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"Dynamic Quality Signaling with Hidden Actions"

by

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Abstract

Asymmetric information is an important source of inefficiency when an asset (such as a firm) is transacted. The two main sources of this asymmetry are the unobserved idiosyncratic characteristics of the asset (such as future profitability) and unobserved idiosyncratic choices (like secret price cuts). Buyers may use noisy signals (such as sales) in order to infer actions and characteristics. In this situation, does the seller prefer to release information fast or slowly? Is it incentive compatible? When the market is pessimistic, is it better to give up or keep signaling? We introduce hidden actions in a dynamic signaling model in order to answer these questions. Separation is found to be fast in equilibrium when sending highly informative signals is more efficient than sending lowly informative signals. When the market is pessimistic about the quality of the asset, depending on the cost structure, the seller either "gives-up" by stopping signaling, or the seller "rushes-out" by increasing the informativeness of the signal. We find that the unobservability of the action causes equilibrium effort to be too low and the seller to stop signaling too early. The model can be applied to education where grades depend on students' effort, which is endogenously related to their skills.

Dynamic Signaling, Dynamic Moral Hazard, Endogenous Effort.

JEL Classification: D82, D83, C73, J24.

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1 Introduction

When a heterogeneous asset (such as a firm) is transacted, the seller usually has private information about its underlying value (or quality.) Potential buyers learn about the quality of the asset through noisy signals (such as sales, profits, etc.) that take place over time. Nevertheless, the seller may be able to take some unobservable action in order to change the distribution of the signal (secret price cuts, more-than-optimal advertising, etc.). The cost of doing so is likely to depend on the underlying value of the asset. In this environment, under what circumstances can the high-quality sellers (partially) separate themselves from the low-quality sellers? Is separation fast or slow? When the market is pessimistic about the quality, is it incentive compatible for the seller to exert high effort, low effort or to stop signaling?

We develop a model of dynamic noisy signaling with hidden actions in order to answer the previous questions. In equilibrium, our privately informed seller is willing to actively engage in investing in his "reputation" in order to receive a high price offer. A seller with a high-quality asset exerts high effort in order to generate good signals in order to get high price offers from the buyers. A seller with a low-quality asset, instead, cannot alter the signal distribution but mimics the high-quality seller on the decision of whether or not to accept offers, making separation more difficult.

Our dynamic game is a repetition of a static noisy-signaling game, where separation occurs through a (noisy) costly message sent by the seller. The (high-quality) seller decides how much effort to put into signaling. Increasing the informativeness of the signal increases its cost, which in our model also includes a fixed cost of signaling. In equilibrium, the different types of sellers pool on the decision to accept or reject equilibrium offers, so we isolate the (hidden) effort choice as the source of separation in a dynamic environment. As a consequence, different ways of intertemporal separation strategies arise, depending on the cost structure. In particular, we find that when the cost function is (not) highly convex, separation happens through low (high) effort choices that take place over long (short) periods of time.

Although signaling and hidden actions have been studied separately in dynamic models (see the literature review below), their dynamic interaction has not been previously analyzed. Our approach allows us to endogenize both the cost of signaling and the information released per unit of time, instead of making them fixed conditional on type. So, we are able to analyze in a unified dynamic framework the two main sources of inefficiency that appear in models with asymmetric information: the non-observability of the type (idiosyncratic characteristics) and the non-observability of effort (idiosyncratic effort choice). We find that, in equilibrium, the effort exerted by the seller is always lower than the optimal choice, and he stops signaling too early.

The equilibrium behavior of the seller is found to be highly dependent on how efficient it is to increase the informativeness of the signal. When the cost of generating highly informative signals is high (i.e., a highly convex cost function), we find the "give-up effect"; that is, when the market is very pessimistic about the quality of the asset, the seller stops signaling. Intuitively, separating from the low type would require revealing a lot of information, which is too costly given the high convexity of the cost function and the fixed cost. We find that, after some histories, signaling is highly inefficient, with the seller incurring a positive fixed cost to generate an almost non-informative signal.

When highly informative signals are less inefficient than lowly informative signals, we find a "rush-out effect." In this case, effort is found to be decreasing in the posterior about the quality of his asset being high, so the signal is more informative when the market is pessimistic about the quality. In particular, we find that the effort is high even for (very) low posteriors, that is, even when updating is potentially slow. The reason is that, in that region, small increases in the posterior generate a big increase in the probability of getting high offers (only) for the high-quality seller. This increase in the expected revenue makes signaling attractive, and therefore, it is incentive compatible to exert high effort that makes Bayes' updating fast enough to compensate the cost. As a consequence, we may have a high degree of separation even when the cost of signaling is high. Similarly, if the noise increases, the effort increases, and even though there is more waste per unit of time, there is more separation due to the increase in the effort. This is in sharp contrast to what is found in static or fixed-action dynamic signaling environments. In those, as in our highly-convex cost function case, players "give up" when beliefs are close to being degenerated toward one of the types because of slow beliefs update in this region and also when the cost of signaling is high.

We characterize the equilibrium structure of all Markov equilibria, and we focus our analysis on most separating equilibria, that is, equilibria where the signal is informative in the largest set of posteriors about the quality of the asset. We show that they are essentially unique; that is, they have the same signal distribution and same distribution over accepted offers. We also show that they both maximize the payoff of the high-quality sellers, making them attractive to market-makers, and are in the spirit of most of the previous refinements.

Additionally to the trade of non-homogeneous assets, high education is another important application of our model. Indeed, high-level education is by nature a dynamic process where information is progressively realized over time. Grades, prizes and test results stochastically depend on the (skill-adjusted) effort of the students. Students, knowing their skills and past history, decide how much effort they exert to affect the new signals to come. On the other side of the market, employers use the observable signals to infer information about the productivity of each student and use it to make her wage offers. If the (utility) cost or the effectiveness of obtaining high grades is correlated with innate skills themselves related to productivity, differently productive students would exert different levels of effort. Therefore, the signal history can be used to infer choices that provide information about individual characteristics.

The organization of the paper is as follows. After this introduction, we review the related literature. Section 2 introduces our model. In Sections 3 and 4 we discuss, respectively, the low and high convex cost function cases. In Section 5 we analyze the observable effort case. Section 6 concludes. An appendix contains the proofs of all lemmas and propositions of the previous sections.

1.1 Literature Review

Our model is closely related to the dynamic signaling literature with preemptive offers, which initially intended to provide a rationale for why unproductive education may last for long periods of time. Indeed, as Weiss (1983) pointed out, if the signal in Spence's (1973) model is interpreted as the length of education, most equilibria can be destroyed if firms make offers on the first day of class, since most of the separation has already taken place, and can then obtain (part of) the reduction in the worker's educational costs.[1] Nöldeke and van Damme (1990) assume that workers have different educational costs per period and receive public offers from firms. They find that (partially) separating equilibria exist. Swinkels (1999) introduces the possibility of private offers into the job market model and

[1]The static analogous to our model is a noisy signaling model, instead of a model such as Spence (1973), where the action is perfectly observed. Noisy signaling was introduced by Matthews and Mirman (1983) in a limit pricing model. Our "stage game" is similar to the game analyzed in de Haan et al. (2011). In line with the dynamic signaling literature, we analyze the implications of repeated signaling in equilibrium behavior, and the dynamics and speed of information transmission. We will point out the differences between our results and previous findings.

finds that no equilibrium with (partial) separation exists when the length of the interval shrinks. Hörner and Vieille (2009) make similar arguments in a lemons environment, finding that bargaining may end up with an impasse or delayed agreement, depending on whether the offers are public or private. Kremer and Skrzypacz (2007) introduce the arrival of news about the asset at some fixed time. Daley and Green (2012) introduce a continuous public stochastic signal in a dynamic signaling model, while Kaya and Kim (2013) introduce private signals in a lemons environment.

There are two important differences between the previous models and our model. First, our model has two sources of payoff-relevant private information: the type and the effort. This makes our model a repeated signaling problem, where at each period an action is chosen by the entepreneur in order to signal the type of his firm, instead of repeated adverse selection, where the only relevant action in each period is to accept or not the curren offer. So, we investigate the incentives of the sellers to increase or decrease the signal informativeness and we characterize the equilibrium speed of learning.[2] Second, in the previous dynamic models, sellers signal their type by waiting in the market (at different costs), by rejecting (public or private) offers and (in some of them) with an exogenous signal that depends on the type. In our model, instead, separation only comes from the different effort choices of the different types of sellers. Therefore, our model focuses on a different channel that generates dynamic signaling effects.

Our paper is also partially related to the literature on reputations. Indeed, our model has one agent with an unobservable type who performs unobservable actions to pool/separate himself with/from other types. In this literature, inaugurated by the seminal works of Kreps and Wilson (1982) and Milgrom and Roberts (1982), the model closest to ours is in Faingold and Sannikov (2011), set in continuous time. More recently, Board and Meyer-ter-Vehn (2010, 2013) and Dilme (2012) also use continuous-time, hidden action models, using a Poisson-arrival news structure. In these models firms repeatedly sell products to customers, so they trade off current "cheating" by producing low-quality goods with future high prices if they build reputation by producing high-quality goods. In our model, instead, there is (at most) one transaction. So, the tradeoff for the seller is saving current costs versus receiving high future price offers. Also, as in other dynamic signaling models, the decision to accept

[2]In the previous literature the speed at which information is released is fixed. In those models it seems natural to wonder about the effects of endogenizing the information revelation speed. For example, Daley and Green (2012) conclude their paper wondering about how optimal the market's revelation of information is. Our model provides answers to some of these questions.

or reject offers acts as an extra signal.

2 The Model

Time is continuous, $t \in \mathbb{R}_+$. There is one (potential) seller who owns an asset. The quality (type) of the asset may be either low (L-asset/L-seller) or high (H-asset/H-seller). At each instant in time $t \in \mathbb{R}_+$, the seller decides how much effort to exert, $e_t \in \mathbb{R}_+$. There is a public noisy signal about the effort exerted. If the effort process that a seller exerts is $(e_t)_{t\in\mathbb{R}_+}$ the signal evolves according to the following stochastic equation

$$\mathrm{d}X_t = e_t\,\mathrm{d}t + \sigma\,\mathrm{d}B_t\ ,$$

where $B_t = \{B_t, \mathcal{F}_t, 0 \le t \le \infty\}$ is standard one-dimensional Brownian motion on the canonical probability space $\{\Omega, \mathcal{F}, \mathcal{Q}\}$. Let $\mathcal{H}_t$ be the σ-algebra generated by $\{X_s, 0 \le s \le t\}$.

The θ-seller values his own asset at $\underline{V}_\theta$, with $\underline{V}_H > \underline{V}_L > 0$. This is interpreted as the value of retaining the asset for himself (i.e., not selling it). The cost of effort is type-dependent. For each type $\theta \in \Theta \equiv \{L, H\}$ it given by

$$c_\theta(e) \equiv \mathbb{I}_{e>0}\,c_{0,\theta} + A_\theta\,e^\alpha \quad \forall\theta \in \{L, H\}\ , \tag{2.1}$$

with $c_{0,L} \ge c_{0,H} > 0$, $\alpha > 1$, $\alpha \neq 2$ and $A_L > A_H$.[3] For simplicity, we will restrict ourselves to the case $A_L = \infty$, that is, when the optimal choice of the L-seller is always $e_t = 0$.[4] Even though most of the results we obtain do not rely on this simplification, it greatly simplifies the results presented in this paper. Since $c_{0,L}$ will be irrelevant for our analysis, we use c_0

[3] Note that if the seller exerts zero effort, he does not incur any cost, and the signal is totally uninformative. Therefore, even though waiting is technically costless, it is also useless, since relevant signaling information is only revealed at a cost.

[4] The assumption that one type of agent is "handicapped" is common in the reputations literature. Indeed, in many models in this literature there is a type that takes an action independently of the history (see, for example, Mailath and Samuelson (2001) or Hörner (2002).) Our setting is equivalent to assuming that if the seller chooses to incur a cost v then the drift of the signal is given by (the type-dependent efficiency-of-signaling function) $g_\theta(v) \equiv \frac{(v-c_{0,\theta})^{1/\alpha}}{A_\theta}$ if $v \ge c_{0,\theta}$ and 0 otherwise. Note that when $A_L = \infty$, the L-seller is totally inept (i.e., no matter what effort he exerts, he is not able to change the drift), but we allow him to act strategically through the decision of accepting or not accepting the offer, as in the standard models of dynamic signaling.

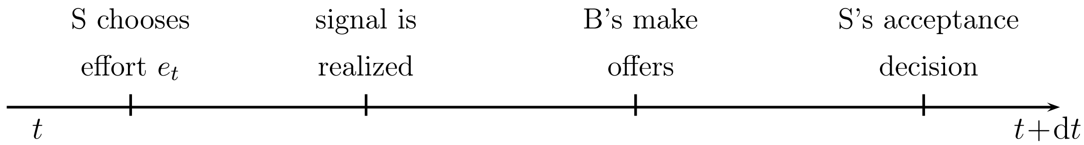


Figure 1: Heuristic timing. "S" refers to the seller, "B's" refers to the buyers.

instead of $c_{H,0}$ to save notation.[5]

There is a competitive market with a continuum of identical competitive (potential) buyers. The value of an asset with a given quality is common across all buyers. The value of an H-asset to a buyer is $\Pi_H \equiv \Pi > \underline{V}_H$, while the value of an L-asset to a buyer is normalized to be $\Pi_L = 0$. They do not observe the type of the seller and share a common prior $p_0 \in (0,1)$ about the asset's quality being high. Buyers are risk-neutral and maximize their expected payoff.

2.1 Strategies, Payoffs and Equilibrium Concept

Strategies and Payoffs

We define the strategies similarly as in Daley and Green (2012). In particular, we do not directly model each buyer. Instead, we model the buyer side using an "offer process" W_t adapted to the filtration $(\mathcal{H}_t)_{t\geq 0}$. As pointed in Daley and Green (2012), this can be micro-found by interpreting W_t as the hidden offers that the seller receives from (two or more) short-lived buyers that only observe the history of (public) signals at time t. The equilibrium conditions on the process W ensure that it reproduces the offer process resulting from the Bertrand competition among buyers as in, for example, Swinkels (1999).

An *effort-choice strategy* for the θ-seller is a stochastic process $e_\theta = \{e_{\theta,t}, 0 \leq t \leq \infty\}$ that is

1. non-negative, right-continuous[6] and adapted to the filtration $(\hat{\mathcal{H}}_t)_{t\geq 0}$, where $\hat{\mathcal{H}}_t$ is the

[5]We can interpret this fixed cost of providing an extra effort to increase sales as an opportunity cost of the time devoted to this. In the education setting, this may be regarded as the cost of attending class (opportunity cost in salaries, for example). L-workers, instead, could already be enjoying their outside option, by just taking the exams.

[6]Right-continuity ensures that when there is a jump in the effort function, the time at which it happens is well defined. This will be particularly important when e_H jumps to 0, since the signal is uninformative there, and therefore the posterior is not updated using the signal.

σ-algebra generated by $\{((X_s, W_s)_{0\le s\le t}, (e_{\theta,s})_{s<t})\}$, and

2. 0 for all t if $\theta = L$.

A *pure acceptance-decision strategy* for the type $\theta \in \{L, H\}$ seller is a $\hat{\mathcal{H}}_t$-adapted stopping time, $\tau_\theta : \Omega' \to \mathbb{R}_+$, where $\Omega' = \{((X_s, W_s)_{0\le s<\infty}, (e_{\theta,s})_{s<\infty})\}$. A *(mixed) acceptance-decision strategy* is a distribution over such times, which can be represented as a stochastic process $(S_t^\theta)_{t\ge 0}$, which is right-continuous, $[0,1]$-valued, increasing and $\hat{\mathcal{H}}_t$-measurable. It is interpreted as the CDF over the type-θ seller's acceptance time on $\mathbb{R}_+ \cup \{\infty\}$. A *strategy* for the θ-seller is a pair (e_θ, S_θ). If the seller accepts an offer at some time t, he leaves the market, and the buyer makes a lump-sum payment W_t to the seller. A heuristic timing is plotted in Figure 1.

For a fixed offer process W, the payoff for the θ-seller is composed of the flow cost of providing effort and the lump-sum payoff when the game stops. Indeed, the θ-seller faces the following problem

$$\sup_{e_\theta, \tau \ge 0} \mathbb{E}_t\big[-\textstyle\int_t^\tau c_\theta(e_{\theta,s})\,\mathrm{d}s + \mathbb{I}_{\tau<\infty} W_{\tau_{\theta,t}} + \mathbb{I}_{\tau=\infty} \underline{V}_\theta \,\big|\, e_\theta \big] \,. \tag{2.2}$$

We say that a pair (e_θ, S_θ) solves (2.2) if for each $\tau \in \operatorname{supp}(S_\theta)$, (e_θ, τ) solves (2.2). Furthermore, if (e_θ, S_θ) solves (2.2), for any $(t, \omega) \in \mathbb{R}_+ \times \Omega'$ such that $S^\theta(\omega) < 1$ (so $\operatorname{supp}(S_\theta) \neq 0$) we have that for all $\tau \in \operatorname{supp}(S_\theta)$ the continuation value for the θ-seller has the following form:

$$V_{\theta,t} \equiv \mathbb{E}\big[-\textstyle\int_t^\tau c_\theta(e_{\theta,s})\,\mathrm{d}s + \mathbb{I}_{\tau<\infty} W_{\tau_{\theta,t}} + \mathbb{I}_{\tau=\infty} \underline{V}_\theta \,\big|\, e_\theta, S_\theta \big] \,, \tag{2.3}$$

which is a $\hat{\mathcal{H}}_t$-measurable function.

Note that $V_{H,t} \ge V_{L,t}$. Indeed, the H-seller has the option of mimicking the strategy of the L-seller. In this case the signal would have the same distribution, so the seller would face the same expected price offers and a higher outside option, and the cost of signaling would be the same. Also, $V_{\theta,t} \ge \underline{V}_\theta$ for all $\theta \in \{L, H\}$, given the option to wait at 0 cost. So, given that the buyers' offers will be no higher than Π, the payoff functions are well-defined.

Since there is no time discounting and no fixed cost of time if no effort is made, the seller's payoff is affected only by the expected price when the asset is sold, the outside option (if he does not sell the asset) and the expected total cost of effort. Therefore, at any moment in time, the H-seller's tradeoff will be to exert high effort and increase the expected price offer or to exert low effort and lower the expected price offer. The channel that translates effort to higher expected prices is the signal, which the buyers will use to update their beliefs about the seller.

Beliefs Process

The payoff to a buyer who makes a price offer is given by the probability of this offer being accepted multiplied by her asset valuation minus the price. So, we need to characterize the buyers' beliefs about the quality of the asset after each history in order to determine their optimal strategy.

Let $(P_t)_{t\in\mathbb{R}_+}$ be a stochastic process adapted to X^t measuring the posterior of the buyers at time t (given the previous public history) about the type of the seller being H.[7] Let $Z_t \equiv \log(\frac{P_t}{1-P_t})$ be the corresponding log-likelihood ratio. Then, following Daley and Green (2012) we can separate belief updating between updating because of the signal ($\hat{Z}_t$) and because of the rejection of offers ($\tilde{Z}_t$):

$$Z_t = \hat{Z}_t + \tilde{Z}_t \ .$$

Since, as we will see, the different types of sellers pool on the decision of rejecting offers, it is convenient to normalize $\tilde{Z}_0 = 0$. Note that, given our definition of the rejection strategy, we have $\tilde{Z}_t = \log(\frac{S_{H,t^-}}{S_{L,t^-}})$. The standard Bayes' rule is used to update $\hat{Z}_t$ (see, for example, Faingold and Sannikov (2011)).

Equilibria

Definition 2.1. An *equilibrium* in our model is a strategy profile $(S_\theta, e_\theta)_{\theta\in\{L,H\}}$, a price offer process W and a beliefs process P such that:

1. Sellers optimality. Given W, (e_θ, S_θ) solves the θ-seller's problem (2.2).
2. Belief Consistency. For all t such that $S_{L,t^-}\, S_{H,t^-} < 1$, Z_t is obtained using Bayes' rule.
3. Zero profit. If there exists a $\tau \in \text{supp}(S^L) \cup \text{supp}(S^H)$ such that $\tau(\omega) = t$ for some $\omega \in \Omega'$, then $W_t = \mathbb{E}[\Pi_\theta | \mathcal{H}_t, \tilde{\tau} = t]$, where $\tilde{\tau}$ is the stopping time induced by $(S_\theta, e_\theta)_{\theta\in\{L,H\}}$.
4. No (Unrealized) Deals. For all θ, t, and ω such that $S_{\theta,t^-}(\omega) < 1$, $V_{\theta,t}(\omega) \geq \mathbb{E}[\Pi_{\theta'} | \mathcal{H}_t, \Pi_{\theta'} \leq \Pi_\theta]$.

As is common in settings where the only payoff-relevant variable for the uninformed part of the market is the type of the informed part, we restrict ourselves to Markov strategies and Markov equilibria with beliefs as the state variable:

[7] We interpret P_t as the posterior of the buyers at the moment of making the offer at time t. Therefore, P_t does not include the information involving the rejection of the offer at time t.

Definition 2.2. A *Markov equilibrium (ME)* is a public equilibrium where both the seller and the buyers follow Markov strategies, the state variable being $p = P_t$, the posterior of the buyers about the type of the seller being H.

2.2 Equilibrium Characterization

Signaling Region

From now on we will focus on Markov equilibria, and therefore we will call them simply equilibria. The following lemma establishes an important property of Markov perfect equilibria:

Lemma 2.1. *In any equilibrium, types pool on the acceptance-decision strategy. Also, the signaling region* $R \equiv \{p \big| V_H(p) > \max\{p\,\Pi, \underline{V}_H\}\}$ *is open in* $[0, 1]$ *and such that*

1. *there is no trade inside* R*:* $\Pr(P_{\tilde{\tau}} \in R) = 0$*; and*
2. *the signal is only useful in* $\bar{R}$*:* $P_t \notin \bar{R} \Rightarrow \Pr(P_{\tilde{\tau}} = P_t) = 1$.

Note that $\Pr(P_{\tau_\theta} \in R|\theta) = 0$ implies that, in equilibrium, the game never ends when $P_t \in R$; that is, both buyer's types reject with probability one all the (equilibrium) offers in this region.

This implies that beliefs updating in R is driven only by the signal realization, not by the rejection of offers. If $P_t \notin R$ either both types accept the corresponding equilibrium offer with probability one, or the signal is not informative, so beliefs remain constant. We say that an equilibrium is *(partially) separating*[8] if $R \neq \emptyset$, and *pooling* otherwise.

Equilibrium Strategies

We first analyze the equilibrium behavior for a given rejection region $R \neq \emptyset$, assuming it exists. After this, we will characterize the existence of equilibria (the conditions for existence are obtained in Sections 3 and 4).

Consider an equilibrium with signaling region R. Let's first consider the case $p_0 \notin R$. The following lemma characterizes the equilibrium offer acceptance:

Lemma 2.2. *Assume* $p_0 \notin R$*. Then,*

[8] As usual, under a separating equilibrium, different types exert different efforts in some beliefs region. So if $p_0 \in R$, the accepted price offer will depend on the signal history, and different types will have different distributions over accepted offers.

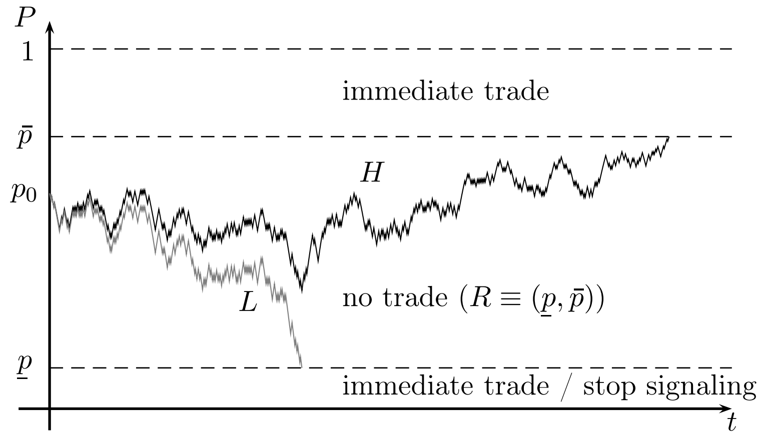


Figure 2: Example of beliefs histories for a signaling region $R = (\underline{p}, \bar{p})$ for the two types. In the simulation, for the same realization of the Brownian motion, the L-seller (gray line) ends up accepting a low offer, while the H-seller (black line) ends up accepting a high offer.

1. *if* $p_0 \Pi > \underline{V}_H$*, all offers equal* $p_0 \Pi$ *and the game ends with probability one;*
2. *if* $p_0 \Pi < \underline{V}_H$ *then there is no trade; and*
3. *if* $p_0 \Pi = \underline{V}_H$ *then either 1. or 2. takes place.*

Note that when pooling is an equilibrium in the static game ($p_0 \Pi > \underline{V}_H$) only the signal (and therefore the effort) is used to separate types and not the decision to accept offers. Intuitively, the L-seller has the option to not accept an offer and wait at no cost, so, in equilibrium, the H-seller will not be able to separate himself from the L-seller by not accepting an offer. The crucial property for this result to hold is not that waiting is costless but that there is no common knowledge of positive gains from trade. In this case, the low-quality seller has the option to wait without incurring any (opportunity) cost.[9] This will help us to identify the new implications of the introduction of imperfectly observable effort as the signaling device.

Let's now assume $p_0 \in R$. Using the results from Lemma 2.1, and given a Markov perfect equilibrium with signaling region R and an initial prior $p_0 \in R$, we define the following limits

$$\underline{p} \equiv \sup \left(\{0\} \cup \left\{ p \leq p_0 \middle| p \notin R \right\} \right) , \tag{2.4}$$

$$\bar{p} \equiv \inf \left(\{1\} \cup \left\{ p \geq p_0 \middle| p \notin R \right\} \right) . \tag{2.5}$$

[9]This is not the case, for example, in Daley and Green (2012), where there is common knowledge of positive gains from trade, and sellers do not pool on rejection strategies.

We know by Lemma 2.2 that the value function of the H-seller at $\underline{p}$ is $\underline{W} \equiv \max\{\underline{p}\,\Pi, \underline{V}_H\}$ and at $\bar{p}$ it is $\overline{W} \equiv \bar{p}\,\Pi > \underline{V}_H$.[10] Note that since R is open, $\underline{p} < p_0 < \bar{p}$. Then, if the initial prior lies in the region $(\underline{p}, \bar{p})$, since P_t moves continuously inside R, the process will stop when P_t reaches either $\underline{p}$ or $\bar{p}$ (where the seller will accept the corresponding price offer). Figure 2 exemplifies two outcomes of the game, with the same realization of the Browinan motion, but with different realizations of the type.

As is usual in the literature on dynamic games in continuous-time, we restrict ourselves to equilibria where $V_H \in \mathcal{C}^2(R) \cup \mathcal{C}^0(\bar{R})$. In this case, for a given equilibrium strategy $e_H(\cdot)$, the optimal effort choice of the H-seller solves the Hamilton-Jacobi-Bellman (HJB) equation, which is given by[11]

$$0 = \max_{\hat{e}_H(p)} \Big(-c_H(\hat{e}_H(p)) + \tilde{\mu}(\hat{e}_H(p), p, e_H(p))\, V_H'(p) + \tfrac{1}{2}\, \tilde{\sigma}(p, e_H(p))^2\, V_H''(p) \Big)\ , \tag{2.6}$$

with boundary conditions $V_H(\bar{p}) = \overline{W}$ and $V_H(\underline{p}) = \underline{W}$, and where

$$\tilde{\mu}(\hat{e}_H, p, e_H) \equiv \frac{(1-p)\, p\, e_H\, (\hat{e}_H - p\, e_H)}{\sigma^2} \quad \text{and} \tag{2.7}$$

$$\tilde{\sigma}(p, e_H) \equiv \frac{(1-p)\, p\, e_H}{\sigma} \tag{2.8}$$

are, respectively, the drift and the volatility of the belief process P_t when $P_t \in R$. Note that when $e_H = e_L = 0$ both the drift and the volatility of the beliefs process are 0, independent of the effort choice $\hat{e}$. This is an important feature of our model that differs from the standard dynamic signaling models: if buyers believe that the signal is uninformative, then the seller cannot change the buyers' beliefs through the signal.

The maximization problem (2.6) is strictly concave in $\hat{e}_H(p)$ for $\hat{e}_H(p) > 0$. So, under the assumption that R is a signaling region of an equilibrium,[12] the first-order condition (FOC) is sufficient for $p \in R$. We differentiate (2.6) with respect to $\hat{e}_H(p)$, for $\hat{e}_H(p) > 0$ to get the FOC and we get

$$A_H\, \alpha\, \hat{e}_H(p)^{\alpha-1} = \frac{V_H'(p)\,(1-p)\, p\, e_H(p)}{\sigma^2}\ . \tag{2.9}$$

[10] Note that if it was the case that $\overline{W} = \underline{V}_H$, then also $\underline{W} = \underline{V}_H$, so the revenue for the seller would be just $\underline{V}_H$. Therefore, the H-seller would not put any effort, what would contradict $p_0 \in R$.

[11] The HJB equation is obtained using the dynamic programming principle. Note that in equilibrium it needs to be the case that $e_H(\cdot)$ itself is a solution of the HJB equation.

[12] Recall that Lemma 2.1 ensures that $e_H(p) > 0$ when $p \in R$. Furthermore, if R is the signaling region of an equilibrium, the solution $V_H(\cdot)$ of the problem (2.6) satisfies $V_H(p) \geq p\,\Pi$ as is required in our equilibrium definition.

The following lemma establishes the functional form of the policy functions:

Lemma 2.3. *There is a unique $e_H(\cdot)$ such that the solution $\hat{e}_H(p)$ of the HJB equation* (2.6) *satisfies $\hat{e}_H(p) = e_H(p)$ for all $p \in (\underline{p}, \bar{p})$, and it is given by*

$$e_H(p) = \left(\frac{C_1 \, (1-p)}{(2-\alpha)\, p} + \frac{2\, c_0}{A_H \, (\alpha - 2)} \right)^{1/\alpha} , \tag{2.10}$$

where C_1 is a constant to be determined by the boundary conditions on $V_H(\cdot)$.

Using the FOC (2.9) and the policy function (2.10), we can find an integral expression for $V_H(\cdot)$:

$$V_H(p) = \underline{W} + \int_{\underline{p}}^{p} \frac{A_H \, \alpha \, \sigma^2 \, e_H(q)^{\alpha - 2}}{(1-q)\, q} \, \mathrm{d}q \, . \tag{2.11}$$

The boundary condition $V_\theta(\bar{p}) = \overline{W}$ determines the value for C_1. The value function for the L-seller is shown at (3.5). Note that $V_L(\cdot)$ depends only on $\underline{p}$, $\bar{p}$ and the payoffs at these points, since it is determined only by the expected revenue.

2.3 Most Separating Equilibria

Dynamic signaling models, in general, feature a high equilibrium multiplicity. In order to focus on some particular equilibrium, different refinements or selection criteria are used, intended to select the equilibrium with the most separation.

Most of the equilibrium multiplicity arises from the so-called beliefs threats. These are given by "punishments" of the buyers by lowering the posterior about the (quality of the asset of the) seller after observing some deviation. In most of the models, the only observable deviation is the rejection of an offer when, in equilibrium, the offer was supposed to be accepted with probability one. In our model there is an additional "beliefs threat" due to the hidden action. It is given by the fact that the buyers may believe that, after a given history, the effort of (both types of) the seller is 0 thereafter, and therefore the signal becomes useless.[13] This makes it convenient to directly focus on the most separating equilibria, that is, equilibria where the signaling region is maximized. We will see afterwards that this equilibrium has properties similar to those of equilibria that pass the selection criteria used in the previous models.

[13] This prevents refinements such as Never-a-Weak-Best-Response (used in Nöldeke and van-Damme 1990) or Belief Monotonicity (used in Swinkels (1999) and Daley and Green (2012)) from working in our model.

Definition 2.3. An equilibrium with signaling region R is a *most separating equilibrium (MSE)* if $R \supset R'$ for any signaling region R' of another equilibrium.

Note that if an MSE exists, it "essentially" is unique, since given our previous results the outcome generated by a signaling region is unique.

Lemma 2.4. *Let $V_H(\cdot)$ and $\tilde{V}_H(\cdot)$ be the payoff functions of an MSE and a non MSE, respectively. Then $V_H(p) \geq \tilde{V}_H(p)$ for all p, and $V_H(p) > \tilde{V}_H(p)$ for some p.*

The previous result makes MSE particularly appealing. Indeed, markets where an MSE is played will attract more H-sellers, since they gain the most. Also, it is easy to show that MSE are in the spirit of most selection criteria, that prevent punishing deviations that are more "likely" to be carried out by high types.[14]

Lemma 2.4 establishes that an(y) MSE is the "most preferred" by the H-sellers. Therefore, an MSE solves the H-seller's problem (2.6) allowing the seller to choose the boundaries of R (and requiring $V_H(p) = \max\{\underline{V}_H, p\,\Pi\}$ for $p \notin R$). This naturally leads to the following useful technical result:

Lemma 2.5. ***(smooth pasting condition)*** *Assume that there exists an MSE, and that $V_H(\cdot)$ is the value function for the H-seller in this equilibrium. Then $V_H(p) \in \mathcal{C}^1(0,1)$.*

3 Not-Very-Convex Cost Function ($\alpha < 2$)

The following proposition establishes the existence of an MSE for all values of c_0 and $\underline{V}_H$:

Proposition 3.1. *When $\alpha < 2$ there exists a unique $\bar{p}_* \in (\frac{\underline{V}_H}{\Pi}, 1)$ such that $(0, \bar{p}_*)$ is the signaling region of all MSE.*

Figure 3 (a) plots the value function for the H-seller in the MSE, for different values of $\underline{V}_H$. Note that as $\underline{V}_H$ gets close to Π, the signaling region increases (it always contains $(0, \frac{\underline{V}_H}{\Pi})$). Hence, signaling takes place in a potentially large region of the beliefs space even when c_0 and A_H are arbitrarily high. This result is not present in static models or dynamic models where the drift is exogenous, in which the size of the region where the signaling takes

[14] More formally, if after the rejection of an offer that in equilibrium is accepted by all the types of the seller a posterior is assigned to the deviator and a new equilibrium is played, only MSE ensure that the H-seller benefits from deviating only when the L-seller also benefits from deviating.

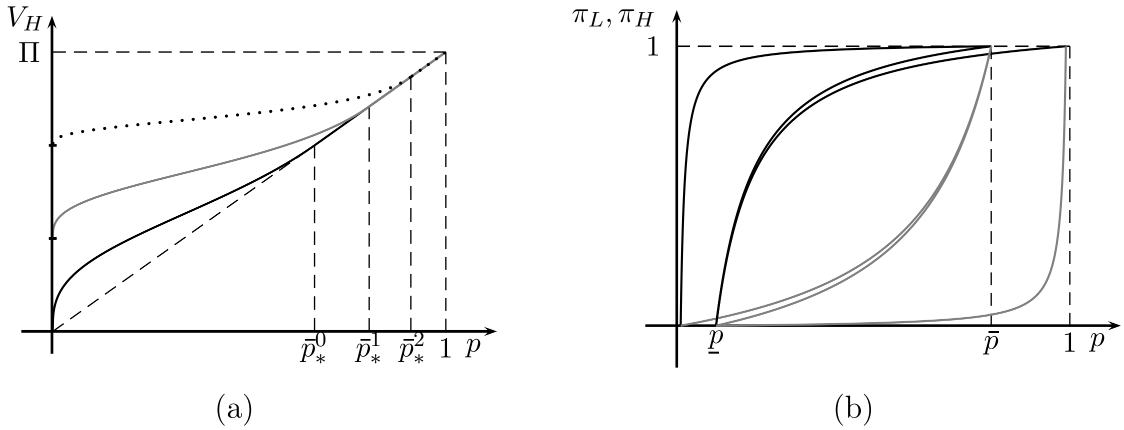


Figure 3: In (a), $V_H(\cdot)$ of an(y) MSE for different values of $\underline{V}_H$, when $\alpha < 2$. In (b), the probability of reaching $\bar{p}$, for different values of $\bar{p}$ and $\underline{p}$ (gray and black lines correspond to the L-seller and the H-seller, respectively.)

place becomes arbitrarily small or disappears when the fixed cost rises. In our case, high costs involve high equilibrium effort, which implies an improvement in the quality of the signal and therefore an acceleration of the signaling process, which compensates the high cost per unit of time. Similarly, we have the following result regarding the noise:

Corollary 3.1. *Assume $\alpha < 2$. Let $(0, \bar{p}_*(\sigma))$ be the signaling region of the MSE for each volatility σ. Then, if $\sigma_1 > \sigma_2$, $\bar{p}_*(\sigma_1) > \bar{p}_*(\sigma_2)$.*

The logic behind the result is clear. When noise increases, separation requires a higher effort. In our model, for a not-very-convex cost function, a high effort makes the signaling less inefficient. So, even though there is more waste per unit of time, the signal is more informative, so the H-seller is able to signal the quality of his asset less inefficiently.[15]

There are two other features of the MSE that are not standard in the previous literature. The first is that the effort function is always decreasing, even when p is small. In particular, this implies that the H-seller has incentives to provide high effort even when belief updating is small. The second is that the signaling region's lower bound is 0; that is, the H-seller prefers to keep signaling even when the market is very pessimistic. These features are not present in models with exogenous information revelation. The reason is that low posteriors

[15] de Haan et al. (2011) show that, in a static signaling model, high types increase their effort when the noise increases. In their model, this increase drives a more wasteful signal, and if the level of noise is high enough separating equilibria cease to exist. In our model effort increases when noise increases only if highly-informative signals are less inefficient. In this case this increases the ability of the H-seller to separate himself from the L-seller, so high noise does not compromise the existence of separating equilibria.

imply slow belief updating, so (at least) one type of seller accepts offers with positive probability and/or stops signaling. Therefore, it is worth analyzing these two features separately.

Corollary 3.2. *When $\alpha < 2$, the effort exerted by the seller ($e_H^*(\cdot)$ in* (2.10)*) is decreasing.*

In order to understand this, let's decompose the value function into two parts, $V_H(p) = E_P(p) - E_c(p)$. The second part, $E_c(p)$, is the expected cost of signaling. The first part, $E_P(\cdot)$, reflects the expected revenue from the sale (or keeping the asset). Using Bayes' rule, we find that the expected revenue takes the following form

$$E_P(p) \equiv \frac{\bar{p}\,(p - \underline{p})}{p\,(\bar{p} - \underline{p})}\,\overline{W} + \frac{\underline{p}\,(\bar{p} - p)}{p\,(\bar{p} - \underline{p})}\,\underline{W} \ . \tag{3.1}$$

If we differentiate the expected revenue we get

$$E_P'(p) = \frac{\underline{p}\,\bar{p}\,(\overline{W} - \underline{W})}{p^2\,(\bar{p} - \underline{p})} \ .$$

Note that when p is small, small increases in p lead to large increases in the expected revenue. This provides high incentives for the seller to exert effort. Nevertheless, as we know, belief updating is, in general, slow when beliefs are close to 0. As we will see in Section 5, high effort makes signaling less inefficient, so the increase in the expected payoff (expected revenue minus cost) compensates the slow updating.

In order to have an intuition as to why the lower bound in the signaling region is $\underline{p} = 0$, we focus our analysis on the limit $c_0 \to 0$. This analysis is relevant since $e_H(p)$ is very high when p is low, so for low posteriors the fixed cost is very small compared with the variable cost.

3.1 No-Fixed-Cost Limit

None of the previous results in Section 3 relies on the fact that $c_0 > 0$. So, we can easily compute the effort policy function for $c_0 = 0$, which now can be explicitly solved for an interval signaling region $(\underline{p}, \bar{p})$:

$$e_H^*(p) = \left(\frac{(\alpha - 2)\,(\overline{W} - \underline{W})}{A_H\,\alpha^2\,\sigma^2\,\big(h(\underline{p}) - h(\bar{p})\big)} \right)^{\frac{1}{\alpha - 2}} \left(\frac{1-p}{p} \right)^{\frac{1}{\alpha}} \tag{3.2}$$

where

$$h(p) \equiv \left(\frac{p}{1-p} \right)^{\frac{2-\alpha}{\alpha}} \ . \tag{3.3}$$

Note that when $\alpha < 2$, h is increasing, $h(0) = 0$ and $\lim_{p\to 1} h(p) = \infty$. The value functions $V_H(\cdot)$ and $V_L(\cdot)$ take the following form

$$V_H(p; \underline{p}, \bar{p}, \underline{W}, \overline{W}) = \frac{h(p) - h(\underline{p})}{h(\bar{p}) - h(\underline{p})} (\overline{W} - \underline{W}) + \underline{W} \ , \tag{3.4}$$

$$V_L(p; \underline{p}, \bar{p}, \underline{W}_L, \overline{W}) = \frac{(p - \underline{p})\,(1 - \bar{p})}{(1 - p)\,(\bar{p} - \underline{p})} (\overline{W} - \underline{W}_L) + \underline{W}_L \ , \tag{3.5}$$

where $\underline{W}_L$ is the value function of the L-seller at $\underline{p}$. Note that V_L is given by the expected revenue conditioning on exerting zero effort (since the cost is 0). We can use the previous formulae to provide a result analogous to Proposition 3.1, with an explicit form of $\bar{p}_*$.

Proposition 3.2. *When $\alpha < 2$ and $c_0 = 0$, the signaling region of the MSE is $R = (0, \bar{p}_*)$, with*

$$\bar{p}_* \equiv \frac{\alpha - 1}{\alpha} \left(1 + \sqrt{1 + \frac{(2 - \alpha)\,\alpha\,\underline{V}_H}{(\alpha - 1)^2\,\Pi}} \right) \quad \in \left[\frac{2\,(\alpha - 1)}{\alpha}, 1 \right] . \tag{3.6}$$

Again, we have that the lower bound of R is $\underline{p} = 0$. One could think that this is a consequence of the assumption that the L-seller does not incur a cost by waiting, so he, with some positive probability, stays in R forever, while the H-seller reaches $\bar{p}$ in finite time with probability 1. Nevertheless this is not the case, since as we show in Appendix B the L-seller reaches $\underline{p} = 0$ in finite time. This may seem counterintuitive since when a signal is random and there is no event that happens with positive probability under one type's strategy and with 0 probability under the other type's strategy, complete information is hardly achievable in finite time.

In order to have an intuition about the previous result, let's define the stochastic process $Y_t \equiv \left(\frac{P_t}{1-P_t}\right)^{1/\beta}$, for some $\beta \in (1, \alpha)$. Note that Y_t is increasing and concave in P_t and $Y_t = 0$ when $P_t = 0$. Using a standard stochastic calculus, it is easy to verify that the drift of Y when the effort is $e_t = 0$ (denoted $\mu^L_{Y,t}$) and the variance of Y ($\sigma^2_{Y,t}$ independently of the type) are given by

$$\mu^L_{Y,t} = -\frac{(\beta - 1)\, C_2^2\, Y_t^{1 - \frac{2\beta}{\alpha}}}{2\,\beta^2\,\sigma^2} \quad \text{and} \quad \sigma^2_{Y,t} = \frac{C_2^2\, Y_t^{2 - \frac{2\beta}{\alpha}}}{\beta^2\,\sigma^2} \ ,$$

where C_2 is the constant in $e^*_H(p) = C_2\,((1 - p)/p)^{1/\alpha}$ for the optimal signaling region (see (3.2)). Note that the drift of Y_t when $e_t = 0$ is negative and increases in absolute value when $p \to 0$. Conversely, the variance shrinks to 0 as $Y_t \to 0$. Therefore, Y_t (and also P_t)

hits 0 in finite time when $e_t = e_{L,t} = 0$. The drift of Y_t when the effort is $e_{H,t}$ is given by

$$\mu^H_{Y,t} = \frac{(\beta+1)\, C_2^2\, Y_t^{1-\frac{2\beta}{\alpha}}}{2\,\beta^2\,\sigma^2}\ .$$

Note that in this case the drift is positive and becomes arbitrarily large when Y_t is small, so the H-seller stays away from 0.[16]

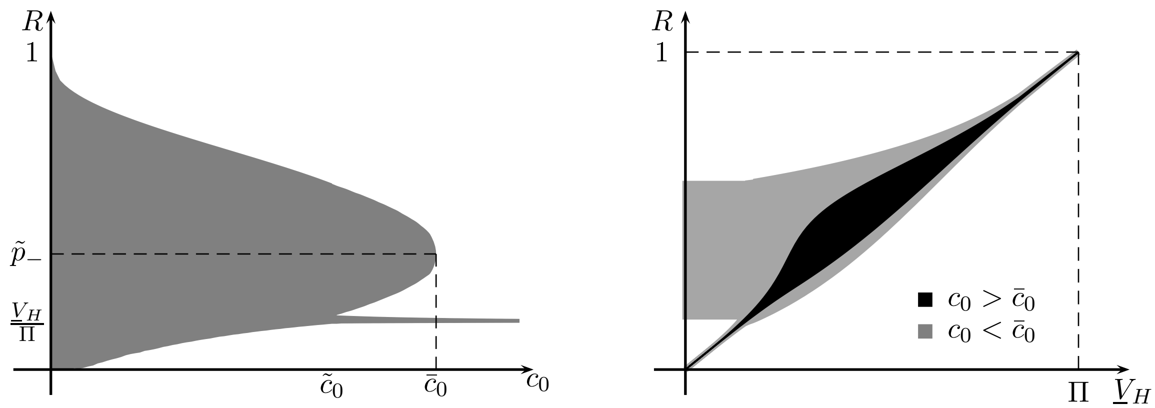
Intuitively, when P_t (or Y_t) gets close to 0, there is a balance between two opposite incentives. The first is to exert low effort due to the slow updating of beliefs. The second is to exert high effort, since the expected revenue function gets steeper. In models with a fixed effort (or signal informativeness), slow beliefs updating forces the seller to "give up." In our model, instead, when the cost function is not very convex, high effort increases the speed of beliefs updating and makes signaling more efficient. In equilibrium this increases the incentive to exert effort and the equilibrium effort. As we will see, this is not the case when the cost function is very convex.

This result should not be considered a continuous-time rarity. Even though in discrete-time versions of this model buyers are never perfectly convinced in finite time about the type of the seller, as the time interval gets short they become arbitrarily convinced. Indeed, numerical simulations show that for each length of the period $\Delta > 0$, the signaling region of the MSE takes the form $R(\Delta) = (\underline{p}(\Delta), \bar{p}(\Delta))$, and is such that $\lim_{\Delta\to 0} \underline{p}(\Delta) = 0$. Since when $\underline{p}(\Delta)$ is low the effort is large for low posteriors, beliefs are updated very quickly, and $\underline{p}(\Delta)$ can be reached in a relatively short time. Also, as can be seen from the explicit effort and value functions (equations (3.2-3.5)), these converge pointwise when we consider signaling regions with $\underline{p} \to 0$.

4 Very-Convex Cost Function ($\alpha > 2$)

The following proposition establishes the existence of equilibria in this case and introduces equilibria where the signaling region is composed of two open intervals instead of one:

[16] It may be helpful for the reader to realize that $\Pr(\int_0^{\min\{\tau_L,t\}} e_H(P_s)\mathrm{d}P_s = \infty | \theta = L) > 0$ for a finite t, but if $\theta = H$, $\Pr(\int_0^{\min\{\tau_L,t\}} e_H(P_t)\mathrm{d}P_t = \infty | \theta = H) = 0$. Intuitively, when P_t decreases, the signal becomes more informative. Therefore, (very) low posteriors are reachable with (very) small probability when $e_t = e_H(P_t)$, and this keeps the expected cost for the H-seller low. When $e_t = 0$, instead, P_t reaches low posteriors with a higher probability, so $e_H(P_t)$ reaches very high values more frequently.



(a) (b)

Figure 4: In (a), R as a function of c_0 for a fixed $\underline{V}_H$. In (b), R as a function of $\underline{V}_H$, for different values of c_0.

Proposition 4.1. *Assume $\alpha > 2$. Then there is a unique MSE. Also, there exist $0 < \tilde{c}_0 < \bar{c}_0$ such that if $c_0 \in (\tilde{c}_0, \bar{c}_0)$ the signaling region for the MSE is the union of two disjoint open non-empty intervals, and it is formed by one open non-empty interval otherwise.*

In Figure 4 (a) we see that when c_0 is low, the signaling region is close to $(0, 1)$; that is, the equilibrium is close to be fully separating. In the limit $c_0 \to 0$ the signaling waste disappears, and the model converges to the full information analogous. As c_0 gets larger, R shrinks, and the seller stops signaling when beliefs are either low or high (that is, when signaling is slow enough that does not compensate to pay the fixed cost of signaling). When the fixed cost passes some given cutoff (depicted as $\tilde{c}_0$), the signaling region splits in two, one containing $\tilde{p}_-$ (defined in the proof of Lemma A.1 in the Appendix) and the other containing $\frac{\underline{V}_H}{\Pi}$. In the first region fast beliefs updating makes the signal valuable, so the seller exerts effort. This region shrinks as c_0 increases and vanishes when $c_0 \geq \bar{c}_0$. In the second region (the lower region), the signal is valuable due to the kink in the boundary conditions. Even of signaling is very costly, the kink in the continuation value of a model without signaling make some signaling worth. As c_0 gets large this region shrinks but never disappears. In this limit, for most initial beliefs, p_0, the asset is sold at $p_0 \Pi$ (if $p_0 > \frac{\underline{V}_H}{\Pi}$) or the seller does not accept any offer (if $p_0 < \underline{V}_H$), independently of his type.

In Figure 4 (b) we plot R as a function of $\underline{V}_H$, for two different values of c_0. Again, we see that the higher the cost, the smaller the region where signaling takes place, for any value of $\underline{V}_H \in (0, \Pi)$. When c_0 is large (larger than $\bar{c}_0$, represented by the black area), R is an interval for all values of $\underline{V}_H$. This interval is small when beliefs are either low or high (beliefs updating is slow), and big for intermediate values of the posterior, where fast beliefs updating makes signaling worthwhile. When c_0 is lower than $\bar{c}_0$ (gray area), instead, Proposition 4.1 establishes that, if $\underline{V}_H$ is small enough, R is split in two parts, as explained in the previous paragraph.

4.1 Increasing Effort

We found in Section 3 that, when $\alpha < 2$, the effort is decreasing in p. The rationale was that the H-seller gets scared when p is low, so he exerts a high (and more efficient) effort in order to avoid a low offer. This is no longer the case when $\alpha > 2$.

Corollary 4.1. *Assume $\alpha > 2$. In any MSE with signaling region R, $e_H(\cdot)$ is increasing on R.*

We find that when the cost function is very convex, the equilibrium effort is increasing in the posterior. Now, exerting a high effort is not efficient (see Section 5.1 for the optimal effort). So, the "rush out" effect is substituted by a "give up" effect; that is, now the seller gets discouraged when p is close to $\underline{p}$. Intuitively, when p gets lower, updating is slow, and because the high convexity of the cost function implies that exerting a high effort is very costly, the seller "gives up" and stops signaling.

In all equilibria when $\alpha < 2$, the infimum of the signaling region R is some $\underline{p} > 0$ where $V^H(\underline{p}) = \underline{V}_H$. By the smooth pasting condition (Lemma 2.5) we have that $\lim_{p \to \underline{p}} V'^H(p) = 0$, and then, by the first-order condition (2.9) we have $\lim_{p \to \underline{p}} e_H(p) = 0$. So, we have a region of the belief space where the effort exerted by the seller is arbitrarily small, even when c_0 is large. This may be surprising, since in our model small effort implies slow revelation of information, and the presence of the fixed cost seems to require information revelation to be fast in order to make it worthwhile. It is easy to see (similar to what we did in Section 3.1) that P_t reaches $\underline{p}$ with positive probability, independently of the type. Also, $\lim_{p \to \underline{p}} e'_H(p) = +\infty$, so the region where the effort is small is itself very small.[17] Hence, when the posterior gets close to $\underline{p}$, the seller "gives up" by choosing low effort. This happens in a region small enough that is abandoned (either by hitting $\underline{p}$ or by p increasing) very quickly, so the fixed cost incurred is not large.

[17]Formally, $e_H(p)$ is $O(\varepsilon)$ only if $p - \underline{p}$ is $O(\varepsilon^\alpha)$. Since $\alpha > 2$, $p - \underline{p}$ converges to 0 faster than $e_H(p)$.

5 Observable Effort

In order to understand the previous results we consider a variation of our model where the effort made by the seller is observable. This will help us to investigate the distortion generated by the unobservability of the effort but keep the unobservability of the type.

We assume that now the effort put into signaling is observable by the buyers. In order to allow the L-seller to mimic the H-seller,[18] in this section we assume that he can (pretend to) make an observable effort at 0 cost, but that effort leaves the drift of X unchanged. Instead, if the H-seller makes an observable effort $e > 0$, he incurs a cost $c_H(e)$, but the drift of X is e as before.[19]

Proposition 5.1. *Assume $\alpha < 2$. Fix a signaling region R and assume effort is observable. Fix a strictly positive policy function $e_H(\cdot) \in \mathcal{C}^1(R)$ and let $V_H(p, e_H)$ be the corresponding value function of the H-seller at p. Then, $V_H(p, \lambda\, e_H) > V_H(p, e_H)$ for all $\lambda > 1$ and $p \in R$.*

The intuition for the previous result is as follows. The HJB equation for the problem with observable effort is the same as that for our main model (given in (2.6)) replacing $e_H(p)$ by $\hat{e}_H(p)$ (since now the seller fully internalizes the effect of his choice on the equilibrium strategy). Since the maximum is reached when the maximand is equal to 0, the solution would be the same if the maximand was divided by $\hat{e}_H(p)$. In this case we would have

$$0 = \max_{\hat{e}_H} \left(-A_H\, \hat{e}_H^{\alpha-2} - c_0\, \hat{e}_H^{-2} + \frac{p\,(1-p)^2}{\sigma^2}\, V_H'(p) + \frac{p^2\,(1-p)^2}{2\,\sigma^2}\, V_H''(p) \right) . \tag{5.1}$$

So, increasing $\hat{e}_H$ does not change the drift of the volatility of this new problem, but reduces the cost (both the fixed and the variable cost, since $\alpha < 2$). Therefore, in the limit where $\hat{e}_H \to \infty$, $V_H(\cdot)$ converges to the expected revenue ($E_P(\cdot)$ defined in (3.1)).

When $\alpha > 2$ the high convexity of the cost function prevents exerting a very high effort from being efficient (notice that, in equation (5.1), if $\alpha > 2$ an increase in $\hat{e}_H$ still lowers the fixed cost but now increases the variable cost). Instead, in the absence of a fixed cost, the seller would want to exert a very low effort and let the information be slowly revealed

[18] Since effort is observable and $e_L(\cdot) \equiv 0$ when $A_L = \infty$, any observation of effort higher than 0 would lead to perfect knowledge of the type.

[19] This construction is similar to Stackelberg action, often used in the reputations literature. The Stackelberg action of a player is the action that he would choose if, at the beginning of the game, he could publicly commit taking this action each period (without considering the incentive constraints in each period's game). Here, we allow for a full strategy, not just a single action.

over time. Nevertheless, the presence of the fixed cost requires a high flow of information in order to make exerting effort worthwhile. The following proposition establishes the optimal effort when $\alpha > 2$.

Proposition 5.2. *Assume $\alpha > 2$. There exists a maximal signaling region $\bar{R}$ such that, for any signaling region $R \subset \bar{R}$, the optimal observable effort choice by H-sellers is given by*

$$e_H(p) = e_*^{OE} \equiv \left(\frac{2\, c_0}{A_H\,(\alpha - 2)} \right)^{\frac{1}{\alpha}} . \tag{5.2}$$

Note that the optimal effort choice is constant. Since $e \mapsto c_H(e)$ is convex for $e > 0$ and the seller puts the same value on all instants of time, it is optimal for him to spread the same effort across all histories.

The optimal cost per unit of time is $\frac{\alpha\, c_0}{\alpha - 2}$. This does not depend on the value of A_H or σ. If, for example, σ increases, the incentive to signal is reduced. This is compensated with a steeper value function, which implies a reduction in the signaling region. So, the signaling technology (given by $g(\cdot)$ in footnote 4 and σ) affects the optimal cost only through the curvature of g. Changes in A_H and σ, instead, only change the decision to stop or keep signaling.

We can compare the functional form of the effort in our main model with unobservable effort (equation (2.10)) with the case when the effort is perfectly observable (equation (5.2)). It is clear that there is a distortion in the effort choice, since $C_1 > 0$ (see the proof of Corollary 4.1). This distortion makes the effort choice inefficiently low when effort is unobservable. The inefficiency is larger when p is small (interpreted as the "give up" effect).

Hence, both when $\alpha < 2$ and when $\alpha > 2$ the effort is unobservable the equilibrium choice is inefficiently low. This is a consequence of the extra information asymmetry given by the unobservability of the effort. Nevertheless, when $\alpha < 2$ the inefficiency is more severe for high posteriors ("rush-out effect" when p is low), while when $\alpha > 2$ it is more severe when p is low ("give-up effect"). As one can expect given the fixed cost of signaling, in both cases the distortion in the effort choice generates inefficiently small signaling regions.

> *Remark* 5.1. Proposition 5.2 allows us to recover the case where the drift is constant but type dependent (e_*^{OE} for H-sellers and 0 for L-sellers) and there is a cost per unit of time. Indeed, models in the literature where the drift depends on the type (but not on the effort), such as Daley and Green (2012), can be reinterpreted as optimal behavior when the effort (but not the type) is observable.

6 Conclusions

We fully characterize the equilibria of a model with dynamic signaling and hidden actions. By introducing unobservable effort, the model provides insights into how the interaction between different sources of asymmetric information affects the signal dynamics.

Our repeated signaling environment allows us to investigate when and how much the seller is willing or able to inform buyers about the quality of the asset. We find that if highly informative signals are very costly, signaling is more efficient when the market is optimistic, and when the market is pessimistic the seller "gives up" on signaling. When the cost function is not very convex, instead, the fear of receiving a bad offer makes exerting high effort more credible, so the seller "rushes out" the low posterior area by increasing the speed of information transmission. In both cases, by comparing our results with an observable-effort model, we find that the unobservability of the effort leads to inefficiently low effort choices and acceptance decisions by the seller.

The endogenous revelation of information has some implications that differ from those obtained in static signaling models or dynamic models without effort. We find that high separation between types may take place even if the cost of signaling, the outside option, or the noise is high. In some equilibria, high-quality sellers avoid for sure low offers by increasing effort after unfortunate bad signals (which make the market pessimistic). Also, despite the presence of a fixed cost, equilibrium effort choices may be arbitrarily low.

Future research shall be devoted to generalizing the results to allow low types to exert effort and to introduce additional types. Introducing productive signaling (such as productive education) may also introduce new tradeoffs, since the uninformed side of the market will value effort as something more than just a separation device.

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A Appendix: Omitted Proofs

Proof of Lemma 2.1 (page 10)

Proof. Assume first that $V_{H,t} = \underline{V}_H$. In this case, no offers higher than $\underline{V}_H$ are made in equilibrium with positive probability (i.e. $\Pr(W_{\tilde{\tau}} > \underline{V}_H) = 0$). Otherwise, the H-seller could wait and with some probability receive such offers, and therefore obtain a payoff strictly higher than $\underline{V}_H$. By the No (Unrealized) Deals condition, we have that $V_{H,t} \geq P_t\,\Pi$. If $P_t < \frac{V_H}{\Pi}$ then it is clear that no offer is made in equilibrium. If, instead, $P_t = \frac{V_H}{\Pi}$, offers may be made with positive probability, and given that $\underline{V}_L < \underline{V}_H$, if such an offer is made, it is accepted with probability one by the two types of the seller.

Assume now $V_{H,t} > \underline{V}_H$. Given that at all moments in time $V_{L,t} \geq \underline{V}_L > \Pi_L = 0$, it is clear that L-assets are never traded in isolation (given the Zero-Profits condition).

This implies that trade will happen only if $W_t = V_{H,t}$. Since, by the No (Unrealized) Deals we have that $V_{H,t} \geq P_t \Pi$, it follows that trade happens inside R only if the H-seller accepts with higher probability than the L-seller. So, there must exist histories where $V_{H,t} = V_{L,t} > P_t \Pi$. Let's normalize $t = 0$ and assume that an equilibrium exists such that $V_{H,0} = V_{L,0} > P_0 \Pi$. Assume first that, in such equilibrium, the probability of one of the players not accepting any offer is 0. In this case, given the zero profits condition, we have that $\mathbb{E}_0[W_{\tilde{\tau}}] = P_0 \Pi$, which is a contradiction. So, at least one of the types has to reject all offers with positive probability. If the L-seller rejects all offers with some positive probability $\phi_L > 0$, her payoff is $V_{0,L} = (1-\phi_L)\,\mathbb{E}_0[W_{\tilde{\tau}}|L] + \phi_L \underline{V}_L$. Nevertheless, in this case the H-seller can mimic the strategy and obtain $(1-\phi_L)\,\mathbb{E}_0[W_{\tilde{\tau}}|L] + \phi_L \underline{V}_H > V_{0,L}$, which is a contradiction. Finally, if only the H-seller rejects all offers with positive probability $\phi_H > 0$, then $\lim_{t\to\infty} \mathbb{E}_0[P_t|\tilde{\tau} = \infty] \to 1$ but $\lim_{t\to\infty} \mathbb{E}_0[V_{H,t}|\tilde{\tau} = \infty] = \underline{V}_H$, which is again a contradiction, since at every instant $V_{H,t} \geq P_t \Pi$. So, $\Pr(P_{\tilde{\tau}} \in R) = 0$. So types pool on the acceptance decision.

When $V_{H,t} = P_t \Pi$ then trade happens only if the proportion both the L-seller and the H-seller accept the offer with at same rate. This implies that the acceptance strategy does not affect the beliefs updating. Given that both types of the seller pool on the acceptance strategy, P_t is a continuous process, and therefore so is $V_{H,t}$ (and the value function $V_H(p)$). This implies that R is open relative to $[0, 1]$. Finally, if there is an equilibrium and t such that $e_{H,t} > 0$ and $P_t \notin \bar{R}$, then $V_H(p) = p$ would be a solution of the equation (2.6), for some $\hat{e}_H = e_H$, and we will show that it is not the case. $\square$

Proof of Lemma 2.2 (page 10)

Proof. Trivial using the fourth paragraph of the proof of Lemma 2.1. $\square$

Proof of Lemma 2.3 (page 13)

Proof. Note that the second derivative of the maximand of (2.6) with respect to $\hat{e}$ is $-\alpha\,(\alpha - 1)\,A_H\,\hat{e}_H^{\alpha-2} < 0$, so the first order condition is sufficient for optimality. Therefore, we impose the equilibrium condition $\hat{e}_\theta(p) \equiv e_\theta(p)$ in the FOC (2.9), so we get

$$-\alpha A_H\, e_H(p)^{\alpha-1} = \frac{p\,(1-p)\,e_H(p)\,V'(p)}{\sigma^2} \quad \Rightarrow \quad V'(p) = \frac{\alpha\,\sigma^2\,A_H\,e_H(p)^{\alpha-1}}{p\,(1-p)}\ .$$

Now, plugging this expression into the maximand of (2.6) we have a first order differential equation for $e_H(p)$, given by

$$0 = -c_0 + \tfrac{\alpha-2}{2}\, A_H\, e_H(p)^{\alpha-1}\, (e_H(p) + \alpha\,(1-p)\,p\, e'_H(p)) \ .$$

The general solution of this equation is given by equation (2.10). $\square$

Proof of Lemma 2.4 (page 14)

Proof. Let R be the signaling region of an MSE, and $\tilde{R} \subset R$ the signaling region of another equilibrium. Note that if $p_0 \notin \tilde{R}$ then $\tilde{V}_H(p) = \max\{\underline{V}_H, p_0\,\Pi\} \leq V_H(p_0)$. So, assume that $p_0 \in \tilde{R}$. Since both R and $\tilde{R}$ are open, there exist some intervals $(\underline{p}, \bar{p}) \subset R$ and $(\tilde{\underline{p}}, \tilde{\bar{p}}) \subset \tilde{R}$ such that $\underline{p}, \bar{p} \notin R$, $\tilde{\underline{p}}, \tilde{\bar{p}} \notin \tilde{R}$ and $p_0 \in (\tilde{\underline{p}}, \tilde{\bar{p}}) \subset (\underline{p}, \bar{p})$. If $\underline{p} = \tilde{\underline{p}}$ and $\bar{p} = \tilde{\bar{p}}$ then trivially $V_H(p_0) = \tilde{V}_H(p_0)$. Otherwise, assume $\tilde{\underline{p}} > \underline{p}$ and $\tilde{\bar{p}} = \bar{p}$. As is shown in the proof of Lemma 2.5, the value function $V_H(\cdot)$ follows the equation (A.1), for some C_1. We can write it explicitly as $V_H(p; \bar{p}, C_1)$. It is easy to verify that $\frac{\partial V'_H(p;\bar{p},C_1)}{\partial C_1} < 0$. Therefore, there exists some $\tilde{C}_1 > C_1$ such that $V_H(\tilde{\underline{p}}; \bar{p}, \tilde{C}_1) = \max\{\tilde{\underline{p}}\,\Pi, \underline{V}_H\}$, so we have that $\tilde{V}_H(p) = V_H(\tilde{\underline{p}}; \bar{p}, \tilde{C}_1)$. Finally, given that $\frac{\partial V_H(p;\bar{p},C_1)}{\partial C_1} < 0$, we have that $\tilde{V}_H(p_0) < V_H(p_0)$. If $\tilde{\underline{p}} \geq \underline{p}$ and $\tilde{\bar{p}} < \bar{p}$ we can repeat the process, keeping $\underline{p}$ constant and decreasing $\bar{p}$ to $\tilde{\bar{p}}$, by writing $V_H(\cdot)$ in therms of $\underline{p}$ instead of $\bar{p}$. Note that if $\tilde{R}$ is a strict subset of R then there exist $p \in R'$ such that $\tilde{V}_H(p) < V_H(p)$. $\square$

Proof of Lemma 2.5 (page 14)

Proof. Assume that an MSE exists and let R be its signaling region. Suppose $p_0 \in R$ and define $\underline{p}$ and $\bar{p}$ as in (2.4) and (2.5). Note that by the definition MSE there is no equilibrium such that its signaling region contains $\bar{p}$. We assume $V_H(\underline{p}) = \underline{p}\,\Pi$ and $V_H(\bar{p}) = \bar{p}\,\Pi$ (the other possible case, when $V_H(\underline{p}) = \underline{V}_H$, is proved analogously). Then, from the FOC (2.9) and the form of the policy function (2.10), there exists some constant C_1 such that

$$V_H(p) \equiv V_H(p, \bar{p}) = \bar{p}\,\Pi - \int_p^{\bar{p}} \frac{A_H\,\alpha\,\sigma^2}{(1-q)\,q} \left(\frac{C_1\,(1-q)}{(2-\alpha)\,q} - \frac{2\,c_0}{(2-\alpha)\,A_H} \right)^{\frac{\alpha-2}{\alpha}} \mathrm{d}q \ . \qquad \text{(A.1)}$$

For $\bar{p}' \in (0,1)$, define $\underline{p}_*(\bar{p}') \equiv \sup\{p < \bar{p}' | V_H(p, \bar{p}') \leq p\,\Pi\}$. Note that $\underline{p} = \underline{p}_*(\bar{p})$.

Note that since $V_H(p) \geq p\,\Pi$ for all $p \in R$ and $V_H(\cdot) \in \mathcal{C}^1(R)$, we have $\lim_{p \downarrow \underline{p}} V'_H(p) \geq \Pi$ and $\lim_{p \uparrow \bar{p}} V'_H(p) \leq \Pi$. We need to show that for the MSE these weak inequalities are equalities, instead.

Assume first $\lim_{p \downarrow \underline{p}} V_H'(p) > \Pi$ and $\lim_{p \uparrow \bar{p}} V_H'(p) < \Pi$. Note that $\frac{\partial V_H(p,\bar{p})}{\partial \bar{p}} = \Pi - V_H'(\bar{p}) > 0$. Therefore, $\underline{p}_*(\bar{p})$ is decreasing, and since $\lim_{p \downarrow \underline{p}} V_H'(p) > \Pi$ exists in a neighborhood of $\bar{p}$. So, since $V_H(p,\bar{p})$ is increasing in $\bar{p}$, for $\varepsilon > 0$ small enough $(\underline{p}_*(\bar{p}+\varepsilon), \bar{p}+\varepsilon) \ni \bar{p}$ is the signaling region of some equilibrium. This contradicts the assumption that $\bar{p}$ is not in the signaling region of any equilibrium.

Now consider the case $V_H'(\underline{p}) > \Pi$ and $V_H'(\bar{p}) = \Pi$ (a similar argument can be used when $V_H'(\underline{p}) = \Pi$ and $V_H'(\bar{p}) < \Pi$). It is easy to see that now $V_H(p) = \overline{V}_H(p,\bar{p})$ where

$$\overline{V}_H(p,\bar{p}) \equiv \bar{p}\,\Pi - \int_p^{\bar{p}} \frac{\Pi\,\bar{p}\,q^{\frac{2}{\alpha}-2}\,(1-\bar{p})^{2/\alpha}\,(1-q)^{-1}}{\left(2\,(2-\alpha)^{-1}\,k\,(\bar{p}-q)((1-\bar{p})\,\bar{p})^{\frac{\alpha}{2-\alpha}} + (1-q)\,\bar{p}\right)^{\frac{2-\alpha}{\alpha}}}\,\mathrm{d}q \tag{A.2}$$

where $k \equiv c_0 \left(\frac{\Pi\,A^{-2/\alpha}}{\alpha\,\sigma^2}\right)^{\frac{\alpha}{2-\alpha}}$. Simple algebra shows that

$$\begin{aligned} \frac{\partial^2 \overline{V}_H(\bar{p},\bar{p})}{\partial \bar{p}^2} > 0 \quad &\Leftrightarrow \quad \frac{\partial \overline{V}_H(p,\bar{p})}{\partial \bar{p}} > 0 \\ &\Leftrightarrow \quad k - (\alpha\,(1-\bar{p}) - 1)\,((1-\bar{p})\,\bar{p})^{\frac{\alpha}{\alpha-2}} > 0\ . \end{aligned}$$

The first condition is a necessary condition for $(\underline{p},\bar{p})$ to be an equilibrium when $\frac{\partial}{\partial p}\overline{V}_H(p,\bar{p}) = \Pi$. Indeed, since $V_H(p,\bar{p}) > p\,\Pi$ for $p \in (\underline{p},\bar{p})$ and $\overline{V}_H(\bar{p},\bar{p}) = \bar{p}\,\Pi$, $\overline{V}_H(\cdot,\bar{p})$ must be convex at $p = \bar{p}$. Using simple algebra we find that when $\alpha < 2$, there exists a unique $\bar{p}^\dagger$ such that $\frac{\partial^2 \overline{V}_H(p,\bar{p})}{\partial p^2} > 0$ iff $\bar{p} > \bar{p}^\dagger$. For $\alpha > 2$, as we will see in the proof of Proposition 4.1 (and Lemma A.1), $\frac{\partial^2 \overline{V}_H(p,\bar{p})}{\partial p^2} < 0$ in a (maybe empty) interval contained in $(0, \frac{\alpha-1}{\alpha})$ that contains $\tilde{p}_-$ (defined in (A.5)). Therefore, since by assumption $V_H(\underline{p}) = \underline{p}\,\Pi$, it must be the case that $\bar{p} > \tilde{p}_-$, so $\frac{\partial^2 \overline{V}_H(\bar{p}',\bar{p}')}{\partial p^2} > 0$ for $\bar{p}' > \bar{p}$.

For $\alpha < 2$, the value function (A.2) is well defined for all $p \in (0,1)$. In this case, $\bar{p}$ can be increased to $\bar{p}+\varepsilon$, for $\varepsilon > 0$ small, such that $\underline{p}_*(\bar{p}+\varepsilon)$ exists, and satisfies $\frac{\partial}{\partial p}V_H(p,\bar{p}+\varepsilon) > \Pi$. Since $V_H(p,\bar{p}+\varepsilon) > V_H(p,\bar{p})$ for all p, $\underline{p}_*(\bar{p}+\varepsilon) < \underline{p}$. This, by a similar argument as before, contradicts the assumption that $\bar{p}$ does not belong to the signaling region of any competitive equilibrium.

When $\alpha > 2$, the term inside the parenthesis of the denominator of (A.2) may not be well defined. It is easy to see that it is well defined for $p \geq \bar{p}$. In particular, given $\bar{p}$, either the denominator is well defined for all p or there exists some function $0 < \tilde{p}_0(\bar{p}) < \bar{p}$ such that it is not well defined for $p < \tilde{p}_0(\bar{p})$ and well defined otherwise. Furthermore, if $\tilde{p}_0(\bar{p})$ exists, it is continuous in $\bar{p}$ and $\lim_{p \to \tilde{p}_0(\bar{p})} \frac{\partial}{\partial p} V_H(p,\bar{p}) = 0$. Since, by assumption, $\frac{\partial}{\partial p}\overline{V}(\underline{p},\bar{p}) >$, then $\underline{p} > \tilde{p}_0(\bar{p})$ if $\tilde{p}_0(\bar{p})$ exists. Now, using the same argument as in the case where $\alpha < 2$, $\bar{p}$ can be increased by $\varepsilon > 0$ small such that $(\underline{p}_*(\bar{p}+\varepsilon), \bar{p}+\varepsilon)$ is the signaling region of an equilibrium. This contradicts our initial assumption. $\square$

Proof of Proposition 3.1 (page 14)

Proof. We will prove this proposition by explicitly constructing the equilibrium. Define $\overline{V}_H(\cdot,\cdot)$ and $\bar{p}^\dagger$ as in the proof of Lemma 2.5. Since $\lim_{\bar{p}\to 0}\overline{V}_H(0,\bar{p}) = 0$ and $\overline{V}_H(p,\bar{p})$ is decreasing in $\bar{p}$ when $\bar{p} < \bar{p}^\dagger$, we have that $\overline{V}_H(0,\bar{p}^\dagger) < 0$. Furthermore, by simple visual inspection we see that $\lim_{\bar{p}\to 1}\overline{V}_H(0,\bar{p}) = \Pi$. Therefore, by continuity and since $\overline{V}_H(p,\bar{p})$ is increasing in $\bar{p}$ when $\bar{p} > \bar{p}^\dagger$, for each $\underline{V}_H$ there exists a unique $\bar{p}_*(\underline{V}_H) \in (\bar{p}^\dagger, 1)$ such that $\overline{V}_H(0,\bar{p}_*(\underline{V}_H)) = \underline{V}_H$.

Let's show that there is an equilibrium with signaling region $R = (0,\bar{p}_*(\underline{V}_H))$. Let's denote $V_{H*}(p) \equiv \overline{V}_H(p,\bar{p}_*(\underline{V}_H))$. Then, since the boundary conditions are satisfied, we only need to show that $V_{H*}(p) \geq \max\{p\,\Pi, \underline{V}_H\}$ for all $p \in (0,\bar{p}_*(\underline{V}_H))$. Since $V'_{H*}(p) > 0$ and $V_{H*}(0) = \underline{V}_H$, we only have to verify that $V_{H*}(p) \geq p\,\Pi$ for all $p \in (0,\bar{p}_*(\underline{V}_H))$. First, taking derivatives in the expression (A.2) we have that

$$V''_{H*}(\bar{p}_*(\underline{V}_H)) > 0 \quad \Leftrightarrow \quad \bar{p}_*(\underline{V}_H) > \bar{p}^\dagger \ .$$

Second, let's find the solutions of the equation $V'_{H*}(p) = \Pi$ other than $p = \bar{p}_*(\underline{V}_H)$. Simple algebra transforms this equation into finding the zeros of $f(\cdot)$, where

$$f(p) \equiv \frac{2\,k\,(\bar{p}-p)}{2-\alpha} - (1-\bar{p})\,p^{\frac{\alpha}{\alpha-2}+1}\,(1-p)^{\frac{\alpha}{\alpha-2}} + (1-p)\,(1-\bar{p})^{\frac{\alpha}{\alpha-2}}\,\bar{p}^{\frac{\alpha}{\alpha-2}+1} \ ,$$

where $\bar{p} \equiv \bar{p}_*(\underline{V}_H)$. Let's show that it has at most one solution lower than $\bar{p}$. In order for the previous equation to have more than one solution in $(0,\bar{p}_*(\underline{V}_H))$, the second derivative must have at least one zero in $(0,\bar{p}_*(\underline{V}_H))$. Nevertheless, if we take the second derivative it is easy to see that it does not have any zero in $(0,\bar{p}_*(\underline{V}_H))$ for $\alpha < 2$. Then, since $V'_{H*}(0) = \infty$, it must be the case that $V_{H*}(p) > p\,\Pi$ for all $p \in (0,\bar{p}_*(\underline{V}_H))$. Indeed, if it was not the case, there must exist $\tilde{p}'_1, \tilde{p}'_2 \in (0,\bar{p}_*(\underline{V}_H))$ such that $V_{H*}(p) < 0$ for $p \in (\tilde{p}'_1,\tilde{p}'_2)$ and

$$V_{H*}(\tilde{p}'_1) = \tilde{p}'_1\,\Pi \ , \quad V_{H*}(\tilde{p}'_2) = \tilde{p}'_2\,\Pi \ , \quad V'_{H*}(\tilde{p}'_1) \leq \Pi \ \text{ and } \ V'_{H*}(\tilde{p}'_2) \geq \Pi \ .$$

Continuity of $V'_{H*}(\cdot)$ implies that there exist $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3 \in (0,\bar{p}_*(\underline{V}_H))$ such that $V'_{H*}(\tilde{p}_1) = V'_{H*}(\tilde{p}_2) = V'_{H*}(\tilde{p}_3) = \Pi$ and $0 < \tilde{p}_1 < \tilde{p}'_1 < \tilde{p}_2 < \tilde{p}'_2 < \tilde{p}_3 < \bar{p}_*(\underline{V}_H)$. But this contradicts the fact that $f(\cdot)$ only has one zero in $(0,\bar{p}_*(\underline{V}_H))$. So, there is an equilibrium with signaling region $R = (0,\bar{p}_*(\underline{V}_H))$.

Using a similar argument as in the proof of Lemma 2.5, we can argue that if there is an equilibrium with signaling region R' and $\sup R' > \sup R$, there must exist another equilibrium with signaling region $\tilde{R}'$ such that $\sup R'' > \sup R'$, satisfying the smooth pasting condition. Nevertheless, as we have just seen, the equilibrium defined is the only one that satisfies them. $\square$

Proof of Corollary 3.1 (page 15)

Proof. The value function for any equilibrium is given by (A.2), and let $V_H(p;\bar{p},k)$ denote the function defined in this equation where we explicitly write k as an argument. Note that when $\alpha < 2$, $k \equiv k(\sigma)$ (defined in the proof of Lemma 2.5) is decreasing in σ. Therefore, if σ increases, the integrand of the expression (A.2) increases for each given q (keeping $\tilde{p}$ the same). So, since $\sigma_1 > \sigma_2$, $V_H(0;\bar{p}_*(\sigma_2),k(\sigma_1)) < V_H(0;\bar{p}_*(\sigma_2),k(\sigma_2)) = \underline{V}_H$. Hence, since $V_H(0;\bar{p},k(\sigma_1))$ is increasing in $\bar{p}$, we have that $\bar{p}_*(\sigma_1) > \bar{p}_*(\sigma_2)$. $\square$

Proof of Corollary 3.2 (page 16)

Proof. Since $\frac{2\,c_0}{A_H\,(\alpha-2)} < 0$, it must be the case that $C_1 < 0$ in order for the term inside the power function to be non-negative. This makes $e_H(\cdot)$ clearly decreasing. $\square$

Proof of Proposition 3.2 (page 17)

Proof. The proof is the same as the one for Proposition 3.1. In Proposition 3.1 the result is proven for $c_0 > 0$, but the argument still applies for $c_0 = 0$. The particular value of $\bar{p}_*$ is obtained by solving the equation $\frac{\partial}{\partial p} V_H(\bar{p}_*;0,\bar{p}_*,\underline{V}_H,\bar{p}_*\,\Pi) = \Pi$ (using the definition (3.4).) $\square$

Proof of Proposition 4.1 (page 18)

Proof. We begin with a technical lemma useful which is to construct the signaling region of the MSE:

Lemma A.1. *There exists $\bar{c}_0 > 0$ such that if $c_0 < \bar{c}_0$ there is a unique non-empty interval $(\underline{p}_*,\bar{p}_*)$ such that if $\underline{V}_H$ is small enough $(\underline{p}_*,\bar{p}_*)$ is the signaling region of an equilibrium with $V_H'(\underline{p}_*) = V_H'(\bar{p}_*) = \Pi$. If $c_0 \geq \bar{c}_0$, no interval with the previous properties exists.*

Proof. Note that in all equilibria with $R \neq \emptyset$ there must be a $\tilde{p} \in R$ such that $V_H'(\tilde{p}) = \Pi$ and $V_H''(\tilde{p}) < 0$. Indeed, consider a $p \in R$ and define $\underline{p}$ and $\bar{p}$ as in (2.4) and (2.5). Note that equilibrium conditions require $V_H(p) > p\,\Pi$ for $p \in (\underline{p},\bar{p})$, $\lim_{p \downarrow \underline{p}} V_H'(p) \geq \Pi$ and

$\lim_{p\uparrow\bar{p}} V_H'(p) \leq \Pi$. So, since $V_H'(\cdot) \in \mathcal{C}^1(\underline{p}, \bar{p})$, using standard calculus (mean value theorem) we know that there must exist at least one $\tilde{p}$ such that $V_H'(\tilde{p}) = \Pi$ and $V_H''(\tilde{p}) < 0$.

Assume that an equilibrium exists and consider $\tilde{p}$ satisfying the previous conditions. Then, given the form of the policy function (2.10), there must exist some $\tilde{v}$ such that the value function $V_H(p) \equiv V_H(p; \tilde{p}, \tilde{v})$ takes the following form:

$$V_H(p; \tilde{p}, \tilde{v}) \equiv \tilde{v} + \int_{\tilde{p}}^{p} \frac{\Pi\, \tilde{p}\, q^{\frac{2}{\alpha}-2}\,(1-\tilde{p})^{2/\alpha}\,(1-q)^{-1}}{\left(2\,(2-\alpha)^{-1}\,k\,(\tilde{p}-q)\,((1-\tilde{p})\,\tilde{p})^{\frac{\alpha}{2-\alpha}} + (1-q)\,\tilde{p}\right)^{-\frac{2-\alpha}{\alpha}}}\, \mathrm{d}q\ , \tag{A.3}$$

where $k \equiv c_0 \left(\frac{\Pi\, A^{-2/\alpha}}{\alpha\, \sigma^2}\right)^{\frac{\alpha}{2-\alpha}}$ as in the proof of Lemma 2.5. Note that $V_H(\tilde{p}; \tilde{p}, \tilde{v}) = \tilde{v}$ and $\frac{\partial}{\partial p} V_H(\tilde{p}; \tilde{p}, \tilde{v}) = \Pi$. If we twice differentiate it with respect to p we find

$$\frac{\partial^2}{\partial p^2} V_H(\tilde{p}; \tilde{p}, \tilde{v}) = \frac{2\,\Pi\,((1-\tilde{p})\,\tilde{p})^{\frac{-\alpha}{\alpha-2}}}{\alpha\,\tilde{p}\,(1-\tilde{p})} \left(k - (\alpha\,(1-\tilde{p})-1)\,((1-\tilde{p})\,\tilde{p})^{\frac{\alpha}{\alpha-2}}\right)\ . \tag{A.4}$$

Note that the first term in the RHS of the expression is clearly positive. The second term in the RHS is k when $\tilde{p} = 0$, $\tilde{p} = \frac{\alpha-1}{\alpha}$ and $\tilde{p} = 1$. If we differentiate this term, we see that it is strictly convex in the region $(0, \frac{\alpha-1}{\alpha})$ and concave otherwise. Therefore, the minimum of the second term of the RHS of (A.4) is in the region $(0, \frac{\alpha-1}{\alpha})$, and it can be shown that it is reached at

$$\tilde{p}_- \equiv \frac{1}{2 + \sqrt{\frac{\alpha-2}{\alpha-1}}}\ . \tag{A.5}$$

Therefore, using the definition of k, a necessary condition for $V_H(p; \tilde{p}, \tilde{v})$ to be concave at $\tilde{p}$ is that

$$c_0 < \bar{c}_0 \equiv \left(\frac{\Pi\, A_H^{-2/\alpha}}{\alpha\, \sigma^2}\right)^{\frac{\alpha}{\alpha-2}} (\alpha\,(1-\tilde{p}_-)-1)\,((1-\tilde{p}_-)\,\tilde{p}_-)^{\frac{\alpha}{\alpha-2}}\ . \tag{A.6}$$

So, the previous condition is necessary for the existence of $\tilde{p} \in R$ satisfying $V_H'(\tilde{p}) = \Pi$ and $V_H''(\tilde{p}) < 0$, that itself is a necessary condition for the existence of equilibria. Then, $c_0 < \bar{c}_0$ is a necessary condition for the existence of equilibria.

Let's show that $c_0 < \bar{c}_0$ is also a sufficient condition for the existence of equilibria. Assume $c_0 < \bar{c}_0$, so $\tilde{p}$ exists such that $\frac{\partial}{\partial p} V_H(\tilde{p}; \tilde{p}, \tilde{v}) = \Pi$ and $\frac{\partial^2}{\partial p^2} V_H(\tilde{p}; \tilde{p}, \tilde{v}) < 0$. If we make $\tilde{v}$ higher (close enough to $\tilde{p}\,\Pi$), standard calculus guarantees that there exist $\underline{p} < \tilde{p}$ and $\bar{p} > \tilde{p}$ such that $V_H(\underline{p}; \tilde{p}, \tilde{v}) = \underline{p}\,\Pi$, $V_H(\bar{p}; \tilde{p}, \tilde{v}) = \bar{p}\,\Pi$ and $V_H(p; \tilde{p}, \tilde{v}) > p\,\Pi$ for all $p \in (\underline{p}, \bar{p})$. Since $R = (\underline{p}, \bar{p})$ satisfying the previous conditions is the signaling region of an equilibrium, $c_0 < \bar{c}_0$ is a sufficient condition for an equilibrium to exist.

Note that $V_H(p;\tilde p,\tilde v)$ is well defined as long as the term inside the parenthesis of the denominator is non-negative. It is easy to verify that it is non-negative if p is in the neighborhood of $\tilde p$, so the previous argument is valid. Note also that if $\tilde p$ is large, the term in the denominator is not well defined for low q. Since the exponent of this term is negative when $\alpha > 2$, this corresponds to the derivative of V_H with respect to p being 0.

Assume that $c_0 < \bar c_0$, fix an equilibrium and $\tilde p$ satisfying the previous properties. Let's define $\tilde v_- \equiv \inf\{\tilde v | V_H(p;\tilde p,\tilde v) > p\,\Pi \quad \forall p < \tilde p\}$ and $\tilde v_+ \equiv \inf\{\tilde v | V_H(p;\tilde p,\tilde v) > p\,\Pi \quad \forall p > \tilde p\}$. Note that since $\lim_{p\to 1} V_H(p;\tilde p,\tilde v) = \infty$, we have $\tilde p\,\Pi < \tilde v_+ < \infty$. Assume $\tilde v_+ \le \tilde v_-$ (the other case is analogous). By continuity, there exists some $\bar p$ such that $V_H(\bar p;\tilde p,\tilde v_+) = \bar p\,\Pi$. Note that $\bar p$ is unique. Indeed, by the previous argument $\frac{\partial^2}{\partial p^2} V_H(p;\tilde p,\tilde v_+)$ has two zeros when $c_0 < \bar c_0$ (one lower than $\tilde p$ and one higher than $\tilde p$), and $\bar p$ must be higher than the higher zero, so $\frac{\partial^2}{\partial p^2} V_H(\bar p;\tilde p,\tilde v_+) > 0$.

Since $V_H(p;\tilde p,\tilde v) \in \mathcal{C}^1(\tilde p, 1)$, it must be the case that $\frac{\partial}{\partial p} V_H(\bar p;\tilde p,\tilde v_+) = \Pi$, and therefore $V_H(\bar p;\tilde p,\tilde v_+) = \overline V_H(p,\bar p)$, where $\overline V_H$ is defined in (A.2). Recall that $\overline V_H(p,\bar p)$ is increasing and continuous in $\bar p$. Furthermore, by assumption (since $\tilde v_- > \tilde v_+$), there exists some $\underline p < \tilde p$ such that $\overline V_H(\underline p,\bar p) = \underline p$. Define $\bar p_* = \inf\{\bar p | \overline V_H(p,\bar p) > p\,\Pi \quad \forall p < \bar p\}$. Using standard calculus, it is easy to prove that there exists some $\underline p_* < \bar p_*$ such that $\overline V_H(\underline p_*,\bar p_*) = \underline p_*\,\Pi$ and $\frac{\partial}{\partial p}\overline V_H(\underline p_*,\bar p_*) = \Pi$. $\square$

(Continuation of the proof of Proposition 4.1)

Let's fix $\underline p \in (0, \frac{V_H}{Y})$ and define

$$\tilde V_H(p,\underline p) \equiv \underline V_H + \int_{\underline p}^{p} \frac{2^{\frac{1}{\alpha}}\,\alpha\,\sigma^2\,A_H^{2/\alpha}\,c_0^{\frac{\alpha-2}{\alpha}}\,(q-\underline p)^{\frac{\alpha-2}{\alpha}}}{(\alpha-2)^{1-\frac{2}{\alpha}}\,(1-\underline p)^{1-\frac{2}{\alpha}}\,(1-q)\,q^{2-\frac{2}{\alpha}}}\,\mathrm{d}q\ . \tag{A.7}$$

It is easy to see that this is the value function corresponding to $C_1 = \frac{2\,\underline p\,c_0}{A_H\,(1-\underline p)}$ in (A.1), using $\underline p$ as the integration limit instead of $\bar p$ and changing $\bar p\,\Pi$ by $\underline V_H$ in the front of the expression. Note that $\tilde V_H(\underline p,\underline p) = \underline V_H$ and $\frac{\partial}{\partial p}\tilde V_H(\underline p,\underline p) = 0$. Note also that $\frac{\partial^2}{\partial p^2}\tilde V_H(p,\underline p) > 0$ when $p > \underline p$ is close to $\underline p$. Therefore, if we choose $\underline p$ close enough to $\frac{V_H}{\Pi}$, it is easy to show that there exists some $\bar p > \frac{V_H}{\Pi}$ such that $\tilde V_H(\bar p,\underline p) = \max\{\underline V_H, \bar p\,\Pi\}$ and $\tilde V_H(p,\underline p) > p\,\Pi$ for all $p \in (\underline p,\bar p)$. Therefore, a competitive equilibrium (with signaling region $R = (\underline p,\bar p)$) exists.

Using simple algebra it is easy to show that $\frac{\partial}{\partial \underline p}\tilde V_H(p,\underline p) < 0$. Furthermore, we see that $\lim_{\underline p\to 0}\tilde V_H(p,\underline p) = \infty$ for all $p > 0$ and $\lim_{p\to 1}\tilde V_H(p,\underline p) = \infty$ for all $\underline p > 0$. Also, twice differentiating (A.7), we see that $\frac{\partial^2}{\partial p^2}\tilde V_H(p,\underline p)$ has at most 2 zeros. Therefore, there exist one and at most two pairs of values $(\underline p_1,\bar p_1)$ and $(\underline p_2,\bar p_2)$, with $\bar p_1 < \bar p_2$, such that $\tilde V_H(\bar p_i,\underline p_i) = \bar p_i\,\Pi$,

$\frac{\partial}{\partial p}\tilde{V}_H(\bar{p}_i, \underline{p}_i) = \Pi$ and $\frac{\partial^2}{\partial p^2}\tilde{V}_H(\bar{p}_i, \underline{p}_i) > 0$ for $i \in \{1, 2\}$. Note also that if $(\underline{p}_i, \bar{p}_i)$ exists for some $i \in \{1, 2\}$, then $\tilde{V}_H(p, \underline{p}_i) = \overline{V}_H(p, \bar{p}_i)$, where $\overline{V}_H$ is defined in (A.2).

Note that two pairs $\{(\underline{p}_i, \bar{p}_i)\}_{i\in\{1,2\}}$ with the previous properties exist only if $c_0 \leq \bar{c}_0$, where $\bar{c}_0$ is defined in Lemma A.1. Indeed, assume otherwise, that is, two pairs exist and $c_0 > \bar{c}_0$. Then, since $\overline{V}_H(p, \bar{p})$ is increasing in $\bar{p}$, is the case that $\overline{V}_H(\bar{p}_j, \bar{p}_i) \leq \bar{p}_j \Pi$ for some i and j such that $i, j \in \{1, 2\}$, $i \neq j$. Assume it is true for $i = 2$ (the other case is analogous). Since $\frac{\partial^2}{\partial p^2}\overline{V}_H(p, \bar{p}_2) > 0$, there exists some $\underline{p} \in [\bar{p}_1, \bar{p}_2)$ such that $\overline{V}_H(\underline{p}, \bar{p}_2) = \underline{p}\,\Pi$ and $\overline{V}_H(p, \bar{p}_2) > p\,\Pi$ for all $p \in (\underline{p}, \bar{p}_2)$. Therefore, there exists a competitive equilibrium with signaling region $(\underline{p}, \bar{p}_2)$, which contradicts Lemma A.1. Furthermore, it is easy to show that if two pairs exist (and therefore $c_0 < \bar{c}_0$), it must be the case that $\bar{p}_1 < \tilde{p}_- < \bar{p}_2$, where $\tilde{p}_-$ is defined in (A.5).

Then, we have the following cases:

1. If $c_0 > \bar{c}_0$, only one pair exists (assume it is $i \in \{1, 2\}$). Note that, by Lemma A.1, no competitive equilibrium with signaling region $(\underline{p}_0, \bar{p}_0)$ exists such that $V_H(\underline{p}_0) = \underline{p}_0\,\Pi$ and $V_H(\bar{p}_0) = \bar{p}_0\,\Pi$. Therefore, the only equilibrium satisfying the smooth pasting condition is $(\underline{p}_i, \bar{p}_i)$.

2. If $c_0 < \bar{c}_0$ let $(\underline{p}_0, \bar{p}_0)$ be the signaling region in Lemma A.1. We have two cases:

   - If $\bar{p}_2 > \bar{p}_0$ then $(\underline{p}_2, \bar{p}_2)$ is the signaling region of a competitive equilibrium. Indeed, since the boundary conditions are satisfied, we only need to verify that $\tilde{V}_H(p, \underline{p}_2) > \max\{\underline{V}_H, p\,\Pi\}$ for all $p \in (\underline{p}_2, \bar{p}_2)$. Since $\frac{\partial}{\partial p}\tilde{V}_H(p, \underline{p}_2) > 0$ for all $p > \underline{p}_2$, we only need to verify that $\tilde{V}_H(p, \underline{p}_2) > p\,\Pi$ for all $p \in (\underline{p}_2, \bar{p}_2)$. Assume otherwise, that is, assume there is some $\tilde{p} \in (\underline{p}_2, \bar{p}_2)$ such that $\tilde{V}_H(\tilde{p}, \underline{p}_2) \leq \tilde{p}\,\Pi$. Since $\tilde{V}_H(p, \underline{p}_2) = \overline{V}_H(p, \bar{p}_2)$ and $\overline{V}_H(p, \bar{p})$ is increasing in $\bar{p}$, a similar argument as the one used in the proof of Lemma A.1 shows that there must an equilibrium with signaling region $(\underline{p}_3, \bar{p}_3)$, with $\bar{p}_3 > \bar{p}_2$, such that $\frac{\partial}{\partial p}\overline{V}_H(\underline{p}_3, \bar{p}_3) = \Pi$. As we see saw in the proof of Lemma A.1, this is unique, which implies that $\bar{p}_3 = \bar{p}_0$. Nevertheless, we have $\bar{p}_0 = \bar{p}_3 > \bar{p}_2$, which is a contradiction. Also, since $\tilde{V}(p, \underline{p})$ is decreasing in $\underline{p}$, we have $(\underline{p}_1, \bar{p}_1) \subset (\underline{p}_2, \bar{p}_2)$.
   - If $\bar{p}_2 \leq \bar{p}_0$, it must be the case that $\bar{p}_1 \leq \underline{p}_0$. Indeed, otherwise there exists some $\tilde{p} \in (\underline{p}_1, \frac{V_H}{\Pi})$ such that $\overline{V}_H(\tilde{p}, \bar{p}_0) = \underline{V}_H$ and $\overline{V}_H(p, \bar{p}_0) > \max\{\underline{V}_H, p\,\Pi\}$ for $p \in (\tilde{p}, \underline{p}_0)$. Since $\overline{V}_H(p, \bar{p})$ is increasing in p, there exists some $\bar{p}_4 = \inf\{\bar{p} > \bar{p}_0 | \overline{V}_H(\tilde{p}_0(\bar{p}), \bar{p}) > \underline{V}_H\}$, where $\tilde{p}_0(\bar{p})$ is defined as in the proof of Lemma 2.5. It is easy to show that $(\tilde{p}_0(\bar{p}_4), \bar{p}_4)$ satisfies the same conditions as $(\tilde{p}_2, \bar{p}_2)$, and

since there are only two pairs that satisfy those conditions and $\bar{p}_4 \geq \bar{p}_0 > \tilde{p}_- > \bar{p}_1$, it must be the case that $\bar{p}_4 = \bar{p}_2$. This contradicts our assumption, since $\bar{p}_2 = \bar{p}_4 > \bar{p}_0 \geq \bar{p}_2$. Furthermore, if $\bar{p}_2 < \bar{p}_0$, there is some $\tilde{p} \in (\underline{p}_2, \bar{p}_2)$ such that $\tilde{V}(\tilde{p}, \underline{p}_2) < \tilde{p}\,\Pi$. The reason is that $\tilde{V}(\cdot, \underline{p}_2) = \overline{V}(\cdot, \bar{p}_2)$, $\underline{p}_2 < \underline{p}_0 < \bar{p}_2$, $\overline{V}(\underline{p}_0, \bar{p}_0) = \underline{p}_0\,\Pi$ and $\overline{V}(p, \bar{p})$ is increasing in $\bar{p}$. Finally, if $\bar{p}_2 = \bar{p}_0$ then $\bar{p}_1 = \underline{p}_0$ and $\underline{p}_1 = \underline{p}_2$. Therefore, in this case, the equilibrium that satisfies the smooth pasting condition with biggest signaling region equal to the union of the (disjoint) intervals $(\underline{p}_0, \bar{p}_0)$ and $(\underline{p}_1, \bar{p}_1)$.

In each case it can be shown that the corresponding is an MSE using a similar argument as in the proof of Lemma 2.5. Indeed, it is easy to see that if any of them is not an MSE, there must exist an equilibrium with a bigger signaling region satisfying the smooth pasting condition. Nevertheless, as we have shown, each of the proposed equilibria is the one with the biggest signaling region among all equilibria satisfying the smooth pasting condition. □

Proof of Corollary 4.1 (page 20)

Proof. First assume $\underline{V}_H$ small enough, and assume $c_0 < \bar{c}_0$. As is shown in the proof of Lemma A.1, equation (A.3) must hold in any equilibrium at some $\tilde{p}$ such that $V_H''(\tilde{p}) < 0$. It is easy, from this equation, to see that C_1 in equation (2.10) takes the following form:

$$C_1 = \frac{4\,c_0\,k\,(1-p)}{A_H\,\tilde{p}\,(\alpha-2)\left((\alpha-2)((1-\tilde{p})\,\tilde{p})^{\frac{\alpha}{\alpha-2}} - 2\,k\right)}$$

where k is defined in the proof of Lemma A.1. Using equation (A.4) (imposing $V_H''(\tilde{p}) < 0$) it is easy to see that $C_1 > 0$. Therefore, $e_H(\cdot)$ defined in (2.10) is increasing.

If $\underline{V}_H > 0$ we have (at least) one interval $(\underline{p}_1, \bar{p}_1)$ such that $V_H(\underline{p}_1) = \underline{V}_H$ and $e_H(\underline{p}_1) = 0$. Therefore, in this interval $C_1 = \frac{2\,c_0\,\underline{p}_1}{A_H\,(1-\underline{p}_1)} > 0$, we have that $e_H(\cdot)$ is increasing in $(\underline{p}_1, \bar{p}_1)$. If we have two intervals, then the limits in the second ones are those when $\underline{V}_H$ is small enough, so the previous part of the proof holds. □

Proof of Proposition 5.1 (page 21)

Proof. Fix a $p_0 \in R$ and define $\bar{p}$ and $\underline{p}$ as in (2.4) and (2.5). Fix $e_H \in \mathcal{C}^1(\underline{p}, \bar{p})$ positive. The equation for the value function for the H-seller exerting effort e_H is given, in $(\underline{p}, \bar{p})$ by the following HJB equation

$$0 = -A_H\,e_H(p)^\alpha + \frac{(p-1)^2\,p\,\left(p\,V_H''(p) + 2\,V_H'(p)\right)}{2\,\sigma^2}\,e_H(p)^2\,, \tag{A.8}$$

and boundary conditions $V_H(\underline{p}) = \underline{W}$ and $V_H(\bar{p}) = \overline{W}$. Let $V_H(p, e_H(\cdot))$ be its solution. Let's consider the following decomposition: $V_H(p, e_H) \equiv V_h(p) + V_t(p, e_H)$. We assume that $p\, V_h''(p) + 2\, V_h'(p) = 0$, and we impose $V_h(\underline{p}) = \underline{W}$ and $V_h(\bar{p}) = \overline{W}$. This leads to

$$V_h(p) = \underline{W} + \frac{(p - \underline{p})\, \bar{p}}{(\bar{p} - \underline{p})\, p} (\overline{W} - \underline{W}) \; .$$

This is exactly the expected payoff when the signaling waste is 0 (note that the homogeneous equation is "as if" $A_H = 0$), which coincides with the expected accepted price offer conditional on being type H (we can see this using the formula (3.1)). Note that $V_t(p; e_H)$ must satisfy (A.8) and $V_t(\underline{p}; e_H) = V_t(\bar{p}; e_H) = 0$. Consider $\lambda > 1$. Then, it is the case that

$$V_t(p, \lambda\, e_H) = \lambda^{\alpha-2}\, V_t(p, e_H) < V_t(p, e_H) \; .$$

This is true because both $V_t(p, \lambda\, e_H)$ and $V_t(p, \lambda\, e_H)$ satisfy the same equations and boundary conditions (equal to 0 at the boundary). Therefore, increasing the effort by a factor $\lambda > 1$, the absolute value of $V_t(p, \lambda\, e)$ is reduced by a factor $\lambda^{\alpha-2} < 1$. Finally, note that $V_t(p, \lambda\, e) < 0$ $\forall p \in (\underline{p}, \bar{p})$. Indeed, it is the solution of a boundary problem with negative flow payoff and with 0-value at the boundary. So, by increasing the effort we increase V_H, we make it asymptotically equal to V_h, that is, signaling waste asymptotically disappears. □

Proof of Proposition 5.2 (page 22)

Proof. The problem of maximizing the value function of the H-seller can be written as a regular stochastic control problem, since now there is no incentive constraint:

$$0 = \max_{e_H(p)} \left(-\, c_0 - A_H\, e_H(p)^\alpha + \frac{(p-1)^2\, p\, \left(p\, V_H''(p) + 2\, V_H'(p)\right)}{2\, \sigma^2}\, e_H(p)^2 \right) \; .$$

The First Order Condition of the previous equation is

$$0 = -\alpha\, A_H\, e_H(p)^{\alpha-1} + \frac{(p-1)^2\, p\, \left(p\, V_H''(p) + 2\, V_H'(p)\right)}{2\, \sigma^2}\, 2\, e_H(p) \; .$$

Note that since $\alpha > 2$ the Second Order Condition is satisfied. Using the previous two equations to solve for $e_H(p)$ it is easy to verify that the statement of the proposition is true (note that the terms of both equations involving p are identical). □

B Expected Stopping Times

In this section we compute the expected stopping times in the case $\alpha < 2$ and $c_0 = 0$. Fix an equilibrium with signaling region $R \equiv (\underline{p}, \bar{p})$. Let $T_\theta(p)$ be the expected time before an

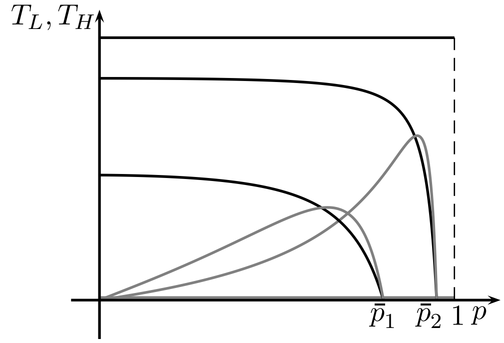

Figure 5: Expected stopping times. Gray and black lines correspond to L-sellers and H-sellers, respectively.

offer is accepted, that is

$$T_\theta(p_0) \equiv \mathbb{E}\big[\, \tau_\theta \,\big|\, p_0, e_t = e_\theta(P_t) \,\big] = \mathbb{E}\big[\textstyle\int_0^{\tau_\theta} \mathrm{d}s \,\big|\, p_0, e_t = e_\theta(P_t) \,\big] \ .$$

Therefore, $T_\theta(\cdot)$ can be thought of as the value function for a flow payoff of 1 while the project is active and 0 when it stops. Hence, $T_\theta(\cdot)$ satisfies the following HJB equation:

$$0 = 1 + \tilde{\mu}(e_\theta(p), p, e(p))\, T_\theta'(p) + \tfrac{1}{2}\, \tilde{\sigma}(p, e(p))^2\, T_\theta''(p) \ ,$$

with boundary conditions $T_\theta(\underline{p}) = T_\theta(\bar{p}) = 0$. The previous equation can be analytically solved. We focus on the limiting case $\underline{p} \to 0$, since this is the relevant case for the MSE. After some amount of algebra, T_H and T_L can be expressed in the following way

$$\begin{aligned} T_H(p|\underline{p}=0) &= \frac{1}{2+\alpha} \left(\frac{(2-\alpha)\,(\bar{p}\,\Pi - \underline{V}_H)}{\alpha^\alpha\, A_H\, \sigma^\alpha} \right)^{\frac{2}{2-\alpha}} \left(1 - \frac{h(p)^{\frac{2}{2-\alpha}}}{h(\bar{p})^{\frac{2}{2-\alpha}}} \right) , \\ T_L(p|\underline{p}=0) &= \frac{1}{2-\alpha} \left(\frac{(2-\alpha)\,(\bar{p}\,\Pi - \underline{V}_H)}{\alpha^\alpha\, A_H\, \sigma^\alpha} \right)^{\frac{2}{2-\alpha}} \left(\frac{(1-\bar{p})\, p}{\bar{p}\,(1-p)} - \frac{h(p)^{\frac{2}{2-\alpha}}}{h(\bar{p})^{\frac{2}{2-\alpha}}} \right) , \end{aligned}$$

where $h(\cdot)$ is defined at (3.3). Figure 5 (b) plots these functions for different values of $\bar{p}$. We see that $T_H(0) \neq 0$. Even though for each $\underline{p} > 0$ we have $T_H(\underline{p}) = 0$, we have $\lim_{\underline{p}\to 0} T_H(p) > 0$ for all $p > 0$. The rationale, as we explained in Section 3.1, is that $\lim_{\underline{p}\to 0} e_H(\underline{p}) = \infty$, so in the limit the unbounded effort around 0 generates a "wall" in the beliefs.[20]

[20] The fact that there is a pointwise convergence when $\underline{p} \to 0$ both for $V_\theta(\cdot)$ and $T_\theta(\cdot)$ reinforces the conjecture (that is verified numerically) that the equilibrium described in Proposition 3.2 is the limit of equilibria in a sequence of discrete-time versions of our model. Even though the fact that $\underline{p} = 0$ can be reached in finite time is only true in the continuous-time model, it is asymptotically true in the sequence of equilibria.