Penn Institute for Economic Research
Department of Economics
University of Pennsylvania
3718 Locust Walk
Philadelphia, PA 19104-6297
pier@econ.upenn.edu
http://economics.sas.upenn.edu/pier

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"Dynamic Education Signaling with Dropout Risk"
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by

Francesc Dilme and Fei Li

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Dynamic Education Signaling with Dropout Risk*

Francesc Dilme　　Fei Li

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Abstract

This paper analyzes a dynamic education signaling model with dropout risk. Workers pay an education cost per unit of time and face an exogenous dropout risk before graduation. Since low-productivity workers' cost of education is high, pooling with early dropouts helps them avoid a high education cost. In equilibrium, low-productivity workers choose to endogenously drop out over time, so the productivity of workers in college increases as the education process progresses. We find that the exogenous dropout risk generates rich dynamics in the endogenous dropout behavior of workers, and the maximum education length is decreasing in the prior about a worker being highly productive. We also extend the baseline model by allowing human capital accumulation.

Keywords: Dynamic Games, Dynamic Signaling, Dropout

JEL Classification Codes: D83, J31

*Francesc Dilme: Department of Economics, University of Bonn, Lennestrasse 35, 53113 Bonn, Germany; Email: fdilme@uni-bonn.de. Fei Li: Department of Economics, 107 Gardner Hall, CB 3305, University of North Carolina, Chapel Hill, NC 27599-3305; Email: lifei@email.unc.edu. We are grateful to Qingmin Liu, George Mailath, and Andrew Postlewaite for insightful instruction and encouragement. We thank the associate editor, Dirk Bergemann, the referees, Douglas Bernheim, David Besanko, Jimmy Chan, Eduardo Faingold, Hanming Fang, Hao Li, Steven Matthews, Guido Menzio, Tymofiy Mylovanov, Mallesh Pai, Andrzej Skrzypacz, Jeroen Swinkels, Jidong Zhou and participants at the UPenn theory lunch, Midwest Theory Conference, The 4th World Congress of the Game Theory Society, The 23rd Stony Brook Game Theory Festival of the Game Theory Society, and the 2012 North America Summer Meeting of the Econometric Society for insightful comments. Any remaining errors are ours.

Contents

1 Introduction

In many markets, such as labor and financial markets, privately informed sellers face a large number of uninformed buyers. Starting with Spence's (1973) seminal work on education, a growing literature explores how each seller signals his type in the market. Such signaling includes wasteful education (Spence, 1973) and excess and inefficient shareholding of a corporation (Leland and Pyle, 1977).[1] In Spence (1973)'s education model, the worker makes a one-shot decision about his graduation year and is able to commit to his initial choice, so he leaves school only after graduation. However, in many markets, including the labor market, the seller makes the signaling decision sequentially. Furthermore, unpredictable exogenous shocks may appear and prevent the seller from fulfilling his intended signaling choice. For example, in the labor market, while some students voluntarily choose to drop out, others drop out because of exogenous reasons such as financial constraints.[2] In other markets such as financial asset markets and housing markets, the owner of the asset may face liquidity shocks which are driven by hedging concerns or changes to his budget constraints.

While the dynamic signaling models have drawn attention to economists since the seminal works by Weiss (1983) and Admati and Perry (1987), the role of exogenous shocks has not been systematically studied in the literature. In this paper, we examine a dynamic signaling model in which the seller may face an exogenous shock in each period. Once the shock arrives, the seller has to trade immediately. Our goal is to show that the presence of such exogenous shocks can generate interesting endogenous trading dynamics. Following Spence (1973), we illustrate our idea in an education model where (1) a worker privately knows his productivity, (2) decides whether to drop out in each period, and (3) faces an exogenous dropout risk. Once the shock arrives, the worker has to go on the job market immediately. We interpret this exogenous dropout process as random shocks faced by the worker, driven by exogenous problems such as financial constraints, family needs and the arrival of utility shocks.

This paper has three main results. First, we show that the presence of the constant exogenous dropout risk can generate time-varying endogenous dropout decisions: A worker's dropout behavior varies over the number of years he has spent in school. Hence, we provide a natural explanation of the dropout rate-grade profile based on information asymmetry. Unfortunately, even though we can derive the equilibrium relation between the worker's dropout rate and their years of education

[1] See Riley (2001) for a survey.

[2] Bound and Turner (2011) report that only about half of students who begin first-level degree programs actually obtain their degrees. According to a survey by the Bill & Melinda Gates Foundation, 52% students drop out of college because they "just couldn't afford the tuition and fees", and 71% mentioned that they "needed to go to work and make money."

in the discrete time model, its dynamics are hard to characterize analytically. Hence, we examine the continuous time limit of our discrete time model. At the continuous time limit, we explicitly show that the dropout rate of a worker decreases the longer he spends in school.

Secondly, we find that the maximum equilibrium education duration is decreasing in the prior about the worker being highly productive. In particular, when the prior approaches one, no wasteful education appears in any equilibrium of the game. This is a consequence of the fact that since early dropout happens in equilibrium, the beliefs of the firms about the productivity of a dropout are disciplined. Then, in equilibrium, the low-productivity worker endogenously drops out with a probability high enough that it balances the incentive to drop out or to stay in education. If there is a low ex-ante probability that the worker is a low-productivity worker, these incentives can only be balanced during a few periods of education.

Last but not least, we extend our pure signaling model by allowing human capital accumulation. In this extension, we consider a model in which education not only separates the high-type worker from low-type worker but also enhances both types' productivity. As a result, both the signaling effect and the human capital accumulation effect contribute to the returns to education. We characterize the equilibrium passing D1 criteria, and we illustrate that the observation of dropout rate is helpful to quantitatively decompose the returns to education into the signaling effect and the human capital accumulation effect.

Even though we present our model in an education signaling environment, our insights are also useful for understanding some other environments in which sending signals is not only costly but also time-consuming. For example, consider a firm owner who is trying to sell his firm. In order to signal the type of the firm, the owner may wait for some time. The opportunity cost of waiting is likely to be low if the quality of the firm is good. The risk of dropping out may be reinterpreted as liquidity shocks or hedging considerations that force the owner to sell the firm early. The observed dropout rate can be interpreted as transaction volume.[3] Another example is given by central banks defending themselves from currency attacks. In this case, the cost of defending may depend on the fundamentals of the economy, known only by the central bank. As time passes, the posterior belief about the economy being healthy increases, so the size of the attacks decreases and the attacks eventually vanish. The exogenous shocks may result from random events in international markets, such as a devaluation of the foreign currency used to defend against attacks.

[3] See Daley and Green (2013a) for an information-based theory to explain time variation in liquidity in the financial market.

Related Literature

This paper is related to a growing literature studying dynamic signaling games with preemptive offers. To the best of our knowledge, the literature springs from Weiss (1983) and Admati and Perry (1987). They argue that the static signaling model (Cho and Kreps, 1987) overlooks the worker dropout behavior. Think of a two-type signaling model. If a separating equilibrium is supposed to be played as predicted by Cho and Kreps (1987), once a worker arrives on the first day of school, the separation has already happened, and firms believe that the worker has high productivity. Hence, the worker should drop out immediately. Cho and Kreps (1987) avoid this challenge by directly assuming that a worker can commit to his decision regarding the duration of his education. In practice, it is hard to see where the commitment power comes from.

Nöldeke and van Damme (1990) formulate an explicitly dynamic game-theoretic version of the signaling model. In their model, long-lived firms simultaneously make *public preemptive offers* to the worker in each period, and the worker decides to accept an offer or to continue his education. As off-equilibrium beliefs may provide incentives to the worker to continue signaling, there are many equilibria with wasteful education, even when the commitment time of the workers is arbitrarily small. Nonetheless, Swinkels (1999) argues that Nöldeke and Van Damme (1990)'s result crucially depends on the fact that job offers are made publicly. Hence, he considers a model where two short-lived firms enter and simultaneously make *private preemptive offers* to the worker in each period before the worker decides on whether to continue his education. Swinkels (1999) finds that, when the interval between consecutive offers goes to zero, the unique sequential equilibrium in this game is a pooling one at no education. Our model is different from Nöldeke and van Damme (1990) and Swinkels (1999) in two respects. First, neither Nöldeke and van Damme (1990) nor Swinkels (1999) study the interaction between dropout rate dynamics and the signaling effect. In the former, some workers do not go to school at all and the rest stay in school until graduation. No worker drops out in between. In the latter, no one goes to school in the first place. Instead, in our model, the return to education and the dynamics of the dropout rate are two main implications. Second, instead of allowing firms to make preemptive offers, we assume that the informed party (the worker) moves first, either going to the job market or not. Consequently, conditional on being in the job market, the uninformed agents (firms) make him offers. Since dropouts cannot return to school, there is no need to distinguish between private offers and public offers in our model.

There is a growing literature that studies dynamic signaling models with extra *type-dependent* signals.[4] Kremer and Skrzypacz (2007) consider a finite horizon model in which an informative (type-dependent) signal about the type of the worker is publicly announced at the deadline. Da-

[4]One example of such signals is students' GPA.

ley and Green (2012, 2013a) and Dilme (2013) study infinite horizon dynamic signaling models in which extra signals continuously arrive. In these models, the dynamics of the dropout rate and wages (trade probability and price in their language) are also characterized. Their characterizations depend on the presence of an extra signal. Instead, we characterize the dropout rate dynamics and wages in a model where the type-dependent extra signal is absent, but the informed party faces a *type-independent* dropout risk. Our model allows one to distinguish exogenous and endogenous dropout behavior and address new policy issues. As we show in Section 2.5.1, as the exogenous dropout risk shrinks, the low type is more likely to choose to drop out in the beginning of the game, which results in new policy implications in many markets.[5] Last, we study the role of the observed dropout rate in a model with productive education: dropout data are helpful in distinguishing the human capital accumulation effect and the signaling effect on the return to education.

Our model is also related to the dynamic adverse selection literature, including Janssen and Roy (2002), Camargo and Lester (2011), Hörner and Vieille (2009) and Kim (2011), in terms of the results. In these papers, the average market quality increases over time as in our model. However, their mechanisms are different from ours. In these models, high-type sellers value their asset more than low-type sellers and, in equilibrium, they gain less than low-type sellers from trade. Consequently, they are more willing to wait, and therefore initial offers are used by buyers to screen low-types out. This contrasts with our model where high-types are willing to wait due to their low waiting costs, while low-types strategically drop out in order to avoid high education costs. Our mechanism crucially depends on the exogenous dropout behavior of the high-types and the endogenous response of the low-types, and allows us to study the dropout dynamics in education.

The rest of this paper is organized as follows. In the next section we present the model with a type-independent dropout rate and characterize the set of equilibria. We consider a model with human capital accumulation in Section 3. In Section 4, we conclude. All omitted proofs are in Appendix A. In Appendix B, we study an extension in which the worker's dropout risk is type-dependent. In Appendix C, we consider a multiple-type extension of the baseline model.

[5]In the labor market, financial constraints are among the main reasons that students drop out. Hence, it is reasonable to believe that the availability of student loans should affect the exogenous dropout rate, and therefore the endogenous dropout choices of students who are not financially constrained. So, it is worthwhile to take such effects into account in the related policy discussion. In financial markets, our insight implies that the government bailout of financial institutions facing a liquidity shock has an additional cleansing effect: it reduces the exogenous liquidity risks faced by all institutions. As a result, most owners of lemon assets hold fire sales as soon as possible in the equilibrium, and thus the average asset quality in the market increases afterwards.

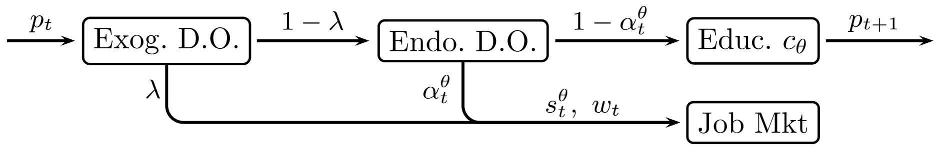


Figure 1: Schematic representation of the timing of the model (D.O. denotes dropout).

2 Model

Time is discrete, $t = 0, 1, 2, \ldots$. There is one worker who has a type $\theta \in \{H, L\}$, which is his private information with a common prior $p_0 = \Pr(\theta = H) \in (0, 1)$. The productivity of a type-θ worker (henceforth, θ-worker) is Y_θ. We normalize $Y_H = 1$ and $Y_L = 0$. In period 0 the worker decides whether to go to school or not. In the rest of the periods, if the worker continues going to school, he pays a type-contingent cost per unit of time, c_θ, where $0 < c_H < c_L$ and $c_H < 1$. The worker, regardless of his type, is subject in each period to an exogenous shock that results in the worker being forced to drop out of school with probability $\lambda \in (0, 1)$. The exogenous dropout behavior is interpreted as being caused by financial or utility shocks. In addition to this exogenous dropout, the worker may decide to endogenously drop out and go on the job market voluntarily.

The timing is summarized as follows. First, nature determines the type of the worker, choosing H with probability p_0. If the worker is still in school, in period t: (1) the worker exogenously drops out with probability λ and, if he does not exogenously drop out, decides whether to endogenously drop out. (2) If the worker decides not to drop out, he pays the education cost and goes to the next period. If the worker drops out, he goes on the job market. (3) In the job market, two or more firms simultaneously make wage offers to the worker who has dropped out. Figure 1 schematically represents the timing of the model.

As in Daley and Green (2012) we do not explicitly model the job market. Instead, we assume that when the worker drops out in some period t, he receives the expected (equilibrium) productivity of a worker that drops out in this period. This can be easily micro-founded by assuming that the profit of a firm that employs a θ-worker at a wage w is given by $Y_\theta - w$, and when the firm hires no worker, its profit is zero. Hence, if there are two or more firms in the job market, they Bertrand-compete, so they end up offering wages equal to the expected productivity of the worker.

The utility of the θ-worker who drops out after t periods of education and accepts a wage of w is $U(w, t) = w - c_\theta t$. If the wage conditional on dropping out at some period t is w_t, the value function of the θ-worker in period t is

$$V_t^\theta = \lambda w_t + (1-\lambda) W_t^\theta \ ,$$

where $W_t^\theta \equiv \max\{w_t, V_{t+1}^\theta - c_\theta\}$ is his continuation value in the complementary event. The worker will decide to endogenously drop out when $w_t > V_{t+1}^\theta - c_\theta$, stay in school when $w_t < V_{t+1}^\theta - c_\theta$, and potentially randomize when $w_t = V_{t+1}^\theta - c_\theta$.

A dropout (behavior) strategy for the θ-worker is $\alpha^\theta : \{0, 1, ...\} \to [0, 1]$, the probability that the type-θ worker chooses to drop out at t conditional on reaching its decision point. It is notationally more convenient to use $s_t^\theta \equiv \lambda + (1-\lambda)\alpha_t^\theta$ (the total probability of the θ-worker dropping out in period t) as the strategy of the worker instead of α_t^θ. So, we will assume that the worker can decide to drop out in period t with any probability $s_t^\theta \in [\lambda, 1]$. Finally, for each strategy profile, let $T^\theta \equiv \min\{t | s_t^\theta = 1\} \in \{0\} \cup \mathbb{N} \cup \infty$, which is the maximum number of education periods the θ-worker may receive under the given strategy profile.

Our equilibrium concept is analogous to that of Daley and Green (2012).[6] In our case, we find it more convenient to define the equilibrium in terms of the posterior about a worker who reached period t being an H-worker (denoted as p_t) instead of the stopping time at which he decides to exit education:

Definition 1. *An* ***equilibrium*** *is a strategy profile* $(s^\theta)_{\theta=L,H}$*, a wage process* w *and a belief sequence* p *such that:*

1. *Worker Optimality. The* θ*-worker chooses* s^θ *to maximize his expected payoff given* w*,*
2. *Zero Profit. If the worker drops out with education* t*, firms offer him the expected productivity of a worker that drops out in period* t*, i.e.*

$$w_t = \frac{p_t s_t^H}{p_t s_t^H + (1-p_t) s_t^L} \ , \tag{1}$$

and

3. *Belief Consistency. When it is well defined,* p_t *is updated following Bayes' rule*

$$p_{t+1} = \frac{p_t(1-s_t^H)}{p_t(1-s_t^H) + (1-p_t)(1-s_t^L)} \ . \tag{2}$$

[6] Unlike Daley and Green (2012), we do not impose the "No (Unrealized) Deals" condition because we do not allow the firms to make preemptive offers. In Section 2.5.2 we consider the preemptive offers case, and we will analyze the effect of including this condition in our equilibrium concept.

2.1 Preliminary Analysis

In our model, pooling at no education is always an equilibrium. The reason is that, as usual, off the path of play, firms may consider that the type of a worker that goes to school is L, so no type has incentives to receive education. However, since our goal is to study the dynamics of the worker's dropout behavior, we mainly focus on equilibria in which wasteful signaling is present. To construct such equilibria, we start by providing some necessary conditions for such equilibria to exist. Lemma 1 characterizes the behavior of the worker before the L-worker drops out for sure.

Lemma 1. *In any equilibrium where $T^L > 0$, in all periods $t < T^L$,*

1. *there is positive voluntary dropout by the L-worker, that is, $s_t^L > \lambda$, and*
2. *there is no voluntary dropout by the H-worker, that is, $s_t^H = \lambda$.*

Proof. The proof is in the appendix on page 23. □

By Lemma 1, when $T^L > 0$, the L-worker randomizes in any equilibrium in every period before T^L. Hence, the L-worker is always indifferent between dropping out and staying in school except (possibly) in his last possible period T^L. So, for all periods $t < T^L$,

$$w_{t+1} - w_t = c_L \ . \tag{3}$$

This fact implies that the wage must increase linearly before T^L. Notice that the constant returns to education are driven by the following assumptions in our model: First, the worker does not discount the future. Second, the marginal cost of education is constant in time, and last, there are worker types.[7] Without any of these assumptions, the returns to education will be time-varying. However, in each case, the (discounted) returns to education are still equal to the marginal education cost of the type of the worker who (1) is still in school with positive probability, and (2) has the lowest productivity among the types in school.

Lemma 2. *In any equilibrium, $T^H \in \{T^L, T^L + 1\}$.*

Proof. The proof is in the appendix on page 23. □

Lemma 2 shows how (exogenous) dropout disciplines the beliefs of the firms about early dropouts. If, for example, in some period the L-worker has already dropped out for sure in a previous period but not the H-worker, the (exogenous and endogenous) dropout must be an H-worker. Therefore, there is no reason for him to wait, because staying in school longer is costly and does not provide any wage increment.

[7] As we will show in Appendix C, when there are more than two types, the returns to education are concave.

Remark 1. The result is implied by the presence of a dropout risk so that dropping out in each period before T^H is on the path of play. When $\lambda = 0$, Lemma 2 does not necessarily hold. The reason is that, similarly to Cho and Kreps (1987) in a static case and and Nöldeke and van Damme (1990) in a dynamic model with public preemptive offers, firms can impose a belief threat off the path of play to force the H-worker to stay in school after $T^L + 1$. Also, this result depends on the fact that signaling is unproductive. In Section 3, we study a productive signaling model in which Lemma 2 is no longer true.

2.2 Equilibrium Analysis

The equilibrium prediction of the game varies depending on the prior p_0, so we first focus on the game in which the worker has high productivity with a probability which is almost one, and then consider the case where p_0 is small. Lemma 3 describes the equilibrium set when the prior p_0 is close to 1.

Lemma 3. *Set*

$$p_0^1 \equiv \frac{1 - c_H}{1 - (1 - \lambda)c_H} \ . \tag{4}$$

Then, if $p_0 > p_0^1$, the only equilibrium outcome is pooling at no education.

Proof. The proof is in the appendix on page 24. □

The intuition behind Lemma 3 is as follows. Because of the presence of the dropout risk, the worker may drop out in the first period on the path of play. Firms' beliefs about the dropout being a high-type are pinned down by equation (1). When the prior p_0 is high, firms' posterior is high as well, and therefore, they offer the dropout a high wage. Hence, when the prior is close to 1, the H-worker would voluntarily drop out to take the high wage offer instead of staying in school.

Remark 2. The result in Lemma 3 and the economic intuition above both rely on the presence of the dropout risk. In a model where $\lambda = 0$, wasteful signaling can be supported even when the prior about the type being high (p_0) is very close to 1. The reason is that there are equilibria in which there is no dropping out in the first period on the path of play. Since, off the path of play, a belief threat can be imposed, early dropouts are punished with low wages so no worker has the incentive to drop out. In our model, since the dropout risk is $\lambda > 0$, on the path of play, the worker may drop out in any period before T^H. Hence, the belief about the dropout being the high-type is pinned down by the equilibrium requirement, $\lambda p_0/(\lambda p_0 + 1 - p_0)$, which is arbitrarily close to one when p_0 is large enough.

Remark 3. Lemma 3 implies that there is no signaling waste when $p_0 \to 1$ in any equilibrium. Consequently, the equilibrium education length converges to that in the symmetric information model as p_0 goes to 1. The continuity result also depends on the positive dropout risk. When the dropout risk is zero, as in Cho and Kreps (1987), a Riley-outcome-like equilibrium always exists for any $p_0 < 1$. In other environments, one can avoid this discontinuity result by (1) imposing a belief-based refinement (see Mailath, Okuno-Fujiwara and Postlewaite, 1993), (2) assuming that firms make preemptive offers (Swinkels, 1999 and Daley and Green, 2012), and (3) allowing an extra informative signal (Daley and Green, 2013b).

Now we consider the model when p_0 is smaller than p_0^1. First, suppose p_0 is slightly smaller than p_0^1. In such a model, the no-education pooling equilibrium trivially exists. However, there is another (semi-separating) equilibrium in which

1. one-period education is supported on the path of play, and
2. the L-worker randomizes between no education and one-period education.

The intuition is as follows. If the L-worker endogenously drops out in period 0 with some positive probability but the H-worker does not, the market belief about the worker who did not drop out in period 0 is strictly greater than p_0, and therefore the wage in period 1 is higher than that in period 0. To ensure that the L-worker is indifferent between dropping out in period 0 and period 1, the wage difference between the two periods must be exactly equal to the L-worker's marginal cost of education, which pins down the posterior and the L-worker's strategy. In fact, there is another cutoff $p_0^2 < p_0^1$ such that, for any $p_0 \in (p_0^2, p_0^1]$, there is an equilibrium where the L-worker randomizes between receiving no education and receiving one period of education. Since p_0 is still large, p_1 becomes greater than p_0^1, so in the continuation game, the unique equilibrium involves all types dropping out immediately. As a result, the maximum equilibrium education is one period.

When p_0 is slightly smaller than p_0^2, repeating a similar argument, we can construct equilibria with (1) pooling at no education, (2) one-period education, and (3) two-period education. By using an induction argument, we can construct a sequence of cutoff values p_0^k where $k = 1, 2, 3, ...$ and $p_0^k > p_0^{k+1}$ such that when $p_0 \in (p_0^{k+1}, p_0^k]$, there exist equilibria with T periods of education where $T \leq k$. The following theorem formalizes the intuition above and characterizes possible education lengths in the set of all equilibria:

Theorem 1. *Let* $T^* \equiv \lceil \frac{1-c_H}{c_L} \rceil$.[8] *There exists a unique strictly decreasing sequence* $\{p_0^k\}_{k=1}^{T^*} \in (0,1)$ *such that for any* $p_0 \in (p_0^{k+1}, p_0^k]$

[8] $\lceil x \rceil = \min\{n \in \mathbb{Z} | n \geq x\}$ denotes the smallest integer no lower than x.

1. *there is a pooling equilibrium at no education;*

2. *for any* $0 < T \leq k$*, there is a semi-separating equilibrium lasting* T *periods, and*

3. *for any* $T > k$*, there is no equilibrium lasting* T *periods.*

Proof. The proof is in the appendix on page 24. □

Theorem 1 implies that the maximum duration of equilibrium education is non-increasing in the prior about the worker being high productivity. As the prior goes to zero, maximum education duration goes to its finite upper bound T^*.

In Lemma 3 we discussed the case where p_0 is close to 1. Now, consider the case in which p_0 is not close to 1. As we have shown in Lemma 1, the low-type endogenously drops out with positive probability and the high-type does not voluntarily drop out; thus, $s_t^L > s_t^H$, which means that p_t is pushed up over time. The low-type indifference condition (3) implies that w_t is linear before T^L. These two observations imply that p_t and w_t will be high (close to 1) after finitely many periods. The smaller the prior p_0, the more periods of education can be supported in an equilibrium. This suggests that the maximum education duration supported by an equilibrium is non-increasing in p_0. In Figure 2, we plot some equilibrium belief sequences p_t and dropout rate ratio sequences s_t^L/s_t^H.[9] In each equilibrium, $T^L = T^H = T$ is the "graduation period." The H-worker's dropout rate is $s_t^H = \lambda$ for all $t < T$ and $s_t^H = 1$ at $t = T$. The L-worker's dropout rate satisfies $s_t^L \in (\lambda, 1)$ for $t < T$, and at $t = T$, $s_t^L = 1$.

Notice that s_t^L/s_t^H may be non-monotone. The intuition can be illustrated as follows. Rewritting equation (1) yields:

$$\frac{s_t^L}{s_t^H} = \frac{1 - w_t}{w_t}\frac{p_t}{1 - p_t},$$

so the worker's equilibrium dropout probability is pinned down by the equilibrium wage and the posterior. Over time, both the posterior p_t and the market wage w_t increase. For a fixed p_t, an increase in w_t pushes s_t^L down, since a lower dropout probability by the L-worker is required to generate a higher overall dropout productivity (dropout composition effect). An increase of p_t for a fixed w_t, instead, pushes s_t^L up, since a higher dropout probability by the L-worker is required to keep the productivity of the dropouts the same (market composition effect). When p_t is large compared with w_t, the market composition effect dominates, so s_t^L/s_t^H may not be monotone.

[9] We plot s_t^L/s_t^H in order to have a nice-looking graph. Indeed, for example, in all plotted equilibria $s_t^H = \lambda = 0.1$ for $t < T^H$ and $s_{T^H}^H = 1$, so s_t^H makes a big jump up at the end. Intuitively, s_t^θ for $t < T^H$ looks like a "flow probability" while $s_{T^H}^\theta$ looks like a "lump-sum probability" (See Section 2.4 for the continuous time limit). Note also that, given p_t, only the ratio s_t^L/s_t^H determines w_t.

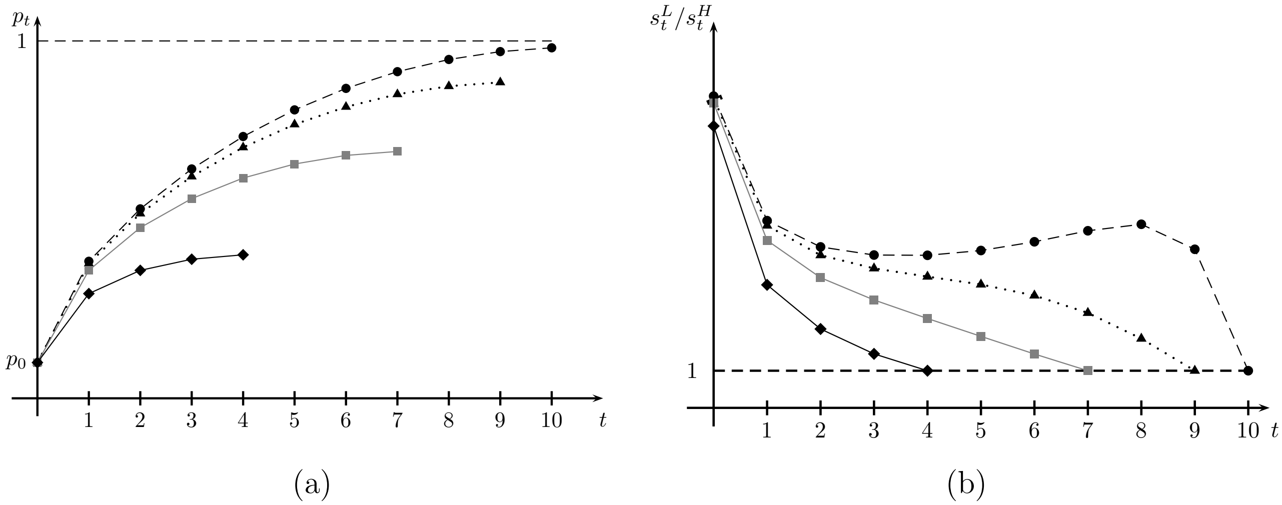


Figure 2: (a) p_t for different equilibria. (b) s_t^L/s_t^H for different equilibria. Dots having the same shape correspond to the same equilibrium, and they are linked with a straight line for visual clarity. The parameter values are $c_H = 0.032$, $c_L = 0.097$, $\lambda = 0.1$ and $p_0 = 0.1$.

The model can predict the worker's equilibrium dropout dynamics. Since the worker's type is his private information, we analyze the dynamics of the unconditional (or observed) dropout probability, defined as follows

$$m_t \equiv p_t s_t^H + (1 - p_t)s_t^L \ .$$

In equilibrium, $s_t^H = \lambda$ for $t < T^L$, so $m_t \equiv p_t\lambda + (1 - p_t)s_t^L$. The dynamics of m_t are driven by two forces. First, p_t increases over time, which pushes the observed dropout rate down. Second, the L-worker's dropout strategy varies over time. As we show in Figure 2(b), s_t^L may not be monotone over time. When s_t^L increases, it pushes m_t up. Hence, the observed dropout rate may go up and down over time, depending on the interaction between two forces. Unfortunately, in a discrete time model, we cannot analytically characterize the dynamics of the worker' dropout rate. In Section 2.4, we analytically characterize the observed dropout dynamics at the continuous time limit of the original model.

2.3 Refinement

Without imposing any refinement, multiple equilibria exist for most p_0. The main reason we do not have equilibrium uniqueness is the arbitrariness of belief after T^H off the path of play, similar to that in Cho and Kreps (1987). Hence, we still have belief threats that push education duration down.

By imposing an appropriate criterion on beliefs off the path of play, for example D1 as defined by Cho and Kreps (1987), one can shrink the equilibrium set.[10] The spirit of these refinements is that, off the path of play, firms put a positive probability only on the type that is most likely to deviate. In our model, since the marginal cost of education of the H-worker is strictly smaller than that of the L-worker, any sequence of wages off the path of play (after T^H) that induces the L-worker to deviate must also induce the H-worker to deviate. As a result, off the path of play, firms put a positive belief only on the H-worker, i.e., $p_t = w_t = 1$ for any $t > T^H$. Given this belief sequence off the path of play, we will say that an equilibrium is eliminated by D1 if $w_T < 1 - c_H$, since otherwise the H-worker would have incentives to stay in school for one more period. If an equilibrium is not eliminated by D1, we say that it passes D1. Similarly to Nöldeke and van Damme (1990), these concepts are not enough to select a unique equilibrium.[11] The key reason for the multiplicity is that, in our model, the education choice is an integer instead of a real number. Consider the following case as an example.

Example 1. *Suppose $p_0 \in (1 - c_H, p_1^0)$. It is easy to show that there is an equilibrium in which $s_0^H = \lambda$ and $s_0^L = 1$. Since, in this equilibrium, $p_1 = w_1 = 1$ is on the path of play, it passes D1. However, there is another equilibrium consisting of pooling at no education, that is, $s_0^H = s_0^L = 1$, so $p_0 = w_0 > 1 - c_H$. Hence, pooling at no education also passes D1.*

Nevertheless, as shown below, when the length of the interval is small, the D1 criterion is essentially unique, in the sense that the outcomes of all equilibria passing D1 become arbitrarily close to each other.

2.4 Frequent Dropout Decision

In this section we consider the limit where the length of the interval is arbitrarily small. This limit exercise allows us to have a clean characterization of the uniquely determined asymptotic behavior for any sequence of equilibria with the same real duration. In particular, it allows us to easily characterize the relationship between the observed dropout rate and years of education.

[10] Belief monotonicity is another refinement concept commonly used in the dynamic signaling literature (see Swinkels, 1999 or Daley and Green, 2012). However, it does not help here. The reason is, given that λ is the same for all players, the increase in the posterior after a deviation can be arbitrarily small, which prevents the H-worker from deviating.

[11] Nöldeke and van Damme (1990) focus on equilibria that satisfy the *never a weak best response* (NWBR), requirement provided by Kohlberg and Mertens (1986), and they find that the equilibrium outcome converges to the Riley outcome when the time interval between two education decision points goes to zero. Nevertheless, in games of this sort the set of equilibria passing NWBR coincides with the set of equilibria passing D1. See Nöldeke and van Damme (1990) and Swinkels (1999) for further discussion.

In our base model, which is parametrized by $\Gamma \equiv (p_0, c_L, c_H, \lambda)$, the implicit time length between two consecutive periods is 1. Hence, to keep the notation simple, in this section we fix a strictly decreasing sequence $\{\Delta_n\}_n$ that converges to 0. In order to increase the dropout decision but keep the same real effects per unit of time, we then consider a sequence of models where the n-th model is parametrized by $\Gamma_n \equiv (p_{0,n}, c_{L,n}, c_{H,n}, \lambda_n) = (p_0, \Delta_n \tilde{c}_L, \Delta_n \tilde{c}_H, \Delta_n \tilde{\lambda})$, for some fixed $\tilde{c}_L, \tilde{c}_H, \tilde{\lambda} \in \mathbb{R}_{++}$.[12]

In our notation, τ corresponds to the real time, that is, it plays the same role as t in our baseline model. Also, κ plays the role of T as the real time-length of education in a given equilibrium. Finally, noting that $\Delta_n T_n^*$ (T^* is defined in Theorem 1) is the maximum real-time length of an equilibrium when the length of the period is Δ_n, we define $\kappa^* \equiv \lim_{n\to\infty} \Delta_n T_n^*$. The following lemma characterizes the maximum real duration of an equilibrium as the decision to drop out becomes more frequent:

Lemma 4. *κ^* exists, belongs to $(0, \frac{1}{\tilde{c}_L})$ and is strictly decreasing in p_0.*

Proof. The proof is in the appendix on page 29. $\square$

Lemma 4 stands in sharp contrast to Swinkels (1999), in which the only equilibrium is pooling at no education when the duration of the period sufficiently short. The contrast between this result and Swinkels' illustrates the critical role of timing in the two models. In Swinkels (1999), firms can make preemptive offers to attract all the workers in school and end the game immediately, so that no wasteful education is present in equilibrium. In our model, firms cannot directly disturb the worker's signaling process by making an in-school offer, and therefore, semi-separating equilibria can survive.

In the next set of results we are going to use a sequence of equilibria, one in each model with period length Δ_n. We will use p_t^n and w_t^n to denote, respectively, the beliefs and wage of the n-th equilibrium (of the model with period length Δ_n) in period t. Since, when Δ_n is small, both s_t^H and s_t^L are $O(\Delta_n)$ (in all periods except possibly the last two), m_t is also $O(\Delta_n)$ (in all periods except possibly the last two). To study the dynamics of the worker's dropout behavior at the continuous time limit, we define the *observed dropout rate* as follows. Given a sequence Δ_n and $m(t)$, define the associated observed dropout rate as $\tilde{m}_t^n \equiv \frac{m_t}{\Delta_n}$, and $\tilde{s}_t^{L,n} \equiv \frac{s_t^L}{\Delta_n}$. In the following theorem, we characterize the continuous time limit of equilibrium belief and the observed dropout rate.

[12] This limit corresponds to interpreting $\tilde{c}_\theta$ to be the flow cost for each $\theta \in \{L, H\}$, and interpreting $\tilde{\lambda}$ as the rate at which the worker is exogenously forced to drop out. Indeed, note that the cost and the probability *per unit of real time* are the same in all models (for the n-th model they are given, respectively, by $\frac{c_{\theta,n}}{\Delta_n} = \tilde{c}_\theta$ and $\frac{\lambda_n}{\Delta_n} = \tilde{\lambda}$). We assume that $\tilde{c}_L > \tilde{c}_H$, $\tilde{c}_H \Delta_1 < 1$ and $\tilde{\lambda}\Delta_1 < 1$, so our parametric assumptions hold for each model in the sequence.

Theorem 2. *1. For any $\kappa \in [0, \kappa^*]$, there exist functions $(p, w, \tilde{m}) : [0, \kappa] \to [0, 1] \times [0, 1] \times \mathbb{R}_+$ such that for any sequence of equilibria with real duration of education converging to κ and for all $\tau \in (0, \kappa)$,*

$$\lim_{n\to\infty} p^n_{\lceil \tau/\Delta_n \rceil} = p(\tau) \ , \quad \lim_{n\to\infty} w^n_{\lceil \tau/\Delta_n \rceil} = w(\tau) \quad \text{and} \quad \lim_{n\to\infty} \tilde{m}^n_{\lceil \tau/\Delta_n \rceil} = \tilde{m}(\tau) \ .$$

2. Furthermore, p solves the following equation, with $p(0) = p_0$:

$$p'(\tau) = \frac{\tilde{\lambda} p(\tau)(p(\tau) - w(\tau))}{w(\tau)} \ ; \tag{5}$$

w is given by $w(\tau) = p(\kappa) - (\kappa - \tau)\tilde{c}_L$ and $\tilde{m}$ by $\tilde{m}(\tau) = \frac{p(\tau)}{w(\tau)}\tilde{\lambda}$.

Proof. The proof is in the appendix on page 30. □

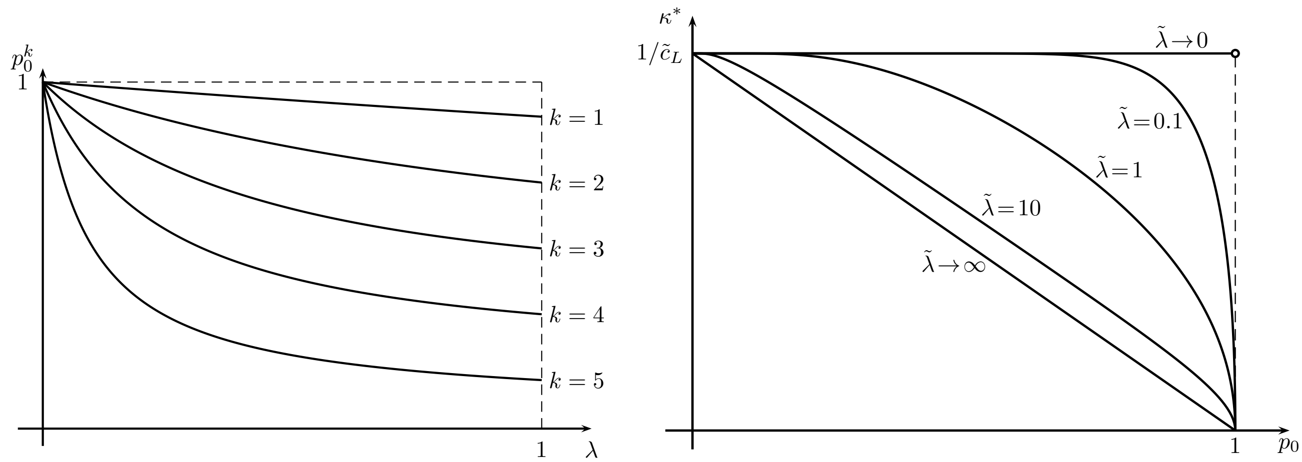
As noted before, D1 selects equilibria where $p^n_{T_n} \in [1 - \Delta_n \tilde{c}_H, 1]$ in the n-th model for $n \in \mathbb{N}$. As Δ_n goes to zero, the last period equilibrium belief converges to 1. So, it is easy to show that equilibria passing D1 have a real duration of $\kappa^* + O(\Delta_n)$ (see the proof of Lemma 4). Indeed, otherwise the last period's beliefs are bounded away from 1 and hence are lower than $1 - \Delta_n \tilde{c}_H$. In the proof of Theorem 1 (see Lemma 10) we explicitly construct for each p_0 equilibria in which the last period's beliefs belong to $[1 - \Delta_n \tilde{c}_H, 1]$. So, for each p_0 and small $\Delta_n > 0$, there are equilibria passing D1, and their duration is close to κ^*. Notice that the limit of the equilibrium outcome is not the least costly separation equilibrium due to the presence of dropout risk.

At the limit where the length of the period becomes small, we can easily characterize the dynamics of the observed dropout rate. As we mentioned before, the dropout rate of the L-worker needs not be monotone. However, the *observed* dropout rate is decreasing, which is consistent with much of the empirical evidence, for example, Hendricks and Leukhina (2013).

Theorem 3. *At the continuous time limit of all equilibria, the observed dropout rate decreases over time.*

Proof. The proof is in the appendix on page 31. □

Since $\tilde{m}'(\tau) = -p'(\tau)(\tilde{s}^L(\tau) - \tilde{\lambda}) + (1 - p(\tau))\tilde{s}'^L(\tau)$, as time goes on, there are two effects on the observed dropout rate. First, there is a skimming effect: the probability of the worker being an L-worker becomes smaller over time, since the L-worker's dropout rate is higher than $\tilde{\lambda}$, the dropout rate of the H-worker. Hence, this skimming effect, measured by $-p'(\tau)(\tilde{s}^L(\tau) - \tilde{\lambda}) < 0$, pushes the observed dropout rate down. Second, there is another effect, measured by $(1 - p(\tau))\tilde{s}'^L(\tau)$: the dropout rate $s^L(\tau)$ for the L-worker who is still in school may go either up or down. When it



(a)

(b)

Figure 3: (a) The role of λ in the discrete time model: $\{p_0^k\}$ as a function of λ. (b) The role of $\tilde{\lambda}$ in the continuous time limit: $\kappa^* = \kappa^*(p_0)$ as a function of p_0, for different values of $\tilde{\lambda}$.

goes up, it pushes the observed dropout rate up as well. However, we can show that the second effect is always dominated by the first one, and thus the observed dropout rate is always declines over time.

2.5 Discussion

2.5.1 The Role of Dropout Risk

The equilibrium characterization crucially depends on the presence of dropout risk. What happens if the dropout risk is arbitrarily small? What if the worker's dropout risk is type-dependent? We address these issues here.

First, we consider the limit case where λ goes to zero in the discrete time model. Figure 3 (a) plots $\{p_0^k\}_{k=1}^{T^*}$ for different values of λ. As we see, when $\lambda \to 0$, p_0^k for all k collapses to 1. This implies that, when λ is low, for almost all priors (1) the H-worker's dropout risk is very small, so he graduates with high probability, (2) the L-worker is unlikely to go to school, so p_1 is close to 1, but w_1 remains low due to the low λ, and (3) the maximum length of an equilibrium is T^*. This is consistent with the canonical signaling model, where $\lambda = 0$. In the other limit, when $\lambda \to 1$, $p_0^k - p_0^{k+1} = c_L$ for all $k > 1$. This is a consequence of the fact that when λ is close to 1, so are s^L and s^H. Therefore, as we see in (1), w_t is close to p_t for all t. Since w_t increases linearly in any equilibrium, this imposes a nearly linear evolution on p_t and therefore also on p_0^k .

Second, we consider the continuous time limit when the dropout rate is small, that is, when $\tilde{\lambda}$ is small. From the equation that κ^* satisfies (equation (9) in the proof of Lemma 4), it is easy to see that $\lim_{\tilde{\lambda}\to 0} \kappa^* = \frac{1}{\tilde{c}_L}$ for all $p_0 \in (0,1)$. Indeed, as we see in Figure 3(b), as $\tilde{\lambda}$ gets small, κ^* converges to $\frac{1}{\tilde{c}_L}$ for all $p_0 \in (0,1)$. Hence, the length of an equilibrium passing D1 approaches $\frac{1}{\tilde{c}_L}$ when the interval gets short and $\tilde{\lambda}$ gets small. This limit is consistent with the finding of Nöldeke and van Damme (1990) that the only equilibrium that passes D1 is the least costly separating equilibrium, found by Riley (1979), which requires an education length equal to $\frac{1}{\tilde{c}_L}$.[13]

Finally, one may wonder whether it is restrictive to assume that the H-worker and the L-worker face the same exogenous dropout risk. Without a second thought, it seems that the low-productivity worker should have a higher probability of dropping out than the high-productivity worker, which seems to conflict with our assumption. However, this naive intuition is based on the total dropping-out behavior, s_t^L, which is driven both by the L-worker's choices (which are related to his productivity) and by exogenous shocks (which may not be related to his productivity). As we have shown, on the equilibrium path, $s_t^L \geq s_t^H$ in each period. Yet, it is still useful to determine whether our equilibrium characterization is robust by relaxing this homogeneous dropout risk assumption. In Appendix B, we consider perturbations of the baseline model by considering heterogeneous dropout risk. The equilibrium characterization is robust to such perturbations.

2.5.2 Preemptive Offers

In our model, we assume that firms cannot make preemptive offers. In a dynamic signaling model in which firms make preemptive offers and students face no exogenous dropout risk, Swinkels (1999) shows that the only equilibrium is pooling at no education when the time interval of a period is sufficiently small. A natural question is what happens if firms can make preemptive offers in the presence of exogenous dropout behavior.

As in Daley and Green (2012) we incorporate the preemptive offers in our model by requiring the "No (Unrealized) Deals" condition in our equilibrium concept. So, for this section only, consider the equilibrium concept to be the same as in Definition 1 with the following extra condition:

4. *No (Unrealized) Deals. For any t and $\theta \in \{L, H\}$, $V_t^\theta \geq \mathbb{E}[Y_{\theta'} | V_t^{\theta'} \leq V_t^\theta]$.*

The economic intuition behind the the No (Unrealized) Deals condition is the following. Consider a model where firms make offers to the worker while he is still in college. If the worker is

[13] The similarity between our findings and that of Nöldeke and van Damme (1990) is rooted in the similar off-equilibrium beliefs discipline, which provides endogenous commitment to the worker. In Nöldeke and van Damme (1990) such a beliefs discipline exists because of the observability of the previous offers, while in our work it is a natural result of the specification of the extensive form of the stage game.

forced to drop out in a given period t, he accepts one of the offers for sure. If the worker is not forced to drop out, he can decide to take the offer or stay in college. Imagine that the condition is violated and, for example, $V_t^H < \mathbb{E}[Y_{\theta'}|V_t^{\theta'} \leq V_t^{\theta}] = p_t$. A firm can then deviate and offer $w_t \in (V_t^H, p_t)$, which will be accepted by the worker with probability one (independently of her type) and gives positive profits to the firm.

In our model, if preemptive offers are allowed, all non-pooling equilibria are destroyed as in Swinkels (1999). The idea is that, from the penultimate period to the last period, belief updating is slow. So firms can post an offer to attract both types of worker and obtain positive profit. In the following, we show that, when firms can privately and frequently make preemptive offers, all of the semi-separating equilibria we constructed do not exist.

We illustrate the idea when Δ is small (following the notation in Section 2.4), because this is where Swinkels' (1999) pooling result holds. Assume that an equilibrium with $T > 0$ periods of education exists. When Δ is small, the H-worker's value V_T^H can be approximated by

$$-\tilde{c}_H\Delta + \tilde{\lambda}\Delta w_{T-1} + (1 - \tilde{\lambda}\Delta)V_T^H = -\tilde{c}_H\Delta + p_T + O(\Delta^2) \ .$$

The approximation holds because $V_T^H = p_T$ and $w_{T-1} = p_T + O(\Delta)$. By Theorem 2, we know that $p_{T-1} = p_T + O(\Delta^2)$ (since $p'(\kappa) = 0$). Hence, for $\Delta > 0$ small, $V_{T-1}^H < p_{T-1}$, so the No (Unrealized) Deals condition is violated. Consequently, when firms can privately make preemptive offers, no semi-separating equilibrium exists.

3 Productive Education

In the baseline model, we assume that education serves as a pure information extraction mechanism and does not affect the worker's productivity. This is clearly a theoretical simplification. In reality, going to school serves not only as a useful signaling device (signaling some of a worker's innate abilities, for example) but also it enhances the worker's productivity (human capital accumulation). Hence, the observed positive education-wage profile comes from both human capital accumulation and the signaling motive of dropping out.

In such a situation, the returns to education incorporate both the signaling effect and the human capital accumulation effect. In this section, we extend the baseline model by assuming that getting an education can enhance the worker's productivity. In the productive education model, both the human capital accumulation effect and the signaling effect contribute to the returns to education.[14]

[14]There is a large body of literature empirically studying how to distinguish human capital accumulation theory and signaling theory, for example, Tyler, Murnane, and Willett (2000), Bedard (2001), Frazis (2002), etc. Fang

We assume that a θ-worker with t periods of education has productivity equaling $Y_\theta = a_\theta + h(t)$, where $\theta = \{H, L\}$, a_θ captures the intrinsic productivity, and $h(t)$ captures productivity accumulated through education. Again, we assume that a worker's intrinsic productivity is a_H with probability p_0 and a_L with probability $1 - p_0$, and we normalize $a_L = 0$ and $a_H = 1$. To illustrate the main idea, we focus on the simplest specification of the human capital accumulation function. We assume that there is some finite number $\hat{T} \in \mathbb{N}$ such that

$$h(t) = \begin{cases} ht & \text{if } t < \hat{T}, \\ h\hat{T} & \text{otherwise,} \end{cases}$$

where $h \in (c_H, c_L)$ is the marginal human capital accumulation coefficient until $\hat{T}$.[15] Notice that the production function for human capital is concave. Also, the socially efficient outcome is that the L-worker gets no education and the H-worker gets $\hat{T}$ periods of education. When $\hat{T} = 0$, getting an education does not enhance any worker's productivity at all, which is the case in our baseline model.

In this section, we focus on the case where $\hat{T} > 0$, and we will show that the obtained equilibria are similar to those in the benchmark model. In this case, the zero-profit condition from our definition of equilibrium needs to be replaced, because now the productivity of a worker (conditional on type) is not constant over time. Let $\hat{p}_t$ be the posterior about the type of a worker that drops out in period t being H, which is equal to the RHS of equation (1). Then, the condition that is analogous to the zero profits condition is

$$w_t = \hat{p}_t (1 + ht) + (1 - \hat{p}_t) ht = \hat{p}_t + ht,$$

so the return to education is given by

$$w_{t+1} - w_t = \underbrace{\hat{p}_{t+1} - \hat{p}_t}_{\text{(signaling)}} + \underbrace{h}_{\text{(human capital)}}$$

where $\hat{p}_{t+1} - \hat{p}_t$ is the contribution of the signaling effect and h is the contribution of the human capital accumulation effect.

Here Lemma 1 still holds; that is, in any equilibrium the L-worker randomizes between dropping out and continuing in school in the first several periods. Hence, we must have $w_{t+1} - w_t = c_L$, which implies that

$$\hat{p}_{t+1} - \hat{p}_t = c_L - h > 0.$$

(2006) estimates a static education choice model with both human capital accumulation and a signaling mechanism and claims that the signaling effect is at most about one third of the actual college wage premium.

[15] More generally, most of our results would also apply if the return varied over time with $h_t \in (c_H, c_L)$ for $t < \hat{T}$ and $h_t < c_H$ for $t \geq \hat{T}$. The equilibrium analysis is similar.

We now focus on equilibria passing D1. First, we consider the case where $\hat{T}$ is large.

Theorem 4. *There exists $T^L \in \mathbb{N}$ such that, when $\hat{T} > T^L$, there is a unique equilibrium passing D1. In the equilibrium,*

1. *the L-worker drops out in each period with probability s_t^L where $s_t^L \in (\lambda, 1)$ for $t < T^L$ and $s_{T^L}^L = 1$, and*
2. *the H-worker does not voluntarily drop out before $\hat{T}$ and drops out for sure in period $\hat{T}$.*

Proof. The proof is in the appendix on page 32. □

In contrast to the baseline model, when $\hat{T}$ is large enough, there is a unique equilibrium passing D1. The intuition for this result is as follows. Since education until $\hat{T}$ is efficient for the H-worker, he prefers to stay in school until $\hat{T}$. The L-worker, instead, keeps dropping out until some T^L. The dropout rate needs to be high enough to imply increases in $\hat{p}_t$ equal to $c_L - h$. Therefore, $T^H > T^L$. This uniquely pins down the dropout rate of the L-worker, so a unique equilibrium exists. In our baseline model, instead, we have different possible behaviors in the last two periods, so in general equilibria passing D1 with $T^L = T^H$ or $T^L = T^H - 1$ may exist.[16]

Theorem 4 implies that when $\hat{T}$ is sufficiently large, there are two phases in the unique equilibrium passing D1. In the first phase, the return to education is c_L, and both the signaling effect and the human capital accumulation effect contribute to it. The observed dropout rate $m(t)$ varies over time. In the second phase, the return to education is h, which purely comes from the human capital accumulation effect. The observed dropout rate is constant.

Remark 4. The specification allows us to distinguish the effect of human capital accumulation and dynamic signaling on return to education. First, from period T^L to $\hat{T}$, only the H-worker is in school, so the return to education is h and the observed dropout rate is λ. From period 0 to T^L, the return to education is c_L, which is different from h. As a result, one can directly estimate $\hat{T}, h, c_L$ and λ from the data on wages and the dropout rate. Second, after recovering the parameters c_L and h one can also calculate the contribution of the signaling effect on the return to education, which is $(c_L - h)/c_L$ for $t < T^L$ and zero for $t \geq T^L$.[17]

[16] Under this specification of human capital accumulation, the weaker refinement of belief monotonicity is enough to select the unique equilibrium in our productive-signaling model as well.

[17] The simple identification strategy works because of the particular specification of the human capital accumulation technology: (1) there are two parameters in the $h(t)$ functions, and (2) human capital accumulation does not depend on the worker's demographic characteristics. In general, one can allow more complicated technology by considering the general function forms of $h(t)$ and other realistic factors, for example, worker' race, IQ test score, etc. However, fully exploring this issue and structurally estimating the model are beyond the scope of this paper.

Similarly to the baseline model, we can examine the continuous time limit of the model and obtain a clean characterization of the equilibrium education returns and the dropout rate dynamics. We proceed similarly to Section 2.4 by fixing a sequence $(\Delta_n)_n$ strictly decreasing to 0. Now, the set of parameters of the model is $\Gamma \equiv (p_0, c_L, c_H, \lambda, h, \hat{T})$, so for each n we consider a model with set of parameters $\Gamma_n \equiv (p_0, \Delta_n \tilde{c}_L, \Delta_n \tilde{c}_H, \Delta_n \tilde{\lambda}, \Delta_n \tilde{h}, \Delta_n \hat{\kappa})$.

Corollary 1. *There exist some $\kappa^L < \hat{\kappa}$ and functions $(p, w, \tilde{m})$ such that for any sequence of equilibria passing D1 and $\tau \in (0, \hat{\kappa})$ we have*

$$\lim_{n\to\infty} p^n_{\lceil \tau/\Delta_n \rceil} = p(\tau), \quad \lim_{n\to\infty} w^n_{\lceil \tau/\Delta_n \rceil} = w(\tau), \quad \lim_{n\to\infty} \tilde{m}^n_{\lceil \tau/\Delta_n \rceil} = \tilde{m}(\tau)$$

and

1. *when $\tau < \kappa^L$, $w'(\tau) = \tilde{c}_L$ and $\tilde{m}'(\tau) < 0$, and*

2. *when $\tau \in (\kappa^L, \hat{\kappa})$, $w'(\tau) = \tilde{h}$ and $m(\tau) = \tilde{\lambda}$.*

The proof is similar to that of the baseline model, so it is omitted here. Notice that the return to education is $\tilde{c}_L$ before $\hat{\kappa}$ and becomes $\tilde{h}$ after $\hat{\kappa}$, and the observed dropout rate initially declines over time and then becomes constant after $\hat{\kappa}$.

When $\hat{T}$ is small, there exist equilibria in which the game ends later than $\hat{T}$. In such equilibria, staying in school more than $\hat{T}$ periods is socially inefficient and the worker does so purely for signaling reasons, so only the signaling effect contributes to the return to education. In this case, similar to the baseline model, there are multiple equilibria passing D1. However, the return to education is c_L for every period, which is observably different from the case in which $\hat{T}$ is large. Since the equilibrium construction and characterization are similar, we only provide the characterization at the continuous time limit.

Corollary 2. *Assume $\hat{\kappa} < \kappa^L$ (defined in Corollary 1). Then, for any sequence of equilibria passing D1 there exist functions $p, w, \tilde{m}$ such that for all $\tau \in (0, \kappa^L)$ we have*

$$\lim_{n\to\infty} p^n_{\lceil \tau/\Delta_n \rceil} = p(\tau), \quad \lim_{n\to\infty} w^n_{\lceil \tau/\Delta_n \rceil} = w(\tau), \quad \lim_{n\to\infty} \tilde{m}^n_{\lceil \tau/\Delta_n \rceil} = \tilde{m}(\tau)$$

and

1. *for all $\tau < \kappa^L$, $w'(\tau) = \tilde{c}_L$ and $\tilde{m}'(\tau) < 0$, and*

2. *at $\tau = \hat{\kappa}$, p and $\tilde{m}$ are continuous but non-differentiable.*

The results in this section highlight the role of the observed dropout rate when estimating the social returns to education. Only when no type is willing to voluntarily drop out (so dropping out has no signaling value) will wages be determined by the increase in productivity due to education. In this case, for example when $t \in \{T^L, T^L+1, ...\hat{T}\}$, the observed dropout rate does not vary over time, and the individual return to education is equal to the social return to education. If, instead, (some) types voluntarily drop out, wages are determined by the education costs of this type. In this case, the observed dropout rate varies over time, and the individual return to education is greater than the social return to education.

4 Concluding Remarks

Our model constitutes a new step towards understanding of dropout behavior and its economic implications. We find that the existence of exogenous dropout induces endogenous dropout by the low-productivity workers. This helps us to rationalize the high observed dropout rates and its variation over the time workers spends in school. The active dropout behavior disciplines the market's beliefs about dropouts. This implies that the maximum length of education is decreasing in the prior about the worker being highly productive. In addition, the expected productivity of workers that go on the job market is increasing in the time the worker spends in school, and the marginal returns to education equals the low productivity worker's marginal cost of education. By incorporating productive education to our model, we highlight the potential role of data on the dropout rate in decomposing the returns to education into the signaling effect and the human capital accumulation effect.

A Appendix: Omitted Proofs

A.1 The Proof of Lemma 1

Let's first prove a preliminary result:

Lemma 5. *(The L-worker does not beat the market) For all equilibria and t, $V_t^L \leq p_t$.*

Proof of Lemma 5. Fix an equilibrium. Let τ be the time at which the game ends. Then,

$$p_t V_t^H + (1 - p_t) V_t^L \leq \mathbb{E}_t[w_\tau | \tau \geq t].$$

Note that, due to the education costs (i.e. signaling waste), there is a (weak) inequality, and it is strict if $t < T^L$. Also,

$$\mathbb{E}_t[w_\tau | \tau \geq t] = \sum_{\tau=t}^{\infty} \Pr(\tau, t) w_\tau = \sum_{\tau=t}^{\infty} \Pr(\tau, t) p_t \frac{\Pr^H(\tau, t)}{\Pr(\tau, t)} = p_t \sum_{\tau=t}^{\infty} \Pr^H(\tau, t) = p_t \; .$$

where $\Pr(\tau, t)$ denotes the conditional probability in period t that the game ends in period τ, and $\Pr^H(\tau, t) = s_\tau^H \prod_{t'=t}^{\tau-1} (1 - s_{t'}^H)$ is further conditioning on the dropout being type H. The last equality holds because the high type has strictly positive dropout rate and therefore he drops out in finite time with probability one. Since $V_t^H \geq V_t^L$ (the H-worker can mimic the L-worker at a cheaper price) the result holds. □

Suppose there is no endogenous dropout by the L-worker in period t, then $p_{t+1} \leq p_t \leq w_t$. But, $w_t \leq W_t^L = V_{t+1}^L - c_L$ due to the fact that the L-worker does not voluntarily drop out. By Lemma 5, $V_{t+1}^L \leq p_{t+1} \leq w_t$; thus $w_t \leq w_t - c_L$, which is a clear contradiction. So part (1) is true. Therefore part (2) is also true, since $W_t^H \geq V_{t+1}^H - c_H \geq w_{t+1} - c_H$ by definition of W_t^H and V_t^H, and $w_{t+1} - c_H = w_t + c_L - c_H > w_t$ by the indifference condition of the L-worker. *Q.E.D.*

A.2 The Proof of Lemma 2

Assume first $T^H > T^L + 1$. In this case, $p_{T^L+1} = 1$. Using equation (1) we know $w_{T^L+1} = 1$. Since the payoff of the worker is bounded by 1, and waiting until next period is costly, the worker is better off dropping out at $T^L + 1$. This is a contradiction.

Lemma 1 implies that $S_{T^L}^H > 0$, and therefore $T^H \geq T^L$. *Q.E.D.*

A.3 The Proof of Lemma 3

The wage in period $t = 1$ is bounded above by 1. This implies that for the H-worker to be (weakly) willing to get one period of education, it must be the case that $w_0 \leq 1 - c_H$. This implies that

$$1 - c_H \geq w_0 = \frac{p_0 s_0^H}{p_0 s_0^H + (1 - p_0) s_0^L} \geq \frac{p_0 \lambda}{p_0 \lambda + 1 - p_0} \ .$$

Solving for p_0 under the equality, we get that the threshold for the existence of an equilibrium with non zero education satisfies equation (4). *Q.E.D.*

A.4 The Proof of Theorem 1

The proof of Theorem 1 is divided into several steps. To make the proof clear to the reader, we note that we will be following this **road map**:

1. We begin defining and proving some properties of the "pull-back functions," which will be used to construct equilibria in the rest of the proof (lemmas 6 and 7).
2. In subsection A.4.1 we define some putative values for p_0^k, denoted $\tilde{p}_0^k$, and we prove by induction that, if $p_0 \in (\tilde{p}_0^{k+1}, \tilde{p}_0^k]$, then there is no equilibrium with more than k periods of education.
3. Then, in subsection A.4.2 we show that, if $p_0 \in (\tilde{p}_0^{k+1}, \tilde{p}_0^k]$, there exists an equilibrium where the L-worker is indifferent on dropping out for all periods except (maybe) the last for all $T \in \{0, ..., k-1\}$.
4. Finally, in subsection A.4.3 we show that, if $p_0 \in (\tilde{p}_0^{k+1}, \tilde{p}_0^k]$, there exists an equilibrium with length k. Therefore, $p_0^k = \tilde{p}_0^k$.

We begin this proof by stating and proving two results that will simplify the rest of the proof and the proofs of other results in our paper. The first one states two properties of the "**pull-back functions**" $\mathcal{S}_\tau(\cdot,\cdot)$ and $\mathcal{M}_\tau(\cdot)$:

Lemma 6. *For any $\tau \in \mathbb{N}$, let $\mathcal{S}_\tau : [0,1]^2 \to [0,1]$ and $\mathcal{M}_\tau : [0,1] \to \mathbb{R}$ be the functions defined by*

$$\mathcal{S}_\tau(p, w) \equiv \frac{\mathcal{S}_{\tau-1}(p, w)\mathcal{M}_\tau(w)}{\mathcal{M}_\tau(w)(1-\lambda) + \mathcal{S}_{\tau-1}(p, w)\lambda} \ , \tag{6}$$

$$\mathcal{M}_\tau(w) \equiv w - \tau c_L \ , \tag{7}$$

with $\mathcal{S}_0(p, w) \equiv p$ and $\mathcal{M}_0(w) = w$. Then, if $w > \tau c_L$, $\mathcal{S}_\tau(p, w)$ is continuous and strictly increasing in both arguments.

Proof of Lemma 6. It is obvious when $\tau = 1$, and it holds when $\tau > 1$ by induction argument. $\square$

The meaning of the pull-back functions is the following. Fix an equilibrium and some $t > 0$ where the L-worker is still present. Then, using equation (1), (2) and the indifference condition $w_t = w_{t-1} + c_L$, we can obtain p_{t-1} and w_{t-1} from p_t and p_{t-1}. These take the form, respectively, of $\mathcal{S}_\tau(p_t, w_t)$ and $\mathcal{M}_\tau(w_t)$. If we apply this iteratively, we can find $p_{t-\tau} = \mathcal{S}_\tau(p_t, w_t)$ and $w_{t-\tau} = \mathcal{M}_\tau(w_t)$ for any $\tau \in \{1, ..., t\}$. So, since by Lemma 1 the L-worker is indifferent between dropping out or not in all periods except the last period, the pull-back functions give us the values of the belief sequences p and w for all periods prior to a given period. The following lemma formalizes this intuition:

Lemma 7. *For any equilibrium with $T > 1$ periods of education and any $T > \tau \geq \tau' \geq 0$ we have*

$$p_{\tau'} = \mathcal{S}_{\tau-\tau'}(p_\tau, w_\tau) \quad \textit{and} \quad w_{\tau'} = \mathcal{M}_{\tau-\tau'}(w_\tau) \ .$$

Proof of Lemma 7. Note that, by Lemma 1, in all periods $t < T - 1$, the L-worker is indifferent between dropping out or not and $s_t^H = \lambda$. This implies that if $t < T - 1$, $w_{t-1} = w_t - c_L$. We can use equations (1) and (2), with $s_t^H = \lambda$, to express the posterior at time t in terms of the posterior of worker in education and in the market at time $t + 1$:

$$p_t = \frac{p_{t+1} w_t}{w_t(1-\lambda) + p_{t+1}\lambda} = \frac{p_{t+1}(w_{t+1} - c_L)}{(w_{t+1} - c_L)(1-\lambda) + p_{t+1}\lambda} = \mathcal{S}_1(p_{t+1}, w_{t+1}) \ .$$

Using this formula recursively and the fact that $\mathcal{S}_\tau(p, w) = \mathcal{S}_{\tau-1}(\mathcal{S}_1(p, w), \mathcal{M}_1(w))$ we obtain the desired result. $\square$

A.4.1 Constructing the Upper Bound on the Length

Define the sequence $\tilde{p}_0^k \equiv \mathcal{S}_{k-1}(p_0^1, 1 - c_H)$, where p_0^1 is defined in (4). Our goal is to show that $\tilde{p}_0^k$ has the same properties as p_0^k (stated in the statement of the theorem), so $p_0^k = \tilde{p}_0^k$. We are going to prove first, by induction, that if $p_0 \in (\tilde{p}_0^{k+1}, \tilde{p}_0^k]$, then there is no equilibrium with more than k periods of education:

Step 1 (proof for $k = 0$ periods of education): By Lemma 3 there is no equilibrium with education for $p_0 > p_0^1$. Also, in the same proof, it is shown that for all equilibria in this region, $w_0 = p_0 \geq p_0^1 > 1 - c_H = \mathcal{M}_0(1 - c_H)$.

Step 2 (proof for $k = 1$ period of education): Assume that p_0 is such that there is an equilibrium with 1 period of education. Then, $w_0 \leq w_0^1 = 1 - c_H$ (at least the H-worker has to be willing to wait). Using Bayes' update (equations (1) and (2)) we can express $w_0 \equiv w_0(p_0, s_0^L, s_0^H)$

and $p_1 = p_1(p_0, s_0^L, s_0^H)$. Therefore, using these equations, we can write p_0 in terms of w_0, p_1 and s_0^H in the following way:

$$p_0 = p_{-1}(p_1, w_0, s_0^H) \equiv \frac{p_1 w_0}{w_0(1 - s_0^H) + p_1 s_0^H} \ .$$

The RHS of the previous expression is maximized when $s_0^H = \lambda$. Therefore, if an equilibrium ends with a length of two periods, the initial prior is at most $p_0^1 \equiv \frac{1-c_H}{1-c_H(1-\lambda)}$.

Step 3 (induction argument for $k > 1$): Here assume the induction hypothesis: assume that k is such that if $p_0 \in (\tilde{p}_0^k, \tilde{p}_0^{k-1}]$ there is no equilibrium with more than $k-1$ periods of education and, if an equilibrium has $k-1$ periods of education, then $w_0 \leq w_0^{k-1} \equiv \mathcal{M}_{k-2}(1-c_H)$.

Assume that p_0 is such that there exists some equilibrium with k periods of education. Denote the beliefs sequences for this equilibrium p and w. Note that, by the induction hypothesis, $p_1 \leq p_0^{k-1}$ and $w_1 \leq w_0^{k-1}$, since the continuation play after 1 is itself an equilibrium with initial prior p_1. Since $k > 2$, by Lemma 1, the H-worker is strictly willing to wait in period 0, so $s_0^H = \lambda$, and the L-worker randomizes in period 0. Then, $w_0 = w_1 - c_L \leq w_0^{k-1} - c_L = w_0^k$. Therefore, by Lemma 7, $p_0 = \mathcal{S}_1(p_1, w_1)$, and that this is increasing in both arguments. So, the maximum value it can take is $\tilde{p}_0^k \equiv \mathcal{S}_1(\tilde{p}_0^{k-1}, w_0^{k-1})$.

Step 4 (T^* is the limit): Note that T^* is such that

$$w_0^{T^*+1} \leq 0 < w_0^{T^*} \ .$$

Then, since $w_0^{T^*+1} \leq 0$, there is no equilibrium longer than T^* periods of education.

A graphical intuition of the proof can be found in Figure 4. It graphically represents both $\tilde{p}_0^T$ and w_0^T used in the proof.

A.4.2 Constructing L-equilibria

We prove a result similar to Theorem 1 about the set of equilibria where the L-worker is indifferent in all periods. For each $p_0 \in (0, 1)$, we use $\tilde{T}^L(p_0)$ to denote the maximum number of education periods of an equilibrium where the L-worker is indifferent to dropping out in all periods except (maybe) the last. We name these equilibria L-equilibria. The following lemma shows that, for any $p_0 \in (0, 1)$, there is a finite integer k such that, for each $T = 0, 1, ..., k$ there is an L-equilibrium that lasts for T periods of education, and no L-equilibrium with a length more than k.

Lemma 8. *Define $T^{**} \equiv \lceil \frac{1-c_L}{c_L} \rceil$, $p_0^{L,k} \equiv \mathcal{S}_k(1, 1)$ for $k = 0, ..., T^{**}$ and $p_0^{L,T^{**}+1} \equiv 0$. If $p_0 \in (p_0^{L,k+1}, p_0^{L,k}]$ for some $k = 0, ..., T^{**}$, then $\tilde{T}^L(p_0) = k$. Furthermore, for each $T \leq \tilde{T}^L(p_0)$, there is a unique L-equilibrium with T periods of education.*

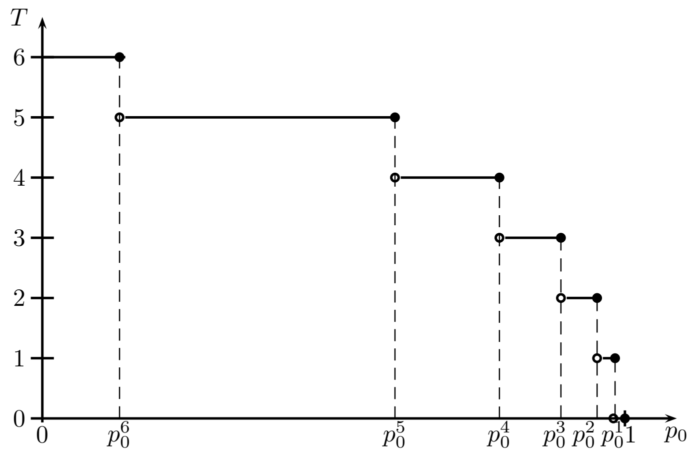


Figure 4: Maximum length of equilibria as a function of the prior p_0. As we see, this function is left continuous and decreasing.

Proof of Lemma 8. Fix some $p_0 \in (0, 1)$. If $p_0 > \mathcal{S}_k(1, 1)$ for some $k \leq T^{**}$ there is no L-equilibrium with k periods of education. Indeed, if there was one (ending at $p_k = w_k$), then $p_0 = \mathcal{S}_k(p_k, p_k)$. But since $\mathcal{S}_k(p_k, p_k)$ is strictly increasing in p_k and $p_0 > \mathcal{S}_T(1, 1)$, then $p_0 > \mathcal{S}_k(p, p)$ for all $p \in [0, 1]$. This is clearly a contradiciton. Note also that, in an L-equilibrium with T periods of education, $w_T - w_0 = Tc_L \leq 1$. Since $(T^{**} + 1)c_L > 1$, we have $\tilde{T}(p_0) < T^{**} + 1$,.

Fix $k < T^{**}$, $p_0 \in (p_0^{L,k+1}, p_0^{L,k}]$ and $T \leq k$. Note that $\mathcal{S}_T(p, p)$ is continuous and strictly increasing when $p > Tc_L$ for any $T \leq T^{**}$ and $\lim_{p \searrow Tc_L} \mathcal{S}_T(p, p) = 0$.[18] So, since $p_0 \leq \mathcal{S}_k(1, 1) \leq \mathcal{S}_T(1, 1)$, there exists a unique $p_T \in (Tc_L, 1)$ such that $p_0 = \mathcal{S}_T(p_T, p_T)$. Furthermore, there is an equilibrium with length T with $p_t = \mathcal{S}_{T-t}(p_T, p_T)$ and $w_t = \mathcal{M}_{T-t}(p_T)$. The argument for $k = T^{**}$ is analogous. $\square$

Lemma 9. *For any $k \leq T^{**}$, we have $p_0^{L,k} \in (\tilde{p}_0^{k+1}, \tilde{p}_0^k)$.*

Proof of Lemma 9. Note first that

$$\overbrace{\frac{1 - c_H}{1 - (1 - \lambda)c_H}}^{p_0^1} > \overbrace{\frac{1 - c_L}{1 - (1 - \lambda)c_L}}^{p_0^{1,L}} = \mathcal{S}_1(1, 1) > \mathcal{S}_1(p_0^1, 1 - c_H)$$

By definition, for $k > 1$, $p_0^k = \mathcal{S}_{k-1}(p_0^1, 1 - c_H) = \mathcal{S}_1(p_0^{k-1}, \mathcal{M}_{k-2}(1 - c_H))$ and $p_0^{k,L} = \mathcal{S}_{k-1}(p_0^{1,L}, 1 - c_L) = \mathcal{S}_1(p_0^{k-1}, \mathcal{M}_{k-2}(1 - c_L))$. Also, note that $\mathcal{M}_k(1 - c_H) > \mathcal{M}_k(1 - c_L) > \mathcal{M}_{k+1}(1 - c_H)$. Therefore, since $\mathcal{S}_1(\cdot, \cdot)$ is strictly increasing in both arguments, we have $p_0^{L,k} \in (\tilde{p}_0^{k+1}, \tilde{p}_0^k)$. $\square$

[18] Note that, if $T \leq T^*{*}$ then $Tc_L < 1$, and, by definition, $\mathcal{M}_T(Tc_L) = 0$. Using the definition of $\mathcal{S}_\tau(\cdot, \cdot)$, we have that $\mathcal{S}_T(c_L T, c_L T) = 0$.

A.4.3 Constructing H-equilibria

Lemma 8 implies that for any $p_0 \in (0,1)$, an L-equilibrium lasting for at most k periods can be constructed, where k satisfies that $p_0 \in (p_0^{L,k+1}, p_0^{L,k}]$. However, Lemma 9 shows that $p_0^{L,k} < \tilde{p}_0^k$. For $p_0 \in (p_0^{L,k}, \tilde{p}_0^k]$, there is no L-equilibrium lasting for k periods. The question now is whether there is any other equilibrium that lasts for k periods in this last region. Lemma 10 shows that the answer to this question is yes.

An equilibrium that lasts for $T > 0$ periods of education is an H-equilibrium if and only if, in equilibrium, the L-worker strictly prefers dropping out in period $T-1$. In other words, in an H-equilibrium $p_T = 1$. Note that each equilibrium is either an L-equilibrium or an H-equilibrium, and never both.

Lemma 10. *If $p_0 \in (p_0^{L,k}, \tilde{p}_0^k]$, there exists an H-equilibrium of length k, for $k \in \{1, ..., T^{**}\}$. If $p_0 \in (\tilde{p}_0^{k+1}, p_0^{L,k}]$, there exists an L-equilibrium of length k, for $k \in \{1, ..., T^* - 1\}$.*

Proof of Lemma 10. For $p_0 \in (\tilde{p}_0^{k+1}, p_0^{L,k}]$ the proof of the previous lemma tells us that there exists an L-equilibrium of length k. To prove the case $p_0 \in (p_0^{L,k}, \tilde{p}_0^k]$, we define the function $g : (p_0^{1,L}, p_0^1] \to (1 - c_L, 1 - c_H]$, which is given by

$$g(p) \equiv \frac{\lambda p}{\lambda p + 1 - p} \ .$$

Then for all $p_0 \in (p_0^{L,k}, p_0^k]$ there exists a unique $f(p_0) \in (p_0^{1,L}, p_0^1]$ such that $p_0 \equiv \mathcal{S}_{k-1}(f(p_0), g(f(p_0)))$. Indeed, we have that $\lim_{p \searrow p_0^{1,L}} g(p) = 1 - c_L$ and $g(p_0^1) = 1 - c_H$. So, we have

$$\lim_{p \searrow p_0^{L,1}} \mathcal{S}_{k-1}(p, g(p)) = p_1^{L,k} \quad \text{and} \quad \mathcal{S}_{k-1}(p_0^1, g(p_0^1)) = \tilde{p}_0^k \ .$$

Since $w(\cdot)$ is continuous and strictly increasing, $\mathcal{S}_{k-1}(\cdot, \cdot)$ is continuous in both arguments and strictly increasing, then there exists such $f(p_0)$, and is unique.

Let's construct one equilibrium with k education periods when $p_0 \in (p_0^{L,k}, \tilde{p}_0^k]$, for $k \leq T^* - 1$. Our claim is that it can be defined by $p_k = w_k = 1$, $p_t = \mathcal{S}_{t-1}(f(p_0), g(f(p_0)))$ and $w_t = g(f(p_0)) - c_L(k - t - 1)$, for $t \in \{0, ..., k-1\}$. To prove that, we show that the corresponding strategies are well defined. Note that, if the L-worker is indifferent in period 0, we have

$$s_t^L = \frac{1}{1 + \frac{(1-\lambda)(1-p_t)w_t}{\lambda p_t (1-w_t)}} = \frac{\lambda}{1 - \frac{(1-\lambda)(p_t - w_t)}{p_t(1-w_t)}} \ .$$

The first equality shows that $s_t^L < 1$. The second equality shows that, if $p_t^1 > w_t$, then $s_t^L > \lambda$, which is equivalent to $p_0^2 < p_0^1$, which is true as long as $w_t > 0$. Since, when $k < T^*$, $w_0 = g(f(p_0)) - c_L(k-1) > 0$, the result holds in this case.

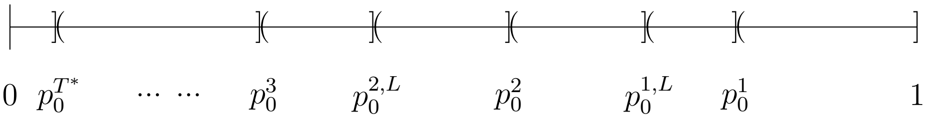


Figure 5: Partition construction

Finally, there are two possible cases. If $T^{**} = T^*$, we know from the previous lemma that there exists an L-equilibrium with length T^{**} in $(0, p_0^{L,T^*})$. If $T^{**} = T^* - 1$ then there exists some $p \in (p_0^{1,L}, p_0^1]$ such that $g(p) = T^{**}c_L$. Indeed, in this case $1 \leq T^* c_L < 1 - c_H + c_L$, so $T^{**}c_L \in (1 - c_H, 1 - c_L]$. Therefore, we can use the same argument as for $p_0 \in (p_0^{L,k}, \tilde{p}_0^k]$, for $k \leq T^* - 1$. The idea of the partition construction can be summarized in Figure 5. $\square$

Finally, note that the set $\{\tilde{p}_0^k\}_{k=0}^{T^*+1}$ is such that $\tilde{p}_0^k > \tilde{p}_0^{k+1}$ for all k. Furthermore, for all $0 \leq k \leq T^*$ and $0 \leq T \leq k$, if $p_0 \in (\tilde{p}_0^{k+1}, \tilde{p}_0^k]$, there exists an equilibrium with T periods of education and no equilibrium with a length larger than k. So, $p_0^k \equiv \tilde{p}_0^k$, for $k = 0, ..., T^* + 1$, satisfies the statement of Theorem 1, and therefore its proof is complete. *Q.E.D.*

A.5 The Proof of Lemma 4

We will do the proof by first fixing the maximum real time and solving for the corresponding p_0, and then showing that for all p_0 there exists a unique limit for the maximum real time. Fix $\bar{\kappa}^* \in (0, \frac{1}{\tilde{c}_L})$. In order to save notation, consider a strictly decreasing sequence Δ_n such that $\frac{\bar{\kappa}^*}{\Delta_n} \in \mathbb{N}$ for all $n \in \mathbb{N}$. Using Bayes' rule, we have the following equation relating $p_0^{\bar{\kappa}^*/\Delta_n, L}$ and $p_0^{\bar{\kappa}^*/\Delta_n - 1, L}$ (defined in Lemma 8):[19]

$$\frac{1}{p_0^{\bar{\kappa}^*/\Delta_n, L}} = \frac{\tilde{\lambda}\Delta_n}{1 - \tilde{c}_L \bar{\kappa}^*} + \frac{1 - \tilde{\lambda}\Delta_n}{p_0^{\bar{\kappa}^*/\Delta_n - 1, L}} = \sum_{m=0}^{\bar{\kappa}^*/\Delta_n} \frac{\tilde{\lambda}\Delta_n(1 - \tilde{\lambda}\Delta_n)^m}{1 - \tilde{c}_L(\bar{\kappa}^* - m\Delta_n)} + (1 - \tilde{\lambda}\Delta_n)^{\bar{\kappa}^*/\Delta_n} . \tag{8}$$

When Δ_n is small, each term of the sum can be approximated as follows

$$\frac{\tilde{\lambda}\Delta_n(1 - \tilde{\lambda}\Delta_n)^m}{1 - \tilde{c}_L(\bar{\kappa}^* - m\Delta_n)} = \frac{\tilde{\lambda}e^{-\tilde{\lambda}s}}{1 - \tilde{c}_L(\bar{\kappa}^* - s)}\Delta_n + O(\Delta_n^2)$$

where $s \equiv m\Delta_n$. The last term of the RHS of equation (8) satisfies $\lim_{n\to\infty}(1 - \tilde{\lambda}\Delta_n)^{\bar{\kappa}^*/\Delta_n} = e^{-\tilde{\lambda}\bar{\kappa}^*}$. Since each term in the sum is a bounded function (note that s ranges from 0 to $\bar{\kappa}^*$) multiplied by

[19] We use $p_0^{\bar{\kappa}^*, L}$ defined in Lemma 8 instead of $p_0^{\bar{\kappa}^*}$ for simplicity. Lemma 9 and the fact that $p_0^{\bar{\kappa}^*/\Delta_n} - p_0^{t/\Delta_n - 1} = O(\Delta_n)$ guarantee that $p_0^{\bar{\kappa}^*/\Delta_n, L}$ and $p_0^{\bar{\kappa}^*/\Delta_n}$ will be asymptotically equal.

Δ_n, at the limit $\Delta_n \searrow 0$ the sum converges to the integral, so we have

$$\frac{1}{\tilde{p}_0(\bar{\kappa}^*)} \equiv \lim_{n\to\infty} \frac{1}{p_0^{\bar{\kappa}^*/\Delta_n,L}} = e^{-\tilde{\lambda}\bar{\kappa}^*} + \int_0^{\bar{\kappa}^*} \frac{e^{-\tilde{\lambda}s}\tilde{\lambda}}{1-\tilde{c}_L(\bar{\kappa}^*-s)}ds \ .$$

Note that the RHS of the previous expression is equal to 1 when $\bar{\kappa}^* = 0$. Differentiating it with respect to $\bar{\kappa}^*$ we find, when $\bar{\kappa}^* > 0$,

$$\frac{d}{d\bar{\kappa}^*}\frac{1}{\tilde{p}_0(\bar{\kappa}^*)} = -e^{-\tilde{\lambda}\bar{\kappa}^*}\tilde{\lambda} + e^{-\tilde{\lambda}\bar{\kappa}^*}\tilde{\lambda} + \int_0^{\bar{\kappa}^*} \frac{e^{-\tilde{\lambda}s}\tilde{\lambda}}{(1-\tilde{c}_L(\bar{\kappa}^*-s))^2}ds > 0 \ .$$

Therefore, $p_0^{\bar{\kappa}^*} \in (0,1)$ when $\bar{\kappa}^* \in (0, \frac{1}{\tilde{c}_L})$.

Note that, for each $p_0 \in (0,1)$, there exists a unique $\bar{\kappa}^*$ such that $\tilde{p}_0(\bar{\kappa}^*) = p_0$. Indeed, $\lim_{\bar{\kappa}^*\to 0}\tilde{p}_0(\bar{\kappa}^*) = 1$, $\lim_{\bar{\kappa}^*\to 1/\tilde{c}_L}\tilde{p}_0(\bar{\kappa}^*) = 0$ and $\tilde{p}_0(\cdot)$ is strictly increasing in $(0, \frac{1}{\tilde{c}_L})$. Therefore, for each p_0 there exists a unique $\kappa^* \equiv \tilde{p}_0^{-1}(p_0)$ that satisfies the conditions of the lemma. It is given by the solution of

$$\frac{1}{p_0} = e^{-\tilde{\lambda}\kappa^*} + \int_0^{\kappa^*} \frac{e^{-\tilde{\lambda}s}\tilde{\lambda}}{1-\tilde{c}_L(\kappa^*-s)}ds \ . \tag{9}$$

Note that since for each set of parameters equilibria passing D1 are those with most periods of education, their real length is $\kappa^* + O(\Delta_n)$. *Q.E.D.*

A.6 The Proof of Theorem 2

Let's first prove that p_{T_n} converges to some $p(\kappa) \in [p_0, 1]$. Proceeding similarly as in the proof of Lemma 4, we use Bayes' rule to write:[20]

$$\frac{1}{p_{T_n-1}} = \frac{(1-\tilde{\lambda}\Delta_n)^{-1}}{p_{T_n-2}} - \frac{\tilde{\lambda}\Delta_n(1-\tilde{\lambda}\Delta_n)^{-1}}{w_{T_n-2}} = \frac{(1-\tilde{\lambda}\Delta_n)^{-(T_n-1)}}{p_0} - \sum_{m=1}^{T_n-1} \frac{\tilde{\lambda}(1-\lambda\Delta_n)^{-m}}{w_{T_n-2}-(m-1)\Delta\tilde{c}_L}\Delta_n \ .$$

Rearranging terms we have

$$\frac{1}{p_0} = \frac{e^{-\tilde{\lambda}\kappa}}{p_{T_n}^n} + \int_0^{\kappa} \frac{e^{\tilde{\lambda}(s-\kappa)}\tilde{\lambda}}{p_{T_n}^n - \tilde{c}_L(\kappa-s)}ds + O(\Delta_n) \ , \tag{10}$$

where we used that $w_{T_n-2}^n = p_{T_n-1}^n + O(\Delta_n)$. Note that the RHS of (10) is decreasing in $p_{T_n}^n$. Also, it is increasing in τ, since $w_0 < p_0$ and its derivative with respect to τ equals to

$$-\frac{\tilde{\lambda}}{p_0} + \frac{\tilde{\lambda}}{w_0} + O(\Delta_n) > 0 \ .$$

[20]This can be easily obtained by writing p_{T_n-1} and w_{T_n-2} in terms of p_{T_n-2}, and then using the fact that for all $t \le T_{n-2}$ we have $s_t^H = \tilde{\lambda}\Delta_n$.

So, the RHS of (10) is asymptotically lower than $\frac{1}{p_0}$ when $\kappa < \kappa^*$ and $p^n_{T_n} = 1$, since the equality holds for $\kappa = \kappa^*$. Also, if $\kappa > 0$ and $p^n_{T_n} = p_0$ the RHS of (10) is larger than p_0. Indeed, it would be equal to p_0 if $\kappa = 0$ and as we know the RHS of (10) is increasing in κ. Therefore, there exists a unique limit of $p^n_{T_n}$, and is strictly lower than 1 when $\kappa < \kappa^*$.

Note that the convergence of w_t is an immediate consequence of the fact that p_{T_n} is convergent. Indeed, we have that $\lim_{n\to\infty} w_{T_n} = \lim_{n\to\infty} p_{T_n} = p(\kappa)$. By Lemma 1, we have $w^n_{\lceil \tau/\Delta_n \rceil} = p^n_{T_n} - (\kappa - \tau)\tilde{c}_L + O(\Delta_n)$ or, in real time, $w(\tau) = p(\kappa) - (\kappa - \tau)\tilde{c}_L$.

To prove the convergence of p is given by just reproducing the same argument that we used in the beginning of this proof to prove the convergence of $p^n_{T_n}$. In this case, replacing T_n by $\lceil \tau/\Delta_n \rceil$, equation (10) is transformed into

$$\frac{1}{p_0} = \frac{e^{-\tilde{\lambda}\tau}}{p^n_{\lceil \tau/\Delta_n \rceil}} + \int_0^\tau \frac{e^{\tilde{\lambda}(s-\tau)}\tilde{\lambda}}{p^n_{T_n} - \tilde{c}_L(\kappa - s)} ds + O(\Delta_n) \ .$$

Therefore, we can solve for $p^n_{\lceil \tau/\Delta_n \rceil}$ in the previous expression in order to have the limit of the sequence $p^n_{\lceil \tau/\Delta_n \rceil}$. Finally, differentiating the previous expression with respect to τ gives us the desired expression (5).

Finally, using Bayes' rule, it is easy to verify that $m_t = \frac{p_t \lambda}{w_t}$ when $t < T_n$. Therefore, trivially, $\tilde{m}(\tau) = \frac{p(\tau)\tilde{\lambda}}{w(\tau)}$ for all $\tau \in (0, \kappa^*)$. *Q.E.D.*

A.7 The Proof of Theorem 3

From Theorem 2 we can differentiate $\tilde{m}(\tau)$ and we get

$$\tilde{m}'(\tau) = \tilde{\lambda} \frac{p'(\tau)w(\tau) - p(\tau)w'(\tau)}{w(\tau)^2} = -\tilde{\lambda} \frac{p(\tau)\big(\tilde{c}_L - (p(\tau) - w(\tau))\tilde{\lambda}\big)}{w(\tau)^2} \ .$$

Note that $\tilde{m}'(\tau)$ is positive only if $p(\tau) - w(\tau) \geq \frac{\tilde{c}_L}{\tilde{\lambda}}$. If we differentiate $p(\tau) - w(\tau)$ we have

$$\frac{d}{d\tau}\big(p(\tau) - w(\tau)\big) = \frac{p(\tau)(p(\tau) - w(\tau))\tilde{\lambda}}{w(\tau)} - \tilde{c}_L \ .$$

Since $p(\tau) - w(\tau)$ is continuous and $p(\kappa^*) - w(\kappa^*) = 0$, if $p(\tau) - w(\tau) \geq \frac{\tilde{c}_L}{\tilde{\lambda}}$ for some $\tau \in (0, \kappa^*)$ it must be the case that $p(\tau^*) - w(\tau^*) = \frac{\tilde{c}_L}{\tilde{\lambda}}$ for some $\tau^* \in (0, \kappa^*)$. This implies that

$$\frac{d}{d\tau}\big(p(\tau) - w(\tau)\big)\Big|_{\tau=\tau^*} = \frac{\tilde{c}_L^2}{\tilde{\lambda} w(\tau^*)} > 0 \ .$$

This clearly implies that such τ^* does not exist. Therefore, $\tilde{m}'(\tau) < 0$ for all $\tau \in (0, \kappa^*)$. *Q.E.D.*

A.8 The Proof of Theorem 4

First note that, in any equilibrium passing D1, the length of education (i.e. $\max\{T^L, T^H\}$) must be no lower than $\hat{T}$. Otherwise, if a worker deviates and drops out at $\max\{T^L, T^H\}+1$, for some $\varepsilon > 0$ small, he should be considered an H-worker, and so should receive a wage offer of $1 + (\max\{T^L, T^H\}+1)h$. Nevertheless, the wage at $\max\{T^L, T^H\}$ is no larger than $1+\max\{T^L, T^H\}h$, so the deviation is profitable for the H-worker.

Let's assume $T^L < \hat{T}$.[21] In any equilibrium passing D1, the H-worker do not voluntarily drop out in period t where $T^L \le t < \hat{T}$. The reason is that when $t > T^L$, $p_t = 1$, so for the H-worker, the marginal return to education is h which is greater than the marginal cost c_H. In the period T^L, we have that $s^L_{T_L} = 1$, which implies

$$\hat{p}_{T^L} = \frac{p_{T^L}\lambda}{p_{T^L}\lambda + 1 - p_{T^L}} \quad \Rightarrow \quad p_{T_L} = \frac{\hat{p}_{T^L}}{\hat{p}_{T^L}(1-\lambda)+\lambda} \equiv f(\hat{p}_{T^L}) \ .$$

Note that $\lim_{\hat{p}_{T^L}\to 1} f(\hat{p}_{T^L}) = 1$. Furthermore, note that $\hat{p}_{T_L}$ needs to be such that the L-worker wants to drop out, so

$$\hat{p}_{T^L} + hT^L \ge 1 + h(T^L+1) - c_L \quad \Rightarrow \quad \hat{p}_{T^L} \ge 1 + h - c_L \ .$$

So, using a technique similar to the one used in Theorem 1, for each $\hat{p} \in [1+h-c_L, 1)$ we can construct a sequence of p and $\hat{p}$ using the pull-back functions defined in Lemma 6, now with $c_L - h$ instead of c_L in equation (7). Indeed, proceeding similarly, it is easy to show that for every $\hat{p} \in [1+h-c_L, 1)$ the sequence $(p^{\hat{p}}_\tau \equiv \mathcal{S}_\tau(f(\hat{p}), \hat{p} - c_L + h))_\tau$ is such that for any $p^{\hat{p}}_\tau \neq p^{\hat{p}'}_{\tau'}$ for all $\hat{p} \neq \hat{p}' \in [1+h-c_L, 1)$ and $\tau, \tau' \in \mathbb{N}$. Also, given that $\mathcal{S}_1(1, 1-c_L+h) = f(1-c_L+h)$, it is easy to show (proceeding similarly to Lemma 10) that for all $p_0 \in (0,1)$ there exists a unique $\hat{p} \in [1+h-c_L, 1)$ and $\tau \ge 0$ such that $p_0 = \mathcal{S}_\tau(f(\hat{p}), \hat{p} - c_L + h)$. *Q.E.D.*

[21] This is true, for example, if $1 + \hat{T}h < \hat{T}c_L$, that is, if L-worker prefer to drop out at 0 than wait until $\hat{T}$.

B Appendix: Type-Dependent Dropout Risk

Here we consider a model in which a worker's dropout rate is correlated with his productivity. It turns out that our predictions in Section 2 are robust. There are three relevant cases: (1) $\lambda_H > \lambda_L \geq 0$, (2) $\lambda_L > \lambda_H > 0$, and (3) $\lambda_L \geq \lambda_H = 0$.

B.1 $\lambda_H > \lambda_L \geq 0$ Case

The first case we consider is $\lambda_H > \lambda_L \geq 0$; that is, the H-worker exogenously drops out at a higher rate than the L-worker. The following lemma implies that the equilibrium set in this case coincides with the base model when $\lambda = \lambda_H$:

Lemma 11. *Assume that $\lambda_H > \lambda_L \geq 0$. Then, (s^L, s^H, w, p) is an equilibrium if and only if it is also an equilibrium in the benchmark model with $\lambda = \lambda_H$.*

Proof. We first prove that Lemma 1 (which holds when $\lambda_H = \lambda_L$) is still valid when $\lambda_H \geq \lambda_L$. Let T be the last period t before T^L where $s_t^L \leq s_t^H$. In this case

$$p_{T+1} \leq p_T \leq w_T \ .$$

Furthermore, since the L-worker is voluntarily dropping out at time $T+1$, this implies $w_T \leq w_{T+1} - c_L$. Nevertheless, since $s_{T+1}^L \geq s_{T+1}^H$, we have $w_{T+1} \geq p_{T+1}$, which is a contradiction, since

$$w_{T+1} \leq p_{T+1} \leq p_T \leq w_T \leq w_{T+1} - c_L \ .$$

So, when $\lambda_H \geq \lambda_L$, it is still true that $s_t^L > s_t^H$ in all periods of all equilibria before T^L. Therefore, relaxing of the constraint $\lambda_L = \lambda_H = \lambda$ to $\lambda_L \leq \lambda_H = \lambda$ does not introduce new equilibria. Trivially, it does not destroy any equilibria, since in the model $\lambda_L = \lambda_H = \lambda$, in all equilibria, $s_t^L > \lambda$ for all equilibria and period $t \leq T^L$. □

The intuition behind this lemma is that, in our original model, by Lemma 1, the endogenous dropout rate of the L-worker is positive in all periods before (maybe) the last. So, the constraint $s_t^L \geq \lambda$ was never binding in equilibrium. Therefore, all equilibria from the base model for $\lambda = \lambda_H$ are also equilibria for the case $\lambda_H > \lambda_L \geq 0$. On the other hand, for any equilibrium in the case where $\lambda_H > \lambda_L$, let $\tilde{\alpha}_t^L$ denote the low type's strategy. It must be true that $\tilde{s}_t^L \geq \lambda_H - \lambda_L$. Define $\hat{s}_t^L = \tilde{s}_t^L - (\lambda_H - \lambda_L) \geq 0$. One can easily verify that $\hat{s}_t^L$ can be supported in an equilibrium of the game with a symmetric exogenous dropout rate, $\lambda = \lambda_H$.

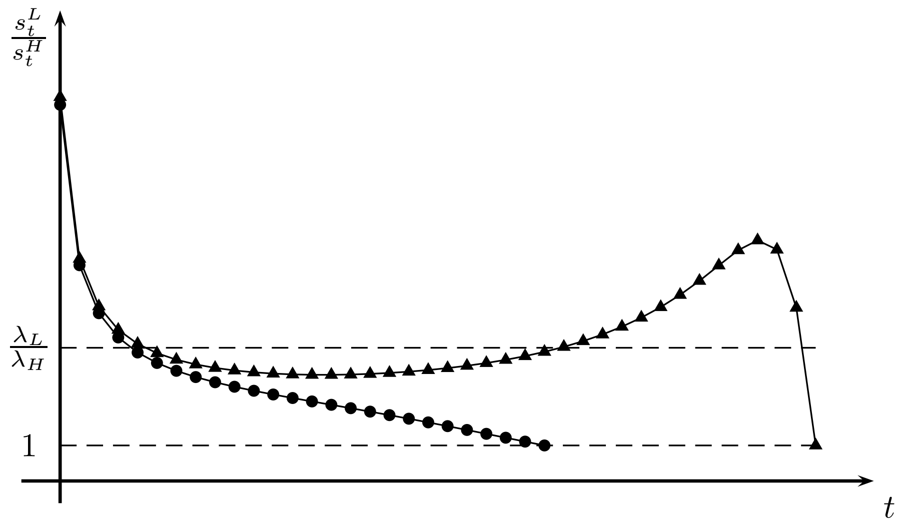


Figure 6: Endogenous dropout rate of the L-worker

B.2 $\lambda_L > \lambda_H > 0$ Case

As we can see in Figure 6, s^L may be non-monotone. In particular, there are some equilibria where it is initially decreasing and then increasing and finally it goes down again. Now, s^L is restricted to be no lower than $\lambda_L > \lambda_H$. We may guess that this constraint will be potentially binding in two connected regions, one for large w and the other for intermediate values. In any equilibrium, when this constraint is binding, both types strictly prefer to wait. Different from the benchmark model, the equilibrium belief p_t still goes up since $\lambda_L > \lambda_H$. After some periods, the constraint may become not binding anymore, and the L-worker starts to play a mixed strategy again. However, the equilibrium characterization in the benchmark model can not survive for some parameters. Fortunately, the following theorem shows that the equilibrium characterization in the benchmark model still works when λ_L is not significantly larger than λ_H.

In other to compare models, we denote the set of parameters as $(p_0, c_L, c_H, \lambda_L, \lambda_H)$. Note that in our original problem the set of parameters is $(p_0, c_L, c_H, \lambda, \lambda)$.

Theorem 5. *Given a set of parameters for our original model $(p_0, c_L, c_H, \lambda, \lambda)$ there exists $\varepsilon > 0$ such that the sets of equilibria of all model with parameters $(p_0, c_L, c_H, \lambda_L, \lambda_H)$ with $\lambda_H = \lambda$ and $\lambda_L = (\lambda, \lambda + \varepsilon]$ are identical.*

Proof. We first prove that if $\varepsilon > 0$ is small enough, the model parameters $(p_0, c_L, c_H, \lambda + \varepsilon, \lambda)$ does not have more equilibria than the model with parameters $(p_0, c_L, c_H, \lambda, \lambda)$. Note that Lemma 5 still holds (the H-worker can imitate the strategy of the L-worker). Now we try to prove a result analogous to Lemma 1. Assume that the L-worker is not voluntarily dropping out in period t, so his dropout rate is $\lambda + \varepsilon$. We then have two cases:

1. First assume that the dropout rate of the H-worker is larger than $\lambda + \varepsilon$, so $w_t > p_t$. In this case, since Lemma 5 is still valid (i.e. $V_t^L \leq p_t$) the L-worker strictly wants to drop out, so we obtain again a contradiction.

2. Assume now that $s^H \in [\lambda, \lambda + \varepsilon]$. In this case $p_{t+1} = p_t + O(\varepsilon)$ and $w_t = p_t + O(\varepsilon)$, so $w_t - p_{t+1} = O(\varepsilon)$. Then, using the same logic as in the proof of Lemma 1, we have

$$w_t \leq W_t^L \leq V_{t+1}^L - c_L \leq p_{t+1} - c_L \ .$$

   Therefore, $w_t - p_{t+1} \leq -c_L$. But this is inconsistent with $w_t - p_{t+1} = O(\varepsilon)$.

That proves that, if $\varepsilon > 0$ is small enough, the model with $\lambda_H = \lambda$ and $\lambda_L = \lambda + \varepsilon$ does not have more equilibria than for the case $\varepsilon = 0$.

Let's prove the converse, that is, that if $\varepsilon > 0$ is small enough model with parameters $(p_0, c_L, c_H, \lambda, \lambda)$ does not have more equilibria than the model with parameters $(p_0, c_L, c_H, \lambda+\varepsilon, \lambda)$. Assume by contradiction that there exists a strictly decreasing sequence $\{\varepsilon_n > 0\}_{n \in \mathbb{N}}$ converging to 0 such that, for each n, there exists an equilibrium in the model with parameters $(p_0, c_L, c_H, \lambda, \lambda)$ and some t_n reached with positive probability such that $s^L_{t_n} \in [\lambda, \lambda + \varepsilon_n)$. This implies $p_{t_n+1} = p_{t_n} + O(\varepsilon_n)$ and $w_{t_n} = p_{t_n} + O(\varepsilon_n)$, so $w_{t_n} - p_{t_n+1} = O(\varepsilon_n)$.[22] So,

$$w_{t_n} = W^L_{t_n} = V^L_{t_n+1} - c_L \leq p_{t_n+1} - c_L \ .$$

This, again, is a contradiction. □

B.3 $\lambda_L \geq \lambda_H = 0$ Case

In this case, there is no exogenous drop out by the H-worker. Our main mechanism in the benchmark model is not present here. Indeed, in our benchmark model, as is proven in Lemma 1, the L-worker uses the fact that the H-worker exogenously drops out to mimic him in order to save the high cost of education. Since the H-worker exogenously drops out, early dropout cannot be punished too much, constraining the belief threats by the firms. This is no longer true when $\lambda_H = 0$, so the set of equilibria is qualitatively different from the $\lambda_H > 0$ case.

[22]Using some abuse of notation, p_{t_n} and w_{t_n} denote the corresponding posteriors in the n-th equilibrium of the sequence.

C Appendix: Multiple Types

Now we consider the $N > 2$ types case in which $\theta \in \{1, 2, 3, ..., N\}$ with a prior p_0^θ, where $\sum_{\theta=1}^N p_0^\theta = 1$. The type θ worker has a cost of waiting c^θ, $c^\theta > c^{\theta+1}$. The productivity of θ is Y^θ, $Y^\theta < Y^{\theta+1}$. All types exogenously drop out with probability λ.

The equilibrium concept is the same as in Definition 1 but adapted to the fact that now we have many types. Note that firms' offers depend only on the expected productivity and not on other moments of the productivity distribution. This fact helps us to keep our definition simple:

Definition 2. *An equilibrium is a strategy profile $(s^\theta)_{\theta=1,...,N}$, a wage process w and N belief sequences p^θ, for $\theta = 1, ..., N$, such that:*

1. *the θ-worker chooses $s^\theta \in [\lambda, 1]$ to maximize her expected payoff given w;*
2. *if a worker drops out with education t, firms offer*
$$w_t = \frac{\sum_{\theta=1}^N p_t^\theta s_t^\theta Y^\theta}{\sum_{\theta=1}^N p_t^\theta s_t^\theta} \ ; \quad \text{and} \tag{11}$$
3. *when it is well defined, p_t^θ is updated according to the Bayes' rule*
$$p_{t+1}^\theta = \frac{p_t^\theta(1 - s_t^\theta)}{\sum_{\theta'=1}^N p_t^{\theta'}(1 - s_t^{\theta'})} \ . \tag{12}$$

Let T^θ be the last time the θ-worker is in school. The following theorem shows that our insight into the binary-type model can be easily extended to a multiple-types model.

Theorem 6. *Under the previous assumptions, in any equilibrium:*

1. *in each period t, there is at most one type, indifferent to dropping out,*
2. *more productive types stay longer in education, $T^\theta \leq T^{\theta+1}$,*
3. *there is positive voluntary dropout in all periods, and*
4. *the wage w_t is concave in t.*

Proof. 1. Assume that, in period t, there are two types $\theta_1, \theta_2 \in \Theta$, with $c^{\theta_1} < c^{\theta_2}$, and both are indifferent between dropping out or not. Let τ_1 and τ_2 denote, respectively, the stopping times of the continuation strategies that make players indifferent on dropping out or not.[23] Then, we have

$$w_t = \mathbb{E}[w_{\tau_{\theta_2}} - c^{\theta_2}\tau_{\theta_2}] \geq \mathbb{E}[w_{\tau_{\theta_1}} - c^{\theta_2}\tau_{\theta_1}] > \mathbb{E}[w_{\tau_{\theta_1}} - c^{\theta_1}\tau_{\theta_1}] = w_t \ .$$

The first (weak) inequality is from the optimality of the θ_2-worker. The strict inequality is because $\mathbb{E}[\tau_{\theta_1}] > 0$ and $c^{\theta_1} < c^{\theta_2}$. The equalities come from the fact that i-worker with $i \in \{1, 2\}$ are indifferent between dropping out (and getting w_t) or staying and following τ_i. Therefore, we have a contradiction.

2. Assume there exists $\theta_1, \theta_2 \in \Theta$ such that $\theta_1 < \theta_2$ and $T^{\theta_1} > T^{\theta_2}$. Let τ_{θ_1} be the stopping time of the continuation strategy after T^{θ_2}, given by the strategy of θ_1. Then, note that

$$w_{T^{\theta_2}} \geq \mathbb{E}[w_{\tau_{\theta_1}} - c^{\theta_2}\tau_{\theta_1}] > \mathbb{E}[w_{\tau_{\theta_1}} - c^{\theta_1}\tau_{\theta_1}] \geq w_{T^{\theta_2}} \ .$$

This is clearly a contradiction. The first inequality comes from the optimality of the θ_2-worker choosing to drop out at T^{θ_2} (since they could deviate to mimic the θ_1-worker). The second inequality is given by the fact that since $\theta_1 < \theta_2$, $c^{\theta_2} < c^{\theta_1}$ and since $T^{\theta_1} > T^{\theta_2}$, $\mathbb{E}[\tau_{\theta_1}] > 0$. The last inequality comes from the optimality of the θ_1-worker choosing to drop out at $T^{\theta_1} > T^{\theta_2}$ (since they could deviate to mimic the θ_2-worker).

3. Define $\Theta_t = \{\theta | T^\theta \geq t\}$ and $\theta_t = \min\{\Theta_t\}$. We proceed as in the proof of Lemma 5. Now we have

$$\begin{aligned}
\mathbb{E}_t[w_\tau | \tau \geq t] &= \sum_{\tau=t}^{\infty} \Pr(\tau, t) w_\tau = \sum_{\tau=t}^{\infty} \Pr(\tau, t) \frac{\sum_\theta Y^\theta s_\tau^\theta p_t^\theta \Pr{}^\theta(\tau, t)}{\Pr(\tau, t)} \\
&= \sum_\theta p_t^\theta Y^\theta \sum_{\tau=t}^{\infty} s_\tau^\theta \Pr{}^\theta(\tau, t) = \sum_\theta p_t^\theta Y^\theta \ ,
\end{aligned}$$

where $\Pr(\tau, t)$ and $\Pr{}^\theta(\tau, t) = s_\tau^\theta \prod_{t'=t}^{\tau-1}(1 - s_{t'}^\theta)$ are defined as in the proof of Lemma 5.

Note that, by the previous result,

$$\sum_{\theta=\theta_t}^{N} p_t^\theta V_t^\theta = \mathbb{E}_t[w_\tau | \tau \geq t] - \sum_{\theta=\theta_t}^{N} p_t^\theta c^\theta \tau^\theta(t) < \mathbb{E}_t[w_\tau | \tau \geq t] \ ,$$

where $\tau^\theta(t)$ is the stopping time for the θ-worker conditional on reaching t. Since $V_t^\theta \leq V_t^{\theta+1}$ (since the $(\theta + 1)$-worker can mimic the θ-worker at a lower cost), and $\sum_{\theta=\theta_t}^{N} p_t^\theta = 1$ we have that $V_t^{\theta_t} < w_t$.

[23] For this proof, for a given strategy, it is convenient to use the random variable τ, which gives the duration of the game.

Assume that in period t there is no voluntary dropout. In this case, $w_t = \sum_\theta p_t^\theta Y_t^\theta$. Since we just showed $V_{\theta_t} < \sum_\theta p_t^\theta Y_t^\theta$, the θ_t-worker strictly prefers to drop out, which is a contradiction.

4. Note that, by part 3 of this theorem, we have that $w_{t+1} - c^{\theta_t} \leq w_t$. Furthermore, $w_{t+1} - c^{\theta_{t+1}} \geq w_t$. This implies that $w_{t+1} - w_{t+1} \in [c^{\theta_{t+1}}, c^{\theta_t}]$. Since c^θ is decreasing in θ and, by part 2 of this theorem, the θ_t-worker is (weakly) increasing in t, w_t is concave in t.

□

Most features of the two-type model are preserved. However, note that under many types we have decreasing returns to education instead of linear ones, since lower types are skimmed out before higher types in equilibria. This pattern of decreasing returns to education is consistent with many empirical studies, for example, Frazis (2002), Habermalz (2003), Heckman *et al.* (2008) and Manoli (2008). The equilibrium construction in multiple-type models is almost identical to that in the two-type model and thus is omitted.

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