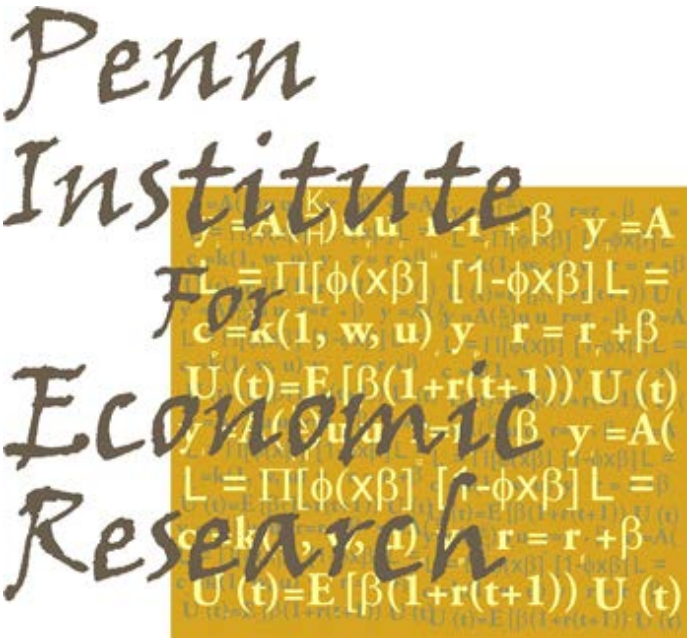

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PIER Working Paper 14-045

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Second Version

http://ssrn.com/abstract=2536358

Using Invalid Instruments on Purpose: Focused Moment Selection and Averaging for GMM*

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This Version: December 9, 2014 First Version: November 9, 2011

Abstract

In finite samples, the use of a slightly endogenous but highly relevant instrument can reduce mean-squared error (MSE). Building on this observation, I propose a moment selection criterion for GMM in which moment conditions are chosen based on the MSE of their associated estimators rather than their validity: the focused moment selection criterion (FMSC). I then show how the framework used to derive the FMSC can address the problem of inference post-moment selection. Treating post-selection estimators as a special case of moment-averaging, in which estimators based on different moment sets are given data-dependent weights, I propose a simulation-based procedure to construct valid confidence intervals for a variety of formal and informal moment-selection and averaging procedures. Both the FMSC and confidence interval procedure perform well in simulations. I conclude with an empirical example examining the effect of instrument selection on the estimated relationship between malaria transmission and income.

Keywords: Moment selection, GMM estimation, Model averaging, Focused Information Criterion, Post-selection estimators

JEL Codes: C21, C26, C52

*I thank Aislinn Bohren, Xu Cheng, Gerda Claeskens, Bruce Hansen, Byunghoon Kang, Toru Kitagawa, Hannes Leeb, Adam McCloskey, Serena Ng, Alexei Onatski, Hashem Pesaran, Benedikt Pötscher, Frank Schorfheide, Neil Shephard, Richard J. Smith, Stephen Thiele, Melvyn Weeks, and seminar participants at Brown, Cambridge, Columbia, George Washington, Oxford, Queen Mary, St Andrews, UPenn, Vienna, and the 2011 Econometric Society European Meetings for their many helpful comments and suggestions. I thank Kai Carstensen for providing data for my empirical example.

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1 Introduction

In finite samples, the addition of a slightly endogenous but highly relevant instrument can reduce estimator variance by far more than bias is increased. Building on this observation, I propose a novel moment selection criterion for generalized method of moments (GMM) estimation: the focused moment selection criterion (FMSC). Rather than selecting only valid moment conditions, the FMSC chooses from a set of potentially mis-specified moment conditions based on the asymptotic mean squared error (AMSE) of their associated GMM estimators of a user-specified scalar target parameter. To ensure a meaningful bias-variance tradeoff in the limit, I employ a drifting asymptotic framework in which mis-specification, while present for any fixed sample size, vanishes asymptotically. In the presence of such *locally mis-specified* moment conditions, GMM remains consistent although, centered and rescaled, its limiting distribution displays an asymptotic bias. Adding an additional mis-specified moment condition introduces a further source of bias while reducing asymptotic variance. The idea behind the FMSC is to trade off these two effects in the limit as an approximation to finite sample behavior.[1]

I consider a setting in which two blocks of moment conditions are available: one that is assumed correctly specified, and another that may not be. This is intended to mimic the situation faced by an applied researcher who begins with a "baseline" set of relatively weak maintained assumptions and must decide whether to impose any of a collection of stronger but also more controversial "suspect" assumptions. When the (correctly specified) baseline moment conditions identify the model, the FMSC provides an asymptotically unbiased estimator of AMSE, allowing us to carry out risk-based selection over the suspect moment conditions. When this is not the case, it remains possible to use the AMSE framework to carry out a sensitivity analysis.[2]

Continuing under the local mis-specification framework, I go on to derive the limit distribution of "moment average estimators," data-dependent weighted averages of estimators based on different moment conditions. These estimators are interesting in their own right and can be used to study the important problem of inference post-selection. I propose a simple, simulation-based procedure for constructing valid confidence intervals that can be applied to a variety of formal moment averaging and post-selection estimators including the FMSC. Using an applied example from development economics, I show that this procedure

[1] When finite-sample mean-squared error (MSE) is undefined or infinite, AMSE comparisons remain meaningful. In this case, one can view AMSE as the limit of a sequence of "trimmed" squared error loss functions, as in Hansen (2013). Trimmed MSE is always well-defined and the trimming fraction can be made asymptotically negligible.

[2] For discussion of this point, see Appendix C.

is well within the ability of a standard desktop computer for problems of a realistic scale.

While the methods described here apply to general GMM models, I focus on two simple but empirically relevant examples: choosing between ordinary least squares (OLS) and two-stage least squares (TSLS) estimators, and selecting instruments in linear instrumental variables (IV) models. In the OLS versus TSLS example the FMSC takes a particularly transparent form, providing a risk-based justification for the Durbin-Hausman-Wu test, and leading to a novel "minimum-AMSE" averaging estimator that combines OLS and TSLS. It is important to note that both the FMSC and related minimum-AMSE averaging estimator considered here are derived for a *scalar* parameter of interest, as this is the most common situation encountered in applied work. As a consequence, Stein-type results do *not* apply: it is impossible to construct an estimator – post-selection, averaging or otherwise – with uniformly lower risk than the "valid" estimator that uses only the baseline moment conditions in estimation. Nevertheless, it remains possible to achieve substantially lower risk than the valid estimator over large regions of the parameter space, particularly in settings where the additional moment conditions are highly informative and *nearly* correct. This is precisely the situation for which the FMSC is designed. Selection and averaging are not a panacea, but the methods presented in this paper can provide substantial gains in realistic settings, as demonstrated in the simulation results presented below.

My approach to moment selection is inspired by the focused information criterion of Claeskens and Hjort (2003), a model selection criterion for maximum likelihood estimation. Like Claeskens and Hjort (2003), I study AMSE-based selection under mis-specification in a drifting asymptotic framework. In contradistinction, however, I consider moment rather than model selection, and general GMM rather than maximum likelihood estimation. Schorfheide (2005) uses a similar approach to select over forecasts constructed from mis-specified vector autoregression models, developed independently of the FIC. Mine is by no means the first paper to consider GMM asymptotics under locally mis-specified moment conditions, an idea that dates at least as far back as Newey (1985). The idea of using this framework for AMSE-based moment selection, however, is novel.

The existing literature on moment selection under mis-specification is primarily concerned with consistent selection: the goal is to select all correctly specified moment conditions while eliminating all invalid ones with probability approaching one in the limit.[3] This idea begins with Andrews (1999) and is extended by Andrews and Lu (2001) and Hong et al. (2003). More recently, Liao (2013) proposes a shrinkage procedure for consistent GMM moment selection and estimation. In contrast to these proposals, which examine only the validity of

[3]Under the local mis-specification asymptotics considered below, consistent moment selection criteria simply choose *all* available moment conditions. For details, see Theorem 4.2.

the moment conditions under consideration, the FMSC balances validity against relevance to minimize AMSE. Although Hall and Peixe (2003) and Cheng and Liao (2013) do consider relevance, their aim is to avoid including redundant moment conditions after consistently eliminating invalid ones. Some other papers that propose choosing, or combining, instruments to minimize MSE include Donald and Newey (2001), Donald et al. (2009), and Kuersteiner and Okui (2010). Unlike the FMSC, however, these proposals consider the *higher-order* bias that arises from including many valid instruments rather than the first-order bias that arises from the use of invalid instruments.

Another important difference between the FMSC and the other proposals from the literature is the "F" – focus: rather than a single moment selection criterion, the FMSC is really a method of constructing application-specific moment selection criteria. To see the potential benefits of this approach consider, for example, a simple dynamic panel model. If your target parameter is a long-run effect while mine is a contemporaneous effect, there is no reason to suppose *a priori* that we should use the same moment conditions in estimation, even if we share the same model and dataset. The FMSC explicitly takes this difference of research goals into account.

Like Akaike's Information Criterion (AIC), the FMSC is a *conservative* rather than consistent selection procedure, as it *remains random* even in the limit. Although consistency is a crucial minimal property in many settings, the situation is more complex for model and moment selection: consistent and conservative selection procedures have different strengths, but these strengths cannot be combined (Yang, 2005). The motivation behind the FMSC is minimum-risk estimation. From this perspective, consistent selection criteria suffer from a serious defect: in general, unlike conservative criteria, they exhibit *unbounded* minimax risk (Leeb and Pötscher, 2008). Moreover, as discussed in more detail below, the asymptotics of consistent selection paint a misleading picture of the effects of moment selection on inference. For these reasons, the fact that the FMSC is conservative rather than consistent is an asset in the present context.

Because it studies inference post-moment selection, this paper relates to a vast literature on "pre-test" estimators. For an overview, see Leeb and Pötscher (2005, 2009). There are several proposals to construct valid confidence intervals post-model selection, including Kabaila (1998), Hjort and Claeskens (2003) and Kabaila and Leeb (2006). To my knowledge, however, this is the first paper to treat the problem in general for post-moment selection and moment average estimators in the presence of mis-specification.[4] While I developed the simulation-based, two-stage confidence interval procedure described below by analogy

[4]Related results appear in Berkowitz et al. (2012), Guggenberger (2010), Guggenberger (2012), and Guggenberger and Kumar (2012).

to a suggestion in Claeskens and Hjort (2008b), Leeb and Pötscher (2014) kindly pointed out that similar constructions have appeared in Loh (1985), Berger and Boos (1994), and Silvapulle (1996). More recently, McCloskey (2012) takes a similar approach to study a class of non-standard testing problems.

The framework within which I study moment averaging is related to the frequentist model average estimators of Hjort and Claeskens (2003). Two other papers that consider weighting estimators based on different moment conditions are Xiao (2010) and Chen et al. (2009). Whereas these papers combine estimators computed using valid moment conditions to achieve a minimum variance estimator, I combine estimators computed using potentially invalid conditions with the aim of reducing estimator AMSE. A similar idea uderlies the combined moments (CM) estimator of Judge and Mittelhammer (2007), who emphasize that incorporating the information from an incorrect specification could lead to favorable bias-variance tradeoff. Unlike the FMSC, however, the CM estimator is not targeted to a particular research goal and does not explicitly aim to minimize AMSE. For a different approach to combining OLS and TSLS estimators, similar in spirit to the Stein-estimator and developed independently of the work presented here, see Hansen (2014). Cheng et al. (2014) provide related results for Stein-type moment averaging in a GMM context with potentially mis-specified moment conditions. Both of these papers consider settings in which the parameter of interest is of sufficiently high dimension that averaging can yield uniform risk improvements. In contrast, I consider a setting with a scalar target parameter in which uniform improvements are unavailable.

A limitation of the results presented here is that they are based upon the assumption of strong identification and a fixed number of moment conditions. When I refer to a bias-variance tradeoff below, either in finite samples or asymptotically, I abstract from weak– and many–instruments considerations. In particular, my asymptotics are based on a classical first-order approximation with the addition of locally invalid moment conditions. Extending the idea behind the FMSC to allow for weak identification or a large number of moment conditions is a challenging topic that I leave for future research.

The remainder of the paper is organized as follows. Section 2 describes the asymptotic framework and Section 3 derives the FMSC, both in general and for two specific examples: OLS versus TSLS and choosing instrumental variables. Section 4 studies moment average estimators and shows how they can be used to construct valid confidence intervals post-moment selection. Section 5 presents simulation results and Section 6 considers an empirical example from development economics. Proofs, computational details and supplementary material appear in the Appendix.

2 Assumptions and Asymptotic Framework

2.1 Local Mis-Specification

Let $f(\cdot,\cdot)$ be a $(p+q)$-vector of moment functions of a random vector Z and an r-dimensional parameter vector θ, partitioned according to $f(\cdot,\cdot) = (g(\cdot,\cdot)', h(\cdot,\cdot)')'$ where $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are p- and q-vectors of moment functions. The moment condition associated with $g(\cdot,\cdot)$ is assumed to be correct whereas that associated with $h(\cdot,\cdot)$ is locally mis-specified. More precisely,

Assumption 2.1 (Local Mis-Specification)**.** *Let $\{Z_{ni}\colon 1 \leq i \leq n, n = 1,2,\ldots\}$ be a triangular array of random vectors defined on a probability space $(\Upsilon, \mathcal{F}, \mathbb{P})$ satisfying*

(a) $E[g(Z_{ni},\theta_0)] = 0$,

(b) $E[h(Z_{ni},\theta_0)] = n^{-1/2}\tau$, *where τ is an unknown constant vector,*

(c) $\{f(Z_{ni},\theta_0)\colon 1 \leq i \leq n, n = 1,2,\ldots\}$ *is uniformly integrable, and*

(d) $Z_{ni} \to_d Z_i$, *where the Z_i are identically distributed.*

For any fixed sample size n, the expectation of h evaluated at the true parameter value θ_0 depends on the unknown constant vector τ. Unless all components of τ are zero, some of the moment conditions contained in h are mis-specified. In the limit however, this mis-specification vanishes, as $\tau/\sqrt{n}$ converges to zero. Uniform integrability combined with weak convergence implies convergence of expectations, so that $E[g(Z_i,\theta_0)] = 0$ and $E[h(Z_i,\theta_0)] = 0$. Because the limiting random vectors Z_i are identically distributed, I suppress the i subscript and simply write Z to denote their common marginal law, e.g. $E[h(Z,\theta_0)] = 0$. It is important to note that local mis-specification is *not* intended as a literal description of real-world datasets: it is merely a device that gives asymptotic bias-variance trade-off that mimics the finite-sample intuition.

2.2 Candidate GMM Estimators

Define the sample analogue of the expectations in Assumption 2.1 as follows:

$$f_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} f(Z_{ni},\theta) = \begin{bmatrix} g_n(\theta) \\ h_n(\theta) \end{bmatrix} = \begin{bmatrix} n^{-1}\sum_{i=1}^{n} g(Z_{ni},\theta) \\ n^{-1}\sum_{i=1}^{n} h(Z_{ni},\theta) \end{bmatrix}$$

where g_n is the sample analogue of the correctly specified moment conditions and h_n is that of the (potentially) mis-specified moment conditions. A candidate GMM estimator $\widehat{\theta}_S$ uses

some subset S of the moment conditions contained in f in estimation. Let $|S|$ denote the number of moment conditions used and suppose that $|S| > r$ so the GMM estimator is unique.[5] Let Ξ_S be the $|S| \times (p+q)$ *moment selection matrix* corresponding to S. That is, Ξ_S is a matrix of ones and zeros arranged such that $\Xi_S f_n(\theta)$ contains only the sample moment conditions used to estimate $\widehat{\theta}_S$. Thus, the GMM estimator of θ based on moment set S is given by

$$\widehat{\theta}_S = \arg\min_{\theta \in \Theta} \left[\Xi_S f_n(\theta)\right]' \widetilde{W}_S \left[\Xi_S f_n(\theta)\right].$$

where $\widetilde{W}_S$ is an $|S| \times |S|$, positive definite weight matrix. There are no restrictions placed on S other than the requirement that $|S| > r$ so the GMM estimate is well-defined. In particular, S may *exclude* some or all of the valid moment conditions contained in g. While this may seem strange, it accommodates a wider range of examples, including choosing between OLS and TSLS estimators.

To consider the limit distribution of $\widehat{\theta}_S$, we require some further notation. First define the derivative matrices

$$G = E\left[\nabla_\theta \, g(Z, \theta_0)\right], \quad H = E\left[\nabla_\theta \, h(Z, \theta_0)\right], \quad F = (G', H')'$$

and let $\Omega = Var\left[f(Z, \theta_0)\right]$ where Ω is partitioned into blocks Ω_{gg}, Ω_{gh}, Ω_{hg}, and Ω_{hh} conformably with the partition of f by g and h. Notice that each of these expressions involves the *limiting random variable* Z rather than Z_{ni}, so that the corresponding expectations are taken with respect to a distribution for which all moment conditions are correctly specified. Finally, to avoid repeatedly writing out pre- and post-multiplication by Ξ_S, define $F_S = \Xi_S F$ and $\Omega_S = \Xi_S \Omega \Xi_S'$. The following high level assumptions are sufficient for the consistency and asymptotic normality of the candidate GMM estimator $\widehat{\theta}_S$.

Assumption 2.2 (High Level Sufficient Conditions)**.**

(a) θ_0 *lies in the interior of* Θ*, a compact set*

(b) $\widetilde{W}_S \to_p W_S$*, a positive definite matrix*

(c) $W_S \Xi_S E[f(Z, \theta)] = 0$ *if and only if* $\theta = \theta_0$

(d) $E[f(Z, \theta)]$ *is continuous on* Θ

(e) $\sup_{\theta \in \Theta} \|f_n(\theta) - E[f(Z, \theta)]\| \to_p 0$

(f) f *is* Z*-almost surely differentiable in an open neighborhood* $\mathcal{B}$ *of* θ_0

[5] Identifying τ requires futher assumptions, as discussed in Section 2.3.

(g) $\sup_{\theta\in\Theta} \|\nabla_\theta f_n(\theta) - F(\theta)\| \to_p 0$

(h) $\sqrt{n} f_n(\theta_0) \to_d M + \begin{bmatrix} 0 \\ \tau \end{bmatrix}$ *where* $M \sim N_{p+q}(0, \Omega)$

(i) $F_S' W_S F_S$ *is invertible*

Although Assumption 2.2 closely approximates the standard regularity conditions for GMM estimation, establishing primitive conditions for Assumptions 2.2 (d), (e), (g) and (h) is slightly more involved under local mis-specification. Low-level sufficient conditions for the two running examples considered in this paper appear in Appendix D. For more general results, see Andrews (1988) Theorem 2 and Andrews (1992) Theorem 4. Notice that identification, (c), and continuity, (d), are conditions on the distribution of Z, the marginal law to which each Z_{ni} converges.

Theorem 2.1 (Consistency)**.** *Under Assumptions 2.1 and 2.2 (a)–(e),* $\widehat{\theta}_S \to_p \theta_0$.

Theorem 2.2 (Asymptotic Normality)**.** *Under Assumptions 2.1 and 2.2*

$$\sqrt{n}(\widehat{\theta}_S - \theta_0) \to_d -K_S \Xi_S \left(\begin{bmatrix} M_g \\ M_h \end{bmatrix} + \begin{bmatrix} 0 \\ \tau \end{bmatrix} \right)$$

where $K_S = [F_S' W_S F_S]^{-1} F_S' W_S$, $M = (M_g', M_h')'$, *and* $M \sim N(0, \Omega)$.

As we see from Theorems 2.1 and 2.2, *any* candidate GMM estimator $\widehat{\theta}_S$ is consistent for θ_0 under local mis-specification. Unless S excludes *all* of the moment conditions contained in h, however, $\widehat{\theta}_S$ inherits an asymptotic bias from the mis-specification parameter τ. The local mis-specification framework is useful precisely because it results in a limit distribution for $\widehat{\theta}_S$ with both a bias *and* a variance. This captures in asymptotic form the bias-variance tradeoff that we see in finite sample simulations. In constrast, fixed mis-specification results in a degenerate bias-variance tradeoff in the limit: scaling up by $\sqrt{n}$ to yield an asymptotic variance causes the bias component to diverge.

2.3 Identification

Any form of moment selection requires an identifying assumption: we need to make clear which parameter value θ_0 counts as the "truth." One approach, following Andrews (1999), is to assume that there exists a unique, maximal set of correctly specified moment conditions that identifies θ_0. In the notation of the present paper[6] this is equivalent to the following:

[6] Although Andrews (1999), Andrews and Lu (2001), and Hong et al. (2003) consider *fixed* mis-specification, we can view this as a version of local mis-specification in which $\tau \to \infty$ sufficiently fast.

Assumption 2.3 (Andrews (1999) Identification Condition)**.** *There exists a subset S_{max} of at least r moment conditions satisfying:*

(a) $\Xi_{S_{max}} E[f(Z_{ni}, \theta_0)] = 0$

(b) For any $S' \neq S_{max}$ *such that* $\Xi_{S'} E[f(Z_{ni}, \theta')] = 0$ *for some* $\theta' \in \Theta$, $|S_{max}| > |S'|$.

Andrews and Lu (2001) and Hong et al. (2003) take the same basic approach to identification, with appropriate modifications to allow for simultaneous model and moment selection. An advantage of Assumption 2.3 is that, under fixed mis-specification, it allows consistent selection of S_{max} without any prior knowledge of *which* moment conditions are correct. In the notation of the present paper this corresponds to having no moment conditions in the g block. As Hall (2005, p. 254) points out, however, the second part of Assumption 2.3 can fail even in very simple settings. When it does fail, the selected GMM estimator may no longer be consistent for θ_0. A different approach to identification is to assume that there is a minimal set of at least r moment conditions *known* to be correctly specified. This is the approach I follow here, as do Liao (2013) and Cheng and Liao (2013).[7]

Assumption 2.4 (FMSC Identification Condition)**.** *Let $\widehat{\theta}_v$ denote the GMM estimator based solely on the moment conditions contained in the g–block*

$$\widehat{\theta}_v = \underset{\theta \in \Theta}{arg\ min}\ g_n(\theta)' \widetilde{W}_v\ g_n(\theta)$$

We call this the "valid estimator" and assume that it satisfies all the conditions of Assumption 2.2. Note that this implies $p \geq r$.

Assumption 2.4 and Theorem 2.2 immediately imply that the valid estimator shows no asymptotic bias.

Corollary 2.1 (Limit Distribution of Valid Estimator)**.** *Let S_v include only the moment conditions contained in g. Then, under Assumption 2.4 we have*

$$\sqrt{n}\left(\widehat{\theta}_v - \theta_0\right) \to_d -K_v M_g$$

by applying Theorem 2.2 to S_v, where $K_v = [G' W_v G]^{-1} G' W_v$ and $M_g \sim N(0, \Omega_{gg})$.

Both Assumptions 2.3 and 2.4 are strong, and neither fully nests the other. In the context of the present paper, Assumption 2.4 is meant to represent a situation in which an applied research chooses between two groups of assumptions. The g–block contains the

[7] For a dicussion of why Assumption 2.4 is necessary and how to proceed when it fails, see Appendix C.

"baseline" assumptions while the h–block contains a set of stronger, more controversial "suspect" assumptions. The FMSC is designed for settings in which the h–block is expected to contain a substantial amount of information beyond that already contained in the g–block. The idea is that, if we knew the h–block was correctly specified, we would expect a large gain in efficiency by including it in estimation. This motivates the idea of trading off the variance reduction from including h against the potential increase in bias. If the h–block assumptions are *nearly correct* we may want to use them in estimation. Not all applications have the structure, but many do. Below, I consider two simple but empirically relevant examples: choosing between OLS and TSLS estimators and choosing instrumental variables.

3 The Focused Moment Selection Criterion

3.1 The General Case

The FMSC chooses among the potentially invalid moment conditions contained in h based on the estimator AMSE of a user-specified scalar target parameter. Denote this target parameter by μ, a real-valued, Z-almost continuous function of the parameter vector θ that is differentiable in a neighborhood of θ_0. Further, define the GMM estimator of μ based on $\widehat{\theta}_S$ by $\widehat{\mu}_S = \mu(\widehat{\theta}_S)$ and the true value of μ by $\mu_0 = \mu(\theta_0)$. Applying the Delta Method to Theorem 2.2 gives the AMSE of $\widehat{\mu}_S$.

Corollary 3.1 (AMSE of Target Parameter)**.** *Under the hypotheses of Theorem 2.2,*

$$\sqrt{n}\left(\widehat{\mu}_S - \mu_0\right) \to_d -\nabla_\theta \mu(\theta_0)' K_S \Xi_S \left(M + \begin{bmatrix} 0 \\ \tau \end{bmatrix} \right)$$

where M is defined in Theorem 2.2. Hence,

$$AMSE\left(\widehat{\mu}_S\right) = \nabla_\theta \mu(\theta_0)' K_S \Xi_S \left\{ \begin{bmatrix} 0 & 0 \\ 0 & \tau\tau' \end{bmatrix} + \Omega \right\} \Xi_S' K_S' \nabla_\theta \mu(\theta_0).$$

For the valid estimator $\widehat{\theta}_v$ we have $K_v = \left[G' W_v G\right]^{-1} G' W_v$ and $\Xi_v = \begin{bmatrix} \mathbf{I}_p & \mathbf{0}_{p\times q} \end{bmatrix}$. Thus, the valid estimator $\widehat{\mu}_v$ of μ has zero asymptotic bias. In contrast, any candidate estimator $\widehat{\mu}_S$ that includes moment conditions from h inherits an asymptotic bias from the corresponding elements of τ, the extent and direction of which depends both on K_S and $\nabla_\theta \mu(\theta_0)$. Adding moment conditions from h, however, generally decreases asymptotic variance. In particular, the usual proof that adding moment conditions cannot increase asymptotic variance under efficient GMM (see for example Hall, 2005, ch. 6) continues to hold under local

mis-specification, because all moment conditions are correctly specified in the limit.[8]

Using this framework for moment selection requires estimators of the unknown quantities: θ_0, K_S, Ω, and τ. Under local mis-specification, the estimator of θ under *any* moment set is consistent. A natural estimator is $\widehat{\theta}_v$, although there are other possibilities. Recall that $K_S = [F_S' W_S F_S]^{-1} F_S' W_S \Xi_S$. Because it is simply the selection matrix defining moment set S, Ξ_S is known. The remaining quantities F_S and W_S that make up K_S are consistently estimated by their sample analogues under Assumption 2.2. Similarly, consistent estimators of Ω are readily available under local mis-specification, although the precise form depends on the situation.[9] The only remaining unknown is τ. Local mis-specification is essential for making meaningful comparisons of AMSE because it prevents the bias term from dominating the comparison. Unfortunately, it also prevents consistent estimation of the asymptotic bias parameter. Under Assumption 2.4, however, it remains possible to construct an *asymptotically unbiased* estimator $\widehat{\tau}$ of τ by substituting $\widehat{\theta}_v$, the estimator of θ_0 that uses only correctly specified moment conditions, into h_n, the sample analogue of the potentially mis-specified moment conditions. In other words, $\widehat{\tau} = \sqrt{n} h_n(\widehat{\theta}_v)$.

Theorem 3.1 (Asymptotic Distribution of $\widehat{\tau}$)**.** *Let $\widehat{\tau} = \sqrt{n} h_n(\widehat{\theta}_v)$ where $\widehat{\theta}_v$ is the valid estimator, based only on the moment conditions contained in g. Then under Assumptions 2.1, 2.2 and 2.4*

$$\widehat{\tau} \to_d \Psi \left(M + \begin{bmatrix} 0 \\ \tau \end{bmatrix} \right), \quad \Psi = \begin{bmatrix} -HK_v & \mathbf{I}_q \end{bmatrix}$$

where K_v is defined in Corollary 2.1. Thus, $\widehat{\tau} \to_d (\Psi M + \tau) \sim N_q(\tau, \Psi \Omega \Psi')$.

Returning to Corollary 3.1, however, we see that it is $\tau\tau'$ rather than τ that enters the expression for AMSE. Although $\widehat{\tau}$ is an asymptotically unbiased estimator of τ, the limiting expectation of $\widehat{\tau}\widehat{\tau}'$ is not $\tau\tau'$ because $\widehat{\tau}$ has an asymptotic variance. Subtracting a consistent estimate of the asymptotic variance removes this asymptotic bias.

Corollary 3.2 (Asymptotically Unbiased Estimator of $\tau\tau'$)**.** *If $\widehat{\Omega}$ and $\widehat{\Psi}$ are consistent for Ω and Ψ, then $\widehat{\tau}\widehat{\tau}' - \widehat{\Psi}\widehat{\Omega}\widehat{\Psi}$ is an asymptotically unbiased estimator of $\tau\tau'$.*

It follows that

$$\text{FMSC}_n(S) = \nabla_\theta \mu(\widehat{\theta})' \widehat{K}_S \Xi_S \left\{ \begin{bmatrix} 0 & 0 \\ 0 & \widehat{\tau}\widehat{\tau}' - \widehat{\Psi}\widehat{\Omega}\widehat{\Psi}' \end{bmatrix} + \widehat{\Omega} \right\} \Xi_S' \widehat{K}_S' \nabla_\theta \mu(\widehat{\theta}) \tag{1}$$

[8] The general result for adding moment conditions in GMM is only relevant in situations where the valid moment set is strictly nested inside of all other candidate moment sets. When this does not hold, such as in the OLS versus IV example, we establish an analogous ordering of asymptotic variances by direct calculation.

[9] See Sections 3.2 and 3.3 for discussion of this point for the two running examples.

provides an asymptotically unbiased estimator of AMSE. Given a set $\mathscr{S}$ of candidate specifications, the FMSC selects the candidate S^* that *minimizes* the expression given in Equation 1, that is $S^*_{FMSC} = \arg\min_{S\in\mathscr{S}} \text{FMSC}_n(S)$.

At this point, it is worth taking a brief pause to survey the ground covered thus far. We began with a target parameter, μ, a risk function, mean-squared error, and a collection of candidate estimators, $\widehat{\mu}_S$ for $S \in \mathscr{S}$. Our goal was to choose the estimator with the lowest risk. Because finite-sample distributions were unavailable, we resorted to an asymptotic experiment, local mis-specification, that preserved the bias-variance tradeoff embodied in our chosen risk function. We then calculated the risk of the *limit distribution* of $\widehat{\mu}_S$ to use as a stand-in for the finite-sample risk. This quantity involved several unknown parameters. We estimated these in such a way that the resulting asymptotic risk estimate would converge in distribution to a random variable with mean equal to the true asymptotic risk. The result was the FMSC: an asymptotically unbiased estimator of the AMSE of $\widehat{\mu}_S$. Viewing the FMSC at this level of abstraction raises two questions. First, could we have chosen a risk function other than mean-squared error? Second, why should we use an *asymptotically unbiased* risk estimator?

The answer to the first question is a straightforward yes. The idea of using asymptotic risk as a stand-in for finite sample risk requires only that we can characterize the limit distribution of each $\widehat{\mu}_S$ and use it to evaluate the chosen risk function. Claeskens et al. (2006) and Claeskens and Hjort (2008a), for example, show how the FIC for model selection in maximum likelihood models can be extended from squared error to L_p and linex loss, respectively, in precisely this way. One could easily do the same for the FMSC although I do not consider this possibility further here. Answering the second question is more difficult. Under local mis-specification it is impossible to consistently estimate AMSE.[10] If we merely use the plug-in estimator of the squared asymptotic bias based on $\widehat{\tau}$, the resulting AMSE estimate will "overshoot" asymptotically. Accordingly, it seems natural to correct this bias as explained in Corollary 3.2. This is the same heuristic that underlies the classical AIC and TIC model selection criteria as well as more recent procedures such as those described in Claeskens and Hjort (2003) and Schorfheide (2005). Nevertheless, there could certainly be situations in which it makes sense to use a risk estimator other than the asymptotically unbiased one suggested here. If one wished to consider risk functions other than MSE, to take a simple example, it may not be possible to derive an asymptotically unbiased risk estimator. The plug-in estimator, however, is always available. Although I do not consider them further below, alternative risk estimators could be an interesting topic for future research.

[10] This is not a defect of the FMSC: there is a fundamental trade-off between consistency and desirable risk properties. See Section 4 for a discussion of this point.

3.2 OLS versus TSLS Example

The simplest interesting application of the FMSC is choosing between ordinary least squares (OLS) and two-stage least squares (TSLS) estimators of the effect β of a single endogenous regressor x on an outcome of interest y. The intuition is straightforward: because TSLS is a high-variance estimator, OLS will have a lower mean-squared error provided that x isn't *too* endogenous.[11] To keep the presentation transparent, I work within an iid, homoskedastic setting for this example and assume, without loss of generality, that there are no exogenous regressors.[12] Equivalently we may suppose that any exogenous regressors, including a constant, have been "projected out." Low-level sufficient conditions for all of the results in this section appear in Assumption D.1 of Appendix D. The data generating process is

$$y_{ni} = \beta x_{ni} + \epsilon_{ni} \tag{2}$$
$$x_{ni} = \mathbf{z}_{ni}'\boldsymbol{\pi} + v_{ni} \tag{3}$$

where β and $\boldsymbol{\pi}$ are unknown constants, $\mathbf{z}_{ni}$ is a vector of exogenous and relevant instruments, x_{ni} is the endogenous regressor, y_{ni} is the outcome of interest, and ϵ_{ni}, v_{ni} are unobservable error terms. All random variables in this system are mean zero, or equivalently all constant terms have been projected out. Stacking observations in the usual way, the estimators under consideration are $\widehat{\beta}_{OLS} = (\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{y}$ and $\widetilde{\beta}_{TSLS} = (\mathbf{x}'P_Z\mathbf{x})^{-1}\mathbf{x}'P_Z\mathbf{y}$ where we define $P_Z = Z(Z'Z)^{-1}Z'$.

Theorem 3.2 (OLS and TSLS Limit Distributions)**.** *Let $(\mathbf{z}_{ni}, v_{ni}, \epsilon_{ni})$ be a triangular array of random variables such that $E[\mathbf{z}_{ni}\epsilon_{ni}] = \mathbf{0}$, $E[\mathbf{z}_{ni}v_{ni}] = \mathbf{0}$, and $E[\epsilon_{ni}v_{ni}] = \tau/\sqrt{n}$ for all n. Then, under standard regularity conditions, e.g. Assumption D.1,*

$$\begin{bmatrix} \sqrt{n}(\widehat{\beta}_{OLS} - \beta) \\ \sqrt{n}(\widetilde{\beta}_{TSLS} - \beta) \end{bmatrix} \xrightarrow{d} N\left(\begin{bmatrix} \tau/\sigma_x^2 \\ 0 \end{bmatrix}, \sigma_\epsilon^2 \begin{bmatrix} 1/\sigma_x^2 & 1/\sigma_x^2 \\ 1/\sigma_x^2 & 1/\gamma^2 \end{bmatrix} \right)$$

where $\sigma_x^2 = \gamma^2 + \sigma_v^2$, $\gamma^2 = \boldsymbol{\pi}'Q\boldsymbol{\pi}$, $E[\mathbf{z}_{ni}\mathbf{z}_{ni}'] \to Q$, $E[v_{ni}^2] \to \sigma_v^2$, and $E[\epsilon_{ni}^2] \to \sigma_\epsilon^2$ as $n \to \infty$.

We see immediately that, as expected, the variance of the OLS estimator is always strictly lower than that of the TSLS estimator since $\sigma_\epsilon^2/\sigma_x^2 = \sigma_\epsilon^2/(\gamma^2 + \sigma_v^2)$. Unless $\tau = 0$, however, OLS shows an asymptotic bias. In contrast, the TSLS estimator is asymptotically unbiased

[11] Because the moments of the TSLS estimator only exist up to the order of overidentificiation (Phillips, 1980) mean-squared error should be understood to refer to "trimmed" mean-squared error when the number of instruments is two or fewer. See, e.g., Hansen (2013).

[12] The homoskedasticity assumption concerns the *limit* random variables: under local mis-specification there will be heteroskedasticity for fixed n. See Assumption D.1 in Appendix D for details.

regardless of the value of τ. Thus,

$$\text{AMSE(OLS)} = \frac{\tau^2}{\sigma_x^4} + \frac{\sigma_\epsilon^2}{\sigma_x^2}, \qquad \text{AMSE(TSLS)} = \frac{\sigma_\epsilon^2}{\gamma^2}.$$

and rerranging, we see that the AMSE of the OLS estimator is strictly less than that of the TSLS estimator whenever $\tau^2 < \sigma_x^2\sigma_\epsilon^2\sigma_v^2/\gamma^2$. To estimate the unknowns required to turn this inequality into a moment selection procedure, I set

$$\widehat{\sigma}_x^2 = n^{-1}\mathbf{x}'\mathbf{x}, \quad \widehat{\gamma}^2 = n^{-1}\mathbf{x}'Z(Z'Z)^{-1}Z'\mathbf{x}, \quad \widehat{\sigma}_v^2 = \widehat{\sigma}_x^2 - \widehat{\gamma}^2$$

and define

$$\widehat{\sigma}_\epsilon^2 = n^{-1}\left(\mathbf{y} - \mathbf{x}\widetilde{\beta}_{TSLS}\right)'\left(\mathbf{y} - \mathbf{x}\widetilde{\beta}_{TSLS}\right)$$

Under local mis-specification each of these estimators is consistent for its population counterpart.[13] All that remains is to estimate τ^2. Specializing Theorem 3.1 and Corollary 3.2 to the present example gives the following result.

Theorem 3.3. *Let $\widehat{\tau} = n^{-1/2}\mathbf{x}'(\mathbf{y} - \mathbf{x}\widetilde{\beta}_{TSLS})$. Then, under the conditions of Theorem 3.2,*

$$\widehat{\tau} \to_d N(\tau, V), \quad V = \sigma_\epsilon^2\sigma_x^2(\sigma_v^2/\gamma^2).$$

It follows that $\widehat{\tau}^2 - \widehat{\sigma}_\epsilon^2\widehat{\sigma}_x^2\,(\widehat{\sigma}_v^2/\widehat{\gamma}^2)$ is an asymptotically unbiased estimator of τ^2 and hence, substituting into the AMSE inequality from above and rearranging, the FMSC instructs us to choose OLS whenever $\widehat{T}_{FMSC} = \widehat{\tau}^2/\widehat{V} < 2$ where $\widehat{V} = \widehat{\sigma}_v^2\widehat{\sigma}_\epsilon^2\widehat{\sigma}_x^2/\widehat{\gamma}^2$. The quantity $\widehat{T}_{FMSC}$ looks very much like a test statistic and indeed it can be viewed as such. By Theorem 3.3 and the continuous mapping theorem, $\widehat{T}_{FMSC} \to_d \chi^2(1)$. Thus, the FMSC can be viewed as a test of the null hypothesis $H_0\colon \tau = 0$ against the two-sided alternative with a critical value of 2. This corresponds to a significance level of $\alpha \approx 0.16$. But how does this novel "test" compare to something more familiar, say the Durbin-Hausman-Wu (DHW) test? It turns out that in this particular example, although not in general, the FMSC is *numerically equivalent* to using OLS unless the DHW test rejects at the 16% level.

Theorem 3.4. *Under the conditions of Theorem 3.2, FMSC selection between the OLS and TSLS estimators is equivalent to a Durbin-Hausman-Wu pre-test with a critical value of* 2.

The equivalence between FMSC selection and a DHW test in this example is helpful for two reasons. First, it provides a novel justification for the use of the DHW test to

[13] While using the OLS residuals to estimate σ_ϵ^2 *also* provides a consistent estimate under local mis-specification, the estimator based on the TSLS residuals should be more robust unless the instruments are quite weak.

select between OLS and TSLS. So long as it is carried out with $\alpha \approx 16\%$, the DHW test is equivalent to selecting the estimator that minimizes an asymptotically unbiased estimator of AMSE. Note that this significance level differs from the more usual values of 5% or 10% in that it leads us to select TSLS *more often*: OLS should indeed be given the benefit of the doubt, but not by so wide a margin as traditional practice suggests. Second, this equivalence shows that the FMSC can be viewed as an *extension* of the idea behind the familiar DHW test to more general GMM environments.

3.3 Choosing Instrumental Variables Example

The OLS versus TSLS example is really a special case of instrument selection: if x is exogenous, it is clearly "its own best instrument." Viewed from this perspective, the FMSC amounts to trading off endogeneity against instrument strength. I now consider instrument selection in general for linear GMM estimators in an iid setting. Consider the following model:

$$y_{ni} = \mathbf{x}_i'\beta + \epsilon_i \tag{4}$$
$$\mathbf{x}_{ni} = \Pi_1'\mathbf{z}_{ni}^{(1)} + \Pi_2'\mathbf{z}_{ni}^{(2)} + \mathbf{v}_{ni} \tag{5}$$

where y is an outcome of interest, $\mathbf{x}$ is an r-vector of regressors, some of which are endogenous, $\mathbf{z}^{(1)}$ is a p-vector of instruments known to be exogenous, and $\mathbf{z}^{(2)}$ is a q-vector of *potentially endogenous* instruments. The r-vector β, $p \times r$ matrix Π_1, and $q \times r$ matrix Π_2 contain unknown constants. Stacking observations in the usual way, we can write the system in matrix form as $\mathbf{y} = X\beta + \boldsymbol{\epsilon}$ and $X = Z\Pi + V$, where $Z = (Z_1, Z_2)$ and $\Pi = (\Pi_1', \Pi_2')'$.

In this example, the idea is that the instruments contained in Z_2 are expected to be strong. If we were confident that they were exogenous, we would certainly use them in estimation. Yet the very fact that we expect them to be strongly correlated with $\mathbf{x}$ gives us reason to fear that they may be endogenous. The exact opposite is true of Z_1: these are the instruments that we are prepared to assume are exogenous. But when is such an assumption plausible? Precisely when the instruments contained in Z_1 are *not especially strong*. Accordingly, the FMSC attempts to trade off a small increase in bias from using a *slightly* endogenous instrument against a larger decrease in variance from increased instrument strength. To this end, consider a general linear GMM estimator of the form

$$\widehat{\beta}_S = (X'Z_S\widetilde{W}_SZ_S'X)^{-1}X'Z_S\widetilde{W}_SZ_S'\mathbf{y}$$

where S indexes the instruments used in estimation, $Z_S' = \Xi_S Z'$ is the matrix containing

only those instruments included in S, $|S|$ is the number of instruments used in estimation and $\widetilde{W}_S$ is an $|S| \times |S|$ positive definite weighting matrix.

Theorem 3.5 (Choosing IVs Limit Distribution)**.** *Let* $(\mathbf{z}_{ni}, v_{ni}, \epsilon_{ni})$ *be a triangular array of random variables such that* $E[\mathbf{z}_{ni}\epsilon_{ni}] = \mathbf{0}$*,* $E[\mathbf{z}_{ni}v_{ni}] = \mathbf{0}$*, and* $E[\epsilon_{ni}v_{ni}] = \tau/\sqrt{n}$ *for all* n*. Suppose further that* $\widetilde{W}_S \to_p W_S > 0$*. Then, under standard regularity conditions, e.g. Assumption D.2,*

$$\sqrt{n}\left(\widehat{\beta}_S - \beta\right) \xrightarrow{d} -K_S \Xi_S \left(\begin{bmatrix} \mathbf{0} \\ \boldsymbol{\tau} \end{bmatrix} + M \right)$$

where

$$-K_S = \left(\Pi' Q_S W_S Q_S' \Pi\right)^{-1} \Pi' Q_S W_S$$

$M \sim N(\mathbf{0}, \Omega)$*,* $Q_S = Q\Xi_S'$*,* $E[\mathbf{z}_{ni}\mathbf{z}_{ni}'] \to Q$ *and* $E[\epsilon_{ni}^2 \mathbf{z}_{ni}\mathbf{z}_{ni}'] \to \Omega$ *as* $n \to \infty$

To implement the FMSC for this example, we simply need to specialize Equation 1. To simplify the notation, let

$$\Xi_1 = \begin{bmatrix} I_p & 0_{p\times q} \end{bmatrix}, \quad \Xi_2 = \begin{bmatrix} 0_{q\times p} & I_q \end{bmatrix} \tag{6}$$

where $0_{m\times n}$ denotes an $m \times n$ matrix of zeros and I_m denotes the $m \times m$ identity matrix. Using this convention, $Z_1 = Z\Xi_1'$ and $Z_2 = Z\Xi_2'$. In this example the valid estimator, defined in Assumption 2.4, is given by

$$\widehat{\beta}_v = \left(X' Z_1 \widetilde{W}_v Z_1' X\right)^{-1} X' Z_1 \widetilde{W}_v Z_1' \mathbf{y} \tag{7}$$

and we estimate $\nabla_\beta \mu(\beta)$ with $\nabla_\beta \mu(\widehat{\beta}_v)$. Similarly,

$$-\widehat{K}_S = n\left(X' Z \Xi_S' \widetilde{W}_S \Xi_S Z' X\right)^{-1} X' Z \Xi_S' \widetilde{W}_S$$

is the natural consistent estimator of $-K_S$ in this setting.[14] Since Ξ_S is known, the only remaining quantities from Equation 1 are $\widehat{\boldsymbol{\tau}}$, $\widehat{\Psi}$ and $\widehat{\Omega}$. The following result specializes Theorem 3.1 to the present example.

Theorem 3.6. *Let* $\widehat{\boldsymbol{\tau}} = n^{-1/2} Z_2'(\mathbf{y} - X\widehat{\beta}_v)$ *where* $\widehat{\beta}_v$ *is as defined in Equation 7. Under the*

[14] The negative sign is squared in the FMSC expression and hence disappears. I write it here only to be consistent with the notation of Theorem 2.2.

conditions of Theorem 3.5 we have $\widehat{\boldsymbol{\tau}} \to_d \boldsymbol{\tau} + \Psi M$ *where* M *is defined in Theorem 3.5,*

$$\begin{aligned} \Psi &= \left[\begin{array}{cc} -\Xi_2 Q \Pi K_v & I_q \end{array} \right] \\ -K_v &= \left(\Pi' Q \Xi_1' W_v \Xi_1 Q' \Pi\right)^{-1} \Pi' Q \Xi_1' W_v \end{aligned}$$

W_v *is the probability limit of the weighting matrix from Equation 7,* I_q *is the* $q \times q$ *identity matrix,* Ξ_1 *is defined in Equation 6, and* $E[\mathbf{z}_{ni}\mathbf{z}_{ni}'] \to Q$*.*

Using this result, I construct the asymptotically unbiased estimator $\widehat{\tau}\widehat{\tau}' - \widehat{\Psi}\widehat{\Omega}\widehat{\Psi}'$ of $\tau\tau'$ from

$$\widehat{\Psi} = \left[\begin{array}{cc} -n^{-1} Z_2' X \left(-\widehat{K}_v\right) & I_q \end{array} \right], \quad -\widehat{K}_v = n \left(X' Z_1 \widetilde{W}_v Z_1' X \right)^{-1} X' Z_1 \widetilde{W}_v$$

All that remains before substituting values into Equation 1 is to estimate Ω. There are many possible ways to proceed, depending on the problem at hand and the assumptions one is willing to make. In the simulation and empirical examples discussed below I examine the TSLS estimator, that is $\widetilde{W}_S = (\Xi_S Z' Z \Xi_S)^{-1}$, and estimate Ω as follows. For all specifications *except* the valid estimator $\widehat{\beta}_v$, I employ the centered, heteroskedasticity-consistent estimator

$$\widehat{\Omega}_S = \frac{1}{n} \sum_{i=1}^{n} u_i(\widehat{\beta}_S)^2 \mathbf{z}_{iS} \mathbf{z}_{iS}' - \left(\frac{1}{n} \sum_{i=1}^{n} u_i(\widehat{\beta}_S) \mathbf{z}_{iS} \right) \left(\frac{1}{n} \sum_{i=1}^{n} u_i(\widehat{\beta}_S) \mathbf{z}_{iS}' \right) \tag{8}$$

where $u_i(\beta) = y_i - \mathbf{x}_i'\beta$, $\widehat{\beta}_S = (X' Z_S (Z_S' Z_S)^{-1} Z_S' X)^{-1} X' Z_S (Z_S' Z_S)^{-1} Z_S' \mathbf{y}$, $\mathbf{z}_{iS} = \Xi_S \mathbf{z}_i$ and $Z_S' = \Xi_S Z'$. Centering allows moment functions to have non-zero means. While the local mis-specification framework implies that these means tend to zero in the limit, they are non-zero for any fixed sample size. Centering accounts for this fact, and thus provides added robustness. Since the valid estimator $\widehat{\beta}_v$ has no asymptotic bias, the AMSE of any target parameter based on this estimator equals its asymptotic variance. Accordingly, I use

$$\widetilde{\Omega}_{11} = n^{-1} \sum_{i=1}^{n} u_i(\widehat{\beta}_v)^2 \mathbf{z}_{1i} \mathbf{z}_{1i}' \tag{9}$$

rather than the $(p \times p)$ upper left sub-matrix of $\widehat{\Omega}$ to estimate this quantity. This imposes the assumption that all instruments in Z_1 are valid so that no centering is needed, providing greater precision.

4 Moment Averaging & Post-Selection Estimators

Because it is constructed from $\widehat{\tau}$, the FMSC is a random variable, even in the limit. Combining Corollary 3.2 with Equation 1 gives the following.

Corollary 4.1 (Limit Distribution of FMSC)**.** *Under Assumptions 2.1, 2.2 and 2.4, we have* $FMSC_n(S) \to_d FMSC_S(\tau, M)$*, where*

$$\begin{aligned} FMSC_S(\tau, M) &= \nabla_\theta \mu(\theta_0)' K_S \Xi_S \left\{ \begin{bmatrix} 0 & 0 \\ 0 & B(\tau, M) \end{bmatrix} + \Omega \right\} \Xi_S' K_S' \nabla_\theta \mu(\theta_0) \\ B(\tau, M) &= (\Psi M + \tau)(\Psi M + \tau)' - \Psi \Omega \Psi'. \end{aligned}$$

This corollary implies that the FMSC is a "conservative" rather than "consistent" selection procedure. While this lack of consistency may sound like serious defect, it is in fact a desirable feature of the FMSC for two reasons. First, as discussed above, the goal of the FMSC is not to select only correctly specified moment conditions: it is to choose an estimator with a low finite-sample MSE as approximated by AMSE. In fact, the goal of consistent selection is very much at odds with that of controlling estimator risk. As explained by Yang (2005) and Leeb and Pötscher (2008), the worst-case risk of a consistent selection procedure *diverges* with sample size.[15] Second, while we know from both simulation studies (Demetrescu et al., 2011) and analytical examples (Leeb and Pötscher, 2005) that selection can dramatically change the sampling distribution of our estimators, invalidating traditional confidence intervals, the asymptotics of consistent selection give the misleading impression that this problem can be ignored. The point is not that conservative criteria are immune to the effects of selection on inference: it is that conservative criteria can be studied using asymptotics that more accurately represent the phenomena encountered in finite samples.

There are two main problems with traditional confidence intervals naïvely applied post-moment selection. First, they ignore model selection uncertainty. If the data had been slightly different, we would have chosen a different set of moment conditions. Accordingly, because traditional intervals condition on the selected model, they are too short. Second, traditional confidence intervals ignore the fact that selection is carried out over potentially invalid moment conditions. Even if our goal were to eliminate all mis-specified moment conditions, for example by using a consistent criterion such as the GMM-BIC of Andrews (1999), in finite-samples we would not always be successful. Because of this, our intervals will be incorrectly centered.

[15] This fact is readily apparent from the results of the simulation study from Section 5.2: the consistent criteria, GMM-BIC and HQ, have the highest worst-case RMSE, while the conservative criteria, FMSC and GMM-AIC, have the lowest.

To account for these two effects, we need a way to represent a *non-normal* sampling distribution in our limit theory, and this rules out consistent selection. The key point is that the post-selection estimator is a *randomly-weighted average* of the individual candidate estimators, some of which are centered away from θ_0. Thus, although the candidate estimators are asymptotically normal, the post-selection estimator follows a *mixture distribution*. Because they choose a single candidate with probability approaching one in the limit, consistent selection procedures make it impossible to represent this phenomenon. In contrast, conservative selection procedures remain random even as the sample size goes to infinity, allowing us to derive a non-normal limit distribution and, ultimately, to carry out valid inference post-moment selection. In the remainder of this section, I derive the asymptotic distribution of generic "moment average" estimators and use them to propose a two-step, simulation-based procedure for constructing valid confidence intervals post-moment selection. I also briefly consider genuine moment average estimators which may have important advantages over selection.

4.1 Moment Average Estimators

A generic moment average estimator takes the form

$$\widehat{\mu} = \sum_{S \in \mathscr{S}} \widehat{\omega}_S \widehat{\mu}_S \tag{10}$$

where $\widehat{\mu}_S = \mu(\widehat{\theta}_S)$ is the estimator of the target parameter μ under moment set S, $\mathscr{S}$ is the collection of all moment sets under consideration, and $\widehat{\omega}_S$ is shorthand for the value of a data-dependent weight function $\widehat{\omega}_S = \omega(\cdot, \cdot)$ evaluated at moment set S and the sample observations $Z_{n1}, \ldots, Z_{nn}$. As above $\mu(\cdot)$ is a $\mathbb{R}$-valued, Z-almost surely continuous function of θ that is differentiable in an open neighborhood of θ_0. When $\widehat{\omega}_S$ is an indicator, taking on the value one at the moment set moment set that minimizes some moment selection criterion, $\widehat{\mu}$ is a post-moment selection estimator. To characterize the limit distribution of $\widehat{\mu}$, I impose the following conditions on $\widehat{\omega}_S$.

Assumption 4.1 (Conditions on the Weights)**.**

(a) $\sum_{S \in \mathscr{S}} \widehat{\omega}_S = 1$*, almost surely*

(b) For each $S \in \mathscr{S}$*,* $\widehat{\omega}_S \to_d \varphi_S(\tau, M)$*, an almost-surely continuous function of* τ*,* M *and consistently estimable constants only.*

Corollary 4.2 (Asymptotic Distribution of Moment-Average Estimators)**.** *Under Assumption 4.1 and the conditions of Theorem 2.2,*

$$\sqrt{n}\left(\widehat{\mu}-\mu_0\right) \to_d \Lambda(\tau) = -\nabla_\theta \mu(\theta_0)' \left[\sum_{S\in\mathscr{S}} \varphi_S(\tau, M) K_S \Xi_S\right] \left(M + \begin{bmatrix} 0 \\ \tau \end{bmatrix}\right).$$

Notice that the limit random variable from Corollary 4.2, denoted $\Lambda(\tau)$, is a *randomly weighted average* of the multivariate normal vector M. Hence, $\Lambda(\tau)$ is non-normal. This is precisely the behavior for which we set out to construct an asymptotic representation. The conditions of Assumption 4.1 are fairly mild. Requiring that the weights sum to one ensures that $\widehat{\mu}$ is a consistent estimator of μ_0 and leads to a simpler expression for the limit distribution. While somewhat less transparent, the second condition is satisfied by weighting schemes based on a number of familiar moment selection criteria. It follows immediately from Corollary 4.1, for example, that the FMSC converges in distribution to a function of τ, M and consistently estimable constants only. The same is true for the J-test statistic, as seen from the following result.

Theorem 4.1 (Distribution of J-Statistic under Local Mis-Specification)**.** *Define the J-test statistic as per usual by* $J_n(S) = n\left[\Xi_S f_n(\widehat{\theta}_S)\right]' \widehat{\Omega}^{-1} \left[\Xi_S f_n(\widehat{\theta}_S)\right]$ *where* $\widehat{\Omega}_S^{-1}$ *is a consistent estimator of* Ω_S^{-1}*. Then, under the conditions of Theorem 2.2, we have* $J_n(S) \to_d J_S(\tau, M)$ *where*

$$J_S(\tau, M) = [\Omega_S^{-1/2}(M_S + \tau_S)]'(I - P_S)[\Omega_S^{-1/2}\Xi_S(M_S + \tau_S)],$$

$M_S = \Xi_S M$, $\tau_S' = (0', \tau')\Xi_S'$, *and* P_S *is the projection matrix formed from the GMM identifying restrictions* $\Omega_S^{-1/2} F_S$.

Hence, normalized weights constructed from almost-surely continuous functions of either the FMSC or the J-test statistic satisfy Assumption 4.1.

Post-selection estimators are merely a special case of moment average estimators. To see why, consider the weight function

$$\widehat{\omega}_S^{MSC} = \mathbf{1}\left\{\text{MSC}_n(S) = \min_{S'\in\mathscr{S}} \text{MSC}_n(S')\right\}$$

where $\text{MSC}_n(S)$ is the value of some moment selection criterion evaluated at the sample observations $Z_{n1}\ldots, Z_{nn}$. Now suppose $\text{MSC}_n(S) \to_d \text{MSC}_S(\tau, M)$, a function of τ, M and consistently estimable constants only. Then, so long as the probability of ties, $P\{\text{MSC}_S(\tau, M) = \text{MSC}_{S'}(\tau, M)\}$, is zero for all $S \neq S'$, the continuous mapping theorem

gives

$$\widehat{\omega}_S^{MSC} \to_d \mathbf{1}\left\{\text{MSC}_S(\tau, M) = \min_{S' \in \mathscr{S}} \text{MSC}_{S'}(\tau, M)\right\}$$

satisfying Assumption 4.1 (b). Thus, post-selection estimators based on the FMSC, a downward J-test procedure, or the GMM moment selection criteria of Andrews (1999) all fall within the ambit of 4.2. The consistent criteria of Andrews (1999), however, are not particularly interesting under local mis-specification.[16] Intuitively, because they aim to select all valid moment conditions w.p.a.1, we would expect that under Assumption 2.1 they simply choose the full moment set in the limit. The following result shows that this intuition is correct.

Theorem 4.2 (Consistent Criteria under Local Mis-Specification)**.** *Consider a moment selection criterion of the form $MSC(S) = J_n(S) - h(|S|)\kappa_n$, where h is strictly increasing, $\lim_{n\to\infty} \kappa_n = \infty$, and $\kappa_n = o(n)$. Under the conditions of Theorem 2.2, $MSC(S)$ selects the full moment set with probability approaching one.*

The preceding result is a special case of a more general phenomenon: consistent selection procedures cannot detect model violations of order $O(n^{-1/2})$.

4.2 Moment Averaging for the OLS versus TSLS Example

Moment selection is a somewhat crude procedure: it gives full weight to the estimator that minimizes the moment selection criterion no matter how close its nearest competitor lies. Accordingly, when competing moment sets have similar criterion values in the population, sampling variation can be *magnified* in the selected estimator. This motivates the idea of averaging estimators based on different moment conditions rather than selecting them. Indeed, in some settings it is possible to derive averaging estimators with uniformly lower risk than the "valid" estimator via Stein-type arguments (e.g. Hansen (2014) and Cheng et al. (2014)). In the case of a scalar target parameter, however, such results are unavailable and hence cannot be used to guide the construction of moment averaging weights for the setting considered in this paper.

So how should one construct weights for a scalar target parameter? One possibility is to adapt a proposal from Buckland et al. (1997), who suggest averaging a collection of competing maximum likelihood estimator with weights of the form $w_k = \exp(-I_k/2)/\sum_{i=1}^K \exp(-I_i/2)$ where I_k is an information criterion evaluated for model k, and i indexes the set of K candidate models. This expression, constructed by an analogy with Bayesian model averaging, gives more weight to models with lower values of the information criterion but non-zero

[16] For more discussion of these criteria, see Section 5.2 below.

weight to all models. Applying a slightly more general form of this idea, suggested by Claeskens and Hjort (2008b), to the moment selection criteria examined above we might consider weights of the form

$$\widehat{\omega}_S = \exp\left\{-\frac{\kappa}{2}\text{MSC}(S)\right\} \Big/ \sum_{S' \in \mathscr{S}} \exp\left\{-\frac{\kappa}{2}\text{MSC}(S')\right\}$$

where $\text{MSC}(\cdot)$ is a moment selection criterion and the parameter $\kappa \geq 0$ varies the uniformity of the weighting. As $\kappa \to 0$ the weights become more uniform; as $\kappa \to \infty$ they approach the moment selection procedure given by minimizing the corresponding criterion. Setting $\kappa = 1$ gives the Buckland et al. (1997) weights.

Some preliminary simulation results, reported in an earlier draft of this paper, suggest that exponential weighting can indeed provide MSE improvements. The difficulty, however, lies in choosing an appropriate value for κ. In at least some applications, however, there is a compelling alternative to the exponential weighting scheme: one can instead derive weights *analytically* to minimize AMSE within the FMSC framework. This immediately suggests a plug-in estimator of the optimal weights along the lines of the FMSC estimate of AMSE. To illustrate this idea, I revisit the OLS versus TSLS example from Section 3.2. Let $\widetilde{\beta}(\omega)$ be a convex combination of the OLS and TSLS estimators, namely

$$\widetilde{\beta}(\omega) = \omega\widehat{\beta}_{OLS} + (1-\omega)\widetilde{\beta}_{TSLS} \tag{11}$$

where $\omega \in [0,1]$ is the weight given to the OLS estimator.

Theorem 4.3. *Under the conditions of Theorem 3.2, the AMSE of the weighted-average estimator* $\sqrt{n}\left[\widehat{\beta}(\omega) - \beta\right]$ *is minimized over* $\omega \in [0,1]$ *by taking* $\omega = \omega^*$ *where*

$$\omega^* = \left[1 + \frac{\tau^2/\sigma_x^4}{\sigma_\epsilon^2(1/\gamma^2 - 1/\sigma_x^2)}\right]^{-1} = \left[1 + \frac{ABIAS(OLS)^2}{AVAR(TSLS) - AVAR(OLS)}\right]^{-1}.$$

The preceding result has several important consequences. First, since the variance of the TSLS estimator is always strictly greater than that of the OLS estimator, the optimal value of ω *cannot* be zero. No matter how strong the endogeneity of x, as measured by τ, we should always give some weight to the OLS estimator. Second, when $\tau = 0$ the optimal value of ω is one. If x is exogenous, OLS is strictly preferable to TSLS. Third, the optimal weights depend on the strength of the instruments $\mathbf{z}$ as measured by γ. All else equal, the stronger the instruments, the less weight we should give to OLS. To operationalize the AMSE-optimal

averaging estimator suggested from Theorem 4.3, I define the plug-in estimator

$$\widehat{\beta}^*_{AVG} = \widehat{\omega}^* \widehat{\beta}_{OLS} + (1 - \widehat{\omega}^*) \widetilde{\beta}_{TSLS} \tag{12}$$

where

$$\widehat{\omega}^* = \left[1 + \frac{\max\left\{0,\ (\widehat{\tau}^2 - \widehat{\sigma}^2_\epsilon \widehat{\sigma}^2_x \left(\widehat{\sigma}^2_x / \widehat{\gamma}^2 - 1\right)) \,/\, \widehat{\sigma}^4_x \right\}}{\widehat{\sigma}^2_\epsilon (1/\widehat{\gamma}^2 - 1/\widehat{\sigma}^2_x)} \right]^{-1} \tag{13}$$

This expression employs the same consistent estimators of σ^2_x, γ and σ_ϵ as the FMSC expressions from Section 3.2. To ensure that $\widehat{\omega}^*$ lies in the interval $[0, 1]$, however, I use a *positive part* estimator for τ^2, namely $\max\{0,\ \widehat{\tau}^2 - \widehat{V}\}$ rather than $\widehat{\tau}^2 - \widehat{V}$.[17] In the following section I show how one can construct a valid confidence interval for $\widehat{\beta}^*$ and related estimators.

4.3 Valid Confidence Intervals

While Corollary 4.2 characterizes the limiting behavior of moment-average, and hence post-selection estimators, the limiting random variable $\Lambda(\tau)$ is a complicated function of the normal random vector M. Because this distribution is analytically intractable, I adapt a suggestion from Claeskens and Hjort (2008b) and approximate it by simulation. The result is a conservative procedure that provides asymptotically valid confidence intervals for moment average and hence post-conservative selection estimators.[18]

First, suppose that K_S, φ_S, θ_0, Ω and τ were known. Then, by simulating from M, as defined in Theorem 2.2, the distribution of $\Lambda(\tau)$, defined in Corollary 4.2, could be approximated to arbitrary precision. To operationalize this procedure one can substitute consistent estimators of K_S, θ_0, and Ω, e.g. those used to calculate FMSC. To estimate φ_S, we first need to derive the limit distribution of $\widehat{\omega}_S$, the data-based weights specified by the user. As an example, consider the case of moment selection based on the FMSC. Here $\widehat{\omega}_S$ is simply the indicator function

$$\widehat{\omega}_S = \mathbf{1}\left\{ \text{FMSC}_n(S) = \min_{S' \in \mathscr{S}} \text{FMSC}_n(S') \right\} \tag{14}$$

To estimate φ_S, first substitute consistent estimators of Ω, K_S and θ_0 into $\text{FMSC}_S(\tau, M)$,

[17] While $\widehat{\tau}^2 - \widehat{V}$ is an asymptotically unbiased estimator of τ^2 it *can* be negative.

[18] Although I originally developed this procedure by analogy to Claeskens and Hjort (2008b), Leeb and Pötscher (2014) kindly pointed out that constructions of the kind given here have appeared elsewhere in the statistics literature, notably in Loh (1985), Berger and Boos (1994), and Silvapulle (1996). More recently, McCloskey (2012) uses a similar approach to study non-standard testing problems.

defined in Corollary 4.1, yielding,

$$\widehat{\text{FMSC}}_S(\tau, M) = \nabla_\theta \mu(\widehat{\theta})' \widehat{K}_S \Xi_S \left\{ \begin{bmatrix} 0 & 0 \\ 0 & \widehat{\mathcal{B}}(\tau, M) \end{bmatrix} + \widehat{\Omega} \right\} \Xi_S' \widehat{K}_S' \nabla_\theta \mu(\widehat{\theta}) \tag{15}$$

where

$$\widehat{\mathcal{B}}(\tau, M) = (\widehat{\Psi} M + \tau)(\widehat{\Psi} M + \tau)' - \widehat{\Psi}\widehat{\Omega}\widehat{\Psi} \tag{16}$$

Combining this with Equation 14,

$$\widehat{\varphi}_S(\tau, M) = \mathbf{1}\left\{ \widehat{\text{FMSC}}_S(\tau, M) = \min_{S' \in \mathscr{S}} \widehat{\text{FMSC}}_{S'}(\tau, M) \right\}. \tag{17}$$

For GMM-AIC moment selection or selection based on a downward J-test, $\varphi_S(\cdot, \cdot)$ may be estimated analogously, following Theorem 4.1.

Although simulating draws from M, defined in Theorem 2.2, requires only an estimate of Ω, the limit φ_S of the weight function also depends on τ. As discussed above, no consistent estimator of τ is available under local mis-specification: the estimator $\widehat{\tau}$ has a non-degenerate limit distribution (see Theorem 3.1). Thus, substituting $\widehat{\tau}$ for τ will give erroneous results by failing to account for the uncertainty that enters through $\widehat{\tau}$. The solution is to use a two-stage procedure. First construct a $100(1-\delta)\%$ confidence region $\mathscr{T}(\widehat{\tau}, \delta)$ for τ using Theorem 3.1. Then, for each $\tau^* \in \mathscr{T}(\widehat{\tau}, \delta)$ simulate from the distribution of $\Lambda(\tau^*)$, defined in Corollary 4.2, to obtain a *collection* of $(1-\alpha) \times 100\%$ confidence intervals indexed by τ^*. Taking the lower and upper bounds of these yields a *conservative* confidence interval for $\widehat{\mu}$, as defined in Equation 10. This interval has asymptotic coverage probability of *at least* $(1-\alpha-\delta) \times 100\%$. The precise algorithm is as follows.

Algorithm 4.1 (Simulation-based Confidence Interval for $\widehat{\mu}$)**.**

1. For each $\tau^* \in \mathscr{T}(\widehat{\tau}, \delta)$

   (i) Generate J independent draws $M_j \sim N_{p+q}(0, \widehat{\Omega})$

   (ii) Set $\Lambda_j(\tau^*) = -\nabla_\theta \mu(\widehat{\theta})' \left[\sum_{S \in \mathscr{S}} \widehat{\varphi}_S(\tau^*, M_j) \widehat{K}_S \Xi_S \right] (M_j + \tau^*)$

   (iii) Using the draws $\{\Lambda_j(\tau^*)\}_{j=1}^J$, calculate $\widehat{a}(\tau^*)$, $\widehat{b}(\tau^*)$ such that

   $$P\left\{ \widehat{a}(\tau^*) \leq \Lambda(\tau^*) \leq \widehat{b}(\tau^*) \right\} = 1 - \alpha$$

2. Set $\widehat{a}_{min}(\widehat{\tau}) = \min_{\tau^* \in \mathscr{T}(\widehat{\tau}, \delta)} \widehat{a}(\tau^*)$ and $\widehat{b}_{max}(\widehat{\tau}) = \max_{\tau^* \in \mathscr{T}(\widehat{\tau}, \delta)} \widehat{b}(\tau^*)$

3. The confidence interval for μ is $\text{CI}_{sim} = \left[\widehat{\mu} - \dfrac{\widehat{b}_{max}(\widehat{\tau})}{\sqrt{n}}, \quad \widehat{\mu} - \dfrac{\widehat{a}_{min}(\widehat{\tau})}{\sqrt{n}}\right]$

Theorem 4.4 (Simulation-based Confidence Interval for $\widehat{\mu}$)**.** *Let $\widehat{\Psi}$, $\widehat{\Omega}$, $\widehat{\theta}$, $\widehat{K}_S$, $\widehat{\varphi}_S$ be consistent estimators of Ψ, Ω, θ_0, K_S, φ_S and define $\Delta_n(\widehat{\tau}, \tau^*) = (\widehat{\tau} - \tau^*)' (\widehat{\Psi}\widehat{\Omega}\widehat{\Psi}')^{-1} (\widehat{\tau} - \tau^*)$ and $\mathscr{T}(\widehat{\tau}, \delta) = \left\{\tau^* \colon \Delta_n(\widehat{\tau}, \tau^*) \leq \chi^2_q(\delta)\right\}$ where $\chi^2_q(\delta)$ denotes the $1-\delta$ quantile of a χ^2 distribution with q degrees of freedom. Then, the interval CI_{sim} defined in Algorithm 4.1 has asymptotic coverage probability no less than $1-(\alpha+\delta)$ as $J, n \to \infty$.*

5 Simulation Results

5.1 OLS versus TSLS Example

I begin by examining the performance of the FMSC and averaging estimator in the OLS versus TSLS example. All calculations in this section are based on the formulas from Sections 3.2 and 4.2 with 10,000 simulation replications. The data generating process is given by

$$y_i = 0.5x_i + \epsilon_i \tag{18}$$
$$x_i = \pi(z_{1i} + z_{2i} + z_{3i}) + v_i \tag{19}$$

with $(\epsilon_i, v_i, z_{1i}, z_{2i}, z_{3i}) \sim$ iid $N(0, \mathcal{S})$

$$\mathcal{S} = \begin{bmatrix} \mathcal{S}_1 & 0 \\ 0 & \mathcal{S}_2 \end{bmatrix}, \quad \mathcal{S}_1 = \begin{bmatrix} 1 & \rho \\ \rho & 1-\pi^2 \end{bmatrix}, \quad \mathcal{S}_2 = I_3/3 \tag{20}$$

for $i = 1, \ldots, N$ where N, ρ and π vary over a grid. The goal is to estimate the effect of x on y, in this case 0.5, with minimum MSE either by choosing between OLS and TSLS estimators or by averaging them. To ensure that the finite-sample MSE of the TSLS estimator exists, this DGP includes three instruments leading to two overidentifying restrictions (Phillips, 1980).[19] This design satisfies regularity conditions that are sufficient for Theorem 3.2 – in particular it satisfies Assumption D.1 – and keeps the variance of x fixed at one so that $\pi = Corr(x_i, z_{1i} + z_{2i} + z_{3i})$ and $\rho = Corr(x_i, \epsilon_i)$. The first-stage R-squared is simply $1 - \sigma_v^2/\sigma_x^2 = \pi^2$ so that larger values of $|\pi|$ *reduce* the variance of the TSLS estimator. Since ρ controls the endogeneity of x, larger values of $|\rho|$ *increase* the bias of the OLS estimator.

Figure 1 compares the root mean-squared error (RMSE) of the post-FMSC estimator to those of the simple OLS and TSLS estimators. For any values of N and π there is a value

[19] Alternatively, one could use fewer instruments in the DGP and use work with trimmed MSE.

of ρ below which OLS outperforms TSLS. As N and π increase this value approaches zero; as they decrease it approaches one. Although the first two moments of the TSLS estimator exist in this simulation design, none of its higher moments do. This fact is clearly evident for small values of N and π: even with 10,000 simulation replications, the RMSE of the TSLS estimator shows a noticable degree of simulation error unlike those of the OLS and post-FMSC estimators. The FMSC represents a compromise between OLS and TSLS. When the RMSE of TSLS is high, the FMSC behaves more like OLS; when the RMSE of OLS is high it behaves more like TSLS. Because the FMSC is itself a random variable, however, it sometimes makes moment selection mistakes.[20] For this reason it does not attain an RMSE equal to the lower envelope of the OLS and TSLS estimators. The larger the RMSE difference between OLS and TSLS, however, the closer the FMSC comes to this lower envelope: costly mistakes are rare. Because this example involves a scalar target parameter, no selection or averaging scheme can provide a uniform improvement over the TSLS estimator. The FMSC is specifically intended for situations in which an applied researcher fears that her "baseline" assumptions may be too weak and consequently considers adding one or more "controversial" assumptions. In this case, she fears that the exogenous instruments z_1, z_2, z_3 are not particularly strong, π is small relative to N, and thus entertains the assumption that x is exogenous. It is precisely in these situations that the FMSC can provide large performance gains over TSLS.

As shown above, the FMSC takes a very special form in this example: it is equivalent to a DHW test with $\alpha \approx 0.16$. Accordingly, Figure 1 compares the RMSE of the post-FMSC estimator to those of DHW pre-test estimators with significance levels $\alpha = 0.05$ and $\alpha = 0.1$, indicated in the legend by DHW95 and DHW90. Since these three procedures differ only in their critical values, they show similar qualitative behavior. When ρ is sufficiently close to zero, we saw from Figure 1 that OLS has a lower RMSE than TSLS. Since DHW95 and DHW90 require a higher burden of proof to reject OLS in favor of TSLS, they outperform FMSC in this region of the parameter space. When ρ crosses the threshold beyond which TSLS has a lower RMSE than OLS, the tables are turned: FMSC outperforms DHW95 and DHW90. As ρ increases further, relative to sample size and π, the three procedures become indistinguishable in terms of RMSE. In addition to comparing the FMSC to DHW pre-test estimators, Figure 2 also presents the finite-sample RMSE of the minimum-AMSE moment average estimator presented in Equations 12 and 13. The performance of the moment average estimator is very strong: it provides the lowest worst-case RMSE and improves uniformly on the FMSC for all but the largest values of ρ.

[20] For more discussion of this point, see Section 4.

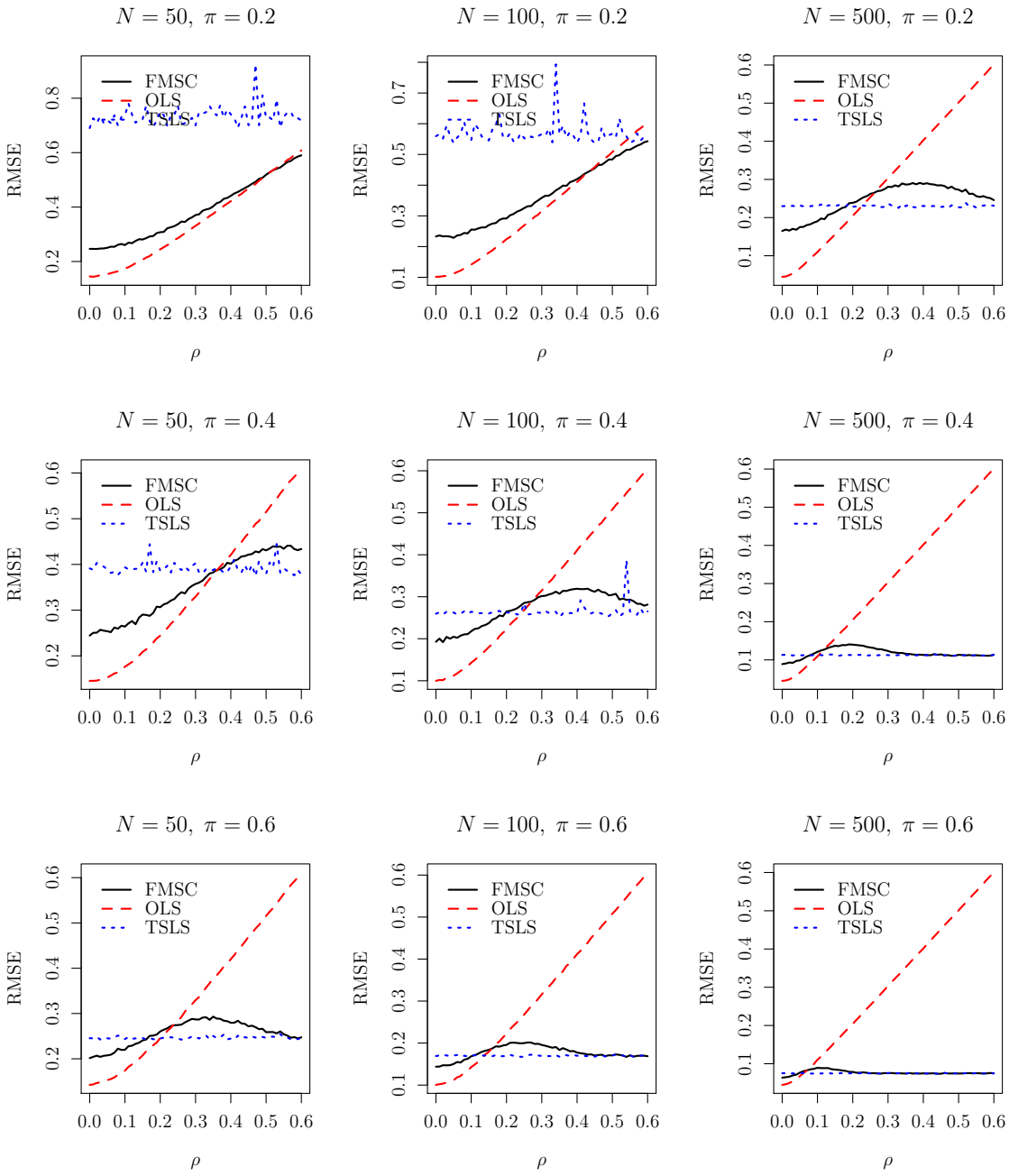

Figure 1: RMSE values for the two-stage least squares (TSLS) estimator, the ordinary least squares (OLS) estimator, and the post-Focused Moment Selection Criterion (FMSC) estimator based on 10,000 simulation draws from the DGP given in Equations 19–20 using the formulas described in Section 3.2.

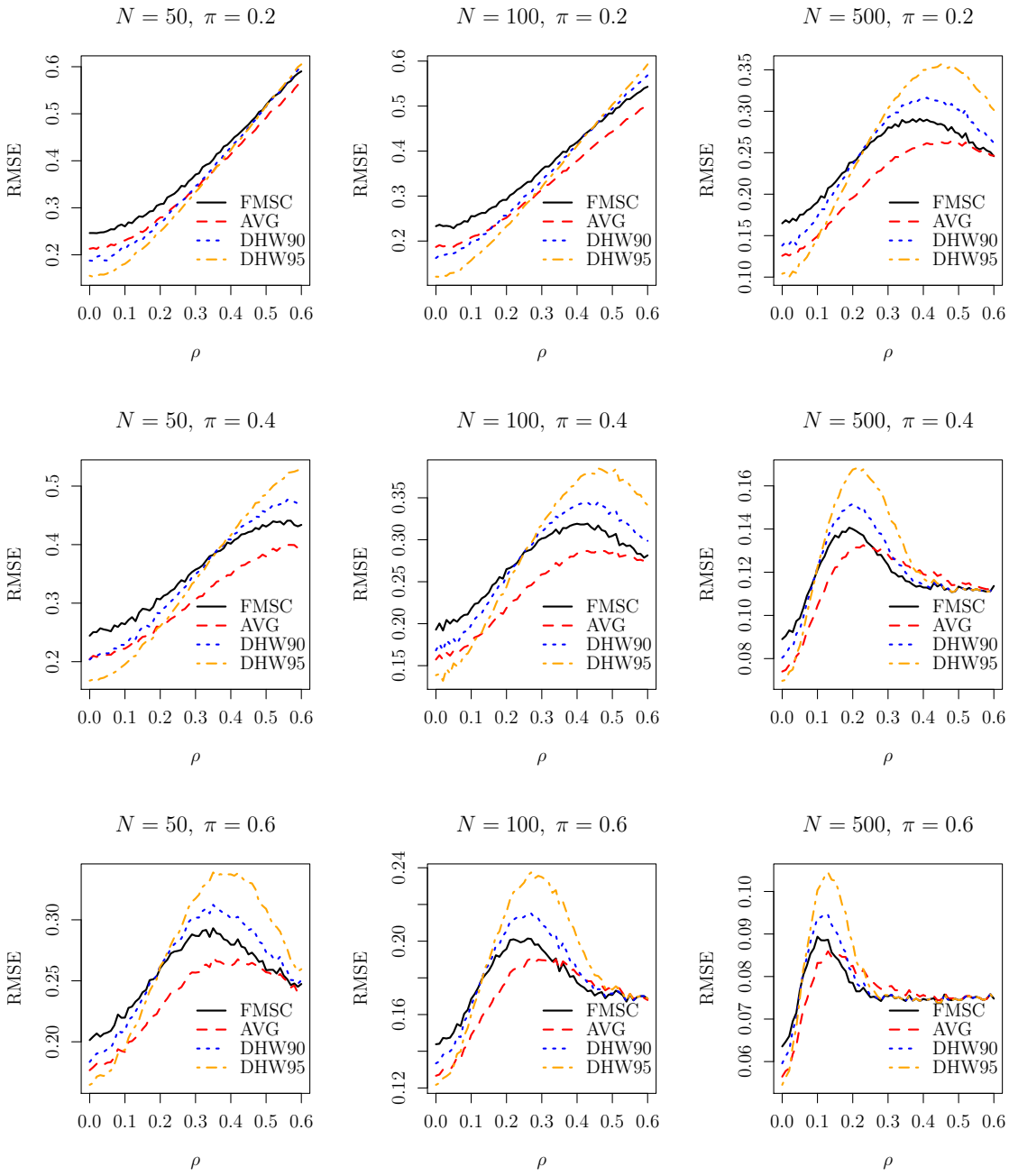

Figure 2: RMSE values for the post-Focused Moment Selection Criterion (FMSC) estimator, Durbin-Hausman-Wu pre-test estimators with $\alpha = 0.1$ (DWH90) and $\alpha = 0.05$ (DHW95), and the minmum-AMSE averaging estimator, based on 10,000 simulation draws from the DGP given in Equations 19–20 using the formulas described in Sections 3.2 and 4.2.

5.2 Choosing Instrumental Variables Example

I now evaluate the performance of FMSC in the instrument selection example described in Section 3.3 using the following simulation design:

$$y_i = 0.5x_i + \epsilon_i \tag{21}$$
$$x_i = (z_{1i} + z_{2i} + z_{3i})/3 + \gamma w_i + v_i \tag{22}$$

for $i = 1, 2, \ldots, N$ where $(\epsilon_i, v_i, w_i, z_{i1}, z_{2i}, z_{3i})' \sim$ iid $N(0, \mathcal{V})$ with

$$\mathcal{V} = \begin{bmatrix} \mathcal{V}_1 & 0 \\ 0 & \mathcal{V}_2 \end{bmatrix}, \quad \mathcal{V}_1 = \begin{bmatrix} 1 & (0.5 - \gamma\rho) & \rho \\ (0.5 - \gamma\rho) & (8/9 - \gamma^2) & 0 \\ \rho & 0 & 1 \end{bmatrix}, \quad \mathcal{V}_2 = I_3/3 \tag{23}$$

This setup keeps the variance of x fixed at one and the endogeneity of x, $Cor(x, \epsilon)$, fixed at 0.5 while allowing the relevance, $\gamma = Cor(x, w)$, and endogeneity, $\rho = Cor(w, \epsilon)$, of the instrument w to vary. The instruments z_1, z_2, z_3 are valid and exogenous: they have first-stage coefficients of $1/3$ and are uncorrelated with the second stage error ϵ. The additional instrument w is only relevant if $\gamma \neq 0$ and is only exogenous if $\rho = 0$. Since x has unit variance, the first-stage R-squared for this simulation design is simply $1 - \sigma_v^2 = 1/9 + \gamma^2$. Hence, when $\gamma = 0$, so that w is irrelevant, the first-stage R-squared is just over 0.11. Increasing γ increases the R-squared of the first-stage. This design satisfies the sufficient conditions for Theorem 3.5 given in Assumption D.2. When $\gamma = 0$, it is a special case of the DGP from Section 5.1.

As in Section 5.1, the goal of moment selection in this exercise is to estimate the effect of x on y, as before 0.5, with minimum MSE. In this case, however, the choice is between two TSLS estimators rather than OLS and TSLS: the *valid* estimator uses only z_1, z_2, and z_3 as instruments, while the *full* estimator uses z_1, z_2, z_3, and w. The inclusion of z_1, z_2 and z_3 in both moment sets means that the order of over-identification is two for the valid estimator and three for the full estimator. Because the moments of the TSLS estimator only exist up to the order of over-identification (Phillips, 1980), this ensures that the small-sample MSE is well-defined.[21] All estimators in this section are calculated via TSLS without a constant term using the expressions from Section 3.3 and 20,000 simulation replications.

Figure 3 presents RMSE values for the valid estimator, the full estimator, and the post-FMSC estimator for various combinations of γ, ρ, and N. The results are broadly similar to those from the OLS versus TSLS example presented in Figure 1. For any combination (γ, N)

[21] Alternatively, one could use fewer instruments for the valid estimator and compare the results using *trimmed* MSE, as in Hansen (2013).

there is a positive value of ρ below which the full estimator yields a lower RMSE than the full estimator. As the sample size increases, this cutoff becomes smaller; as γ increases, it becomes larger. As in the OLS versus TSLS example, the post-FMSC estimator represents a compromise between the two estimators over which the FMSC selects. Unlike in the previous example, however, when N is sufficiently small there is a range of values for ρ within which the FMSC yields a lower RMSE than *both* the valid and full estimators. This comes from the fact that the valid estimator is quite erratic for small sample sizes. Such behavior is unsurprising given that its first stage is not especially strong, R-squared $\approx 11\%$, and it has only two moments. In contrast, the full estimator has three moments and a stronger first stage. As in the OLS versus TSLS example, the post-FMSC estimator does not uniformly outperform the valid estimator for all parameter values, although it does for smaller sample sizes. The FMSC never performs much worse than the valid estimator, however, and often performs substantially better, particularly for small sample sizes.

I now compare the FMSC to the GMM moment selection criteria of Andrews (1999), which take the form $MSC(S) = J_n(S) - h(|S|)\kappa_n$, where $J_n(S)$ is the J-test statistic under moment set S and $-h(|S|)\kappa_n$ is a "bonus term" that rewards the inclusion of more moment conditions. For each member of this family we choose the moment set that *minimizes* $MSC(S)$. If we take $h(|S|) = (p + |S| - r)$, then $\kappa_n = \log n$ gives a GMM analogue of Schwarz's Bayesian Information Criterion (GMM-BIC) while $\kappa_n = 2.01 \log\log n$ gives an analogue of the Hannan-Quinn Information Criterion (GMM-HQ), and $\kappa_n = 2$ gives an analogue of Akaike's Information Criterion (GMM-AIC). Like the maximum likelihood model selection criteria upon which they are based, the GMM-BIC and GMM-HQ are consistent provided that Assumption 2.3 holds, while the GMM-AIC, like the FMSC, is conservative. Figure 4 gives the RMSE values for the post-FMSC estimator alongside those of the post-GMM-BIC, HQ and AIC estimators. I calculate the J-test statistic using a centered covariance matrix estimator, following the recommendation of Andrews (1999). For small sample sizes, the GMM-BIC, AIC and HQ are quite erratic: indded for $N = 50$ the FMSC has a uniformly smaller RMSE. This problem comes from the fact that the J-test statistic can be very badly behaved in small samples.[22] As the sample size becomes larger, the classic tradeoff between consistent and conservative selection emerges. For the smallest values of ρ the consistent criteria outperform the conservative criteria; for moderate values the situation is reversed. The consistent criteria, however, have the highest worst-case RMSE. For a discussion of a combined strategy based on the GMM information criteria of Andrews (1999) and the canonical correlations information criteria of Hall and Peixe (2003), see Appendix E.2. For a comparison with the downward J-test, see Appendix E.1.

[22] For more details, see Appendix E.1.

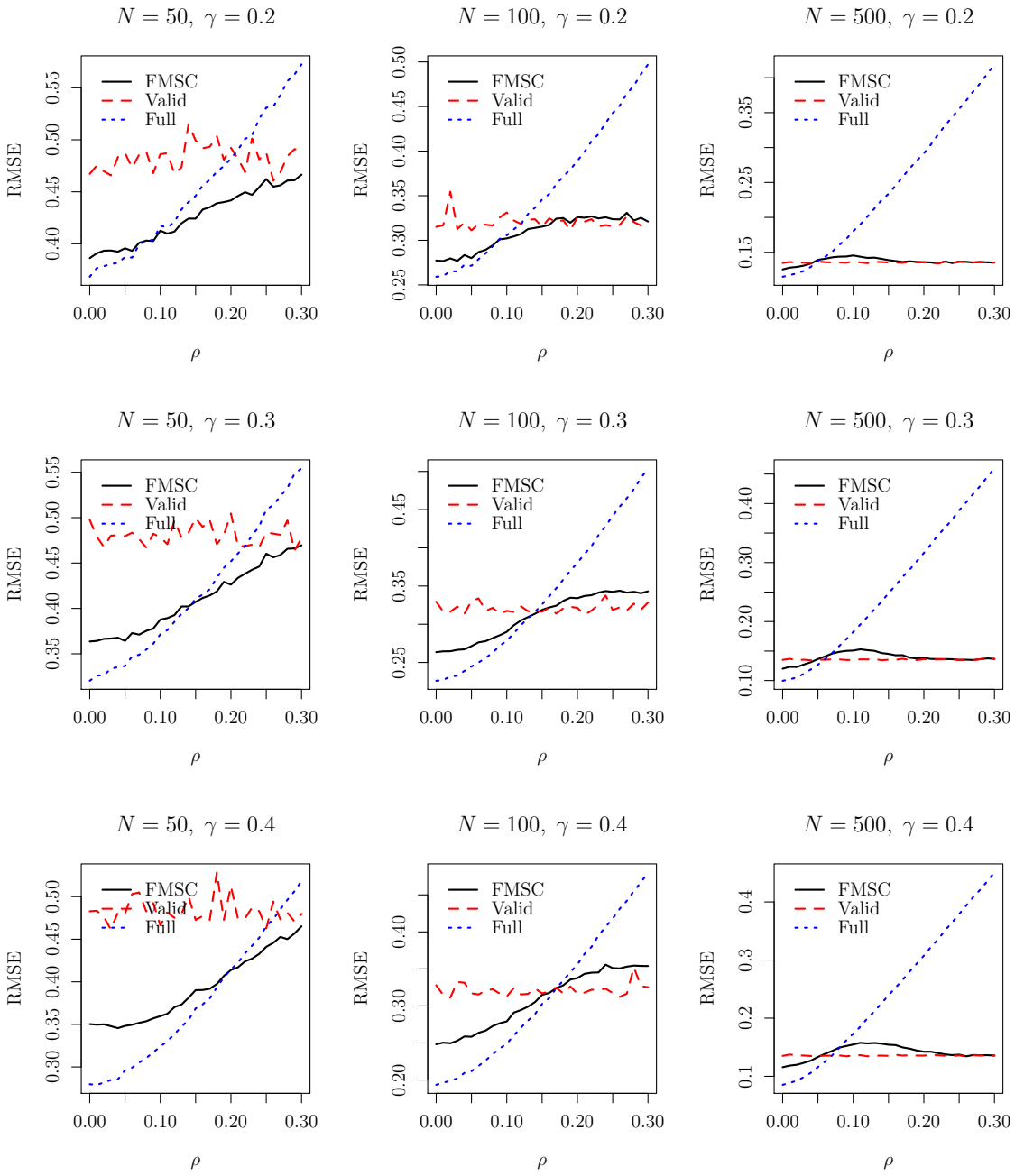

Figure 3: RMSE values for the valid estimator, including only (z_1, z_2, z_3), the full estimator, including (z_1, z_2, z_3, w), and the post-Focused Moment Selection Criterion (FMSC) estimator based on 20,000 simulation draws from the DGP given in Equations 22–23 using the formulas described in Sections 3.3.

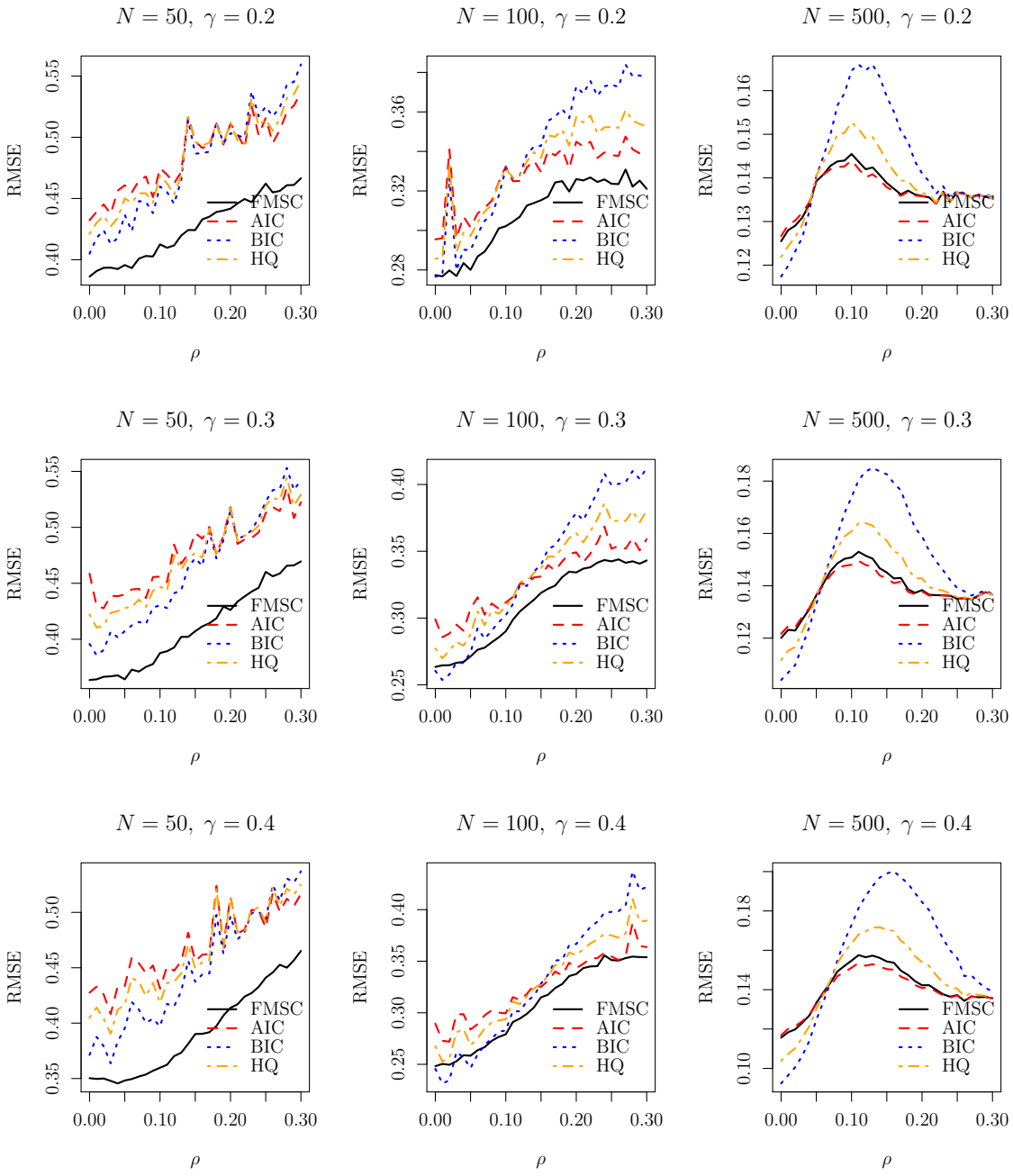

Figure 4: RMSE values for the post-Focused Moment Selection Criterion (FMSC) estimator and the GMM-BIC, HQ, and AIC estimators based on 20,000 simulation draws from the DGP given in Equations 22–23 using the formulas described in Sections 3.3.

5.3 Valid Confidence Intervals

I now revisit the simulation experiments introduced above in Sections 5.1 and 5.2 to evaluate the finite-sample performance of confidence intervals contructed according to Algorithm 4.1. All results are based on 10,000 simulation replications from the appropriate DGP with $\alpha = \delta = 0.05$. For more computational details, see Appendix B. Coverage probabilities and relative widths are all given in percentage points, rounded to the nearest whole percent.

Table 1b shows the problem of ignoring moment selection by presenting the the actual coverage probability of a naïve 90%, post-FMSC confidence interval for the OLS versus TSLS simulation experiment. The naïve procedure simply constructs a textbook 90% interval around the FMSC-selected estimator. Unsurprisingly, it performs poorly: coverage probabilities can be made *arbitrarily* close to zero by choosing appropriate parameter values, a problem that persists even for large N. At other parameter values, however, the intervals are close to their nominal level. This is precisely the lack of uniformity described by Leeb and Pötscher (2005). A similar pattern emerges in the choosing instrumental variables simulation: see Table 9b in Appendix E.3.

Table 2a gives the actual coverage probability of the conservative, 90% post-FMSC confidence interval, constructed according to Algorithm 4.1, for the OLS versus TSLS example. These intervals achieve their nominal minimum coverage for all parameter values but can be quite conservative, particularly for smaller values of π, ρ and N. In particular, coverage never falls below 94% but occasionally exceeds 99.5%. Some conservatism is inevitable given the procedure, which takes which takes *worst-case* bounds over a collection of intervals. The real culprit in this example, however, is the TSLS estimator, as we see from Table 1a. Although this estimator is correctly specified and is not subject to model selection uncertainty, its textbook 90% confidence interval dramatically overcovers for smaller values of π even if N is fairly large. This is a manifestation of the weak instruments problem. This additional source of conservatism is inherited by the two-step post-FMSC intervals. Results for the minimum-AMSE moment average estimator, given in Table 2b, are similar.

The worry, of course, is not conservatism as such but the attendant increase in confidence interval width. Accordingly, Tables 3a and 3b compare the median width of the simulation-based post-FMSC and minimum-AMSE intervals to that of the TSLS estimator. A value of 25, for example indicates that the simulation-based interval is 25% wider than the corresponding interval for the TSLS estimator. This comparison shows us the inferential cost of carrying out moment selection relative to simply using the correctly-specified TSLS estimator and calling it a day. Moment selection is not a free lunch: the averaging and post-selection intervals are wider than those of the TSLS estimator, sometimes considerably so. Intriguingly, the minimum-AMSE intervals are generally much shorter than the post-FMSC

(a) Two-Stage Least Squares

$N = 50$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	98	98	96	93	89	82
	0.2	97	97	95	93	88	83
π	0.3	96	96	94	92	88	85
	0.4	94	93	93	91	89	87
	0.5	92	92	92	91	90	88
	0.6	91	91	90	90	90	88

$N = 100$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	98	98	97	94	89	83
	0.2	96	96	95	92	89	85
π	0.3	94	94	93	91	89	87
	0.4	92	92	92	91	90	88
	0.5	91	91	90	90	89	89
	0.6	90	90	90	90	90	89

$N = 500$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	96	96	94	93	90	86
	0.2	92	92	91	91	90	89
π	0.3	91	91	91	91	90	90
	0.4	90	90	91	90	90	90
	0.5	90	90	90	90	90	90
	0.6	90	91	90	90	90	90

(b) Naïve post-FMSC

$N = 50$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	88	80	58	30	11	4
	0.2	88	79	59	34	15	10
π	0.3	87	81	62	39	25	23
	0.4	86	80	66	46	38	43
	0.5	86	81	68	56	54	62
	0.6	85	81	72	66	67	75

$N = 100$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	88	72	36	10	4	4
	0.2	87	74	40	17	13	19
π	0.3	86	74	45	29	32	45
	0.4	85	74	51	43	54	70
	0.5	85	76	59	57	70	84
	0.6	85	78	66	68	81	88

$N = 500$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	87	31	8	12	16	24
	0.2	84	35	24	42	62	80
π	0.3	83	42	43	70	87	90
	0.4	84	49	62	86	90	90
	0.5	84	57	76	89	90	90
	0.6	86	66	84	90	90	90

Table 1: Coverage probabilities of nominal 90% CIs for the OLS versus TSLS simulation experiment from Section 5.1. Values are given in percentage points, rounded to the nearest whole percent, based on 10,000 simulation draws from the DGP given in Equations 19–20.

intervals in spite of being somewhat more conservative.

Turning our attention now to the choosing instrumental variables simulation experiment from Section 5.2, Table 4a gives the coverage probability and Table 4b the median relative width of the conservative, 90%, simulation-based, post-FMSC confidence interval. In this case, the width calculation is relative to the valid estimator, the TSLS estimator that includes the exogenous instruments z_1, z_2, z_3 but excludes the potentially endogenous instrument w. Here the simulation-based intervals are far less conservative and occasionally undercover slightly. The worst case, 81% actual coverage compared to 90% nominal coverage, occurs when $N = 50, \gamma = 0.6, \rho = 0.5$. This problem stems from the fact that traditional interval for the valid estimator systematically under-covers when $N = 50$ or 100.[23] Nevertheless, the simulation-based interval works well in this example: in the worst case, its median width is

[23]For details, see Table 9a in Appendix E.3.

(a) FMSC

$N = 50$		ρ = 0	0.1	0.2	0.3	0.4	0.5
	0.1	100	100	99	99	98	97
	0.2	99	99	99	99	98	97
π	0.3	99	99	99	99	98	96
	0.4	98	98	98	98	98	95
	0.5	97	98	98	98	97	94
	0.6	97	97	97	97	96	94

$N = 100$		ρ = 0	0.1	0.2	0.3	0.4	0.5
	0.1	100	99	99	99	99	98
	0.2	99	99	99	99	99	97
π	0.3	98	98	99	99	98	95
	0.4	97	97	98	98	97	94
	0.5	97	97	98	97	95	95
	0.6	97	97	97	96	95	96

$N = 500$		ρ = 0	0.1	0.2	0.3	0.4	0.5
	0.1	99	99	99	99	99	96
	0.2	97	97	98	99	97	94
π	0.3	96	97	98	97	95	98
	0.4	96	97	97	95	98	98
	0.5	96	97	96	97	98	98
	0.6	96	97	95	97	97	96

(b) AMSE-Averaging Estimator

$N = 50$		ρ = 0	0.1	0.2	0.3	0.4	0.5
	0.1	100	100	100	99	98	96
	0.2	100	100	100	99	98	96
π	0.3	100	100	99	99	98	95
	0.4	99	99	99	98	97	94
	0.5	99	99	98	98	96	93
	0.6	98	98	98	97	96	93

$N = 100$		ρ = 0	0.1	0.2	0.3	0.4	0.5
	0.1	100	100	100	100	99	97
	0.2	100	100	100	99	98	95
π	0.3	99	99	99	99	97	94
	0.4	99	99	99	98	96	94
	0.5	98	99	98	97	95	94
	0.6	98	98	97	96	95	95

$N = 500$		ρ = 0	0.1	0.2	0.3	0.4	0.5
	0.1	100	100	100	99	98	95
	0.2	99	99	99	98	96	94
π	0.3	98	98	98	97	95	97
	0.4	98	98	97	96	96	97
	0.5	98	98	96	96	97	97
	0.6	98	97	96	96	97	96

Table 2: Coverage probabilities of simulation-based conservative 90% CIs for the OLS versus TSLS simulation experiment from Section 5.1. Values are given in percentage points, rounded to the nearest whole percent, based on 10,000 simulation draws from the DGP given in Equations 19–20.

only 22% greater than that of the valid estimator.

Although the simulation-based intervals work fairly well, two caveats are in order. First, when the usual first-order asymptotic theory begins to break down, such as a weak instruments example, the simulation-based intervals can inherit an under– or over–coverage problem from the valid estimator. Second, moment selection comes with a cost: the simulation-based intervals are on average wider than a textbook confidence interval for the valid estimator, as we would expect given the impossibility results for post-selection inference outlined in Leeb and Pötscher (2005).[24] As described above, the primary goal of the the FMSC is *estimation* rather than inference. Once the decision to carry out moment selection has been

[24]The intervals presented here could potentially be shortened by optimizing width over α while holding $\alpha + \delta$ fixed at 0.1. For more discussion of this idea, see Claeskens and Hjort (2008b) and McCloskey (2012).

(a) post-FMSC Estimator

$N = 50$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	40	41	41	41	42	42
	0.2	41	42	42	43	42	42
π	0.3	42	43	43	43	43	43
	0.4	43	43	43	43	43	43
	0.5	43	42	42	42	42	41
	0.6	41	41	40	40	39	38

$N = 100$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	40	41	41	41	41	42
	0.2	42	42	42	42	43	44
π	0.3	43	43	43	44	45	46
	0.4	43	43	43	44	44	44
	0.5	43	43	42	42	42	42
	0.6	42	41	40	39	39	39

$N = 500$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	40	41	42	42	43	46
	0.2	42	43	45	47	49	51
π	0.3	43	44	46	48	49	49
	0.4	43	44	45	46	46	44
	0.5	43	43	42	42	39	27
	0.6	42	40	39	37	28	20

(b) AMSE-Averaging Estimator

$N = 50$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	32	33	33	33	33	33
	0.2	33	34	34	34	34	34
π	0.3	34	35	34	34	35	34
	0.4	35	35	35	35	34	34
	0.5	36	36	35	34	34	33
	0.6	36	35	35	33	32	32

$N = 100$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	33	32	32	32	32	33
	0.2	34	33	34	33	34	35
π	0.3	35	35	34	35	35	35
	0.4	35	35	35	35	35	35
	0.5	36	36	35	34	33	33
	0.6	36	35	34	32	32	32

$N = 500$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	31	32	33	33	34	35
	0.2	33	34	35	36	37	39
π	0.3	35	35	36	36	37	38
	0.4	35	35	35	36	36	36
	0.5	36	35	34	34	34	31
	0.6	36	33	32	32	30	25

Table 3: Median width of two-step, simulation-based conservative 90% CI relative to that of a traditional 90% CI for the TSLS estimator in the OLS versus TSLS example from Section 5.1. Values are given in percentage points, rounded to the nearest whole percent, based on 10,000 simulation draws from the DGP given in Equations 19–20.

taken, however, one cannot simply ignore this fact and report the usual confidence intervals. Algorithm 4.1 provides a way to carry out honest inference post-selection and construct confidence intervals for complicated objects such as the minimum-AMSE averaging estimator from Section 4.2. More to the point, although formal moment selection is relatively rare, *informal* moment selection is extremely common in applied work. Downward J-tests, DHW tests and the like are a standard part of the applied econometrician's toolkit. Because it can be employed to construct confidence intervals that account for the effects of specification searches, Algorithm 4.1 can provide a valuable robustness check, as I explore in the empirical example that follows.

(a) Coverage Probability

$N = 50$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	89	88	88	87	89	89
	0.2	90	88	87	86	88	89
γ	0.3	91	89	87	85	86	88
	0.4	91	91	87	84	83	87
	0.5	92	91	88	84	82	84
	0.6	92	92	90	85	82	81

$N = 100$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	92	90	90	91	91	91
	0.2	92	90	88	89	91	91
γ	0.3	93	90	87	88	90	91
	0.4	94	92	86	84	88	90
	0.5	94	93	87	83	84	89
	0.6	94	93	89	82	82	86

$N = 500$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	95	93	92	92	92	92
	0.2	96	91	92	91	92	93
γ	0.3	96	89	92	92	92	93
	0.4	96	89	90	91	92	93
	0.5	96	90	87	91	91	92
	0.6	95	92	83	91	91	92

(b) Relative Median Width

$N = 50$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	20	18	18	17	17	19
	0.2	20	18	17	16	16	16
γ	0.3	21	17	15	14	14	17
	0.4	21	17	13	13	13	15
	0.5	22	16	13	12	12	13
	0.6	22	17	13	10	10	12

$N = 100$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	20	18	16	15	14	14
	0.2	22	17	15	13	12	14
γ	0.3	22	16	13	12	12	14
	0.4	21	16	11	10	10	14
	0.5	21	15	11	9	9	12
	0.6	21	15	11	8	8	11

$N = 500$		ρ 0	0.1	0.2	0.3	0.4	0.5
	0.1	22	17	13	10	8	7
	0.2	22	15	10	7	6	9
γ	0.3	21	13	8	6	7	12
	0.4	20	13	7	6	7	11
	0.5	19	13	8	6	6	9
	0.6	20	13	8	5	6	8

Table 4: Performance of the simulation-based, conservative 90% post-FMSC confidence interval in the choosing instrumental variables simulation from Section 5.2. The left panel gives coverage probabilities, and the right panel gives median widths relative to that of a traditional 90% interval for the valid estimator. Values are given in percentage points, rounded to the nearest whole percent, based on 10,000 simulation draws from the DGP given in Equations 22–23.

6 Empirical Example: Geography or Institutions?

Carstensen and Gundlach (2006) address a controversial question from the development literature: does geography directly effect income after controlling for institutions? A number of well-known studies find little or no direct effect of geographic endowments (Acemoglu et al., 2001; Easterly and Levine, 2003; Rodrik et al., 2004). Sachs (2003), on the other hand, shows that malaria transmission, a variable largely driven by ecological conditions, directly influences the level of per capita income, even after controlling for institutions. Because malaria transmission is very likely endogenous, Sachs uses a measure of "malaria ecology," constructed to be exogenous both to present economic conditions and public health interventions, as an instrument. Carstensen and Gundlach (2006) address the robustness of Sachs's results using the following baseline regression for a sample of 44 countries:

$$\ln gdpc_i = \beta_1 + \beta_2 \cdot institutions_i + \beta_3 \cdot malaria_i + \epsilon_i \tag{24}$$

This model extends the baseline specification of Acemoglu et al. (2001) to include a direct effect of malaria transmission which, like institutions, is treated as endogenous.[25] Considering a variety of measures of both institutions and malaria transmission, and a number of instrument sets, Carstensen and Gundlach (2006) find large negative effects of malaria transmission, lending support to Sach's conclusion.

In this section, I revisit and expand upon the instrument selection exercise given in Table 2 of Carstensen and Gundlach (2006) using the FMSC and corrected confidence intervals described above. I consider two questions. First, based on the FMSC methodology, which instruments should we choose if our target parameter is β_3, the effect of malaria transmission on per capita income? Does the selected instrument set change if our target parameter is β_2, the effect of institutions? Second, are the results robust to the effects of instrument selection on inference? All results are calculated by TSLS using the formulas from Section 3.3 and the variables described in Table 5, with ln*gdpc* as the outcome variable and *rule* and *malfal* as measures of institutions and malaria transmission.

To apply the FMSC to the present example, we require a minimum of two valid instruments besides the constant term. Based on the arguments given by Acemoglu et al. (2001) and Sachs (2003), I proceed under the assumption that ln*mort* and *maleco*, measures of early settler mortality and malaria ecology, are exogenous. Rather than selecting over all 128 possible instrument sets, I consider eight specifications formed from the four instrument blocks defined by Carstensen and Gundlach (2006). The baseline block contains ln*mort*, *maleco* and

[25] Due to a lack of data for certain instruments, Carstensen and Gundlach (2006) work with a smaller sample of countries than Acemoglu et al. (2001).

Name	Description	
ln*gdpc*	Real GDP/capita at PPP, 1995 International Dollars	Outcome
rule	Institutional quality (Average Governance Indicator)	Regressor
malfal	Fraction of population at risk of malaria transmission, 1994	Regressor
ln*mort*	Log settler mortality (per 1000 settlers), early 19th century	Baseline
maleco	Index of stability of malaria transmission	Baseline
frost	Prop. of land receiving at least 5 days of frost in winter	Climate
humid	Highest temp. in month with highest avg. afternoon humidity	Climate
latitude	Distance from equator (absolute value of latitude in degrees)	Climate
eurfrac	Fraction of pop. that speaks major West. European Language	Europe
engfrac	Fraction of pop. that speaks English	Europe
coast	Proportion of land area within 100km of sea coast	Openness
trade	Log Frankel-Romer predicted trade share	Openness

Table 5: Description of variables for Empirical Example.

a constant; the climate block contains *frost*, *humid*, and *latitude*; the Europe block contains *eurfrac* and *engfrac*; and the openness block contains *coast* and *trade*. Full descriptions of these variables appear in Table 5. Table 6 lists the eight instrument sets considered here, along with TSLS estimates and traditional 95% confidence intervals for each.[26]

Table 7 presents FMSC and "positive-part" FMSC results for instrument sets 1–8. The positive-part FMSC sets a negative squared bias estimate to zero when estimating AMSE. If the squared bias estimate is positive, FMSC and positive-part FMSC coincide; if the squared bias estimate is negative, positive-part FMSC is strictly greater than FMSC. Additional simulation results for the choosing instrumental variables experiment from Section 5.2, available upon request, reveal that the positive-part FMSC never performs worse than the ordinary FMSC and sometimes performs slightly better, suggesting that it may be preferable in real-world applications. For each criterion the table presents two cases: the first takes the effect of *malfal*, a measure of malaria transmission, as the target parameter while the second uses the effect of *rule*, a measure of institutions. In each case the two best instrument sets are numbers 5 (baseline, climate and Europe) and 8 (all instruments). When the target parameter is the coefficient on *malfal*, 8 is the clear winner under both the plain-vanilla and positive-part FMSC, leading to an estimate of -1.08 for the effect of malaria transmission on per-capita income. When the target parameter is the coefficient on *rule*, however, instrument sets 5 and 8 are virtually indistinguishable. Indeed, while the plain-vanilla FMSC selects instrument set 8, leading to an estimate of 0.84 for the effect of instutitions on per-capita income, the positive-part FMSC selects instrument set 5, leading to an estimate of 0.93. Thus the FMSC methodology shows that, while helpful for estimating the effect of malaria

[26]The results for the baseline instrument presented in panel 1 of Table 6 are slightly different from those in Carstensen and Gundlach (2006) as I exclude Vietnam to keep the sample fixed across instrument sets.

	1		2		3		4	
	rule	*malfal*	*rule*	*malfal*	*rule*	*malfal*	*rule*	*malfal*
coeff.	0.89	−1.04	0.97	−0.90	0.81	−1.09	0.86	−1.14
SE	0.18	0.31	0.16	0.29	0.16	0.29	0.16	0.27
lower	0.53	−1.66	0.65	−1.48	0.49	−1.67	0.55	−1.69
upper	1.25	−0.42	1.30	−0.32	1.13	−0.51	1.18	−0.59
	Baseline		Baseline		Baseline		Baseline	
			Climate					
					Openness			
							Europe	

	5		6		7		8	
	rule	*malfal*	*rule*	*malfal*	*rule*	*malfal*	*rule*	*malfal*
coeff.	0.93	−1.02	0.86	−0.98	0.81	−1.16	0.84	−1.08
SE	0.15	0.26	0.14	0.27	0.15	0.27	0.13	0.25
lower	0.63	−1.54	0.59	−1.53	0.51	−1.70	0.57	−1.58
upper	1.22	−0.49	1.14	−0.43	1.11	−0.62	1.10	−0.58
	Baseline		Baseline		Baseline		Baseline	
	Climate		Climate				Climate	
			Openness		Openness		Openness	
	Europe				Europe		Europe	

Table 6: Two-stage least squares estimation results for all instrument sets.

transmission, the openness instruments *coast* and *trade* provide essentially no additional information for studying the effect of institutions.

Table 8 presents three alternative post-selection confidence intervals for each of the instrument selection exercises from Table 7: Naïve, 1-Step, and 2-Step. The Naïve intervals are standard, nominal 95% confidence intervals that ignore the effects of instrument selection. These are constructed by identifying the selected instrument set from Table 7 and simply reporting the corresponding nominal 95% interval from Table 6 unaltered. The 1-Step intervals are simulation-based nominal 95% intervals constructed using a simplified, and less computationally intensive, version of the procedure given in Algorithm 4.1. Rather than taking the minimum lower confidence limit and maximum upper confidence limit over all values in a given confidence region for τ, this procedure simply assumes that the estimated value $\widehat{\tau}$ is exactly correct, and generates simulations for Λ under this assumption. Neither the Naïve nor the 1-Step procedures yield valid 95% confidence intervals. They are provided merely for comparison with the 2-Step procedure, which fully implements Algorithm 4.1 with $\alpha = \delta = 0.05$ and $J = 10,000$. As explained above, the 2-Step interval is guaranteed to have asymptotic coverage probability of at least $1 - \alpha - \delta$, in this case 90%. From the

	$\mu = malfal$			$\mu = rule$		
	FMSC	posFMSC	$\widehat{\mu}$	FMSC	posFMSC	$\widehat{\mu}$
(1) Valid	3.03	3.03	−1.04	1.27	1.27	0.89
(2) Climate	3.07	3.07	−0.90	1.00	1.00	0.97
(3) Openness	2.30	2.42	−1.09	1.21	1.21	0.81
(4) Europe	1.82	2.15	−1.14	0.52	0.73	0.86
(5) Climate, Europe	0.85	2.03	−1.02	0.25	0.59	0.93
(6) Climate, Openness	1.85	2.30	−0.98	0.45	0.84	0.86
(7) Openness, Europe	1.63	1.80	−1.16	0.75	0.75	0.81
(8) Full	0.53	1.69	−1.08	0.23	0.62	0.84

Table 7: FMSC and and positive-part FMSC values corresponding to the instrument sets from Table 6

2-Step intervals, we see that the results of Carstensen and Gundlach (2006) are extremely robust. There is no evidence that accounting for the effects of instrument selection changes our conclusions about the sign or significance of *malfal* or *rule*.

	$\mu =$*malfal*		$\mu =$*rule*	
	FMSC	posFMSC	FMSC	posFMSC
Naïve	$(-1.58, -0.58)$	$(-1.58, -0.58)$	$(0.57, 1.10)$	$(0.63, 1.22)$
1-Step	$(-1.52, -0.67)$	$(-1.51, -0.68)$	$(0.57, 1.08)$	$(0.68, 1.17)$
2-Step	$(-1.62, -0.55)$	$(-1.62, -0.55)$	$(0.49, 1.18)$	$(0.58, 1.27)$

Table 8: Post-selection CIs for the instrument selection exercise from Table 7.

Although this example uses a simple model and a relatively small number of observations, it nevertheless provides a realistic proof of concept for FMSC instrument selection and post-selection inference because the computational complexity of the procedures described above is determined almost *entirely* by the dimension, q, of τ. This is because the 2-Step confidence interval procedure requires us to carry out two q-dimensional constrained optimization problems with a stochastic objective function: one for each confidence limit. Fixing q, the number of instrument sets under consideration is far less important because we can pre-compute any quantities that do not depend on τ^*. With $q = 7$, this example presents the kind of computational challenge that one would reasonably expect to encounter in practice yet is well within the ability of a standard desktop computer using off-the-shelf optimization routines. Running on a single core it took just over ten minutes to generate all of the results for the empirical example in this paper. For more computational details, including a description of the packages used, see Appendix B.

7 Conclusion

This paper has introduced the FMSC, a proposal to choose moment conditions using AMSE. The criterion performs well in simulations, and the framework used to derive it allows us to construct valid confidence intervals for post-selection and moment-average estimators. Although simulation-based, this procedure remains feasible for problems of a realistic scale without the need for specialized computing resources, as demonstrated in the empirical example above. Moment selection is not a panacea, but the FMSC and related confidence interval procedures can yield sizeable benefits in empirically relevant settings, making them a valuable complement to existing methods. While the discussion here concentrates on two cross-section examples, the FMSC could prove useful in any context in which moment conditions arise from more than one source. In a panel model, for example, the assumption of contemporaneously exogenous instruments may be plausible while that of predetermined instruments is more dubious. Using the FMSC, we could assess whether the extra information contained in the lagged instruments outweighs their potential invalidity. Work in progress explores this idea in both static and dynamic panel settings by extending the FMSC to allow for simultaneous moment and model selection. Other potentially fruitful extensions include the consideration of risk functions other than MSE, and an explicit treatment of weak identification and many moment conditions.

A Proofs

Proof of Theorems 2.1, 2.2. Essentially identical to the proofs of Newey and McFadden (1994) Theorems 2.6 and 3.1. □

Proof of Theorems 3.2, 3.5. The proofs of both results are similar and standard, so I provide only a sketch of the argument for Theorem 3.5. First substitute the DGP into the expression for $\widehat{\beta}_S$ and rearrange so that the left-hand side becomes $\sqrt{n}(\beta_S - \beta)$. The right-hand side has two factors: the first converges in probability to $-K_S$ by an L_2 argument and the second converges in distribution to $M + (0', \tau')'$ by the Lindeberg-Feller Central Limit Theorem. □

Proof of Theorem 3.1. By a mean-value expansion:

$$\begin{aligned}\widehat{\tau} &= \sqrt{n}h_n\left(\widehat{\theta}_v\right) = \sqrt{n}h_n(\theta_0) + H\sqrt{n}\left(\widehat{\theta}_v - \theta_0\right) + o_p(1) \\ &= -HK_v\sqrt{n}g_n(\theta_0) + \mathbf{I}_q\sqrt{n}h_n(\theta_0) + o_p(1) \\ &= \Psi\sqrt{n}f_n(\theta_0) + o_p(1)\end{aligned}$$

The result follows since $\sqrt{n}f_n(\theta_0) \to_d M + (0', \tau')'$ under Assumption 2.2 (h). □

Proof of Corollary 3.2. By Theorem 3.1 and the Continuous Mapping Theorem, we have $\widehat{\tau}\widehat{\tau}' \to_d UU'$ where $U = \Psi M + \tau$. Since $E[M] = 0$, $E[UU'] = \Psi\Omega\Psi' + \tau\tau'$. □

Proof of Theorem 3.4. By Theorem 3.3, $\sqrt{n}(\widehat{\beta}_{OLS} - \widetilde{\beta}_{TSLS}) \to_d N\left(\tau/\sigma_x^2, \Sigma\right)$ where $\Sigma = \sigma_\epsilon^2\left(1/\gamma^2 - 1/\sigma_x^2\right)$. Thus, under $H_0\colon \tau = 0$, the DHW test statistic

$$\widehat{T}_{DHW} = n\,\widehat{\Sigma}^{-1}(\widehat{\beta}_{OLS} - \widetilde{\beta}_{TSLS})^2 = \frac{n(\widehat{\beta}_{OLS} - \widetilde{\beta}_{TSLS})^2}{\widehat{\sigma}_\epsilon^2\left(1/\widehat{\gamma}^2 - 1/\widehat{\sigma}_x^2\right)}$$

converges in distribution to a $\chi^2(1)$ random variable. Now, rewriting $\widehat{V}$, we find that

$$\widehat{V} = \widehat{\sigma}_\epsilon^2\widehat{\sigma}_x^2\left(\frac{\widehat{\sigma}_v^2}{\widehat{\gamma}^2}\right) = \widehat{\sigma}_\epsilon^2\widehat{\sigma}_x^2\left(\frac{\widehat{\sigma}_x^2 - \widehat{\gamma}^2}{\widehat{\gamma}^2}\right) = \widehat{\sigma}_\epsilon^2\widehat{\sigma}_x^4\left(\frac{1}{\widehat{\gamma}^2} - \frac{1}{\widehat{\sigma}_x^2}\right) = \widehat{\sigma}_x^4\,\widehat{\Sigma}$$

using the fact that $\widehat{\sigma}_v = \widehat{\sigma}_x^2 - \widehat{\gamma}^2$. Thus, to show that $\widehat{T}_{FMSC} = \widehat{T}_{DHW}$, all that remains is to establish that $\widehat{\tau}^2 = n\widehat{\sigma}_x^4(\widehat{\beta}_{OLS} - \widetilde{\beta}_{TSLS})^2$, which we obtain as follows:

$$\widehat{\tau}^2 = \left[n^{-1/2}\mathbf{x}'(\mathbf{y} - \mathbf{x}\widetilde{\beta})\right]^2 = n^{-1}\left[\mathbf{x}'\mathbf{x}\left(\widehat{\beta} - \widetilde{\beta}\right)\right]^2 = n^{-1}\left[n\widehat{\sigma}_x^2\left(\widehat{\beta} - \widetilde{\beta}\right)\right]^2.$$

□

Proof of Corollary 4.2. Because the weights sum to one

$$\sqrt{n}\left(\widehat{\mu}-\mu_0\right)=\sqrt{n}\left[\left(\sum_{S\in\mathscr{S}}\widehat{\omega}_S\widehat{\mu}_S\right)-\mu_0\right]=\sum_{S\in\mathscr{S}}\left[\widehat{\omega}_S\sqrt{n}\left(\widehat{\mu}_S-\mu_0\right)\right].$$

By Corollary 3.1, we have

$$\sqrt{n}\left(\widehat{\mu}_S-\mu_0\right)\to_d -\nabla_\theta\mu(\theta_0)'K_S\Xi_S\left(M+\begin{bmatrix}0\\ \tau\end{bmatrix}\right)$$

and by the assumptions of this Corollary we find that $\widehat{\omega}_S\to_d\varphi_S(\tau,M)$ for each $S\in\mathscr{S}$, where $\varphi_S(\tau,M)$ is a function of M and constants only. Hence $\widehat{\omega}_S$ and $\sqrt{n}\left(\widehat{\mu}_S-\mu_0\right)$ converge jointly in distribution to their respective functions of M, for all $S\in\mathscr{S}$. The result follows by application of the Continuous Mapping Theorem. □

Proof of Theorem 4.3. Since the weights sum to one, by Theorem 3.2

$$\sqrt{n}\left[\widehat{\beta}(\omega)-\beta\right]\xrightarrow{d}N\left(\text{Bias}\left[\widehat{\beta}(\omega)\right],Var\left[\widehat{\beta}(\omega)\right]\right)$$

where

$$\begin{aligned}\text{Bias}\left[\widehat{\beta}(\omega)\right]&=\omega\left(\frac{\tau}{\sigma_x^2}\right)\\ Var\left[\widehat{\beta}(\omega)\right]&=\frac{\sigma_\epsilon^2}{\sigma_x^2}\left[(2\omega^2-\omega)\left(\frac{\sigma_x^2}{\gamma^2}-1\right)+\frac{\sigma_x^2}{\gamma^2}\right]\end{aligned}$$

and accordingly

$$\text{AMSE}\left[\widehat{\beta}(\omega)\right]=\omega^2\left(\frac{\tau^2}{\sigma_x^4}\right)+(\omega^2-2\omega)\left(\frac{\sigma_\epsilon^2}{\sigma_x^2}\right)\left(\frac{\sigma_x^2}{\gamma^2}-1\right)+\frac{\sigma_\epsilon^2}{\gamma^2}.$$

The preceding expression is a globally convex function of ω. Taking the first order condition and rearranging, we find that the unique global minimizer is

$$\omega^*=\left[1+\frac{\tau^2/\sigma_x^4}{\sigma_\epsilon^2(1/\gamma^2-1/\sigma_x^2)}\right]^{-1}=\left[1+\frac{\text{ABIAS(OLS)}^2}{\text{AVAR(TSLS)}-\text{AVAR(OLS)}}\right]^{-1}.$$

□

Proof of Theorem 4.1. By a mean-value expansion,

$$\sqrt{n}\left[\Xi_S f_n\left(\widehat{\theta}_S\right)\right]=\sqrt{n}\left[\Xi_S f_n(\theta_0)\right]+F_S\sqrt{n}\left(\widehat{\theta}_S-\theta_0\right)+o_p(1).$$

Since $\sqrt{n}\left(\widehat{\theta}_S - \theta_0\right) \to_p -\left(F_S' W_S F_S\right)^{-1} F_S' W_S \sqrt{n}\left[\Xi_S f_n(\theta_0)\right]$, we have

$$\sqrt{n}\left[\Xi_S f_n(\widehat{\theta}_S)\right] = \left[I - F_S\left(F_S' W_S F_S\right)^{-1} F_S' W_S\right] \sqrt{n}\left[\Xi_S f_n(\theta_0)\right] + o_p(1).$$

Thus, for estimation using the efficient weighting matrix

$$\widehat{\Omega}_S^{-1/2}\sqrt{n}\left[\Xi_S f_n\left(\widehat{\theta}_S\right)\right] \to_d \left[I - P_S\right]\Omega_S^{-1/2}\Xi_S\left(M + \begin{bmatrix} 0 \\ \tau \end{bmatrix}\right)$$

by Assumption 2.2 (h), where $\widehat{\Omega}_S^{-1/2}$ is a consistent estimator of $\Omega_S^{-1/2}$ and P_S is the projection matrix based on $\Omega_S^{-1/2} F_S$, the identifying restrictions.[27] The result follows by combining and rearranging these expressions. □

Proof of Theorem 4.2. Let S_1 and S_2 be arbitrary moment sets in $\mathscr{S}$ and let $|S|$ denote the cardinality of S. Further, define $\Delta_n(S_1, S_2) = MSC(S_1) - MSC(S_2)$ By Theorem 4.1, $J_n(S) = O_p(1)$, $S \in \mathscr{S}$, thus

$$\begin{aligned} \Delta_n(S_1, S_2) &= \left[J_n(S_1) - J_n(S_2)\right] - \left[h\left(p + |S_1|\right) - h\left(p + |S_2|\right)\right]\kappa_n \\ &= O_p(1) - C\kappa_n \end{aligned}$$

where $C = \left[h\left(p + |S_1|\right) - h\left(p + |S_2|\right)\right]$. Since h is strictly increasing, C is positive for $|S_1| > |S_2|$, negative for $|S_1| < |S_2|$, and zero for $|S_1| = |S_2|$. Hence:

$$\begin{aligned} |S_1| > |S_2| &\implies \Delta_n(S_1, S_2) \to -\infty \\ |S_1| = |S_2| &\implies \Delta_n(S_1, S_2) = O_p(1) \\ |S_1| < |S_2| &\implies \Delta_n(S_1, S_2) \to \infty \end{aligned}$$

The result follows because the full moment set contains more moment conditions than any other moment set S. □

Proof of Theorem 4.4. By Theorem 3.1 and Corollary 4.2,

$$P\left\{\mu_0 \in \text{CI}_{sim}\right\} \to P\left\{a_{min} \leq \Lambda(\tau) \leq b_{max}\right\}$$

where $a(\tau^*), b(\tau^*)$ define a collection of $(1-\alpha)\times 100\%$ intervals indexed by τ^*, each of which

[27] See Hall (2005), Chapter 3.

is constructed under the assumption that $\tau = \tau^*$

$$P\left\{a(\tau^*) \leq \Lambda(\tau^*) \leq b(\tau^*)\right\} = 1 - \alpha$$

and we define the shorthand a_{min}, b_{max} as follows

$$\begin{aligned}
a_{min}(\Psi M + \tau) &= \min\left\{a(\tau^*)\colon \tau^* \in \mathscr{T}(\Psi M + \tau, \delta)\right\} \\
b_{max}(\Psi M + \tau) &= \max\left\{b(\tau^*)\colon \tau^* \in \mathscr{T}(\Psi M + \tau, \delta)\right\} \\
\mathscr{T}(\Psi M + \tau, \delta) &= \left\{\tau^*\colon \Delta(\tau, \tau^*) \leq \chi^2_q(\delta)\right\} \\
\Delta(\tau, \tau^*) &= (\Psi M + \tau - \tau^*)'(\Psi\Omega\Psi')^{-1}(\Psi M + \tau - \tau^*)
\end{aligned}$$

Now, let $A = \left\{\Delta(\tau, \tau) \leq \chi^2_q(\delta)\right\}$ where $\chi^2_q(\delta)$ is the $1 - \delta$ quantile of a χ^2_q random variable. This is the event that the *limiting version* of the confidence region for τ contains the true bias parameter. Since $\Delta(\tau, \tau) \sim \chi^2_q$, $P(A) = 1 - \delta$. For every $\tau^* \in \mathscr{T}(\Psi M + \tau, \delta)$ we have

$$P\left[\left\{a(\tau^*) \leq \Lambda(\tau^*) \leq b(\tau^*)\right\} \cap A\right] + P\left[\left\{a(\tau^*) \leq \Lambda(\tau) \leq b(\tau^*)\right\} \cap A^c\right] = 1 - \alpha$$

by decomposing $P\left\{a(\tau^*) \leq \Lambda(\tau^*) \leq b(\tau^*)\right\}$ into the sum of mutually exclusive events. But since

$$P\left[\left\{a(\tau^*) \leq \Lambda(\tau^*) \leq b(\tau^*)\right\} \cap A^c\right] \leq P(A^c) = \delta$$

we see that

$$P\left[\left\{a(\tau^*) \leq \Lambda(\tau^*) \leq b(\tau^*)\right\} \cap A\right] \geq 1 - \alpha - \delta$$

for every $\tau^* \in \mathscr{T}(\Psi M + \tau, \delta)$. Now, by definition, if A occurs then the true bias parameter τ is contained in $\mathscr{T}(\Psi M + \tau, \delta)$ and hence

$$P\left[\left\{a(\tau) \leq \Lambda(\tau) \leq b(\tau)\right\} \cap A\right] \geq 1 - \alpha - \delta.$$

But when $\tau \in \mathscr{T}(\Psi M + \tau, \delta)$, $a_{min} \leq a(\tau)$ and $b(\tau) \leq b_{max}$. It follows that

$$\left\{a(\tau) \leq \Lambda(\tau) \leq b(\tau)\right\} \cap A \subseteq \left\{a_{min} \leq \Lambda(\tau) \leq b_{max}\right\}$$

and therefore

$$1 - \alpha - \delta \leq P\left[\left\{a(\tau^*) \leq \Lambda(\tau^*) \leq b(\tau^*)\right\} \cap A\right] \leq P\left[\left\{a_{min} \leq \Lambda(\tau) \leq b_{max}\right\}\right]$$

as asserted. □

B Computational Details

This paper is fully replicable using freely available, open-source software. For full source code and replication details, see https://github.com/fditraglia/fmsc. Results for the simulation studies and empirical example were generated using R version 3.1.0 (R Core Team, 2014) and C++ via the Rcpp (Eddelbuettel, 2013; Eddelbuettel and François, 2011) and RcppArmadillo (Eddelbuettel and Sanderson, 2014) packages, versions 0.11.2 and 0.4.300.8.0, respectively. RcppArmadillo version 0.4.300.8.0 provides an interface to version 4.300 of the Armadillo C++ linear algebra library (Sanderson, 2010). All figures in the paper were converted to tikz using version 0.7.0 of the tikzDevice package (Sharpsteen and Bracken, 2013). The simulation-based confidence intervals from Section 5.3 were calculated using Algorithm 4.1 with $J = 1000$ by searching over a grid of 100 equally spaced values within a 95% confidence interval for the scalar τ. In constrast, the simulation-based intervals for the empirical example from Section 6 were constructed with $J = 10,000$ using a mesh-adaptive search algorithm provided by version 3.6 of the NOMAD C++ optimization package (Abramson et al., 2013; Audet et al., 2009; Le Digabel, 2011), called from R using version 0.15-22 of the crs package (Racine and Nie, 2014). TSLS results for Table 6 were generated using version 3.1-4 of the sem package (Fox et al., 2014).

C Failure of the Identification Condition

When there are fewer moment conditions in the g-block than elements of the parameter vector θ, i.e. when $r > p$, Assumption 2.4 fails: θ_0 is not estimable by $\widehat{\theta}_v$ so $\widehat{\tau}$ is an infeasible estimator of τ. A naïve approach to this problem would be to substitute another consistent estimator of θ_0 and proceed analogously. Unfortunately, this approach fails. To understand why, consider the case in which all moment conditions are potentially invalid so that the g–block is empty. Letting $\widehat{\theta}_f$ denote the estimator based on the full set of moment conditions in h, $\sqrt{n}h_n(\widehat{\theta}_f) \to_d \Gamma\mathcal{N}_q(\tau, \Omega)$ where $\Gamma = \mathbf{I}_q - H\left(H'WH\right)^{-1}H'W$, using an argument similar to that in the proof of Theorem 3.1. The mean, $\Gamma\tau$, of the resulting limit distribution does not equal τ, and because Γ has rank $q - r$ we cannot pre-multiply by its inverse to extract an estimate of τ. Intuitively, $q - r$ over-identifying restrictions are insufficient to estimate a q-vector: τ cannot be estimated without a minimum of r valid moment conditions. However, the limiting distribution of $\sqrt{n}h_n(\widehat{\theta}_f)$ partially identifies τ even when we have no valid moment conditions at our disposal. A combination of this information with prior restrictions on the magnitude of the components of τ allows the use of the FMSC framework to carry out a sensitivity analysis when $r > p$. For example, the worst-case estimate of

AMSE over values of τ in the identified region could still allow certain moment sets to be ruled out. This idea shares similarities with Kraay (2012) and Conley et al. (2012), two recent papers that suggest methods for evaluating the robustness of conclusions drawn from IV regressions when the instruments used may be invalid.

D Low-Level Sufficient Conditions

Assumption D.1 (Sufficient Conditions for Theorem 3.2)**.** *Let* $\{(\mathbf{z}_{ni}, v_{ni}, \epsilon_{ni})\colon 1 \leq i \leq n, n = 1, 2, \ldots\}$ *be a triangular array of random variables such that*

(a) $(\mathbf{z}_{ni}, v_{ni}, \epsilon_{ni}) \sim$ *iid and mean zero within each row of the array (i.e. for fixed* n*)*

(b) $E[\mathbf{z}_{ni}\epsilon_{ni}] = \mathbf{0}$*,* $E[\mathbf{z}_{ni}v_{ni}] = \mathbf{0}$*, and* $E[\epsilon_{ni}v_{ni}] = \tau/\sqrt{n}$ *for all* n

(c) $E[|\mathbf{z}_{ni}|^{4+\eta}] < C$*,* $E[|\epsilon_{ni}|^{4+\eta}] < C$*, and* $E[|v_{ni}|^{4+\eta}] < C$ *for some* $\eta > 0$*,* $C < \infty$

(d) $E[\mathbf{z}_{ni}\mathbf{z}_{ni}'] \to Q > 0$*,* $E[v_{ni}^2] \to \sigma_v^2 > 0$*, and* $E[\epsilon_{ni}^2] \to \sigma_\epsilon^2 > 0$ *as* $n \to \infty$

(e) As $n \to \infty$*,* $E[\epsilon_{ni}^2\mathbf{z}_{ni}\mathbf{z}_{ni}'] - E[\epsilon_{ni}^2]E[\mathbf{z}_{ni}\mathbf{z}_{ni}'] \to 0$*,* $E[\epsilon_i^2 v_{ni}\mathbf{z}_{ni}'] - E[\epsilon_{ni}^2]E[v_{ni}\mathbf{z}_{ni}'] \to 0$*, and* $E[\epsilon_{ni}^2 v_{ni}^2] - E[\epsilon_{ni}^2]E[v_{ni}^2] \to 0$

(f) $x_{ni} = \mathbf{z}_{ni}'\boldsymbol{\pi} + v_i$ *where* $\boldsymbol{\pi} \neq \mathbf{0}$*, and* $y_{ni} = \beta x_{ni} + \epsilon_{ni}$

Parts (a), (b) and (d) correspond to the local mis-specification assumption, part (c) is a set of moment restrictions, and (f) is simply the DGP. Part (e) is the homoskedasticity assumption: an *asymptotic* restriction on the joint distribution of v_{ni}, ϵ_{ni}, and $\mathbf{z}_{ni}$. This condition holds automatically, given the other asssumptions, if $(\mathbf{z}_{ni}, v_{ni}, \epsilon_{ni})$ are jointly normal, as in the simulation experiment described in the paper.

Assumption D.2 (Sufficient Conditions for Theorem 3.5.)**.** *Let* $\{(\mathbf{z}_{ni}, \mathbf{v}_{ni}, \epsilon_{ni})\colon 1 \leq i \leq n, n = 1, 2, \ldots\}$ *be a triangular array of random variables with* $\mathbf{z}_{ni} = (\mathbf{z}_{ni}^{(1)}, \mathbf{z}_{ni}^{(1)})$ *such that*

(a) $(\mathbf{z}_{ni}, \mathbf{v}_{ni}, \epsilon_{ni}) \sim$ *iid within each row of the array (i.e. for fixed* n*)*

(b) $E[\mathbf{v}_{ni}\mathbf{z}_{ni}'] = \mathbf{0}$*,* $E[\mathbf{z}_{ni}^{(1)}\epsilon_{ni}] = \mathbf{0}$*, and* $E[\mathbf{z}_{ni}^{(2)}\epsilon_{ni}] = \boldsymbol{\tau}/\sqrt{n}$ *for all* n

(c) $E[|\mathbf{z}_{ni}|^{4+\eta}] < C$*,* $E[|\epsilon_{ni}|^{4+\eta}] < C$*, and* $E[|\mathbf{v}_{ni}|^{4+\eta}] < C$ *for some* $\eta > 0$*,* $C < \infty$

(d) $E[\mathbf{z}_{ni}\mathbf{z}_{ni}'] \to Q > 0$ *and* $E[\epsilon_{ni}^2\mathbf{z}_{ni}\mathbf{z}_{ni}'] \to \Omega > 0$ *as* $n \to \infty$

(e) $\mathbf{x}_{ni} = \Pi_1'\mathbf{z}_{ni}^{(1)} + \Pi_2'\mathbf{z}_{ni}^{(2)} + \mathbf{v}_{ni}$ *where* $\Pi_1 \neq \mathbf{0}$*,* $\Pi_2 \neq \mathbf{0}$*, and* $y_i = \mathbf{x}_{ni}'\beta + \epsilon_{ni}$

These conditions are similar to although more general than those contained in Assumption D.1 as they do not impose homoskedasticity.

E Supplementary Simulation Results

This section discusses additional simulation results for the choosing instrumental variables example, as a supplement to those given in Section 5.2.

E.1 Downward J-Test

The downward J-test is an informal but fairly common procedure for moment selection in practice. In the context of the simulation example from Section 5.2 it amounts to simply using the full estimator unless it is rejected by a J-test. Table 5 compares the RMSE of the post-FMSC estimator to that of the downward J-test with $\alpha = 0.1$ (J90), and $\alpha = 0.05$ (J95). For robustness, I calculate the J-test statistic using a centered covariance matrix estimator, as in the FMSC formulas from section 3.3. Unlike the FMSC, the downward J-test is very badly behaved for small sample sizes, particularly for the smaller values of γ. For larger sample sizes, the relative performance of the FMSC and the J-test is quite similar to what we saw in Figure 1 for the OLS versus TSLS example: the J-test performs best for the smallest values of ρ, the FMSC performs best for moderate values, and the two procedures perform similarly for large values. These results are broadly similar to those for the GMM moment selection criteria of Andrews (1999) considered in Section 5.2, which should not come as a surprise since the J-test statistic is an ingredient in the construction of the GMM-AIC, BIC and HQ.

E.2 Canonical Correlations Information Criterion

Because the GMM moment selection criteria suggested by Andrews (1999) consider only instrument exogeneity, not relevance, Hall and Peixe (2003) suggest combining them with their canonical correlations information criterion (CCIC), which aims to detect and eliminate "redundant instruments." Including such instruments, which add no information beyond that already contained in the other instruments, can lead to poor finite-sample performance in spite of the fact that the first-order limit distribution is unchanged. For the choosing instrumental variables simulation example, presented in Section 5.2, the CCIC takes the following simple form

$$\text{CCIC}(S) = n \log\left[1 - R_n^2(S)\right] + h(p + |S|)\kappa_n \tag{25}$$

where $R_n^2(S)$ is the first-stage R^2 based on instrument set S and $h(p+|S|)\kappa_n$ is a penalty term (Jana, 2005). Instruments are chosen to *minimize* this criterion. If we define $h(p + |S|) = (p + |S| - r)$, setting $\kappa_n = \log n$ gives the CCIC-BIC, while $\kappa_n = 2.01 \log\log n$ gives the

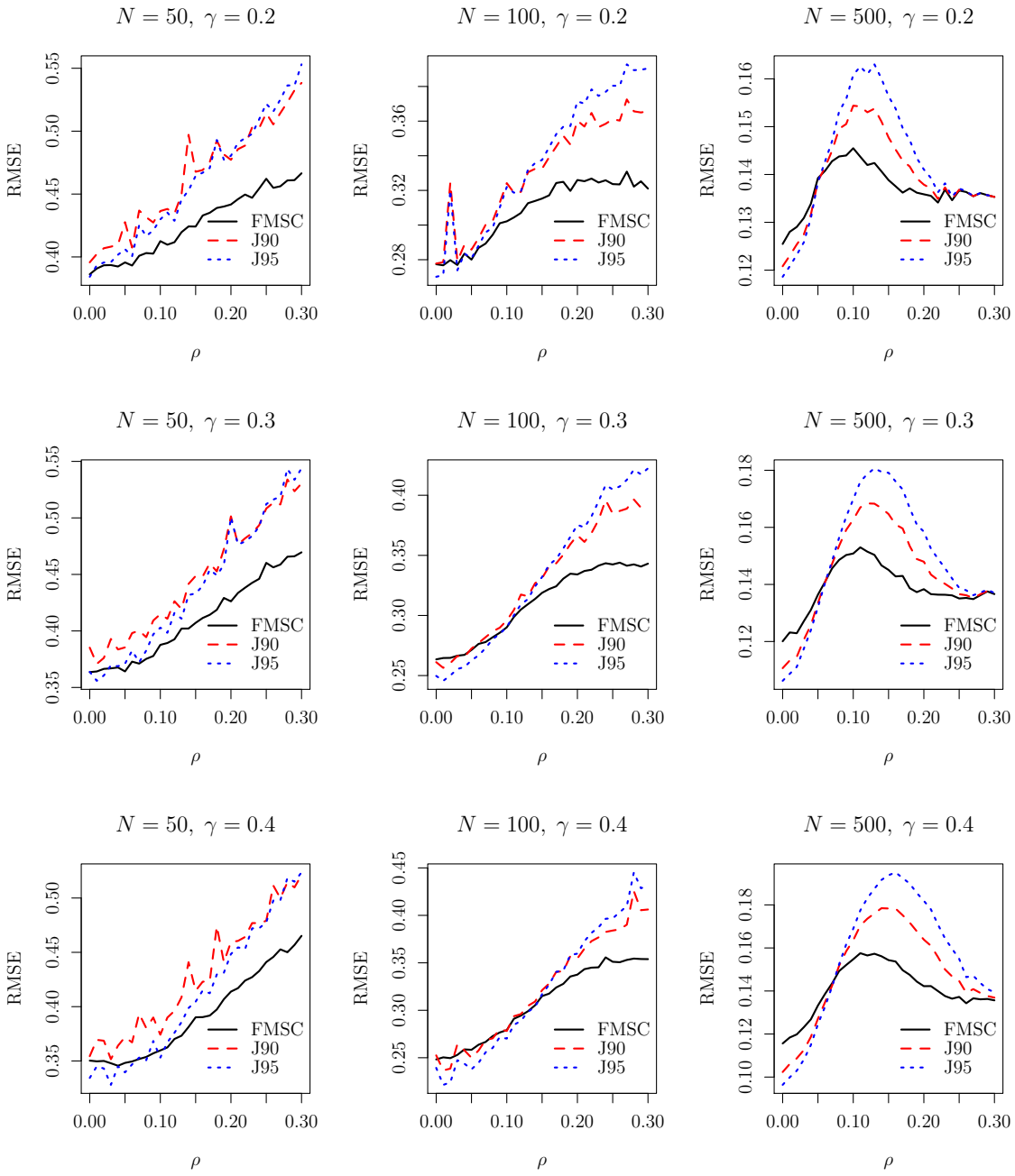

Figure 5: RMSE values for the post-Focused Moment Selection Criterion (FMSC) estimator and the downward J-test estimator with $\alpha = 0.1$ (J90) and $\alpha = 0.05$ (J95) based on 20,000 simulation draws from the DGP given in Equations 22–23 using the formulas described in Sections 3.3.

CCIC-HQ and $\kappa_n = 2$ gives the CCIC-AIC. By combining the CCIC with an Andrews-type criterion, Hall and Peixe (2003) propose to first eliminate invalid instruments and then redundant ones. A combined GMM-BIC/CCIC-BIC criterion for the simulation example from section 5.2 uses the valid estimator unless both the GMM-BIC *and* CCIC-BIC select the full estimator. Combined HQ and AIC-type procedures can be defined analogously. In the simulation design from this paper, however, *each* of these combined criteria gives results that are practically identical to those of the valid estimator. This hold true across all parameter values and sample sizes. Full details are available upon request.

E.3 Additional Confidence Interval Simulations

(a) Valid Estimator

$N = 50$		ρ = 0	0.1	0.2	0.3	0.4	0.5
	0.1	83	83	83	82	83	83
	0.2	83	83	83	83	84	83
γ	0.3	83	82	83	83	83	83
	0.4	82	83	84	83	83	84
	0.5	84	83	83	83	83	83
	0.6	83	83	83	83	82	82

$N = 100$		ρ = 0	0.1	0.2	0.3	0.4	0.5
	0.1	86	87	87	87	86	86
	0.2	86	86	86	86	87	86
γ	0.3	86	86	87	87	87	87
	0.4	86	87	86	86	87	87
	0.5	87	87	86	86	86	86
	0.6	87	86	86	87	86	87

$N = 500$		ρ = 0	0.1	0.2	0.3	0.4	0.5
	0.1	90	89	89	89	90	90
	0.2	89	90	90	90	90	90
γ	0.3	89	90	90	90	90	90
	0.4	89	90	90	89	90	90
	0.5	90	90	90	90	90	89
	0.6	89	89	89	90	90	90

(b) Naïve post-FMSC

$N = 50$		ρ = 0	0.1	0.2	0.3	0.4	0.5
	0.1	80	78	77	79	81	82
	0.2	79	75	73	74	79	81
γ	0.3	79	72	66	67	73	78
	0.4	78	71	62	59	66	75
	0.5	78	68	57	52	57	67
	0.6	77	68	54	47	50	60

$N = 100$		ρ = 0	0.1	0.2	0.3	0.4	0.5
	0.1	84	82	84	86	86	86
	0.2	83	77	78	84	87	86
γ	0.3	82	74	71	79	85	86
	0.4	82	71	63	71	82	86
	0.5	81	69	56	62	76	84
	0.6	81	66	51	52	69	81

$N = 500$		ρ = 0	0.1	0.2	0.3	0.4	0.5
	0.1	89	88	89	89	90	90
	0.2	87	84	90	90	90	90
γ	0.3	86	77	89	90	90	90
	0.4	85	67	88	89	90	90
	0.5	84	59	84	90	90	89
	0.6	83	52	77	89	90	90

Table 9: Coverage probabilities of nominal 90% CIs for the choosing instrumental variables simulation experiment described in Section 5.2. All values are given in percentage points, rounded to the nearest whole percent, based on 10,000 simulation draws from the DGP given in Equations 22–23.

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