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“Product Upgrades and Posted Prices”

by

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# Product Upgrades and Posted Prices

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## ABSTRACT

We consider the dynamic pricing problem of a durable good monopolist with full commitment power, when a new version of the good is expected at some point in the future. The new version of the good is superior to the existing one, bringing a higher flow utility. If the arrival is a stationary stochastic process, then the corresponding optimal price path is shown to be constant for both versions of the good, hence there is no delay on purchases and time is not used to discriminate over buyers, which is in line with the literature. However, if the arrival of the new version occurs at a commonly known deterministic date, then the optimal price path may be decreasing over time, resulting in delayed purchases. For both stochastic and deterministic arrival processes, posted prices is not the optimal mechanism, which on the other hand, involves into bundling of both new and old versions of the good and selling them only together.

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# 1 INTRODUCTION

The literature on durable good monopolies assumes a population of forward looking buyers with heterogeneous valuations that are strategically timing their purchases. There is a unique product that the monopolist sells over time. Buyers have unit demand for the product, and so leave the market after purchasing. The main channel governing the optimal pricing policy originates from two counterbalancing factors. First, to be able to sell the product to the agents with lower valuations, the monopolist must decrease the price of the good over time. Second, customers with high valuations delay their purchases since they predict that the monopolist will decrease the price over time. Hence, decreasing prices is beneficial as it allows the firm to capture the surplus from the demand of the agents with lower valuations, but at the same time it is costly as it causes a delay in the purchases.

The pioneering paper of the literature, [Stokey \(1979\)](#), shows that, if the firm can commit to a price path before starting to sell, then the optimal price path is constant and equal to the static monopoly price. Consequently, all the purchases take place at the beginning (at  $t=0$ ) and so there is no delay. This result is significant as it asserts that the monopolist should not use time to discriminate over people with different valuations, which would have occurred if the price path had been decreasing and purchases had been occurring throughout time.

In this paper, we turn our attention to the optimal pricing problem when a new version of the existing durable good is expected to arrive at some point in the future. In contrast to one of the main assumptions of the existing literature, durable goods, for many real life examples, do not persistently stay in the market. Rather, newer and better functioning versions are taking place of the older ones over time. Technology companies such as Apple, Intel, Samsung, and Microsoft are good examples for the problem that we address. For instance a phone is replaced by a new version with better features. This situation does not hurt the durability of the product that is replaced as customers can still use it after the new version is launched. However it alters the consumer's preferences as the newer product might offer a higher flow utility. In such an environment, the structure of the buyers' incentives would be different than the ones of the classical framework. In particular, a buyer does not necessarily leave the market after purchasing a version of the good. He may rather prefer to wait for the newer one. Or he may abandon to purchase a version and wait for the newer one. Therefore the price path of a version of the good not only affects its own sales, but also the sales of the other versions.

We consider a monopolist which is selling two consecutive versions of the same good, with a restriction that it only sells the most current version at a given period. We assume

that it has full commitment power, and can commit to a price path for both versions of the good at the beginning. At the beginning, the monopolist is endowed with the initial version and the upgrade will take place over time. The timing of the upgrade is not a choice variable in this paper, rather we take it as exogenously given.

The analysis is divided into two parts depending on the specification of the arrival process of the new version of the good. In the first part, the arrival time is stochastic and follows a Poisson process. The optimal price path is shown to be constant for both versions of the good (at different levels) and the price level for the second version of the good is independent from the realized arrival time. Consequently, there are no delays in any purchases. Any purchase of the first version occurs immediately at the beginning, and any purchase of the second version occurs as soon as it arrives into the market. This result is in line with [Stokey \(1979\)](#) as it also suggests that the time can not be used as a tool to discriminate over buyers with heterogeneous valuations. Stationary structure of the environment, which originates from the Poisson arrival process, plays a crucial role for this result.

In the second part of the analysis, we consider the same question when the newer version arrives at a certain time period which is commonly known by the participants of the market. Here the structure is not stationary anymore, and this comes with a cost on tractability. Therefore, to simplify the analysis we assume a discrete type space consisting of buyers only with two possible valuations for this case. The results indicate that a decreasing price path is possible for some parameters and this gives rise to delayed purchases. Therefore, in contrast to the main theme of the literature, the monopolist may use time to discriminate when the arrival of the new version occurs at a commonly known date.

Unlike the existing literature on durable good pricing with full commitment power, in our environment, a sale mechanism with posted prices is not an optimal mechanism. The reason is that the anonymous structure of the posted prices comes at a cost for the monopolist. More precisely, the sales of the new version of the good cannot be conditioned on the first version sales. The optimal selling mechanism, on the other hand, requires to bundle both versions of the good and selling them only together so that only some groups of buyers purchase and they purchase both versions of the good. The resulting allocation of this policy cannot be implemented with posted price path which is anonymous by definition.

## 1.1 RELATED LITERATURE

Our paper mainly fits into the literature of durable good pricing with full commitment power. The classic reference is [Stokey \(1979\)](#), which shows that the optimal price path is

constant and equal to the static monopoly price.<sup>1</sup> Therefore all the buyers either purchase immediately or leave without any purchase, hence the time is not used by monopolist to discriminate over buyers with different valuations.

In our paper, there are both favoring and contradicting results with this mainstream result. We show that, under some circumstances, the optimal posted price path may be decreasing over time. Some other papers have also shown that the optimal posted price path, under full commitment power, may fluctuate over time. In particular, [Board \(2008\)](#) shows that, in a setting with stochastic population, if the incoming population of buyers are heterogeneous in terms of their demand, then the optimal price path displays a fluctuating structure. This leads some buyers to delay their purchases, hence time is an effective discriminatory tool in such a situation. The reason for this is not the dynamic population but the heterogeneous demand structure. Another important paper showing a fluctuating price path is [Garrett \(2011\)](#) in which the flow valuations of buyers are stochastic due to the private circumstances. The environment is stationary, in the sense that neither the value distributions of the entering buyers nor the stochastic process governing the valuations are time dependent. However it has been shown that the optimal price path includes fluctuations. Like our paper, in [Garrett \(2011\)](#), the posted prices is not an optimal selling mechanism. The optimal mechanism involves selling option contracts to purchase the good at future dates. However, the source of the inefficiency on posted prices is different than the one in this paper.

In addition, our paper is also related to the literature on dynamic auctions.<sup>2</sup> A common feature of this literature is that there is a certain time period  $T$ , until when the seller must sell his multi-unit indivisible goods. The buyers are entering into and leaving from the market over time. [Pai and Vohra \(2008\)](#), also [Board and Skrzypacz \(2010\)](#) are good examples. They first define the optimal allocations, which follows a simple index rule, and then show how to implement this via monetary transfers. The main difference of our paper from this literature is that, in ours, the number of goods that the seller can sell is not limited whereas in dynamic auction literature the seller is endowed with a certain number of goods.

Finally, there are some papers that are incorporating structures similar to the product upgrade story in our paper. They are different than ours not only in terms of many aspects of the modeling structure, but also in terms of the questions that they want to answer. For instance, in the literature of planned obsolescence, the lifetime of the durable good is an endogenous choice variable for the firm. The firm, by producing goods with shorter lifetime,

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<sup>1</sup>Actually optimal price path is not unique, there are infinitely many price paths resulting with the same allocation. The important thing is that the initial price level is equal to the static monopoly price and the price level never falls below that.

<sup>2</sup> [Bergemann and Said \(2011\)](#) is a good survey about dynamic auctions.

leads consumers to repurchase again rather than leaving the market after purchasing once. [Bulow \(1986\)](#) is a good example of this literature. Also there is an old literature investigating the effect of vintage capital on the aggregate growth of the economy. The basic story of these papers is that, in each period, the machines of production are being improved because of technological progress. And firms are deciding how much to invest in replacing old machines with the newer ones. An example of this literature is [Benhabib and Rustichini \(1991\)](#). Last, there is a literature on *R&D* decisions of a durable monopoly to improve the quality of the good, see for example [Swan \(1970\)](#), [Fishman and Rob \(2000\)](#). The main concern of these papers is to understand the product development decisions of the firms given a non-strategic buyers side, which is represented with a demand function on the quality of the durable good.

## 2 A MODEL OF DURABLE GOOD PRICING

Time is continuous and  $r$  is the common discount factor. There is a monopoly firm that is selling its good which is durable. An initial version of the product exists in the marketplace at the beginning of the time ( $t=0$ ), and a newer version will eventually take the place of the existing one. We describe the process of the arrival of the new version later. The firm sells only the most current version of the good at a given time; in other words, when the new version arrives the firm is no longer able to sell the earlier version. The cost of the production is normalized to 0 for both versions of the good.

There is a continuum of buyers of measure 1 with differing valuations for the product, and each has a unit demand so that they can use at most one unit of good at a time. The new version is valued more by all of the buyers and we assume that the ratio of valuations for two versions of the good is same for each buyer. If a buyer has a flow utility  $x$  from the first version of the good then his flow utility is equal to  $\beta x$  for the second version of the good where  $\beta > 1$ . We call  $x$  as the type of the buyer which follows a continuous and differentiable distribution function  $F(x)$  on  $[0, 1]$  with full support and  $f(x)$  is the corresponding density function which we assume continuous. The buyers are strategically deciding the time of their purchase(s), and also which version(s) of the good to buy. The good is durable and so a buyer may use it forever after the purchase. However, since the flow utility of the new version is higher, a buyer may want to purchase the newer version even though he already own the older one. Therefore, buyers do not necessarily leave the market upon purchase unlike the existing literature. On the other hand, version-wise strategic delays of purchases may appear. In particular, a buyer may prefer to wait for the arrival of the new version rather than buying the current version of the good.

The firm commits to a price path for both versions of the good at  $t = 0$ . The price path

is consisting of a price level for the first version of the good for each time until the arrival of the new version and also a price level for the new version for each time after its arrival. Note that this implies an anonymous pricing structure in the sense that the monopolist has to charge the same price for every buyer that is purchasing at the same time. In other words, the possibility of conditioning the new version sales to the buyers' ownership status of the old version is ruled out. However, as a benchmark, we also consider the case where the firm can commit to a non-anonymous selling policy. As we mentioned in the introduction, there are two parts of this paper depending on the arrival process of the new version.

## 2.1 STOCHASTIC ARRIVAL

The second version of the good arrives stochastically with a Poisson arrival process with rate  $\lambda$  and the realized arrival time is denoted by  $T$ . The price path that the firm commits to at  $t = 0$  is contingent on  $T$ . More precisely, it is of the form  $\left\{ \{p_t\}_{t \in [0, \infty)}, \{ \{p_t^T\}_{t \in [0, \infty)} \}_{T \in [0, \infty)} \right\}$ . The price level of the first version at a given time  $t$ , conditional on the event that the second version has not arrived until  $t$ , is  $p_t$ . On the other hand,  $p_t^T$  is the price level of the second version of the good  $t$  period after the arrival time, contingent on the realized arrival time  $T$ . Therefore  $p_t^T$  is the price level at time  $t + T$  contingent on the arrival time  $T$ .

Each buyer has a continuum of decision variables. In particular the buyer decides whether and when to purchase the first version of the good, also whether and when to purchase the second version for each possible arrival time  $T$ . More precisely, buyer  $x$ 's decisions are of the form  $\{t_x, \{t_{x,T}\}_{T \in (0, \infty)}\}$ , where  $t_x$  is the time of the purchase of the first version of the good. If arrival occurs before the purchase of the first version  $t_x$ , then buyer  $x$  does not buy the first version. If  $t_x = \infty$ , then the buyer never purchases the first version of the good. The term  $t_{x,T}$ , on the other hand, specifies how long after the realized arrival time  $T$ , buyer  $x$  purchases the second version. Hence, the corresponding calendar time of the purchase is  $T + t_{x,T}$ . Again if  $t_{x,T} = \infty$  then the agent does not purchase the second version when arrival occurs at  $T$ .

The utility of buyer  $x$  is denoted by  $U(x)$ , which linearly depends on the sum of the flow values and the amount paid to the monopolist. If arrival occurs before  $t_x$ , then the utility of buyer  $x$  depends only on the flow values acquired from the second version. If, on the other hand, the arrival takes place after  $t_x$ , then the utility is consisting of the flow values acquired from both versions of the good. More precisely,  $U(x)$  is of the following form:

$$\begin{aligned}
U(x) &= \int_0^{t_x} e^{-\lambda T} \lambda \left( \int_{T+t_x, T}^{\infty} e^{-rt} \beta x dt - e^{-r(T+t_x, T)} p_{t_x, T}^T \right) dT \\
&+ \int_{t_x}^{\infty} e^{-\lambda T} \lambda \left( \int_{t_x}^{T+t_x, T} e^{-rt} x dt + \int_{T+t_x, T}^{\infty} e^{-rt} \beta x dt - e^{-rt_x} p_{t_x} - e^{-r(T+t_x, T)} p_{t_x, T}^T \right) dT
\end{aligned}$$

The first line captures the contingencies in which arrival occurs before  $t_x$ . For each arrival time  $T \in [0, t_x]$ , the corresponding utility is the expression inside the parenthesis. Then, after weighting them with probability of arrival at T (i.e with  $e^{-\lambda T} \lambda$ ), we integrate it to get the expectation. The second line accounts for the late arrivals that are after the purchase of the first version, where the expression inside the parenthesis represents the utility corresponding to a specific arrival time  $T \in [t_x, \infty)$ . In each contingency the agent gets  $x$  flow utility until the second version purchase, and  $\beta x$  afterwards. We also discount the payments and integrate them after weighting the probability of arrival. Note that, if  $t_x$  is  $\infty$  i.e the agent never purchases the first generation of the good, then his utility is equal to the first line of the above expression, and if  $t_x = 0$  then only the second line of the above expression is relevant.

The profit of the firm, which we denote by  $\Pi$ , can be written as follows

$$\Pi = \int_0^1 e^{-\lambda t_x} e^{-rt_x} p_{t_x} f(x) dx + \int_0^{\infty} e^{-\lambda T} \lambda \left( \int_0^1 e^{-r(T+t_x, T)} p_{t_x, T}^T f(x) dx \right) dT.$$

The first term is the profit from the first version sales where we integrate the discounted payment of each agent over the type space of buyers. To discount the payment of agent  $x$  (i.e  $p_{t_x}$ ), we multiply it by  $e^{-rt_x}$  and also by the probability of the event that the arrival does not occur until  $t_x$ , which is  $e^{-\lambda t_x}$ . The second term is the profit from the second version sales. And the inner integral is the level of profit for a specific arrival time  $T$ , while we then compute the expectation.

## 2.2 DETERMINISTIC ARRIVAL

In the second part of the paper we assume that the arrival of the new version of the good occurs at a commonly known time period  $T$ . In this case, the price path that the firm commits to at the beginning is of the form  $\{p_t\}_{t \in [0, \infty)}$  where  $p_t$  is the price level of the first(second) version of the good at time  $t$  if  $t < T$  ( $t \geq T$ ). The decision variables for buyers are defined accordingly. The model that we consider here is mainly the analogous of the



model with stochastic arrival, but with some further modifications to the model, which we will detail in section 4.

### 3 OPTIMAL POSTED PRICES: STOCHASTIC ARRIVAL

So far we have described the basics of the model including the decision variables of the monopolist and the buyers and the corresponding utility and profit functions. In this section we characterize the optimal contingent posted prices. Some benchmark cases are presented first to develop a better understanding of the general framework. The next subsection illustrates the problem of the canonical durable good monopoly model, in which there is no product upgrade, and then we consider the case where there is a product upgrade but the monopolist is not restricted to use posted prices. It can rather use any selling mechanism including the non-anonymous ones where the sales of the second version of the good are conditional on the first version purchases.

#### 3.1 BENCHMARK I: CANONICAL DURABLE GOOD MONOPOLY

This benchmark is analyzed in [Stokey \(1979\)](#).<sup>3</sup> There is only one version of the durable good staying in the market forever. Note that this can be seen as a special case of our model when  $\lambda = 0$ , since the newer version of the good never arrives in this case. Another way to produce this canonical model from ours is to assume  $\beta = 1$  so that there is no distinction between the first and the second versions of the good for buyers. The monopolist chooses a unique price path  $\{p_t\}_{t \in [0, \infty)}$ , and agents decide the timing of their purchases which is denoted by  $t_x$  for agent  $x$ . Corresponding utility of the agent  $x$  is

$$U(x) = \int_{t_x}^{\infty} e^{-rt} x dt - e^{-rt_x} p_{t_x} = e^{-rt_x} \left( \frac{x}{r} - p_{t_x} \right)$$

and the profit of the firm is

$$\Pi = \int_0^1 e^{-rt_x} p_{t_x} f(x) dx.$$

Since the monopolist has full commitment power, his profit maximization problem boils down to a mechanism design problem where the incentive constraints of the agents are originating from the strategic timing of their purchases. Thanks to the revelation principle

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<sup>3</sup>Here unlike the analysis presented in [Stokey \(1979\)](#) we follow the general mechanism design approach. We first characterize the incentive constraints and then rewrite the firm's problem as an optimal allocation problem.

we can restrict attention to the direct mechanisms WLOG. The firm asks agents to report their types, and then decides the allocations i.e. a purchase time  $t_x$  and a payment  $p_x$  to be paid for each report  $x$ . Note that the payment must be same for the agents that are purchasing at the same time for otherwise there would be a profitable deviation for one of the types from truthful reporting and this would violate the incentive compatibility. Therefore, for each allocation time there is a corresponding payment level. Hence we can denote the payment rule of the mechanism by  $p_{t_x}$ .

First we need to understand the nature of the incentive constraints. The following Lemma, proved in the appendix, is an adapted version of the fundamental IC Lemma corresponding to the durable good framework that characterizes the incentives.

LEMMA 1. *The direct mechanism is incentive compatible iff:*

- i)  $t_x$  is non-increasing with  $x$ .
- ii)  $U(x) = U(0) + \frac{1}{r} \int_0^x e^{-rt_{\tilde{x}}} d\tilde{x}$

Lemma 1 states that a higher type will not purchase the good at a later time than a lower type and so the purchase times are monotonic. It also states that the derivative of the buyer's utility with respect to his type is proportional to the effective discounting up to his purchase time ( $\frac{1}{r}e^{-rt_x}$ ).

Since the lemma states both necessary and sufficient conditions for the incentive compatibility, we can write the monopolist's problem as follows.

$$\max_{\{p_t\}_{t \in [0, \infty)}, \{t_x\}_{x \in [0, 1]}} \int_0^1 e^{-rt_x} p_{t_x} f(x) dx \quad s.t \quad t_x \text{ is non-increasing with } x.$$

$$U(x) = U(0) + \frac{1}{r} \int_0^x e^{-rt_{\tilde{x}}} d\tilde{x} \quad \forall x$$

Then, to simplify the objective function and the constraints we rewrite the optimization problem. The resulting form consisting of only allocation terms  $t_x$ 's and its solution defines the optimal allocation rule. Then by using the incentive constraints again we get the corresponding optimal pricing rule implementing the optimal allocation. In particular, by incentive compatibility we know that

$$U(x) = e^{-rt_x} \left( \frac{x}{r} - p_{t_x} \right) = \frac{1}{r} \int_0^x e^{-rt_{\tilde{x}}} d\tilde{x} \quad ^4$$

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<sup>4</sup>In optimal mechanism  $U(0) = 0$ , hence we can omit it.

hence we have

$$e^{-rt_x} p_{t_x} = e^{-rt_x} \frac{x}{r} - \frac{1}{r} \int_0^x e^{-rt_{\tilde{x}}} d\tilde{x}.$$

Therefore, the profit of the firm is equal to

$$\Pi = \int_0^1 e^{-rt_x} p_{t_x} f(x) dx = \int_0^1 \left( e^{-rt_x} \frac{x}{r} - \frac{1}{r} \int_0^x e^{-rt_{\tilde{x}}} d\tilde{x} \right) f(x) dx.$$

After integrating by parts we get

$$\Pi = \int_0^1 e^{-rt_x} p_{t_x} f(x) dx = \frac{1}{r} \int_0^1 e^{-rt_x} \left( x - \frac{1 - F(x)}{f(x)} \right) f(x) dx.$$

Hence the new form of the problem is

$$\max_{\{t_x\}_{x \in [0,1]}} \frac{1}{r} \int_0^1 e^{-rt_x} \left( x - \frac{1 - F(x)}{f(x)} \right) f(x) dx \quad \text{s.t. } t_x \text{ is non-increasing.} \quad (1)$$

The term  $(x - \frac{1-F(x)}{f(x)})$  is referred as the virtual value of type  $x$  and the monopolist's problem boils down to maximize the sum of the discounted virtual values by arranging the  $t_x$ 's accordingly. The corresponding optimal allocation rule maximizing the monopolist's profit is then of the following form.

$$t_x = \begin{cases} 0 & \text{if } x \in [x^*, 1] \\ \infty & \text{otherwise} \end{cases}$$

In words the monopolist chooses a threshold value  $x^*$  and immediately allocates the good to the buyers with a value above the threshold level and does not allocate the good to the other buyers. If the virtual valuation function,  $x - \frac{1-F(x)}{f(x)}$ , is increasing in  $x$ , then the threshold  $x^*$  would be equal to the point above which the virtual valuation function is non-negative. In other words, the optimal rule would involve allocating the good to the agents with positive virtual valuations immediately and never allocating to the rest. Also, for the case of non-monotonic virtual valuation function, the optimal allocations follows a cutoff rule with immediate allocations as well. But in this case the monopolist will choose  $x^*$  in such a way that the integral of the virtual valuation function above  $x^*$  is maximized. This is a result of the monotonicity constraint on  $t_x$ . This does not mean that all the buyers of a type higher than  $x^*$  have a positive virtual valuation. An important thing to note here is that this allocation rule is the optimal allocation rule of the static monopoly.

Now, we need to find a price path to implement the optimal allocation rule we specified. For this we use the fact that, at the optimal mechanism,  $U(0) = 0$  and hence  $U(x^*) =$

$e^{-rt_{x^*}} \left( \frac{x^*}{r} - p_{t_{x^*}} \right) = U(0) + \frac{1}{r} \int_0^{x^*} e^{-rt_{\tilde{x}}} d\tilde{x} = 0$ , which requires  $p_{t_{x^*}} = p_0 = \frac{x^*}{r}$ . Therefore an optimal price path to implement the optimal allocation is a constant price path at level  $\frac{x^*}{r}$ . The importance of this price level is that the buyer with type  $x^*$  is indifferent between purchasing and not.<sup>5</sup> The significance of this result is that, for the dynamic setting, the optimal price is constant and equal to the static monopoly price. Therefore the monopolist does not use the time to discriminate over the buyers with different valuations.

### 3.2 BENCHMARK II: PRODUCT REPLACEMENT AND NO ANONYMITY

In this benchmark we turn back to the case in which there is an upgrade of the current version of the good, following a Poisson process but we do not restrict monopolist to use posted prices. In other words, the mechanism that the monopolist commits to does not necessarily have to be anonymous for the second version sales, rather it can be conditional on the first version sales. Note that, under posted prices, the decisions of the buyer of type  $x$  owning the first version at the time of arrival  $T$  and the buyer of type  $(\beta - 1)x$  not owning the first version at  $T$  will be same. This stems from the fact that both of them have the same marginal benefit (which is equal to  $(\beta - 1)x$ ) from the purchase of the new version. Therefore, the monopolist, to find the optimal posted prices, must treat these agents in the same manner under the mechanism with posted prices.

The situation is different when the monopolist is not restricted to use posted prices. In particular, the direct mechanism in which the buyers are asked to report their types leads to the allocation of both versions and a payment rule for each type. In other words, the monopolist can bundle the goods of version 1 and version 2 and can define a non-linear payment rule. The payment rule is denoted by  $P(x)$  for a given reported type  $x$  and is to be paid at time  $t = 0$  without loss of generality, since we can also think of it as any dynamic payment rule with present value equal to  $P(x)$ .

Contrary to the change in the payment scheme the allocation rule will be the same as introduced in the beginning where the monopolist is restricted to use posted prices. Hence a buyer after reporting its type as  $x$ , will get the goods at  $t_x$ , and  $\{t_{x,T}\}_{T \in [0, \infty)}$  as we defined earlier after reporting its type  $x$ . Therefore we can write the corresponding utility of agent  $x$  from this direct mechanism (without anonymity) is as follows:

$$U(x) = Q(x)x - P(x).$$

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<sup>5</sup>Even though the optimal allocation is unique, there are infinitely many price paths that can implement it. The important thing is to fix the initial price level to  $\frac{x^*}{r}$  and always keep it above or equal to the initial level.

where

$$Q(x) = \int_0^{t_x} e^{-\lambda T} \lambda e^{-r(T+t_{x,T})} dT + \int_{t_x}^{\infty} e^{-\lambda T} \lambda \left( \int_{t_x}^{T+t_{x,T}} e^{-rt} dt + \beta \int_{T+t_{x,T}}^{\infty} e^{-rt} dt \right) dT$$

More precisely,  $xQ(x)$  is the sum of the flow utilities from  $t = 0$  to  $t = \infty$ , without including the monetary payment. Note that the maximum value of  $Q(x)$  is acquired by arranging  $t_x = 0$  and  $t_{x,T} = 0$  for every  $T$  (immediate allocation of both versions), in which the value of  $Q(x)$  will be  $\frac{r+\beta\lambda}{r(r+\lambda)}$ . The minimum value of  $Q(x)$  is 0 which means that there is no allocation of any version of the good at any time (i.e the allocation times are  $\infty$  for both versions of the good). Hereafter we can follow the same steps as in the first benchmark case. To this end, the following lemma defines the incentive constraints. Its proof is omitted as it is same with the one of 1.

LEMMA 2. *The direct mechanism without anonymity restriction is incentive compatible iff:*

- i)  $Q(x)$  is non-decreasing
- ii)  $U(x) = U(0) + \int_0^x Q(\tilde{x}) d\tilde{x}$

Then the monopolist's problem can be written as

$$\max_{\{Q(x)\}_{x \in [0,1]}} \frac{1}{r} \int_0^1 Q(x) \left( x - \frac{1 - F(x)}{f(x)} \right) f(x) dx \quad s.t \quad Q(x) \text{ is non-decreasing with } x$$

As can be seen, this problem is in a very similar form to the one in the first benchmark. The only difference is now that, rather than  $t_x$ , we have  $Q(x)$  as a choice variable. Hence the solution will be analogous to the one of the canonical model. There is a threshold  $x^*$ , which is equal to the threshold that is defined in the first benchmark and, therefore equal to the static monopoly cutoff value. And the optimal allocation will be such that if a buyer has a valuation higher than  $x^*$ , then  $Q(x)$  will be arranged as high as possible, and will be set to its minimum value, which is 0, otherwise. More precisely the following is the optimal allocation rule:

- i)  $t_x = t_{x,T} = 0 \quad \forall x \in [x^*, 1]$
- ii)  $t_x = t_{x,T} = \infty \quad \forall x \in [0, x^*)$

Therefore, in the optimal mechanism there is no buyer acquiring only one version of the good. In other words, the monopolist is bundling two generations of the good and selling them only together. This result resembles the discounted upgrading policies that are used

in real life examples. For instance, some companies, like Microsoft, offer discounts to their customers in case they have an old version and want to upgrade to a newer one. Here we see an extreme version of this policy in the sense that the price of the second version for those who already own the first version of the good is equal to zero.

The payment rule to implement this allocation requires the monopolist to charge the same amount from all the agents in  $[x^*, 1]$  since all of them have the same  $Q(x)$ . That amount is equal to  $x^* \frac{r+\beta\lambda}{r(r+\lambda)}$  which leaves the marginal agent  $x^*$  indifferent between purchasing and not. This mechanism is the optimal selling mechanism. However, it is impossible to implement the corresponding allocation rule by using posted prices. To see this, suppose there is a contingent price path that can implement it anonymously. Then, under these prices, marginal return from the second version purchase for agent  $x^*$  is larger than or equal to the price at the corresponding time period. But since his marginal benefit is equal to  $(\beta - 1)x^*$ , the agent  $x^* - \epsilon$ , for  $\epsilon$  sufficiently small, would prefer to purchase the second version as well. Hence we get a contradiction.

### 3.3 SALES WITH POSTED PRICES: ANONYMITY

Here, the focus is on the characterization of the optimal posted prices rather than the optimal selling mechanism which is already considered in the second benchmark. In this case there are some additional constraints due to the anonymous structure of the posted prices, and monopolist needs to incorporate these into the maximization problem. Therefore the corresponding direct mechanism would be complicated as there are more constraints. To cope with this complication we rather use a different approach than the direct mechanism. In this new approach, which we call as "two-step mechanism", the allocations of the two versions of the good are done by two independent stages of reporting.

The anonymity restriction leads agents, having the same marginal benefit from the purchase of the second version, to have the same decision in terms of second version purchase regardless of their ownership of the first version of the good. To this end, we define the following concept which we incorporate into the structure of our two-step mechanism.

**DEFINITION 1.** *The **Effective type** of agent  $x$  at realized arrival time  $T$ , is equal to his marginal flow utility from the second version purchase. Particularly, it is equal to  $\beta x$  if he does not own the first version and it is equal to  $(\beta - 1)x$  if he does.*

By using this concept, we will interpret the model in a different way. Specifically, in our framework the durable good have two consecutive versions and an agent can use at most one version of the good at a given time. Therefore, if an agent, at some point, is owning both versions of the good, he is using only the current one as it gives more flow utility, he stops

to use the first version after the purchase of the second one. However, we can also interpret this story in such a way that rather than two consecutive versions of the same good there are two separate goods, first and second good respectively. If agent  $x$  purchases the first good, he will use it forever and gain a flow utility  $x$ . If he also purchases the second good, he uses both goods forever and gains flow utility from both. Finally, the flow utility of agent  $x$  from the second good is equal to his effective type, and hence it depends on the first version purchasing decision. From this point on we follow this interpretation in our analysis, as it simplifies the exposition. We define two-step mechanism as follows:

**DEFINITION 2.** *The two-step mechanism is a mechanism in which buyers are asked to report twice. First, at  $t = 0$ , buyers are asked to report their types. Then the allocations and payments for the first version of the good occur according to the first step reports. Second, at the realized arrival time  $T$ , buyers are asked to report their effective types at  $T$  and afterwards the allocations and the payments of the second version of the good occur according to the second step reports independent from the first step reports.*

Note that, finding the optimal mechanism among this class of mechanisms will give us the optimal price path that the monopolist can commit to.

We slightly modify the notation specified earlier since the mechanism structure is different now. For the second version allocations the relevant information is the reported effective type. Contingent on the realized arrival time  $T$ , the amount of time after which the effective type  $x$  purchases the second version of the good is denoted by  $t_x^T$  and so the corresponding purchase time is  $T + t_x^T$ .<sup>6</sup> For the first step allocations the relevant information is the initial type and so we keep the old notation, where  $t_x$  is the purchase time for type  $x$ .

The expressions for the utility of the agents and the profit of the firm is also slightly modified. The utility of an agent is consisting of two main parts, one for each version purchases. We start with the second version contingent on the arrival time  $T$ . The discounted expected utility calculated at  $T$  of the effective type  $x$ , from purchasing the second version of the good, is denoted by  $V_x^T$ , and it is equal to

$$V_x^T = \int_{t_x^T}^{\infty} e^{-rt} x dt - e^{-rt_x^T} p_{t_x^T}^T = e^{-rt_x^T} \left( \frac{x}{r} - p_{t_x^T}^T \right).$$

Similarly, expected discounted utility of buyer of type  $x$ , from the first version purchase calculated at  $t=0$ , is denoted by  $V_x$ , and is equal to

$$V_x = e^{-\lambda t_x} \int_{t_x}^{\infty} e^{-rt} x dt - e^{-(r+\lambda)t_x} p_{t_x} = e^{-(r+\lambda)t_x} \left( \frac{x}{r} - p_{t_x} \right).$$

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<sup>6</sup>Note that the previous notation was  $t_{x,T}$ , for type  $x$ . We now take the effective types as our basis.

Note that this expression reflects the alternative interpretation depicted above. In particular the good of version 1 is used forever and the flow utility acquired from it is equal to  $x$ , and we multiply the integral with the probability of no arrival until  $t_x$ . Hence the total utility of buyer  $x$ , which is denoted by  $U(x)$ , can be written as follows

$$U(x) = V_x + \int_0^{t_x} e^{-(r+\lambda)T} \lambda V_{\beta x}^T dT + \int_{t_x}^{\infty} e^{-(r+\lambda)T} \lambda V_{(\beta-1)x}^T dT.$$

The term  $V_x$  is the expected utility from the first version of the good, and the rest of the expression is the expected utility from the second version. In particular, if the arrival of the second version occurs before  $t_x$ , then the effective type of the agent  $x$  is  $\beta x$  as he has not purchased the first version yet. Therefore, we are taking the expectation of  $V_{\beta x}^T$  for the time interval  $[0, t_x)$ . On the other hand, for an arrival time larger than  $t_x$  the effective type of buyer  $x$  is equal to  $(\beta - 1)x$ . Hence we take the expectation of  $V_{(\beta-1)x}^T$  for the time interval  $[t_x, \infty)$ . An important thing to note here is that the timing of first version purchase  $t_x$ , affects not only the value of  $V_x$  but also the expected utility acquired from the second version as it alters the boundaries of the integrals.

Similarly, the firm's profit function,  $\Pi$ , is formed by two components, specifically

$$\Pi = \int_0^1 e^{-rt_x} e^{-\lambda t_x} p_{t_x} f(x) dx + \int_0^{\infty} e^{-(r+\lambda)T} \lambda \Pi^T dT. \quad (2)$$

The first term is the expected profit from the first version sales, and second term is from the second version sales, where each  $\Pi^T$  is the discounted profit (calculated at time  $T$ ) corresponding to the specific arrival time  $T$ , and is equal to

$$\Pi^T = \int_0^1 e^{-rt_x^T} p_{t_x^T}^T f_T(x) dx.$$

Where  $f_T(\cdot)$  is the distribution of effective types contingent on the realized arrival time  $T$  and  $F_T(\cdot)$  is the corresponding cdf. The distributions  $f_T(\cdot)$ , for each  $T$ , is depending on the first version allocations  $t_x$ 's. Therefore, we can think of it as monopolist is altering the demand for the second version of the good by altering the allocations of the first version of the good. This is an important feature of our model.

To characterize the incentive constraints, we need to consider both reporting stages separately. Since buyers are forward looking, while reporting in the initial stage they will internalize the effect of their report on the second stage of the mechanism. To this respect, we start to characterize the incentives from the second stage, which requires us to consider all the possible arrival realizations for the second reporting stage. The following lemma



considers the second stage incentives for a given arrival time  $T$ .

LEMMA 3. *For any arrival time  $T$ , the second step of the mechanism is incentive compatible iff the following two are satisfied:*

$$\begin{aligned} i) & \ t_x^T \text{ is non-increasing with } x \\ ii) & \ V_x^T = V_0^T + \frac{1}{r} \int_0^x e^{-rt_{\tilde{x}}^T} d\tilde{x}. \end{aligned}$$

PROOF. Follows from the same arguments with the proof of Lemma 1.

Before moving on to the first step incentives we assume a restriction on the parameters.

ASSUMPTION 1. The arrival rate is sufficiently small so that  $\lambda$  satisfies  $\lambda \leq \frac{r}{\beta-1}$ .

Now we turn to the incentive constraint of the first stage reporting. Any deviation from truthful reporting at this stage will also alter the optimal behavior in the second stage as it changes the corresponding effective types. Hence defining the incentives of the first stage is more complicated.

LEMMA 4. *Fix a two-step mechanism that is incentive compatible for the second stage reports and assume that assumption 1 is satisfied. If the mechanism is incentive compatible in the first stage then:*

$$\begin{aligned} i) & \ t_x \text{ is non-increasing with } x \\ ii) & \ V_x = V_0 + \frac{1}{r} \int_0^x e^{-(r+\lambda)t_{\tilde{x}}} d(\tilde{x}) - \lambda \int_0^x e^{-(r+\lambda)t_{\tilde{x}}} \frac{\partial t_{\tilde{x}}}{\partial x} \left( V_{\beta\tilde{x}}^{t_{\tilde{x}}} - V_{(\beta-1)\tilde{x}}^{t_{\tilde{x}}} \right) d\tilde{x} \end{aligned}$$

PROOF. i) Monotonicity: Take arbitrarily two agents of type  $x$ , and  $x'$ , where  $x > x'$  without loss of generality, we want to show that  $t_x \leq t_{x'}$ . Showing that purchasing the good at time  $t > t_{x'}$  is worse than purchasing it at  $t_{x'}$  for agent  $x$  is sufficient to prove monotonicity. To this end, take an arbitrary  $t$  satisfying  $t > t_{x'}$ . We know by revealed preferences of agent  $x'$  that:

$$U(x') \geq e^{-(r+\lambda)t} \left( \frac{x'}{r} - p_t \right) + \int_0^t e^{-(r+\lambda)T} \lambda V_{\beta x'}^T dT + \int_t^\infty e^{-(r+\lambda)T} \lambda V_{(\beta-1)x'}^T dT$$

Then we get:

$$\frac{x'}{r} (e^{-(r+\lambda)t_{x'}} - e^{-(r+\lambda)t}) - (e^{-(r+\lambda)t_{x'}} p_{t_{x'}} - e^{-(r+\lambda)t} p_t) \geq \int_{t_{x'}}^t e^{-(r+\lambda)T} \lambda (V_{\beta x'}^T - V_{(\beta-1)x'}^T) dT$$

We want show that the symmetric version of the above expression holds for agent  $x$  as well. Hence we need show

$$\frac{x - x'}{r}(e^{-(r+\lambda)t_{x'}} - e^{-(r+\lambda)t}) \geq \int_{t_{x'}}^t e^{-(r+\lambda)T} \lambda ((V_{\beta x}^T - V_{(\beta-1)x}^T) - (V_{\beta x'}^T - V_{(\beta-1)x'}^T)) dT. \quad (3)$$

Now to show the inequality above is correct we need to consider two cases.

- Case 1 :  $[x' < \frac{\beta-1}{\beta}x]$

Incentive compatibility in the second step is satisfied by hypothesis. Therefore, by the second condition in Lemma 3 we know that the highest possible value of  $((V_{\beta x}^T - V_{(\beta-1)x}^T) - (V_{\beta x'}^T - V_{(\beta-1)x'}^T))$  can be attained by arranging  $t_z^T = 0$ , for all  $z \in [(\beta-1)x, \beta x]$ , and  $t_z^T = \infty$ , for all  $z \in [0, (\beta-1)x)$ . Therefore

$$((V_{\beta x}^T - V_{(\beta-1)x}^T) - (V_{\beta x'}^T - V_{(\beta-1)x'}^T)) \leq \frac{x}{r},$$

which leads to

$$\int_{t_{x'}}^t e^{-(r+\lambda)T} \lambda ((V_{\beta x}^T - V_{(\beta-1)x}^T) - (V_{\beta x'}^T - V_{(\beta-1)x'}^T)) dT \leq x \frac{\lambda}{r(r+\lambda)} (e^{-(r+\lambda)t_{x'}} - e^{-(r+\lambda)t}).$$

However, since  $x' < \frac{\beta-1}{\beta}x$ , and  $\lambda < \frac{r}{\beta-1}$ , we know that

$$\frac{x - x'}{r}(e^{-(r+\lambda)t_{x'}} - e^{-(r+\lambda)t}) > x \frac{\lambda}{r(r+\lambda)} (e^{-(r+\lambda)t_{x'}} - e^{-(r+\lambda)t}).$$

Therefore equation (3) is satisfied and we are done for this case.

- Case 2 :  $[x' \geq \frac{\beta-1}{\beta}x]$

Again by Lemma 3 the highest possible value of  $((V_{\beta x}^T - V_{(\beta-1)x}^T) - (V_{\beta x'}^T - V_{(\beta-1)x'}^T))$  can be attained by arranging  $t_z^T = 0$ , for all  $z \in [\beta x', \beta x]$ , and  $t_z^T = \infty$ , for all  $z \in [0, \beta x')$ . Therefore,

$$((V_{\beta x}^T - V_{(\beta-1)x}^T) - (V_{\beta x'}^T - V_{(\beta-1)x'}^T)) \leq (x - x') \frac{\beta}{r}.$$

Hence

$$\int_{t_{x'}}^t e^{-(r+\lambda)T} \lambda ((V_{\beta x}^T - V_{(\beta-1)x}^T) - (V_{\beta x'}^T - V_{(\beta-1)x'}^T)) dT \leq (x - x') \frac{\beta \lambda}{r(r+\lambda)} (e^{-(r+\lambda)t_{x'}} - e^{-(r+\lambda)t}).$$

However, since  $\lambda < \frac{r}{\beta-1}$ , we know that

$$\frac{x - x'}{r} (e^{-(r+\lambda)t_{x'}} - e^{-(r+\lambda)t}) > (x - x') \frac{\lambda\beta}{r(r+\lambda)} (e^{-(r+\lambda)t_{x'}} - e^{-(r+\lambda)t}).$$

So equation (3) is valid for this case as well. Hence we are done to show monotonicity.

ii) Derivative of  $V_x$ : By truthfully reporting, agent get the utility

$$U(x) = V_x + \int_0^{t_x} e^{-(r+\lambda)T} \lambda V_{\beta x}^T dT + \int_{t_x}^{\infty} e^{-(r+\lambda)T} \lambda V_{(\beta-1)x}^T dT.$$

Now, for a given type  $x'$  with  $x > x'$ , what would happen if the agent  $x$  reports his type as  $x'$  at first stage reports? He would be allocated the first version of the good at time  $t_{x'}$  rather than  $t_x$  where, from part *i*), we know that  $t_x \leq t_{x'}$ . This deviation from truth-telling will affect his utility via two different channels. The first channel is a direct effect as he now acquires the first version at a different time.<sup>7</sup> The second channel is an indirect effect due to the change on second stage utility.

We know that the second step of the mechanism is incentive compatible and so the agent  $x$  reports his effective type truthfully in the second stage. Because of this, misreporting in the first stage alters the reports of the second stage only if it alters the effective types at the realized arrival time. This happens only if the arrival occurs between  $t_x$  and  $t_{x'}$ . In particular, after truthful reporting, the effective type of agent  $x$  would be  $(\beta - 1)x$  inside the time interval  $(t_x, t_{x'})$ , and it would be  $\beta x$  if he deviates and misreports its type as  $x'$ . Therefore, the incentive constraint of agent  $x$  preventing him to not to mimic  $x'$  is

$$\begin{aligned} U(x) &\geq e^{-(r+\lambda)t_{x'}} \left( \frac{x}{r} - p_{t_{x'}} \right) + \int_0^{t_{x'}} e^{-(r+\lambda)T} \lambda V_{\beta x}^T dT + \int_{t_{x'}}^{\infty} e^{-(r+\lambda)T} \lambda V_{(\beta-1)x}^T dT \\ &= V_{x'} + e^{-(r+\lambda)t_{x'}} \frac{(x - x')}{r} + \int_0^{t_{x'}} e^{-(r+\lambda)T} \lambda V_{\beta x}^T dT + \int_{t_{x'}}^{\infty} e^{-(r+\lambda)T} \lambda V_{(\beta-1)x}^T dT. \end{aligned}$$

The incentive constraint of the agent  $x'$  preventing him to mimic  $x$  is a symmetric version of the above expression. Then by combining these two inequalities we get

$$\frac{e^{-(r+\lambda)t_x}}{r} + \frac{\int_{t_x}^{t_{x'}} e^{-(r+\lambda)T} \lambda (V_{\beta x'}^T - V_{(\beta-1)x'}^T) dT}{x - x'} \geq \frac{V_x - V_{x'}}{x - x'} \geq \frac{e^{-(r+\lambda)t_{x'}}}{r} + \frac{\int_{t_x}^{t_{x'}} e^{-(r+\lambda)T} \lambda (V_{\beta x}^T - V_{(\beta-1)x}^T) dT}{x - x'}.$$

First of all, we know  $t_x$  is monotone. Therefore it is continuous almost everywhere, and so when  $x' \rightarrow x$ ,  $\frac{e^{-(r+\lambda)t_{x'}}}{r} \rightarrow \frac{e^{-(r+\lambda)t_x}}{r}$ , almost everywhere. Moreover, when  $x' \rightarrow x$ , by using Leibniz Rule, L'Hopital's Rule, almost everywhere continuity of  $t_x$  and incentive constraints

<sup>7</sup>If  $t_x = t_{x'}$  then we do not need to worry about first stage incentive constraints.

of second step reports which we have proven in previous Lemma 3, we get the following :

$$\begin{aligned} \lim_{x' \rightarrow x} \frac{\int_{t_x}^{t_{x'}} e^{-(r+\lambda)T} \lambda (V_{\beta x'}^T - V_{(\beta-1)x'}^T) dT}{x - x'} &= \lim_{x' \rightarrow x} \frac{\int_{t_x}^{t_{x'}} e^{-(r+\lambda)T} \lambda (V_{\beta x'}^T - V_{(\beta-1)x'}^T) dT}{x - x'} \\ &= -\lambda e^{-(r+\lambda)t_x} \frac{\partial t_x}{\partial x} (V_{\beta x}^{t_x} - V_{(\beta-1)x}^{t_x}) \quad a.e \end{aligned}$$

Therefore we can conclude that

$$\frac{\partial V_x}{\partial x} = \frac{e^{-(r+\lambda)t_x}}{r} - \lambda e^{-(r+\lambda)t_x} \frac{\partial t_x}{\partial x} (V_{\beta x}^{t_x} - V_{(\beta-1)x}^{t_x}) \quad a.e$$

Then by integrating it we get the result ii).  $\square$

To sum up, we have four conditions, which are defined in lemmas 3 and 4, that the optimal two step mechanism needs to satisfy. We have proven that the two conditions given in Lemma 3 are necessary and sufficient for the second step incentive compatibility, whereas the two conditions of Lemma 4 are just necessary conditions for the first stage incentive compatibility.<sup>8</sup> We take only these four conditions into account. Note that, as the latter two conditions are not sufficient for the second stage incentive compatibility, the resulting solution of the problem, that is written with only these four constraints, does not necessarily leads us to the mechanism that we are looking for as it may not be incentive compatible. However the resulting allocation rule and the corresponding payment scheme, as will be shown, is incentive compatible. Therefore the resulting solution is also the one that we are looking for. As usual we reformulate the maximization problem by embedding the incentive constraints into the objective function of the monopolist so that it only contains the allocation terms . Hence

$$V_x^T = e^{-rt_x^T} \left( \frac{x}{r} - p_{t_x^T}^T \right) = V_0^T + \frac{1}{r} \int_0^x e^{-rt_{\tilde{x}}^T} d\tilde{x}$$

In an optimal solution, the firm sets  $V_0^T = 0$ , therefore we have

$$e^{-rt_x^T} p_{t_x^T}^T = e^{-rt_x^T} \frac{x}{r} - \frac{1}{r} \int_0^x e^{-rt_{\tilde{x}}^T} d\tilde{x}.$$

Integrating by parts gives us  $\Pi_T$  as

$$\Pi^T = \int_0^1 e^{-rt_x^T} p_{t_x^T}^T f_t(x) dx = \frac{1}{r} \int_0^1 e^{-rt_x^T} \left( x - \frac{1 - F_T(x)}{f_T(x)} \right) f_T(x) dx.$$

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<sup>8</sup>Our conjecture is that they are also sufficient but since we do not need the sufficiency in the general result we did not show it formally.

For the profit from the first version sales

$$\begin{aligned} V_x &= e^{-(r+\lambda)t_x} \left( \frac{x}{r} - p_{t_x} \right) \\ &= V_0 + \frac{1}{r} \int_0^x e^{-(r+\lambda)t_{\tilde{x}}} d(\tilde{x}) - \lambda \int_0^x e^{-(r+\lambda)t_{\tilde{x}}} \frac{\partial t_{\tilde{x}}}{\partial x} \left( V_{\beta \tilde{x}}^{t_{\tilde{x}}} - V_{(\beta-1)\tilde{x}}^{t_{\tilde{x}}} \right) d\tilde{x}. \end{aligned}$$

Again in optimal mechanism  $V_0 = 0$ , hence

$$e^{-(r+\lambda)t_x} p_{t_x} = e^{-(r+\lambda)t_x} \frac{x}{r} - \frac{1}{r} \int_0^x e^{-(r+\lambda)t_{\tilde{x}}} d(\tilde{x}) + \lambda \int_0^x e^{-(r+\lambda)t_{\tilde{x}}} \frac{\partial t_{\tilde{x}}}{\partial x} \left( V_{\beta \tilde{x}}^{t_{\tilde{x}}} - V_{(\beta-1)\tilde{x}}^{t_{\tilde{x}}} \right) d\tilde{x}.$$

After integrating by parts, we get

$$\begin{aligned} \int_0^1 e^{-rt_x} e^{-\lambda t_x} p_{t_x} f(x) dx &= \frac{1}{r} \int_0^1 e^{-(r+\lambda)t_x} \left( x - \frac{1-F(x)}{f(x)} \right) f(x) dx \\ &\quad + \lambda \int_0^1 e^{-(r+\lambda)t_x} \frac{\partial t_x}{\partial x} (1-F(x)) \left( V_{\beta x}^{t_x} - V_{(\beta-1)x}^{t_x} \right) dx \end{aligned}$$

Therefore the monopolist's optimization problem is the following:

$$\begin{aligned} \max_{\{t_x\}_{x \in [0,1]}, \{t_x^T\}_{x \in [0,1]}}_{T > 0} &\quad \frac{1}{r} \int_0^1 e^{-(r+\lambda)t_x} \left( x - \frac{1-F(x)}{f(x)} \right) f(x) dx \\ &\quad + \lambda \int_0^1 e^{-(r+\lambda)t_x} \frac{\partial t_x}{\partial x} (1-F(x)) \left( V_{\beta x}^{t_x} - V_{(\beta-1)x}^{t_x} \right) dx \\ &\quad + \frac{\lambda}{r} \int_0^\infty e^{-(r+\lambda)T} \left( \int_0^1 e^{-rt_x^T} \left( x - \frac{1-F_T(x)}{f_T(x)} \right) f_T(x) dx \right) dT \end{aligned}$$

$$\begin{aligned} \text{subject to} &\quad \bullet \quad t_x \text{ is non-increasing in } x \\ &\quad \bullet \quad t_x^T \text{ is non-increasing in } x, \quad \forall T \in [0, \infty) \end{aligned} \tag{4}$$

An important thing to notice here is that the distribution  $f_T(\cdot)$ , for each  $T$  is a function of  $\{t_x\}_{x \in [0,1]}$ . Therefore, the monopolist, while choosing the allocations for the first version of the good, needs to internalize this effect. The first line of the objective function is the sum of the discounted virtual valuations for the first version of the good which also appeared in the canonical model as described in the first benchmark case. Similarly, the third line is its analogous for the second version of the good where virtual valuations are based on the effective type distributions for every possible arrival time  $T$ . The second line, which is always non-positive due to the monotonicity of  $t_x$  on  $x$ , is the cost of inter-versional incentives on the monopolist. While purchasing the first version of the good, the buyers are concerned

not only with the possibility of purchasing the first version cheaper at a later time but also with the possibility of the arrival of the second version to the marketplace. If the second version arrives then the buyer can purchase it and better off. The second line of the objective function reflects the effect of this concern on the monopolist's profit. This is more noticeable, if we look at the term  $(V_{\beta x}^{t_x} - V_{(\beta-1)x}^{t_x})$  appearing inside the integral. It is the difference on the utility from the second version of the good at a specific arrival time  $T$  for agent  $x$  depending on his ownership of the first version at that moment.

When  $\lambda \rightarrow 0$ , the optimization problem (4) boils down to optimization problem (1) of canonical model, because the second and the third terms of objective function in (4) are equal to 0 if  $\lambda = 0$ . This is intuitive because as  $\lambda$  approaches to 0 the arrival of the second version of the good never happens. Hence we get back to the canonical model in which there is only one version of the durable good. Theorem 1 characterizes the optimal allocation rule of the monopolist problem.

**THEOREM 1.** *The optimal allocation rule, maximizing problem (4) is described by a cut-off policy with immediate allocation for both generations of the good. There are two cutoff values  $x_1$ , and  $x_2$  such that any agent with (effective) type higher then  $x_1$  ( $x_2$ ) is allocated with first (second) version at  $t = 0$  ( $t = T$  i.e. as soon as it arrives).*

$$t_x = \begin{cases} 0 & x \geq x_1 \\ \infty & x < x_1 \end{cases} \quad \forall T, t_x^T = \begin{cases} 0 & x \geq x_2 \\ \infty & x < x_2 \end{cases}$$

**PROOF.** For now, rather than the problem in (4), we consider an auxiliary problem in which the second term of the objective function is omitted. Precisely

$$\begin{aligned} \max_{\{t_x\}_{x \in [0,1]}, \{t_x^T\}_{x \in [0,1]}} \{T > 0\} & \frac{1}{r} \int_0^1 e^{-(r+\lambda)t_x} \left( x - \frac{1 - F(x)}{f(x)} \right) f(x) dx \\ & + \frac{\lambda}{r} \int_0^\infty e^{-(r+\lambda)T} \left( \int_0^1 e^{-rt_x^T} \left( x - \frac{1 - F_T(x)}{f_T(x)} \right) f_T(x) dx \right) dT \end{aligned}$$

$$\begin{aligned} \text{subject to} \quad & \bullet t_x \text{ is non-increasing in } x \\ & \bullet t_x^T \text{ is non-increasing in } x, \quad \forall T \in [0, \infty) \end{aligned} \tag{5}$$

In this problem, the contingent allocation terms  $\{t_x^T\}_{x \in [0,1]}$ , for the second version of the good, appear only on the last term of the objective function. Then the optimal  $\{t_x^T\}_{x \in [0,1]}$  for this problem will be similar to the one of the canonical model. Therefore it follows a cutoff rule, where the value of the cutoff is a function of the distribution  $f_T(\cdot)$ . Hence we

denote the cutoff value by  $x^*(f_T)$ , and its value is exactly the same as the value of the cutoff for the static monopoly with distribution  $f_T(\cdot)$ . So that for a given  $f_T$  the allocations are of the form:

$$t_x^T = \begin{cases} 0 & x \geq x^*(f_T) \\ \infty & x < x^*(f_T) \end{cases}$$

On the other hand, the first version allocations  $\{t_x\}_{x \in [0,1]}$  are affecting both lines of the objective function as they alter the distribution functions  $f_T(\cdot)$  of the effective types. If this indirect effect did not exist, then the optimal allocation rule would be the immediate allocation for those agents having a type higher than  $x^*$  i.e the static monopoly allocation. Despite this additional effect, the optimal allocations have a similar structure to the one of the static monopoly in the sense that it also follows a cutoff rule. This is because of the stationary structure of the environment

*Claim:* There is a cutoff value  $\hat{x}$ , depending on the values of  $\lambda, \alpha, r$ , such that the optimal solution of the program (5) satisfies:

$$t_x = \begin{cases} 0 & x \geq \hat{x} \\ \infty & x < \hat{x} \end{cases}$$

*Proof of the Claim:* This is due to the stationary structure resulting from the Poisson arrival process. In particular, if at  $t \neq 0$  an agent is allocated the first version of the good then it must be the case that the total effect of allocating the first version to this agent on the objective function is positive. But then it must be positive  $t=0$  as well since the environment is stationary. Therefore it is better for the monopolist to allocate the good to this agent at the beginning  $t = 0$ . Hence we know that the term  $t_x$  must be either 0 or  $\infty$  for every  $x$ . Furthermore, since  $t_x$  is restricted to be monotone with respect to  $x$ , optimal solution must incorporate a structure as given above.  $\diamond$

Then we have the solution of the problem (5), as:

$$t_x = \begin{cases} 0 & x \geq \hat{x} \\ \infty & x < \hat{x} \end{cases} \quad \forall T, t_x^T = \begin{cases} 0 & x \geq x^*(\hat{x}) \\ \infty & x < x^*(\hat{x}) \end{cases}$$

Since the allocation of the first version only occurs at  $t = 0$ , the distribution of the effective types is independent of the realized arrival time  $T$ , and just depending on the cutoff of the first version allocations. Moreover, the allocation of the second version is same with the static monopoly allocations corresponding to the effective type distribution.

Turning back to the original problem of the monopolist as defined in (4) we know that the second term, which is omitted in the relaxed problem, would be equal to zero under the allocation rule that is specified above. This is because of the fact that  $\frac{\partial t_x}{\partial x} = 0$  almost everywhere. Furthermore, we also know that the highest possible value of this term is also zero, since  $t_x$  must be non-increasing and hence its derivative is never strictly positive. Therefore the solution of program (5), which is defined as above, is also the solution for the original problem (4) as it is maximizing the second term as well. Then we have  $x_1 = \hat{x}$ , and  $x_2 = x^*(\hat{x})$ .  $\square$

We have characterized the optimal allocation rule. The monopolist can implement this allocation rule by setting a constant price level for both versions  $p_1$ , and  $p_2$ , since allocations only occur at  $t=0$  and at the realized arrival time  $T$ .<sup>9</sup> The constant price of the second version does not depend on the realized arrival time. Specifically,  $p_2 = \frac{x_2}{r}$  so that the agent of effective type  $x_2$  is indifferent between purchasing and not purchasing the second version, so that  $V_{x_2}^T = 0$ . And  $p_1$  is the price level at which the agent of type  $x_1$  is indifferent on his purchase decision of version 1. In particular, if  $(\beta - 1)x_1 \geq x_2$  he is indifferent between purchasing both versions of the good and purchasing only the second one. And if  $(\beta - 1)x_1 < x_2$ , then he is indifferent between purchasing only the first version and purchasing only the second version.

The important thing to note here is that the monopolist is not using time to discriminate agents with different valuation which is in line with the canonical durable good monopoly setting. The only thing that changes here is the cutoff types. The concerns about the consecutive version sales lead the monopolist to alter the allocation rule compared to the benchmark. However we still have immediate allocation for both versions which gives us a constant price path version-wise.

*REMARK 1. As we mentioned earlier the solution of the optimization problem 4 does not necessarily be the optimal solution of the monopolist's problem, because the conditions in Lemma 4 are only necessary and not sufficient for the incentive compatibility of the first step of the mechanism. Hence to conclude that the solution of problem 4 is also a solution of the main problem we need to make sure that it is incentive compatible. Fortunately the solution that we specified above is obviously incentive compatible, so the analysis goes through.*

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<sup>9</sup>Version-wise constant price path is not the only price path to implement the optimal allocation rule. In particular, any increasing contingent price path, that is satisfying  $p_0 = p_1$ , and  $p_0^T = p_2$ , would also implement the optimal allocation rule. We would need a decreasing price path to implement, if the optimal allocation were to occur throughout time rather than immediately.



## 4 DETERMINISTIC ARRIVAL

So far, the arrival process of the new version of the good was stochastic and so the environment was stationary. Consequently, we showed that the optimal price path for both versions of the good is constant, and there is no delay on purchases. This result is in line with [Stokey \(1979\)](#) as the monopolist does not use time to discriminate over with different valuations. The main goal of this section is to claim that, if the arrival of the second version of the good occurs at a deterministic and commonly known date, then the monopolist with full commitment power may use time as a discriminatory tool contrary to the stochastic arrival case.

There is a time period  $T$  at which a new version of the durable good arrives into the marketplace. As time goes on, the arrival becomes closer which results in un-stationary incentives for the second version purchases. In other words, incentives for the second version purchases are time-dependent, contrary to the stochastic arrival case.

To achieve our goal of showing the possibility of a decreasing price path, we simplify the exposition as much as possible and focus on a discrete type space consisting of only two types, namely H (High) and L (Low), where  $H > L$ .<sup>10</sup> There are continuum of buyers and the measure of the H-type buyers is  $\mu \in (0, 1)$  while the measure of the L-type buyers is  $1 - \mu$ . This distribution, which is consisting of two-atoms, is a special case of the distribution  $F(\cdot)$ .

The utility structure that we specified in the previous case with stochastic arrival carries into here. Therefore the flow valuation of the second version of the good is  $\beta$  times the flow valuation from the first version of the good, where  $\beta > 1$ .

We mainly focus on the price path of the first version of the good. To this respect we further simplify the analysis by assuming that the price path for the second version is constant, and hence any purchase of the second version occurs only at time  $T$ .<sup>11</sup> More precisely, the price path is of the form  $\{\{p_t\}_{t \in [0, T)} \cup p_T\}$ , where  $p_t$  is the price level of the first version at  $t \in [0, T)$ , and  $p_T$  is the constant price level for the second version good. The price path here is not contingent since the arrival occurs at a deterministic time. We have the following assumption the parameters.

ASSUMPTION 2.  $\beta e^{-rT} < 1$ .

This assumption is basically a restriction on restriction on arrival time  $T$  and the flow utility multiplier  $\beta$ . More precisely, if this assumption is satisfied, then the total flow utility resulting

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<sup>10</sup> To omit the discount factor  $r$  that appears due to the integration of the flow utilities, say the types are  $h, l$ , and we have  $H = \frac{h}{r}$ , and  $L = \frac{l}{r}$ .

<sup>11</sup>We can rather think of this as a restriction so that the markets close down after  $T$ .

from purchasing only the first good at  $t = 0$  is higher than the total flow utility resulting from purchasing only the second version at time  $T$ .

Given any price path of the form above, suppose there is only the first version of the good and take the corresponding part of the price path i.e. the part corresponding to  $[0, T)$ . In such a situation there will be an optimal purchase time of the first version for each buyer (where as usual purchase time that is equal to  $\infty$  means not purchasing). Then the following lemma shows that this optimal time is also the optimal purchasing time when there is a second version of the good under the same price path. This does not mean that a buyer purchasing the first version of the good, when there is only the first version with corresponding prices, will also purchase it when the second version is added to the environment. She may rather choose to purchase only the second version of the good.

LEMMA 5. *For a given price path  $\{\{p_t\}_{t \in [0, T)} \cup p_T\}$ , consider a buyer that is purchasing the first version of the good at time  $t \in [0, T)$ . Then, purchasing at time  $t$  is also optimal for the case in which only the first version exists with the corresponding prices  $\{p_t\}_{t \in [0, T)}$ .*

PROOF. To show it by contradiction, suppose that there exists a time  $\tilde{t} \in [0, T)$  which strictly dominates purchasing at  $t$  for the case in which there is only the first version of the good and the price path is  $\{p_t\}_{t \in [0, T)}$ .

If this buyer does not purchase the second version, we would directly get the contradiction. Hence he must be purchasing the second version as well. Then by revealed preferences:

$$e^{-r\tilde{t}}(X - p_{\tilde{t}}) > e^{-rt}(X - p_t)$$

$$X(e^{-rt} + (\beta - 1)e^{-rT}) - e^{-rt}p_t - e^{-rT}P_T \geq X(e^{-r\tilde{t}} + (\beta - 1)e^{-rT}) - e^{-r\tilde{t}}p_{\tilde{t}} - e^{-rT}P_T$$

The first line compares the total utilities from purchasing only the first version at times  $\tilde{t}$ , and  $t$  respectively. On the other hand, the second line compares the total utilities from purchasing both goods with distinction on the first version purchasing times  $t$  and  $\tilde{t}$  respectively.

The second line boils down to:

$$e^{-rt}(X - p_t) \geq e^{-r\tilde{t}}(X - p_{\tilde{t}})$$

which is a direct contradiction to the first line. □

The next lemma shows that, until a certain period of time, the purchases of the first version of the good must be monotonic with the buyer's type. Precisely, if L-type buyers purchase the first version of the good until this specific time then H-type buyers would purchase earlier. For the purchases occurring later than this specific time period we do not

have this monotonicity. This is due to the fact that the arrival of the second version of the good becomes closer as time goes on, and the incentive to wait for the newer version of the good becomes strengthened, and these strengthened incentives is stronger for H-type buyers.

LEMMA 6. *For a given price path*

- i If the L-type buyers purchase both versions of the good then the H-type buyers would also purchase both versions of the good.*
- ii Let  $t^*$  be defined by  $e^{-rt^*} = \beta e^{-rT}$ .<sup>12</sup> If the L-type buyers purchase the first version of the good at time  $t < t^*$  then the H-type buyers also purchase the first version of the good and the purchase time is not later than time  $t$ .*

PROOF. To start with the first part suppose that the L-type buyers purchase both versions of the good and so  $p_T < (\beta - 1)L$  due to the sequential rationality. Then the H-type buyers purchase the second version as well, since  $p_T < (\beta - 1)H$ . Moreover, since purchasing the first version conditional on purchasing the second version has a positive return for the L-type buyers, it must have a positive return for the H-type buyers as well hence H-type buyers also purchase the first version. This proves our claim.

For the second part, suppose the L-type buyers purchase the first version at  $t < t^*$ . Then, to prove the statement, we just need to show that the H-type buyers purchase the first version of the good due to the lemma 5. Suppose not to get a contradiction. Then it must be the case that the L-type buyers are only purchasing the first version of the good while H-type buyers are only purchasing the second version, because otherwise if the L-type buyers were purchasing the second version, then, from the first part of the lemma, the H-type buyers would purchase both versions of the good. Also, if a H-type buyers is not purchasing the second version, then it means that he is not purchasing any versions of the good which would also be a contradiction. Then by the revealed preferences:

$$\begin{aligned} e^{-rt}(L - p_t) &\geq e^{-rT}(\beta L - p_T) \\ e^{-rT}(\beta L - p_T) &\geq e^{-rt}(H - p_t) \end{aligned}$$

Which is a direct contradiction since  $H > L$  and  $e^{-rt} > e^{-rt^*} = \beta e^{-rT}$ . □

The following lemma shows that, if assumption 2 is satisfied then in an optimal price path, the H-type buyers immediately purchase the first version of the good.

LEMMA 7. *Suppose assumption 2 is satisfied. Then in an optimal posted price mechanism, H-type buyers purchase the first version of the good immediately at  $t = 0$ .*

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<sup>12</sup>The existence of  $t^*$  is guaranteed by assumption 2.

PROOF. Showing that under the optimal price path there must be an agent of some type that is purchasing at  $t = 0$  would be sufficient to prove this lemma due to the second part of Lemma 6.

Assume that nobody purchases at  $t=0$  to get a contradiction. There must be a sale of the first version at some time before  $T$ , because otherwise, if there is a sale of only the second version good, we would get a contradiction immediately, as the monopolist could deviate and sell only the first version of the good at  $t = 0$  to the agents that are purchasing the second version. This is better for the firm as it can get a higher discounted payment due to assumption 2.

Denote the earliest time period at which a sale of the first version occurs by  $\underline{t}$ . We want to show that it is equal to 0. Suppose  $\underline{t} > 0$  to get a contradiction. Then there must be a sale of the second version of the good, because otherwise there exists an obvious profitable deviation, which is selling at  $t = 0$  with the price level  $p_{\underline{t}}$ . If the agent purchasing the first version at  $\underline{t}$  also purchases the second version, then it must be a H-type from lemma 6. Then the monopolist can be made better off by changing the price level at  $t = 0$  so that the H-type is indifferent between purchasing at 0 and  $\underline{t}$  as that does not alter the incentives of the L-type buyers. We get a similar contradiction for the other case in which the agent purchasing the first version at  $\underline{t}$  is not purchasing the second version.  $\square$

The characterization of the optimal posted prices depends on the values of  $H$  and  $L$ . In particular, the relation between  $(\beta - 1)H$  and  $\beta L$  is crucial. When  $(\beta - 1)H > \beta L$  the H-type agent owning the first version of the good acquires higher additional utility from the second version purchase compared to a L-type buyer that does not own the first version of the good and vice versa for the other case. From now on we will consider the case  $(\beta - 1)H > \beta L$ . The analysis of the other case follows from similar arguments and are not given here.

Theorem 2 characterizes the optimal posted prices. As usual there is not a unique price path that implements the optimal allocation. However, we say that the prices are constant for the first version of the good as long as all of the purchases occurs at time  $t = 0$ . Because in this case time is not used to discriminate over buyers for the sale of the first version of the good. On the other hand, if the optimal allocations requires buyers to purchase the first version of the good at different times than the monopolist is using time to discriminate over buyers. The monopolist is using a decreasing price path for the first version of the good to implement the optimal allocation.

**THEOREM 2.** *If  $(\beta - 1)H > \beta L$  and assumption 2 is satisfied, then the optimal posted prices and the corresponding purchases must be one of the following.*

- 1)  $p_t = H \forall t \in [0, T)$ , and  $p_T > (\beta - 1)H$ . Only H-type buyers purchase the first version of the good, and no one purchases the second version.
- 2)  $p_t = L \forall t \in [0, T)$ , and  $p_T = (\beta - 1)H$ . Both type of buyers purchase the first version of the good at  $t = 0$ , and only H-type buyers purchase the second version.
- 3)  $p_t = \begin{cases} (1 - e^{-rT})H & \forall t \in [0, \bar{t}) \\ L & \forall t \in [\bar{t}, T) \end{cases}$  and  $p_T = (\beta - 1)H$  where  $\bar{t}$  satisfies  $e^{-r\bar{t}} = e^{-rT} \frac{H}{H-L}$ . Both H-type and L-type buyers purchase the first version of the good at times  $t = 0$  and  $t = \bar{t}$  respectively. And only H-type buyers purchase the second version.
- 4)  $p_t = (1 - e^{-rT})H \forall t \in [0, T)$ , and  $p_T = \beta L$ . Only H-type buyers purchase the first version of the good, and both types purchase the second version.
- 5)  $p_t = (1 - e^{-rT})L \forall t \in [0, T)$ , and  $p_T = (\beta - 1)L$ . Both types purchase the first version at  $t = 0$  and they also purchase the second version of the good.

It is easy to calculate the corresponding profit of each policy for the monopolist. Then we can see that each of these policies is the optimal one for some values of the parameters of the model.<sup>13</sup> In other words, for each policy there is a subset of parameters, which are also satisfying the condition  $(\beta - 1)H > \beta L$  and assumption 2, such that the policy is optimal. We are particularly interested on the third policy, because it displays a decreasing price path for the first version of the good. In particular, the purchases of the first version of the good occur throughout time and hence the corresponding price path implementing this allocation must be decreasing. Therefore, as opposed to the stochastic arrival, when the arrival of the new version occurs at a deterministic time, the optimal price path for the first version of the good might display a decreasing pattern.

In the case of a deterministic arrival, the buyers' incentives are not stationary since the value of the first version of the good decreases over time until the arrival of the new version of the good. This situation strengthens the ability of the monopolist to sort out the buyers with lower valuations. More precisely, the existence of the second version of the good limits the price that the H-type buyers are willing to pay at  $t = 0$  for the first version of the good,<sup>14</sup> unless the price of the second version of the good is higher than H so that no one would like to purchase. At this maximum price the net return to the H-type buyers from the first version purchase is positive. Therefore the monopolist can sell the first version of the good to the L-type buyers at a later time with a price that is equal to the maximum price that L-type buyers would like to pay. This situation does not hurt the incentives of the H-type

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<sup>13</sup> The parameters are  $\mu, r, T, H, L$ .

<sup>14</sup>Because, now there is an additional option for buyers which is just to purchase the second version.

buyers on their first version purchases at  $t = 0$ . The critical time  $\bar{t}$  in the third policy of Theorem 2 is defined such that the H-type buyers are indifferent between purchasing the first version of the good at  $\bar{t}$ , by paying the maximum amount that the L-type buyers are willing to pay at that time, and purchasing at  $t = 0$  by paying their own maximum willingness.

The proof of Theorem 2, which is given in the appendix, follows a backward analysis. In particular, we first define the optimal sales policy for the first version of the good for a given  $p_T$ . Then we chose the optimal  $p_T$  by considering the corresponding optimal price path for first version of the good for each  $p_T$ .

## 5 CONCLUSION

We analyzed the optimal pricing problem of a durable good monopolist with full commitment when there is product replacement. In our model there are two consecutive versions of the same good, one of which arrives and replaces the existing one in the market. The main assumption is that the sales are anonymous, so that the seller cannot condition the sale of the second version of the good on the sales of the first version. The analysis is divided into two parts. In the first part we consider the case in which the arrival follows a Poisson process. In this case the environment is stationary, and the optimal price path is constant for both versions of the good. In other words, all the purchases occurs immediately and, hence, the monopolist does not use time to discriminate over buyers with different valuations. This result is in the same line with the classical result of [Stokey \(1979\)](#), and the main reason for this is the stationary environment.

In the second part we considered the case in which the arrival is commonly known time period. Then we have shown that, when there are only two types of buyers, depending on the parameters of the model, the monopolist may want to decrease the price throughout time. Purchases do not necessarily occur immediately and it may display a delay. In this case the monopolist may use time as a tool to discriminate over buyers for some sets of the parameter values.

For both cases the optimal selling mechanism without the restriction of anonymity involves bundling of both versions and selling them together. The corresponding allocations in this case cannot be implemented by posted prices which is anonymous by definition.

## APPENDIX: PROOFS

**PROOF OF LEMMA 1.** The “only if part” of the statement follows directly from revealed preferences. In other words, if the mechanism is incentive compatible, then the buyer of

type  $x$  does not want to mimic type  $x'$  and vice versa. More precisely take  $x > x'$ , then:

$$\begin{aligned} U(x) &\geq e^{-rt_{x'}} \left( \frac{x}{r} - p_{t_{x'}} \right) = U(x') + e^{-rt_{x'}} \left( \frac{x}{r} - \frac{x'}{r} \right) \\ U(x') &\geq e^{-rt_x} \left( \frac{x'}{r} - p_{t_x} \right) = U(x) - e^{-rt_x} \left( \frac{x}{r} - \frac{x'}{r} \right) \end{aligned}$$

Then we get,

$$\frac{e^{-rt_x}}{r} \geq \frac{U(x) - U(x')}{x - x'} \geq \frac{e^{-rt_{x'}}}{r} \quad (6)$$

which requires  $t_x \leq t_{x'}$ . Therefore we get *i*). Now, since  $t_x$  is monotone it is differentiable and continuous almost everywhere. Therefore  $e^{-rt_x}$  is differentiable and continuous a.e. and hence,  $\lim_{x' \rightarrow x} \frac{e^{-rt_{x'}}}{r} = \frac{e^{-rt_x}}{r}$  a.e. We also know that  $U(x)$  is continuous and differentiable a.e. so by taking the limit of the expression (6), when  $x' \rightarrow x$ , we get

$$\frac{\partial U(x)}{\partial x} = \frac{1}{r} e^{-rt_x} \text{ a.e.}$$

Hence, *ii*) follows immediately.

For the “if” part, suppose for a given mechanism conditions *i*) and *ii*) are satisfied, and we want to show that this mechanism is incentive compatible. Take any two arbitrary types  $x$ , and  $x'$  and WLOG assume  $x > x'$ . First, we want to show that  $x$  does not want to report his type as  $x'$ . In other words the following must be true

$$U(x) \geq e^{-rt_{x'}} \left( \frac{x}{r} - p_{t_{x'}} \right) = U(x') + e^{-rt_{x'}} \left( \frac{x}{r} - \frac{x'}{r} \right). \quad (7)$$

However by *ii*) we know that

$$U(x) - U(x') = \frac{1}{r} \int_{x'}^x e^{-rt_{\tilde{x}}} d\tilde{x}.$$

Hence, expression (7) boils down to:

$$\int_{x'}^x e^{-rt_{\tilde{x}}} d\tilde{x} \geq e^{-rt_{x'}} (x - x').$$

But this is correct given monotonicity in *i*). Similar arguments follow for the reports of  $x'$  as well. Hence the statement is true. □

**PROOF OF THEOREM 2.** To prove the theorem, we first treat the price of the second version of the good as a fixed value. We then find the corresponding optimal price path  $\{p_t\}_{t \in [0, T]}$

of the first version of the good for any given value of  $p_T$  and we finally optimize  $p_T$  at the end. There are 5 cases to consider for  $p_T$ .

Case 1:  $[\beta H < p_T]$

In this case there is no sale of the second version of the good. We know that in an optimal policy H-type buyers purchase the first version at  $t=0$ , and the maximum amount that they are willing to pay at  $t=0$  is  $H$ . On the other hand, at any time  $t > T$ , the L-type buyers are willing to pay at most  $L$  (given that there is no sale of the second version). Therefore, if the monopolist is going to sell the first version of the good to L-type buyers at a time  $t$ , then he should arrange the price as  $p_t = L$ . However this will affect the incentives of the H-type buyers that are purchasing at  $t = 0$ . Given that  $p_t = L$  for some  $t$ , the maximum amount that the H-type buyers are willing to pay at  $t = 0$ , which we denote by  $\bar{p}$ , satisfies

$$\begin{aligned} H - \bar{p} &= e^{-rt}(H - L) \\ \bar{p} &= (1 - e^{-rt})H + e^{-rt}L \end{aligned}$$

And the corresponding profit of the monopolist, when L-type buyers purchase at time  $t$ , is

$$\Pi_t = \mu((1 - e^{-rt})H + e^{-rt}L) + (1 - \mu)e^{-rt}L. \quad (8)$$

Note that the expression above is linear in  $e^{-rt}$ , hence it is maximized either at  $t = 0$  or at  $t = \infty$ <sup>15</sup>. If  $t = 0$  is optimal, then both types purchase the good at  $t = 0$  and the price level is equal to  $L$ . On the other hand, if  $t = \infty$  is optimal, then only the H-type buyers purchase the good (at  $t = 0$ ) at price  $H$ .

Therefore, for the first case there are two candidates of the optimal policy.

- A1: Sell the first version of the good to agents of both types at  $t=0$  at a price level  $L$  and have no sales of the second version.
- A2: Sell the first version only to the H-types at  $t=0$  at a price level  $H$ , and have no sales of the second version.

Actually, this is analogous to the result of [Stokey \(1979\)](#) and intuitively follows because if there is no sales of the second version, we turn back to canonical model.

Case 2:  $[(\beta - 1)H < p_T \leq \beta H]$

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<sup>15</sup>It is also possible have that any  $t > 0$  is a maximizer of the expression 8. In such a case restricting  $t$  to be either 0 or  $\infty$  is wlog.



At the optimal policy, the marginal benefit from the second version of the good is  $(\beta - 1)H$  for H-type buyers since they purchase the first version at  $t = 0$ . Therefore, there is no sale of the second version in this case as well. However, the situation is different than the previous case in the sense that now there is an additional option for H-type buyers. In other words, by purchasing only the second version of the good, they can guarantee a non-negative utility. As a result, the maximum amount that H-type buyers are willing to pay at  $t=0$  is less than  $H$ . Denote this maximum price level by  $\bar{p}$ , which satisfies:

$$\begin{aligned} H - \bar{p} &= e^{-rT} \beta H - e^{-rT} p_T. \\ \bar{p} &= (1 - \beta e^{-rT})H + e^{-rT} p_T \end{aligned}$$

Where the LHS of the first line is the utility from the purchase of the first version of the good, and the RHS is from the purchase of the second version. Similar to the previous case, a candidate optimal policy is selling the first version good at  $t=0$ , to both types of agents at a price  $L$ <sup>16</sup>. However, the corresponding policy would be equivalent to A1, so that we do not write it again here.

We can think of another candidate, which is a modified version of the policy A2, that is selling version 1 at  $t=0$  only to the H-type buyers but now with a payment  $\bar{p}$ , rather than  $H$ . However, this policy is strictly dominated since the payment is less than the one of A2.

Finally, in policy A2, there is no sale of the first version to the L-types, which is due to the fact that the monopolist needs to decrease the price level at  $t = 0$  (which was equal to  $H$ ) to be able to sell to the L-type buyers at any time. However, in this case, by departing from the case 1, there may exist a time period earlier than  $T$ , at which selling the good to the L-type buyers at the maximum price that they are willing to pay (which is equal to  $L$ ) does not hurt the incentives of the H-types. Hence it does not require to decrease the price at  $t=0$ <sup>17</sup>, because, now H-type buyers are having a positive utility from the first version purchase at price  $\bar{p}$ . Nevertheless, even if such a period of time exists, doing any better than both of the policies A1 and A2 is not possible due to the fact that the resulting policy would be equivalent to one of the intermediate policies in the expression 8. Therefore, the firm cannot do any better here in this case.

Case 3:  $[\beta L < p_T \leq (\beta - 1)H]$

In this case, H-type buyers purchase the second version of the good, since  $p_T$  is always smaller

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<sup>16</sup>Note that  $L$  is less than  $\bar{p}$ , since  $p_T \leq \beta H$ ,  $\beta e^{-rT} < 1$ , and  $(\beta - 1)H > \beta L$ . Therefore H-type buyers are willing to buy at this price level.

<sup>17</sup>We can find such a time period by finding a  $t$  so that the H-type buyers are indifferent between purchasing at  $t$  at price  $L$  and purchasing at time 0 with price  $\bar{p}$ . Time period  $t$  satisfying this indifference condition should be less than  $T$ .

then their marginal benefit and L-type buyers do not purchase. The maximum amount that H-type buyers are willing to pay at  $t=0$  for the first version of the good,  $\bar{p}$ , in this case satisfies the following:

$$\begin{aligned} H - \bar{p} + e^{-rT}((\beta - 1)H - p_T) &= e^{-rT}(\beta H - p_T) \\ \bar{p} &= (1 - e^{-rT})H \end{aligned}$$

where the LHS of the first line is the utility of the H-type buyers from purchasing both versions of the good, and the RHS is the utility from purchasing only the second version of the good. Note that  $\bar{p}$  is higher than  $L$ .<sup>18</sup> Hence the L-type buyers are not willing to purchase at  $t=0$  with price  $\bar{p}$ . Then the monopolist should either set a smaller price than  $\bar{p}$  at  $t = 0$  to be able to sell the L-type buyers at  $t = 0$ , or he can sell it at a later time to them. Note that, at any time  $t$ , the highest amount that L-type buyers are willing to pay for the first version of the good is  $L$  as they do not purchase the second version of the good in this case. Like in the previous cases, charging a price level that is equal to  $L$  will affect the incentives of the H-type buyers.

One possible optimal policy here is to sell the first version of the good at  $t=0$  to both types of buyers at a price  $L$ , and set  $p_T = (\beta - 1)H$ . Another possibility is to sell the first version of the good to the H-type buyers at a price  $\bar{p}$  at  $t = 0$ , and to the L-type buyers at a later time (before  $T$ ) without hurting the incentives of the H-type buyers. As we discussed in case 2, finding such a time period is possible here, because H-type buyers are having a positive utility from purchase of the first version of the good. To find this time period, let's denote  $p_t^H$  as the price level, which leaves the H-type buyers indifferent between purchasing good at  $t=0$  with payment  $\bar{p}$ , and purchasing at  $t$  with payment  $p_t^H$ . In particular

$$\begin{aligned} H - \bar{p} &= e^{-rt}(H - p_t^H) \\ p_t^H &= \frac{(e^{-rt} - e^{-rT})H}{e^{-rt}} \end{aligned}$$

Then we can find the earliest possible time period  $\bar{t}$  at which the firm can sell the first version of the good to the L-type buyers at price  $L$  without hurting the incentives of the H-type by using the equality  $p_{\bar{t}}^H = L$ . In particular,

$$\begin{aligned} (e^{-r\bar{t}} - e^{-rT})H &= e^{-r\bar{t}}L \\ e^{-r\bar{t}} &= e^{-rT} \frac{H}{H - L} \end{aligned}$$

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<sup>18</sup>This is because  $\beta e^{-rT} < 1$ , and  $(\beta - 1)H > \beta L$ .

Note that  $\bar{t} < T$  is always satisfied due to assumption 2 and  $\beta L < (\beta - 1)H$ . Hence selling the first version of the good to the L-type buyers at  $\bar{t}$  at a price  $L$  is a feasible policy. To sum up, we have the following two candidates for this case.

- A3: Sell the first version to both type of buyers at  $t=0$  at price  $L$ , and sell the second version to only to the H-type buyers at a price  $(\beta - 1)H$ .<sup>19</sup>
- A4: Sell the first version of the good to the H-type buyers at  $t=0$  with price  $\bar{p} = (1 - e^{-rT})H$ , and also sell to L-type buyers at  $t = \bar{t}$  with payment  $p_{\bar{t}}^H = L$ . Sell the second version to H-type buyers at price  $(\beta - 1)H$ .

Note that, in this case there can not be any better policy then these two due to the linearity of the profit function as we have discussed in case 1. For instance, take the policy A4, if it is better to decrease the price at  $t=0$  to sell to the L-type buyers earlier than  $\bar{t}$ , then the monopolist should continue to decrease price level at 0 until it reaches  $L$  at which L-type is willing to buy; and this corresponds to the policy A3.

Case 4 :  $[(\beta - 1)L < p_T \leq \beta L]$

In this case, the H-type buyers always purchase the second version of the good while L-type buyers purchase the second version only if they have not purchased the first one. We can easily see that the maximum amount that the H-type buyers are willing to pay at  $t = 0$  for the first version of the good is same as in case 3 and so it is equal to  $\bar{p} = (1 - e^{-rT})H$ .

There are three candidates for the optimal policy in this case. The first one is to sell the first version of the good to both types of buyers at  $t=0$  at a price level that leaves the L-type buyers indifferent between purchasing only the first version and purchasing only the second version, and to sell the second version only to the H-type buyers. However, this policy is strictly dominated by A3. In particular, the maximum amount of the payment that L-type buyers are willing to pay at  $t = 0$  is less than  $L$ , and the amount charged for the second version of the good at time  $T$  is strictly less than the one of A3. The second policy, is to sell the first version of the good to the H-type buyers at  $t=0$  with price  $\bar{p} = (1 - e^{-rT})H$ , and to the L-type buyers at a later time, and to sell the second version only to the H-type buyers. This is dominated by the policy A4 for the same reason above. Then the final candidate is:

- A5: Sell the first version of the good only to the H-type buyers at  $t=0$  with price  $\bar{p} = (1 - e^{-rT})H$ , and sell the second version of the good to both type of buyers at price  $\beta L$ .

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<sup>19</sup>Note that this policy is strictly dominating the policy A1.

CASE 5:  $[(p_T \leq (\beta - 1)L)]$

In this case, both types of the buyers purchase the second version of the good regardless of their decision on the first version sales. From the same reasoning as above the maximum amount that the H type buyers are willing to pay at  $t=0$  is  $\bar{p} = (1 - e^{-rT})H$ , and he is indifferent between purchasing the first version at  $t = 0$  with payment  $\bar{p}$  and purchasing at  $t$  with payment  $p_t^H = \frac{(e^{-rt} - e^{-rT})H}{e^{-rt}}$ . Similarly, the maximum amount that the L-type buyers can are willing to pay for the first version of good at time  $t$  is  $p_t^L = \frac{(e^{-rt} - e^{-rT})L}{e^{-rt}}$ .

Since  $p_t^L < p_t^H$ , there does not exist a time period in which the monopolist can sell the first version of the good to the L-type buyers at price  $p_t^L$  without hurting the incentives of the H-type buyers when they are purchasing at  $t = 0$  with price  $\bar{p}$ . Therefore the monopolist must decrease the initial price to be able to sell the L-type agents at any time. Then, again due to the linearity of the monopolist profit, there are two possibilities for the optimal policy in this case. However one of them, which is selling the first version of the good only to the H-type buyers with price  $\bar{p}$  and selling the second version of the good to both types with a price  $(\beta - 1)L$  is strictly dominated by the policy A5. Therefore the only option that we are left with is:

- A6: Sell the first version of the good to both type of buyers at  $t=0$  with a payment  $(1 - e^{-rT})L$ , and sell the second version of the good to both types of buyers at a price  $(\beta - 1)L$ .

We have considered all of the possible optimal policies for the monopolist. First note that the policy A1 is strictly dominated by the policy A3, (hence we omit A1). Then the corresponding profit level for each are listed below as follows:

TABLE 1: Sale Policies and Corresponding Profits.

Policy	Profit
A2	$\mu H$
A3	$L + \mu e^{-rT}(\beta - 1)H$
A4	$\mu[1 + e^{-rT}(\beta - 2)]H + (1 - \mu)e^{-rT} \frac{LH}{H-L}$
A5	$\mu(1 - e^{-rT})H + e^{-rT}\beta L$
A6	$[1 + e^{-rT}(\beta - 2)]L$

Each of these 5 policies may be the optimal one depending on the values of  $\beta$ ,  $\mu$ ,  $H$ ,  $L$  and  $r$ . All the policies except A4 involves immediate allocations like in the stochastic arrival case. Hence time is not used to discriminate over people in those policies. On the contrary, in policy A4 the price of the first version of the good is decreasing over time. As a result,

purchase times of agents for the first version of the good are different for L and H-types of buyers. More precisely, the H-type buyers purchase at the beginning, whereas the L-type buyers purchase at a later time (before T). The reason for such a pattern is based on the anonymous structure of the posted prices for the second version sales. The existence of the second version puts a restriction on the amount that a H-type buyer is willing to pay for the first version of the good since it is possible for him to give up from the purchase of the first version of the good and purchase only the second one. As a result there exists a time period so that the monopolist can sell the first version of the good to the L-type buyers without hurting the incentives of the H-type buyers. This is not possible in the canonical durable good monopoly model.  $\square$

## REFERENCES

- Benhabib, J. and A. Rustichini (1991). Vintage capital, investment, and growth. *Journal of Economic Theory* 55(2), 323–339.
- Bergemann, D. and M. Said (2011). Dynamic auctions. *Wiley Encyclopedia of Operations Research and Management Science*.
- Board, S. (2008). Durable-goods monopoly with varying demand. *The Review of Economic Studies* 75(2), 391–413.
- Board, S. and A. Skrzypacz (2010). Revenue management with forward-looking buyers. *Unpublished manuscript, Stanford University*.
- Bulow, J. (1986). An economic theory of planned obsolescence. *The Quarterly Journal of Economics* 101(4), 729–749.
- Fishman, A. and R. Rob (2000). Product innovation by a durable-good monopoly. *The RAND journal of Economics*, 237–252.
- Garrett, D. (2011). Durable goods sales with dynamic arrivals and changing values. *Unpublished manuscript, Northwestern University*.
- Pai, M. and R. V. Vohra (2008). Optimal auctions with financially constrained bidders. Technical report, Discussion paper//Center for Mathematical Studies in Economics and Management Science.
- Stokey, N. L. (1979). Intertemporal price discrimination. *The Quarterly Journal of Economics*, 355–371.
- Swan, P. L. (1970). Market structure and technological progress: The influence of monopoly on product innovation. *The Quarterly Journal of Economics* 84(4), 627–638.