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"Near Feasible Stable Matchings with Complementarities"

by

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Near Feasible Stable Matchings with Complementarities

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Abstract

The National Resident Matching program strives for a stable matching of medical students to teaching hospitals. With the presence of couples, stable matchings need not exist. For any student preferences, we show that each instance of a stable matching problem has a 'nearby' instance with a stable matching. The nearby instance is obtained by perturbing the capacities of the hospitals. Specifically, given a reported capacity k_h for each hospital h, we find a redistribution of the slot capacities k'_h satisfying $|k_h - k'_h| \leq 4$ for all hospitals h and $\sum_h k_h \leq \sum k'_h \leq \sum_h k_h + 9$, such that a stable matching exists with respect to k'. Our approach is general and applies to other type of complementarities, as well as matchings with side constraints and contracts.

Keywords: stable matching, complementarities, Scarf's lemma

JEL classification: C78, D47

1 Introduction

The problem of finding a stable matching was introduced by Gale and Shapley [1962] who also identified a setting where such matchings always exist. Since then, matching theory has become a central topic of market design, and changed the way centralized markets such as medical matching and school choice are organized. Motivated by these applications, there has been a flood of work, including Fleiner [2003], Hatfield and Milgrom [2005], Ostrovsky [2008] and Hatfield and Kojima [2010], enlarging the settings in which a stable matching is

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guaranteed to exist. A common feature of these generalizations is the absence of complementarities in the preferences of each side of the market. Yet, many settings, claimed as natural applications of these models exhibit such complementarities. Indeed, Che et al. [2014] write:

> "... complementarities of preferences are a pervasive feature of many matching markets. Firms often seek to hire workers with complementary skills. For instance, in professional athletic leagues, teams demand athletes that complement one another in skills as well as in the positions they play. Some public schools in New York City seek diversity of their student bodies in their skill levels. US colleges tend to exhibit a desire to assemble a class that is complementary and diverse in terms of their aptitudes, life backgrounds, and demographics.
>
> Unless we can get a handle on complementarities, we would not know how to organize such markets, and the applicability of centralized matching will remain severely limited...."

Perhaps the most well known instance of this problem is the matching of medical students to teaching hospitals in the US. Each year at least 20,000 medical school graduates participate in the National Resident Match Program (NRMP).[1] In the absence of couples and complementarities in hospital's preferences, a stable matching is guaranteed to exist. Adding couples who submit joint preference lists rules out the existence of stable matches in some cases (see Roth [1984]).[2] Nevertheless, in the NRMP it has been observed that in spite of the presence of couples, the resulting matches are stable with respect to the reported preferences. This phenomenon is explained in Kojima et al. [2013] and also in Ashlagi et al. [2014]. These papers identify restrictions on applicant preferences and the proportion of couples for which the probability a stable matching exists is high or the number of blocking coalitions is small as the number of agents and slots gets large. Specifically, applicant preferences are assumed to be drawn independently from a distribution, which rules out a

[1] `http://www.nrmp.org/wp-content/uploads/2014/04/Main-Match-Results-and-Data-2014.pdf`

[2] In fact, the problem of determining whether a stable matching exists in this setting is $\mathcal{NP}$-complete (see Ronn [1990]).

certain degree of heterogeneity. Moreover, the results in Kojima et al. [2013] and Ashlagi et al. [2014] hold only when the proportion of couples approaches 0 as the market gets large.[3]

In this paper we propose an alternative measure of closeness to stability motivated by Budish [2011] and Dean et al. [2006]. Specifically, how many additional resources must be added or removed to ensure the existence of a stable matching? In the context of the NRMP this would be how many additional residency slots are needed to ensure the existence of a stable matching? The answer is *at most 9*, independent of the number of hospital slots and medical school graduates. Importantly, unlike Kojima et al. [2013] our answer to this question does not rely on restrictions on the proportions of couples, probabilistic assumptions about doctor's preferences or asymptotic arguments.

Our result applies to settings more general than the NRMP.[4] To describe them, label one side of a two-sided market, supply (teaching hospitals, schools) and the other demand (medical students, school students). The supply side has capacity constraints. A matching is feasible if the supply of each agent on the supply side is allocated to the demand side so as to comply with the constraints of the application without exceeding its capacity. Call a matching α-feasible if the supply of each agent on the supply side allocated to the demand side differs (up or down) from its capacity by at most α.

The contribution of this paper is to establish the existence of near feasible stable matchings when preferences on the demand side exhibit a limited degree of complementarity. The degree of complementarity in preferences is measured by a parameter α. Given α, we establish the existence of a $(2\alpha - 1)$-feasible stable matching. We also give an algorithm for identifying it. In the context of the NRMP with couples, $\alpha = 2$. Our result implies a stable matching in which each hospital is assigned a number of residents that differs from its re-

[3]In the National Resident Match Program the proportion of couples is between 5% and 10%. There are settings where the proportion of couples is high. Biró and Klijn [2013], for example, identify a setting where the the proportion of paired applications is almost 40%. Biró et al. [2013] shows that when the proportion of couples is high, the method developed by Roth and Peranson [1999] for the the NRMP does not find a stable matching most of the time.

[4]The model described in Section 3 captures matching with contracts and applies to matching with side constraints, generalizing the setting in Kamada and Kojima [2014b]. This generalization is discussed in Section 5.

ported capacity by at most 3. This seems like a small additional number of positions to lay on or do without. Every additional resident, according to the AMA costs a hospital about \$100,000 on average of which 40% is covered by the US government.[5] Thus, it is unlikely that any one hospital will be willing to spend the money to increase its capacity or suffer a reduction to ensure stability. However, the total number of residencies and their distribution is determined by the Federal government (via formulae of various kinds). One might speculate that, if in *total*, the extra number of positions needed is small, it might be willing to spend for it. Hence, our next result. There is a 4-feasible stable matching such that in total, across all hospitals, we do not reduce the number of slots and the additional number of slots needed is at most 9. As the total number of first year residency positions in the US is presently around 30,000, an additional 9 slots, is, in the aggregate, a small proportion of the total.

The technique used to arrive at the result is a combination of Scarf's lemma (Scarf [1967]) and a combinatorial optimization method, called iterative rounding, developed in Lau et al. [2011] and Nguyen et al. [2014]. Both Scarf's lemma and iterative rounding are constructive, therefore, the near feasible matches we identify can be obtained by a finite time algorithm.[6]

In the following, we first discuss the related literature, then, rather than describe the most general setting in which the technique applies, we begin in Section 2, for expositional purposes, by considering the stable matching problem with couples. In Section 3 we introduce a more general model, that might be useful in the context of matching with contracts, and prove the main results. Section 4 further extends our results to a broader class of choice functions. Section 5 applies the main results in Section 3 to matching problems with side constraints. Section 6 concludes. Proofs are given in the Appendix.

[5] These numbers are from an AMA pamphlet in support of the current approach to funding residency programs. `http://savegme.org/wp-content/uploads/2013/01/graduate-medical-education-action-kit.2-3.pdf`

[6] Our approach though constructive does rely on Scarf's lemma which is known to be PPAD complete, Kintali [2008]. Thus, it has a worst-case complexity equivalent to that of computing a fixed point. We do not see this as an obvious barrier to implementation. For example, building on Budish [2011], a course allocation scheme that relies on a fixed point computation has been proposed and implemented at the Wharton School.

Related work

Roth [1984] is one of the first papers to consider complementarities in matching problems. That paper shows non-existence of a stable matching when some agents are couples. Subsequently, the design of matching in the presence of complementarities has become an important topic. See Biró and Klijn [2013] for a brief survey. The literature has focused on the couples setting and taken three approaches to circumventing the problem of non-existence.

The first, is by imposing restrictions on the preferences of the agents to ensure existence of a stable matching. Examples of this are Cantala [2004], Klaus and Klijn [2005], Pycia [2012] and Sethuraman et al. [2006]. These restrictions impose some kind of strong alignment in agent preferences. In Klaus and Klijn [2005], for example, which focuses on matchings with couples, it is assumed that a unilateral improvement of one partners job is considered beneficial for the couple as well.[7] Cantala [2004] and Sethuraman et al. [2006] consider an alternative restriction motivated by geographical considerations (couples prefer to be in close proximity to each other).

The second is to argue that instances of non-existence are rare in large markets. For example, Kojima et al. [2013], Ashlagi et al. [2014] and Che et al. [2014] consider large (in some cases the continuum), random markets, where agent's preference are independently drawn from a distribution and the fraction of couples compared with the size of the market approaches zero. They prove that a stable matching exists with high probability in these environments. The assumption that couples form a vanishing proportion of the population is crucial as Ashlagi et al. [2014] show that the result does not hold when the fraction of couples is a constant.

The third approach is to 'ignore' the indivisibility of agents, identify and provide interpretations of 'fractional' stable matchings. Examples are Dean et al. [2006], Aharoni and Holzman [1998], Aharoni and Fleiner [2003], Király and Pap [2008] and Biro and Fleiner [2012]. Dean et al. [2006], for instance, examines a machine scheduling problem that em-

[7] See also the bilateral substitutes condition of Hatfield and Kojima [2010].

beds, as a special case, a restricted instance of the stable matching problem with couples. In that instance, couples prefer to be together rather than apart and a hospital cannot accept just one member of the couple. Under these conditions they show how to find a 2-feasible stable matching. The remaining papers we cite, establish the existence of a fractional stable matching via Scarf's lemma Scarf [1967]. Biró et al. [2013] reports on numerical experiments that show in many cases the fractional solutions obtained from Scarf's lemma are actually integral and thus are stable matchings.

This paper builds on the technique of Lau et al. [2011] to show how to round the fractional stable matchings returned by Scarf's lemma into integral stable matchings that are near feasible. The degree of infeasibility in these stable matchings depends, as noted earlier, on the degree of complementarity exhibited in the preferences of one side of the market. Our approach extends beyond matching with couples to include the general setting of multilateral matching with contracts as well as matching with side constraints.

2 Matching with Couples

To fix ideas we begin with a description of the standard matching model with couples, that is studied, for example, in Roth [1984] and Kojima et al. [2013]. Let H be the set of hospitals, D^1 the set of single medical graduates (who are doctors) and D^2 the set of couples. Each couple $c \in D^2$ is denoted by $c = (f, m)$. For each couple $c \in D^2$ we denote by f_c and m_c the first and second member of c. The set of all doctors, D is given by $D^1 \cup \{m_c | c \in D^2\} \cup \{f_c | c \in D^2\}$.

Each single doctor $s \in D^1$ has a strict preference relation $\succ_s$ over $H \cup \{\emptyset\}$ where $\emptyset$ denotes the outside option for each doctor. If $h \succ_s \emptyset$, we say that hospital h is acceptable for s. Each couple $c \in D^2$ has a strict preference relation $\succ_c$ over $H \cup \{\emptyset\} \times H \cup \{\emptyset\}$, i.e., over pairs of hospitals including the outside option.

Each hospital $h \in H$ has a fixed capacity $k_h > 0$. The preference of a hospital h over subsets of D is summarized by h's choice function $ch_h(.) : 2^D \to 2^D$. While a choice function can be associated with every strict preference ordering over subsets of D, the converse is not true. The information contained in a choice function is sufficient to recover a partial order, only, over the subsets of D. We assume $ch_h(.)$ is responsive. This means that h has a strict priority ordering $\succ_h$ over elements of $D \cup \{\emptyset\}$. If $\emptyset \succ_h d$, we say d is not feasible for h. For any set $D^* \subset D$, hospital h's choice from that subset, $ch_h(D^*)$, consists of the (upto) k_h highest priority doctors among the feasible doctors in D^*. Formally, $d \in ch_h(D^*)$ iff $d \in D^*; d \succ_h \emptyset$ and there exists no set $D' \subset D^* \setminus \{d\}$ such that $|D'| = k_h$ and $d' \succ_h d$ for all $d' \in D'$.

A matching μ in this model describes an assignment of each single doctor to a hospital or his/her outside option, an assignment of couples to at most two positions (in the same or different hospitals) or their outside option, and such that the total number of doctors assigned to any hospital h does not exceed its capacity k_h. Given matching μ, let μ_h denote the subset of doctors matched to h; μ_s and μ_{f_c}, μ_{m_c} denote the position(s) that the single doctor s, and the female member, the male member of the couple c obtain in the matching, respectively.

We say μ is individual rational if $ch_h(\mu_h) = \mu_h$ for any hospital h; $\mu_s \succeq_s \emptyset$ for any single doctor s and $(\mu_{f_c}, \mu_{m_c}) \succeq_c (\emptyset, \mu_{m_c})$; $(\mu_{f_c}, \mu_{m_c}) \succeq_c (\mu_{f_c}, \emptyset)$; $(\mu_{f_c}, \mu_{m_c}) \succeq_c (\emptyset, \emptyset)$ for any couple c.

We list the the ways in which different small coalitions can block a matching μ.

1. A pair $s \in D^1$ and $h \in H$ can block μ if $h \succ_s \mu(s)$ and $s \in ch_h(\mu(h) \cup s)$.

2. A triple $(c, h, h') \in D^2 \times (H \cup \{\emptyset\}) \times (H \cup \{\emptyset\})$ with $h \neq h'$ can block μ if $(h, h') \succ_c \mu(c)$, $f_c \in ch_h(\mu(h) \cup f_c)$ when $h \neq \emptyset$ and $m_c \in ch_{h'}(\mu(h') \cup m_c)$ when $h' \neq \emptyset$.

3. A pair $(c, h) \in D^2 \times H$ can block μ if $(h, h) \succ_c \mu(c)$ and $(f_c, m_c) \subseteq ch_h(\mu(h) \cup c)$.

Notice, restricting attention to blocking by the small coalitions listed above, is, as shown in Roth and Sotomayor [1992] without loss when each hospital's preferences are responsive.

Given preference lists for single doctors and couples; a matching μ is **stable with respect to a capacity vector** k if under the responsive choice functions of hospitals defined above, μ is individually rational and cannot be blocked in any of the three ways listed above.

THEOREM 2.1 *Suppose each doctor in D^1 has a strict preference ordering over the elements of $H \cup \{\emptyset\}$, each couple in D^2 has a strict preference ordering over $H \cup \{\emptyset\} \times H \cup \{\emptyset\}$ and each hospital has responsive preferences. Then, for any reported capacity vector k, there exists a k' and a stable matching with respect to k', such that $\sum_{h \in H} k_h \leq \sum_{h \in H} k'_h$ and* $\max_{h \in H} |k_h - k'_h| \leq 3$.

Theorem 2.1 shows that one can perturb any reported capacity vector up *or* down slightly to guarantee the existence of stable matching.[8] However, in the aggregate we do not decrease the total number of slots.

Theorem 2.1 does not limit the *aggregate* excess demand or supply for positions. For example, if each hospital is over allocated by 3 slots, one would require an additional $3|H|$ positions in total. The next theorem controls for this possibility.

THEOREM 2.2 *Suppose each doctor in D^1 has a strict preference ordering over the elements of $H \cup \{\emptyset\}$, each couple in D^2 has a strict preference ordering over $H \cup \{\emptyset\} \times H \cup \{\emptyset\}$ and each hospital has responsive preferences. Then, for any reported capacity vector k, there exists a k' and a stable matching with respect to k', such that* $\max_{h \in H} |k_h - k'_h| \leq 4$. *Furthermore,* $\sum_{h \in H} k_h \leq \sum_{h \in H} k'_h \leq \sum_{h \in H} k_h + 9$.

Under Theorem 2.1 and 2.2 it is possible for a hospital to be assigned fewer doctors than its reported capacity. However, the condition $\sum_{h \in H} k_h \leq \sum_{h \in H} k'_h$, ensures that in aggregate the matching does not 'burn' positions to ensure stability.

We delay the proofs of Theorem 2.1 and 2.2 to the next section, where we derive more general results.

[8]If we are not permitted to decrease the number of slots in each hospital, we can apply Theorem 2.1 to the capacities $k^*_h := k_h + 3$. In this case we obtain k' satisfying $|k^*_h - k'_h| \leq 3$, which implies $k_h \leq k'_h \leq k_h + 6$, such that a stable matching with respect to k' exists.

3 Matching with Complementaries

We now generalize the couples models. Let A, B represent the sets of agents on the demand and supply side of the market respectively. Associated with each $b \in B$ is a capacity k_b. We allow each agent $a \in A$ to 'consume' a bundle of agents in B. In the context of matching with couples, A would represent the set of doctors and B the set of hospitals. A single doctor $a \in A$ is interested in consuming bundles of agents in B of size at most 1. A couple in A, however is interested in consuming a bundle of size 2, either 2 copies of the same agent in B or two distinct agents in B.

Denote by $\mathbb{N}^{|B|}$ the set of bundles. Any set containing multiple copies of different agents in B is considered to be a bundle. For each $S \in \mathbb{N}^{|B|}$, let S_b denote the number of copies of $b \in B$ contained in S. The *size* of a bundle S is denoted $size(S)$ and $size(S) = \sum_{b \in B} S_b$.

Given $a \in A$ and $S \in \mathbb{N}^{|B|}$, denote by $F_{a,S}$ the finite set of possible ways in which agent a can consume the bundle S. In the matching with couples context, suppose a is a couple and S consists of one copy each of a pair of hospitals. Then, $F_{a,S}$ denotes all possible ways in which each member of the couple a can be assigned to exactly one of the hospitals in S. A *matching* in this general environment will assign to each $a \in A$ a pair (f, S) where $S \in \mathbb{N}^{|B|}$ and $f \in F_{a,S}$. More generally, one can think of $F_{a,S}$ as a set of possible contracts between $a \in A$ and the agents in B that provide resources to form the bundle S, i.e, $\{b \in B | S_b > 0\}$.[9]

For each agent $a \in A$ let $\mathcal{S}_a$ be the set of feasible bundles that a can be assigned to and let $\succ_a$ be her strict preference ordering over $\{f, S\}$; where $S \in \mathcal{S}_a; f \in F_{a,S}$. We assume agent a prefers any of the triples in $\mathcal{S}_a$ to her outside option.

Notice that the formulation above imposes no restriction on the degree of complementarity that agent a's preferences can exhibit. We now restrict it by assuming that each $a \in A$ is only interested in bundles of size at most α. In the couples example, $\alpha = 2$. In other words, the set of feasible bundles $\mathcal{S}_a$ only contains bundles of size at most α.

[9]See Hatfield and Milgrom [2005] for a fuller description.

Let Φ denote the set of all triples, i.e.,

$$\Phi = \{(a, f, S) | a \in A, S \in \mathcal{S}_a, f \in F_{a,S}\}.$$

Given $\mathcal{H} \subset \Phi$, let $\mathcal{H}_a = \{(a, f, S) : (a, f, S) \in \mathcal{H}\}$, that is, $\mathcal{H}_a$ contains all triples in $\mathcal{H}$ that involve a. Similarly, for each agent $b \in B$, let $\mathcal{H}_b$ be the set of all triples whose bundle contains at least one good owned by b, i.e., $\mathcal{H}_b = \{(a, f, S) \in \mathcal{H} | S_b > 0\}$. Similarly,

$$\Phi_a = \{(a, f, S) | S \in \mathcal{S}_a, f \in F_{a,S}\} \text{ is the set of all triples involving } a,$$

and

$$\Phi_b = \{(a, f, S) | S_b > 0\} \text{ is the set of all triples involving } b.$$

Agent b's preference is modeled by a *choice function* $CH_b(.)$. For each $\mathcal{H} \subset \Phi_b$, $CH_b(\mathcal{H})$ returns a subset of $\mathcal{H}$ (that does not violate b's capacity constraint) representing b's choice when the elements in $\mathcal{H}$ are available.[10]

Definition 3.1 *A set of triples $\mathcal{M} \subset \Phi$ is a* **feasible matching with respect to** k *if $|\mathcal{M}_a| \leq 1$ for each agent $a \in A$ and and for every $b \in B$ the matching $\mathcal{M}$ does not allocate more than k_b copies of agent b to agents in A, i.e.,*

$$\sum_{(a,f,S) \in \mathcal{M}_b} S_b \leq k_b.$$

The following is the analog of blocking introduced in Section 2.

Definition 3.2 *A feasible matching $\mathcal{M}$ is* blocked *by a triple $(a, f, S) \notin \mathcal{M}$ if*

(i) $(a, f, S) \succ_a \mathcal{M}_a$,

(ii) for all $b \in B$ such that $S_b > 0$ $(a, f, S) \in CH_b(\mathcal{M}_b \cup \{(a, f, S)\})$.

A feasible matching $\mathcal{M}$ is **stable** *if the following hold:*

[10]We will introduce a specific class of choice function that we consider in this section in Definition 3.4.

1)[Individual rationality] $\forall\, b \in B\ CH_b(\mathcal{M}_b) = \mathcal{M}_b$,

2)[No blocking] *It is not blocked by any triple.*

A stable matching is robust to blocking by a single triple, that is a single agent $a \in A$ and a coalition of agents in B. One might also wish to consider the possibility of blocking by subsets of triples.

DEFINITION 3.3 *Given a feasible matching $\mathcal{M}$, a nonempty, disjoint set of triples $\mathcal{H} \subset \Phi$ are said to be a* blocking coalition *of $\mathcal{M}$ if the following hold:*

(i) For every $a \in A$ $|\mathcal{H}_a| \leq 1$ and if $|\mathcal{H}_a| = 1$, then $\mathcal{H}_a \succ_a \mathcal{M}_a$,

(ii) For every $b \in B$, $\mathcal{H}_b \subset CH_b(\mathcal{M}_b \cup \mathcal{H}_b)$.

A feasible matching $\mathcal{M}$ is **group-stable** *if the following hold:*

1) [Individual rationality] $\forall\, b \in B\ CH_b(\mathcal{M}_b) = \mathcal{M}_b$,

2)[No blocking] *There does not exist a blocking coalition $\mathcal{H}$.*

As in the case of couples, by limiting the preferences of agents in B we can restrict attention to blocking by smaller coalitions. In the remainder of this section we restrict to the class of choice function we call generalized responsive.[11]

DEFINITION 3.4 *A choice function CH_b is* **generalized responsive** *if the following holds. There is a strict 'priority' ordering $\succ^*_b$ over the elements in Φ_b. Given a $\mathcal{H} \subset \Phi_b$, order the elements of $\mathcal{H}$ by $\succ^*_b$. $CH_b(\mathcal{H})$ selects the elements of $\mathcal{H}$ in order of priority as long as doing so is compatible with the elements already selected. In other words, the triple $(a, f, S) \in \mathcal{H}$ will be selected by b if no triple containing a has already been selected or the capacity constraint of b is satisfied.*

Notice that CH_b allows $b \in B$ to express preferences over what agents $b' \in B$ an agent a is matched to. Under generalized responsive preferences, if a matching is not blocked by any one triple, it is not blocked by any coalition of triples. This property follows directly from the definitions above, thus, we have the following claim.

[11] In Section 4 we show that the results of this section continue to hold for a much larger class of preferences.

Claim 3.1 *Assuming generalized responsive preferences for all agents in B, a matching is stable if and only if it is group-stable.*

Proof. Assume $\mathcal{M}$ is stable, but not group-stable. Then, there exists a blocking coalition, $\mathcal{H}$ disjoint from $\mathcal{M}$. Let $(a, f, S) \in \mathcal{H}$, we have:

(i) $(a, f, S) \succ_a \mathcal{M}_a$.

(ii) For every b such that $S_b > 0$, $(a, f, S) \in \mathcal{H}_b \in CH_b(\mathcal{M}_b \cup \mathcal{H}_b)$. Now because $CH_b(.)$ is responsive, this implies $(a, f, S) \in CH_b(\mathcal{M}_b \cup (a, f, S))$.

This shows that (a, f, S) blocks $\mathcal{M}$, contradicting the stability of $\mathcal{M}$. ■

Now we show how the matching problem with couples can be encoded in this general set up. Let A be the set of doctors listed as single or couples, i.e., $A = D^1 \cup D^2$. Let B be the set of hospitals. The main difficulty will be to represent a hospitals b's priority ordering, $\succ_b$, over individual doctors in terms of a priority ordering, $\succ_b^*$, over triples (a, S, f). Fix a hospital $b \in B$ and a triple (a, f, S).

- If a is a single doctor and S a single position in hospital b, then, f is redundant and taken to be a NULL element. Set $(a, f, S)|_b := a$.

- If a represents a couple and S a bundle that contains 1 position at hospital b and 1 position at hospital b', then, f represents which member of the couple is assigned to b and b'. Let $(a, f, S)|_b$ be the member assigned to b .

- If a represents a couple and S a bundle that contains 2 positions at hospital b, then f is redundant and is taken to be a NULL element. Among the two members of a, let $(a, f, S)|_b$ denote the lower ranked of the pair according to h's priority ordering $\succ_h$.

We extend the ordering $\succ_b$ over doctors to an ordering $\succ_b^*$ over triples $(a, f, S) \in \Phi_b$ as follows: given two triples (a, f, S), $(a', f', S') \in \Phi_b$ where $a \neq a'$ we set $(a, f, S) \succ_b^* (a', f', S')$ if $(a, f, S)|_b \succ_b (a', f', S')|_b$. However, here it is possible that indifference occur when we combine the two members. In this case we will use the preference of the couple to break tie. That is, for (a,f,S), $(a,f',S') \in \Phi_b$ and $(a,f,S)|_b = (a,f',S')|_b$ we set $(a,f,S) \succ_b^* (a,f',S')$ if $(a,f,S) \succ_a (a,f',S')$. Under this priory ordering, we obtain the following result.

CLAIM 3.2 (BIRÓ ET AL. [2013]) *Let $\mathcal{M}$ be a stable matching with respect to the priory ordering $\{\succ_b^*\}_{b\in B}$. Then, $\mathcal{M}$ is stable in the couples problem with respect to the ordering $\{\succ_b\}_{b\in B}$.*

Proof. See Appendix A.1.

The converse of Lemma 3.2 is false in the sense that a matching stable with respect $\{\succ_b\}_{b\in B}$ need not be stable with respect to $\{\succ_b^*\}_{b\in B}$.

3.1 Existence of Near Feasible Stable Matches

In the remainder of this section we show the existence of near feasible stable matches. Our main result is the following.

THEOREM 3.1 *Suppose each agent $b \in B$ has generalized responsive preferences, and each agent $a \in A$ is interested in bundles of size at most α. For any capacity vector $\{k_b | b \in B\}$ the following are true:*

- *There exists a capacity vector $\{\bar{k}_b | b \in B\}$ satisfying $\max_{b\in B} |k_b - \bar{k}_b| \le 2\alpha - 1$ and $\sum_{b\in B} k_b \le \sum_{b\in B} \bar{k}_b$, such that a stable matching with respect to $\bar{k}$ exists.*
- *There exists a capacity vector $\{\hat{k}_b | b \in B\}$ satisfying $\max_{b\in B} |k_b - \hat{k}_b| \le 2\alpha$ and $\sum_{b\in B} k_b \le \sum_{b\in B} \hat{k}_b \le \sum_{b\in B} k_b + (2\alpha+1)\alpha - 1$ such that a stable matching with respect to $\hat{k}$ exists.*

Moreover, the near feasible matches identified above can be determined by a finite time algorithm.

COROLLARY 3.1 *Theorems 2.1 and 2.2 are implied by Theorem 3.1.*

Proof. In the residency matching with couples, $\alpha = 2$. Substituting this into Theorem 3.1 and according to Claim 3.2, we obtain Theorems 2.1 and 2.2. ■

Remark. In some applications, the agents in B (i.e. hospitals) are partitioned into different classes that may represent different regions and their regional capacity constraints in addition to capacity constraints for each $b \in B$.[12] The proof of Theorem 3.1 extends to this case. Namely, if there is a partition of B in to regions, then there is a stable matching that allocates at most $(2\alpha + 1)\alpha - 1$ items more than the total reported capacity in *each* region.

The proof of Theorem 3.1 employs Scarf's lemma. To state the lemma we need the following definition.

DEFINITION 3.5 *Let Q be an $n \times m$ nonnegative matrix and $r \in \mathbb{R}^n_+$. Denote $\mathcal{P} = \{x \in \mathbb{R}^m_+ : Qx \leq r\}$. Associated with each row $i \in [n]$ of Q is a strict order $\succ_i$ over the set of columns j for which $q_{i,j} > 0$. A vector $x \in \mathcal{P}$* **dominates** *column j if there exists a row i such that $q_i x = r_i$ and $k \succeq_i j$ for all $k \in [m]$ such that $q_{i,k} > 0$ and $x_k > 0$. Here q_i represents the i^{th} row of matrix Q. In this case, we also say x* **dominates column** *j* **at row** *i.*

We use the following version of Scarf's lemma due to Király and Pap [2008]:

LEMMA 3.1 (SCARF [1967]) *Let Q be an $n \times m$ nonnegative matrix, $r \in \mathbb{R}^n_+$ and $\mathcal{P} = \{x \in \mathbb{R}^m_+ : Qx \leq r\}$. Then, $\mathcal{P}$ has a vertex that dominates every column of Q.*

To apply Scarf's lemma we require a linear inequality representation of the set of matchings $\mathcal{M}$. Let $x_{a,f,S} = 1$ if the bundle S is assigned to a according to $f \in F_{a,S}$ and zero otherwise. To ensure each agent a is assigned at most one bundle:

$$\sum_{(f,S)} x_{a,f,S} \leq 1 \text{ for every agent } a \in A. \qquad (1)$$

[12] See Kamada and Kojima [2014b] for an example.

To ensure we satisfy b's capacity constraint:

$$\sum_{(a,f,S)\in\Phi} S_b \cdot x_{a,f,S} \leq k_b \text{ for every agent } b \in B. \tag{2}$$

Now, relax the condition that $x_{a,f,S} \in \{0,1\}$ to $x_{a,f,S} \geq 0$ and let $Qx \leq r$ be the matrix representation of (2) and (1). Notice, each row corresponds to an element of $A \cup B$ and each column of Q corresponds to a triple (a, f, S). Each row $a \in A$ orders the columns (a, f, S) according to $\succ_a$. Each $a \in B$ orders the columns (a, f, S) according to the priority ordering $\succ_b^*$.

According to Lemma 3.1 there exists a vertex $x^* \geq 0$ that dominates all columns of Q. Thus, for every triple (a^0, f^0, S^0), at least one of the following is true:

1. $\sum_{(f,S)} x^*_{a^0,f,S} = 1$, and there is no triple (a^0, f, S) for which $x^*_{a^0,f,S}$ is positive but $(a^0, f^0, S^0) \succ_{a^0} (a^0, f, S)$

2. There exists $b \in B$, such that S^0 contain at least one copy of b (i.e, $S^0_b \geq 1$) and

$$\sum_{(a,f,S)\in\Phi} S_b \cdot x^*_{a,f,S} = k_b.$$

   Furthermore, b assigns weakly higher priority to all triples $(a, f, S) \in \Phi_b$ such that $x^*_{a,f,S} > 0$ than to (a^0, f^0, S^0).

Therefore, if x^* is an integral vector, then, x^* is a stable matching because the conditions above rule out a blocking triple (a^0, f^0, S^0). Unfortunately, the polytope defined by (1) and (2) is not integral. We take advantage of x^*'s structure to construct a near feasible stable matching. The following observation is important for our results.

Lemma 3.2 *Let $x^* \in \mathcal{P}$ dominate every column of Q. Let $\bar{x}$ be a $\{0,1\}$ vector whose support* *is a subset of x^*'s support, i.e, $x^*_{a,f,S} = 0 \Rightarrow \bar{x}_{a,f,S} = 0$, such that*

$$\left\lfloor \sum_{(f,S)} {x^*}_{a,f,S} \right\rfloor \leq \sum_{(f,S)} \bar{x}_{a,f,S} \leq \left\lceil \sum_{(f,S)} {x^*}_{a,f,S} \right\rceil \quad \forall a \in A. \tag{3}$$

Let $\bar{k}$ be the following capacity vector

$$\bar{k}_b = \begin{cases} \sum_{(a,f,S)\in\Phi} S_b \cdot \bar{x}_{a,f,S} & \text{if } \sum_{(a,f,S)\in\Phi} S_b \cdot x^*_{a,f,S} = k_b \\ k_b, & \text{otherwise.} \end{cases}$$

Then, $\bar{x}$ is a stable matching with respect to $\bar{k}$.

Proof. Using the fact that x^* dominates all columns of Q, we will show that under the new capacity $\bar{k}$, $\bar{x}$ dominates all columns of Q.

Given any triple (a^0, f^0, S^0), x^* dominates (a^0, f^0, S^0) either at a^0 or at a $b \in B$ where $S^0_b > 0$. Suppose first x^* dominates (a^0, f^0, S^0) at a^0. Then, $\sum_{(f,S)} {x^*}_{a,f,S} = 1$, and a^0 weakly prefers all the triples containing a^0 in the support of x^* to (a^0, f^0, S^0). Because $\bar{x}$ is a $0-1$ vector on the support of x^*, a^0 also weakly prefers all the triples containing a^0 in the support of $\bar{x}$ to (a^0, f^0, S^0). However, because $\bar{x}$ is an integral vector rounded from x^*, it is possible that there is no triples containing a^0 in $\bar{x}$. But this cannot be because (3) guarantees that $\sum_{(f,S)} {x^*}_{a,f,S} = 1 \Rightarrow \sum_{(f,S)} \bar{x}_{a,f,S} = 1$. Hence, $\bar{x}$ dominates (a^0, f^0, S^0) at a^0.

Next, suppose x^* dominates (a^0, f^0, S^0) at b. This implies $\sum_{(a,f,S)\in\Phi} S_b \cdot x^*_{a,f,S} = k_b$, and b weakly prefers all triples (a, f, S) in the support of x^* for which S contains at least one copy of b to (a^0, f^0, S^0). Hence, when the capacity at b is $\bar{k}_b$, $\bar{x}$ dominates (a^0, f^0, S^0) at b.

Therefore, when the capacity of b is $\bar{k}_b$, $\bar{x}$ dominates all triples (a, f, S), which shows that the matching corresponding to $\bar{x}$ is a stable matching with respect to the new capacity vector $\bar{k}$. ∎

Lemma 3.2 provides the main ingredient to prove our result. The next lemma shows that a given fractional vector x^* can be rounded into a 0-1 vector $\bar{x}$ whose support is a subset of x^*'s support such that (3) is satisfied and $\bar{k}$ is close to k. Furthermore, this rounding procedure can accommodate aggregate constraints as well.

LEMMA 3.3 *Let x^* be a (fractional) nonnegative vector satisfying*

$$\sum_{(f,S)\in\Phi} x^*_{a,f,S} \leq 1 \text{ for every } a \in A. \tag{4}$$

$$\sum_{(a,f,S)\in\Phi} S_b \cdot x^*_{a,f,S} \leq k_b \text{ for every } b \in B \tag{5}$$

Suppose $x^*_{a,f,S} = 0$ whenever $\sum_{b\in B} S_b > \alpha$, then for **every** *cost vector c the following are true.*

(A): *There exists a $0-1$ vector $\bar{x}$ such that*

$$x^*_{a,f,S} = 0 \Rightarrow \bar{x}_{a,f,S} = 0 \tag{6}$$

$$c \cdot \bar{x} \geq c \cdot x^* \tag{7}$$

$$\left\lfloor \sum_{(f,S)} x^*{}_{a,f,S} \right\rfloor \leq \sum_{(f,S)} \bar{x}_{a,f,S} \leq \left\lceil \sum_{(f,S)} x^*{}_{a,f,S} \right\rceil \quad \forall a \in A \tag{8}$$

$$\left\lfloor \sum_{(a,f,S)\in\Phi} S_b \cdot x^*_{a,f,S} \right\rfloor - 2\alpha + 1 \leq \sum_{(a,f,S)\in\Phi} S_b \cdot \bar{x}_{a,f,S} \leq \left\lceil \sum_{(a,f,S)\in\Phi} S_b \cdot x^*_{a,f,S} \right\rceil + 2\alpha - 1 \quad \forall b \in B \tag{9}$$

(B): *There exists a $0-1$ vector $\hat{x}$ such that*

$$x^*_{a,f,S} = 0 \Rightarrow \hat{x}_{a,f,S} = 0 \tag{10}$$

$$c \cdot \hat{x} \geq c \cdot x^* \tag{11}$$

$$\left\lfloor \sum_{(f,S)} x^*{}_{a,f,S} \right\rfloor \leq \sum_{(f,S)} \hat{x}_{a,f,S} \leq \left\lceil \sum_{(f,S)} x^*{}_{a,f,S} \right\rceil \quad \forall a \in A \tag{12}$$

$$\left\lfloor \sum_{(a,f,S)\in\Phi} S_b \cdot x^*_{a,f,S} \right\rfloor - 2\alpha \leq \sum_{(a,f,S)\in\Phi} S_b \cdot \hat{x}_{a,f,S} \leq \left\lceil \sum_{(a,f,S)\in\Phi} S_b \cdot x^*_{a,f,S} \right\rceil + 2\alpha \quad \forall b \in B \tag{13}$$

$$\sum_{(a,f,S)\in\Phi} \hat{x}_{(a,f,S)} \cdot size(S) \leq \left\lceil \sum_{(a,f,S)\in\Phi} x^*{}_{(a,f,S)} \cdot size(S) \right\rceil + (2\alpha+1)\alpha - 1 \tag{14}$$

Proof. See Appendix B.

Remark. Notice, the $\hat{x}$ obtained in part (B) of Lemma 3.3 satisfies an additional constraint compared to part (A): all agents in A do not consume more than $(2\alpha+1)\alpha-1$ units in excess of the total supply. On the other hand, for each individual $b \in B$, the bound in (13) is slightly worse than the bound in (9) of part (A).

Proof of Theorem 3.1 The proof follows from Lemma 3.2 and 3.3. In particular, starting from the vertex x^* found using Scarf's lemma, we use Lemma 3.3 to obtain $\bar{x}$ and $\hat{x}$ and construct $\bar{k}$ and $\hat{k}$, respectively, according to Lemma 3.2. In fact, the proof is constructive. Scarf [1967] gives a finite algorithm for identifying a dominating vertex in Lemma 3.1. Our Lemma 3.3 is established via a polynomial time algorithm. Therefore, the near feasible matches we identify can be constructed in finite time.

However, Lemma 3.3 does not guarantee that $\sum_b \bar{k}_b \geq \sum_b k_b$ and $\sum_b \hat{k}_b \geq \sum_b k_b$. In particular, in order to apply Lemma 3.3, we need to specify a cost vector c. We will carefully choose c so that those conditions are satisfied.

Given a fractional solution x^* found using Scarf's lemma, let B^* be a subset of B where the capacity constraint for agent $b \in B^*$ binds. That is,

$$\sum_{(a,f,S)\in\Phi} S_b \cdot x^*_{a,f,S} = k_b \text{ for } b \in B^*.$$

Let the cost of the triple (a,f,S), denoted $c_{a,f,S}$, be the total number of items in B^* contained in S, i.e,

$$c_{a,f,S} := \sum_{b\in B^*} S_b.$$

The total cost of x^* is

$$c \cdot x^* = \sum_{(a,f,S)\in\Phi} \sum_{b\in B^*} S_b \cdot x^*_{(a,f,S)} = \sum_{b\in B^*} \sum_{(a,f,S)\in\Phi} S_b \cdot x^*_{(a,f,S)} = \sum_{b\in B^*} k_b.$$

According to Lemma 3.3, the rounding procedure does not reduce the total cost, thus

$$c \cdot \bar{x} \geq c \cdot x^* = \sum_{b \in B^*} k_b \text{ and } c \cdot \hat{x} \geq c \cdot x^* = \sum_{b \in B^*} k_b.$$

Notice, in Lemma 3.2, for $b \in B^*$ $\bar{k}_b = \sum_{(a,f,S) \in \Phi} S_b \cdot \bar{x}_{(a,f,S)}$, therefore,

$$c \cdot \bar{x} = \sum_{(a,f,S) \in \Phi} \sum_{b \in B^*} S_b \cdot \bar{x}_{(a,f,S)} = \sum_{b \in B^*} \sum_{a,f,S \in \Phi} S_b \cdot \bar{x}_{(a,f,S)} = \sum_{b \in B^*} \bar{k}_b.$$

Thus,

$$\sum_{b \in B^*} \bar{k}_b \geq \sum_{b \in B^*} k_b.$$

Furthermore, we do not alter the capacities for agents outside of B^*, i.e., $\bar{k}_b = k_b$ for $b \in B \setminus B^*$. Hence,

$$\sum_{b \in B} \bar{k}_b = \sum_{b \in B^*} \bar{k}_b + \sum_{b \in B \setminus B^*} \bar{k}_b \geq \sum_{b \in B^*} k_b + \sum_{b \in B \setminus B^*} k_b = \sum_{b \in B} k_b.$$

Similarly, for the rounding procedure of $\hat{k}$, we also obtain $\sum_b \hat{k}_b \geq \sum_b k_b$. ∎

4 Optimization Based Choice Function

In this section we introduce a new class of choice functions that subsumes the responsive choice function in Section 3. The example below motivates this class.

Example 1 *There is one hospital with capacity 2, one couple of doctors* (m, f) *and two single doctor* s_1, s_2. *The preferences of the hospital are* $(m, f) \succ s_1 \succ s_2 \succ (m, \emptyset) \succ (\emptyset, f)$. *Here* $(m, f), (m, \emptyset)$ *and* $(\emptyset, f)$ *represent the hospital hiring both members, only the male and only the female, respectively. Given a subset of the available choices, i.e.,* $\mathcal{H} \subset \{(m, f), s_1, s_2, (m, \emptyset), (\emptyset, f)\}$, *what would be reasonable choices of the hospital?*

First of all, under the responsiveness assumption as in Section 3,

$$CH(\{(m,f),s_1,s_2,(m,\emptyset),(\emptyset,f)\}) = \{(m,f)\} \quad \textit{and} \quad CH(\{(m,f),s_1\} = \{(m,f)\}.$$

Now suppose these preferences corresponded to the following cardinal utilities:

$$V((m,f)) = 4; V(s_1) = 3; V(s_2) = 2; V((m,\emptyset)) = 1; V((\emptyset,f)) = 1/2.$$

If utility is additive between s_1, s_2, a more reasonable choice would give

$$CH(\{(m,f),s_1,s_2,(m,\emptyset),(\emptyset,f)\}) = \{s_1,s_2\} \quad \textit{and} \quad CH(\{(m,f),s_1\}) = \{(m,f)\}.$$

The choice function consistent with utility maximization can be represented in the following way. Given a subset of candidates $\mathcal{H} \subset \{(m,f),s_1,s_2,(m,\emptyset),(\emptyset,f)\}$, the hospital's choice, $CH(\mathcal{H})$, is a subset of $\mathcal{H}$ and defined as $\{\tau \in \mathcal{H} : x_\tau = 1\}$, where x is the solution of the following optimization problem.

$$\begin{aligned} \max \quad & \sum_{\tau\in\mathcal{H}} V(\tau)\cdot x_\tau \\ s.t: \quad & \sum_{\tau\in\mathcal{H}} c_\tau \cdot x_\tau \leq 2 \\ & x_{(m,f)} + x_{(m,\emptyset)} + x_{(\emptyset,f)} \leq 1 \\ & x_\tau \in \{0,1\} \end{aligned}$$

Here c_τ is the number of positions needed for τ. In particular, $c_{(m,f)} = 2, c_{s_1} = c_{s_2} = 1$; and $c_{(m,\emptyset)} = c_{(\emptyset,f)} = 1$.

Notice the choice function defined above does not only captures the complementarity between m and f, but also among $(m,f), s_1$ and s_2. In particular, $CH(\{(m,f),s_1,s_2\}) = \{s_1,s_2\}$ and $CH(\{(m,f),s_1\}) = CH(\{(m,f),s_2\}) = \{(m,f)\}$, thus, $CH(.)$ does not satisfy *substitutablity.*[13]

Using the same notation as in Section 3, we define the class of optimization based choice functions for each $b \in B$. First, for a triple $(a, f, S) \in \Phi_b$,[14] let $V^b_{a,f,S} \gneqq 0$ be the utility that b enjoys when a consumes $\{S, f\}$.

DEFINITION 4.1 *Given a subset $\mathcal{H} \subset \Phi_b$, an* **optimization based** *choice function, $CH_b(\mathcal{H})$, is defined as $\{(a, f, S) \in \mathcal{H} | x_{a,f,S} = 1\}$, where $x_{a,f,S} \in \{0, 1\}$ is the solution of the following problem.*

$$
\begin{aligned}
\max \quad & \sum_{(a,f,S)\in\mathcal{H}} V^b_{a,f,S} \cdot x_{a,f,S} \\
s.t: \quad & \sum_{(f,S):(a,f,S)\in\mathcal{H}} x_{a,f,S} \leq 1 \ \forall \ a \in A \\
& \sum_{(a,f,S)\in\mathcal{H}} x_{a,f,S} \cdot S_b \leq k_b \\
& x_{a,f,S} \in \{0, 1\}
\end{aligned}
$$

If there are multiple solutions, we break ties lexicographically. Choose the first solution in a lexicographical order for a fixed linear order $\succ^{tie}_b$ over Φ_b.

It can be shown that generalized responsive preferences are a special case of optimization based choice functions. Optimization-based choice functions can encode knapsack problem, thus given $\mathcal{H}$, finding $CH(\mathcal{H})$ is an $\mathcal{NP}$-hard problem. In the following we give a necessary condition to check if $\mathcal{H}^* = CH(\mathcal{H})$, which we use later.

LEMMA 4.1 *Given $\mathcal{H}^* \subset \mathcal{H} \subset \Phi_b$ such that for every a there is at most 1 triple $(a, f^*, S^*) \in \mathcal{H}^*$, i.e, $|\mathcal{H}^*_a| \leq 1$, furthermore, $\sum_{(a^*,f^*,S^*)\in\mathcal{H}^*} S^*_b = k_b$; in addition, for every $(a^*, f^*, S^*) \in$*

[13] A choice function models substitute preference if $x \in CH(H)$ and $x \neq y \in H$, then $x \in C(H \setminus y)$. See Hatfield and Kominers [2012], for example, for a discussion.

[14] Recall that Φ_b contains all the triples, for which the third member, S, contains at least 1 copy of the good owned by b.

$\mathcal{H}^*$ *and every* $(a, f, S) \in \mathcal{H} \setminus \mathcal{H}^*$

$$\frac{V^b_{(a^*,f^*,S^*)}}{S^*_b} > \frac{V^b_{(a,f,S)}}{S_b}$$

or

$$\frac{V^b_{(a^*,f^*,S^*)}}{S^*_b} = \frac{V^b_{(a,f,S)}}{S_b} \text{ and } (a^*, f^*, S^*) \succ^{tie}_b (a, f, S),$$

then $CH_b(\mathcal{H}) = \mathcal{H}^*$.

Proof. Appendix A.2

We introduce a stability concept stronger than stable, but weaker than group-stable, as defined in Definition 3.2 and 3.3.

DEFINITION 4.2 *Given a feasible matching* $\mathcal{M}$*, an agent* $b \in B$ *and a nonempty set of triples* $\mathcal{H} \subset \Phi_b$ *are said to be a* blocking coalition *of* $\mathcal{M}$ *if the following conditions hold*

1) $\mathcal{H} \subset CH_b(\mathcal{M}_b \cup \mathcal{H})$

2) For all $(a, f, S) \in \mathcal{H}$*:* $(a, f, S) \succ_a \mathcal{M}_a$

3) For all $(a, f, S) \in \mathcal{H}$ *and all* b' *such that* $S_{b'} > 0$*:* $(a, f, S) \in CH_{b'}(\mathcal{M}_{b'} \cup \{(a, f, S)\})$.

A feasible matching $\mathcal{M}$ *is* **strongly stable** *if the following conditions hold:*

1) [Individual rationality] $\forall\, b \in B$, $CH_b(\mathcal{M}_b) = \mathcal{M}_b$.

2)[No blocking] There does not exist a blocking coalition $(b, \mathcal{H})$.

Strong stability differs from stability in that considers blocking by subsets of triples. However, unlike group-stability, the coalition of triples must share a common agent in B. The interpretation of such coalition is that the agent b is the "initiating" blocker, and other agents $b' \in B$ involved with the potential deviation of agents in $\mathcal{H}$ are passive. Thus, strong stability allows certain joint deviations to happen, but rules out deviations more complex involving larger groups. [15]

[15] In the application of matching with couples our notion of strong stability is similar, but slightly stronger than the one defined in Kojima et al. [2013].

Under optimization based preferences, stable matches are not necessarily strongly stable. The example below illustrates this.

EXAMPLE 2 *There are two hospitals h_1 and h_2, each with 2 positions. There is one couple (m, f) and two single doctors s_1, s_2. Both s_1, s_2 prefer h_1 over h_2. The couple's preference ordering is*

$$(h_1, h_1) \succ_{mf} (h_2, h_2) \succ_{mf} (h_1, h_2) \succ_{mf} (h_2, h_1) \succ_{mf}$$

$$(h_1, \emptyset) \succ_{mf} (h_2, \emptyset) \succ_{mf} (\emptyset, h_1) \succ_{mf} (\emptyset, h_2) \succ_{mf} (\emptyset, \emptyset).$$

The choice function of the hospital h_1 is defined as in example 1. Hospital 2's has generalized responsive preferences (as defined in Definition 3.4), where the preference order is

$$s_1 \succ_{h2} s_2 \succ_{h2} (m, f) \succ_{h2} (m, \emptyset) \succ_{h2} (\emptyset, f).$$

Consider the matching in which (m, f) is matched with h_1 and s_1, s_2 are matched with h_2. This matching is unblocked but not stable. In particular, if both s_1, s_2 would like to switch to h_1, then h_1 would choose s_1, s_2 over (m, f). However if only one of the agents s_1 or s_2 would like to switch to h_1, hospital h_1 still prefers the couple.

Example 2 shows that by generalizing the choice function we further restrict the set of stable matches. Nevertheless, if we relax the capacity constraints slightly, strongly stable matchings still exist.

THEOREM 4.1 *If each agent $b \in B$ has an optimization based choice function and each agent $a \in A$ is interested of bundles of size at most α, then for any capacity vector $\{k_b | b \in B\}$ the following is true.*

- *There exists a capacity vector $\{\bar{k}_b | b \in B\}$ satisfying $\max_{b\in B} |k_b - \bar{k}_b| \leq 2\alpha - 1$ and $\sum_b k_b \leq \sum_b \bar{k}_b$ such that a strongly stable matching with respect to $\bar{k}$ exists.*

- *There exists a capacity vector* $\{\hat{k}_b | b \in B\}$ *satisfying* $\max_{b\in B} |k_b - \hat{k}_b| \leq 2\alpha$ *and* $\sum_b k_b \leq \sum_b \hat{k}_b \leq \sum_b k_b + (2\alpha+1)\alpha - 1$ *such that a strongly stable matching with respect to* $\hat{k}$ *exists.*

Proof. We use the optimization based choice function of each $b \in B$ to induce a priority ordering over Φ_b for all $b \in B$. Subsequently, we invoke Scarf's lemma to obtain a fractional dominating solution x^* and use the rounding Lemma 3.3 to get an approximate stable matching as in the proof of Theorem 3.1.

Given $(a, f, S), (a', f', S') \in \Phi_b$ (therefore $S_b, S'_b > 0$), we define the priority order $\succ_b$ as follows:

$$\text{If } \frac{V^b_{(a,f,S)}}{S_b} > \frac{V^b_{(a',f',S')}}{S'_b}, \text{ then } (a,f,S) \succ_b (a',f',S').$$
$$\text{If } \frac{V^b_{(a,f,S)}}{S_b} = \frac{V^b_{(a',f',S')}}{S'_b} \text{ and } (a,f,S) \succ^{tie}_b (a',f',S'), \text{ then } (a,f,S) \succ_b (a',f',S'),$$

where $\succ^{tie}_b$ is a fixed order that is used for a tie-breaking rule in Definition 4.1. For simplicity, in the remaining of the proof, we assume $V^b_{a,f,S}$ are generic, and thus, we do not need to use the tie-breaking rule. The proof easily extends to include such tie-breaking conditions.

Consider

$$\sum_{(f,S)} x_{a,f,S} \leq 1 \text{ for every agent } a \in A$$
$$\sum_{(a,f,S)\in\Phi} S_b \cdot x_{a,f,S} \leq k_b \text{ for every agent } b \in B$$
$$x_{a,f,S} \geq 0.$$

Scarf's lemma tells us that the linear system above has a dominating vertex x^* with respect to $\{\succ_a, a \in A\}$ and $\{\succ_b, b \in B\}$ as defined above.

We need the following lemma.

Lemma 4.2 *Let $\bar{x}$ be a $\{0,1\}$ vector whose support is a subset of x^*'s support, i.e, $x^*_{a,f,S} = 0 \Rightarrow \bar{x}_{a,f,S} = 0$, such that*

$$\left\lfloor \sum_{(f,S)} {x^*}_{a,f,S} \right\rfloor \leq \sum_{(f,S)} \bar{x}_{a,f,S} \leq \left\lceil \sum_{(f,S)} {x^*}_{a,f,S} \right\rceil \quad \forall a \in A. \tag{15}$$

Let $\bar{k}$ be the following capacity vector

$$\bar{k}_b = \begin{cases} \sum_{(a,f,S)\in\Phi} S_b \cdot \bar{x}_{a,f,S} & \text{if } \sum_{(a,f,S)\in\Phi} S_b \cdot x^*_{a,f,S} = k_b \\ k_b, & \text{otherwise.} \end{cases}$$

If each agent in B has an optimization based choice function, $\bar{x}$ is a feasible matching with respect to $\bar{k}$ that is strongly stable.

Once Lemma 4.2 is proved, we can again use Lemma 3.3 to show the existence of an approximate stable matching as in Theorem 3.1. Thus, it remains to show Lemma 4.2.

Proof of Lemma 4.2 The proof of Lemma 4.2 is similar to that of Lemma 3.2. However, in Lemma 3.2 preferences of $b \in B$ are assumed to be responsive. Thus, according to Claim 3.1 it was enough to check that $\bar{x}$ is not blocked by any triple. Here, because of optimization-based choice functions, we will need a more elaborate argument.

Let $\mathcal{M}$ be the matching corresponding to $\bar{x}$, that is $\mathcal{M} = \{(a,f,S) : \bar{x}_{a,f,S} = 1\}$. If $\mathcal{M}$ is not strongly stable, there exists a blocking coalition $(b, \mathcal{H})$, where $b \in B$ and non-empty $\mathcal{H} \subset \Phi_b$ disjoint from $\mathcal{M}$ such that

(i) $\mathcal{H} \subset CH_b(\mathcal{M}_b \cup \mathcal{H})$.

(ii) $(a, f, S) \succ_a \mathcal{M}_a$ for all $(a, f, S) \in \mathcal{H}$, and

(iii) for all $(a, f, S) \in \mathcal{H}$ as well as $b \neq b' \in B$ such that $S_{b'} > 0$: $(a, f, S) \in CH_{b'}(\mathcal{M}_{b'} \cup (a, f, S))$.

We first show that if (ii) and (iii) hold then x^* dominates all triples in $\mathcal{H}$ at b.

Now (ii) implies that x^* cannot dominate (a, f, S) at a for any $(a, f, S) \in \mathcal{H}$. This is true because if x^* dominates (a, f, S) at a, then $\sum_{f',S'} x^*_{a,f',S'} = 1$ and for all $x^*_{a,f',S'} > 0$: $(a, f', S') \succ_a (a, f, S)$ which implies that $\sum_{f',S'} \bar{x}_{a,f',S'} = 1$, and thus $\emptyset \neq \mathcal{M}_a \succ_a (a, f, S)$, contradicting (ii).

Similarly, because of (iii) x^* cannot dominate (a, f, S) at b' for any $b' \neq b$ such that $S_{b'} > 0$. To see why, assume not. Then, $\sum_{a',f',S' \in \Phi} S'_{b'} \cdot x^*_{a',f',S'} = k_{b'}$ and for all $x^*_{a',f',S'} > 0, S_{b'} > 0$: $(a', f', S') \succ_{b'} (a, f, S)$. This implies

$$\sum_{(a',f',S') \in \mathcal{M}} S'_{b'} = \bar{k}_{b'} \ \forall \ (a', f', S') \succ_{b'} (a, f, S) \text{ for all} (a', f', S') \in \mathcal{M}_{b'}.$$

Thus

$$\frac{V^b_{(a',f',S')}}{S'_{b'}} > \frac{V^b_{(a,f,S)}}{S_{b'}} \ \forall \ (a', f', S') \in \mathcal{M}_{b'}.$$

However, according to Lemma 4.1 $(a, f, S) \notin CH_{b'}(\mathcal{M}_{b'} \cup (a, f, S))$, which contradicts (iii).

Now, because x^* dominates all triples in $\mathcal{H}$, x^* dominates $\mathcal{H}$ at b. This implies

$$\sum_{(a,f,S) \in \Phi} S_b \cdot x^*_{a,f,S} = k_b \tag{16}$$

$$(a', f', S') \succ_b (a'', f'', S'') \ \ \forall (a', f', S') \in \Phi_b \text{ with } x^*_{a',f',S'} > 0 \text{ and } \forall (a'', f'', S'') \in \mathcal{H}. \tag{17}$$

From (16), (17), we prove that $\mathcal{H} \not\subset CH_b(\mathcal{M}_b \cup \mathcal{H})$, contradicting (i), which proves the Lemma.

Because the matching $\mathcal{M}$ is on the support of x^*, if (17) is true then

$$(a', f', S') \succ_b (a'', f'', S'') \ \ \forall (a', f', S') \in \mathcal{M}_b \text{ and } \forall (a'', f'', S'') \in \mathcal{H}.$$

This implies

$$\frac{V^b_{(a',f',S')}}{S'_b} > \frac{V^b_{(a'',f'',S'')}}{S''_b} \ \ \forall (a', f', S') \in \mathcal{M}_b \text{ and } \forall (a'', f'', S'') \in \mathcal{H}. \tag{18}$$

Furthermore because of (16), and the definition of $\bar{k}$, we have

$$\sum_{(a,f,S)\in\mathcal{M}} S_b = \bar{k}_b. \tag{19}$$

From (18) and (19) if the capacity of b is $\bar{k}_b$, and the choice function of b is defined as in Definition 4.1 then according to Lemma 4.1, $CH_b(\mathcal{M}_b \cup \mathcal{H}) = \mathcal{M}_b$. This shows the contradiction with (i). ■

5 Matching with Side Constraints

In this section we show how our methods can be applied to matching problems with side constraints. To illustrate, we consider a residency matching problem with regional caps, and will show how the approach extends to other types of side constraints. The matching problem with regional caps was identified by Kamada and Kojima [2014b] and is motivated by a need to ensure a sufficiency of medical residents in less 'attractive' locations. We discuss it because the obstacle to existence of a stable matching here does not arise from complementarity in preferences but the interaction between the regional capacity constraints and the capacity constraints of each hospital.

We describe the model of Kamada and Kojima [2014b] in our notation. As before let A be the set of doctors, B the set of hospitals; k_b is the capacity of hospital b. Denote by $\{\succ_a\}_{a\in A}$ the preferences of doctors over $B \cup \{\emptyset\}$. Note, there are no couples.

Denote by $\{\succ_b\}_{b\in B}$ the preferences of hospitals over $A \cup \{\emptyset\}$.[16] Hospitals preferences are assumed to be responsive.

There is a partition of the set of hospitals $B = B_1 \biguplus \cdots \biguplus B_p$. Each element of the partition B_i corresponds to a region and has a capacity K_i. For each $b \in B$ let $\pi(b)$ be the index of the region that b is in, that is, $b \in B_{\pi(b)}$.

[16] If $\emptyset \succ_a b$ then b is not feasible for a; if $\emptyset \succ_b a$ then a is not feasible for b.

If $\mathcal{M} \subset \{A \times B\}$ let $\mathcal{M}_{a^0} = \{b|(a^0, b) \in \mathcal{M}\}$; $\mathcal{M}_{b^0} = \{a|(a, b^0) \in \mathcal{M}\}$.

Definition 5.1 *$\mathcal{M}$ is a* **feasible matching** *if :*

$$\mathcal{M} \subset \{(a, b)|a \in A; \in B; b \succ_a \emptyset; a \succ_b \emptyset\}$$

$$|\mathcal{M}_a| \leq 1 \; \forall a \in A; |\mathcal{M}_b| \leq k_b \; \forall b \in B; \sum_{b \in B_i} |\mathcal{M}_b| \leq K_i \; \forall i \in [p]$$

Strong stability in this setting is defined as follows.[17]

Definition 5.2 *A doctor-hospital pair (a, b)* **strongly blocks** *a feasible matching $\mathcal{M}$ if:*

i. $b \succ_a \mathcal{M}_a$

ii. $a \in CH_b(\mathcal{M}_b \cup \{a\})$

iii. If under $\mathcal{M}$, a is currently unmatched, or matched outside region $\pi(b)$, then $\sum_{b'} \in B_{\pi(b)} |\mathcal{M}_{b'}| \leq K_{\pi(b)} - 1$.

A feasible matching $\mathcal{M}$ is **strongly stable** *if there is no strong blocking pair.*

Kamada and Kojima [2014b] show that a strongly stable matching need not exist in this setting. Furthermore, a mechanism that chooses a strongly stable match when it exists will not be strategy proof. For this reason they propose the following relaxation:

Definition 5.3 *A doctor-hospital pair (a, b)* **weakly blocks** *a feasible matching $\mathcal{M}$ if:*

i. a prefers b to the current hospital $\mathcal{M}_a$: $b \succ_a \mathcal{M}_a$

ii. $a \in CH_b(\mathcal{M}_b \cup \{a\})$

iii. the cap for the region of the hospital is not full: $\sum_{b'} \in B_{\pi(b)} |\mathcal{M}_{b'}| \leq K_{\pi(b)} - 1$.

A feasible matching $\mathcal{M}$ is **weakly stable** *if there is no weak blocking pair.*

A weakly stable matching $\mathcal{M}$ allows for a blocking pair (a, b) where b is in the same region as the hospital, $\mathcal{M}_a$, that a is currently matched with. A strongly stable matching rules out such blocking pairs. Here we show the existence of a near feasible strongly stable matching.

[17] Strong stability in this section is defined as in Kamada and Kojima [2014b] and is different from that notion defined in Section 4.

- For each region i, introduce a dummy hospital r_i with capacity K_i.
- Extend each doctor-hospital pair (a, b) to a triple $(a, \{b, r_{\pi(b)}\})$ that is $\{b, r_{\pi(b)}\}$ is the bundle that a consumes.
- Extend the preferences of agents in A and B over the triples $(a, \{b, r_{\pi(b)}\})$ in the obvious way.
- Endow each dummy hospital r_i with responsive preferences. The underlying priority ordering is constructed as follows. For each $a \in A$, order the elements $\{(a, \{b, r_i\}) | b \in B_i; a \succ_b \emptyset; b \succ_a \emptyset\}$ according to $\succ_a$. Subsequently, concatenate these lists.

There is a one-to-one correspondence between the matching defined in Definition 5.1 and the matching in our general framework. Given a feasible matching $\mathcal{M}$ let $\hat{\mathcal{M}}$ be the corresponding match involving triples $(a, \{b, r_{\pi(b)}\})$.

CLAIM 5.1 *If $\hat{\mathcal{M}}$ is a stable matching in this framework with responsive preferences, then its corresponding matching $\mathcal{M}$ is a strongly stable as in Definition 5.2.*

Proof. Given a stable matching $\hat{\mathcal{M}}$ in the generalized framework, assume for a contradiction that its corresponding $\mathcal{M}$ is not strongly stable as in Definition 5.2.

Let (a, b) be a strong blocking pair. We show $(a, \{b, r_{\pi(b)}\})$ is a blocking triple for $\hat{\mathcal{M}}$. This is true because the preferences of a and b do not change, therefore both a, b will 'accept' $(a, \{b, r_{\pi(b)}\})$.

Now, if under matching $\mathcal{M}$ region $\pi(b)$'s capacity is not full, then so is the hospital $r_{\pi(b)}$ under $\hat{\mathcal{M}}$. Thus, $r_{\pi(b)}$ can continue to accept $(a, \{b, r_{\pi(b)}\})$.

If under matching $\mathcal{M}$ region $\pi(b)$ is at capacity, then in order for (a, b) to be a strong blocking pair of $\mathcal{M}$, a's current match, b^* must be in the same region as b, that is $b^* = \mathcal{M}_a$ and $\pi(b) = \pi(b^*)$. Because a's preferences are aligned and $r_{\pi(b)}$'s, $r_{\pi(b)}$ will also prefers $(a, \{b, r_{\pi(b)}\})$ to $(a, \{b^*, r_{\pi(b)}\})$. However, $r_{\pi(b)}$'s preferences are responsive, $r_{\pi(b)}$ will reject the latter to accept the former.

This shows that $(a, \{b, r_{\pi(b)}\})$ is a blocking triple for $\hat{\mathcal{M}}$, a contradiction. ■

From this and Theorem 3.1 we obtain:

THEOREM 5.1 *Given a hospital capacity vector k and a regional capacity vector K; there exists a strongly stable matching with respect to capacity vectors (k', K') such that* $\max_b |k_b - k'_b| \leq 3$ *and* $\max_i |K_i - K'_i| \leq 3$.

The regional cap example shows how one may handle other types of side constraints with upper quotas. For example, limits on numbers enrolled in particular specializations, or numbers with certain characteristics (gender, race) admitted. One simply introduces dummy hospitals for each such side constraint, and extends the bundle in the appropriate way. For example, if in addition to the cap constraints above, a constraint on gender is also considered, then add two dummy hospital m (male) and f (female) and extend $(a, \{b, r_{\pi(b)}\})$ to either $(a, \{b, r_{\pi(b)}, m\})$ or $(a, \{b, r_{\pi(b)}, f\})$ depending on the gender of a. Notice, unlike Budish et al. [2013] and Kamada and Kojima [2014a], we do not need to impose the hierarchical structure on these side constraints. If the number of such constraints is small relative to the size of the market, the 'bundles' will be small and we will obtain near feasible stable matchings.

In the residency matching problem with regional caps, hospitals are divided into disjoint set of regions. This additional structure allows us to get better handle on which hospital's capacity needs to be modified.

Define a hospital to be either urban or rural depending on the relation of the cap constraint with the total capacities of all hospitals in the region. Call a region i urban if $K_i < \sum_{b \in B_i} k_b$ and rural otherwise.[18]Hospitals in urban and rural regions are called urban and rural, respectively.

We show that only the capacities of the urban hospitals and the regions that contain them need to be modified to guarantee the existence of a strongly stable match. Thus, this

[18]Kamada and Kojima [2014b] provides data for the cap constraints in the Japanese Residency Matching Program. For major cities such as Tokyo and Osaka the government imposes a cap constraint that is significantly smaller than the total available positions in these areas.

might suggest a method for a better implementation of the basic idea behind the Japanese Residency Matching Program, which introduces cap constraints for urban regions to ensure a sufficiency of medical residents in promote areas. Namely, the unavoidable instability of a matching caused by strict regions cap constraints can be avoided with more flexible constraints.

THEOREM 5.2 *Given a hospital capacity vector k and a regional capacity vector K; there exists a strongly stable matching with respect to capacity vectors (k', K') such that* $\max_b |k_b - k'_b| \leq 3$ *and* $\max_i |K_i - K'_i| \leq 3$*, furthermore* $k'_b = k_b$ *for all rural hospitals.*

Proof. See appendix C.

6 Conclusion

A key goal in the design of centralized matching markets is to eliminate the incentive for participants to contract outside of the market. This is formalized as stability and is considered crucial for the long-term sustainability of a market. In the presence of complementarities, stable matchings need not exist and limits the applicability of centralized matching. Others have responded to this challenge by weakening the notion of stability. We instead, weaken 'feasibility' and establish the existence of near feasible stable matchings in the presence of complementarities.

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Appendix

A Omitted Proofs

A.1 Proof of Claim 3.2

The proof is by contradiction. Let $\mathcal{M}$ be a stable matching in the general set-up, assume the corresponding assignment μ in the residency matching with couples is not stable. This means that at least one of the three items below obtains.

1. A pair $s \in D^1$ and $h \in H$ can block μ if $h \succ_s \mu(s)$ and $s \in ch_h(\mu(h) \cup s)$.
2. A triple $(c, h, h') \in D^2 \times H \times H$ with $h \neq h'$ can block μ if $(h, h') \succ_c \mu(c)$, $f_c \in ch_h(\mu(h) \cup f_c)$ and $m_c \in ch_{h'}(\mu(h') \cup m_c)$.
3. A pair $(c, h) \in D^2 \times H$ can block μ if $(h, h) \succ_c \mu(c)$ and $(f_c, m_c) \subseteq ch_h(\mu(h) \cup \{f_c, m_c\})$.

The first type of blocking coalition corresponds to a triple (a, f, S) where a is a single student s, f is redundant (NULL), and S is a single slot in the hospital h. We need to show that if $s \in ch_h(\mu(h) \cup s)$, then in the general setup $(a, f, S) \in CH_h(\mathcal{M}_h \cup (a, f, S))$.

Now, because $ch_h(.)$ is a responsive choice function over *individual* doctors, $s \in ch_h(\mu(h) \cup s)$ implies that s is among the best k_h candidates among $\mu(h) \cup s$. Therefore, even when some other members of $\mu(h)$ form pairs they cannot improve their rank in the new $\succ_b^*$ order, because $\succ_b^*$ ranks these couples according to their worst member. Hence, the corresponding triple (a, f, S) would still be selected in the choice function, that is $(a, f, S) \in CH_h(\mathcal{M}_h \cup (a, f, S))$.

For the second type of blocking coalition, the proof is exactly the same as above.

In the third type of blocking coalition, the pair (f_c, m_c) and a hospital h correspond to (a, f, S), where a represents the couple, S contains 2 positions in h, and f is NULL. Because $(f_c, m_c) \subseteq ch_h(\mu(h) \cup c)$, *both* f_c and m_c are among the k_h best candidates, thus even when represented by the triple (a, f, S), both members are still ranked highly among $\mathcal{M}_h \cup (a, f, S)$. Hence, if CH_h is generalized responsive, (a, f, S) will be selected, that is $(a, f, S) \in CH_h(\mathcal{M}_h \cup (a, f, S))$. This shows that if $(f_c, m_c), h$ blocks a matching μ, then (a, f, S) blocks $\mathcal{M}$. ∎

A.2 Proof of Lemma 4.1

According to Definition 4.1, $CH_b(\mathcal{H}) = \{(a, f, S) \in \mathcal{H} | x_{a,f,S} = 1\}$, where $x_{a,f,S} \in \{0, 1\}$ is the solution of the following problem.

$$\begin{aligned} \max \quad & \sum_{(a,f,S)\in\mathcal{H}} V^b_{a,f,S} \cdot x_{a,f,S} \\ s.t: \quad & \sum_{(f,S):(a,f,S)\in\mathcal{H}} x_{a,f,S} \leq 1 \ \forall \ a \in A \\ & \sum_{(a,f,S)\in\mathcal{H}} x_{a,f,S} \cdot S_b \leq k_b \\ & x_{a,f,S} \in \{0, 1\} \end{aligned} \tag{20}$$

If there are multiple solutions, we break ties lexicographically. Choose the first solution in a lexicographical order for a fixed linear order $\succ_b^{tie}$ over Φ_b.

Let

$$\beta = \min_{(a^*,f^*,S^*)\in\mathcal{H}^*} \frac{V^b_{(a^*,f^*,S^*)}}{S^*_b} > 0$$

and

$$\lambda_{a,f,S} := \min\{0, V^b_{(a,f,S)} - \beta S_b\}$$

Let x^* be the solution corresponding to $\mathcal{H}^*$, that is $x^*_{a,f,S} = 1$ iff $(a, f, S) \in \mathcal{H}^*$. It is straightforward to check that the primal solution x^* and the dual (λ, β) satisfy the complementary slackness condition of the linear relaxation of (20), where we replace $x_{a,f,S} \in \{0, 1\}$ by $0 \leq x_{a,f,S} \leq 1$. Thus, x^* is the optimal solution of the above optimization problem.

It remains to verify that among all optimal solutions, $\mathcal{H}^*$ is the first according to the tie breaking rule $\succ_b^{tie}$. First, if $\frac{V^b_{(a,f,S)}}{S_b} > \beta$ then (a, f, S) is in all optimal solutions of (20). Second, if $\frac{V^b_{(a,f,S)}}{S_b} < \beta$, then (a, f, S) cannot be in any optimal solution of (20). Third, for an $(a, f, S) \in \mathcal{H} \backslash \mathcal{H}^*$ such that $\frac{V^b_{(a,f,S)}}{S_b} = \beta$, by the assumption, we know that if $(a^*, f^*, S^*) \in \mathcal{H}^*$ such that

$$\frac{V^b_{(a^*,f^*,S^*)}}{S^*_b} = \frac{V^b_{(a,f,S)}}{S_b},$$

then, $(a^*, f^*, S^*) \succ_b^{tie} (a, f, S)$. Therefore, $\mathcal{H}^* = CH_b(\mathcal{H})$. ∎

B Rounding Technique: Proof of Lemma 3.3

B.1 Iterative Rounding

The proof is based on a combinatorial optimization method called iterative rounding. The iterative rounding algorithm tries to identify an $x \in \arg\max\{c \cdot x : Dx \leq d, x \geq 0\}$ that is integral. Choose an $x^{opt} \in \arg\max\{c \cdot x : Dx \leq d, x \geq 0\}$. If it is integral, we are done. If not, the iterative rounding method will eliminate one or more constraints and resolve the linear program.

The algorithm starts from a given $x^* \geq 0$ such that $Dx^* \leq d$, and executes the following steps.

Step 0: Initiate $x^{opt} := x^*$.

Step 1: If x^{opt} is integral, stop and output x^{opt}, otherwise continue to either Step 2a or 2b.

Step 2a: If any coordinate of x^{opt} is integral, fix the value of those coordinates, and update the linear program.

To describe the updated linear program, let C and $\overline{C}$ be the set of columns of D that correspond to the non-integer and integer valued coordinates of x^{opt}, respectively. Let D_C, and $D_{\overline{C}}$ be the sub-matrix of D that consists of columns in C and the complement $\overline{C}$, respectively. Similarly, for a vector x, let x_C and $x_{\overline{C}}$ be the sub-vector of x that consists of all coordinates in C and $\overline{C}$.

The updated linear program is:

$$\begin{aligned} \max \quad & c_C \cdot x_C \\ \text{s.t} \quad & D_C \cdot x_C \leq d - D_{\overline{C}} \cdot x_{\overline{C}}^{opt}. \end{aligned}$$

In other words, we replace c by c_C; x by x_C; D by D_C and d by $d - D_{\overline{C}} \cdot x_{\overline{C}}^{opt}$, and move to Step 3.

Step 2b If all coordinates of x^{opt} are fractional, then delete *certain* rows of D (to be specified later) and the corresponding constraints from the linear program. Update the linear program, move to Step 3.

Step 3 Solve the updated linear program $\{\max c \cdot x \text{ subject to } Dx \leq d\}$ to get an extreme point solution. Let this be the new x^{opt} and return to Step 1.

LEMMA B.1 *Assume that whenever the algorithm passes Step 1 and has not terminated, it can either enter Step 2a, or will find at least one row of the current D to delete in Step 2b. Then, the algorithm will terminate in a finite number of steps and output a 0-1 vector. Furthermore, if x^{OUT} is the output, then, $c \cdot x^{OUT} \geq c \cdot x^*$.*

Proof. In Step 2a, we fix at least one coordinate and update the linear program, thus at least one column of the matrix is eliminated. In Step 2b, on the other hand, we delete at least one row. D is a finite matrix. Thus, the algorithm can only execute Step 2a and Step 2b a finite number of times. Therefore, if the assumption in Lemma B.1 holds, then, the algorithm has to terminate.

Observe that after each iteration of the algorithm, we eliminate some constraints and resolve the linear program, thus the objective function cannot decrease.

Therefore, $c \cdot x^{OUT} \geq c \cdot x^*$. ■

To prove Lemma 3.3, given the fractional vector x^* we will set up two linear programs corresponding to part A and part B of Lemma 3.3. We first show that x^* is a feasible solution of each of these programs. Then, apply the iterative rounding algorithm that uses certain row elimination rules for Step 2b. We describe these rules later.

In order to show that the iterative rounding algorithm will terminate, we will need the following result.

LEMMA B.2 *Given nonnegative integers* $\underline{d}_a \leq \bar{d}_a$ *for all* $a \in A$,

$$\mathcal{F} = \{x : \underline{d}_a \leq \sum_{(f,S)} x_{a,f,S} \leq \bar{d}_a \text{ for all } a \in A\}$$

is an integral polytope. That is for any c, *the linear program* $\max_{x \in \mathcal{P}} c \cdot x$ *has an integral optimal solution.*

Proof. Each variable appears with a non-zero coefficient in at most one constraint. The non-zero coefficient has the value 1, making the constraint matrix totally unimodular. ■

B.2 Proof of part A of Lemma 3.3:

Consider the following program.

$$\text{PROGRAM A: } \max c \cdot x \text{ such that :}$$

$$x_{a,f,S} = 0 \text{ if } x^*_{a,f,S} = 0$$

$$x_{a,f,S} \leq 1$$

$$\left\lfloor \sum_{(f,S)} x^*{}_{a,f,S} \right\rfloor \leq \sum_{(f,S)} x_{a,f,S} \leq \left\lceil \sum_{(f,S)} x^*{}_{a,f,S} \right\rceil \quad \forall a \in A \tag{A1}$$

$$\sum_{(a,f,S)} S_b \cdot x_{a,f,S} \leq \left\lceil \sum_{(a,f,S)} S_b \cdot x^*_{a,f,S} \right\rceil \ \forall b \in B \tag{A2}$$

$$\sum_{(a,f,S)} S_b \cdot x_{a,f,S} \geq \left\lfloor \sum_{(a,f,S)} S_b \cdot x^*_{a,f,S} \right\rfloor \ \forall b \in B. \tag{A3}$$

We apply the iterative rounding algorithm to this program, and start with x^* at Step 0. Clearly, x^* satisfies all the constraints in PROGRAM A. Our row elimination rule we will only delete constraints of type (A2) and (A3).

Consider an intermediate step of the algorithm. Assume the current linear program is

$$\text{PROGRAM A': } \max \ c \cdot x \text{ such that :}$$

$$x_{a,f,S} = 0 \text{ if } (a,f,S) \notin \mathcal{H}$$

$$x_{a,f,S} \leq 1$$

$$\underline{d}_a \leq \sum_{(f,S)} x_{a,f,S} \leq \bar{d}_a \quad \forall a \in A \tag{A1'}$$

$$\sum_{(a,f,S)\in\mathcal{H}} S_b \cdot x_{a,f,S} \leq \bar{d}_b \ \ \forall b \in B' \tag{A2'}$$

$$\sum_{(a,f,S)\in\mathcal{H}} S_b \cdot x_{a,f,S} \geq \underline{d}_b \ \ \forall b \in B''. \tag{A3'}$$

Here $\mathcal{H}$ corresponds to the set of active columns; $B' \cup B''$ corresponds to the set of remaining constraints of type (A2) and (A3). Notice all $\underline{d}_a, \bar{d}_a, \underline{d}_b, \bar{d}_b$ are integral.

Let x^{opt} be an optimal solution of PROGRAM A'. Without loss of generality assume none of the coordinates of x^{opt} are integral; otherwise by fixing those integral values we can continue to update and reduce the size of the linear program.

CLAIM B.1 *Given an optimal solution x^{opt} of PROGRAM A'. If NONE of the coordinates of x^{opt} are integral, then, there exists a b such that either*[19]

$$\sum_{(a,f,S)\in\mathcal{H}} S_b \cdot \lceil x^{opt}_{a,f,S} \rceil \le \bar{d}_b + 2\alpha - 1$$

or

$$0 = \sum_{(a,f,S)\in\mathcal{H}} S_b \cdot \lfloor x^{opt}_{a,f,S} \rfloor \ge \underline{d}_b - (2\alpha - 1).$$

Assume that Claim B.1 is true. Apply the iterative rounding algorithm with the following row elimination rule for Step 2b:

- Eliminate any constraint b of type (A2') if $\sum_{(a,f,S)} S_b \cdot \lceil x^{opt}_{a,f,S} \rceil \le \bar{d}_b + 2\alpha - 1$.
- Eliminate any constraint b of type (A3') if $0{=}\sum_{(a,f,S)} S_b \cdot \lfloor x^{opt}_{a,f,S} \rfloor \ge \underline{d}_b - (2\alpha - 1)$.

To show that with this elimination rule the algorithm will terminate, notice that if no constraints of type (A2') or (A3') remain, according to Lemma B.2, the linear program has an integral solution, and thus the algorithm terminates at Step 1. Assume the algorithm does not terminates at Step 1. Because of Claim B.2, we can always proceed either to Step 2a or 2b. Thus according to Lemma B.1 the algorithm has to end in a finite number of steps.

Let $\bar{x}$ be the output of this algorithm. We show that $\bar{x}$ satisfies (6)-(9).

Once coordinate $x_{a,f,S}$ is fixed, it is never changed in subsequent iterations. Thus, (6) is true because if $x^*_{a,f,S} = 0$, then that variable is fixed to be 0 in all subsequent iterations of the algorithm.

Because of Lemma B.1, $c \cdot \bar{x} \ge c \cdot x^*$, thus (C1) is true.

[19]Notice, because all coordinates of x^{opt} are fractional, they are less than 1. Therefore, $\lfloor x^{opt}_{a,f,S} \rfloor = 0$.

(8) is not violated because we do not eliminate such constraints during the algorithm.

Finally we need to show (9), that is

$$\Big\lfloor \sum_{(a,f,S)} S_b \cdot x^*_{a,f,S} \Big\rfloor - 2\alpha + 1 \leq \sum_{(a,f,S)} S_b \cdot \bar{x}_{a,f,S} \leq \Big\lceil \sum_{(a,f,S)} S_b \cdot x^*_{a,f,S} \Big\rceil + 2\alpha - 1 \quad \forall b \in B.$$

Notice that the constraints that were eliminated during the execution of the algorithm satisfy the conditions in Claim B.1, and once coordinate $x_{a,f,S}$ is fixed, it is never changed in subsequent iterations. Therefore, the above inequalities must hold.

As $\bar{x}$ satisfies (6)-(9), this suffices to prove part A of Lemma 3.3.

It remains to prove Claim B.1.

Proof of Claim B.1 This proof is an extension of Király et al. [2012].[20] Given the extreme point x^{opt}, assume none its coordinates are integral, i.e, $0 < x^{opt}_{a,f,S} < 1$.

We credit each non-zero variable $x^{opt}_{a,f,S}$ with a single token. We then redistribute these tokens to the binding, linearly independent constraints in a particular way.

Suppose for a contradiction that the conclusion of Claim B.1 is false. Because all $\underline{d}_b, \bar{d}_b$ are integral, the contracdiction assumption means that for all b:

$$\sum_{(a,f,S)\in\mathcal{H}} S_b \cdot \lceil x^{opt}_{a,f,S} \rceil \geq \bar{d}_b + 2\alpha \text{ and } 0 \leq \underline{d}_b - 2\alpha.$$

We show that each binding constraint will get at least one token, and there will be one token left over. This shows that the number of non-zero variables in x^{opt} exceeds the number of binding, linearly independent constraints, which contradicts the fact that x^{opt} is an extreme point.

We redistribute the tokens as follows:

1. Credit 1/2 of the tokens of $x^{opt}_{a,f,S}$ to the constraint corresponding to agent a (A1').

[20]We generalize Király et al. [2012] to allow $S_b > 1$. Our counting technique is also different, which allows us to extend the proof to part B of Lemma 3.3.

2. Credit $S_b[\frac{1-x^{opt}_{a,f,S}}{2\alpha}]$ to each constraint of type (A2') corresponding to each $b \in B'$.

3. Credit $S_b[\frac{x^{opt}_{a,f,S}}{2\alpha}]$ to each constraint of type (A3') corresponding to each $b \in B''$.

Notice that this is feasible because the size of each bundle is at most α. The total of number of tokens debited from each $x^{opt}_{a,f,S}$ is

$$1/2 + \sum_b S_b \frac{1 - x^{opt}_{a,f,S}}{2\alpha} + \sum_b S_b \frac{x^{opt}_{a,f,S}}{2\alpha} \leq 1/2 + \frac{(\sum_b S_b)}{2\alpha} = 1/2 + \frac{size(S)}{2\alpha} \leq 1.$$

Consider an agent a such that $\sum_{(f,S)} x^{opt}_{a,f,S} = 1$. As all components of $x^{opt}_{a,f,S}$ are fractional, there are at least 2 positive $x^{opt}_{a,f,S}$. Each of them contributes 1/2 a token, thus, the constraint corresponding to agent a gets at least 1 token.

Consider an agent $b \in B''$ and suppose its corresponding constraint of type (A3') binds. Then, the number of tokens it gets is

$$\sum_{a,f,S} S_b \frac{x^{opt}_{a,f,S}}{2\alpha} = \frac{\underline{d}_b}{2\alpha}.$$

Because of the contradiction assumption, $\underline{d}_b \geq 2\alpha$, this constraint gets at least 1 token.

Now suppose a constraint of type (A2') for agent $b \in B'$ binds. Then, the number of tokens it gets is

$$\sum_{a,f,S} S_b \frac{1 - x^{opt}_{a,f,S}}{2\alpha} = \frac{\sum_{(a,f,S)} S_b \cdot \lceil x^{opt}_{a,f,S} \rceil - \bar{d}_b}{2\alpha}.$$

As assumed, $\sum_{(a,f,S)} S_b \cdot \lceil x^{opt}_{a,f,S} \rceil \geq \bar{d}_b + 2\alpha$, this constraint gets at least 1 token.

Hence, the amount of tokens given at the beginning (which is the dimension of x^{opt}) have been redistributed to the binding constraints, so that each is credited with at least 1 token. Thus the dimension of x^{opt} is at least the number of binding constraints.

Now, equality obtains only if for every nonzero $x^{opt}_{a,f,S}$, $size(S) = \alpha$. Furthermore, the constraint corresponding to agent a as well as all constraints corresponding to all $b \in B$ bind. In this case one can show that the set of binding constraints is not linearly independent. To see this, consider the sum of all the binding constraints of type (A1'). The coefficient of each variable will be 1. On the other hand, in the sum of all the constraints of type (A2'), the coefficient of each variable $x_{a,f,S} > 0$ is $size(S)$. However, each bundle consumed contains exactly α items. Thus each coefficient is α.

By this we have shown that the number of nonzero variables in an extreme point solution is larger than the number of linearly independent binding constraints. ■

B.3 Proof of part B of Lemma 3.3

The proof of part B is similar to the previous proof of part A.

Consider the following program.

PROGRAM B: $\max c \cdot x$ such that :

$$x_{a,f,S} = 0 \text{ if } x^*_{a,f,S} = 0$$

$$x_{a,f,S} \leq 1$$

$$\left\lfloor \sum_{(f,S)} {x^*}_{a,f,S} \right\rfloor \leq \sum_{(f,S)} x_{a,f,S} \leq \left\lceil \sum_{(f,S)} {x^*}_{a,f,S} \right\rceil \quad \forall a \in A \tag{B0}$$

$$\sum_{(a,f,S)} S_b \cdot x_{a,f,S} \leq \left\lceil \sum_{(a,f,S)} S_b \cdot x^*_{a,f,S} \right\rceil \quad \forall b \in B \tag{B1}$$

$$\sum_{(a,f,S)} S_b \cdot x_{a,f,S} \geq \left\lfloor \sum_{(a,f,S)} S_b \cdot x^*_{a,f,S} \right\rfloor \quad \forall b \in B. \tag{B2}$$

$$\sum_{(a,f,S)} x_{(a,f,S)} \cdot size(S) \leq \left\lceil \sum_{(a,f,S)} {x^*}_{(a,f,S)} \cdot size(S) \right\rceil \tag{B3}$$

We apply the iterative rounding algorithm to this program, and start with x^* at Step 0. Clearly, x^* satisfies all the constraints in PROGRAM B. In the row elimination rule we will only delete the constraints of type (B1), (B2) and (B3).

Consider an intermediate step of the algorithm. The current linear program is either

$$\text{PROGRAM B': } \max c \cdot x \text{ such that :}$$

$$x_{a,f,S} = 0 \text{ if } (a, f, S) \notin \mathcal{H}$$

$$x_{a,f,S} \leq 1$$

$$\underline{d}_a \leq \sum_{(f,S)} x_{a,f,S} \leq \bar{d}_a \quad \forall a \in A \tag{B0'}$$

$$\sum_{(a,f,S)\in\mathcal{H}} S_b \cdot x_{a,f,S} \leq \bar{d}_b \ \ \forall b \in B' \tag{B1'}$$

$$\sum_{(a,f,S)\in\mathcal{H}} S_b \cdot x_{a,f,S} \geq \underline{d}_b \ \ \forall b \in B''. \tag{B2'}$$

$$\sum_{(a,f,S)\in\mathcal{H}} x_{(a,f,S)} \cdot size(S) \leq d \tag{B3'}$$

or one without constraint (B3').[21] Without loss of generality, we assume that the current linear program has such a constraint, otherwise the proof is the same as in part A.

As in the proof of part A, $\mathcal{H}$ corresponds to the set of active columns; $B' \cup B''$ corresponds to the set of remaining constraints of type (B1) and (B2).

Again, we assume x^{opt} to be the optimal solution of the PROGRAM B' and all the coordinates of x^{opt} are fractional, that is, $0 < x^{opt} < 1$. With this we have the following.

Claim B.2 *Let x^{opt} be the optimal solution of the PROGRAM B' such that all coordinates of x^{opt} are fractional, then, one of the following is true:*

C1: *there exists $b \in B'$ such that* $\sum_{(a,f,S)\in\mathcal{H}} S_b \cdot \lceil x^{opt}_{a,f,S} \rceil \leq \bar{d}_b + 2\alpha$

C2: *there exists $b \in B''$ such that* $0 = \sum_{(a,f,S)\in\mathcal{H}} S_b \cdot \lfloor x^{opt}_{a,f,S} \rfloor \geq \underline{d}_b - 2\alpha$

C3: $\sum_{(a,f,S)\in\mathcal{H}} \lceil {x^{opt}}_{(a,f,S)} \rceil \cdot size(S) \leq d + (2\alpha + 1)\alpha - 1.$

First assume that Claim B.2 is true, then we can prove part (B) as follows. Apply the iterative

[21] Notice all $d, \underline{d}_a, \bar{d}_a, \underline{d}_b, \bar{d}_b$ are integral.

rounding algorithm to PROGRAM B above with the row elimination rule corresponding to the conditions C1, C2 and C3 as in Claim B.2.

First, if there are no constraints of type (B1'), (B2') or (B3') left, then according to Lemma B.2, the linear program has an integral solution, and thus the algorithm terminates at Step 1. Second, according to Claim B.2, after Step 1, if has not terminated, the algorithm can always proceed to either Step 2a or 2b. Therefore because of Lemma B.1 the algorithm will end in a finite number of steps.

Let $\hat{x}$ be the output of the algorithm. The remainder of the proof shows that $\hat{x}$ must satisfy (10)–(14) is analogous to the proof of part (A). It remains to show Claim B.2.

Proof of Claim B.2 The proof is similar to that of Claim B.1, but a more elaborate redistribution of tokens is employed. We first assume that the claim is not true. Because $d, \underline{d}_b, \bar{d}_b$ are all integral, this means that for all remaning constraints:

- $0 = \sum_{(a,f,S)\in\mathcal{H}} S_b \cdot \lfloor x^{opt}_{a,f,S} \rfloor \leq \underline{d}_b - 2\alpha - 1$
- $\sum_{(a,f,S)} S_b \cdot \left\lceil x^{opt}_{a,f,S} \right\rceil \geq \bar{d}_b + 2\alpha + 1$
- $\sum_{(a,f,S)\in\mathcal{H}} \lceil x^{opt}{}_{(a,f,S)} \rceil \cdot size(S) \geq d + (2\alpha + 1)\alpha$

Again, credit each fractional $x^{opt}_{a,f,S}$ with one token. We redistribute this token as follows.

Credit $\frac{\alpha+1}{2\alpha+1} x^{opt}_{a,f,S} + \frac{\alpha}{2\alpha+1}(1 - x^{opt}_{a,f,S})$ tokens to the constraint corresponding to agent a in the constraints of type (B0').

Credit $S_b \frac{1-x^{opt}_{a,f,S}}{2\alpha+1}$ tokens to each constraint of type (B1') corresponding to each b

Credit $S_b \frac{x^{opt}_{a,f,S}}{2\alpha+1}$ tokens to each constraint of type (B2') corresponding to each b

Credit $\frac{1-x^{opt}_{a,f,S}}{2\alpha}$ to each constraint of type (B3').

It is easy to check that the sum of these tokens is at most 1, because for every S, $size(S) = \sum_b S_b \leq \alpha$.

As in the the proof of Claim B.1, consider an agent $b \in B$. If its corresponding constraint of type (B2') binds then the amount of tokens it gets is

$$\sum_{a,f,S} S_b \frac{x^{opt}_{a,f,S}}{2\alpha+1} = \frac{\underline{d}_b}{2\alpha+1}.$$

Thus, if $\underline{d}_b \geq 2\alpha + 1$, then, this constraint gets at least 1 token.

Consider a constraint of type (B1') for an agent $b \in B$. If this constraint is binding, then, the amount of tokens it gets is

$$\sum_{a,f,S} S_b \frac{1 - x^{opt}_{a,f,S}}{2\alpha+1} = \frac{\sum_{(a,f,S)} S_b \cdot \left\lceil x^{opt}_{a,f,S} \right\rceil - \bar{d}_b}{2\alpha+1}.$$

Notice the above equality is true because we assume $0 < x^{opt}_{a,f,S} < 1$, and thus $\lceil x^{opt}_{a,f,S} \rceil = 1$. Thus, if $\sum_{(a,f,S)} S_b \cdot \left\lceil x^{opt}_{a,f,S} \right\rceil \geq \bar{d}_b + 2\alpha + 1$, then this constraint gets at least one token.

Consider constraint (B3'). If it binds, the amount of tokens it gets is

$$\sum_{a,f,S} \frac{1 - x^{opt}_{a,f,S}}{2\alpha+1} = \sum_{a,f,S} \frac{size(S) - size(S) \cdot x^{opt}_{a,f,S}}{size(S)(2\alpha+1)} \geq \frac{\sum_{(a,f,S)} size(S) \cdot \left\lceil x^{opt}_{a,f,S} \right\rceil - d}{\alpha(2\alpha+1)}.$$

Therefore, if $\sum_{(a,f,S)} size(S) \cdot \lceil x^{opt}_{a,f,S} \rceil - d \geq (2\alpha+1)\alpha$, this constraint also gets at least one token.

Finally, for a constraint corresponding to an agent a. The total tokens that it gets is

$$\frac{\alpha+1}{2\alpha+1} \sum_{f,S} x^{opt}_{a,f,S} + \frac{\alpha}{2\alpha+1} \sum_{f,S} (1 - x^{opt}_{a,f,S}).$$

Since, all $x^{opt}_{a,f,S}$ are fractional, if the constraint corresponding to agent a binds, then $\sum_{(f,S)} x_{a,f,S} = 1$. Thus,

$$\frac{\alpha+1}{2\alpha+1} \sum_{f,S} x^{opt}_{a,f,S} = \frac{\alpha+1}{2\alpha+1}.$$

Furthermore, for such a, there are at least 2 positive $x^{opt}_{a,f,S}$, otherwise $x^{opt}_{a,f,S} = 1$, therefore

$$\frac{\alpha}{2\alpha+1}\sum_{f,S}(1 - x^{opt}_{a,f,S}) \geq \frac{\alpha}{2\alpha+1}(2 - \sum_{f,S} x^{opt}_{a,f,S}) = \frac{\alpha}{2\alpha+1}.$$

Combining these inequalities, we obtain that the constraint corresponding to agent a also gets at least one token. The remainder of the proof shows if all the inequalities above hold at equality, then these binding constraints are linearly dependent as in the the proof of Claim B.1. ■

C Proof of Theorem 5.2

The proof of this theorem is analogous to the proof of Theorem 3.1. Using the construction of dummy hospitals as described in Section 5, we transform the problem into our general framework described in Section 3.

For a rural region B_i, because $\sum_{b \in B_i} k_b \leq K_i$, we can ignore the cap for this region, and therefore omit the dummy hospital corresponding to this constraint.

Thus, a possible match between a doctor a and a urban hospital b corresponds to the triple $(a, \{b, r_{\pi(b)}\})$. However, a possible match between a doctor a and a rural hospital b will correspond to $(a, \{b\})$, that is the match between a and b in the usual sense.

Let x^* be the fractional solution found using Scarf's lemma for this setting. All we need to show is that x^* can be rounded into an integral vector $\bar{x}$ such that:

For every urban hospital b:

$$\left\lfloor \sum_{(a,\{b,r_{\pi(b)}\})} x^*_{(a,\{b,r_{\pi(b)}\})} \right\rfloor - 3 \leq \sum_{(a,\{b,r_{\pi(b)}\})} \bar{x}_{(a,\{b,r_{\pi(b)}\})} \leq \left\lceil \sum_{(a,\{b,r_{\pi(b)}\})} x^*_{(a,\{b,r_{\pi(b)}\})} \right\rceil + 3,$$

which guarantees that we do not change the capacity of any urban hospital by more than 3.

For every dummy hospital r_i:

$$\Big\lfloor \sum_{(a,\{b,r_i\})} x^*_{(a,\{b,r_i\})} \Big\rfloor - 3 \leq \sum_{(a,\{b,r_i\})} \bar{x}_{(a,\{b,r_i\})} \leq \Big\lceil \sum_{(a,\{b,r_i\})} x^*_{(a,\{b,r_i\})} \Big\rceil + 3,$$

which guarantees that we do not change the cap of any urban region by more than 3. For every rural hospital b:

$$\Big\lfloor \sum_{(a,\{b\})} x^*_{(a,\{b\})} \Big\rfloor \leq \sum_{(a,\{b\})} \bar{x}_{(a,\{b\})} \leq \Big\lceil \sum_{(a,\{b\})} x^*_{(a,\{b\})} \Big\rceil,$$

which guarantees that we do not change the capacity of any rural hospital. Finally, for every doctor a:

$$\Big\lfloor \sum_{(a,\{b,r_{\pi(b)}\})} x^*_{(a,\{b,r_{\pi(b)}\})} + \sum_{(a,\{b\})} x^*_{(a,\{b\})} \Big\rfloor \leq \sum_{(a,\{b,r_{\pi(b)}\})} \bar{x}_{(a,\{b,r_{\pi(b)}\})} + \sum_{(a,\{b\})} \bar{x}_{(a,\{b\})}$$

$$\sum_{(a,\{b,r_{\pi(b)}\})} \bar{x}_{(a,\{b,r_{\pi(b)}\})} + \sum_{(a,\{b\})} \bar{x}_{(a,\{b\})} \leq \Big\lceil \sum_{(a,\{b,r_{\pi(b)}\})} x^*_{(a,\{b,r_{\pi(b)}\})} + \sum_{(a,\{b\})} x^*_{(a,\{b\})} \Big\rceil,$$

which guarantees that if in x^* a doctor a is always matched to a hospital, that is $\sum_{(a,\{b,r_{\pi(b)}\})} x^*_{(a,\{b,r_{\pi(b)}\})} + \sum_{(a,\{b\})} x^*_{(a,\{b\})} = 1$, then doctor a will be matched in $\bar{x}$.

The existence of such a $\bar{x}$ can be derived by the iterative rounding procedure. For this procedure to work the following lemma is at the core of the argument.

CLAIM C.1 *Consider the following linear program.*

PROGRAM C: $\max c \cdot x$ *such that :*

$$x_{(a\{b,r_{\pi(b)})} = 0; x_{(a,\{b\})} = 0 \text{ if } (a\{b,r_{\pi(b)}) \text{ and } (a,\{b\}) \notin \mathcal{H}$$

$$x_{(a\{b,r_{\pi(b)})} \leq 1; x_{(a,\{b\})} \leq 1$$

$$\underline{d}_a \leq \sum_{(a\{b,r_{\pi(b)})} x_{(a\{b,r_{\pi(b)})} + \sum_{(a,\{b\})} x_{(a,\{b\})} \leq \bar{d}_a \quad \forall a \in A \tag{C0}$$

$$\underline{d}_b \leq \sum_{(a,\{b\})} x_{(a,\{b\})} \leq \bar{d}_b \quad \text{for all rural hospital } b \tag{C1}$$

$$\sum_{(a\{b,r_{\pi(b)})\in\mathcal{H}} x_{(a\{b,r_{\pi(b)}\})} \leq \bar{d}_b \quad \text{for urban hospital } b \in B' \tag{C2}$$

$$\sum_{(a\{b,r_{\pi(b)})\in\mathcal{H}} x_{(a\{b,r_{\pi(b)}\})} \geq \underline{d}_b \quad \text{for urban hospital } b \in B" \tag{C3}$$

$$\sum_{(a\{b,r_i)\in\mathcal{H}} x_{(a\{b,r_i\})} \leq \bar{d}_{r_i} \quad \text{for dummy hospital } r_i \in I' \tag{C4}$$

$$\sum_{(a\{b,r_i)\in\mathcal{H}} x_{(a\{b,r_i\})} \geq \underline{d}_{r_i} \quad \text{for dummy hospital } r_i \in I'' \tag{C5}$$

where $\mathcal{H}$ *is the set of indexes (columns)of active variables, and* $B'; B''; I'; I''$ *are the set of active constraints.*[22] *Let* x^{opt} *be an extreme solution of this program, and assume all active coordinates of* x^{opt} *are fractional, then there exists* $b \in B' \cup B"$ *or* $r_i \in I' \cup I''$ *such that at least one of the following is true.*

- $\sum_{(a\{b,r_{\pi(b)}\})\in\mathcal{H}} \lceil x^{opt}_{(a\{b,r_{\pi(b)}\})} \rceil \leq \bar{d}_b + 3$
- $\sum_{(a\{b,r_{\pi(b)}\})\in\mathcal{H}} \lfloor x^{opt}_{(a\{b,r_{\pi(b)}\})} \rfloor \geq \underline{d}_b - 3$
- $\sum_{(a\{b,r_i\})\in\mathcal{H}} \lceil x^{opt}_{(a\{b,r_i\})} \rceil \leq \bar{d}_{r_i} + 3$
- $\sum_{(a\{b,r_i\})\in\mathcal{H}} \lfloor x^{opt}_{(a\{b,r_i\})} \rfloor \geq \underline{d}_{r_i} - 3$

The proof of this claim is similar to Claim B.1, and so is omitted.

[22] This corresponds to the updated linear program obtained at the beginning of stage 3 during the iterative algorithm.

With Claim C.1, the proof of Theorem 5.2 is as follows. Given a fractional solution x^* corresponding to the solution of Scarf lemma, we apply the iterative rounding algorithm to the linear program C, where

$$\underline{d}_a = \Big\lfloor \sum_{(a\{b,r_{\pi(b)})} x^*_{(a\{b,r_{\pi(b)})} + \sum_{(a,\{b\})} x^*_{(a,\{b\})} \Big\rfloor ; \bar{d}_a = \Big\lceil \sum_{(a\{b,r_{\pi(b)})} x^*_{(a\{b,r_{\pi(b)})} + \sum_{(a,\{b\})} x^*_{(a,\{b\})} \Big\rceil ;$$

$$\underline{d}_b = \Big\lfloor \sum_{(a,\{b\})} x^*_{(a,\{b\})} \Big\rfloor ; \bar{d}_b = \Big\lceil \sum_{(a,\{b\})} x^*_{(a,\{b\})} \Big\rceil \text{ for all rural hospitals } b;$$

$$\underline{d}_b = \Big\lfloor \sum_{(a\{b,r_{\pi(b)})} x*_{(a\{b,r_{\pi(b)}\})} \Big\rfloor ; \bar{d}_b = \Big\lceil \sum_{(a\{b,r_{\pi(b)})} x*_{(a\{b,r_{\pi(b)}\})} \Big\rceil \text{ for all urban hospitals } b;$$

$$\text{and } \underline{d}_{r_i} = \Big\lfloor \sum_{(a\{b,r_i)} x^*_{(a\{b,r_i\})} \Big\rfloor ; \bar{d}_{r_i} = \Big\lceil \sum_{(a\{b,r_i)} x^*_{(a\{b,r_i\})} \Big\rceil \text{ for all dummy hospitals } r_i.$$

Clearly x^* is a feasible solution to this linear program. According to Claim C.1, if there still exists a constraint of type (C2); (C3); (C4) or (C5), and the algorithm has not terminated, we can always eliminate one such constraint without violating it by more than 3. When no such constraint exists, only constraints of type (C0) and (C1) remain. However, these are standard bipartite matching constraints, and thus an integral solution exists. Therefore, the iterative rounding procedure will terminate in this case and gives us the desired integral vector $\bar{x}$.

■