\right) > r_i \log\left(1-\frac{1}{r_i}\right)$$

which is true since the function $f(x) = x\log(1-\frac{1}{x})$ is strictly increasing. Therefore, the average payoff is strictly less than $U(r_i-1) \leq U(d^*) = \bar{u}$.

Case 3: $w_i = r_i$. (This covers the case in which C^i is complete as well.)
Then the average payoff per node is less than $U(r_i) \leq \bar{u}$, and the inequality is strict unless C^i is a $(d^*+1)-$complete graph. □

All stable networks other than the core network are, thus, inefficient.[18] This suggests that some inefficiencies that arise in the observed networks may stem from the inability of large groups to coordinate at the network formation stage.

In order to focus on the benchmark case we economize on the proofs of other results by sketching lengthy ones or omitting entirely, proofs similar to previous proofs, in the paper. The fundamental techniques we use are contained in the proofs provided thus far.

5.1.2 Relaxing the strength of contagion

In this subsection only, we relax the assumption governing the strength of contagion to provide better intuition for why agents may or may not form efficient networks.

In Assumption 5, $\theta_1 + (N-2)\theta_0 < 0$ ensures anyone incident to an edge subject to a bad shock defaults, whatever his degree $d \leq N-1$ is. This allows a single bad shock to start a contagion, and we keep this unchanged here. The condition $(\theta_0 - 1) + (N-2)\theta_0 < 0$ ensures that a node, even when all incident edges are good, has to default if at least one neighbor defaults, whatever his degree $d \leq N-1$ is. This governs the spread of contagion, and we relax this condition here.

First note that under $\theta_1 + (N-2)\theta_0 < 0$, a realized network is Nash [19] only if the degrees of all nodes are less than or equal to k^*.

[18] Blume et al. (2013) find that their stable networks are not efficient. However, their notion of efficient is a worst-case one, very different from the one employed here. Farboodi (2014) also finds that formed networks are inefficient, despite having the core as the solution concept.

[19] A network in which no node has a profitable unilateral deviation

If $2(\theta_0 - 1) + (N-3)\theta_0 < 0$ that would mean, a node incident to a bad edge, and has degree exactly N, defaults if she has two defaulting neighbors. But, it is unlikely for relatively large N that any node will have degree N since in any Nash, hence stable, hence core network, all degrees must be less than or equal to k^*. What is actually relevant for a node with degree d is $2(\theta_0 - 1) + (d-3)\theta_0 < 0$, hence we could safely relax the assumption by many degrees, especially for large N.

For this reason, we consider the other extreme, as a way of retarding contagion: if all a node's incident edges are good, she defaults only when all her neighbors default. As long as one neighbor does not default, she does not default either. Formally: $(N-2)(\theta_0 - 1) + \theta_0 > 0$.

In this case, the expected payoff of an agent who has degree d, and whose neighbors have degrees $n_1, n_2, ..., n_d$ is

$$\frac{1}{\alpha}\alpha^d \left(\alpha^{n_1} + \alpha^{n_2} + ... + \alpha^{n_d}\right).$$

Define $k^{**} := argmax_{d \in \mathbb{N}} d\alpha^{2d-2}$. Note that $\frac{k^*}{2} - 1 \leq k^{**} \leq \frac{k^*}{2} + 1$.

Proposition 6. *A network is efficient if and only if it is k^{**}-regular.*

Proof. See appendix. □

Proposition 7. *Consider any stable network. There are at most $\left(\frac{k^*+1}{2}\right)^2$ many nodes with degree different than $k^* - 1$. The remainder have degree $k^* - 1$. In this sense, any sufficiently large stable network is almost $k^* - 1$ regular, hence inefficient.*

Proof. See appendix. □

Note that stable networks exhibit almost double the efficient level of degree per node. In this sense, there is excess interconnection at any stable network when the risk of contagion is low. There are other properties of stable networks which are not of first order importance, thus omitted.[20]

Proposition 8. *If $N \geq \left(\frac{k^*}{2} + 1\right)^2$ and $\alpha > \frac{2}{\mathbf{e}}$, the core is empty.*

Proof. See appendix. □

Recall that $k^* = \lfloor \frac{1}{1-\alpha} \rfloor$. When α is such that $k^* < \frac{1}{1-\alpha} - \alpha$, there are no stable networks for large N. For α such that $\frac{1}{1-\alpha} - \alpha < k^* < \frac{1}{1-\alpha}$, we have the following.

[20] For example, nodes with degrees other than $k^* - 1$ are in close proximity to each other.

Proposition 9. *If $N \equiv 0\,(mod\, k^*)$ and α such that $\frac{1}{1-\alpha} - \alpha < k^* < \frac{1}{1-\alpha}$, then, a network that consists of disjoint complete subgraphs of order k^* is stable.*

Proof. See appendix. □

When contagion is very strong, any externality imposed on another at any distance, comes back to 'bite one.' The strength of contagion ensures nodes internalize their externalities. Hence, they form efficient structures, in the form of complete subgraphs. When contagion is very weak, nodes no longer internalize the externalities they impose on others. Therefore, efficiency is lost. This highlights the risk of contagion (conditional on it being initiated) as a source of *efficiency* (but not necessarily higher welfare with respect to the weak contagion case), rather than inefficiency, in our main result.

5.1.3 Comparative Statics

We return to the benchmark model with strong contagion, and provide some comparative statics on efficiency.

Note that the total payoff in a network which consists of disjoint complete subgraphs of order $d+1$ is $N \times U(d)$. The figures below illustrate the differences in connectivity and efficiency between core and stable networks. [21]

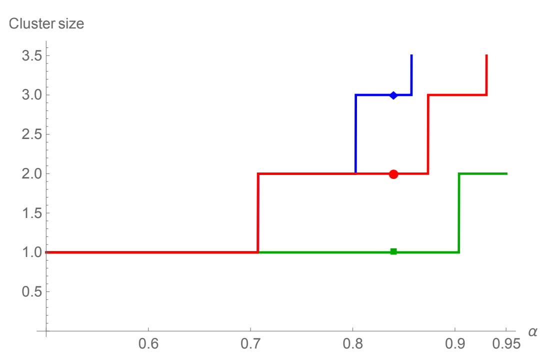

Figure 1(a): For $0.5 < \alpha < 0.9$

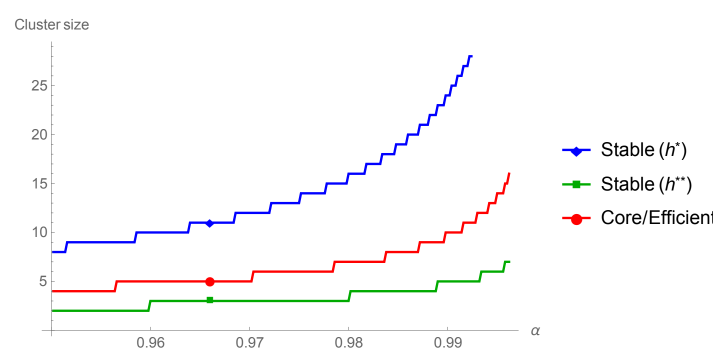

Figure 1(b): For $\alpha > 0.9$

Figure 1: Cluster Sizes of Stable and Core Networks vs. α

[21] We plot the properties of the the most and the least interconnected uniform-stable networks, the ones with cluster size h^* and h^{**}.

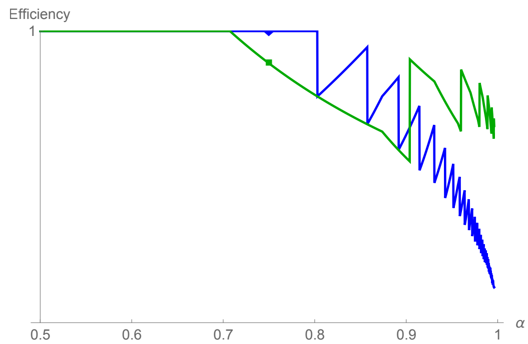

Figure 2(a): Efficiency in Stable Networks vs. α

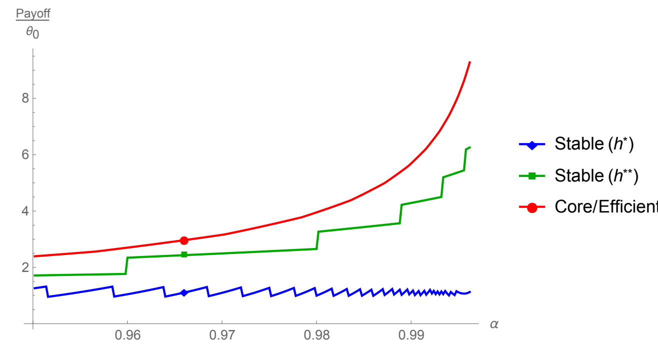

Figure 2(b): Payoff Per Node in Stable and Core/Efficient Networks vs. α

Figure 2: Payoffs and Efficiency in Stable Networks

5.2 Systemic risk

5.2.1 Systemic risk at the core/efficient network

Fix $N \equiv 0\,(mod\, d^* + 1)$ and consider the core network. Recall that all nodes of a maximal complete subgraph play D if at least one of the edges in the complete subgraph is in a 'bad' state; otherwise they all choose action B. The probability that any node/all nodes in a maximal complete subgraph chooses D is $1 - \alpha^{(0.5)d^*(d^*+1)}$. Hence, the probability that everybody defaults, i.e. systemic risk, is

$$\left(1 - \alpha^{(0.5)d^*(d^*+1)}\right)^{\frac{N}{d^*+1}}.$$

For fixed α, the above expression is increasing in $d^* < N$. An increase in d^* leads to fewer but larger complete subgraphs. Thus, for fixed α higher interconnectedness translates into higher systemic risk. For fixed d^*, the expressiong decreases in α. However it is not apriori clear whether systemic risk increases or decreases with a change in α. Note that as α increases, the core consists of fewer but larger clusters. As one can see in Figure 3, it turns out, systemic risk of the core/efficient network increases with α. In our model, d^* (weakly) increases with α. It increases at such a rate that systemic risk of the core/efficient network also increases with α.[22] This is displayed in Figure 3.

[22] Since d^* is a step function of α, in intervals where d^* stays constant the probability decreases. However, this is an artifact of discreteness. When α hits $\left(\frac{d-1}{d}\right)^{\frac{1}{d}}$, d^* jumps from $d-1$ to d. If one considers these jumping points of α, the probability is increasing. In order to clarify further, recall the definition of $d^* = argmax_{d \in \mathbb{N}} d\alpha^{(0.5)d(d+1)}$. For a "smooth version" of d^* as a function of α, a real number $d^* = argmax_{d \in \mathbb{R}} d\alpha^{(0.5)d(d+1)}$, the probability is strictly increasing.

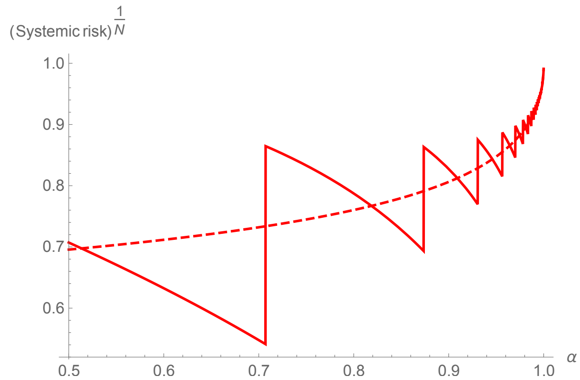

Figure 3: Systemic Risk of the Core Network vs. α

Intuitively, as the economy gets fundamentally safer, agents form much larger clusters. That is in their individual interest and furthermore the outcome is efficient. However, the risk from interconnectedness dominates the safety from α, and this results in *increased* systemic risk: catastrophic events become more frequent. Note that once α becomes too large and hits $\left(\frac{N-1}{N}\right)^{\frac{1}{N}}$, d^* becomes N and the clusters cannot get any larger. Hence the systemic risk cannot get any larger and it starts decreasing again.

Figures 4 below show how the expected number of defaults, $N \times \left(1 - \alpha^{(0.5)d^*(d^*+1)}\right)$ varies with α.

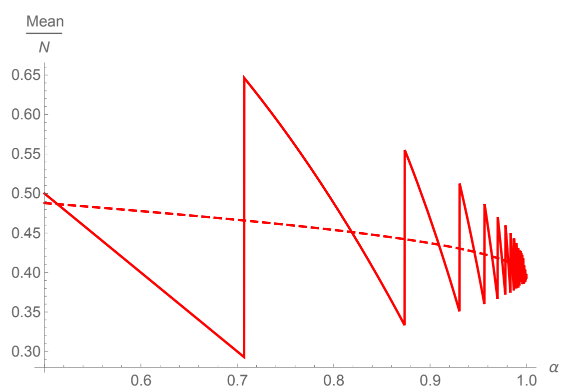

Figure 4: Mean of the Number of Defaults at Core Network vs. α

We can actually pin down the exact distribution of the number of nodes that default. Given α, the number of maximal complete subgraphs that fail is k with probability

$$\binom{\frac{N}{d^*+1}}{k}\left(1 - \alpha^{(0.5)d^*(d^*+1)}\right)^k \left(\alpha^{(0.5)d^*(d^*+1)}\right)^{\frac{N}{d^*+1}-k}.$$

This is also the probability that $(d^*+1)k$ agents default and the rest do not. For $N=100$, Figure 5 illustrates the distribution.

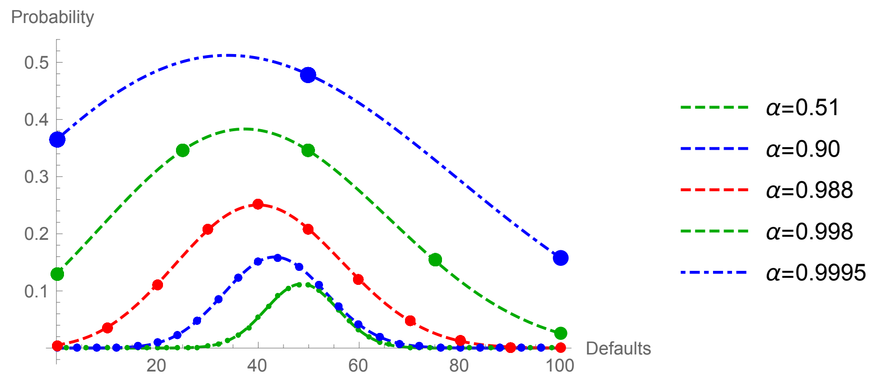

Figure 5: Probability Distribution of the Number of Defaults at Core Network (For $N=100$)

There is no first order stochastic dominance order among these distributions. However, the distributions with larger α's second order stochastically dominate those with smaller α's.

5.2.2 Systemic risk at stable and core/efficient networks

Next, we compare the systemic risk of stable networks with core/efficient networks. Call uniform stable networks whose maximal complete subgraphs all have order larger than or equal to d^*+1 be called **upper-uniform stable** networks, and those with all maximal complete subgraphs having order smaller than d^*+1 be called **lower-uniform stable** networks.

Proposition 10. *Take $N \equiv 0 \,(\mathrm{mod}\, d^*+1)$. Upper-uniform (lower-uniform) stable networks have higher (lower) systemic risk than the core/efficient network.*

Proof. Recall that $\left(1-\alpha^{(0.5)x(x+1)}\right)^{1/x}$ is increasing in x. Take any complete subgraph with order $d+1 \geq d^*+1$.

$$1-\alpha^{(0.5)d(d+1)} = \left(1-\alpha^{(0.5)d(d+1)}\right)^{(d+1)/(d+1)} \geq \left(1-\alpha^{(0.5)d^*(d^*+1)}\right)^{(d+1)/(d^*+1)}.$$

Let d_t+1's be the orders of maximally complete subgraphs of a upper-uniform stable network. Then

$$\prod_t \left(1-\alpha^{(0.5)d_t(d_t+1)}\right) \geq \left(1-\alpha^{(0.5)d^*(d^*+1)}\right)^{\frac{1}{d^*+1}\sum d_t+1} = \left(1-\alpha^{(0.5)d^*(d^*+1)}\right)^{\frac{N}{d^*+1}}.$$

The case for lower-uniform stable networks have the similar proof. $\square$

Figure 6 illustrates the difference in systemic risk between stable and core networks for various values of α.

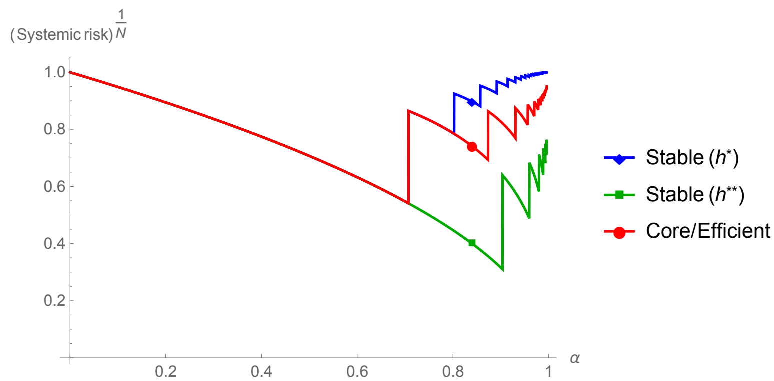

Figure 6: Systemic Risk in Stable and Core Networks vs. α

These findings suggest that some inefficiencies in observed networks may stem from the inability of parties to coordinate. However, systemic risk of these inefficient networks can be more or less than that of the core network. Thus, systemic risk is not a good indicator of inefficiency. The frequency of catastrophic events can be more or less at inefficient networks than the efficient network.

6 Correlation

We noted earlier a debate about whether interconnectedness of nodes is a significant contributor to systemic risk. An alternative theory is that the risk faced is via common exposures, i.e., popcorn. Observed outcomes might be similar in both scenarios but the dynamics can be significantly different.

We model the popcorn story as perfect correlation in states of edges through ϕ. Thus, ϕ is such that with probability σ *all* edges have state θ_0, with probability $1 - \sigma$ *all* edges are in state θ_1.

There is no change in the analysis of stage three. As for stage one, now there is no risk of contagion.

Theorem 4. *Under 'popcorn', the unique core (and unique stable) network is the complete graph on N nodes, denoted K_N.*

Proof. In any given realized network, if all states are θ_0 then everybody play B and if all states are θ_1 then everybody play D. The payoff of an agent with d edges is $d\theta_0$ or 0 respectively.

Thus, the expected payoff of each agent is $d\sigma\theta_0$. Then, it is clear that in a core (or stable) network there cannot be any missing edges because that would lead to a profitable pairwise deviation. The only candidate is K_N which is as clearly in the core. $\square$

When agents anticipate common exposures (popcorn) rather than contagion, they form highly interconnected networks in order to reap the benefits of trade.

In an independent shocks world, the probability that everybody defaults in K_N is $1 - \alpha^N$, which is the highest systemic risk that any network can achieve in this world. However, K_N is as safe as all the other possible realized networks in the correlated shocks world. This highlights the importance of identifying the shock structure before investigating a given network. A specific network and a particular shock structure might very well be incompatible.

6.1 More general correlation

Perfect correlation and complete independence are two extremes. Here we extend the benchmark model to allow for a correlation structure that is in between. With some probability the economy operates as 'normal' and edges are subject to their own idiosyncratic shocks, while with complementary probability a common exposure to risk is realized and all edges have bad states. Formally, with probability $1 - \sigma$ all edges are θ_1, while with probability σ all states of edges are i.i.d.: θ_0 with probability α and θ_1 with probability $1 - \alpha$. Notice that '$\sigma = 1$, $\alpha > 0$' is the extreme case of 'independence with α being the probability of an edge being in a good state'. The case '$\alpha = 0$, $\sigma < 1$' is the extreme case of 'perfect correlation with σ being the probability of all edges being in a good state'.

In this setting, the expected payoff of an agent is $d\alpha^e\sigma\theta_0$. Clearly, the identical analysis in section 4 goes through for any σ. Notice that as α tends to 1, d^* diverges to ∞. For some $\bar{\alpha} < 1$, $\alpha > \bar{\alpha}$ implies that $d^* > N$. Then, by Theorem 2, the unique core is K_N. This illustrates that Theorem 4 is not an anomaly due to perfect correlation. In fact, it is a corollary of Theorem 2; the same result holds for sufficiently strong correlation not just perfect correlation.

7 Extensions

We summarize three variations to our model to illustrate robustness of our results. The first considers weaker notions of network formation. The second allows for shocks to nodes in addition to edge shocks. Lastly, we consider different forms of asymmetries between nodes and see how the results are altered.

7.1 Weaker notions of network formation

The results above about the core assume the ability of any coalition to get together and 'block'. Networks that survive weaker notions of blocking are also of interest. Two natural candidates are Nash networks and stable networks. The first preclude deviations by single nodes only, while the second by pairs only. All core networks are pairwise stable, and all pairwise stable networks are Nash networks.

Robustness to unilateral deviations is too permissive. Most (permutation classes of) graphs with degree less than k^* are Nash networks. This is because no node can add an edge in a feasible Nash deviation. As for deleting edges, for graphs that are sufficiently well connected a unilateral deletion will not reduce the cluster size very much. Hence, agents are not going to delete edges since they already have less than k^* edges. We have already studied stability before in the benchmark model.

Here we consider the middle ground between the core and stable networks. Call a network (N, E) **t-stable** if no coalition of size t or less has a profitable deviation. Notice that N-stable is equivalent to the core, and 2-stable is equivalent to stable.

Proposition 11. *For any $t \geq d^* + 1$, the unique t-stable network is the core.*

Keeping in mind that we typically think of $d^* + 1$ as being relatively small with respect to N, this proposition shows us that the results in the paper don't need the full power of the core that precludes profitable deviations by any coalition. A restriction on relatively small sized coalitions is sufficient. The next theorem concerns $t \leq d^*$.

Proposition 12. *Take any $t \leq d^*$. Let $h^*(t) \geq d^*$ be the largest integer such that $U(t) \leq U(h^*(t))$. Any network that consists of disjoint complete subgraphs, each with order between $d^* + 1$ and $h^*(t) + 1$, is t-stable. Call these **upper-uniform t-stable** networks.*

Notice that as $t \leq d^*$ gets smaller, upper-uniform t-stable networks become similar to upper-uniform stable networks. As $t \leq d^*$ gets larger, $h^*(t)$ approaches d^*+1, so that upper-uniform t-stable networks become closer to core networks. After d^*, for $t \geq d^* + 1$ the only t-stable network is the core itself (the upper-uniform $(d^* + 1)$-stable network). These results bridge the gap between the core and stability.

As t gets larger, t-stable-complete networks become more efficient in a sense. Networks are subjected to further constraints by precluding deviations by larger coalitions, and the remaining set of networks get closer to the efficient/core networks, increasing the efficiency. Similarly, systemic risk of upper-uniform t-stable networks decline with larger values of t.

7.2 Node shocks

We now consider shocks to individual nodes. There are two ways to think about such shocks. The first is an idiosyncratic shock that affects an institution without any direct effect to any other institution, such as liquidity shocks. The second is one in which the financial sector has ties with the real sector and these ties are subject to shocks as well. In the model, each node (financial institution) is incident to an (imaginary) edge outside of the network. The shock to this edge is effectively an idiosyncratic shock to the node itself.

These shocks can be correlated but we consider the case of independent node-shocks only. Formally, after stage two has ended and before we move on to stage three, each 'imaginary' edge independently defaults with probability $1-\beta$ or proceed as **normal** with probability β.

In stage three, ex-post cohesive sets are maximally connected sets all of whose edges are in state θ_0 and nodes are normal. In this case members of such a set play B and get θ_0 for each edge they have. Otherwise they play D and get 0.

In stage two, the expected payoff of a node with degree d in a maximally connected component with e edges and f nodes has payoff, $d\alpha^e\beta^f\theta_0$.

As for stage one, the earlier results apply. A core network will consist of disjoint complete subgraphs. Let $d^{**} := \arg\max_{d\in\mathbb{N}} d\alpha^{(0.5)d(d+1)}\beta^{d+1}$. Theorems and comparative statics concerning the core apply with d^* replaced by d^{**}.

Note that d^{**} is smaller than d^*. This tells us that when agents are exposed to new types of risks, which effectively increases their overall risk, they form less interconnected networks.

7.3 Different Types of Agents

The ex-ante symmetry of agents leads to symmetric realized network as well. Here, we allow one agent to differ from the others in its exposure to risks from states of edges.

This one agent, named C, has a utility function which does not depend on the state of its incident edges. In particular, for some fixed $p \in (0,1)$, $u_C(B,B;\theta) = p$, $u_C(B,D;\theta) = p-1$,

$u_C(D,B;\theta) = u_C(D,D;\theta) = 0$ for every θ. On the other hand, the other agents enjoy the same payoffs as in the benchmark model from all their incident edges, except the edges with C. The payoffs associated with edges incident with C have the form: $u(B,B;\theta) = \theta + \varepsilon$, $u(B,D;\theta) = \theta - 1$, $u(D,B;\theta) = u(D,D;\theta) = 0$.[23] In particular, the game played on the edges of C is given by

	B	D
B	p, $\theta + \varepsilon$	$p-1$, 0
D	0, $\theta - 1$	0, 0

For technical convenience, we take p such that $\frac{1}{1-p}$ is an integer: $s^* := \frac{1}{1-p} \in \mathbb{N}$, and $p \geq \alpha^* := \alpha^{(0.5)d^*(d^*+1)}$.[24] Subsequently we will provide an interpretation of agent C as a 'lender'.

Call a set of nodes not containing C a **group** if these nodes are connected without using paths going through C. If a group is connected to C, call it a C**-group**, otherwise an NC**-group**. If C defaults, everybody in all C-groups default in any strategy profile that survives iterated dominance. If strictly more than p portion of C's neighbors play D, node C's only best response is to play D. If at most fraction p of C's neighbors play D, then B is a best response of C to the belief that the remaining nodes play B. Therefore, the unique cooperating equilibrium is given by: 1) all NC-groups behaving as in the benchmark case, 2) if more than p portion of C's neighbors have at least one bad edge in their group, all C-groups and C play D, 3) if more than or equal to $1-p$ portion of C's neighbors have all good edges in their group, then those groups and C play B, the other C-groups play D.

Proposition 13. *Any stable network consists of some complete subgraphs each containing vertex C but are otherwise disjoint, and some other disjoint complete subgraphs.*

Proof. See appendix. □

Thus, C becomes a central node with many clusters around it, which are still internally densely connected. The number of attached clusters can be large at stable networks, so that C serves as a channel through which contagion might spread from one cluster to the other. In this sense, this 'favored' node becomes too central and contributes excessively to systemic risk.

[23] ε can be thought of as a robustness or selection tool. Without this slight perturbation, indifferences lead to many candidates for core which are less intuitive than the unique candidate for the core with this perturbation. We don't provide explicit bounds on ε but it can be chosen to be bounded away from 0 as N diverges to infinity.

[24] It is easy to check that $\alpha^* > 0.5$, indeed very close to 0.6 independently of α.

Proposition 14. *Take N such that $N > 1+(d^*+1)s^*$. Any core network consists of exactly s^* many complete subgraphs of order d^*+1 that include C and are otherwise disjoint, and some isolated disjoint complete subgraphs of order d^*+1, and possibly one more left-over isolated complete subgraph of order less than d^*+1.*

Proof. See appendix. □

In stable networks, there can be many complete subgraphs, possibly more than s^* many, that include C. However, in the core, there are at most s^* complete subgraphs that contain C. When s^* or fewer complete subgraphs contain C, a contagion that starts at some complete subgraph cannot cause C to default. In fact, even if all but one of the complete subgraphs that contain C defaults it is still a best response for C not to default. If, however, there are at least s^*+1 complete subgraphs containing C, if all but one default, then, C will default. Thus, no complete subgraph will want to connect to C once C is contained in too many complete subgraphs as this would increase the risk of contagion from other complete subgraphs.

The comparison of stable and core networks here reinforces the previous intuition that the inability of large groups to coordinate leads to inefficiencies. Moreover, we see here that the number of firms matter for the global properties of the network. In an economy where there are a few firms, the result resembles networks with highly interconnected central nodes. However, if the number of firms keeps growing, while the number of risk free nodes remain bounded, the network is going to look more and more like the core in the benchmark model.

7.3.1 Borrowing and lending

Here we illustrate how C can be interpreted as a lender. Every investment, in the benchmark case, requires two partners. Now, suppose that the agents can undertake these ventures solo only if they can find outside funding. Node C represents this outside funding source. No other node can serve in this role. Without borrowing from C, agents must form partnerships for the investments.

An investment undertaken by a single agent n with the backing of C will involve two funding rounds, at the amounts $x \geq 1$ and $y > 0$ respectively. After the initial investment x, C and n are informed what the stochastic gross return R will be on the investment. Execution requires a second stage infusion of y. Lending x involves risk and requires a gross rate of return $r > 1$ determined exogenously. Lending y is optional and decided after R is observed. This is riskless and the gross rate of return on y is 1.

An edge between n and C represents a decision by C to extend to n the initial amount of x. After the edge is formed, x is a sunk cost for C. After R is determined by nature in the second stage, both C and n must decide whether to continue with the project.

If both C and n choose to continue (this will correspond to action B), C lends n the extra y and the investment is complete. Node n obtains R and pays C back $rx + y$. Hence the payoff to C is $rx + y - x - y = (r - 1)x$ and to n is $R - rx - y$.

If C chooses to continue (action B) but n defaults (play D), then C does not give y, and n does not return the initial x. The payoffs to C in this case is $-x$ and to n is 0.

If C chooses to stop (action D), but n chooses B, n pays C back rx which he owes (C uses these funds to pay its other debts and still defaults), but does not obtain y, and hence cannot complete the project. Therefore, the payoffs are 0 for C and $-rx$ for n. If both play D, both get 0. The game form is given by

	B	D
B	$rx - x$, $R - rx - y$	$-x$, 0
D	0, $-rx$	0, 0

Define p to be $1 - \frac{1}{r}$. Since all edges of C have the same payoff structure, his payoffs can be scaled for normalization. Multiply C's payoffs by $\frac{1-p}{x}$. Assume that the uncertainty in R is tied to the state of the edge θ in the form $R = \varepsilon + y + rx + \theta$. Then the game form on the edges of C becomes:

	B	D
B	p, $\theta + \varepsilon$	$p - 1$, 0
D	0, $-c$	0, 0

Here $c > 1$. This is identical to the extension outlined above, modulo c. Notice this does not effect our results as long as $c > 1 - \theta$ for all θ, which is true. The interest rate r could be determined endogenously via $\frac{1}{1-p^*}$ where p^* is the endogenous probability of default for n. That is beyond the scope of this paper.

7.3.2 Other forms of asymmetry

There can be many forms of asymmetries between nodes and edges. For example α's could be different. Indeed, if all α's are in an interval (α_0^2, α_0) for some $\alpha_0 \in (0, 1)$, then stable networks still consist of disjoint complete subgraphs.

Alternatively, consider the benchmark model with node shocks with differing individual default probabilities.

Proposition 15. *If there is one firm with a different node shock probability, say $\beta' > \beta$, everything follows similarly. The core exists and is unique and consists of disjoint cliques of order $d^* + 1$ for appropriate modularity of N.*

If there are several groups of people such that each group has number of people divisible by d^*+1 and members of each group have the same β among themselves, possibly different across groups, then there is assortative matching in the core: 'safer' firms cluster with 'safer' firms from top to bottom.

8 Future Work

The model we introduce is tractable and rich. We have considered some extensions, and many more important extensions are possible. We list some of them here.

A major extension is allowing for government intervention in the contagion and/or network formation stages. Would the anticipation of government intervention be harmful due to moral hazard costs, or would the ex-post gains from intervention outweigh moral hazard costs ? Should there be caps on the ability of a government to intervene ? What are the welfare implications of specific policies ? Furthermore, government reputation can be considered when the model is cast into a dynamic framework.

As we have illustrated in the asymmetry section, borrowing and lending can be incorporated into the model and endogenous prices can be tractably determined.

Another important but difficult extension is introducing asymmetric information. For example in stage three, nodes could be modeled to know the states of their incident edges but not the rest. It is important to see the what happens in that case, yet it is significantly harder to solve for technical reasons.

In the network formation stage, we have introduced a proposal game to micro-found the solution concepts. The agents could have started off with an existing status-quo network, and build extra edges on top of the the existing ones. It would be interesting to see how this will alter the resulting network. Furthermore, one can think of a dynamic proposal game to see whether first-movers tend to become too central.

Recall that the maximal cohesive sets protect themselves from contagion, and this result is independent of the particular coordination game later embedded. Network formation is driven by the utility functions, and it is important to see what other utility functions, symmetric or asymmetric among agents, lead to. Some that are of particular interest would be those that resemble borrowing and lending correspondences.

Other extensions can include allowing for more than two actions; allowing for moderate strength of contagion; allowing for heterogeneous volumes of edges; allowing for bilateral transfers between neighbors and allowing for different forms of correlations of shocks.

9 Conclusion

In our model, rational agents who anticipate the possibility of system wide failure during network formation, guard against it by segregating themselves into densely connected clusters that are sparsely connected to each other. As the economy gets fundamentally safer, they organize into larger clusters which results in an increase in systemic risk.

Whether the networks formed efficiently trade-off the benefits of surplus generation against systemic risk depends on two factors. First is the ability of agents to coordinate among themselves during network formation. If the networks formed are robust to bilateral deviations only, they are inefficient. If robust to deviations by relatively larger subsets, they are fully efficient. Second, is the infectiousness of counter-party risk, which serves as a natural mechanism for agents to internalize externalities. With strong contagion, agents recognize they are in the same boat during network formation.

Our model highlights that assessing the vulnerability of a network to system wide failure cannot be done in ignorance of the beliefs of agents who formed that network. Efficient markets generate structures that are safe under the correct specification of shocks, which will appear fragile under the wrong specification of the shock structure. Thus, mistakes in policy can arise from a misspecification in the correlation of risks.

Asymmetries between firms can lead to the emergence of 'central' institutions. However, it does not follow that they are 'too-big' or 'too-interconnected' if the networks formed are in the core. If the networks are robust to bilateral deviations only, then, there can be excess interconnectedness around these central institutions which can generate an excessive risk of contagion. However, in a large enough economy, these central groups become marginal and isolated.

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10 Appendix

Proof of Proposition 6

Proof. (Sketch) By Cauchy-Schwarz inequality, the total expected payoff is less than $\sum_{i \in N} d_i \alpha^{2d_i}$, which is less than $Nk^{**}\alpha^{2k^{**}}$. This bound is achieved if and only if the network is k^{**}-regular. □

Proof of Proposition 7

Proof. (Sketch)

1. A realized network is Nash if and only if the degree of all nodes is less than or equal to k^*.
2. Take a stable network. Let $\underline{d}$ be the smallest degree of any node in the network. If two nodes have degrees d_1 and d_2 such that $k^* + \underline{d} \geq d_1 + d_2 + 2$, then they are adjacent.
3. Any node with degree $d_i \leq k^* - 2$ must be adjacent to each node with degree $\underline{d}$. Thus, there can be at most $\underline{d}$ nodes with degrees less than or equal to $k^* - 2$. ($\underline{d} \leq k^* - 1$).
4. Take a stable network. Take two nodes v', v'' with degrees at most d, which are adjacent to a third node with degree at least $d + 1$. Then v' and v'' are adjacent.
5. Nodes with degree at most $k^* - 1$ who have neighbors of degree k^* form a clique. Then, there can be at most $max\{m(k^* - 1 - m)\} \leq \left(\frac{k^*-1}{2}\right)^2$ nodes with degree k^* that have neighbors with degrees smaller than k^*.
6. If two nodes v', v'' have degrees k^*, and all their neighbors also have degree k^*, then v' and v'' are adjacent. Such nodes form a clique, so that there can be at most $k^* + 1$ many such nodes.
7. Bringing all the pieces together: there can be at most $k^* - 1 + \left(\frac{k^*-1}{2}\right)^2 + k^* + 1 = \left(\frac{k^*+1}{2}\right)^2$ many nodes with degree different from $k^* - 1$.

□

Proof of Proposition 8

Proof. (sketch)

1. All nodes with degree less than or equal to $k^* - 2$ form a clique.
2. There are at most $\left(\frac{k^*-1}{2}\right)^2$ nodes which have degree at most $k^* - 2$ or have neighbors with degree less than equal to $k^* - 2$. The remainder have degree at least $k^* - 1$ and all neighbors with degree $k^* - 1$ or k^*. Hence all these others have payoff at most $\frac{1}{\alpha}\left(k^*\alpha^{k^*}\right) \times \alpha^{k^*-1}$.

3. k^{**} is very close to $k^*/2$. Hence there are at least k^{**} people who would like to deviate and form an isolated clique. The core is empty.

□

Proof of Proposition 9

Proof. (sketch)

1. As nodes are already in cliques, no two non-adjacent nodes have a common neighbor. Hence, no node is willing to delete an edge to gain at most one other edge
2. No two nodes from disjoint cliques are willing to connect due to their already high degree.

□

Proof of Proposition 13

Proof. (sketch)

1. Any group has to be a clique.
2. All nodes of a C-group has to be adjacent to C.

□

Proof of Proposition 14

Proof. (sketch)

1. Let the C-groups be indexed by $t = 1, 2, ..., c$ and P_t be the probability that C plays B conditional on '-t has no bad edges'.
2. Among all C-groups, at most one can have $P_t < 1$.
3. Among all C-groups, at most one can have nodes less than d^*, all the rest have exactly d^* nodes.
4. If among all C-groups at most one has order less than d^*, all the rest have exactly d^* nodes, then $P_t = 1$ for all but at most one if and only if $c \leq s^*$.
5. If $c \leq s^*$, and all but one C-group has order d^*, then the remainder C -group is also of order d^*.
6. If all C-groups are of order d^*, and $c \leq s^*$, then $c = s^*$.
7. Among all NC-groups, at most one can have nodes less than d^*, all the rest have exactly d^* nodes.

□