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by

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Cautious Expected Utility and the Certainty Effect*

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Abstract

Many violations of the Independence axiom of Expected Utility can be traced to subjects' attraction to risk-free prospects. The key axiom in this paper, Negative Certainty Independence (Dillenberger, 2010), formalizes this tendency. Our main result is a utility representation of all preferences over monetary lotteries that satisfy Negative Certainty Independence together with basic rationality postulates. Such preferences can be represented as if the agent were unsure of how to evaluate a given lottery p; instead, she has in mind a set of possible utility functions over outcomes and displays a cautious behavior: she computes the certainty equivalent of p with respect to each possible function in the set and picks the smallest one. The set of utilities is unique in a well-defined sense. We show that our representation can also be derived from a 'cautious' completion of an incomplete preference relation.

JEL: *D80, D81*

Keywords: *Preferences under risk, Allais paradox, Negative Certainty Independence, Incomplete preferences, Cautious Completion, Multi-Utility representation.*

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1 Introduction

Despite its ubiquitous presence in economic analysis, the paradigm of Expected Utility is often violated in choices between risky prospects. While such violations have been documented in many different experiments, a specific preference pattern emerges as one of the most prominent: the tendency of people to favor certain (risk-free) options– the so-called Certainty Effect (Kahneman and Tversky, 1979). This is shown, for example, in the classic Common Ratio Effect, one of the Allais paradoxes, in which subjects face the following two choice problems:

1. A choice between A and B, where A is a degenerate lottery which yields \$3000 for sure and B is a lottery that yields \$4000 with probability 0.8 and \$0 with probability 0.2.

2. A choice between C and D, where C is a lottery that yields \$3000 with probability 0.25 and \$0 with probability 0.75, and D is a lottery that yields \$4000 with probability 0.2 and \$0 with probability 0.8.

The typical result is that the vast majority of subjects choose A in problem 1 and D in problem 2,[1] in violation of Expected Utility and in particular of its key postulate, the Independence axiom. To see this, note that prospects C and D are the 0.25:0.75 mixture of prospects A and B, respectively, with the lottery that gives \$0 for sure. This means that the only pairs of choices consistent with Expected Utility are (A, C) and (B, D). Additional evidence of the Certainty Effect includes, among many others, Allais' Common Consequence Effect, as well as the experiments of Cohen and Jaffray (1988), Conlisk (1989), Dean and Ortoleva (2012b), and Andreoni and Sprenger (2012).[2]

Following these observations, Dillenberger (2010) suggests a way to define the Certainty Effect behaviorally, by introducing an axiom, called *Negative Certainty Independence* (NCI), that is designed precisely to capture this tendency. To illustrate, note that in the example above, prospect A is a risk-free option and it is chosen by most subjects over the risky prospect B. However, once both options are mixed,

[1] This example is taken from Kahneman and Tversky (1979). Of 95 subjects, 80% choose A over B, 65% choose D over C, and more than half choose the pair A and D. These findings have been replicated many times (see footnote 2).

[2] A comprehensive reference to the evidence on the Certainty Effect can be found in Peter Wakker's annotated bibliography, posted at http://people.few.eur.nl/wakker/refs/webrfrncs.doc.

leading to problem 2, the first option is no longer risk-free, and its mixture (C) is now judged worse than the mixture of the other option (D). Intuitively, after the mixture one of the options is no longer risk-free, which reduces its appeal. Following this intuition, axiom NCI states that for any two lotteries p and q, any number λ in $[0,1]$, and any lottery δ_x that yields the prize x for sure, if p is preferred to δ_x then $\lambda p + (1-\lambda)\,q$ is preferred to $\lambda\delta_x + (1-\lambda)\,q$. That is, if the sure outcome x is not enough to compensate the decision maker (henceforth DM) for the risky prospect p, then mixing it with any other lottery, thus eliminating its certainty appeal, will not result in the mixture of δ_x being more attractive than the corresponding mixture of p. NCI is weaker than the Independence axiom, and in particular it permits Independence to fail when the Certainty Effect is present – allowing the DM to favor certainty, but ruling out the converse behavior. For example, in the context of the Common Ratio questions above, NCI allows the typical choice (A, D), but is not compatible with the opposite violation of Independence, (B, C). It is easy to see how NCI is also consistent with most other experiments that document the Certainty Effect.

The goal of this paper is to characterize the class of continuous, monotone, and complete preference relations, defined on lotteries over some interval of monetary prizes, that satisfy NCI. That is, we aim to characterize a new class of preferences that are consistent with the Certainty Effect, together with very basic rationality postulates. A characterization of all preferences that satisfy NCI could be useful for a number of reasons. First, it provides a way of categorizing some of the existing decision models that can accommodate the Certainty Effect (see Section 5 for details). Second, a representation for a very general class of preferences that allow for the Certainty Effect can help to delimit the potential consequences of such violations of Expected Utility. For example, a statement that a certain type of strategic or market outcomes is not possible for any preference relation satisfying NCI would be easier to obtain given a representation theorem. Lastly, while Dillenberger (2010) did not provide a utility representation of the preferences that satisfy NCI, he showed that this property is linked not only to the Certainty Effect, but also to behavioral patterns in dynamic settings, such as preferences for one-shot resolution of uncertainty. Hence, characterizing this class will have implications also in different domains of choice.

Our main result is that any continuous, monotone, and complete preference relation over monetary lotteries satisfies NCI if and only if it can be represented as follows: there exists a set $\mathcal{W}$ of strictly increasing (Bernoulli) utility functions over outcomes, such that the value of any lottery p is given by

$$V(p) = \inf_{v \in \mathcal{W}} c(p, v),$$

where $c(p, v)$ is the certainty equivalent of lottery p calculated using the utility function v. That is, if we denote by $\mathbb{E}_p(v)$ the expected utility of p with respect to v, then $c(p, v) = v^{-1}(\mathbb{E}_p(v))$.

We call this representation a Cautious Expected Utility representation and interpret it as follows. The DM acts as if she were unsure how to exactly evaluate each given lottery: she does not have one, but a set of possible utility functions over monetary outcomes. She then reacts to this multiplicity using a form of *caution*: she evaluates each lottery according to the lowest possible certainty equivalent corresponding to some function in the set.[3] The use of certainty equivalents allows the DM to compare different evaluations of the same lottery, each corresponding to a different utility function, by bringing them to the same "scale" – dollar amounts. Note that, by definition, $c(\delta_x, v) = x$ for all $v \in \mathcal{W}$. Therefore, while the DM acts with caution when evaluating general lotteries, such caution does not play a role when evaluating degenerate ones. This leads to the Certainty Effect.

While the representation theorem discussed above characterizes a complete preference relation, the interpretation of Cautious Expected Utility is linked with the notion of a completion of incomplete preferences. Consider a DM who has an incomplete preference relation over lotteries, which is well-behaved (that is, a reflexive, transitive, monotone, and continuous binary relation that satisfies Independence). Suppose that the DM is asked to choose between two options that the original preference relation is unable to compare; she then needs to choose a rule to complete her ranking. While there are many ways to do so, the DM may like to follow what we call a Cautious Completion: if the original relation is unable to compare a lottery p with a degenerate lottery δ_x, then in the completion the DM opts for the latter – "when in doubt, go with certainty." Theorem 5 shows that there always exists a unique Cautious Completion, which admits a Cautious Expected Utility representation.[4] That is, it shows

[3] As we discuss in Section 5, this interpretation of cautious behavior is also provided in Cerreia-Vioglio (2009).

[4] In fact, we will show that it admits a Cautious Expected Utility representation with the same set of utilities that can be used to represent the original incomplete preference relation (in the sense

that Cautious Completions must generate preferences that satisfy NCI – leading to another way to interpret this property.

We conclude by assessing the empirical performance of our model and its relation with existing literature. We will argue that not only our model suggests a useful way of interpreting existing empirical evidence and derives new theoretical predictions, but it can also accommodate some additional evidence on the Certainty Effect (e.g., the presence of Allais-type behavior with large stakes but not with small ones), which poses difficulties to many popular alternative models.

The remainder of the paper is organized as follows. Section 2 presents the axiomatic structure, states the main representation theorem, and discusses the uniqueness properties of the representation. Section 3 characterizes risk attitudes and comparative risk aversion. Section 4 presents the result on the completion of incomplete preference relations. Section 5 surveys related theoretical models. Section 6 discusses experimental evidence. Section 7 concludes. All proofs appear in the Appendices.

2 The Model

2.1 Framework

Consider a compact interval $[w, b] \subset \mathbb{R}$ of monetary prizes. Let Δ be the set of lotteries (Borel probability measures) over $[w, b]$, endowed with the topology of weak convergence. We denote by x, y, z generic elements of $[w, b]$ and by p, q, r generic elements of Δ. We denote by $\delta_x \in \Delta$ the degenerate lottery (Dirac measure at x) that gives the prize $x \in [w, b]$ with certainty. The primitive of our analysis is a binary relation $\succcurlyeq$ over Δ. The symmetric and asymmetric parts of $\succcurlyeq$ are denoted by $\sim$ and $\succ$, respectively. The certainty equivalent of a lottery $p \in \Delta$ is a prize $x_p \in [w, b]$ such that $\delta_{x_p} \sim p$.

We start by imposing the following basic axioms on $\succcurlyeq$.

Axiom 1 (Weak Order)**.** *The relation $\succcurlyeq$ is complete and transitive.*

Axiom 2 (Continuity)**.** *For each $q \in \Delta$, the sets $\{p \in \Delta : p \succcurlyeq q\}$ and $\{p \in \Delta : q \succcurlyeq p\}$ are closed.*

Axiom 3 (Weak Monotonicity)**.** *For each $x, y \in [w, b]$, $x \geq y$ if and only if $\delta_x \succcurlyeq \delta_y$.*

of the Expected Multi-Utility representation of Dubra et al. (2004)).

The three axioms above are standard postulates. Weak Order is a common assumption of rationality. Continuity is needed to represent $\succcurlyeq$ through a continuous utility function. Finally, under the interpretation of Δ as monetary lotteries, Weak Monotonicity simply implies that more money is better than less.

2.2 Negative Certainty Independence (NCI)

We now discuss the axiom which is the core assumption of our work. As we have previously mentioned, a bulk of evidence against Expected Utility arises from experiments in which one of the lotteries is degenerate, that is, yields a certain prize for sure. For example, recall Allais' Common Ratio Effect: subjects choose between A and B, where $A = \delta_{3000}$ and $B = 0.8\delta_{4000} + 0.2\delta_0$. They also choose between C and D, where $C = 0.25\delta_{3000} + 0.75\delta_0$ and $D = 0.2\delta_{4000} + 0.8\delta_0$. The typical finding is that the majority of subjects tend to *systematically* violate Expected Utility by choosing the pair A and D. Kahneman and Tversky (1979) called this pattern of behavior the Certainty Effect. The next axiom, introduced in Dillenberger (2010), captures the Certainty Effect with the following relaxation of the Independence axiom.[5]

Axiom 4 (Negative Certainty Independence)**.** *For each $p, q \in \Delta$, $x \in [w, b]$, and $\lambda \in [0, 1]$,*

$$p \succcurlyeq \delta_x \;\Rightarrow\; \lambda p + (1-\lambda)q \succcurlyeq \lambda\delta_x + (1-\lambda)q. \qquad \text{(NCI)}$$

The NCI axiom states that if the sure outcome x is not enough to compensate the DM for the risky prospect p, then mixing it with any other lottery, thus eliminating its certainty appeal, will not result in the mixture of x being more attractive than the corresponding mixture of p. In particular, x_p, the certainty equivalent of p, might not be enough to compensate for p when part of a mixture.[6] In this sense NCI captures the Certainty Effect. When applied to the Common Ratio experiment, NCI only posits that if B is chosen in the first problem, then D must be chosen in the second one. Specifically, it allows the DM to choose the pair A and D, in line with the typical pattern of choice. Coherently with this interpretation, NCI captures the Certainty Effect as defined by Kahneman and Tversky (1979) – except that, as opposed to the

[5]Recall that a binary relation $\succcurlyeq$ satisfies *Independence* if and only if for each $p, q, r \in \Delta$ and for each $\lambda \in (0, 1]$, we have $p \succcurlyeq q$ if and only if $\lambda p + (1-\lambda)\, r \succcurlyeq \lambda q + (1-\lambda)\, r$.

[6]We show in Appendix A (Proposition 5) that our axioms imply that $\succcurlyeq$ preserves First Order Stochastic Dominance and thus for each lottery $p \in \Delta$ there exists a unique certainty equivalent x_p.

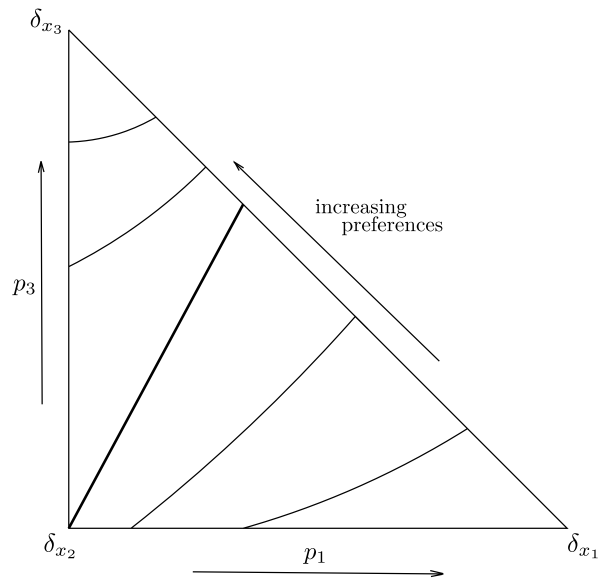

Figure 1: NCI and the Marschak-Machina triangle

latter, NCI applies also to lotteries with three or more possible outcomes, and can thus be applied to broader evidence such as Allais' Common Consequence Effect.[7]

Besides capturing the Certainty Effect, NCI has static implications that put additional structure on preferences over Δ. For example, NCI (in addition to the other basic axioms) implies Convexity: for each $p, q \in \Delta$, if $p \sim q$ then $\lambda p + (1-\lambda)\, q \succcurlyeq q$ for all $\lambda \in [0,1]$. To see this, assume $p \sim q$ and apply NCI twice to obtain that $\lambda p + (1-\lambda)\, q \succcurlyeq \lambda \delta_{x_p} + (1-\lambda)\, q \succcurlyeq \lambda \delta_{x_p} + (1-\lambda)\, \delta_{x_q} = \delta_{x_q} \sim q$. NCI thus suggests weak preference for randomization between indifferent lotteries. Furthermore, note that by again applying NCI twice we obtain that $p \sim \delta_x$ implies $\lambda p + (1-\lambda)\, \delta_x \sim p$ for all $\lambda \in [0,1]$, which means neutrality towards mixing a lottery with its certainty equivalent.

To further illustrate the restrictions on preferences imposed by NCI, it will be useful to discuss its implications on the shape of indifference curves in any Marschak-Machina triangle, which represents all lotteries over fixed three outcomes $x_3 > x_2 > x_1$ (see Figure 1). NCI implies three restrictions on these curves: (i) Convexity implies that all curves must be *convex*; (ii) the bold indifference curve through the

[7]Kahneman and Tversky (1979, p. 267) define the Certainty Effect as the requirement that for each $x, y \in [w, b]$ and $\alpha, \beta \in (0,1)$, if $\alpha\delta_y + (1-\alpha)\,\delta_0$ is indifferent to δ_x then $\alpha\beta\delta_y + (1-\alpha\beta)\,\delta_0$ is preferred to $\beta\delta_x + (1-\beta)\,\delta_0$. Notice that this immediately follows from NCI.

origin (which represents the lottery δ_{x_2}) is *linear*, due to neutrality towards mixing a lottery with its certainty equivalent; and (iii) this bold indifference curve is also the *steepest*, that is, its slope relative to the (p_1, p_3) coordinates exceeds that of any other indifference curve through any point in the triangle.[8] Since, as explained by Machina (1982), the slope of an indifference curve expresses local attitude towards risk (with greater slope corresponds to higher local risk aversion), this last property captures the Certainty Effect by, loosely speaking, requiring that local risk aversion is at its peak when it involves a degenerate lottery. In Section 6 we show that this pattern of indifference curves is consistent with a variety of experimental evidence on decision making under risk.[9]

2.3 Representation Theorem

Before stating our representation theorem, we introduce some notation. We say that a function $V : \Delta \to \mathbb{R}$ represents $\succcurlyeq$ when $p \succcurlyeq q$ if and only if $V(p) \geq V(q)$. Denote by $\mathcal{U}$ the set of continuous and strictly increasing functions v from $[w, b]$ to $\mathbb{R}$. We endow $\mathcal{U}$ with the topology induced by the supnorm. For each lottery p and function $v \in \mathcal{U}$, we denote by $\mathbb{E}_p(v)$ the expected utility of p with respect to v. The certainty equivalent of lottery p calculated using the utility function v is thus $c(p, v) = v^{-1}(\mathbb{E}_p(v)) \in [w, b]$.

Definition 1. *Let $\succcurlyeq$ be a binary relation on Δ and $\mathcal{W}$ a subset of $\mathcal{U}$. The set $\mathcal{W}$ is a Cautious Expected Utility representation of $\succcurlyeq$ if and only if the function $V : \Delta \to \mathbb{R}$, defined by*

$$V(p) = \inf_{v \in \mathcal{W}} c(p, v) \qquad \forall p \in \Delta,$$

represents $\succcurlyeq$. We say that $\mathcal{W}$ is a Continuous Cautious Expected Utility representation if and only if V is also continuous.

We now present our main representation theorem.

[8] The *steepest middle slope* property is formally derived in Lemma 3 of Dillenberger (2010).

[9] NCI also has implications in non-static settings. Dillenberger (2010) shows that in the context of recursive and time-neutral, non-Expected Utility preferences over compound lotteries, NCI is equivalent to an intrinsic aversion to receiving partial information – a property which he termed preferences for one-shot resolution of uncertainty. Dillenberger (2010) also shows that in the context of preferences over information structures, NCI characterizes all non-Expected Utility preferences for which, when applied recursively, perfect information is always the most valuable information system.

Theorem 1. *Let $\succcurlyeq$ be a binary relation on Δ. The following statements are equivalent:*

(i) The relation $\succcurlyeq$ satisfies Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence;

(ii) There exists a Continuous Cautious Expected Utility representation of $\succcurlyeq$.

According to a Cautious Expected Utility representation, the DM has a set $\mathcal{W}$ of possible utility functions over monetary outcomes. Each of these functions is strictly increasing, i.e., agrees that "more money is better". These utility functions, however, may have different curvatures: it is as if the DM is unsure how to evaluate each lottery. The DM then reacts to this multiplicity with *caution*: she evaluates each lottery p by using the utility function that returns the lowest *certainty equivalent*. That is, there are two key components in the representation: (i) the agent is unsure about how to evaluate each lottery (having a non-degenerate set $\mathcal{W}$); and (ii) she acts conservatively and uses the most cautious criterion at hand (captured by the inf aggregator).

As a concrete example, suppose that the DM needs to evaluate the lottery p, which pays either \$0 or \$10,000, both equally likely. The DM may reasonably find it difficult to give a precise answer, but, instead, finds it conceivable that her certainty equivalent of p falls in the range $[\$3,500, \$4,500]$, and that this interval is tight (that is, the end points are plausible evaluations). This is the first component of the representation: the DM has a set of plausible valuations that she considers. Nevertheless, when asked how much she would be willing to pay in order to obtain p, she is cautious and answers *at most* \$3,500. This is the second component of the representation. Note that if $\mathcal{W}$ contains only one element then the model reduces to standard Expected Utility. Note also that, since each $u \in \mathcal{W}$ is strictly increasing, the model preserves monotonicity with respect to First Order Stochastic Dominance.

An important feature of the representation above is that the DM uses the utility function that minimizes the certainty equivalent of a lottery, instead of just minimizing its expected utility.[10] The reason is that comparing certainty equivalents means bringing each evaluation with each utility function to a unified measure, amounts of money, where a meaningful comparison is possible. It is then easy to see how

[10] This alternative model is studied in Maccheroni (2002). We discuss it in detail in Section 5.

the representation in Theorem 1 leads to the Certainty Effect: while the DM acts with caution when evaluating general lotteries, caution does not play a role when evaluating degenerate ones – no matter which utility function is used, the certainty equivalent of a degenerate lottery that yields the prize x for sure is simply x.

Example 1. *Let $[w,b] \subseteq [0,\infty)$ and $\mathcal{W} = \{u,v\}$ where*

$$u(x) = -\exp(-\beta x), \ \beta > 0; \text{ and } v(x) = x^{\alpha}, \ \alpha \in (0,1).$$

That is, u (resp., v) displays constant absolute (resp., relative) risk aversion. Furthermore, if the interval $[w,b]$ is large enough ($b > \frac{1-\alpha}{\beta} > w$), then u and v are not ranked in terms of risk aversion, that is, there exist p and q such that the smallest certainty equivalent for p (resp., q) corresponds to u (resp., v). This functional form can easily address the Common Ratio Effect.[11]

Example 1 shows that one could address experimental evidence related to the Certainty Effect using a set $\mathcal{W}$ that includes only two utility functions. The key feature of this example is not the specific functional forms used, but rather the fact that there is no unique v in $\mathcal{W}$ which minimizes the certainty equivalents for all lotteries. If this were the case, then only v would matter and behavior would coincide with Expected Utility. This implies, for example, that if all utilities in $\mathcal{W}$ have constant relative risk aversion (that is, $v_i \in \mathcal{W}$ only if $v_i(x) = x^{\alpha_i}$ for some $\alpha_i \in (0,1)$), then preferences will be indistinguishable from Expected Utility with coefficient of relative risk aversion equals $1 - \min_j \alpha_j$. (See Section 2.6, where we suggest some convenient parametric class of utility functions that can be used in applications.)

The discussion above further suggests that the Cautious Expected Utility model treats the Certainty Effect as a local property, which is not necessarily invariant to changes in the stakes involved. To illustrate, consider again Example 1 and note that v has a higher coefficient of (absolute or relative) risk aversion than u for all outcomes below $\frac{1-\alpha}{\beta}$. Therefore, when restricted to lotteries with outcomes only in

[11] For example, let $\alpha = 0.8$ and $\beta = 0.0002$. Let $p = 0.8\delta_{4000} + 0.2\delta_0$, $q = 0.2\delta_{4000} + 0.8\delta_0$ and $r = 0.25\delta_{3000} + 0.75\delta_0$. Direct calculations show that

$$V(p) = c(p,u) \simeq 2904 < 3000 = V(\delta_{3000}),$$

but

$$V(q) = c(q,v) \simeq 535 > 530 \simeq c(r,v) = V(r).$$

We have $\delta_{3000} \succ p$ but $q \succ r$.

$\left[w, \frac{1-\alpha}{\beta}\right]$, preferences are Expected Utility with Bernoulli index v, but they violate Expected Utility for larger stakes. This is compatible with experimental evidence that suggests that Allais-type behavior is mostly prominent when the stakes are high (or, more precisely, when there is a large gap between the best and worst possible outcome). The fact that our model can accommodate this evidence is one of its distinctive features, which we discuss in detail in Section 6.

The interpretation of the Cautious Expected Utility representation is different from some of the most prominent existing non-Expected Utility models. For example, the common interpretation of the Rank Dependent Utility model of Quiggin (1982) is that the DM knows her utility function but she distorts probabilities. By contrast, in a Cautious Expected Utility representation the DM takes probabilities at face value, but she is unsure of which utility function to use, and applies caution by using the most conservative one in the set. In Section 5, we point out that not only the two models have a different interpretation but they entail stark differences in behavior: the only preference relation that is compatible with both models is Expected Utility.

Lastly, we note that the use of the most conservative utility in a set is reminiscent of the Maxmin Expected Utility of Gilboa and Schmeidler (1989) under ambiguity, in which the DM has not one, but a set of probabilities, and evaluates acts using the worst probability in the set. Our model can be seen as a corresponding model under risk. This analogy with Maxmin Expected Utility will then be strengthened by our analysis in Section 4, where we argue that both models can be derived from extending incomplete preferences using a cautious rule.

In the next subsection we will outline the main steps in the proof of Theorem 1. There is one notion that is worth discussing independently, since it plays a major role in the analysis of all subsequent sections. We introduce a *derived* preference relation, denoted $\succcurlyeq'$, which is the largest subrelation of the original preference $\succcurlyeq$ that satisfies the Independence axiom. Formally, define $\succcurlyeq'$ on Δ by

$$p \succcurlyeq' q \Longleftrightarrow \lambda p + (1-\lambda) r \succcurlyeq \lambda q + (1-\lambda) r \qquad \forall \lambda \in (0,1], \forall r \in \Delta. \tag{1}$$

In the context of choice under risk, this derived relation was proposed and characterized by Cerreia-Vioglio (2009). It parallels a notion introduced in the context of choice under ambiguity by Ghirardato et al. (2004) (see also Cerreia-Vioglio et al. (2011a)). This binary relation, which contains the comparisons over which the DM abides by the precepts of Expected Utility, is often interpreted as including the comparisons that the DM is confident in making. We refer to $\succcurlyeq'$ as the Linear Core of $\succcurlyeq$. Note that, by definition, if the original preference relation $\succcurlyeq$ satisfies NCI, then $p \not\succcurlyeq' \delta_x$ implies $\delta_x \succ p$. That is, whenever the DM is not confident to declare p better than the certain outcome x, the original relation will rank δ_x strictly above p. This intuition will be our starting point in Section 4, where we discuss the idea of Cautious Completions. Lastly, as we will see in Section 2.5, $\succcurlyeq'$ will allow us to identify the uniqueness properties of the set $\mathcal{W}$ in a Cautious Expected Utility representation.

2.4 Proof Sketch of Theorem 1

In what follows we discuss the main intuition of the proof of Theorem 1; a complete proof, which includes the many omitted details, appears in Appendix B. We focus here only on the sufficiency of the axioms for the representation.

Step 1. Define the Linear Core of $\succcurlyeq$ *.* As we have discussed above, we introduce the binary relation $\succcurlyeq'$ on Δ defined in (1).

Step 2. Find the set $\mathcal{W} \subseteq \mathcal{U}$ *that represents* $\succcurlyeq'$*.* By Cerreia-Vioglio (2009), $\succcurlyeq'$ is reflexive and transitive (but possibly incomplete), continuous, and satisfies Independence. In particular, there exists a set $\mathcal{W}$ of continuous functions on $[w, b]$ that constitutes an Expected Multi-Utility representation of $\succcurlyeq'$, that is, $p \succcurlyeq' q$ if and only if $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}$ (see Dubra et al., 2004). Since $\succcurlyeq$ satisfies Weak Monotonicity and NCI, $\succcurlyeq'$ also satisfies Weak Monotonicity. For this reason, the set $\mathcal{W}$ can be chosen to be composed only of strictly increasing functions.

Step 3. Representation of $\succcurlyeq$*.* We show that $\succcurlyeq$ admits a certainty equivalent representation, i.e., there exists $V : \Delta \to \mathbb{R}$ such that V represents $\succcurlyeq$ and $V(\delta_x) = x$ for all $x \in [w, b]$.

Step 4. Relation between $\succcurlyeq$ *and* $\succcurlyeq'$*.* We note that *(i)* $\succcurlyeq$ is a completion of $\succcurlyeq'$, i.e., $p \succcurlyeq' q$ implies $p \succcurlyeq q$; and *(ii)* for each $p \in \Delta$ and for each $x \in [w, b]$, $p \not\succcurlyeq' \delta_x$ implies $\delta_x \succ p$. The latter is an immediate implication of NCI.

Step 5. Final step. We conclude the proof by showing that we must have $V(p) = \inf_{v \in \mathcal{W}} c(p, v)$ for all $p \in \Delta$. For each p, find $x \in [w, b]$ such that $p \sim \delta_x$, which means $V(p) = V(\delta_x) = x$. First note that we must have $V(p) = x \leq \inf_{v \in \mathcal{W}} c(p, v)$. If this was not the case, then we would have that $x > c(p, v)$ for some $v \in \mathcal{W}$, which means, by *Step 2*, $p \not\succcurlyeq' \delta_x$. But by *Step 4(ii)* we would obtain $\delta_x \succ p$, contradicting $\delta_x \sim p$.

Second, we must have $V(p) = x \geq \inf_{v \in \mathcal{W}} c(p, v)$: if this was not the case, then we would have $x < \inf_{v \in \mathcal{W}} c(p, v)$. We could then find y such that $x < y < \inf_{v \in \mathcal{W}} c(p, v)$, which, by *Step 2*, would yield $p \succcurlyeq' \delta_y$. By *Step 4 (i),* we could conclude that $p \succcurlyeq \delta_y \succ \delta_x$, contradicting $p \sim \delta_x$.

2.5 Uniqueness and Properties of the Set of Utilities

We now discuss the uniqueness properties of a set of utilities $\mathcal{W}$ in a Cautious Expected Utility representation of $\succcurlyeq$. To do so, we define the set of *normalized* utility functions $\mathcal{U}_{\text{nor}} = \{v \in \mathcal{U} : v(w) = 0,\ v(b) = 1\}$, and, without loss of generality, confine our attention to a normalized Cautious Expected Utility representation, that is, we further require $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$. Even with this normalization, we are bound to find uniqueness properties only 'up to' the closed convex hull: if two sets share the same closed convex hull, then they must generate the same representation, as proved in the following proposition. Denote by $\overline{\text{co}}\,(\mathcal{W})$ the closed convex hull of a set $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$.

Proposition 1. *If $\mathcal{W}, \mathcal{W}' \subseteq \mathcal{U}_{\text{nor}}$ are such that $\overline{\text{co}}\,(\mathcal{W}) = \overline{\text{co}}\,(\mathcal{W}')$ then*

$$\inf_{v \in \mathcal{W}} c(p, v) = \inf_{v \in \mathcal{W}'} c(p, v) \qquad \forall p \in \Delta.$$

Moreover, it is easy to see how $\mathcal{W}$ will in general not be unique, even up to the closed convex hull, as we can always add redundant utility functions that will never achieve the infimum. In particular, consider any set $\mathcal{W}$ in a Cautious Expected Utility representation and add to it a function $\bar{v}$ which is a continuous, strictly increasing, and strictly convex transformation of some other function $u \in \mathcal{W}$. The set $\mathcal{W} \cup \{\bar{v}\}$ will give a Cautious Expected Utility representation of the same preference relation, as the function $\bar{v}$ will never be used in the representation.[12]

Once we remove these redundant utilities, we can identify a unique (up to the closed convex hull) set of utilities. In particular, for each preference relation that admits a Continuous Cautious Expected Utility representation, there exists a set $\widehat{\mathcal{W}}$ such that any other Cautious Expected Utility representation $\mathcal{W}$ of these preferences is such that $\overline{\text{co}}(\widehat{\mathcal{W}}) \subseteq \overline{\text{co}}(\mathcal{W})$. In this sense $\widehat{\mathcal{W}}$ is a 'minimal' set of utilities. Moreover, the set $\widehat{\mathcal{W}}$ will have a natural interpretation in our setup: it constitutes a unique

[12] Since $u \in \mathcal{W}$ and $c(p, u) \leq c(p, \overline{v})$ for all $p \in \Delta$, there will not be a lottery p such that $\inf_{v \in \mathcal{W} \cup \{\bar{v}\}} c(p, v) = c(p, \overline{v}) < \inf_{v \in \mathcal{W}} c(p, v)$.

(up to the closed convex hull) Expected Multi-Utility representation of the Linear Core $\succcurlyeq'$, the derived preference relation defined in (1). In terms of uniqueness, if two sets constitute a Continuous Cautious Expected Utility representation of $\succcurlyeq$ and an Expected Multi-Utility representation of $\succcurlyeq'$, then their closed convex hull must coincide. This is formalized in the following result.

Theorem 2. *Let $\succcurlyeq$ be a binary relation on Δ that satisfies Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence. Then there exists $\widehat{\mathcal{W}} \subseteq \mathcal{U}_{\text{nor}}$ such that*

(i) The set $\widehat{\mathcal{W}}$ is a Continuous Cautious Expected Utility representation of $\succcurlyeq$;

(ii) If $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$ is a Cautious Expected Utility representation of $\succcurlyeq$, then $\overline{\text{co}}(\widehat{\mathcal{W}}) \subseteq \overline{\text{co}}(\mathcal{W})$;

(iii) The set $\widehat{\mathcal{W}}$ is an Expected Multi-Utility representation of $\succcurlyeq'$, that is,

$$p \succcurlyeq' q \Longleftrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \widehat{\mathcal{W}}.$$

Moreover, $\widehat{\mathcal{W}}$ is unique up to the closed convex hull.

2.6 Parametric Sets of Utilities and Elicitation

In applied work, it is common to specify a parametric class of utility functions and estimate the relevant parameters. The purpose of this subsection is to suggest some parametric classes that are compatible with Cautious Expected Utility representation. We then remark on the issue of how to elicit the set of utilities from a finite data set.

The discussion in Section 2.3 suggests that our model coincides with Expected Utility if we focus on a set $\mathcal{W}$ that includes only utilities with constant absolute (or relative) risk aversion. More generally, if preferences are not Expected Utility and $\mathcal{W}$ contains only functions from the same parametric class, then the level of risk aversion within this class must depend on more than a single parameter. We now suggest two examples of parsimonious families of utility functions for which risk attitude is characterized by the values of only two parameters; this property could be useful in empirical estimations.

The first example is the increasingly popular family of Expo-Power utility functions (Saha (1993)), which generalizes both constant absolute and constant relative risk aversion, given by

$$u(x) = 1 - \exp\left(-\lambda x^{\theta}\right), \text{ with } \lambda \neq 0,\ \theta \neq 0, \text{ and } \lambda\theta > 0.$$

This functional form has been applied in a variety of fields, such as finance, intertemporal choices, and agriculture economics. Holt and Laury (2002) show that this functional form fits well experimental data that involve both low and high stakes. The second example is the set of Pareto utility functions, given by

$$u(x) = 1 - \left(1 + \frac{x}{\gamma}\right)^{-\kappa}, \text{ with } \gamma > 0 \text{ and } \kappa > 0.$$

Ikefuji et al. (2012) show that a Pareto utility function has some desirable properties. If u is Pareto, then the coefficient of absolute risk aversion is $-\frac{u''(x)}{u'(x)} = \frac{\kappa+1}{x+\gamma}$, which is increasing in κ and decreasing in γ. Therefore, for a large enough interval $[w, b]$, if $\kappa_u > \kappa_v$ and $\gamma_u > \gamma_v$ then u and v are not ranked in terms of risk aversion.

We conclude with a brief remark on the issue of elicitation. If one could observe the certainty equivalents for all lotteries, then the whole preference relation would be recovered and the set $\mathcal{W}$ identified (up to its uniqueness properties) – but this requires an infinite number of observations. With a finite data set, one can approximate, or partially recover, the set $\mathcal{W}$ as follows. Note that if a function v assigns to some lottery p a certainty equivalent that is *smaller* than the one observed in the data (i.e., $c(p, v) < x_p$), then v cannot belong to $\mathcal{W}$. Therefore, by observing the certainty equivalents of a finite number of lotteries, one could exclude a set of possible utility functions and approximate the set $\mathcal{W}$ 'from above.' It is easy to see that the set thus obtained would necessarily contain the 'true' one, and that as the number of observations increases, the set will shrink to coincide with $\mathcal{W}$ (or, more precisely, with a version of $\mathcal{W}$ up to uniqueness). Such elicitation would be significantly faster if, as is often the case in empirical work, one is willing to assume that utility functions come from a specific parametric class, such as the ones described above.

3 Cautious Expected Utility and Risk Attitudes

In this section we explore the connection between Theorem 1 and standard definitions of risk attitude, and characterize the comparative notion of "more risk averse than". Throughout this section, we mainly focus on a 'minimal' representation $\widehat{\mathcal{W}}$ as in Theorem 2.

Remark. *If $\mathcal{W}$ is a Continuous Cautious Expected Utility representation of a preference relation $\succcurlyeq$, we denote by $\widehat{\mathcal{W}}$ a set of utilities as identified in Theorem 2 (which is unique up to the closed convex hull). More formally, we can define a correspondence T that maps each set $\mathcal{W}$ that is a Continuous Cautious Expected Utility representation of some $\succcurlyeq$ to a class of subsets of $\mathcal{U}_{\text{nor}}$, $T(\mathcal{W})$, each element of which satisfies the properties of points (i)-(iii) of Theorem 2 and is denoted by $\widehat{\mathcal{W}}$.*

3.1 Characterization of Risk Attitudes

We adopt the following standard definition of risk aversion/risk seeking.

Definition 2. *We say that $\succcurlyeq$ is risk averse if $p \succcurlyeq q$ whenever q is a mean preserving spread of p. Similarly, $\succcurlyeq$ is risk seeking if $q \succcurlyeq p$ whenever q is a mean preserving spread of p.*

Theorem 3. *Let $\succcurlyeq$ be a binary relation that satisfies Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence. The following statements are true:*

(i) The relation $\succcurlyeq$ is risk averse if and only if each $v \in \widehat{\mathcal{W}}$ is concave.

(ii) The relation $\succcurlyeq$ is risk seeking if and only if each $v \in \widehat{\mathcal{W}}$ is convex.

Theorem 3 shows that the relation found under Expected Utility between the concavity/convexity of the utility function and the risk attitude of the DM holds also for the more general Continuous Cautious Expected Utility model – although it now involves *all* utilities in the set $\widehat{\mathcal{W}}$. In turn, this shows that our model is compatible with many types of risk attitudes. For example, despite the presence of the Certainty Effect, when all utilities are convex the DM would be risk seeking.

3.2 Comparative Risk Aversion

We now proceed to compare the risk attitudes of two individuals.

Definition 3. *Let $\succcurlyeq_1$ and $\succcurlyeq_2$ be two binary relations on Δ. We say that $\succcurlyeq_1$ is more risk averse than $\succcurlyeq_2$ if and only if for each $p \in \Delta$ and for each $x \in [w, b]$,*

$$p \succcurlyeq_1 \delta_x \quad \Longrightarrow \quad p \succcurlyeq_2 \delta_x.$$

Theorem 4. *Let $\succcurlyeq_1$ and $\succcurlyeq_2$ be two binary relations with Continuous Cautious Expected Utility representations, $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively. The following statements are equivalent:*

(i) *$\succcurlyeq_1$ is more risk averse than $\succcurlyeq_2$;*

(ii) *Both $\mathcal{W}_1 \cup \mathcal{W}_2$ and $\mathcal{W}_1$ are Continuous Cautious Expected Utility representations of $\succcurlyeq_1$;*

(iii) $\overline{\mathrm{co}}\left(\widehat{\mathcal{W}_1 \cup \mathcal{W}_2}\right) = \overline{\mathrm{co}}(\widehat{\mathcal{W}}_1)$.

Theorem 4 states that DM1 is more risk averse than DM2 if and only if all the utilities in $\mathcal{W}_2$ are redundant when added to $\mathcal{W}_1$.[13,14] This result compounds two conceptually different channels that in a Cautious Expected Utility representation lead one decision maker to be more risk averse than another. The first channel is related to the *curvatures* of the functions in each set of utilities. For example, if each $v \in \mathcal{W}_2$ is a strictly increasing and strictly convex transformation of some $\hat{v} \in \mathcal{W}_1$, then DM2 assigns a strictly higher certainty equivalent than DM1 to any nondegenerate lottery $p \in \Delta$ (while the certain outcomes are, by construction, treated similarly in both). In particular, as we discussed in Section 2.5, no member of $\mathcal{W}_2$ will be used in the representation corresponding to the union of the two sets. The second channel corresponds to comparing the *size* of the two sets of utilities. Indeed, if $\mathcal{W}_2 \subseteq \mathcal{W}_1$ then for each $p \in \Delta$ the certainty equivalent under $\mathcal{W}_2$ is weakly greater than that under $\mathcal{W}_1$, implying that $\succcurlyeq_1$ is more risk averse than $\succcurlyeq_2$.

[13] We thank Todd Sarver for suggesting point (iii) in Theorem 4.

[14] Note that if both $\succcurlyeq_1$ and $\succcurlyeq_2$ are Expected Utility preferences, then there are v_1 and v_2 such that $\{v_1\} = \mathcal{W}_1 = \widehat{\mathcal{W}}_1$, $\{v_2\} = \mathcal{W}_2 = \widehat{\mathcal{W}}_2$, and points (ii) and (iii) in Theorem 4 are equivalent to v_1 being an increasing concave transformation of v_2.

We can distinguish between these two different channels, and characterize the behavioral underpinning of the second one. To do so, we focus on the notion of Linear Core and its representation as in Theorem 2.

Definition 4. *Let $\succcurlyeq_1$ and $\succcurlyeq_2$ be two binary relations on Δ with corresponding Linear Cores $\succcurlyeq'_1$ and $\succcurlyeq'_2$. We say that $\succcurlyeq_1$ is more indecisive than $\succcurlyeq_2$ if and only if for each $p, q \in \Delta$*

$$p \succcurlyeq'_1 q \quad \Longrightarrow \quad p \succcurlyeq'_2 q.$$

Since we interpret the derived binary relation $\succcurlyeq'$ as capturing the comparisons that the DM is confident in making, Definition 4 implies that DM1 is more indecisive than DM2 if whenever DM1 can confidently declare p weakly better than q, so does DM2. The following result characterizes this comparative relation and links it to the comparative notion of risk aversion.

Proposition 2. *Let $\succcurlyeq_1$ and $\succcurlyeq_2$ be two binary relations that satisfy Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence. The following statements are true:*

(i) $\succcurlyeq_1$ is more indecisive than $\succcurlyeq_2$ if and only if $\overline{\text{co}}(\widehat{\mathcal{W}}_2) \subseteq \overline{\text{co}}(\widehat{\mathcal{W}}_1)$;

(ii) If $\succcurlyeq_1$ is more indecisive than $\succcurlyeq_2$, then $\succcurlyeq_1$ is more risk averse than $\succcurlyeq_2$.

Proposition 2 establishes the relationship between indecisiveness, which is akin to incompleteness of the Linear Core, and risk aversion. The more indecisive the DM is, the more possible evaluations of each lottery she considers; and since she is cautious and uses only the lowest of such evaluations, a more indecisive DM has a lower certainty equivalent for each lottery and thus is also more risk averse.

4 Cautious Completions of Incomplete Preferences

While our analysis thus far has focused on the characterization of a complete preference relation that satisfies NCI (in addition to the other basic axioms), we will now show that this analysis is deeply related to that of a 'cautious' completion of an incomplete preference relation over lotteries.

Consider a DM who has an *incomplete* preference relation over the set of lotteries. We can see this relation as representing the comparisons that the DM feels comfortable making. There might be occasions, however, in which the DM is asked to choose among lotteries she cannot compare, and to do this she has to complete her preferences. Suppose that the DM wants to do so applying *caution*, i.e., when in doubt between a sure outcome and a lottery, she opts for the sure outcome. Which preferences will she obtain after the completion?

This analysis parallels the one of Gilboa et al. (2010), who consider an environment with ambiguity instead of risk, although with one minor formal difference: while in Gilboa et al. (2010) both the incomplete relation and its completion are a primitive of the analysis, in our case the primitive is simply the incomplete preference relation over lotteries, and we study the properties of all possible completions of this kind.[15]

Since we analyze an incomplete preference relation, the analysis in this section requires a slightly stronger notion of continuity, called Sequential Continuity.[16]

Axiom 5 (Sequential Continuity)**.** *Let* $\{p_n\}_{n\in\mathbb{N}}$ *and* $\{q_n\}_{n\in\mathbb{N}}$ *be two sequences in* Δ*. If* $p_n \to p$*,* $q_n \to q$*, and* $p_n \succcurlyeq q_n$ *for all* $n \in \mathbb{N}$ *then* $p \succcurlyeq q$*.*

In the rest of the section, we assume that $\succcurlyeq'$ is a reflexive and transitive (though potentially incomplete) binary relation over Δ, which satisfies Sequential Continuity, Weak Monotonicity, and Independence. We look for a Cautious Completion of $\succcurlyeq'$, as formalized in the following definition.

Definition 5. *Let* $\succcurlyeq'$ *be a binary relation on* Δ*. We say that the relation* $\hat{\succcurlyeq}$ *is a Cautious Completion of* $\succcurlyeq'$ *if and only if the following hold:*

1. *The relation* $\hat{\succcurlyeq}$ *satisfies Weak Order, Weak Monotonicity, and for each* $p \in \Delta$ *there exists* $x \in [w, b]$ *such that* $p \mathrel{\hat{\sim}} \delta_x$*;*

2. *For each* $p, q \in \Delta$*, if* $p \succcurlyeq' q$ *then* $p \mathrel{\hat{\succcurlyeq}} q$*;*

3. *For each* $p \in \Delta$ *and* $x \in [w, b]$*, if* $p \not\succcurlyeq' \delta_x$ *then* $\delta_x \mathrel{\hat{\succ}} p$*.*

Point 1 imposes few minimal requirements of rationality on $\hat{\succcurlyeq}$, most notably, the existence of a certainty equivalent for each lottery p. Weak Monotonicity will imply

[15] Riella (2013) develops a more general treatment that encompasses the result in this section and the one in Gilboa et al. (2010); he shows that a combined model could be obtained starting from a preference relation over acts that admits a Multi-Prior Expected Multi-Utility representation, as in Ok et al. (2012) and Galaabaatar and Karni (2013), and constructing a Cautious Completion.

[16] This notion coincides with our Continuity axiom if the binary relation is complete and transitive.

that this certainty equivalent is unique. In point 2, we assume that the relation $\hat{\succcurlyeq}$ extends $\succcurlyeq'$. Finally, point 3 requires that such a completion of $\succcurlyeq'$ is done with caution.

Theorem 5. *If $\succcurlyeq'$ is a reflexive and transitive binary relation on Δ that satisfies Sequential Continuity, Weak Monotonicity, and Independence, then $\succcurlyeq'$ admits a unique Cautious Completion $\hat{\succcurlyeq}$ and there exists a set $\mathcal{W} \subseteq \mathcal{U}$ such that for all $p, q \in \Delta$*

$$p \succcurlyeq' q \Longleftrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \mathcal{W}$$

and

$$p \,\hat{\succcurlyeq}\, q \Longleftrightarrow \inf_{v \in \mathcal{W}} c(p, v) \geq \inf_{v \in \mathcal{W}} c(q, v).$$

Moreover, $\mathcal{W}$ is unique up to the closed convex hull.

Theorem 5 shows that, given a binary relation $\succcurlyeq'$ which satisfies all the tenets of Expected Utility except completeness, not only a Cautious Completion $\hat{\succcurlyeq}$ is always possible, but it is also unique. Most importantly, such completion $\hat{\succcurlyeq}$ admits a Cautious Expected Utility representation, using the same set of utilities as in the Expected Multi-Utility Representation of the original preference $\succcurlyeq'$. This result shows that our model could also represent the behavior of a subject who might be unable to compare some of the available options and, when asked to extend her ranking, does so by being cautious. Together with Theorem 1, Theorem 5 shows that this behavior is indistinguishable from that of a subject who starts with a complete preference relation and satisfies Axioms 1–4. In turn, this shows that incomplete preferences followed by a Cautious Completion could *generate* the Certainty Effect.

Finally, Theorem 5 strengthens the link between the Cautious Expected Utility model and the Maxmin Expected Utility model of Gilboa and Schmeidler (1989). Gilboa et al. (2010) show that the latter could be derived as a completion of an incomplete preference relation over Anscombe-Aumann acts that satisfies the same normative assumptions as $\succcurlyeq'$ (adapted to their domain), by applying a form of caution according to which, when in doubt, the DM chooses a constant act. Similarly, here we derive the Cautious Expected Utility model by extending an incomplete preference over lotteries using a form of caution according to which, when in doubt, the DM chooses a risk-free lottery.[17]

[17]The condition in Gilboa et al. (2010) is termed Default to Certainty; Point 3 of Definition 5 is

5 Related Literature

Dillenberger (2010) introduces the NCI axiom and discusses its implication in dynamic settings. Under some specific assumptions on preferences over two-stage lotteries, he shows that NCI is a necessary and sufficient condition to a property called "preference for one-shot resolution of uncertainty". Dillenberger, however, does not provide a utility representation as in Theorem 1. Dillenberger and Erol (2013) provide an example of a continuous, monotone, and complete preference relation that satisfies NCI but not a property called Betweenness (see below), suggesting that indeed these are different properties. The present paper provides a complete characterization of all binary relations that satisfy NCI (in addition to the other three basic postulates) and clarifies the precise relationship with Betweenness.

Cerreia-Vioglio (2009) characterizes the class of continuous and complete preference relations that satisfy Convexity, that is, $p \sim q$ implies $\lambda p + (1-\lambda)\, q \succcurlyeq q$ for all $\lambda \in (0,1)$. Loosely speaking, Cerreia-Vioglio shows that there exists a set $\mathcal{V}$ of normalized Bernoulli utility functions, and a real function U on $\mathbb{R} \times \mathcal{V}$, such that preferences are represented by

$$V(p) = \inf_{v \in \mathcal{V}} U\left(\mathbb{E}_p\left(v\right), v\right).$$

Using this representation, Cerreia-Vioglio interprets Convexity as a behavioral property that captures a preference for hedging; such preferences may arise in the face of uncertainty about the value of outcomes, future tastes, and/or the degree of risk aversion. He suggests the choice of the minimal certainty equivalent as a criterion to resolve uncertainty about risk attitudes and as a completion procedure. (See also Cerreia-Vioglio et al. (2011b) for a risk measurement perspective.) As we discussed in Section 2, NCI implies Convexity, which means that the preferences we study in this paper are a subset of those studied by Cerreia-Vioglio. Indeed, this is apparent also from Theorem 1: our preferences correspond to the special case in which $U\left(\mathbb{E}_p\left(v\right), v\right) = v^{-1}(\mathbb{E}_p\left(v\right)) = c\left(p, v\right)$. Furthermore, our representation theorem establishes that NCI is the *exact* strengthening of convexity needed to characterize the minimum certainty equivalent criterion.

A popular generalization of Expected Utility is the Rank Dependent Utility (RDU)

the translation of this condition to the context of choice under risk.

model of Quiggin (1982), also used within Cumulative Prospect Theory (Tversky and Kahneman, 1992). According to this model, individuals weight probability in a nonlinear way. Specifically, if we order the prizes in the support of the lottery p, with $x_1 < x_2 < ... < x_n$, then the functional form for RDU is:

$$V(p) = u(x_n)f(p(x_n)) + \sum_{i=1}^{n-1} u(x_i)[f(\sum_{j=i}^{n} p(x_j)) - f(\sum_{j=i+1}^{n} p(x_j))],$$

where $f : [0,1] \to [0,1]$ is strictly increasing and onto, and $u : [w,b] \to \mathbb{R}$ is increasing. If $f(p) = p$ then RDU reduces to Expected Utility. If f is convex, then larger weight is given to inferior outcomes; this corresponds to a pessimistic probability distortion suitable to explain the Allais paradoxes. Apart from the different interpretation of RDU compared to our Cautious Expected Utility representation, as discussed in Section 2.3, the two models have completely different behavioral implications: Dillenberger (2010) demonstrates that the only RDU preference relations that satisfy NCI are Expected Utility. That is, RDU is generically incompatible with NCI.[18]

Another popular and broad class of continuos and monotone preferences is the Betweenness class introduced by Dekel (1986) and Chew (1989). The central axiom in this class is a weakening of the Independence axiom which implies neutrality toward randomization among equally-good lotteries.[19] That is, if $\succcurlyeq$ satisfies Betweenness, then its indifference curves in the Marschak-Machina triangle are linear, but not necessarily parallel as under Expected Utility. One of the most prominent examples of preference relations that satisfy Betweenness is Gul (1991)'s model of Disappointment Aversion (denoted DA in Figure 2). For some parameter $\beta \in (-1, \infty)$ and a strictly increasing function $u : [w,b] \to \mathbb{R}$, the disappointment aversion value of a simple lottery p is the unique v that solves

$$v = \frac{\sum_{\{x_i | u(x_i) \geq v\}} p(x_i) u(x_i) + (1+\beta) \sum_{\{x_i | u(x_i) < v\}} p(x_i) u(x_i)}{1 + \beta \sum_{\{x_i | u(x_i) < v\}} p(x_i)}.$$

In most applications, attention is confined to the case where $\beta > 0$, which corre-

[18] Bell and Fishburn (2003) showed that Expected Utility is the only RDU with the property that for each binary lottery p and $x \in [w,b]$, $p \sim \delta_x$ implies $\alpha p + (1-\alpha)\delta_x \sim \delta_x$. This property is implied by NCI (see Section 2.2). Geometrically, it corresponds to the linear indifference curve through the origin in any Marschak-Machina triangle (Figure 1 in Section 2.2).

[19] More precisely, the Betweenness axiom states that for each $p, q \in \Delta$ and $\lambda \in (0,1)$, $p \succ q$ (resp., $p \sim q$) implies $p \succ \lambda p + (1-\lambda) q \succ q$ (resp., $p \sim \lambda p + (1-\lambda) q \sim q$).

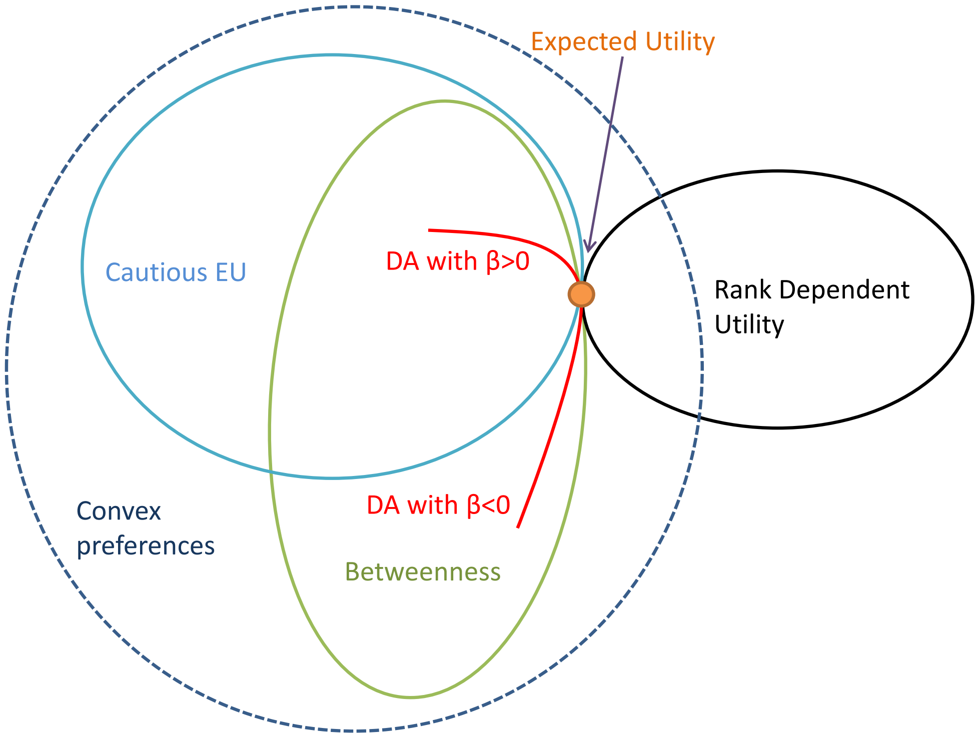

Figure 2: Cautious Expected Utility and other models

sponds to "Disappointment Aversion" (the case of $\beta \in (-1, 0)$ is referred to "Elation Seeking", and the model reduces to Expected Utility when $\beta = 0$). Artstein-Avidan and Dillenberger (2011) show that Gul's preferences satisfy NCI if and only if $\beta \geq 0$. Combining with the example given in Dillenberger and Erol (2013) mentioned above, we conclude that preferences in our class neither nest, nor are nested in, those that satisfy Betweenness.

Figure 2 summarizes our discussion thus far about the relationship between the various models.[20]

Maccheroni (2002) (see also Chatterjee and Krishna, 2011) derives a utility function over lotteries of the following form: there exists a set $\mathcal{T}$ of utilities over outcomes, such that the value of every lottery p is the lowest Expected Utility, calculated with

[20] Chew and Epstein (1989) show that there is no intersection between RDU and Betweenness other than Expected Utility (see also Bell and Fishburn, 2003). Whether or not RDU satisfies Convexity depends on the curvature of the distortion function f; in particular, concave f implies Convexity. In addition to Disappointment Aversion with negative β, an example of preferences that satisfy Betweenness but do not satisfy NCI is Chew (1983)'s model of Weighted Utility.

respect to members of $\mathcal{T}$, that is,

$$V(p) = \min_{v \in \mathcal{T}} \mathbb{E}_p(v).$$

Maccheroni's interpretation of this functional form, according to which "the most pessimist of her selves gets the upper hand over the others" is closely related to our idea that "the DM acts conservatively and uses the most cautious criterion at hand." In addition, both models satisfy Convexity. Despite these similarities, the two models are very different. First, Maccheroni's model cannot (and it was not meant to) address the Certainty Effect: since certainty equivalents are not used, also degenerate lotteries have multiple evaluations (see Section 2.3). Second, one of Maccheroni (2002)'s key axioms is Best Outcome Independence, which states that for each $p, q \in \Delta$ and each $\lambda \in (0,1)$, $p \succ q$ if and only if $\lambda p + (1-\lambda)\delta_b \succ \lambda q + (1-\lambda)\delta_b$. This axiom is conceptually and behaviorally distinct from NCI.

Schmidt (1998) develops a model in which the value of any nondegenerate lottery p is $\mathbb{E}_p(u)$, whereas the value of the degenerate lottery δ_x is $v(x)$. The Certainty Effect is captured by requiring $v(x) > u(x)$ for all x. Schmidt thus captures the Certainty Effect while maintaining Expected Utility if no certain outcomes are involved. The key difference with Cautious Expected Utility is that Schmidt's model violates both Continuity and Monotonicity, while in this paper we confine our attention to preferences that satisfy both of these basic properties. In addition, his model also violates NCI: For example, take $u(x) = x$ and $v(x) = 2x$, and note that $V(\delta_3) = 6 > 4 = V(\delta_2)$, but $V(\delta_2) = 4 > 2.5 = V(0.5\delta_3 + 0.5\delta_2)$.[21] Other discontinuous specifications of the Certainty Effect include Gilboa (1988) and Jaffray (1988). Both are models which may be dubbed 'expected utility with a security level'. Roughly speaking, the security level of a lottery is a function of the worst outcome in its support. These models are also derived from variants of the Independence axiom, but will generically violate NCI.

Dean and Ortoleva (2012a) present a model which, when restricted to preferences over lotteries, generalizes pessimistic RDU. In their model, the DM has a single utility function and a set of pessimistic probability distortions; she then evaluates each lottery using the most pessimistic of these distortions. This property is derived by an axiom, Hedging, that captures the intuition of preference for hedging of Schmeidler

[21] The statement in Dillenberger and Erol (2013) that Schmidt's model satisfies NCI is incorrect.

(1989), but applies it to preferences over lotteries. Dean and Ortoleva (2012a) show how this axiom captures the behavior in the Allais paradoxes. The exact relation between NCI and Hedging remains an open question.

Machina (1982) studies a model with minimal restrictions imposed on preferences apart from requiring them to be smooth (in the sense of Fréchet differentiability). One of the main behavioral assumptions proposed by Machina is Hypothesis II, which implies that indifference curves in the Marschak-Machina triangle *Fan Out*, that is, they become steeper as one moves in the north-west direction. The steepest middle slope property (Section 2.2) implies that our model can accommodate Fanning Out in the lower-right part of the triangle (from where most evidence on Allais-type behavior had come), while global Fanning Out is ruled out.

6 Experimental Evidence

Over the last decades, a large amount of experimental evidence has documented not only the existence of violations of Expected Utility, as in the Allais paradoxes, but also other regularities of preferences over lotteries. Based on the comprehensive surveys of Camerer (1995) and Starmer (2000), the following could be considered the three most established stylized empirical findings:[22]

1. Indifference curves in the Marschak-Machina triangle exhibit *Mixed Fanning*: indifference curves become first steeper (Fanning Out) and then flatter (Fanning In) as we move towards the north-west direction;[23]

2. Violations of Expected Utility are much less frequent when all options are non-degenerate lotteries and have similar support, i.e., inside the triangle;[24]

3. Indifference curves are typically nonlinear, meaning that Betweenness is often violated.[25]

[22] The following are documented for lotteries involving only positive outcomes. As it is well know, behavior may be very different when losses are involved (Camerer, 1995).

[23] Chew and Waller (1986); Camerer (1989); Conlisk (1989); Starmer and Sugden (1989); Battalio et al. (1990); Prelec (1990); Sopher and Gigliotti (1993); Wu (1994).

[24] Conlisk (1989); Camerer (1992); Harless (1992); Sopher and Gigliotti (1993); Harless and Camerer (1994); Andreoni and Sprenger (2012).

[25] Chew and Waller (1986); Bernasconi (1994); Camerer and Ho (1994); Prelec (1990).

To these, Camerer (1995) and more recent experimental studies add the following robust finding:

4. Allais-type behavior is significantly less frequent when stakes are small rather than large.[26]

These stylized facts, taken together, pose difficulties for most models of non-Expected Utility, including those discussed in Section 5. In the discussion below, we confine our attention to the two most popular alternatives to Expected Utility, namely Rank Dependent Utility (RDU) and Betweenness.

RDU is compatible with Mixed Fanning in the interior of the triangle (fact #1),[27] with the vast reduction of violations of Expected Utility inside the triangle (fact #2), and, as is evident from Figure 2 of Section 5, with non-Betweenness (fact #3). The compatibility of RDU with facts 1-3 has led some authors to consider it the most empirically supported of existing generalization of Expected Utility. However, as its name suggests, one of the key defining features of RDU is that the only thing that matters for probability distortion is the rank of an outcome within the support of a lottery, not its size: thus the presence of Allais-type behavior should be fully independent of the stakes. This is in contrast with the evidence that Allais-type behavior tends to disappear as stakes become lower (fact #4).[28] In addition, there is evidence of frequent violations of RDU's main behavioral underpinning, Comonotonic/Ordinal Independence.[29] Finally, empirical works that have focused on RDU and estimated the shape of the probability distortion functions have found strong empirical support for it having an S-shape (Wu and Gonzalez, 1996). However, RDU with an S-shaped probability distortion has further behavioral implications that are rejected by numerous studies.[30]

[26] Conlisk (1989); Camerer (1989); Burke et al. (1996); Fan (2002); Huck and Müller (2012); Agranov and Ortoleva (2013).

[27] According to RDU, indifference curves along the hypotenuse of the triangle should be parallel to each other – thus it is not compatible with Mixed Fanning in a strict sense. At the same time, depending on the parameters of the models, RDU can generates indifference curves with the Mixed Fanning property in the interior of the triangle.

[28] RDU implies that if we detect an Allais-type violation of Expected Utility in some range of prizes, e.g., with $x_1 < x_2 < x_3$, then similar violations of Expected Utility can be produced in any range of prizes. That is, for any $y_1 < y_3$ there exists $y_2 \in (y_1, y_3)$ and $a, b \in (0,1)$ such that $\delta_{y_2} \succ a\delta_{y_3} + (1-a)\,\delta_{y_1}$ but $b\delta_{y_2} + (1-b)\,\delta_{y_1} \prec ab\delta_{y_3} + (1-ab)\,\delta_{y_1}$.

[29] Wu (1994); Wakker et al. (1994).

[30] Battalio et al. (1990); Harless and Camerer (1994); Starmer and Sugden (1989); Andreoni and Sprenger (2012).

Models based on Betweenness may be consistent with a less frequent presence of Allais-type behavior with smaller stakes (fact #4). They can also exhibit Mixed Fanning (fact #1), with the most prominent example being Gul's model of Disappointment Aversion. The fact that the Betweenness axiom is often violated (fact #3), however, poses the greatest challenge to the empirical validity of this class. The linearity of indifference curves also means that fact #2 is violated: behavior should be invariant to translation of the corresponding prospects into the interior of the triangle.

By contrast, the Cautious Expected Utility model is compatible with all four stylized facts above. First, the steepest middle slope property (Section 2.2) implies that the model is compatible with Mixed Fanning (fact #1) and rules out global fanning out. It is also compatible with fact #2 since indifference curves could be linear and parallel to each other as we move inside the triangle, but also convex – thus allowing for non-Expected Utility behavior – as we approach the boundaries. As we mentioned in Section 5, Cautious Expected utility does not imply, and is not implied by, Betweenness, hence is consistent with fact #3.

But perhaps the most distinctive feature of the Cautious Expected Utility model is its compatibility with the presence of Allais-type behavior with large stakes but not small ones (fact #4): this would be the case if, for example, all utility functions in the representation agree on an initial smaller range and then start disagreeing as stakes become larger; or if, as in Example 1, one of the utility functions in $\mathcal{W}$ is the most risk averse for a range of outcomes below a threshold. More generally, this property, to some degree, will be implied whenever the set $\mathcal{W}$ is finite.

This leads us to discuss a more general point. Camerer (1995), Tversky and Kahneman (1992) and Wakker (2010) all mention how the RDU model is "unlikely to be accurate in detail," mostly for its full separability of probability weights and outcomes. At the same time, they were skeptical about possible generalizations, as the benefit of achieving a better fit may be outweighed by the loss of parsimony and predictive power. The Cautious Expected Utility model takes a different route: instead of generalizing RDU, it suggests a different approach to capture violations of Expected Utility based on multiple utilities rather than probability distortions. While keeping tractability, as we have seen above this different approach can accommodate most prominently observed behavioral patterns.

We conclude by noting that the Cautious Expected Utility model has additional behavioral implications which may or not find empirical support, and that have not been subject to similar scrutiny yet. For example, while consistent with many of the findings on the Certainty Effect, to our knowledge no direct tests of NCI have been conducted thus far. Indeed, the simplicity of the axiom should make such direct tests easy to implement. In addition, as we have already discussed, our model implies Convexity of preferences, a property that has been tested experimentally, albeit possibly with smaller scrutiny. The existing evidence is mixed: while the experimental papers that documents violations of betweenness found deviations in both directions (that is, either preference or aversion to mixing), both Sopher and Narramore (2000) and Dwenger et al. (2013) find explicit evidence in favor of Convexity.

7 Concluding Remarks

This paper characterizes a new class of preferences over lotteries that capture the Certainty Effect, together with very basic rationality postulates. We show that this type of violation of Expected Utility, which is one of the most prominently observed behavioral patterns, can be interpreted as reflecting cautious behavior in the evaluation of lotteries. We also demonstrate that the Certainty Effect can be generated by a Cautious Completion of incomplete preferences.

We conclude the paper by assessing our model in light of the goals it aimed to achieve and its relationship with the existing literature. There are three dimensions in which models are typically assessed and compared (the relative merit of which are often debated). First, in terms of their underlying assumptions: models derived from more transparent and well-grounded assumptions are often preferred. Second, in terms of implications: better empirical performance is typically valued. Third, and more loosely, in terms of their plausibility: does the model provide a sound story?

In terms of *assumptions*, our model imposes, together with three very basic postulates, only a single axiom, NCI, that is designed precisely to capture the Certainty Effect.[31] Thus, the model is constructed to study the consequences of the Certainty Effect and basic rationality alone. This is in contrast with most other prominent

[31] As we have mentioned in Section 2.2, NCI can be thought of as a natural extension of the definition of the Certainty Effect in Kahneman and Tversky (1979), which permits same type of violations but applies only to binary lotteries.

models that, while consistent with Allais-type behavior, are derived by either imposing additional properties that are not directly related to the Certainty Effect (e.g., Comonotonic/Ordinal Independence of RDU, or Betweenness) or by violating one or more of the basic assumptions (e.g., Monotonicity or Continuity).

In terms of testable *implications*, in Section 6 we show that our model is consistent with many well-established empirical findings on preferences under risk. In addition, it allows the Certainty Effect to be substantially less prominent when stakes are small rather than large – a robust finding in the experimental literature that is hard to reconcile with most popular alternative models. We believe this goes to show how the different way to think of the Certainty Effect and other violations of Expected Utility suggested by our model could be used to organize, and re-interpret, a broad class of existing empirical evidence.

Lastly, in terms of *interpretation*, our model differs from prominent alternatives as it involves a decision maker who does not distort probabilities, but rather is unsure about how to evaluate lotteries and applies a criterion of *caution*. Whether such interpretation is to be considered more plausible is of course a subjective matter – but it offers a novel way of reading existing evidence and their implications.

Appendix A: Preliminary Results

We begin by proving some preliminary results that will be useful for the proofs of the main results in the text. In the sequel, we denote by $C([w,b])$ the set of all real valued continuous functions on $[w,b]$. Unless otherwise specified, we endow $C([w,b])$ with the topology induced by the supnorm. We denote by $\Delta = \Delta([w,b])$ the set of all Borel probability measures endowed with the topology of weak convergence. We denote by Δ_0 the subset of Δ which contains only the elements with finite support. Since $[w,b]$ is closed and bounded, Δ is compact with respect to this topology and Δ_0 is dense in Δ. Given a binary relation $\succcurlyeq$ on Δ, we define an auxiliary binary relation $\succcurlyeq'$ on Δ by

$$p \succcurlyeq' q \Longleftrightarrow \lambda p + (1-\lambda) r \succcurlyeq \lambda q + (1-\lambda) r \qquad \forall \lambda \in (0,1], \forall r \in \Delta.$$

Lemma 1. *Let $\succcurlyeq$ be a binary relation on Δ that satisfies Weak Order. The following statements are true:*

1. *The relation $\succcurlyeq$ satisfies Negative Certainty Independence if and only if for each $p \in \Delta$ and for each $x \in [w, b]$*

$$p \succcurlyeq \delta_x \Longrightarrow p \succcurlyeq' \delta_x. \qquad \text{(Equivalently } p \not\succcurlyeq' \delta_x \Longrightarrow \delta_x \succ p.\text{)}$$

2. *If $\succcurlyeq$ also satisfies Negative Certainty Independence then $\succcurlyeq$ satisfies Weak Monotonicity if and only if for each $x, y \in [w, b]$*

$$x \geq y \Longleftrightarrow \delta_x \succcurlyeq' \delta_y,$$

that is, $\succcurlyeq'$ satisfies Weak Monotonicity.

Proof. It follows from the definition of $\succcurlyeq'$.

We define

$$\begin{aligned} \mathcal{V}_{in} &= \{v \in C([w, b]) : v \text{ is increasing}\}, \\ \mathcal{V}_{inco} &= \{v \in C([w, b]) : v \text{ is increasing and concave}\}, \\ \mathcal{U} = \mathcal{V}_{s-in} &= \{v \in C([w, b]) : v \text{ is strictly increasing}\}, \\ \mathcal{U}_{\text{nor}} &= \{v \in C([w, b]) : v(b) - 1 = 0 = v(w)\} \cap \mathcal{V}_{s-in}. \end{aligned}$$

Consider a binary relation $\succcurlyeq^*$ on Δ such that

$$p \succcurlyeq^* q \Longleftrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \mathcal{W} \tag{2}$$

where $\mathcal{W}$ is a subset of $C([w, b])$. Define $\mathcal{W}_{\max}$ as the set of all functions $v \in C([w, b])$ such that $p \succcurlyeq^* q$ implies $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$. Define also $\mathcal{W}_{\max - \text{nor}}$ as the set of all functions $v \in \mathcal{U}_{\text{nor}}$ such that $p \succcurlyeq^* q$ implies $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$. Clearly, we have that $\mathcal{W}_{\max - \text{nor}} = \mathcal{W}_{\max} \cap \mathcal{U}_{\text{nor}}$ and $\mathcal{W}_{\max - \text{nor}}, \mathcal{W} \subseteq \mathcal{W}_{\max}$.

Proposition 3. *Let $\succcurlyeq^*$ be a binary relation represented as in (2) and such that $x \geq y$ if and only if $\delta_x \succcurlyeq^* \delta_y$. The following statements are true:*

1. *$\mathcal{W}_{\max}$ and $\mathcal{W}_{\max - \text{nor}}$ are convex and $\mathcal{W}_{\max}$ is closed;*

2. *$\emptyset \neq \mathcal{W}_{\max - \text{nor}}$;*

3. *$\mathcal{W}_{\max} \subseteq \mathcal{V}_{in}$, $\emptyset \neq \mathcal{W}_{\max} \cap \mathcal{V}_{s-in}$, and $cl(\mathcal{W}_{\max} \cap \mathcal{V}_{s-in}) = \mathcal{W}_{\max}$;*

4. *$p \succsim^* q$ if and only if $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for each $v \in \mathcal{W}_{\max-\text{nor}}$;*

5. *If $\mathcal{W}$ is a convex subset of $\mathcal{U}_{\text{nor}}$ that satisfies (2) then $cl(\mathcal{W}) = cl(\mathcal{W}_{\max-\text{nor}})$.*

Proof. 1. Consider $v_1, v_2 \in \mathcal{W}_{\max-\text{nor}}$ (resp., $v_1, v_2 \in \mathcal{W}_{\max}$) and $\lambda \in (0,1)$. Since both functions are continuous, strictly increasing, and normalized (resp., continuous), it follows that $\lambda v_1 + (1-\lambda) v_2$ is continuous, strictly increasing, and normalized (resp., continuous). Since $v_1, v_2 \in \mathcal{W}_{\max-\text{nor}}$ (resp., $v_1, v_2 \in \mathcal{W}_{\max}$), if $p \succsim^* q$ then $\mathbb{E}_p(v_1) \geq \mathbb{E}_q(v_1)$ and $\mathbb{E}_p(v_2) \geq \mathbb{E}_q(v_2)$. This implies that

$$\begin{aligned}\mathbb{E}_p(\lambda v_1 + (1-\lambda) v_2) &= \lambda \mathbb{E}_p(v_1) + (1-\lambda) \mathbb{E}_p(v_2) \\ &\geq \lambda \mathbb{E}_q(v_1) + (1-\lambda) \mathbb{E}_q(v_2) = \mathbb{E}_q(\lambda v_1 + (1-\lambda) v_2),\end{aligned}$$

proving that $\mathcal{W}_{\max-\text{nor}}$ (resp., $\mathcal{W}_{\max}$) is convex. Next, consider $\{v_n\}_{n\in\mathbb{N}} \subseteq \mathcal{W}_{\max}$ such that $v_n \to v$. It is immediate to see that v is continuous. Moreover, if $p \succsim^* q$ then $\mathbb{E}_p(v_n) \geq \mathbb{E}_q(v_n)$ for all $n \in \mathbb{N}$. By passing to the limit, we obtain that $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$, that is, that $v \in \mathcal{W}_{\max}$, hence $\mathcal{W}_{\max}$ is closed.

2. By Dubra et al. (2004, Proposition 3), it follows that there exists $\hat{v} \in C([w,b])$ such that

$$\begin{aligned}&p \sim^* q \Longrightarrow \mathbb{E}_p(\hat{v}) = \mathbb{E}_q(\hat{v}) \\ &\quad \textit{and} \\ &p \succ^* q \Longrightarrow \mathbb{E}_p(\hat{v}) > \mathbb{E}_q(\hat{v}).\end{aligned}$$

By assumption, we have that $x \geq y$ if and only if $\delta_x \succsim^* \delta_y$. This implies that $x \geq y$ if and only if $\hat{v}(x) \geq \hat{v}(y)$, proving that $\hat{v}$ is strictly increasing. Since $\hat{v}$ is strictly increasing, by taking a positive and affine transformation, $\hat{v}$ can be chosen to be such that $\hat{v}(w) = 0 = 1 - \hat{v}(b)$. It is immediate to see that $\hat{v} \in \mathcal{W}_{\max-\text{nor}}$.

3. By definition of $\mathcal{W}_{\max}$, we have that if $p \succsim^* q$ then $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}_{\max}$. On the other hand, by assumption and since $\mathcal{W} \subseteq \mathcal{W}_{\max}$, we have that if $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}_{\max}$ then $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}$ which, in turn, implies that $p \succsim^* q$. In other words, $\mathcal{W}_{\max}$ satisfies (2) for $\succsim^*$. By assumption, we can thus conclude that

$$x \geq y \implies \delta_x \succsim^* \delta_y \implies \mathbb{E}_{\delta_x}(v) \geq \mathbb{E}_{\delta_y}(v) \ \ \forall v \in \mathcal{W}_{\max} \Longrightarrow v(x) \geq v(y) \ \ \forall v \in \mathcal{W}_{\max},$$

proving that $\mathcal{W}_{\max} \subseteq \mathcal{V}_{in}$. By point 2 and since $\mathcal{W}_{\max-\text{nor}} \subseteq \mathcal{W}_{\max}$, we have that $\emptyset \neq \mathcal{W}_{\max} \cap \mathcal{V}_{s-in}$. Since $\mathcal{W}_{\max} \cap \mathcal{V}_{s-in} \subseteq \mathcal{W}_{\max}$ and the latter is closed, we have that $cl\left(\mathcal{W}_{\max} \cap \mathcal{V}_{s-in}\right) \subseteq \mathcal{W}_{\max}$. On the other hand, consider $\dot{v} \in \mathcal{W}_{\max} \cap \mathcal{V}_{s-in}$ and $v \in \mathcal{W}_{\max}$. Define $\{v_n\}_{n\in\mathbb{N}}$ by $v_n = \frac{1}{n}\dot{v} + \left(1 - \frac{1}{n}\right) v$ for all $n \in \mathbb{N}$. Since $v, \dot{v} \in \mathcal{W}_{\max}$ and the latter set is convex, we have that $\{v_n\}_{n\in\mathbb{N}} \subseteq \mathcal{W}_{\max}$. Since $\dot{v}$ is strictly increasing and v is increasing, v_n is strictly increasing for all $n \in \mathbb{N}$, proving that $\{v_n\}_{n\in\mathbb{N}} \subseteq \mathcal{W}_{\max} \cap \mathcal{V}_{s-in}$. Since $v_n \to v$, it follows that $v \in cl\left(\mathcal{W}_{\max} \cap \mathcal{V}_{s-in}\right)$, proving that $\mathcal{W}_{\max} \subseteq cl\left(\mathcal{W}_{\max} \cap \mathcal{V}_{s-in}\right)$ and thus the opposite inclusion.

4. By assumption, we have that there exists a subset $\mathcal{W}$ of $C([w, b])$ such that $p \succsim^* q$ if and only if $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}$. By point 3 and its proof, we can replace first $\mathcal{W}$ with $\mathcal{W}_{\max}$ and then with $\mathcal{W}_{\max} \cap \mathcal{V}_{s-in}$. Consider $v \in \mathcal{W}_{\max} \cap \mathcal{V}_{s-in}$. Since v is strictly increasing, there exist (unique) $\gamma_1 > 0$ and $\gamma_2 \in \mathbb{R}$ such that $\bar{v} = \gamma_1 v + \gamma_2$ is continuous, strictly increasing, and satisfies $\bar{v}(w) = 0 = 1 - \bar{v}(b)$. For each $v \in \mathcal{W}_{\max} \cap \mathcal{V}_{s-in}$, it is immediate to see that $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ if and only if $\mathbb{E}_p(\bar{v}) \geq \mathbb{E}_q(\bar{v})$. Define $\bar{\mathcal{W}} = \{\bar{v} : v \in \mathcal{W}_{\max} \cap \mathcal{V}_{s-in}\}$. Notice that $\bar{\mathcal{W}} \subseteq \mathcal{U}_{\text{nor}}$. From the previous part, we can conclude that $p \succsim^* q$ if and only if $\mathbb{E}_p(\bar{v}) \geq \mathbb{E}_q(\bar{v})$ for all $\bar{v} \in \bar{\mathcal{W}}$. It is also immediate to see that $\bar{\mathcal{W}} \subseteq \mathcal{W}_{\max-\text{nor}}$. By construction of $\mathcal{W}_{\max-\text{nor}}$, notice that

$$p \succsim^* q \Longrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \mathcal{W}_{\max-\text{nor}}.$$

On the other hand, since $\bar{\mathcal{W}} \subseteq \mathcal{W}_{\max-\text{nor}}$, we have that

$$\mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \mathcal{W}_{\max-\text{nor}} \Longrightarrow \mathbb{E}_p(\bar{v}) \geq \mathbb{E}_q(\bar{v}) \qquad \forall \bar{v} \in \bar{\mathcal{W}} \Longrightarrow p \succsim^* q.$$

We can conclude that $\mathcal{W}_{\max-\text{nor}}$ represents $\succsim^*$.

5. Consider $v \in \mathcal{W}$. By assumption, v is a strictly increasing and continuous function on $[w, b]$ such that $v(w) = 0 = 1 - v(b)$. Moreover, since $\mathcal{W}$ satisfies (2), it follows that $p \succsim^* q$ implies that $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$. This implies that $v \in \mathcal{W}_{\max-\text{nor}}$. We can conclude that $\mathcal{W} \subseteq \mathcal{W}_{\max-\text{nor}}$, hence, $cl(\mathcal{W}) \subseteq cl(\mathcal{W}_{\max-\text{nor}})$. In order to prove the opposite inclusion, we argue by contradiction. Assume that there exists $v \in cl(\mathcal{W}_{\max-\text{nor}}) \setminus cl(\mathcal{W})$. Since $v \in cl(\mathcal{W}_{\max-\text{nor}})$, we have that $v(w) = 0 = 1 - v(b)$. By Dubra et al. (2004, p. 123–124) and since both $\mathcal{W}$ and $\mathcal{W}_{\max-\text{nor}}$ satisfy (2), we

also have that

$$cl\left(cone\left(\mathcal{W}\right)+\left\{\theta 1_{[w,b]}\right\}_{\theta\in\mathbb{R}}\right)=cl\left(cone\left(\mathcal{W}_{\max-\mathrm{nor}}\right)+\left\{\theta 1_{[w,b]}\right\}_{\theta\in\mathbb{R}}\right).$$

We can conclude that $v\in cl\left(cone\left(\mathcal{W}\right)+\left\{\theta 1_{[w,b]}\right\}_{\theta\in\mathbb{R}}\right)$. Observe that there exists $\left\{\hat{v}_n\right\}_{n\in\mathbb{N}}\subseteq cone\left(\mathcal{W}\right)+\left\{\theta 1_{[w,b]}\right\}_{\theta\in\mathbb{R}}$ such that $\hat{v}_n\to v$. By construction and since $\mathcal{W}$ is convex, there exist $\left\{\lambda_n\right\}_{n\in\mathbb{N}}\subseteq[0,\infty)$, $\left\{v_n\right\}_{n\in\mathbb{N}}\subseteq\mathcal{W}$, and $\left\{\theta_n\right\}_{n\in\mathbb{N}}\subseteq\mathbb{R}$ such that $\hat{v}_n=\lambda_n v_n+\theta_n 1_{[w,b]}$ for all $n\in\mathbb{N}$. It follows that

$$0=v\left(w\right)=\lim_n\hat{v}_n\left(w\right)=\lim_n\left\{\lambda_n v_n\left(w\right)+\theta_n 1_{[w,b]}\left(w\right)\right\}=\lim_n\theta_n$$

$$\text{and}$$

$$1=v\left(b\right)=\lim_n\hat{v}_n\left(b\right)=\lim_n\left\{\lambda_n v_n\left(b\right)+\theta_n 1_{[w,b]}\left(b\right)\right\}=\lim_n\left\{\lambda_n+\theta_n\right\}.$$

This implies that $\lim_n\theta_n=0=1-\lim_n\lambda_n$. Without loss of generality, we can thus assume that $\left\{\lambda_n\right\}_{n\in\mathbb{N}}$ is bounded away from zero, that is, that there exists $\varepsilon>0$ such that $\lambda_n\geq\varepsilon>0$ for all $n\in\mathbb{N}$. Since $\left\{\theta_n\right\}_{n\in\mathbb{N}}$ and $\left\{\hat{v}_n\right\}_{n\in\mathbb{N}}$ are both convergent, both sequences are bounded, that is, there exists $k>0$ such that

$$\left\|\hat{v}_n\right\|\leq k\text{ and }\left|\theta_n\right|\leq k\qquad\forall n\in\mathbb{N}.$$

It follows that

$$\begin{aligned}\varepsilon\left\|v_n\right\|\leq\lambda_n\left\|v_n\right\|=\left\|\lambda_n v_n\right\|&=\left\|\lambda_n v_n+\theta_n 1_{[w,b]}-\theta_n 1_{[w,b]}\right\|\\&\leq\left\|\lambda_n v_n+\theta_n 1_{[w,b]}\right\|+\left\|-\theta_n 1_{[w,b]}\right\|\\&\leq\left\|\lambda_n v_n+\theta_n 1_{[w,b]}\right\|+\left|\theta_n\right|\\&\leq\left\|\hat{v}_n\right\|+\left|\theta_n\right|\leq 2k\qquad\forall n\in\mathbb{N},\end{aligned}$$

that is, $\left\|v_n\right\|\leq\frac{2k}{\varepsilon}$ for all $n\in\mathbb{N}$. We can conclude that

$$\begin{aligned}\left\|v-v_n\right\|&=\left\|v-\hat{v}_n+\hat{v}_n-v_n\right\|\leq\left\|v-\hat{v}_n\right\|+\left\|\hat{v}_n-v_n\right\|\\&=\left\|v-\hat{v}_n\right\|+\left\|\lambda_n v_n+\theta_n 1_{[w,b]}-v_n\right\|\\&\leq\left\|v-\hat{v}_n\right\|+\left|\lambda_n-1\right|\left\|v_n\right\|+\left|\theta_n\right|\\&\leq\left\|v-\hat{v}_n\right\|+\left|\lambda_n-1\right|\frac{2k}{\varepsilon}+\left|\theta_n\right|\qquad\forall n\in\mathbb{N}.\end{aligned}$$

Passing to the limit, it follows that $v_n \to v$, that is, $v \in cl(\mathcal{W})$, a contradiction. $\blacksquare$

We next provide a characterization of $\succcurlyeq'$ which is due to Cerreia-Vioglio (2009). Here, it is further specialized to the particular case where $\succcurlyeq$ satisfies Weak Monotonicity and NCI in addition to Weak Order and Continuity. Before proving the statement, we need to introduce a piece of terminology. We will say that $\succcurlyeq''$ is an integral stochastic order if and only if there exists a set $\mathcal{W}'' \subseteq C([w,b])$ such that

$$p \succcurlyeq'' q \Longleftrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \mathcal{W}''.$$

Proposition 4. *Let $\succcurlyeq$ be a binary relation on Δ that satisfies Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence. The following statements are true:*

(a) There exists a set $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$ such that $p \succcurlyeq' q$ if and only if $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}$.

(b) For each $p, q \in \Delta$ if $p \succcurlyeq' q$ then $p \succcurlyeq q$.

(c) If $\succcurlyeq''$ is an integral stochastic order that satisfies (b) then $p \succcurlyeq'' q$ implies $p \succcurlyeq' q$.

(d) If $\succcurlyeq''$ is an integral stochastic order that satisfies (b) and such that $\mathcal{W}''$ can be chosen to be a subset of $\mathcal{U}_{\text{nor}}$ then $\overline{\text{co}}(\mathcal{W}) \subseteq \overline{\text{co}}(\mathcal{W}'')$.

Proof. (a). By Cerreia-Vioglio (2009, Proposition 22), there exists a set $\mathcal{W} \subseteq C([w,b])$ such that $p \succcurlyeq' q$ if and only if $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}$. By Lemma 1, we also have that $x \geq y$ if and only if $\delta_x \succcurlyeq' \delta_y$. By point 4 of Proposition 3, if $\succcurlyeq^* = \succcurlyeq'$ then $\mathcal{W}$ can be chosen to be $\mathcal{W}_{\max-\text{nor}}$.

(b), (c), and (d). The statements follow from Cerreia-Vioglio (2009, Proposition 22 and Lemma 35). $\blacksquare$

The next proposition clarifies what is the relation between our assumption of Weak Monotonicity and First Order Stochastic Dominance. Given $p, q \in \Delta$, we write $p \succcurlyeq_{FSD} q$ if and only if p dominates q with respect to First Order Stochastic Dominance.

Proposition 5. *If $\succcurlyeq$ is a binary relation on Δ that satisfies Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence then*

$$p \succcurlyeq_{FSD} q \Longrightarrow p \succcurlyeq q.$$

Proof. By Proposition 4, there exists $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$ such that

$$p \succcurlyeq' q \Longleftrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \mathcal{W}.$$

By Proposition 4 and since $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}} \subseteq \mathcal{V}_{in}$, it follows that

$$\begin{aligned} p \succcurlyeq_{FSD} q &\Longrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \mathcal{V}_{in} \\ &\Longrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \mathcal{W} \\ &\Longrightarrow p \succcurlyeq' q \Longrightarrow p \succcurlyeq q, \end{aligned}$$

proving the statement. ■

Appendix B: Proof of the Results in the Text

Proof of Theorem 1. Before starting, we point out that in proving (i) implies (ii), we will prove the existence of a Continuous Cautious Expected Utility representation $\mathcal{W}$ which is convex and normalized, that is, a subset of $\mathcal{U}_{\text{nor}}$. This will turn out to be useful in the proofs of other results in this section. The normalization of $\mathcal{W}$ will play no role in proving (ii) implies (i).

(i) implies (ii). We proceed by steps.

Step 1. There exists a continuous certainty equivalent utility function $V : \Delta \to \mathbb{R}$.

Proof of the Step.

Since $\succcurlyeq$ satisfies Weak Order and Continuity, there exists a continuous function $\bar{V} : \Delta \to \mathbb{R}$ such that $\bar{V}(p) \geq \bar{V}(q)$ if and only if $p \succcurlyeq q$. By Weak Monotonicity, we have that

$$x \geq y \Longleftrightarrow \delta_x \succcurlyeq \delta_y \Longleftrightarrow \bar{V}(\delta_x) \geq \bar{V}(\delta_y). \tag{3}$$

Next, observe that $\delta_b \succcurlyeq_{FSD} q \succcurlyeq_{FSD} \delta_w$ for all $q \in \Delta$. By Proposition 5 and since $\succcurlyeq$

satisfies Weak Order, Continuity, Weak Monotonicity, and NCI, this implies that

$$\delta_b \succcurlyeq q \succcurlyeq \delta_w \qquad \forall q \in \Delta. \tag{4}$$

Consider a generic $q \in \Delta$ and the sets

$$\{\delta_x : \delta_x \succcurlyeq q\} = \{p \in \Delta : p \succcurlyeq q\} \cap \{\delta_x\}_{x \in [w,b]}$$

and

$$\{\delta_x : q \succcurlyeq \delta_x\} = \{p \in \Delta : q \succcurlyeq p\} \cap \{\delta_x\}_{x \in [w,b]} .$$

By (4), Continuity, and Aliprantis and Border (2005, Theorem 15.8), both sets are nonempty and closed. Since $\succcurlyeq$ satisfies Weak Order, it follows that the sets

$$\{x \in [w,b] : \delta_x \succcurlyeq q\} \text{ and } \{x \in [w,b] : q \succcurlyeq \delta_x\}$$

are nonempty, closed, and their union coincides with $[w,b]$. Since $[w,b]$ is connected, there exists an element x_q in their intersection. In other words, there exists $x_q \in [w,b]$ such that $\delta_{x_q} \sim q$. Since q was chosen to be generic and by (3) and (4), such element is unique and we further have that

$$\bar{V}(\delta_b) \geq \bar{V}(q) = \bar{V}\left(\delta_{x_q}\right) = \bar{V}(q) \geq \bar{V}(\delta_w) \qquad \forall q \in \Delta. \tag{5}$$

Next, define $f : [w,b] \to \mathbb{R}$ by $f(x) = \bar{V}(\delta_x)$ for all $x \in [w,b]$. By (3), Aliprantis and Border (2005, Theorem 15.8), and (5), f is strictly increasing, continuous, and such that $f([w,b]) = \bar{V}(\Delta)$. It follows that $V : \Delta \to \mathbb{R}$ defined by $u = f^{-1} \circ \bar{V}$ is a well defined continuous function such that $p \succcurlyeq q$ if and only if $V(p) \geq V(q)$ and $V(\delta_x) = x$ for all $x \in [w,b]$, proving the statement. $\square$

Step 2. $\succcurlyeq'$ is represented by a set $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$, that is,

$$p \succcurlyeq' q \Longleftrightarrow c(p,v) \geq c(q,v) \qquad \forall v \in \mathcal{W}. \tag{6}$$

Proof of the Step.

It follows by point (a) of Proposition 4. Recall that $\mathcal{W}$ can be chosen to be $\mathcal{W}_{\max - \text{nor}}$ for $\succcurlyeq'$. $\square$

Step 3. For each $p \in \Delta$ we have that $\inf_{v \in \mathcal{W}} c(p,v) \in [w,b]$.

Proof of the Step.

Fix $p \in \Delta$. By construction, we have that $b \geq c(p, v) \geq w$ for all $v \in \mathcal{W}$. It follows that $c = \inf_{v \in \mathcal{W}} c(p, v)$ is a real number in $[w, b]$.

Step 4. For each $p \in \Delta$ we have that

$$V(p) \leq \inf_{v \in \mathcal{W}} c(p, v).$$

Proof of the Step.

Fix $p \in \Delta$. By Step 3, $c = \inf_{v \in \mathcal{W}} c(p, v)$ is a real number in $[w, b]$. Since $V(\Delta) = [w, b]$, if $c = b$ then we have that $V(p) \leq b = c$. Otherwise, pick d such that $b > d > c$. Since $d > c$, we have that there exists $\tilde{v} \in \mathcal{W}$ such that

$$c(p, \widetilde{v}) < d = c(\delta_d, \widetilde{v}).$$

By Step 2, it follows that $p \not\succcurlyeq' \delta_d$. By Lemma 1, this implies that $\delta_d \succ p$, that is, $V(p) < V(\delta_d) = d$. Since d was chosen to be generic and strictly greater than c, we have that $V(p) \leq c$, proving the statement. $\square$

Step 5. For each $p \in \Delta$ we have that

$$V(p) \geq \inf_{v \in \mathcal{W}} c(p, v).$$

Proof of the Step.

Fix $p \in \Delta$. By Step 3, $c = \inf_{v \in \mathcal{W}} c(p, v)$ is a real number in $[w, b]$. By construction, we have that

$$c(p, v) \geq c = c(\delta_c, v) \qquad \forall v \in \mathcal{W}.$$

By Step 2, it follows that $p \succcurlyeq' \delta_c$. By Proposition 4 point (b), this implies that $p \succcurlyeq \delta_c$, that is, $V(p) \geq V(\delta_c) = c$, proving the statement. $\square$

The implication follows from Steps 1, 2, 4, and 5.

(ii) implies (i). Assume there exists a set $\mathcal{W} \subseteq \mathcal{U}$ such that $V : \Delta \to \mathbb{R}$, defined by

$$V(p) = \inf_{v \in \mathcal{W}} c(p, v) \qquad \forall p \in \Delta,$$

is a continuous utility function for $\succcurlyeq$. Since $\succcurlyeq$ is represented by a continuous utility function, it follows that it satisfies Weak Order and Continuity. By construction, it is also immediate to see that $V(\delta_x) = x$ for all $x \in [w,b]$. In light of this fact, Weak Monotonicity follows immediately. Finally, consider $p \in \Delta$ and $x \in [w,b]$. Assume that $p \succcurlyeq \delta_x$. It follows that for each $\lambda \in [0,1]$ and for each $q \in \Delta$

$$\begin{aligned}
c(p,v) \geq V(p) \geq V(\delta_x) = x = c(\delta_x, v) \qquad \forall v \in \mathcal{W} \\
&\Longrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_{\delta_x}(v) \qquad \forall v \in \mathcal{W} \\
&\Longrightarrow \mathbb{E}_{\lambda p+(1-\lambda)q}(v) \geq \mathbb{E}_{\lambda\delta_x+(1-\lambda)q}(v) \qquad \forall v \in \mathcal{W} \\
&\Longrightarrow c(\lambda p + (1-\lambda)q, v) \geq c(\lambda\delta_x + (1-\lambda)q, v) \qquad \forall v \in \mathcal{W} \\
&\Longrightarrow V(\lambda p + (1-\lambda)q) \geq V(\lambda\delta_x + (1-\lambda)q) \\
&\Longrightarrow \lambda p + (1-\lambda)q \succcurlyeq \lambda\delta_x + (1-\lambda)q,
\end{aligned}$$

proving that $\succcurlyeq$ satisfies NCI. ■

Proof of Proposition 1. Consider $\mathcal{W}$ and $\mathcal{W}'$ in $\mathcal{U}_{\text{nor}}$ such that $\overline{\text{co}}(\mathcal{W}) = \overline{\text{co}}(\mathcal{W}')$. Notice first that if both $\mathcal{W}$ and $\mathcal{W}'$ are convex, the proposition follows trivially. To prove the proposition, it will therefore suffice to show that for each $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$ we have that

$$\inf_{v\in\mathcal{W}} c(p,v) = \inf_{v\in co(\mathcal{W})} c(p,v) \qquad \forall p \in \Delta.$$

Consider $p \in \Delta$. It is immediate to see that

$$\inf_{v\in\mathcal{W}} c(p,v) \geq \inf_{v\in co(\mathcal{W})} c(p,v).$$

Conversely, consider $\bar{v} \in co(\mathcal{W})$. It follows that there exist $\{v_i\}_{i=1}^n \subseteq \mathcal{W}$ and $\{\lambda_i\}_{i=1}^n \subseteq [0,1]$ such that $\sum\limits_{i=1}^n \lambda_i = 1$ and $\sum\limits_{i=1}^n \lambda_i v_i = \bar{v}$. Define $x \in [w,b]$ and $\{x_i\}_{i=1}^n \subseteq [w,b]$ by $x = c(p,\overline{v})$ and $x_i = c(p,v_i)$ for all $i \in \{1,...,n\}$. By contradiction, assume that $x < \min_{i\in\{1,...,n\}} x_i$. Since $\{v_i\}_{i=1}^n \subseteq \mathcal{W} \subseteq \mathcal{V}_{s-in}$, we have that

$$\mathbb{E}_p(\bar{v}) = \mathbb{E}_p\left(\sum_{i=1}^n \lambda_i v_i\right) = \sum_{i=1}^n \lambda_i \mathbb{E}_p(v_i) = \sum_{i=1}^n \lambda_i v_i(x_i) > \sum_{i=1}^n \lambda_i v_i(x) = \bar{v}(x),$$

that is, $x = c(p,\overline{v}) > x$, a contradiction. This implies that

$$c(p,\overline{v}) = x \geq \min_{i\in\{1,...,n\}} x_i = \min_{i\in\{1,...,n\}} c(p,v_i) \geq \inf_{v\in\mathcal{W}} c(p,v).$$

Since $\bar{v}$ was chosen to be generic in $co\left(\mathcal{W}\right)$, we can conclude that

$$c\left(p,\overline{v}\right) \geq \inf_{v\in\mathcal{W}} c\left(p,v\right) \qquad \forall \bar{v} \in co\left(\mathcal{W}\right),$$

proving that $\inf_{v\in\mathcal{W}} c\left(p,v\right) \leq \inf_{v\in co(\mathcal{W})} c\left(p,v\right)$ and thus the statement. ∎

Proof of Theorem 2. By the proof of Theorem 1 (Steps 1, 2, 4, and 5), we have that there exists a set $\widehat{\mathcal{W}} \subseteq \mathcal{U}_{\text{nor}}$ such that

$$p \succcurlyeq' q \Longleftrightarrow \mathbb{E}_p\left(v\right) \geq \mathbb{E}_q\left(v\right) \qquad \forall v \in \widehat{\mathcal{W}} \tag{7}$$

and such that $V : \Delta \to \mathbb{R}$, defined by

$$V\left(p\right) = \inf_{v\in\widehat{\mathcal{W}}} c\left(p,v\right) \qquad \forall p \in \Delta, \tag{8}$$

is a continuous utility function for $\succcurlyeq$. This proves points (i) and (iii). Next consider a subset $\mathcal{W}$ of $\mathcal{U}_{\text{nor}}$ such that the function $V : \Delta \to \mathbb{R}$ defined by $V\left(p\right) = \inf_{v\in\mathcal{W}} c\left(p,v\right)$ for all $p \in \Delta$ represents $\succcurlyeq$. Define $\succcurlyeq''$ by

$$p \succcurlyeq'' q \Longleftrightarrow \mathbb{E}_p\left(v\right) \geq \mathbb{E}_q\left(v\right) \qquad \forall v \in \mathcal{W}.$$

It is immediate to see that if $p \succcurlyeq'' q$ then $p \succcurlyeq q$. By point (d) of Proposition 4, this implies that $\overline{\text{co}}\left(\widehat{\mathcal{W}}\right) \subseteq \overline{\text{co}}\left(\mathcal{W}\right)$, proving point (ii).

Finally, consider two sets $\widehat{\mathcal{W}}_1$ and $\widehat{\mathcal{W}}_2$ in $\mathcal{U}_{\text{nor}}$ that satisfy (7) and (8). By point 5 of Proposition 3, it follows that $\overline{\text{co}}(\widehat{\mathcal{W}}_1) = \overline{\text{co}}(\widehat{\mathcal{W}}_2)$. ∎

Proof of Theorem 3. We just prove point (i) since point (ii) follows by an analogous argument. Given $p \in \Delta$, we denote by $e(p)$ its expected value. We say that $p \succcurlyeq_{MPS} q$ if and only if q is a mean preserving spread of p.[32] Recall that $\succcurlyeq$ is risk averse if and

[32] Recall that, by Rothschild and Stiglitz (1970), if p and q are elements of Δ_0 and q is a mean preserving spread of p, then p and q have the same mean and they give the same probability to each point in their support with the exception of four ordered points $x_1 < x_2 < x_3 < x_4$. There the following relations hold:

$$\begin{gathered} q\left(x_1\right) - p\left(x_1\right) = p\left(x_2\right) - q\left(x_2\right) \geq 0 \\ \textit{and} \\ q\left(x_4\right) - p\left(x_4\right) = p\left(x_3\right) - q\left(x_3\right) \geq 0. \end{gathered}$$

only if $p \succcurlyeq_{MPS} q$ implies $p \succcurlyeq q$. Assume that $\succcurlyeq$ is risk averse. Let $p, q \in \Delta_0$.[33] Since Δ_0 is dense in Δ and $\succcurlyeq$ satisfies Weak Order and Continuity, we have that

$$\begin{aligned} p \succcurlyeq_{MPS} q &\Longrightarrow \lambda p + (1-\lambda) r \succcurlyeq_{MPS} \lambda q + (1-\lambda) r \qquad \forall \lambda \in (0,1], \forall r \in \Delta_0 \\ &\Longrightarrow \lambda p + (1-\lambda) r \succcurlyeq \lambda q + (1-\lambda) r \qquad \forall \lambda \in (0,1], \forall r \in \Delta_0 \\ &\Longrightarrow \lambda p + (1-\lambda) r \succcurlyeq \lambda q + (1-\lambda) r \qquad \forall \lambda \in (0,1], \forall r \in \Delta \\ &\Longrightarrow p \succcurlyeq' q. \end{aligned}$$

This implies that

$$p \succcurlyeq_{MPS} q \Longrightarrow p \succcurlyeq' q \Longrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \widehat{\mathcal{W}}.$$

We can conclude that each v in $\widehat{\mathcal{W}}$ is concave. For the other direction, assume that each v in $\widehat{\mathcal{W}}$ is concave. Since $\widehat{\mathcal{W}} \subseteq \mathcal{V}_{inco}$, we have that

$$\begin{aligned} p \succcurlyeq_{MPS} q &\Longrightarrow e(p) = e(q) \text{ and } \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \mathcal{V}_{inco} \\ &\Longrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \widehat{\mathcal{W}} \Longrightarrow p \succcurlyeq' q \Longrightarrow p \succcurlyeq q, \end{aligned}$$

proving that $\succcurlyeq$ is risk averse. ■

Proof of Theorem 4. Before proceeding, we make a few remarks. Fix $i \in \{1,2\}$. By the proof of Theorem 1 and since $\succcurlyeq_i$ satisfies Weak Order, Continuity, Weak Monotonicity, and NCI, it follows that $\mathcal{W}^i_{\max-\mathrm{nor}}$ for $\succcurlyeq'_i$ constitutes a Continuous Cautious Expected Utility representation of $\succcurlyeq_i$. Since $\mathcal{W}^i_{\max-\mathrm{nor}}$ is convex, if $\widehat{\mathcal{W}}_i$ is chosen as in Theorem 2 then we have that $\overline{\mathrm{co}}(\widehat{\mathcal{W}}_i)$ coincides with the closure of $\mathcal{W}^i_{\max-\mathrm{nor}}$. Also recall that for each $p \in \Delta$, we denote by x^i_p the element in $[w, b]$ such that $p \sim_i \delta_{x^i_p}$. We also have that $V_i : \Delta \to \mathbb{R}$, defined by

$$V_i(p) = \inf_{v \in \mathcal{W}^i_{\max-\mathrm{nor}}} c(p, v) = \inf_{v \in \widehat{\mathcal{W}}_i} c(p, v) = \inf_{v \in \mathcal{W}_i} c(p, v) \qquad \forall p \in \Delta,$$

represents $\succcurlyeq_i$, yielding that $x^i_p = V_i(p)$.

(i) implies (ii) and (i) implies (iii). Since $\succcurlyeq_1$ is more risk averse than $\succcurlyeq_2$, we have that $p \sim_1 \delta_{x^1_p}$ implies $p \succcurlyeq_2 \delta_{x^1_p}$. Since $\succcurlyeq_2$ satisfies Weak Order and Weak Monotonicity,

[33] Recall that Δ_0 is the subset of Δ which contains just the elements with finite support.

it follows that $x_p^2 \geq x_p^1$ for all $p \in \Delta$. This implies that

$$V_1(p) = \min\{V_1(p), V_2(p)\} = \inf_{v \in \mathcal{W}_1 \cup \mathcal{W}_2} c(p, v) \qquad \forall p \in \Delta,$$

that is, $\mathcal{W}_1 \cup \mathcal{W}_2$ is a Continuous Cautious Expected Utility representation of $\succcurlyeq_1$. By the remark in Section 3, it follows that $\widehat{\mathcal{W}_1 \cup \mathcal{W}_2}$ is also a Continuous Cautious Expected Utility representation of $\succcurlyeq_1$. By the initial part of the proof, we can conclude that $\overline{\mathrm{co}}(\widehat{\mathcal{W}}_1) = cl\left(\mathcal{W}^1_{\max-\mathrm{nor}}\right) = \overline{\mathrm{co}}(\widehat{\mathcal{W}_1 \cup \mathcal{W}_2})$.

(iii) implies (i). Since $\overline{\mathrm{co}}(\widehat{\mathcal{W}}_1) = cl\left(\mathcal{W}^1_{\max-\mathrm{nor}}\right) = \overline{\mathrm{co}}(\widehat{\mathcal{W}_1 \cup \mathcal{W}_2})$, it follows that

$$\begin{aligned} V_1(p) &= \inf_{v \in \widehat{\mathcal{W}}_1} c(p, v) = \inf_{v \in \widehat{\mathcal{W}_1 \cup \mathcal{W}_2}} c(p, v) \\ &= \inf_{v \in \mathcal{W}_1 \cup \mathcal{W}_2} c(p, v) = \min\{V_1(p), V_2(p)\} \ \leq \ V_2(p) \quad \forall p \in \Delta, \end{aligned}$$

proving that $x_p^2 \geq x_p^1$ for all $p \in \Delta$. It follows that $\succcurlyeq_1$ is more risk averse than $\succcurlyeq_2$.

(ii) implies (i). Since $\mathcal{W}_1 \cup \mathcal{W}_2$ is a Continuous Cautious Expected Utility representation of $\succcurlyeq_1$, it follows that

$$V_1(p) = \inf_{v \in \mathcal{W}_1 \cup \mathcal{W}_2} c(p, v) \leq \inf_{v \in \mathcal{W}_2} c(p, v) \leq V_2(p) \qquad \forall p \in \Delta,$$

proving that $x_p^2 \geq x_p^1$ for all $p \in \Delta$. It follows that $\succcurlyeq_1$ is more risk averse than $\succcurlyeq_2$. ■

Proof of Proposition 2. (i) We first prove necessity. By Theorem 2 and since $\succcurlyeq_1$ and $\succcurlyeq_2$ satisfy Weak Order, Continuity, Weak Monotonicity, and NCI, we have that

$$p \succcurlyeq_i' q \Longleftrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \widehat{\mathcal{W}}_i \Longleftrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \overline{\mathrm{co}}(\widehat{\mathcal{W}}_i). \quad (9)$$

By Proposition 4 point (b) and since $\succcurlyeq_1$ is more indecisive than $\succcurlyeq_2$, we have that

$$p \succcurlyeq_1' q \Longrightarrow p \succcurlyeq_2' q \Longrightarrow p \succcurlyeq_2 q.$$

By Proposition 4 point (d) and (9), we can conclude that $\overline{\mathrm{co}}(\widehat{\mathcal{W}}_2) \subseteq \overline{\mathrm{co}}(\widehat{\mathcal{W}}_1)$. We next prove sufficiency. By (9) and since $\overline{\mathrm{co}}(\widehat{\mathcal{W}}_2) \subseteq \overline{\mathrm{co}}(\widehat{\mathcal{W}}_1)$, we have that

$$p \succcurlyeq_1' q \Longrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \ \forall v \in \overline{\mathrm{co}}(\widehat{\mathcal{W}}_1) \Longrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \ \forall v \in \overline{\mathrm{co}}(\widehat{\mathcal{W}}_2) \Longrightarrow p \succcurlyeq_2' q,$$

proving point (i).

(ii) By (9) and Proposition 3, we have that $\overline{\mathrm{co}}(\widehat{\mathcal{W}}_i) = cl\left(\mathcal{W}^i_{\max - \mathrm{nor}}\right)$ for $i \in \{1, 2\}$. Since $\succcurlyeq_1$ is more indecisive than $\succcurlyeq_2$, it follows that $cl\left(\mathcal{W}^2_{\max - \mathrm{nor}}\right) \subseteq cl\left(\mathcal{W}^1_{\max - \mathrm{nor}}\right)$. By definition of $\mathcal{W}^1_{\max - \mathrm{nor}}$ and $\mathcal{W}^2_{\max - \mathrm{nor}}$, it follows that $\mathcal{W}^2_{\max - \mathrm{nor}} \subseteq \mathcal{W}^1_{\max - \mathrm{nor}}$. By the proof of Theorem 1, this implies that

$$V_1(p) = \inf_{v \in \mathcal{W}^1_{\max - \mathrm{nor}}} c(p, v) \leq \inf_{v \in \mathcal{W}^2_{\max - \mathrm{nor}}} c(p, v) = V_2(p) \qquad \forall p \in \Delta.$$

Since each V_i is a continuous certainty equivalent utility function, it follows that $\succcurlyeq_1$ is more risk averse than $\succcurlyeq_2$. ■

Proof of Theorem 5. Let $\succcurlyeq'$ be a reflexive and transitive binary relation on Δ that satisfies Sequential Continuity, Weak Monotonicity, and Independence. We first prove the existence of a Cautious Completion. In doing this, we show that this completion has a Cautious Expected Utility representation. By Dubra et al. (2004), there exists a set $\mathcal{W} \subseteq C([w, b])$ such that $p \succcurlyeq' q$ if and only if $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}$. By Proposition 3, without loss of generality, we can assume that $\mathcal{W} \subseteq \mathcal{U}_{\mathrm{nor}} \subseteq \mathcal{U}$.

Next define the binary relation $\hat{\succcurlyeq}$ as

$$p \mathrel{\hat{\succcurlyeq}} q \Longleftrightarrow \inf_{v \in \mathcal{W}} c(p, v) \geq \inf_{v \in \mathcal{W}} c(q, v). \tag{10}$$

Notice that $\hat{\succcurlyeq}$ is well defined and it satisfies Weak Order, Weak Monotonicity, and clearly for each $p \in \Delta$ there exists $x \in [w, b]$ such that $p \mathrel{\hat{\sim}} \delta_x$. Next, we show $\hat{\succcurlyeq}$ is a completion of $\succcurlyeq'$. Since each $v \in \mathcal{W}$ is strictly increasing, we have that

$$p \succcurlyeq' q \Longleftrightarrow c(p, v) \geq c(q, v) \qquad \forall v \in \mathcal{W}. \tag{11}$$

This implies that if $p \succcurlyeq' q$ then $\inf_{v \in \mathcal{W}} c(p, v) \geq \inf_{v \in \mathcal{W}} c(q, v)$, that is, if $p \succcurlyeq' q$ then $p \hat{\succcurlyeq} q$. Finally, let x be an element of $[w, b]$ and $p \in \Delta$ such that $p \not\succcurlyeq' \delta_x$. By (11), it follows that there exists $\tilde{v} \in \mathcal{W}$ such that $c(\delta_x, \widetilde{v}) = x > c(p, \widetilde{v})$. By (10), this implies that $\inf_{v \in \mathcal{W}} c(\delta_x, v) = x > \inf_{v \in \mathcal{W}} c(p, v)$, hence $\delta_x \mathrel{\hat{\succ}} p$. This concludes the proof of the existence of a Cautious Completion.

We are left with proving uniqueness. Let $\succcurlyeq^\circ$ be a Cautious Completion of $\succcurlyeq'$. By point 1 of Definition 5, $\succcurlyeq^\circ$ satisfies Weak Order, Weak Monotonicity, and for each $p \in \Delta$ there exists $x \in [w, b]$ such that $p \sim^\circ \delta_x$. This implies that there exists $V : \Delta \to \mathbb{R}$ such that V represents $\succcurlyeq^\circ$ and $V(\delta_x) = x$ for all $x \in [w, b]$. Moreover, we have that $V(\Delta) = [w, b]$. Let $p \in \Delta$. Define $c = \inf_{v \in \mathcal{W}} c(p, v) \in [w, b]$. If $c = b$ then $V(p) \leq b = c = \inf_{v \in \mathcal{W}} c(p, v)$. If $c < b$ then for each $d \in (c, b)$ there exists $\tilde{v} \in \mathcal{W}$ such that $c(\delta_d, \widetilde{v}) = d > c(p, \widetilde{v})$, yielding that $p \not\succcurlyeq' \delta_d$. By point 3 of Definition 5, we can conclude that $\delta_d \succ^\circ p$, that is, $d = V(\delta_d) > V(p)$. Since d was arbitrarily chosen in (c, b), it follows that $V(p) \leq c = \inf_{v \in \mathcal{W}} c(p, v)$. Finally, by definition of c and (11), we have that $c(p, v) \geq c = c(\delta_c, v)$ for all $v \in \mathcal{W}$, that is, $p \succcurlyeq' \delta_c$. By point 2 of Definition 5, it follows that $p \succcurlyeq^\circ \delta_c$, that is, $V(p) \geq V(\delta_c) = c = \inf_{v \in \mathcal{W}} c(p, v)$. In other words, we have shown that $V(p) = \inf_{v \in \mathcal{W}} c(p, v)$ for all $p \in \Delta$. By (10) and since V represents $\succcurlyeq^\circ$, we can conclude that $\succcurlyeq^\circ = \hat{\succcurlyeq}$, proving the statement. ■

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