# PIER Working Paper 14-029

"Network Formation and Systemic Risk"

by

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http://ssrn.com/abstract=2500846

# Network Formation and Systemic Risk[*]

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August 29, 2014

## Abstract

This paper introduces a model of endogenous network formation and systemic risk. In it, agents form networks that efficiently trade-off the possibility of systemic risk with the benefits of trade. Second, fundamentally 'safer' economies generate higher interconnectedness, which in turn leads to higher systemic risk. Third, the structure of the network formed depends on whether the shocks to the system are believed to be correlated or independent of each other. In particular, when shocks are perfectly correlated, the network formed is a complete graph, i.e., a link between every pair of agents. This underlines the importance of specifying the shock structure before investigating a given network because a given network and shock structure could be incompatible.

*JEL classification*: D85, G01.

*Keywords*: Network Formation, Systemic Risk, Contagion, Rationalizability, Core.

# 1 Introduction

The awkward chain of events that upset the bankers, began with the collapse of Lehmann Brothers in 2008. Panic spread, the dollar wavered and world markets fell. Interconnectedness of the financial system, some suggested, allowed Lehmann's fall to threaten the stability of the entire system. Thus prompted, scholars have sought to characterize the networks that would allow shocks to one part of the financial network to be spread and amplified. Blume et al. (2013) as well as Vivier-Lirimonty (2006), for example, argue that dense interconnections pave the way to systemic failures. In contrast, Allen and Gale (2000) as well as Freixas et al. (2000), argue that a more interconnected architecture protects the system against contagion because the losses of a distressed institution can be divided among many creditors. A common feature of many of these and other papers (Acemoglu et al. (2013), Eboli (2013), Elliott et al. (2014), Gai et al. (2011), Glasserman and Young (2014)) is that a network is given exogenously.[1] A node (or subset of them) is subjected to a shock and its propagation studied as the size of the shock varies. Absent are reasons for the presence of links between agents. After all, one could eliminate the possibility of a system-wide failure by never forming links. This is, of course, silly. The presence of a link between two agents or institutions represents a potentially lucrative joint opportunity. Yet, every link increases the possibility of contagion. If agents form links anticipating the possibility of system-wide failure, what kinds of networks would they form? In particular, do they form networks that are susceptible to contagion?

In the model we use to answer these questions, agents form links. The payoff to each party that shares a link is uncertain and depends upon the future realization of a random variable (which we will call a shock) and actions taken contingent on the shock. Specifically, there are three stages. In stage one, agents form links which can be interpreted as partnerships or joint ventures. In stage two, each link formed is subjected to a shock. In stage three, with full knowledge of the realized shocks, each agent decides whether to 'default' or not. The payoff of an agent depends on the action she takes in the third stage as well the actions of her neighbors and the realized shocks. The default decision corresponds to exiting from every partnership formed in stage 1. If the only Nash equilibrium of the game in stage three is that everyone defaults, we call that a system wide failure. In our model, default is the result of a 'loss of confidence' rather than simple 'spillover' effects.[2]

In this model we show that the network formed in stage one, efficiently trades-off the risk of system wide failure with the benefits of trade. A second contribution is to examine how

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[1] An exception is Blume et al. (2013) .

[2] Glasserman and Young (2014) argue that spillover effects have only a limited impact. They suggest that the "mere possibility (rather than the actuality) of a default can lead to a general and widespread decline in valuations....."

the probability of system wide failure varies with a change in the distribution of shocks. In a setting where shocks are independent and binary (good or bad), the probability of system wide failure *increases* with an increase in the probability of a good shock up to a point and then it declines. Intuitively, as partnerships become less risky, agents are encouraged to form more partnerships increasing interconnectedness which increases the probability of system wide failure.

Our third contribution is to show that the structure of the network formed in stage one depends on whether the shocks to the system are believed to be correlated or independent of each other. In particular, when shocks are perfectly correlated, the network formed in stage one is a complete graph, i.e., a link between every pair of agents. We think this finding relevant to the debate between two theories of financial destruction advanced to explain the 2008 financial crisis. The first, mentioned above, is dubbed the 'domino theory'. The alternative, advocated most prominently by Edward Lazear[3], is dubbed 'popcorn'. Lazear describes it thusly in a 2011 opinion piece in the Wall Street Journal:

> "The popcorn theory emphasizes a different mechanism. When popcorn is made (the old fashioned way), oil and corn kernels are placed in the bottom of a pan, heat is applied and the kernels pop. Were the first kernel to pop removed from the pan, there would be no noticeable difference. The other kernels would pop anyway because of the heat. The fundamental structural cause is the heat, not the fact that one kernel popped, triggering others to follow.
>
> Many who believe that bailouts will solve Europe's problems cite the Sept. 15, 2008 bankruptcy of Lehman Brothers as evidence of what allowing one domino to fall can do to an economy. This is a misreading of the historical record. Our financial crisis was mostly a popcorn phenomenon. At the risk of sounding defensive (I was in the government at the time), I believe that Lehman's downfall was more a result of the factors that weakened our economic structure than the cause of the crisis."

Our model suggests that underlying structural weaknesses (as modeled by strong correlations between shocks) and greater interconnectedness can coexist. Therefore, it would be incorrect to highlight the interconnectedness of the system and suggest it *alone* as the cause of instability.

Our model differs from the prior literature in three important ways.

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[3]Chair of the US President's Council of Economic Advisers during the 2007-2008 financial crisis.

1. The networks we study are formed endogenously.

    Babus (2013) also has a model of network formation, but one in which agents share the goal of minimizing the probability of system wide default. In our model agents are concerned with their own expected payoffs and only indirectly with the possibility of system wide failure. Acemoglu et al. (2013) also discusses network formation, but the links permitted to form are constrained by an exogenously given 'opportunity' network is exogenously given. If the opportunity network is complete then all links are permitted. However, the networks that are compared in their paper arise from different opportunity networks; one complete and the other a ring. When the opportunity network is complete, agents from a complete network. When the opportunity network is a ring, agents form a ring. In our model, agents are free to partner with anyone and we compare networks that arise within the same world of possibilities. In this sense our network formation is genuinely endogenous. Zawadowski (2013) models the decision of agents to purchase default insurance on their counterparties. This can be interpreted as a model of network formation, but it is not a model of an agent choosing a particular counterparty. Rather, the counterparties are fixed, and default insurance serves to change the terms of trade with an existing counterparty. Blume et al. (2013), is closest to our paper, in that networks form endogenously. However, in their model, the risk of a node defaulting is independent of the network formed. In our model, the likelihood of a node defaulting depends on the structure of the network formed.

2. We examine the effects of a distribution that generates the shocks rather than the effects of fixed shocks applied to particular nodes.

    Glasserman and Young (2014) is the only exception we are aware of, but the networks they consider are exogenously given.

3. The decision to default in our model is strategic.

    We are not aware of prior work that incorporates this. In Blume et al. (2013), Acemoglu et al. (2013), Eboli (2013), Elliott et al. (2014), Gai et al. (2011), Glasserman and Young (2014), for example, default is triggered when a parameter beyond the control of an agent falls below some threshold.

In the next section we give a formal description of our model. The subsequent section characterizes the set of agents that choose to default in stage three for a given realized network and realization of shocks. The following section uses these results to characterize the structure of the realized networks. Section 5 discusses correlated shocks and section 6 describes some extensions to the basic model.

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# 2    The Model

Denote by $N$ a finite set of agents.[4]  Each pair of agents in $N$ can form a joint venture. For this reason we frequently refer to agents as **nodes** and each potential partnership as a **potential edge**.

A potential edge $e$, a subset of $N$ with two elements, represents a bilateral contract whose payoff to each party is contingent on some future realized state $\theta^e$ and actions that each incident node can take upon realization of $\theta^e$. The set of possible values of $\theta^e$ is $\Theta$, a finite set of real numbers.

The model has three stages. In stage one, network formation, agents, by mutual consent, decide which potential edges to pick. The edges picked are called realized. The set of **realized edges** is denoted $E$. The corresponding network denoted $(N, E)$, is called a **realized network**.

In stage two, for each realized edge $e$, $\theta^e$ is chosen by nature identically and independently across edges via a distribution $\phi$ over $\Theta$. We relax the independence assumption in section 5. We denote by $(N, E, \boldsymbol{\theta})$ the realized network and vector of realized $\theta$'s.

In stage three, with full knowledge of $(N, E, \boldsymbol{\theta})$ each agent $n$ chooses one of two possible actions called $B$ (business as usual) or $D$ (default), denoted by $a_n$. Agent $n$ enjoys the sum of payoffs $u_n(a_n, a_m; \theta^{\{n,m\}})$ over all of his neighbors $m$ in $(N, E)$.

We make two assumptions about payoff functions. The first is that if an agent $n$ in $(N, E)$ has degree one and the counterparty defaults, it is the unique best response for agent $n$ to default as well. Formally

**Assumption 1.** $u_n(D, D; \theta) > u_n(B, D; \theta)$ *for all $n$ and $\theta$.*

The second assumption is a supermodularity which can be interpreted as a form increasing returns in fulfilling the terms of the contract.

**Assumption 2.** $u_n(D, D; \theta) + u_n(B, B; \theta) > u_n(B, D; \theta) + u_n(D, B; \theta)$ *for all $n$ and $\theta$.*

If we focus on a pair of agents $(n, m)$ and denote by $e$ the realized edge between them, the payoff matrix of the game they are engaged in stage three is the following (player $n$ is the row player and $m$ the column player):

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[4]We abuse notation by using $N$ to denote the cardinality of the set when appropriate.

|  | $B$ | $D$ |
|---|---|---|
| $B$ | $u_n(B,B;\theta^e),\ u_m(B,B;\theta^e)$ | $u_n(B,D;\theta^e),\ u_m(D,B;\theta^e)$ |
| $D$ | $u_n(D,B;\theta^e),\ u_m(B,D;\theta^e)$ | $u_n(D,D;\theta^e),\ u_m(D,D;\theta^e)$ |

A special case of this game is the coordination game of Carlsson and van Damme (1993) reproduced below that will be considered in section 4:

|  | $B$ | $D$ |
|---|---|---|
| $B$ | $\theta^e,\ \theta^e$ | $\theta^e - 1,\ 0$ |
| $D$ | $0,\ \theta^e - 1$ | $0,\ 0$ |

It is clear from this last table that a pair of agents that share a realized edge play a coordination game whose payoffs depend upon the realized state variable $\theta^e$. Following Carlsson and van Damme (1993), the game has a natural interpretation. In stage one the agents get together to pursue a joint investment. In stage two, $\theta^e$ is realized, i.e. new information arrives about the profitability of the project. In stage three, agents are allowed to reassess their decision to continue with the project or not. For other examples of games of this kind and their applications in finance see Morris and Shin (2003).

Two features of the model deserve discussion. First, in contrast to prior literature, shocks, in the form of realized states, apply to edges rather than nodes. In section 6 we describe how easy it is to extend our model to allow for shocks to nodes as well. However, we believe shocks to edges to be more reasonable. An agent's solvency depends on the outcomes of the many investments she has chosen to make. The interesting case is when these investments required coordination with at least one other agent, a joint venture if you will. It is the outcome of this joint venture that will determine whether the participants decide to continue or walk away.

The second is that agents *cannot* default on some partnerships and not others. An agent must default on all or none. While extreme, it is not, as we argue, unreasonable. Were an agent free to default on any subset of its partnerships, we could model this by splitting each node in $(N, E)$ into as many copies of itself as its degree.[5] Each copy would be incident to exactly one of the edges that were previously incident to the original node. Thus, our model could easily accommodate this possibility. However, this has the effect of treating a single entity as a collection of independent smaller entities which we think inaccurate. Institutions considering default face liquidity constraints, which restrict, at best, the number of parties

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[5] The degree of a node in a graph is the number of edges incident to it.

they can repay. While entities like countries can indeed selectively default, there is a knock-on effect. Countries that selectively default, have their credit ratings downgraded which raise their borrowing costs for the other activities they are engaged in. When a company fails to pay sufficiently many of its creditors, the creditors will force the company into bankruptcy. Thus, it is entirely reasonable to suppose that the default decisions associated with the edges a node is incident to should be linked. Ours is an extreme, but simple, version of such a linking.

## 2.1   Solution Concepts

Here we describe the solution concepts to be employed in stages one and three. We begin with stage three as the outcomes in stage three will determine the choices made by agents in stage one.

Agents enter stage three knowing $(N, E, \boldsymbol{\theta})$. With this knowledge, each simultaneously chooses action $B$ or $D$. We do not allow actions chosen in stage three to be conditioned on what happens in earlier stages. The outcome in stage three is assumed to be a Nash equilibrium. While 'everybody plays $D$' is a Nash equilibrium, by Assumption 1, it need not be the only one. We focus on the Nash equilibrium in which largest (with respect to set inclusion) set of agents, among all Nash equilibria, play $B$. Call this the **cooperating equilibrium**. The proposition below shows that the cooperating equilibrium is well-defined and unique, by using rationalizable strategies.

A realized network along with realized states, $(N, E, \boldsymbol{\theta})$, exhibits **system wide failure** if in the cooperating equilibrium of the game all agents in $N$ choose $D$.[6] In this case, agents can coordinate on nothing but action $D$. The probability of system wide failure of a realized network is called its **systemic risk**.

**Proposition 1.** *A cooperating equilibrium is well-defined and unique.*

*Proof.* Fix $(N, E, \boldsymbol{\theta})$. The profile where all agents in $N$ play $D$ is a Nash equilibrium by Assumption 1. Hence, $D$ is rationalizable for everyone. Let $M$ be the set of agents who have the unique rationalizable action $D$. For agents in $N \setminus M$, both $B$ and $D$ are rationalizable.

Consider an agent $n \notin M$. $B$ is rationalizable, i.e., $B$ is a best response to some strategy profile, say $a_{-n}$, of agents $-n$ in which agents in $M$ play $D$. Let $\Delta(s_{-n})$ be the difference in payoffs for $n$ between playing $B$ and $D$ against strategy profile $s_{-n}$ of $-n$. $\Delta(a_{-n}) \geq 0$ since $B$ is a best reply to $a_{-n}$.

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[6]This is equivalent to saying that 'everybody plays $D$' is the only Nash equilibrium.

Now consider the strategy profile $b_{-n}$ of agents $-n$ such that agents in $M$ play $D$ and the rest play $B$. We will prove that $\Delta(b_{-n}) \geq \Delta(a_{-n})$. In $a_{-n}$, players in $N \setminus M$ could be playing $B$ or $D$. Let $K \subseteq N \setminus M$ be those agents who play $D$ in $a_{-n}$ and let $\Gamma_n$ be the set of neighbors of $n$ in the realized network $(N, E)$. Then $\Delta(b_{-n}) - \Delta(a_{-n}) =$

$$\sum_{k \in K \cap \Gamma_n} \left( u_n \left( B, B; \theta^{\{n,k\}} \right) - u_n \left( D, B; \theta^{\{n,k\}} \right) \right) - \left( u_n \left( B, D; \theta^{\{n,k\}} \right) - u_n \left( D, D; \theta^{\{n,k\}} \right) \right)$$

which is positive by Assumption 2.

As $\Delta(b_{-n}) \geq \Delta(a_{-n}) \geq 0$ it follows that $B$ is a best reply by $n$ to $b_{-n}$. This argument works for every agent in $N \setminus M$, not only $n$. Also, recall that $D$ is the unique rationalizable action for agents in $M$ so that $D$ is the unique best reply to any strategy profile in which all agents in $M$ play $D$. Therefore, a profile where all agents in $M$ play $D$ and all agents in $N \setminus M$ choose $B$ is a Nash equilibrium.

Note that in any Nash equilibrium, everyone in $M$ must play $D$ since it is their unique rationalizable action. Therefore, "$M$ plays $D$, $M^c$ plays $B$" is the unique cooperating equilibrium. $\qquad\square$

The proof suggests an *equivalent definition* of a cooperating equilibrium: the rationalizable strategy profile in which those who have the unique rationalizable action $D$ play $D$, while the remainder play $B$.

Recall that rationalizable actions are those which remain after the iterated elimination of strictly dominated actions. The iteration is as follows. Those agents who have a strictly dominant action $D$ play $D$. Then, knowing that these agents play $D$, it becomes strictly dominant to play $D$ for other agents to do so. This iteration stops in a finite number of steps as $N$ is finite. The remaining action profiles are the rationalizable ones.

One conceptual contribution is to establish a natural analogy between contagion of sequential defaults and rationalizable strategies. First, agents whose incident edges have realized states that force them to default, no matter what their counterparties do, default. Then, more agents, anticipating that some of their counterparties will default, choose to default, and so on. This resembles the fictitious sequential default idea put forth by Eisenberg and Noe (2001). But, in our model rational agents foresee the sequential default and act strategically, whereas in their model it is an iterative process on auto-pilot that leads to the outcome.

In stage one, agents know the distribution by which nature assigns states and the equilibrium selection in stage three. Therefore, agents are in a position to evaluate their expected payoff

in each possible realized network. Using this knowledge they decide which links to form. Here we describe how the realized network is formed.

Consider a candidate network $(N, E)$ and a coalition of agents $V \subset N$. A **feasible deviation** by $V$ allows agents in $V$

1. to add any absent edges within $V$, and

2. to delete any edges incident to at least one vertex in $V$.

A **profitable deviation** by $V$ is a feasible deviation in which all members of $V$ receive strictly higher expected payoff.

A realized network $(N, E)$ is called **pairwise stable** if there are no profitable deviations by any $V \subset N$ with $|V| \leq 2$ (see Jackson (2010)). $G$ is in the **core** if there are no profitable deviations for any $V \subset N$. We assume that the network formed in stage one is in the core. In the sequel we discuss how our main results change if we employ weaker notions of stability.

Our use of the core can be justified as the *strong* Nash equilibrium of a non-cooperative network formation game played between the members of $N$. Each agent simultaneously proposes to a subset of agents to form an edge. The cost of each proposal is $b > 0$. If a proposal is reciprocated, the corresponding edge is formed. The owners of the edge are refunded $b$. If a proposal is not reciprocated, $b$ is not refunded and the edge is not formed.

Notice that in any Nash equilibrium of this game, all proposals must be mutual. Consider a *strong* Nash equilibrium of the proposal game. A coalition $V$ can make mutual proposals between themselves to form a missing edge, or undo a proposal by any member which would delete the corresponding edge. Therefore, a *strong* Nash equilibrium of this game is a core network as we have defined it.

# 3  Structure of the Cooperating Equilibrium

For a given $(N, E, \boldsymbol{\theta})$ we characterize the structure of a cooperating equilibrium. In what follows, the following notation will be useful.

Let $\Theta = \{\theta_0, \theta_1, ..., \theta_k\}$ be the set of possible states. For each $v \in N$, let $\Delta_v(\theta) = u_v(B, D; \theta) - u_v(D, D; \theta)$, $\Delta'_v(\theta) = u_v(B, B; \theta) - u_v(D, B; \theta)$; be the gains to $v$ from deviating to $B$ from from $D$. Denote the vector of these gains by

$$\Delta_v = (\Delta_v(\theta_0), \Delta_v(\theta_1), ..., \Delta_v(\theta_k), \Delta'_v(\theta_0), \Delta'_v(\theta_1), ...\Delta'_v(\theta_k)) \in \mathbb{R}^{2k+2}.$$

Let $V^c = N \setminus V$ be the complement of $V$ in $N$ for $V \subset N$. For a given $(N, E, \boldsymbol{\theta})$ let $d(v)$ be the degree of $v \in N$ and $d(v, V, \theta_s)$ be the number of edges in state $\theta_s$ which are incident to $v$ and $V$. Let

$$\pi_s(V|v) = \frac{d(v, V, \theta_s)}{d(v)}.$$

Denote the vector of these ratios respectively for $V^c$ and $V$ by

$$\pi^v(V) = (\pi_0(V^c|v), \pi_1(V^c|v), ..., \pi_k(V^c|v), \pi_0(V|v), \pi_1(V|v), ..., \pi_k(V|v)) \in \mathbb{R}^{2k+2}.$$

Strictly speaking our notation should depend upon $(N, E, \boldsymbol{\theta})$. However, as these are all fixed in stage three we omit doing so.

Notice that $\Delta_v(\theta) < 0$ and $\Delta_v(\theta) \leq \Delta_v'(\theta)$ for all $\theta$ and $v$ (by Assumptions 1 and 2). The following lemma characterizes an agent's best response to the actions of other agents.

**Lemma 1.** *Consider a $V \subset N$ and $v \in N$. Suppose that agents in $V \setminus \{v\}$ play $B$, and agents in $(N \setminus V) \setminus \{v\}$ play $D$. Then $D$ ($B$) is the unique best reply of $v$ if and only if $\Delta_v \cdot \pi^v(V) < 0$ ($\Delta_v \cdot \pi^v(V) > 0$).*

*Proof.* Straightforward. ◻

Call a $V \subset N$ **strategically cohesive** if for all $v \in V$

$$\Delta_v \cdot \pi^v(V) \geq 0.$$

**Proposition 2.** *In the cooperating equilibrium, an agent $v$ plays $B$ if and only if there exists a strategically cohesive set $V$ with $v \in V$.*

*Proof.* (If part) By the 1, $\Delta_v \cdot \pi^v(V) \geq 0$ implies that $B$ is a best reply by $v$ if all players in $V$ play $B$ and others play $D$. $D$ is rationalizable for every player, therefore, $B$ can never be eliminated for players in $V$. For all players in $V$ playing $B$ is rationalizable. Hence in the cooperating equilibrium, all of $V$, in particular $v$, play $B$.

(Only if part) Suppose not. Then $N$ is not strategically cohesive (since $v \in N$) and there exists $v_1 \in N$ such that $\Delta_{v_1} \cdot \pi^{v_1}(N) < 0$. Notice that $\pi^{v'}(V') = \pi^{v'}(V'/\{v'\}) = \pi^{v'}(V' \cup \{v'\})$ for any $V'$ and $v'$ since nodes are not adjacent to themselves. Then, $\Delta_{v_1} \cdot \pi^{v_1}(N/\{v_1\}) < 0$. By Lemma 1, $v_1$'s best response to $N/\{v_1\}$ playing $B$ is $D$. By Assumption 2, $v_1$'s best response to any strategy profile, in particular any strategy profile not eliminated, is $D$. Thus, $v_1$ plays $D$ in a cooperating equilibrium. Hence $v_1 \neq v$. Let $N_1 = N/\{v_1\}$. $v \in N_1$. Therefore, by supposition, $N_1$ is not strategically cohesive. Hence, there exists $v_2 \in N$ such

that $\Delta_{v_2} \cdot \pi^{v_2}(N_1) < 0$. Similarly, by Lemma 1 and Assumption 2, $v_2$'s best response to any profile in which $N_1/\{v_2\}$ plays $B$, in particular any strategy profile not eliminated, is $D$. Thus $v_2$ plays $D$ in the cooperating equilibrium, and $v_2 \neq v$. Let $N_2 = N_1/\{v_2\}$. Since $N$ is finite and $v$ plays $D$ in the cooperating equilibrium, we reach a contradiction in a finite number of steps. $\qquad\square$

**Lemma 2.** *If $V$ and $V'$ are both strategically cohesive, then $V \cup V'$ is also strategically cohesive.*

*Proof.* Consider a $v \in V$. We show that $\Delta_v \cdot [\pi^v(V \cup V') - \pi^v(V)] \geq 0$. In this summation the $t$'th component is $\Delta_v(\theta_t) \times [\pi_t((V \cup V')^c | v) - \pi_t(V^c | v)]$ and $k + t$'th component is $\Delta'_v(\theta_t) \times [\pi_t((V \cup V') | v) - \pi_t(V | v)]$. The terms in the brackets add up to 0. Hence the sum of these two terms is equal to $[\Delta'_v(\theta_t) - \Delta_v(\theta_t)] \times [\pi_t((V \cup V') | v) - \pi_t(V | v)] \geq 0$ by Assumption 2.

Therefore, $\Delta_v \cdot \pi^v(V \cup V') = \Delta_v \cdot \pi^v(V) + \Delta_v \cdot [\pi^v(V \cup V') - \pi^v(V)] \geq \Delta_v \cdot \pi^v(V) \geq 0$. $\qquad\square$

Call a set $V \subset N$ **maximally cohesive** if it is the largest strategically cohesive set. This is well-defined by Lemma 2.

**Corollary 1.** *In the cooperating equilibrium, all members of the maximally cohesive set play $B$, all the others play $D$.*

Resilience to system wide failure is determined by the existence of a strategically cohesive set.[7] Strategic cohesiveness is in turn determined by both $\Delta_v$ and $\pi^v(V)$. The first captures the effect of payoffs, while the second captures the effect of the structure of the realized network with states. This suggests that the correct notion of fragility cannot rely on purely network centric measures such as too-interconnected-to-fail, or degree sequences. To separate the effects of network and payoff structure we make some simplifying assumptions and examine their consequences below.

## 3.1 Separating network and payoff effects

We suppose that $\theta < 1$ for all $\theta \in \Theta$ and that payoff functions are the same across agents: $u_v \equiv u$. In particular:

**Assumption 3.** $u(B, B; \theta) = \theta$, $u_v(B, D; \theta) = \theta - 1$, $u(D, B; \theta) = u(D, D; \theta) = 0$ *(in line with Carlsson and van Damme (1993)).*

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[7]One can think of strategic cohesive sets as 'firebreaks'.

For each $V \subset N$ and $v \in N$ let $d(v, V)$ be the number of $v$'s neighbors that are in $V$. Let

$$\pi(V|v) = \frac{d(v, V)}{d(v)}.$$

Let

$$\pi(v) = (\pi_0(N|v), \pi_1(N|v), ..., \pi_k(N|v)).$$

Given $(N, E, \boldsymbol{\theta})$, a set $V \subset N$ is said to be **ex-post cohesive** if $\pi(V|v) + \theta \cdot \pi(v) \geq 1$ for all $v \in V$. The term $\theta \cdot \pi(v)$ captures $v$'s individual resilience from his payoffs, $\pi(V|v)$ captures the collective resilience of $V$ as a function of network structure. If $V$ is sufficiently resilient individually and collectively, then it is ex-post payoff cohesive. Notice that under Assumption 3, strategic cohesiveness reduces to ex-post cohesiveness.

For a given $(N, E, \boldsymbol{\theta})$, vertices within an ex-post cohesive set all play $B$. Thus, they resist default through mutual 'support'. To illustrate, suppose a vertex $v$'s incident edges all have states that are negative valued. Then, $1 - \theta \cdot \pi(v) > 1$ so that $v$ cannot be part of any ex-post cohesive set. Thus, $v$ defaults for sure. As another example, suppose all elements of $\Theta$ are positive. Then, the maximally cohesive set would be $N$ itself for *any* possible case of $(N, E, \boldsymbol{\theta})$. Thus, in any realization, all agents play $B$. Similarly, if all states in $\Theta$ were negative, the maximally cohesive set would be the empty set. In every realization all agents would choose $D$, i.e., there would be certainty of system wide failure.

### 3.1.1 $p$-cohesiveness

Ex-post cohesiveness is closely related to $p$-cohesiveness introduced in Morris (2000). Given $p \in \mathbb{R}$, a set $V$ is $p$-**cohesive** if for all $v \in V$, $\pi(V|v) \geq p$. $p$-cohesiveness imposes a uniform bound on the number of neighbors each vertex in $V$ has in $V$. Ex-post cohesiveness imposes heterogeneous bounds on the same quantity that depend solely on the realized characteristics of $v$, particularly how $v$'s edge states are distributed. Notice that if $\Theta$ was a singleton, say $\{\theta_0\}$, ex-post cohesiveness would be equivalent to $(1 - \theta_0)$-cohesiveness.

$p$-cohesiveness is an ex-ante concept relying only on the structure of $(V, E)$ only. Ex-post cohesiveness, as its name suggests, is an ex-post concept that depends on $(N, E, \boldsymbol{\theta})$. To illustrate, consider a realized edge with a "bad state" $\theta < 0$ in which $\Delta_v(\theta)$ and $\Delta'_v(\theta)$ are very small. The presence of such an edge would help a set 'containing' the edge become "more" $p$-cohesive, however it makes it "less" ex-post cohesive. In this sense, strategic cohesiveness is the appropriate ex-post notion of cohesiveness taking into account the variety in states, while $p$-cohesiveness is possibly an appropriate ex-ante notion of cohesiveness when the states of

edges are not yet realized.

## 3.2 Two states

We introduce a further simplification, $|\Theta| = 2$, with one state being positive and the other negative. This will be convenient for the analysis of the network formation stage and is sufficient to capture most of the essential intuition.

**Assumption 4.** $\Theta = \{\theta_0, \theta_1\}$, $\theta_1 < -1 < 0 < \theta_0 < 1$.

In addition:

**Assumption 5.** $0 < \theta_0 < \frac{1}{N}$.

Assumption 5 ensures that the maximum possible sum of gains from trade scale linearly with $N$.

Call a set $V \subset N$ **maximally** connected[8] if $V$ is connected and there is no $i \in N \setminus V$ adjacent to at least one vertex in $V$.

**Proposition 3.** *Fix $(N, E, \boldsymbol{\theta})$. A set $V \subset N$ of nodes is ex-post cohesive if and only if it is (ex-ante) maximally connected and (ex-post) all edges with endpoints in $V$ have state $\theta_0$.*

*Proof.* Choose any $V \subset N$ and any $v \in V$. Observe that $\pi(V|v) = 1$ if and only if all of $v$'s neighbors are in $V$. Otherwise $\pi(V|v) \leq 1 - \frac{1}{N-1}$. Also, $\theta \cdot \pi(v) = \theta_0$ if and only if all edges of $v$ are $\theta_0$. Otherwise $\theta \cdot \pi(v) \leq \frac{N-1}{N}\theta_0 + \frac{1}{N}\theta_1 < 0$. Note that $1 - \frac{1}{N-1} + \theta_0 < 1$ and $1 + \frac{N-1}{N}\theta_0 + \frac{1}{N}\theta_1 < 1$. Therefore, $\pi(V|v) + \theta \cdot \pi(v) \geq 1$ if only if both $\pi(V|v) = 1$ and $\theta \cdot \pi(v) = \theta_0$ hold. Equivalently, $V$ is ex-post cohesive if and only if for any $v \in V$ all of $v$'s neighbors are in $V$ and all edges incident to $v$ are in state $\theta_0$. $\square$

In the cooperating equilibrium, an agent defaults even if only one edge in the agent's maximally connected component is in the bad state. Without Assumption 5 the condition for an agent's default is not so simple. If, for example, $\frac{1}{N} < \theta_0 < \frac{2}{N}$, an agent defaults if two edges in the component are in bad state which must be placed in a certain distance to the agent, or three edges placed in a certain distance to agent,... The entire position of 'bad edges' on the rooted graph around the node becomes relevant, and the analysis gets very complicated without Assumption 5.

---

[8] A subset of nodes, $V$, is connected if there is a path between any two nodes in $V$ using only nodes in $V$.

# 4 Network Formation

In this section we characterize the set of core networks when all agents have the same payoff functions and $|\Theta| = 2$. We show that a core network consists of a collection of node disjoint complete subgraphs. By forming into complete subgraphs agents increase the benefits they enjoy from partnerships. However, the complete subgraphs formed are limited in size and disjoint. In this way agents ensure that a default in one portion of the realized network does not spread to the entire network. This extreme structure is a consequence of the spareness of our model. However, it suggests that more generally we should expect to see collections of densely connected clusters that are themselves sparsely connected to each other. Blume et al. (2013) have a similar finding in their paper.

The second main result of this section is that a core network is efficient. Call a realized network $(N, E)$ **efficient** if it maximizes the sum of expected payoffs of agents among all realized networks.

## 4.1 Core Networks

Recall that nature determines states identically and independently across edges. Let $\alpha$ be the probability that an edge has state $\theta_0$ and $1 - \alpha$ be the probability that it has state $\theta_1$.

To identify core networks we need to determine an agent's expected payoff in various realized networks. Consider $v \in N$ and suppose that in a realized network, $v$ has degree $d$ and the maximally connected component that contains $v$ has $e$ edges. By virtue of Proposition 3 we need only consider the case where everyone in the maximally connected component defaults or no one does. The probability that every node in the relevant component defaults is $1 - \alpha^e$. In this case, $v$ gets 0. The probability that no one in the relevant component defaults is $\alpha^e$. In this case, $v$ gets $d\theta_0$. So $v$'s expected payoff in stage two is $d\alpha^e\theta_0$. Using this, we can find what happens in stage one.

**Lemma 3.** *Any pairwise stable network, hence core network, consists of disjoint complete subgraphs.*

*Proof.* Take a stable network and two nodes $v'$ and $v''$ in the same connected component. Suppose they are not adjacent. Take a path $v' = v_1, v_2, ... \ v_t = v''$ between $v'$ and $v''$. Let $v'$ and $v''$ join $\{v', v''\}$, delete $\{v', v_2\}$, and delete $\{v_{t-1}, v''\}$. Their degrees are the same and the number of edges in the component strictly decrease. Hence this is a profitable pairwise deviation by $v'$ and $v''$ which contradicts stability. Therefore, in any stable network all nodes within the same connected component are adjacent, which completes the proof. □

Let $d^* = \arg\max_{d \in \mathbb{N}} d\alpha^{(0.5)d(d+1)}$. For generic $\alpha$ this is well defined. Note that $d\alpha^{(0.5)d(d+1)}$ is strictly increasing in $d \in \mathbb{N}$ up to $d^*$. Further, $d^*$ is an increasing step function of $\alpha$.

**Lemma 4.** *If a network is in the core, it consists of a collection of disjoint complete subgraphs, all but one of order $(d^* + 1)$. The remaining complete subgraph is of order at most $(d^* + 1)$.*

*Proof.* By Lemma 3 a core network (if it exists) is composed of disjoint complete subgraphs. The payoff to an agent in a $(d+1)$-complete subgraph is $U(d) = d\alpha^{(0.5)d(d+1)}\theta_0$. This is strictly increasing up to $d^*$.

First, no complete subgraph can have order $d+1 > d^*+1$ in the realized network. Otherwise, $d^* + 1$ members could deviate by forming a $(d^* + 1)$-complete subgraph and cutting all other edges. This would be a strict improvement since $d^*$ is the unique maximizer of $U(d)$.

Second, there cannot be two complete subgraphs of order $d + 1 < d^* + 1$. Suppose not. Let there be $d' + 1$ nodes all together in these two complete subgraphs. Then $min\{d' + 1, d^* + 1\}$ nodes would have a profitable deviation by forming an isolated complete subgraphs since $U(d)$ is increasing in $d$ up to $d^*$. $\qquad\square$

Lemma 4 says that a realized network in the core will consist of a collection of complete subgraphs of order $d^* + 1$ and one 'left-over' complete subgraph with order different from $(d^* + 1)$. To avoid having to deal with the 'left-over' we make a parity assumption about $N$. For the remainder of the analysis we assume $N \equiv 0 \,(mod\, d^* + 1)$. In fact, without this assumption, the core may be empty. To see why, suppose not and assume that the 'left-over' complete subgraph is of order 1. The intuition is as follows. This single left-over agent would like to have an edge, and any other agent would be happy to form that edge since that extra edge does not carry excess risk. We would expect a pairwise deviation which would contradict the initial hypothesis. However, even in this case, $N - 1$ agents don't have a deviation among themselves without using the single left over node. In section 6 we replace the notion of core by the weaker notion of bilaterally stable (which is always non-empty) and show essentially similar results hold.

**Theorem 1.** *For $N \equiv 0 \,(mod\, d^* + 1)$, the core is non-empty, unique (up to permutations) and consists of disjoint $(d^* + 1)$-complete subgraphs.*

*Proof.* Assuming non-emptiness of the core and the parity assumption, Lemma 4 suffices to yield uniqueness once we have existence. It remains to show that a realized network $G = (N, E)$ consisting of disjoint complete subgraphs $C_1, C_2, ..., C_k$ all of order $(d^* + 1)$ (therefore, $N = k\,(d^* + 1)$) is a core network.

For any profitable deviation by $V'$ from $G$ to $G'$, define $\phi(V', G')$ as the number of edges between $V'$ and $N/V'$ in $G'$. Let the minimum of $\phi$ be attained at $(V^*, G^*)$.

Consider $G^*$. Take a node $v' \in V^*$ that is adjacent to $N/V^*$. Suppose that there exists $v'' \in V^*$ such that $v'$ is connected but not adjacent to. Cut one edge connecting $v'$ to $N/V^*$ and join the missing edge between $v'$ and $v''$. This new graph, say $G''$, is also a profitable deviation by $V^*$ from $G$. This is because when we move from $G^*$ to $G''$, the degrees of all nodes in $V^*$ weakly increase, and their component sizes weakly decreases. However, $\phi(V^*, G'') < \phi(V^*, G^*)$, which is a contradiction. Therefore, any node in $V^*$ that is connected to $v'$ is adjacent to it. Same holds for any node that is adjacent to $N/V^*$.

Take a node in $V^*$ with minimal degree, say $v$ with degree $d$. Let $d' \geq 0$ be the number of $v$'s neighbors in $N/V^*$. Suppose $d' \geq 1$. By the last paragraph, a node in $V^*$ that is connected to a neighbor of $v$ can only be a neighbor of $v$. Therefore, any neighbor of $v$ in $V^*$ has at most $d - d'$ neighbors in $V^*$, hence at least $d' \geq 1$ neighbors in $N/V^*$. So by the last paragraph, $v$ and his $d - d'$ neighbors in $V^*$ are all adjacent to each other, forming $(0.5)(d - d' + 1)(d - d')$ edges. Each of them have at least $d'$ edges to $N/V^*$, so that makes $d'(d - d' + 1)$ edges. Finally, since nodes in $N/V^*$ have not deviated from $G$ and are connected to each other, they are all adjacent to each other, forming $(0.5)d'(d' - 1)$ edges. Therefore, in $v$'s maximally connected component, there are at least $(0.5)d(d + 1)$ edges, so that his payoff is at most $U(d)$. Now suppose $d' = 0$. Then all $v$'s $d$ neighbors are in $V^*$, hence all have degree at least $d$. Then again, $v$'s component has at least $d(d + 1)/2$ edges, so that his payoff is at most $U(d)$. In both cases, $v$'s payoff in $G^*$ is at most $U(d) \leq U(d^*)$; contradiction with profitable deviation from $G$.

$\square$

**Theorem 2.** *For $N < d^* + 1$, the unique core is the $N$-complete subgraph.*

*Proof.* Recall that $d\alpha^{(0.5)d(d+1)}$ is increasing in $d$ up to $d^* > N$. The remainder of the proof follows the proof of Theorem 1 by replacing $d^* + 1$ with $N$. We omit the details. $\square$

## 4.2 Efficiency

Consider a connected graph with $e$ edges. A node in the graph with degree $d$ enjoys an expected payoff of $d\alpha^e\theta_0$. Therefore, the sum of payoffs of nodes within the graph is $2e\alpha^e\theta_0$. Here we use the well known fact that the sum of degrees is twice the number of edges. It follows then, that the problem of finding an efficient network devolves into two parts: how

to partition nodes into maximally connected components, and how many edges to put into each component.

Let $k^* = \arg\max_{y \in \mathbb{N}} y\alpha^y$. For generic $\alpha$ this is well defined.[9] Note that $y\alpha^y$ is strictly increasing in $y \in \mathbb{N}$ up to $k^*$ and strictly decreasing after $k^*$. Note also that when maximizing $y\alpha^y$ over the non-negative reals, the maximum occurs at a number $y^*$ where $\alpha^{y^*} = \mathbf{e}^{-1}$. Here $\mathbf{e}$ is Euler's constant and $y^*$ lies in the interval $(\frac{\alpha}{1-\alpha}, \frac{1}{1-\alpha})$.

**Theorem 3.** *If $N \equiv 0 \,(mod\, d^* + 1)$, a network is efficient if and only if it is in the core.*

*Proof.* Recall that $U(x) = x\alpha^{(0.5)x(x+1)}$. Let $\mathcal{U} = \{u \in \mathbb{R} \mid u = U(x) \text{ for some } x \in \mathbb{N}\}$. The maximum of $\mathcal{U}$ is achieved, uniquely, at $x = d^*$. Let $\bar{u} = U(d^*)$. Notice that this is the average payoff at the core network. We will prove that the average is strictly less in any other network.

Consider an efficient network $G$ and suppose it to be made up of a collection of disjoint connected components: $C^1, C^2, C^3, \ldots$. Consider component $C^i$ and suppose it has $q_i$ edges. The total payoff of $C^i$ scales with $2q_i\alpha^{q_i}$. If $q_i \neq k^*$ we can improve total payoff by deleting or adding (if not complete graph) edges to $C^i$. Therefore, we can assume that $q_i = k^*$, or that $C^i$ is complete.

Let $r_i$ be the largest integer such that $r_i(r_i - 1)/2 < q_i \le r_i(r_i + 1)/2$. Let $w_i$ be such that $q_i = r_i(r_i - 1)/2 + w_i$, where $1 \le w_i \le r_i$. Note that there must be at least $r_i + 1$ nodes in $C^i$.

**Case 1:** $1 \le w_i \le \frac{r_i - 1}{2}$.

The average degree of nodes in $C^i$ is at most

$$\frac{2k^*}{r_i + 1} = \frac{r_i(r_i - 1) + 2w_i}{r_i + 1} \le r_i - 1.$$

Note that $k^* = q_i \ge (r_i - 1)r_i/2 + 1$. Hence the average payoff per node is at most

$$(r_i - 1)\alpha^{k^*} < (r_i - 1)\alpha^{\frac{(r_i - 1)r_i}{2}} \le \bar{u}.$$

So the average payoff is strictly less than $\bar{u}$.

**Case 2:** $r_i - 1 \ge w_i \ge \frac{r_i}{2}$.

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[9]For $\alpha$ such that $(1-\alpha)^{-1}$ is integral, there are two integers in the $\arg\max$: $\frac{\alpha}{1-\alpha}$ and $\frac{1}{1-\alpha}$. In other cases, the $\arg\max$ is unique: it is the unique integer in the open interval $(\frac{\alpha}{1-\alpha}, \frac{1}{1-\alpha})$.

Since $w_i < r_i$, $k^* = q_i \leq r_i(r_i+1)/2 - 1$. The average degree of nodes in $C^i$ is at most

$$\frac{2k^*}{r_i+1} \leq \frac{r_i(r_i+1)-2}{r_i+1} \leq r_i - \frac{2}{r_i+1}.$$

Note that $k^* = q_i = (r_i-1)r_i/2 + w_i \geq r_i^2/2$. Hence the average payoff per node is at most

$$\left(r_i - \frac{2}{r_i+1}\right)\alpha^{\frac{r_i^2}{2}}.$$

Now we show that this is strictly less than

$$(r_i-1)\alpha^{\frac{r_i^2-r_i}{2}} = U(r_i-1).$$

This is equivalent to showing that

$$\alpha < \left(\frac{r_i+1}{r_i+2}\right)^{\frac{2}{r_i}}.$$

Recall that $k^*$ is the unique integer between $\alpha/(1-\alpha)$ and $1/(1-\alpha)$. Therefore,

$$\alpha \leq 1 - \frac{1}{k^*+1} \leq 1 - \frac{2}{r_i(r_i+1)}.$$

Hence, it suffices to verify that

$$1 - \frac{2}{r_i(r_i+1)} < \left(\frac{r_i+1}{r_i+2}\right)^{\frac{2}{r_i}} \iff \left(\frac{r_i+1}{r_i+2}\right)^{\frac{2}{r_i}} > \frac{(r_i+2)(r_i-1)}{(r_i)(r_i+1)}$$

$$\iff (r_i+2)\log\left(1 - \frac{1}{r_i+2}\right) > r_i \log\left(1 - \frac{1}{r_i}\right)$$

which is true since the function $f(x) = x\log(1 - \frac{1}{x})$ is strictly increasing. Therefore, the average payoff is strictly less than $U(r_i-1) \leq U(d^*) = \bar{u}$.

**Case 3:** $w_i = r_i$. (This covers the case in which $C^i$ is complete as well.)
Then the average payoff per node is less than $U(r_i) \leq \bar{u}$ , and the inequality is strict unless $C^i$ is a $(d^*+1)-$complete graph.

$\square$

## 4.3 Systemic Risk

Fix $N \equiv 0 \, (mod \, d^* + 1)$ and consider the core network. Recall that all nodes of a maximal complete subgraph play $D$ if at least one of the edges in the complete subgraph is in a 'bad' state; otherwise they all choose action $B$. The probability that all nodes in a maximal complete subgraph choose $D$ is $\alpha^{(0.5)d^*(d^*+1)}$, hence the probability that everybody defaults, i.e. systemic risk, is

$$\left(1 - \alpha^{(0.5)d^*(d^*+1)}\right)^{\frac{N}{d^*+1}}.$$

For fixed $\alpha$, the probability of an edge being in a 'good' state, the above expression is increasing in $d^* < N$. An increase in $d^*$ leads to fewer but larger complete subgraphs. Thus, for fixed $\alpha$ higher interconnectedness translates into higher systemic risk. Figure 1 shows a plot of systemic risk versus cluster size for various values of $\alpha$. The dotted line shows how systemic risk of the core varies with $\alpha$. Note that as $\alpha$ increases, the core consists of fewer but larger clusters.
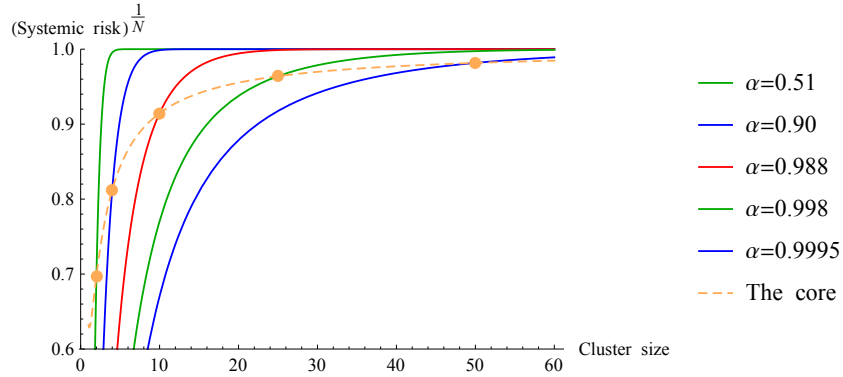


Figure 1: Interpolated Systemic Risk vs. Cluster Size

As one can see in Figure 1, systemic risk of the core/efficient network increases with $\alpha$. In our model, $d^*$ (weakly) increases with $\alpha$. It increases at such a rate that systemic risk of the core/efficient network also increases with $\alpha$.[10] This is displayed in Figure 2 below.

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[10]Since $d^*$ is a step function of $\alpha$, in intervals where $d^*$ stays constant the probability decreases. However, this is an artifact of discreteness. When $\alpha$ hits $\left(\frac{d-1}{d}\right)^{\frac{1}{d}}$, $d^*$ jumps from $d-1$ to $d$. If one considers these jumping points of $\alpha$, the probability is increasing. In order to clarify further, recall the definition of $d^* = argmax_{d \in \mathbb{N}} d\alpha^{(0.5)d(d+1)}$. For a "smooth version" of $d^*$ as a function of $\alpha$, a real number $d^* = argmax_{d \in \mathbb{R}} d\alpha^{(0.5)d(d+1)}$, the probability is increasing.
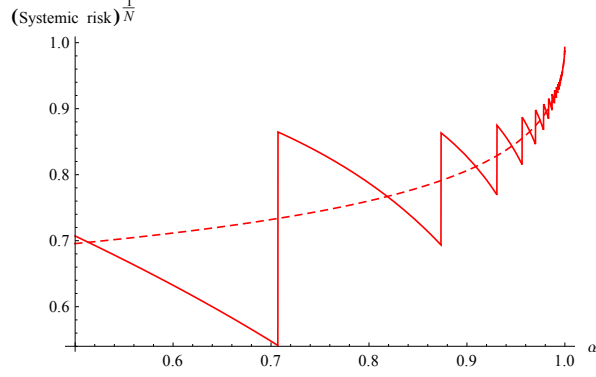
Figure 2: Systemic Risk of the Core Network vs. $\alpha$

Intuitively, as the economy gets fundamentally safer, agents form much larger clusters. That is in their individual interest and furthermore the outcome is efficient. However, this results in *increased* systemic risk: catastrophic events become more frequent. Note that once $\alpha$ becomes too large and hits $\left(\frac{N-1}{N}\right)^{\frac{1}{N}}$, $d^*$ becomes $N$ and the clusters cannot get any larger. Hence the systemic risk cannot get any larger and it starts decreasing again.

Figures 3 below show how the expected number of defaults, $N \times \left(1 - \alpha^{(0.5)d^*(d^*+1)}\right)$ varies with $\alpha$.
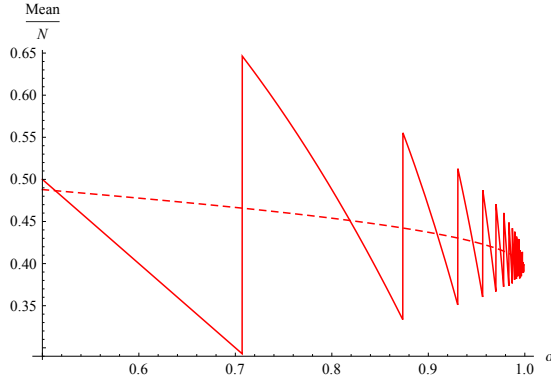


Figure 3: Mean of the Number of Defaults at Core Network vs. $\alpha$

We can actually pin down the exact distribution of the number of nodes that default. Given $\alpha$, the number of maximal complete subgraphs that fail is $k$ with probability

$$\binom{\frac{N}{d^*+1}}{k} \left(1 - \alpha^{(0.5)d^*(d^*+1)}\right)^k \left(\alpha^{(0.5)d^*(d^*+1)}\right)^{\frac{N}{d^*+1}-k}.$$

This is also the probability that $(d^* + 1)k$ agents default and the rest do not. For $N = 100$, Figure 4 illustrates the distribution.
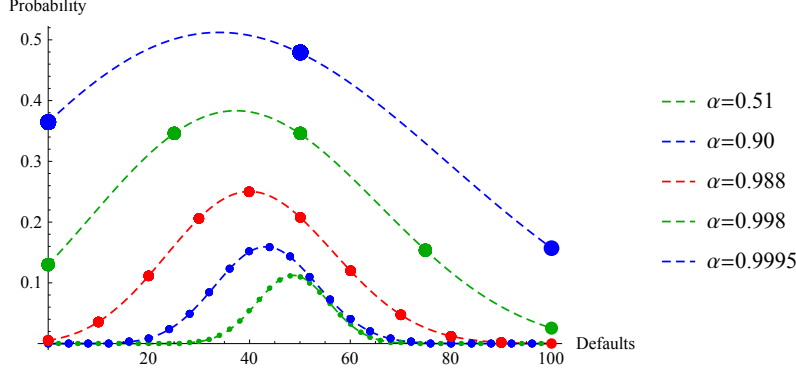
Figure 4: Probability Distribution of the Number of Defaults at Core Network (For $N = 100$)

# 5 Correlation

We noted earlier a debate about whether interconnectedness of nodes is a significant contributor to systemic risk. An alternative theory is that the risk faced is via common exposures, i.e., popcorn. Observed outcomes might be similar in both scenarios but the dynamics can be significantly different.

We model the popcorn story as perfect correlation in states of edges through $\phi$. Thus, $\phi$ is such that with probability $\sigma$ *all* edges have state $\theta_0$, with probability $1 - \sigma$ *all* edges are in state $\theta_1$.

There is no change in the analysis of stage three. As for stage one, now there is no risk of contagion.

**Theorem 4.** *Under 'popcorn', the unique core is the $N$- complete graph, denoted $K_N$.*

*Proof.* In any given realized network, if all states are $\theta_0$ then everybody play $B$ and if all states are $\theta_1$ then everybody play $D$. The payoff of an agent with $d$ edges is $d\theta_0$ or $0$ respectively. Thus, the expected payoff of each agent is $d\sigma\theta_0$. Then, it is clear that in a core network there cannot be any missing edges because that would lead to a profitable pairwise deviation. The only candidate is $K_N$ which is as clearly in the core. □

When agents anticipate common exposures (popcorn) rather than contagion, they form highly interconnected networks in order to reap the benefits of trade.

In an independent shocks world, the probability that everybody defaults in $K_N$ is $1 - \alpha^N$, which is the highest systemic risk that any network can achieve in this world. However, $K_N$ is as safe as all the other possible realized networks in the correlated shocks world.

This highlights the importance of identifying the shock structure before investigating a given network. A specific network and a particular shock structure might very well be incompatible.

## 5.1 More general correlation

Perfect correlation and complete independence are two extremes. Here we extend the benchmark model to allow for a correlation structure that is in between. With some probability the economy operates as 'normal' and edges are subject to their own idiosyncratic shocks, while with complementary probability a common exposure to risk is realized and all edges have bad states. Formally, with probability $1 - \sigma$ all edges are $\theta_1$, while with probability $\sigma$ all states of edges are i.i.d.: $\theta_0$ with probability $\alpha$ and $\theta_1$ with probability $1 - \alpha$. Notice that '$\sigma = 1$, $\alpha > 0$' is the extreme case of 'independence with $\alpha$ being the probability of an edge being in a good state'. The case '$\alpha = 0$, $\sigma < 1$' is the extreme case of 'perfect correlation with $\sigma$ being the probability of all edges being in a good state'.

In this setting, the expected payoff of an agent is $d\alpha^e \sigma \theta_0$. Clearly, the identical analysis in section 4 goes through for any $\sigma$. Notice that as $\alpha$ tends to 1, $d^*$ diverges to $\infty$. For some $\bar{\alpha} < 1$, $\alpha > \bar{\alpha}$ implies that $d^* > N$. Then, by Theorem 2, the unique core is $K_N$. This illustrates Theorem 4 is not an anomaly due to perfect correlation. In fact, it is a corollary of Theorem 2; the same result derives for sufficiently strong correlation not just the extreme correlation.

# 6 Extensions

We summarize two extensions to our model. The first allows shocks to nodes, the second will consider weaker notions of network formation.

## 6.1 Node shocks

We extend the benchmark model to allow for shocks to individual nodes. There are two ways to think about such shocks. The first is an idiosyncratic shock that affects an institution without any direct effect to any other institution. The second is one in which the financial sector has ties with the real sector and these ties are subject to shocks as well. In the model, each node (financial institution) would be connected with an (imaginary) edge to the outside of the network. The shock to this edge is effectively an idiosyncratic shock to the node itself.

These shocks can be correlated but we consider the case of independent node-shocks only. Formally, after stage two has ended and before we move on to stage three, each 'imaginary' edge independently defaults with probability $1 - \beta$ or proceed as **normal** with probability $\beta$.

In stage three, ex-post cohesive sets are maximally connected sets all of whose edges are in state $\theta_0$ and nodes are normal. In this case members of such a set play $B$ and get $\theta_0$ for each edge they have. Otherwise they play $D$ and get 0.

In stage two, the expected payoff of a node with degree $d$, and $e$ number of edges and $f$ number of nodes in its maximally connected set has payoff, $d\alpha^e\beta^f\theta_0$.

As for stage one, the earlier results apply. A core network will consist of disjoint complete subgraphs. Let $d^{**} := \arg\max_{d\in\mathbb{N}} d\alpha^{(0.5)d(d+1)}\beta^{d+1}$. Theorems and comparative statics concerning the core apply with $d^{**}$ replacing $d^*$.

Note that $d^{**}$ is smaller than $d^*$. This tells us that when agents are exposed to further new types of risks, which effectively increases their overall risk, they form less interconnected networks.

## 6.2 Weaker notions of network formation

### 6.2.1 Pairwise Stability

The results above concerning the core assume the ability of any coalition to get together and 'block'. Networks that survive weaker notions of blocking are also of interest. Two natural candidates are **Nash networks** and pairwise stable networks.[11] The first preclude deviations by single entities only, while the second by pairs only. All core networks are pairwise stable, and all pairwise stable networks are Nash networks.

Robustness to unilateral deviations is too permissive. Most (isomorphism classes of) graphs with degree less than $k^*$ are Nash networks. The reason is that there is no way anyone can add an edge in a feasible Nash deviation. As for deleting edges, for graphs that are sufficiently well connected a unilateral deletion will not reduce the cluster size very much. Hence agents are not going to delete edges since they already have less than $k^*$ edges.

Concerning pairwise stability, henceforth stability, some of our earlier results continue to hold. For example, $K_N$ is the unique stable network in the popcorn model. However, in

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[11]Blume et al. (2013) find that their stable networks are *not* efficient. However, their notion of efficient is a worst-case one, very different from the one employed here.

the independent shocks model there can be stable networks which are not in the core and therefore not efficient.

**Proposition 4.** *Let $h^* \geq d^*$ be the largest integer such that $U(1) \leq U(h^*)$. Any network that consists of disjoint complete subgraphs each with order between $d^* + 1$ and $h^* + 1$, is stable. Call these* **stable-complete networks.**

*Proof.* Consider an agent in a stable-complete network. Let her have degree $d$, and let her have $e = d(d+1)/2$ edges in her complete subgraph. Suppose that she has a pairwise or unilateral profitable deviation in which she deletes $x$ of her existing edges in her complete subgraph, and adds $t \in \{0, 1\}$ new edges.

If $x = d$, her payoff is at most $\alpha = U(1) \leq U(h^*) \leq U(d)$ (since $d^* \leq d \leq h^*$) which cannot be a profitable deviation. So $x < d$, which means she is still connected to $e - x$ edges in her old component. Then her payoff is at most $(d - x + t)\alpha^{e-x+t}$. If $t - x \leq 0$, this is less than $d\alpha^e$ since $y\alpha^y$ is strictly increasing up to $k^*$ in $y \in \mathbb{N}$ and $h \leq k^*$. Then $t - x > 0$, which is possible only when $t = 1$ and $x = 0$.

By adding an edge, her component gains one adjacent edge, and $m \geq 0$ extra edges that are connected to the new edge. Therefore, her new payoff is $(d + 1)\alpha^{e+1+m}$. Since this is a profitable deviation, $d\alpha^e < (d + 1)\alpha^{e+1+m}$. Since $d \geq d^*$, we have $U(d) > U(d + 1)$. So $d > (d + 1)\alpha^{d+1}$. Therefore, we have $d\alpha^e < d\alpha^{e+m-d}$, i.e. $m < d$.

All in all, a profitable deviation must be pairwise, in which deviating nodes get connected to a number of edges, $m_1$ and $m_2$, say, less than their original degrees, $d_1$ and $d_2$, say. This is not possible for both of the deviating nodes: $m_1 \geq d_2 > m_2 \geq d_1 > m_1$ which is a contradiction. □

Stable networks can be inefficiently too interconnected. Figures 5 and 6 illustrate the differences between core and stable networks[12] on various metrics.

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[12]We plot the properties of the the most interconnected stable-complete network, the one with cluster size $h^*$. The properties of the other stable-complete networks lie in the regions between the plots for the core network and the most interconnected stable network in the figures.
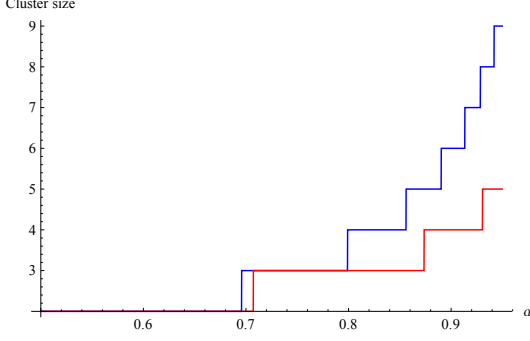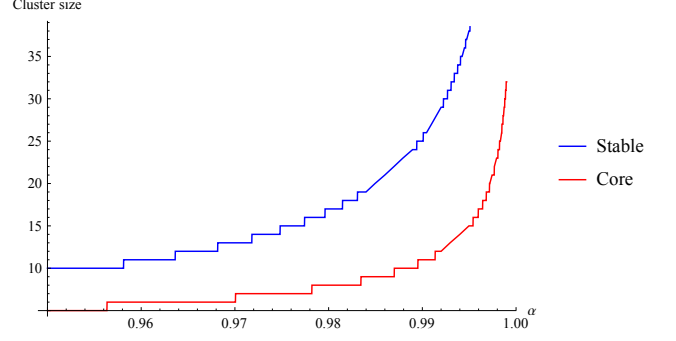
Figure 5(a): For $0.5 < \alpha < 0.9$



Figure 5(b): For $\alpha > 0.9$

Figure 5: Cluster Sizes of Stable and Core Networks vs. $\alpha$
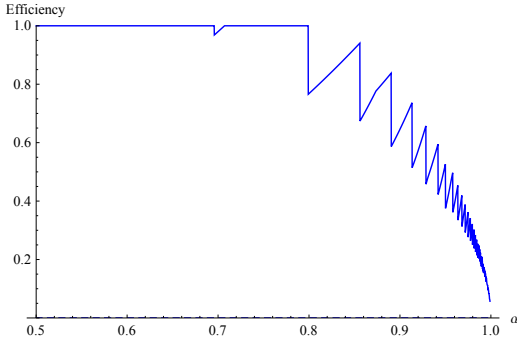


Figure 6(a): Efficiency in Stable
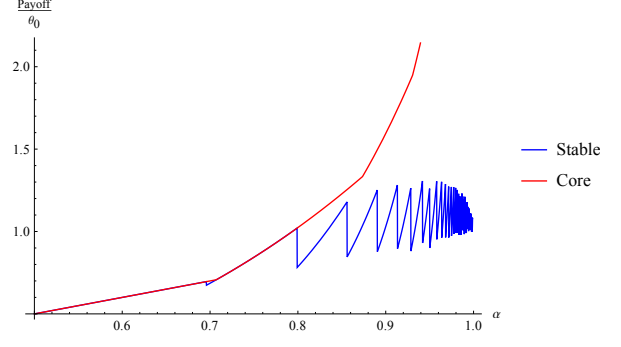Networks vs. $\alpha$



Figure 6(b): Payoff Per Node in Stable
and Core/Efficient Networks vs. $\alpha$

Figure 6: Payoffs and Efficiency in Stable Networks

Next we show that these particular stable networks have higher systemic risk than the core/efficient networks due to higher interconnectedness.

**Proposition 5.** *Take $N \equiv 0 \, (mod \, d^* + 1)$. Stable-complete networks have higher systemic risk than core/efficient networks.*

*Proof.* Recall that $\left(1 - \alpha^{(0.5)x(x+1)}\right)^{1/x}$ is increasing in $x$. Take any complete subgraph with order $d + 1 \geq d^* + 1$.

$$1 - \alpha^{(0.5)d(d+1)} = \left(1 - \alpha^{(0.5)d(d+1)}\right)^{(d+1)/(d+1)} \geq \left(1 - \alpha^{(0.5)d^*(d^*+1)}\right)^{(d+1)/(d^*+1)}.$$

Let $d_t + 1$'s be the orders of maximally complete subgraphs, indexed by $t$, of a stable-complete network. Then

$$\prod_t \left(1 - \alpha^{(0.5)d_t(d_t+1)}\right) \geq \left(1 - \alpha^{(0.5)d^*(d^*+1)}\right)^{\frac{1}{d^*+1}\sum d_t + 1} = \left(1 - \alpha^{(0.5)d^*(d^*+1)}\right)^{\frac{N}{d^*+1}}$$

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which completes the proof. □

Figure 7 illustrates the difference in systemic risk between stable and core networks for various values of $\alpha$.
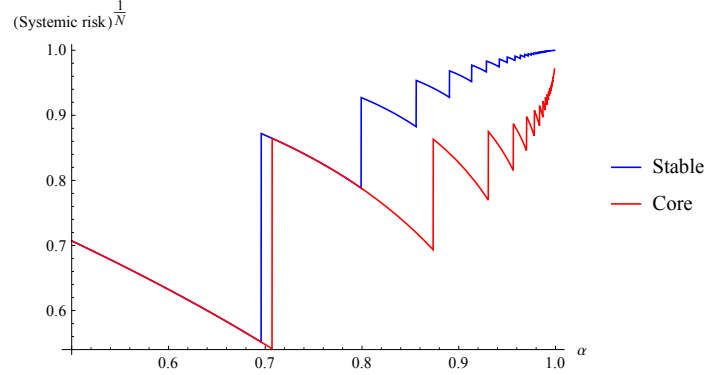


Figure 7: Systemic Risk in Stable and Core Networks vs. $\alpha$

These findings on stable networks suggest that some inefficiencies in observed networks that contribute to systemic risk may stem from the inability of parties to coordinate.

### 6.2.2  $t$-stability

In this section we consider the middle ground between the core and pairwise stable networks. The results in this section use proofs similar to the ones above and so are omitted. Call a network $(N, E)$ $t$-**stable** if no coalition of size $t$ or less has a profitable deviation. Notice that $N$-stable is equivalent to the core, and 2-stable is equivalent to pairwise stable.

**Theorem 5.** *For any $t \geq d^* + 1$, the unique $t$-stable network is the core.*

Keeping in mind that we typically think of $d^* + 1$ as being relatively small with respect to $N$, this theorem shows us that the results in the paper don't need the full power the core that precludes deviation by any coalition. A restriction on relatively small sized coalitions is sufficient. The next theorem concerns $t \leq d^*$.

**Theorem 6.** *Take any $t \leq d^*$. Let $h^*(t) \geq d^*$ be the largest integer such that $U(t) \leq U(h^*(t))$. Any network that consists of disjoint complete subgraphs each with order between $d^* + 1$ and $h^*(t) + 1$, is $t$-stable. Call these $t$-**stable-complete** networks.*

Notice that as $t \leq d^*$ gets smaller, $t$-stable-complete networks become similar to stable-complete networks. As $t \leq d^*$ gets larger, $h^*(t)$ approaches $d^* + 1$, so that $t$-stable-complete

networks become closer to core networks. After $d^*$, for $t \geq d^* + 1$ the only $t$-stable network is the core itself (the $(d^* + 1)$-stable-complete network). These results bridge the gap between core results and pairwise stability results.

For an entirely complete picture, the only remaining open question is finding the exact set of $t$-stable networks, which we know must consist of disjoint complete subgraphs. We leave it to future work.

One further important thing to note is that as $t$ gets larger, $t$-stable-complete networks become more efficient. Similarly, systemic risk of $t$-stable networks are smaller for larger values of $t$.

# 7    Conclusion

Our model finds that forward looking rational agents who rely on bilateral contracts for surplus generation, anticipate the possibility of system wide failure during network formation. They guard against it by segregating themselves into densely connected clusters that are sparsely connected to each other. As the economy gets fundamentally safer, agents organize into much larger clusters which results in an increase in systemic risk.

Whether the networks formed optimally trade-off the benefits of surplus generation against systemic risk depends on the ability of agents to coordinate among themselves during network formation. If the networks formed are robust to bilateral deviations only, they are inefficient. If robust to deviations by any subset, they are fully efficient.

Our model also emphasizes that assessing the susceptibility of a network to system wide failure should not be done in ignorance of the beliefs of agents who formed that network. Efficient markets generate structures that are safe under the correct specification of shocks, which will appear fragile under the wrong specification of the shock structure. Thus, mistakes in policy can arise from a misspecification in the correlation of risks.

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