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“Skewed Noise”

by

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Skewed Noise*

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Experimental evidence suggests that individuals who face an asymmetric distribution over the likelihood of a specific event might actually prefer not to know the exact value of this probability. We address these findings by studying a decision maker who has recursive, non-expected utility preferences over two-stage lotteries. For a binary lottery that yields the better outcome with probability p , we identify noise around p with a compound lottery that induces a probability distribution over the exact value of the probability and has an average value p . We first propose and characterize a new notion of skewed distributions. We then use this result to provide conditions under which a decision maker who always rejects symmetric noise around p will always reject skewed to the left noise, but might accept skewed to the right noise. The model can be applied to the areas of investment under risk, medical decision making, and criminal law procedures, and can also be used to address the phenomenon of ambiguity seeking in the context of decision making under uncertainty.

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1 Introduction

The standard model of decision making under risk assumes that individuals obey the reduction of compound lotteries axiom. According to this axiom, a decision maker is indifferent between any multi-stage lottery and its single-stage counterpart, that is, the simple lottery that induces the same probability distribution over final outcomes. Experimental and empirical evidence suggest, however, that this axiom is often violated (see, among others, Kahneman and Tversky [23], Bernasconi and Loomes [3], Conlisk [9], Harrison, Martinez-Correa, and Swarthout [19], and Abdellaoui, Klibanoff, and Placido [1]). Individuals may have preferences between gambles with identical probability distributions over final outcomes if they differ in the timing of resolution of uncertainty. Alternatively, individuals may simply like or dislike not knowing the exact values of the probabilities. For example, subjects are typically not indifferent between betting on a known probability p_0 and betting on a known distribution over the value of that probability even when the mean probability of the distribution is p_0 .

Halevy [18] and recently Miao and Zhong [29], for example, consider preferences over two-stage lotteries and demonstrate that individuals are averse to introducing symmetric noises, that is, symmetric mean-preserving spread into the first-stage lottery. One rationale for this kind of behavior is that symmetric noises that cancel out each other simply create an undesired confusion in evaluation. On the other hand, asymmetric noises, and in particular positively skewed ones, may be desirable. Boney [4], for example, conducted an experiment in which decision makers had to choose one of three investment plans. In all three prospects, the overall probability of success (which results in a prize $M = \$200$) is $p = 0.2$, and with the remaining probability the investment fails and the decision maker receives $m = \$0$. Option *A* represents an investment plan in which the decision maker is confident about the probability of success. This corresponds to the simple binary lottery that yields M with probability 0.2. In *B* and *C*, on the other hand, the probability of success is uncertain. Prospect *B* represents a negatively skewed distribution around p in which it is very likely that the true probability slightly exceeds p but it is also possible (albeit unlikely) that the true probability is much lower. Specifically, the true probability is either 0.22, with probability $\frac{9}{10}$ or 0.02, with probability $\frac{1}{10}$. Prospect *C* represents a positively skewed distribution where the true probability is either 0.18, with probability $\frac{9}{10}$ or 0.38, with probability $\frac{1}{10}$. Boney's [4] main finding is that decision makers are not

indifferent between the three prospects and that most prefer C to A to B . Moreover, these preferences are robust to different values of m , M , and p . One possible explanation of this behavior is that option B entails a possibility of extreme disappointment, which might enforce the basic, intrinsic aversion to symmetric noise. Option C , on the other hand, admits possible elation, which, if strong enough, might make it desirable despite its underlying noise.

In Boney’s experiment, the underlying probability of success p was the same in all three options. In a recent experiment, which we discuss in more detail in Section 5, Abdellaoui, Klibanoff, and Placido [1] found strong evidence that aversion to compound risk (noise) is an increasing function of p . In particular, their results are consistent with a greater aversion to negatively skewed noises around high probabilities than to positively skewed noises around small probabilities. Our goal in this paper is to suggest a model that can accommodate the behavioral patterns discussed above. We use Segal’s [34] framework of two-stage lotteries without reduction and outline conditions under which a decision maker who always rejects (small) symmetric noises will also reject negative skewed asymmetric noises, but nevertheless might seek positive skewed asymmetric noises.

Our first step is to introduce a new notion of skewness and to show that a distribution F is skewed to the left [right] if and only if it is the limit of a sequence of distributions originating at a degenerate distribution and in which each distribution is obtained from its predecessor by a symmetric split of an outcome below [above] its average. We then apply this result to our study of attitudes toward two-stage lotteries in the face of asymmetric noise.

Fix two outcomes $x > y$ and denote by $\langle p_1, q_1; \dots, p_n, q_n \rangle$ the two-stage lottery which yields with probability q_i the simple lottery $(x, p_i; y, 1 - p_i)$. Note that the total probability of receiving the prize x in this compound lottery is $\sum_i p_i q_i$. We say that the decision maker is averse to symmetric noise at p if there is $\bar{\alpha}$ such that for any $\alpha \leq \bar{\alpha}$ and $\varepsilon \leq 0.5$, he prefers the simple lottery p to the two-stage lottery $\langle p + \alpha, \varepsilon; p, 1 - 2\varepsilon; p - \alpha, \varepsilon \rangle$. Our main result states conditions under which if the decision maker is averse to symmetric noise, then he must also be averse to negatively skewed noise. These conditions do not imply specific attitude towards skewed to the right distributions. In this case, it may, but doesn’t have to be, that a distribution over the exact value of p is desirable. This result is consistent with evidence from lab experiments, and can be applied to the areas of investment under risk, medical decision making, and criminal law procedures. It can also be used to address the phenomenon of ambiguity seeking in the context of

decision making under uncertainty (see Section 5 for details).

In this paper, we confine our attention to the analysis of attitudes to noise related to the probability p in the binary lottery $(x, p; y, 1 - p)$ with $x > y$. In reality the decision maker may, and probably will, face lotteries with many outcomes and the probabilities of receiving each of them may be uncertain. We prefer to deal only with binary lotteries for two reasons. The first is that many situations involve just two outcomes (or so they are perceived). For example, marriages succeed or fail; candidates are elected or not; exams are passed or not, etc. The second reason is that when there are many outcomes their probabilities depend on each other and therefore skewed noise over the probability of one event may affect noises over other probabilities in too many ways. This complication is avoided when there are only two outcomes. Whatever we believe about the behavior of the probability of receiving x completely determines our beliefs regarding the probability of receiving y . Note that while the underlying lottery is binary, the noise itself (that is, the distribution over the value of p) may have many possible values or may even be continuous.

The rest of the paper is organized as follows: Section 2 introduces notations and definitions that will be used in our main analysis. Section 3 defines and characterizes our notion of negatively skewed distributions. Section 4 studies attitudes towards asymmetric noises and states our main behavioral result. Section 5 describes numerous applications and connect our analysis to empirical evidence. Section 6 comments on the relation of our paper to a version of the recursive model in which the two stages are evaluated using different expected utility functionals (Kreps and Porteus [24], Klibanoff, Marinacci, and Mukerjis [25]). All proofs are relegated to the appendix.

2 Preliminaries

Let $I = [\underline{x}, \bar{x}]$ be some compact segment of \Re . A lottery X over I is simple if it has a finite support. Denote by \mathcal{F} the set of all cumulative distribution functions of simple lotteries over I and let \succeq be a preference relation over \mathcal{F} . We will identify simple lotteries with their distributions. That is, for lotteries X and Y with the distributions F and G , we write $X \succeq Y$ iff $F \succeq G$. We assume throughout that \succeq is monotonic with respect to first-order stochastic dominance and continuous with respect to the weak topology.

We analyze noise as two-stage lotteries, that is, lotteries over lotteries

over I . A typical two-stage lottery $(X_1, q_1; \dots; X_m, q_m)$ yields the simple lottery X_i with probability q_i . As in Segal [34], there are two ways in which such lotteries can be transformed into simple lotteries: using the reduction of compound lotteries axiom or using the compound independence axiom.

Definition 1 *The preference relation \succeq is extended to two-stage lotteries via the reduction of compound lotteries axiom if for $X_i = (x_i^1, p_i^1; \dots; x_i^{n_i}, p_i^{n_i})$, $i = 1, \dots, m$,*

$$(X_1, q_1; \dots; X_m, q_m) \sim (\dots; x_i^j, q_i p_i^j; \dots)$$

Definition 2 *The preference relation \succeq is extended to two-stage lotteries via the compound independence axiom if for $X_i = (x_i^1, p_i^1; \dots; x_i^{n_i}, p_i^{n_i})$, $i = 1, \dots, m$,*

$$(X_1, q_1; \dots; X_m, q_m) \sim (c_1, q_1; \dots; c_m, q_m)$$

where c_i is the certainty equivalent of X_i , that is, $(c_i, 1) \sim X_i$.¹

The reduction axiom suggests an objective way to transform two-stage lotteries into simple ones, using probability calculus. Compound independence, on the other hand, allows the transformation to be subjective, since the corresponding simple lottery is defined via preferences. In line with many experimental evidence (e.g. Conlisk [9] and Kahneman and Tversky [23]), we assume in this paper the compound independence but not the reduction axiom. Our notion of aversion (affinity) towards noise, which we introduce in Section 4, relies on the difference in the evaluation of $(X_1, q_1; \dots; X_m, q_m)$ and its objective single stage counterpart, $(\dots; x_i^j, q_i p_i^j; \dots)$.

In the rest of the section we introduce some of the concepts we will use for our analysis in subsequent sections. We say that the preference relation is risk averse if $F \succeq G$ whenever G is a mean preserving spread of F . It is quasi-concave (quasi-convex) if for all $F, G \in \mathcal{F}$, $F \sim G$ implies $\frac{1}{2}F + \frac{1}{2}G \succeq F$ ($\preceq F$). Let V represent \succeq . In Section 4.1 we follow Machina [26] and assume that \succeq is smooth, in the sense that V is Fréchet differentiable, defined as follows.

¹Observe that by continuity and monotonicity of preferences, the certainty equivalent of every simple lottery is well defined.

Definition 3 *The function $V : \mathcal{F} \rightarrow \mathfrak{R}$ is Fréchet differentiable if for every $F \in \mathcal{F}$ there exists a “local utility” function $u_F(\cdot) : \mathfrak{R} \rightarrow \mathfrak{R}$ such that for every $G \in \mathcal{F}$,*

$$V(G) - V(F) = \int u_F(x) d[G(x) - F(x)] + o(\|G - F\|) \quad (1)$$

where $\|\cdot\|$ is the L_1 -norm.

For fixed $x > y > z$, let (p, q) represent the distribution of the lottery $(z, p; y, 1 - p - q; x, q)$, where $(p, q) \in \mathfrak{R}_+^2$ and $p + q \leq 1$. Such lotteries are represented in a Marschak-Machina [26] triangle (see panel (i) of Fig. 1). The dotted parallels to the tangent line at the distribution F to the indifference curve through this point are indifference curves of the local utility function $u_F(\cdot)$.

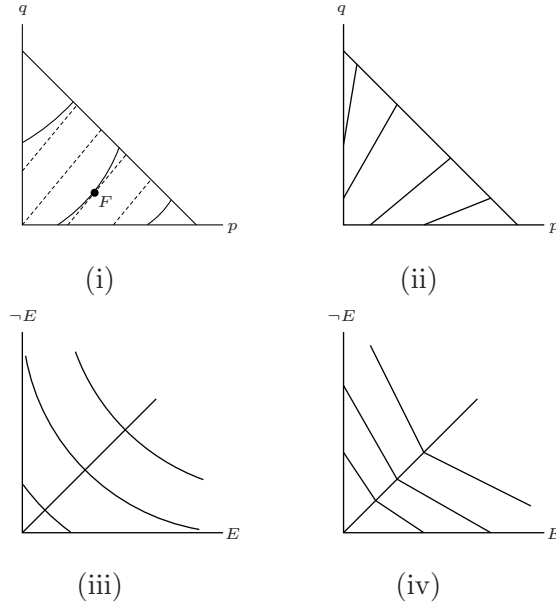


Figure 1: Indifference curves

We use in Section 4.1 the following assumption, taken from Machina [26]:

Definition 4 *The preferences \succeq that are represented by the Fréchet differentiable functional V satisfy hypothesis II if for every x*

$$-\frac{u_G''(x)}{u_G'(x)} \geq -\frac{u_F''(x)}{u_F'(x)}$$

whenever G dominates F by first-order stochastic dominance.

In the Marschak-Machina triangle, hypothesis II means that indifference curves are “fanning out” (panel (ii)). The term $-\frac{u_F''(\cdot)}{u_F'(\cdot)}$ is analogous to the Arrow-Pratt measure of risk aversion and has the same interpretation as in expected utility theory. Loosely speaking, hypothesis II implies that if G dominates F by first-order stochastic dominance, then $u_G(\cdot)$ represents a higher degree of risk aversion than $u_F(\cdot)$.

For $F \in \mathcal{F}$, let $s + F$ and $t \times F$ be the distributions $F(x - s)$ and $F(\frac{x}{t})$, respectively. For $F \in \mathcal{F}$ with expected value zero and for $x^* \in \mathfrak{R}$, define the risk premium function π implicitly by $\delta_{x^* - \pi(x^*, F)} \sim x^* + F$ where δ_x is the distribution of the lottery that yields x with probability one. Following Segal and Spivak [35] (see also Epstein and Zin [12]), we say that the preference relation \succeq satisfies second-order risk aversion if for every x^* and for every F with expected value zero,

$$\left. \frac{\partial}{\partial t} \pi(x^*, t \times F) \right|_{t=0^+} = 0 \quad \text{and} \quad \left. \frac{\partial^2}{\partial t^2} \pi(x^*, t \times F) \right|_{t=0^+} > 0$$

If V is Fréchet differentiable, then \succeq satisfies second-order risk aversion. There are interesting preference relations that are not Fréchet differentiable (and that do not satisfy second-order risk aversion). The most famous of them are Quiggin’s [30] rank-dependent utility (see Chew, Karni, and Safra [6]) and Gul’s [17] theory of disappointment aversion. These preference relations, which we study in Section 4.2, satisfy first-order risk aversion: for any x^* and every non-trivial F with expected value zero,

$$\left. \frac{\partial}{\partial t} \pi(x^*, t \times F) \right|_{t=0^+} > 0$$

First-order risk aversion implies kinked indifference curves along the main diagonal in a states-of-the-world representation (fig. 1, panel (iv)), while second-order risk aversion implies smooth such indifference curves (panel (iii), see Segal and Spivak [35]). The concept of orders of risk aversion has some economic applications. See for example Segal and Spivak [35] for applications to insurance purchases and Epstein and Zin [12] for applications to the equity premium puzzle.

3 Sequences of Symmetric Shifts

As stated in the introduction, our aim in this paper is to analyze preferences for skewed noise, that is, for noise that is not symmetric around its mean. For that we need first to formally define the notion of a skewed distribution. The analysis of this section is interesting by its own right as it turns out that a natural manipulation of lotteries leads to a new, and non-trivial definition of skewness. To simplify notation and terminology we only analyze skewness to the left, but all definitions and results can be made with skewness to the right.

For a distribution F on $[\underline{x}, \bar{x}]$ with expected value μ and for $\delta > 0$, let

- $\eta_1(F, \delta) = \int_{\underline{x}}^{\mu - \delta} F(x) dx$
- $\eta_2(F, \delta) = \int_{\mu + \delta}^{\bar{x}} [1 - F(x)] dx$.

Definition 5 *The lottery X with the distribution F on $[\underline{x}, \bar{x}]$ and expected value μ is skewed to the left if for every $\delta > 0$, $\eta_1(F, \delta) \geq \eta_2(F, \delta)$, that is, if the area below F between \underline{x} and $\mu - \delta$ is larger than the area above F between $\mu + \delta$ and \bar{x} (see Fig. 2).*

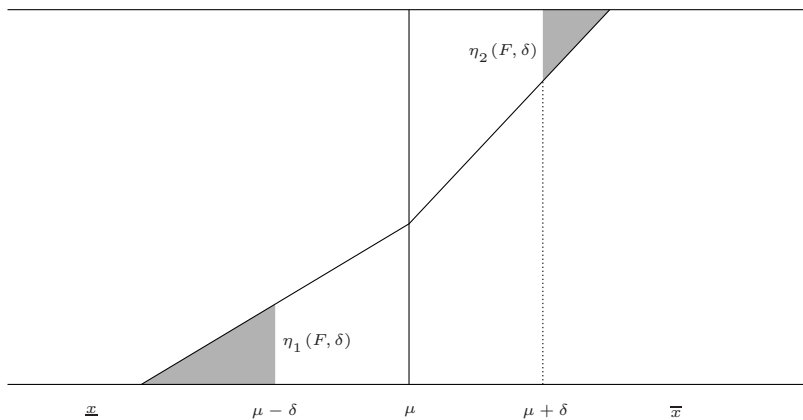


Figure 2: Definition 5: $\eta_1(F, \delta) \geq \eta_2(F, \delta)$

Definition 6 Let μ be the expected value of a lottery X . Lottery Y is obtained from X by a left symmetric split if Y is the same as X , except for that one of the outcomes x of X such that $x \leq \mu$ was split into $x + \alpha$ and $x - \alpha$, each with half of the probability of x .

Our aim is to show that a lottery Y with expected value μ is skewed to the left iff it is the limit of a sequence of lotteries originating at $(\mu, 1)$ and in which each lottery is obtained from its predecessor by a left symmetric split (Theorem 1 below). A by-product of this theorem is that it will enable us to show that our definition of skewness is stronger than other definitions in the literature (see Section 3.1 below). Formally,

Theorem 1

1. If the lottery $Y = (y_1, p_1; \dots; y_n, p_n)$ with expected value μ is skewed to the left, then there is a sequence of lotteries X_i , each with expected value μ , such that $X_1 = (\mu, 1)$, $X_i \rightarrow Y$, and X_{i+1} is obtained from X_i by a left symmetric split. Moreover, it can be done such that in each step the size of the spread is bounded by $\max_i y_i - \mu$.
2. Consider the sequence $\{X_i\}$ of lotteries where $X_1 = (\mu, 1)$ and X_{i+1} is obtained from X_i by a left symmetric split. Then the distributions F_i of X_i converge and the limit distribution F is skewed to the left.

To illustrate the theorem, consider two examples, one where the procedure terminates in a finite number of steps and one where it does not. In both cases we move from a degenerate lottery X to a skewed to the left binary lottery Y with the same expected value as X . For the first example, let $X = (3, 1)$ and $Y = (0, \frac{1}{4}; 4, \frac{3}{4})$ and obtain

$$X = (3, 1) \rightarrow (2, \frac{1}{2}; 4, \frac{1}{2}) \rightarrow (0, \frac{1}{4}; 4, \frac{1}{4} + \frac{1}{2}) = Y$$

For a sequence that does not terminate let $X = (5, 1)$ and $Y = (0, \frac{1}{6}; 6, \frac{5}{6})$. Here we obtain

$$\begin{aligned} X &= (5, 1) \rightarrow (4, \frac{1}{2}; 6, \frac{1}{2}) \rightarrow (2, \frac{1}{4}; 6, \frac{3}{4}) \rightarrow (0, \frac{1}{8}; 4, \frac{1}{8}; 6, \frac{3}{4}) \rightarrow \dots \\ &\quad (0, \frac{1}{2} \sum_1^n \frac{1}{4^i}; 4, \frac{1}{2^{4^n}}; 6, \frac{1}{2} + \sum_1^n \frac{1}{4^i}) \rightarrow \dots (0, \frac{1}{6}; 6, \frac{5}{6}) = Y \end{aligned}$$

The two parts of the theorem do not create a simple if and only if statement, because the support of the limit distribution F in part 2 need not be

finite. On the other hand, part 1 of the theorem does not hold for continuous distributions. By the definition of left symmetric splits, if the probability of $x > \mu$ in X_i is p , then for all $j > i$, the probability of x in X_j must be at least p . It thus follows that the distribution F cannot be continuous above μ . However, the following claim will enable us to use Theorem 1 even for continuous distributions.

Claim 1 *If F with expected value μ is skewed to the left, then there is a sequence of finite skewed to the left distributions F_n , each with expected value μ , such that $F_n \rightarrow F$.*

Theorem 1 is proved in Appendix A. Claim 1 and the rest of the paper's formal statements are proved in Appendix B. The main difficulty in the proof of Theorem 1 is the simple fact that whereas outcomes to the left of μ can be manipulated, any split of probabilities that lands some probability to the right of μ must hit its exact place according to Y , as we will not be able to touch it later again.

We use Theorem 1 in section 4 to prove our main behavioral result — attitudes towards symmetric noise that is added to the value of the probability of an outcome can be linked with attitudes towards asymmetric noises.

3.1 Other Definitions of Skewness

We show here that our definition of skewness (Definition 5 above) is stronger than other definitions in the literature.

Claim 2 *If X with expected value μ is skewed to the left, then the highest outcome in the median $\bar{m}(X)$ of X satisfies $\bar{m}(X) \geq \mu$.*

Another possible definition of skewness suggests that the lottery X with the distribution F and expected value μ is skewed to the left if $\int_{\underline{x}}^{\bar{x}} (y - \mu)^3 dF(y) \leq 0$.

Claim 3 *If X with distribution F and expected value μ is skewed to the left as in Definition 5, then for all odd n , $\int_{\underline{x}}^{\bar{x}} (y - \mu)^n dF(y) \leq 0$.*

The converse of Claim 3 is false. For example, let F be the distribution of the lottery $(-10, \frac{1}{10}; -2, \frac{1}{2}; 0, \frac{4}{35}; 7, \frac{2}{7})$. Note that its expected value μ is zero. Moreover, $E[(X - \mu)^3] = -6 < 0$, which means that F is skewed to

the left according to the sign of its third moment. However, the area below the distribution from -10 to -5 is $\frac{1}{2}$, but the area above the distribution from 5 to 10 is $\frac{4}{7} > \frac{1}{2}$, which means that F is not skewed to the left according to Definition 5.

Menezes, Geiss, and Tressler [28] characterize a notion of increasing downside risk by combining a mean-preserving spread of an outcome below the mean followed by a mean-preserving contraction of an outcome above the mean, in a way that the overall result is a transfer of risk from the right to the left of a distribution, keeping the variance intact. Distribution F has more downside risk than distribution G if one can move from G to F in a sequence of such mean-variance-preserving transformations. Menezes et al. [28] do not provide a definition (and a characterization as in our Theorem 1) of a skewed to the left distribution. Observe that our characterization involves a sequence of only symmetric left splits, starting in the degenerate lottery that puts all the mass on the mean. In particular, our splits are not mean-variance-preserving and occur only in one side of the mean.

4 Asymmetric Noise

In this section we discuss the main topic of the paper, namely the attitude of decision makers to two-stage lotteries of the form $(X_1, q_1; \dots; X_m, q_m)$ where $X_i = (x, p_i; y, 1 - p_i)$ with $x > y$. We denote this two-stage lottery by $\langle p_1, q_1; \dots; p_m, q_m \rangle$ and discuss the case where this distribution is not necessarily symmetric around $\sum q_j p_j$. We divide the analysis into two sub-cases: Fréchet differentiable (Machina [26]) and non-differentiable functionals. Note that we keep the two outcomes x and y in the underlying lottery fixed throughout our analysis.

4.1 Fréchet Differentiable Functionals

To illustrate the analysis of Theorem 2 below, we start with a (relatively) simple numerical example. The purpose of the example is to illustrate that under some conditions, rejection of all symmetric noises implies rejection of negatively skewed noise as well.

Suppose that we know that for every $p \in [\frac{7}{10}, \frac{8}{10}]$, for every $\alpha \leq \frac{2}{10}$, and for every $q \leq \bar{q}$ the decision maker prefers the lottery $(100, p; 0, 1 - p)$ to

the two-stage lottery that yields the three lotteries $(100, p - \alpha; 0, 1 - p + \alpha)$, $(100, p; 0, 1 - p)$, and $(100, p + \alpha; 0, 1 - p - \alpha)$, with the probabilities q , $1 - 2q$, and q , respectively (see Fig. 3 for an example where $p = \frac{8}{10}$, $\alpha = \frac{1}{10}$, and $q = \frac{1}{7}$). By the compound independence axiom, the two-stage lottery is indifferent to $(c_{p-\alpha}, q; c_p, 1 - 2q; c_{p+\alpha}, q)$, where c_r is the certainty equivalent of the lottery $(100, r; 0, 1 - r)$.

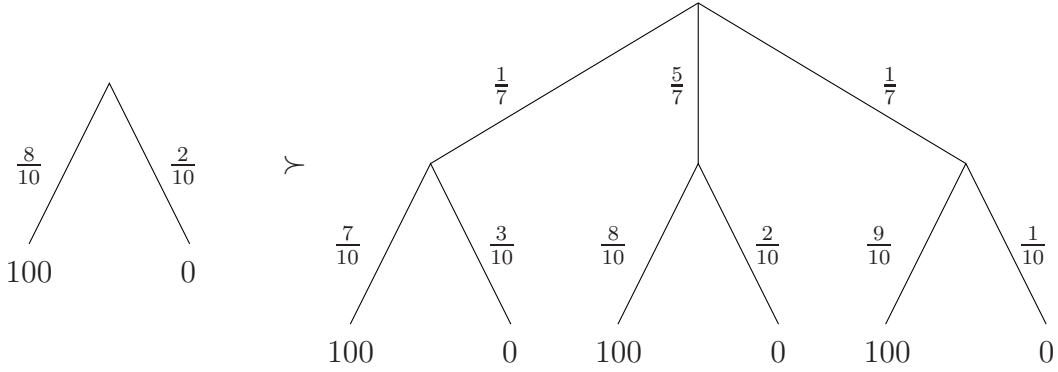


Figure 3: Symmetric noise around $\frac{8}{10}$

Suppose now that the preferences \succeq are strictly quasi-concave, that they can be represented by a Fréchet differentiable functional V , and that they satisfy Hypothesis II (see Definitions 3 and 4 in Section 2). Observe that the distribution of $(c_{p-\alpha}, q; c_p, 1 - 2q; c_{p+\alpha}, q)$ is a $2q : 1 - 2q$ mixture of the distributions of $(c_{p-\alpha}, \frac{1}{2}; c_{p+\alpha}, \frac{1}{2})$ and $(c_p, 1)$. Since for every $q \in (0, \frac{1}{2}]$ the decision maker strictly prefers $(c_p, 1)$ to $(c_{p-\alpha}, q; c_p, 1 - 2q; c_{p+\alpha}, q)$, it follows by strict quasi-concavity that for all $q < q' \leq \frac{1}{2}$ the desirability of $(c_{p-\alpha}, q; c_p, 1 - 2q; c_{p+\alpha}, q)$ is declining with q , that is

$$(c_{p-\alpha}, q; c_p, 1 - 2q; c_{p+\alpha}, q) \succ (c_{p-\alpha}, q'; c_p, 1 - 2q'; c_{p+\alpha}, q') \quad (2)$$

In particular, preferences strictly decrease if $(100, 0.8; 0, 0.2)$ is replaced by an even chance lottery over $(100, 0.7; 0, 0.3)$ and $(100, 0.9; 0, 0.1)$, and if $(100, 0.7; 0, 0.3)$ is replaced by an even chance lottery over $(100, 0.5; 0, 0.5)$ and $(100, 0.9; 0, 0.1)$. We are now interested in the path that takes us from $(c_{0.7}, \frac{1}{2}; c_{0.9}, \frac{1}{2})$ to $(c_{0.5}, \frac{1}{4}; c_{0.9}, \frac{3}{4})$. For every $q \leq \frac{1}{4}$, the lottery $(c_{0.5}, q; c_{0.7}, \frac{1}{2} - 2q; c_{0.9}, \frac{1}{2} + q)$ with the distribution $A(q)$ stochastically dominates $(c_{0.5}, 2q; c_{0.7}, 1 - 4q; c_{0.9}, 2q)$ with the distribution $B(q)$ (observe that $A(q)$ is the midpoint

on the path between $(c_{0.9}, 1)$ and $B(q)$ which is dominated by $(c_{0.9}, 1)$). By hypothesis II, for every x , the local utility $u_{A(q)}(x)$ represents a higher degree of risk aversion than the local utility $u_{B(q)}(x)$. By eq. (2), for all $q \leq \frac{1}{4}$ the local utility $u_{B(q)}$ prefers $(c_{0.7}, 1)$ to $(c_{0.5}, \frac{1}{2}; c_{0.9}, \frac{1}{2})$, hence by hypothesis II so does the local utility $u_{A(q)}$. It therefore follows that $(c_{0.7}, \frac{1}{2}; c_{0.9}, \frac{1}{2}) \succ (c_{0.5}, \frac{1}{4}; c_{0.9}, \frac{3}{4})$ and by transitivity, $(c_{0.8}) \succ (c_{0.5}, \frac{1}{4}; c_{0.9}, \frac{3}{4})$.

Consider two-stage lotteries of the form $((x, p_1; y, 1 - p_1), q_1; \dots; (x, p_n; y, 1 - p_n), q_n)$ where $x > y$, and let $p = \sum q_i p_i$. Using recursive utility and assuming the compound independence axiom, this lottery is indifferent to $(c_{p_1}, q_1; \dots; c_{p_n}, q_n)$. Theorem 2 generalizes the above procedure and shows conditions under which the decision maker will prefer the average probability p to the distribution. This theorem does *not* say what happens if its assumptions are satisfied but beliefs are skewed to the right. In that case it may happen (even if it doesn't have to) that a distribution over the value of p is actually desired.

Theorem 2 *Suppose that the decision maker's preferences over lotteries can be represented by the Fréchet differentiable functional V , that his preferences satisfy Machina's [26] Hypothesis II, and that he evaluates two-stage lotteries using the compound independence axiom. Consider a distribution over the value of the probability p in the lottery $(x, p; y, 1 - p)$ where $x > y$.*

If V is quasi-concave and for all $p \in [p_1, p_2]$, for all $\alpha \leq \alpha^$, and for all $\varepsilon \leq \frac{1}{2}$, the decision maker rejects the symmetric distribution $\langle p - \alpha, \varepsilon; p, 1 - 2\varepsilon; p + \alpha, \varepsilon \rangle$, then he also rejects any skewed to the left distribution over the value of the probability in this segment.*

In a similar way we can prove that if V is quasi-convex and for all $p \in [p_1, p_2]$, for all $\alpha \leq \alpha^*$, and for all $\varepsilon \leq \frac{1}{2}$, the decision maker accepts the symmetric distribution $\langle p - \alpha, \varepsilon; p, 1 - 2\varepsilon; p + \alpha, \varepsilon \rangle$, then he also accepts any skewed to the right distribution over the value of the probability in this segment.

Remark: Theorem 2 does not restrict the location of the skewed distribution, but it is reasonable to find skewed to the left distributions over the value of the probability p when p is high, and skewed to the right distributions when p is low. By Chew and Waller [8] and Chew, Epstein, and Segal [5], preferences tend to be quasi-concave at higher levels of utility and

quasi-convex at lower levels. Thus the theorem is consistent with the empirical observation that decision makers reject skewed to the left distributions concerning high probability of the good event, but seek such distributions when the probability of the good event is low.

4.2 Non-Differentiable Functionals

In this subsection we analyze functionals representing first-order risk aversion, e.g. rank-dependent utility (Quiggin [30]), dual theory (Yaari [38]), or disappointment aversion (Gul [17]). As mentioned in Section 2, indifference curves of such functionals have kinks along the certainty line in a state-of-nature diagram. Therefore, even for small lotteries, the order of magnitude of the risk premium a decision maker is willing to pay to avoid a lottery is the same as the order of magnitude of the lottery. We will use the following result, taken from Segal and Spivak [35, section 5].

Fact 1 *Let \succeq satisfy first-order risk aversion and let φ be a differentiable function such that $\varphi(0) = 0$. Let X be a lottery with expected value 0 and let $F_{\varphi(tX)}$ be the distribution function of the lottery $(\dots; \varphi(tx_i), p_i; \dots)$. Then*

$$\left. \frac{\partial}{\partial t} \pi(x^*, F_{\varphi(tX)}) \right|_{t=0^+} > 0$$

Consider now the two-stage lottery $(X_1, q_1; \dots; X_m, q_m)$ where $X_i = (x, p_i; y, 1 - p_i)$ with $x > y$ and assume that it is evaluated recursively using the compound independence axiom. Let $\bar{p} = \sum q_j p_j$ and let $r_j = p_j - \bar{p}$, hence the expected value of R is 0 where $R = (r_1, q_1; \dots; r_m, q_m)$. Define the function $\varphi(r)$ to be the certainty equivalent of the lottery $(x, \bar{p} + r; y, 1 - \bar{p} - r)$ and consider the case where φ is differentiable.² By fact 1 it follows that if preferences represent first-order risk aversion, then for a sufficiently small t , the decision maker prefers the lottery $(x, \bar{p}; y, 1 - \bar{p})$ to the two-stage lottery $(\dots; (x_j, \bar{p} + tr_j; y, 1 - \bar{p} - tr_j), q_j; \dots)$ regardless of whether the distribution of r is symmetric or not. In particular, for a sufficiently small $\bar{\varepsilon} > 0$ the decision maker rejects the symmetric distribution $(p - \alpha, \varepsilon; p, 1 - 2\varepsilon; p + \alpha, \varepsilon)$ for all $\varepsilon \leq \bar{\varepsilon}$. If on top of it the preferences \succeq are quasi-concave, then the decision maker rejects this lottery for all ε .

²This is the case if u and f are differentiable in Quiggin [30], if f is differentiable in Yaari [38], and if u is differentiable in Gul [17].

Using recursive analysis of two-stage lotteries, Dillenberger [10] proved that in Gul's [17] disappointment aversion model, risk averse decision makers always prefer the one-shot lottery $(x, p; y, 1 - p)$ to the compound lottery $\langle p_1, q_1; \dots; p_m, q_m \rangle$ where $\sum q_j p_j = p$, regardless of its skewness. Risk loving decision makers have the opposite preferences. Segal [33] showed, by means of an example, that in the rank-dependent model it is possible to have aversion to some symmetric noise together with preferences for some skewed to the right noise. This paper also offered sufficient conditions for the *rejection* of (some) asymmetric noise.

Consider the rank-dependent [30] model, where for $x_1 \leq \dots \leq x_n$, the value of the lottery $(x_1, p_1; \dots; x_n, p_n)$ is

$$u(x_n)f(p_n) + \sum_{i=1}^{n-1} u(x_i) \left[f\left(\sum_{j=i}^n p_j\right) - f\left(\sum_{j=i+1}^n p_j\right) \right] \quad (3)$$

This functional represents risk aversion iff u is concave and f convex (see Chew, Karni, and Safra [6]). Our aim here is to offer sufficient conditions under which the decision maker is rejecting symmetric binary noise as well as skewed to the left binary noise while still *accepting* (some) skewed to the right noise. We do not claim that these conditions are necessarily appealing or that they have a simple normative justification. And being sufficient (rather than necessary and sufficient) conditions they do not cover all the cases where skewed to the right noise may be desired even though all symmetric and skewed to the left noises are not (see Example 1 in Appendix C for a function that belongs to the desired set of functionals even though it does not satisfy the sufficient conditions). Our goal is to show that our analysis is not vacuous when preferences represent first-order risk aversion, and moreover, that it does not refer just to pathological examples.

Claim 4 *Consider the rank-dependent model and assume risk aversion. If*

1. $f''' < 0$
2. $\frac{1}{2^n} \geq f(\frac{1}{2})$ for the first n such that the n -th order derivative $f^{(n)}(0) \neq 0$
3. $f(p)/f(2p)$ is a concave function

Then the decision maker rejects symmetric noise as well as skewed to the left noise, but there are skewed to the right noises that he will accept.

Since preferences are continuous, we get these acceptance and rejection in open neighborhoods. The only restriction on the utility function u is that it is (weakly) concave (for risk aversion).

5 Applications

In this section we discuss several applications of Theorem 2 and connect our results to some empirical studies.

5.1 Ambiguity Aversion and Seeking

Ambiguity aversion is one of the most investigated phenomenon in decision theory. Consider the classic Ellsberg [11] thought experiment: subjects are presented with two urns. Urn 1 contains 100 red and black balls, but the exact color composition is unknown. Urn 2 has exactly 50 red and 50 black balls. Subjects are asked to choose an urn to draw a ball from, and to bet on the color that will be drawn. If a bet on a specific urn is correct the subject wins \$100. Otherwise, the subject gets nothing. Let C_i be the bet on a color (Red or Black) draw from Urn i . Ellsberg predicted that most subjects will be indifferent between R_1 and B_1 as well as between B_2 and R_2 , but will strictly prefer R_2 to R_1 and B_2 to B_1 . While, based on symmetry arguments, it seems plausible that the number of red balls in urn 1 equals the number of black balls, Urn 1 is ambiguous in the sense that the exact distribution is unknown whereas urn 2 is risky, as the probabilities are known. An ambiguity averse decision maker will prefer to bet on the risky urn to bet on the ambiguous one. Ellsberg's predictions were confirmed in many experiments.³

The recursive model we study here was first suggested by Segal [33] as a formal way to capture ambiguity aversion.⁴ Under this interpretation, ambiguity is identified as two-stage lotteries. The first stage captures the decision maker's uncertainty about the true probability distribution over the states

³A comprehensive reference to the evidence on Ellsberg-type behavior, and on attitude towards ambiguity in general, can be found in Peter Wakker's annotated bibliography, posted at his website under <http://people.few.eur.nl/wakker/refs/webrfnscs.doc>.

⁴There are many other ways to model ambiguity aversion. Prominent examples include Choquet expected utility (Schmeidler [32], Gilboa [14]), maximin expected utility (Gilboa and Schmeidler [15]), variational preferences (Maccheroni, Marinacci, and Rustichini [27]), α -maxmin (Ghirardato, Maccheroni, and Marinacci [13]), and the smooth model of ambiguity aversion (Klibanoff, Marinacci, and Mukerji [25]).

of the world (the true composition of the urn in Ellsberg's example), and the second stage determines the probability of each outcome, conditional on the probability distribution that has been realized. For example, if the decision maker believes that in Urn 1 all possible combinations of red and black are equally likely, then he perceives the bet on Urn 1 as a two-stage lottery, in which, for $i = 0, 1, \dots, 100$, the second-stage lottery $(100, \frac{i}{101}; 0, 1 - \frac{i}{101})$ is realized with probability $\frac{1}{101}$. More generally, this notion of ambiguity aversion corresponds to the case where there is some set of states of the world, and the decision maker does not know the exact probability distribution over the states. Instead, he has in mind a set of conceivable distributions and, furthermore, he is able to assign (subjective) probabilities to the different distributions in this set. Holding the prior probability distribution over states fixed, an ambiguity averse decision maker prefers the objective (unambiguous) simple lottery to any (ambiguous) compound one. Note that according to Segal's model, preferences over ambiguous prospects are induced from preferences over the compound lotteries that reflect the decision maker's beliefs. That is, the first stage is imaginary and corresponds to the decision maker's subjective beliefs over the true probabilities.

While Ellsberg-type behavior seems intuitive and is widely documented, there are situations where decision makers actually prefer not to know the probabilities with much preciseness. Becker and Bronwson [2, footnote 4] describe a conversation where Ellsberg himself suggested that people may prefer ambiguity with respect to a low probability good event. There is indeed a growing experimental literature that challenges the assumption that people are globally ambiguity averse. See, among others, Chew, Miao, and Zhong [7], and van de Kuilen and Wakker [36]. Consider the following situation. A person suspects that there is a high probability that he will face a bad outcome (severe loss of money, serious illness, criminal conviction, etc.). Yet he believes that there is some (small) chance things are not as bad as they seem (Federal regulations will prevent the bank from taking possession of his home, it is really nothing, they won't be able to prove it). These beliefs might emerge, for example, from consulting with a number of experts (such as accountants, doctors, lawyers) who disagree in their opinions; the vast majority of which are negative but some believe the risk is much less likely. Does the decision maker really want to know the exact probabilities of these events? The main distinction between the sort of ambiguity in Ellsberg's experiment and the ambiguity in the last examples is that the latter is asymmetric and, in particular, is positively skewed. On the other hand, if the decision maker

expects a good outcome with high probability, he would probably prefer to know this probability for sure, rather than knowing that there is actually a small chance that things are not that good. In other words, asymmetric but negatively skewed ambiguity may well be undesired.

To illustrate attitudes towards skewed ambiguity, consider the following example.⁵ A decision maker can choose one of three investment plans, denoted A , B , and C . For each of these plans, the decision maker believes that the overall probability that the investment be successful (which results in a prize M) is p , and with the remaining probability the investment fails and the decision maker will get $m \leq 0$. Option A represents an investment plan in which the decision maker is confident about its probability of success, but the decision maker is uncertain about the true probability of success in both options B and C . Option B describes a negatively skewed distribution around p , where the decision maker believes it is very likely that the true probability slightly exceeds p but it is also possible (albeit unlikely) that the true probability is actually very low. Prospect C , on the other hand, describes a positively skewed distribution around p , where the decision maker believes it is very likely that the true probability falls slightly below p but there is a small chance it is actually a decent probability. It is plausible that the decision maker will not be indifferent between the three options and, in particular, that he displays the ranking $C \succ A \succ B$. One possible explanation is that option B entails a possibility of extreme disappointment which might enforce the basic, intrinsic ambiguity aversion. Option C , on the other hand, admits possible elation, which, if strong enough, might make it desirable despite its underlying uncertainty. Experimental evidence (e.g. Boiney [4]) supports this intuition.

Our model is consistent with this type of behavior. Let $(M, p; m, 1 - p)$ denotes a conceivable probability distribution over the two possible states. As before, denote by c_p the certainty equivalent of the lottery $(M, p; m, 1 - p)$. For some $q_1, q_2 \in (\frac{1}{2}, 1)$ and $p_1, p_2, p_3, p_4 \in (0, 1)$ such that $p_1 < p_2 < p < p_3 < p_4$ and $q_1 p_3 + (1 - q_1) p_1 = q_2 p_2 + (1 - q_2) p_4 = p$, the three investment options A , B , and C can be described as the two-stage lotteries $\langle p, 1 \rangle$, $\langle p_3, q_1; p_1; 1 - q_1 \rangle$, and $\langle p_2, q_2; p_4, 1 - q_2 \rangle$, respectively. The recursive utility descriptions of these compound lotteries are $\tilde{A} = (c_p, 1)$, $\tilde{B} = (c_{p_3}, q_1; c_{p_1}; 1 - q_1)$, and

⁵This example is slightly different from the intuition given in the previous paragraph; it demonstrates different ambiguity attitude towards different patterns of noise around a given average value, rather than towards skewed to the right (left) noise around low (high) average probability.

$\tilde{C} = (c_{p_2}, q_2; c_{p_4}, 1 - q_2)$, respectively. Since B is skewed to the left, the hypothesis of Theorem 2 implies that \tilde{A} is preferred to \tilde{B} . And since C is skewed to the right, we have no implication for the ranking of \tilde{C} and \tilde{A} . In particular, the model permits having strict preference for \tilde{C} , in line with our intuition and with the observed data.⁶

5.2 Further Applications and Experimental Evidence

In this section we discuss additional implications and potential applications of the model. Our model can be applied to medical decision making. Consider, for example, a patient who needs to choose between two possible treatments. The first is an established pharmaceutical drug, whose probability of being effective is known. The other is a new type of surgery, on which there is not enough data and hence its exact probability of curing the disease is uncertain. Our model can accommodate different choices which are based on the initial severity of the problem. For example, if the disease is terminal, medicine can help only in a small proportion of cases (say 10 percent). The effectiveness of a complicated surgery at this stage, on the other hand, is unknown. It is believed to be highly likely to fail, but anecdotal evidence suggests that if properly done it might be quite effective. The patient may opt for it due to the small chance that it will actually be successful. The surgery in this example offers a skewed to the right noise, which might be desirable when the base rate is low. But if both the medicine and the surgery are known to be effective with the same high average rate of success, only that the surgery involves some little uncertainty emerging from unobtainable information regarding the patient's exact condition (in which case the surgery is much less likely to succeed), it is plausible that he and his doctor will opt for the drug, rather than facing the uncertainty related to the effectiveness of the surgery. This is an example of aversion to skewed to the left noise.

Another application is offered by Horowitz and Segal [20]. In order to achieve maximal deterrence, society should choose criminal procedures which are disfavored by potential criminals. The authors argue that as the probability of detection is often low while the probability of court conviction is

⁶Boiney [4] used a two-stage lottery as an operational definition of ambiguity. His experiment involves prospects like the ones described in the text, with the specification $M = \$200$, $m = \$0$, $p = 0.2$, $q_1 = q_2 = 0.9$, $p_1 = 0.02$, $p_2 = 0.18$, $p_3 = 0.22$, and $p_4 = 0.38$. His data strongly supports the ranking $C \succ A \succ B$. He further showed that this ranking is robust for a wide range of parameters.

high, and as decision makers, i.e. potential criminals prefer unambiguous probabilities of likely good events and ambiguous probabilities of likely bad events, society ought to make the former ambiguous and the latter unambiguous. For an analysis of court procedures that is based on similar preferences, see [21].

Our results can explain some of the findings in a recent paper by Abdellaoui, Klibanoff, and Placido [1]. For three different compound lotteries, subjects were asked for their compound lottery premium (as in Dillenberger [10]), that is, the amount they will be willing to pay to replace a compound lottery with its binary, single-stage counterpart. The underlying binary lottery yields €50 with probability p and 0 otherwise. The three two-stage lotteries were $\langle 0.5, \frac{1}{6}; 0, \frac{5}{6} \rangle$, $\langle 1, \frac{5}{22}; 0.5, \frac{12}{22}; 0, \frac{5}{22} \rangle$, and $\langle 1, \frac{5}{6}; 0.5, \frac{1}{6} \rangle$, with base probabilities of winning $p = \frac{1}{12}$, $p = \frac{1}{2}$, and $p = \frac{11}{12}$, respectively. (As before, the lottery $\langle p_1, q_1; \dots, p_n, q_n \rangle$ obtains the true probability p_i with probability q_i , $i = 1, \dots, n$). They found that the compound lottery premium is an increasing function of p . They also refer to some other studies that provide evidence for the pattern of more compound risk aversion for high probabilities than for low probabilities, and even for compound risk seeking for low probabilities (see, for example, Kahn and Sarin [22] and Viscusi and Chesson [37]). We argue that it is not only the magnitude of the probabilities that drive their results, but the fact that in the three lotteries above, noise is positively skewed, symmetric, and negatively skewed, respectively. As we have pointed out after the statement of Theorem 2, it is more plausible to find skewed to the left distributions over the value of the probability p when p is high, and skewed to the right distributions when p is low. Our theoretical predictions indeed suggest that individuals may be the least averse to positively skewed noise, and that such noise may even be desirable.

6 Relaxing Time Neutrality

In our model, the function V used in the recursive evaluation of a two-stage lottery is the same in both stages. This is known as the *time neutrality* assumption (Segal [34]). Consider instead a model in which the decision maker is an expected utility maximizer in each of the two stages, but he is not time neutral. More specifically, let u and v be the vNM utility functions over outcomes in the first and in the second stages. The decision maker

evaluates a two-stage lottery $(X_1, q_1; \dots; X_n, q_n)$ recursively by

$$V(X_i, q_i; \dots; X_n, q_n) = \sum_i q_i u(v^{-1}(E_v[X_i]))$$

where $v^{-1}(E_v[X])$ is the certainty equivalent of lottery X calculated using the function v . In the context of temporal lotteries, this model is a special case of the one studied by Kreps and Porteus [24]. In the context of ambiguity, this is the model of Klibanoff, Marinacci, Mukerji [25]. We now argue that this model cannot address the phenomenon of rejecting all symmetric noise but still accepting some (positively skewed) noise. To see this, fix $x > y$ and note that for any p , the value of the noise $\langle p + \alpha, \frac{1}{2}; p - \alpha, \frac{1}{2} \rangle$ is

$$\begin{aligned} & \frac{1}{2}u(v^{-1}[(p + \alpha)v(x) + (1 - p - \alpha)v(y)]) + \\ & \frac{1}{2}u(v^{-1}[(p - \alpha)v(x) + (1 - p + \alpha)v(y)]) \end{aligned}$$

Without time neutrality, the simple lottery $(x, p; y, 1 - p)$ can be translated into either $((x, p; y, 1 - p), 1)$, with the recursive value

$$u(v^{-1}[pv(x) + (1 - p)v(y)])$$

Or into $(\delta_x, p; \delta_y, 1 - p)$,⁷ with the recursive value

$$pu(v^{-1}[v(x)]) + (1 - p)u(v^{-1}[v(y)]) = pu(x) + (1 - p)u(y)$$

Assume first that the decision maker views it as the lottery $((x, p; y, 1 - p), 1)$. Then rejection of symmetric noise implies that for any p and α in the relevant range,

$$\begin{aligned} & u(v^{-1}[pv(x) + (1 - p)v(y)]) \geq \\ & \frac{1}{2}u(v^{-1}[(p + \alpha)v(x) + (1 - p - \alpha)v(y)]) + \\ & \frac{1}{2}u(v^{-1}[(p - \alpha)v(x) + (1 - p + \alpha)v(y)]) \end{aligned} \tag{4}$$

Pick any two numbers $a > b$ in $[0, 1]$ and note that by setting $p = \frac{1}{2}(a + b)$ and $\alpha = \frac{1}{2}(a - b)$, inequality (4) is equivalent to the requirement that

$$\begin{aligned} & u(v^{-1}[\frac{1}{2}(a + b)v(x) + (1 - \frac{1}{2}(a + b))v(y)]) \geq \\ & \frac{1}{2}u(v^{-1}[av(x) + (1 - a)v(y)]) + \frac{1}{2}u(v^{-1}[bv(x) + (1 - b)v(y)]) \end{aligned}$$

⁷Recall that δ_x is the distribution of the lottery that yields x with probability one.

Since a and b were arbitrary, this inequality should hold for any such pair. This is the case if and only if the function $u \circ v^{-1}$ is mid-point concave, which by continuity implies that $u \circ v^{-1}$ is concave. But then the decision maker would reject any noise.

If, on the other hand, the decision maker translates the simple lottery $(x, p; y, 1-p)$ into $(\delta_x, p; \delta_y, 1-p)$, then, after repeating the same steps above, rejection of symmetric noise implies that for any $a > b$ in $[0, 1]$,

$$\begin{aligned} \frac{1}{2}(a+b)u(v^{-1}[v(x)]) + (1 - \frac{1}{2}(a+b))u(v^{-1}[v(y)]) \geq \\ \frac{1}{2}u(v^{-1}[av(x) + (1-a)v(y)]) + \frac{1}{2}u(v^{-1}[bv(x) + (1-b)v(y)]) \end{aligned}$$

Since the distribution $(\frac{1}{2}(a+b), v(x); 1 - \frac{1}{2}(a+b), v(y))$ is a mean preserving spread of the distribution $(\frac{1}{2}, av(x) + (1-a)v(y); \frac{1}{2}, bv(x) + (1-b)v(y))$, this inequality can be satisfied only if $u \circ v^{-1}$ is convex, which, yet again, implies that the decision maker will reject any noise.

Appendix A: Proof of Theorem 1

Lemma 1 proves part 1 of the theorem for binary lotteries Y . After a preparatory claim (Lemma 2), the general case of this part is proved in Lemma 3 for lotteries Y with $F_Y(\mu) \geq \frac{1}{2}$, and for all lotteries in Lemma 4. That this can be done with bounded shifts is proved in Lemma 5. Part 2 of the theorem is proved in Lemma 6.

Lemma 1 *Let $Y = (x, r; z, 1-r)$ with mean $E[Y] = \mu$, $x < z$, and $r \leq \frac{1}{2}$. Then there is a sequence of lotteries X_i with expected value μ such that $X_1 = (\mu, 1)$, $X_i \rightarrow Y$, and X_{i+1} is obtained from X_i by a left symmetric split. Moreover, if r_i and r'_i are the probabilities of x and z in X_i , then $r_i \uparrow r$ and $r'_i \uparrow 1-r$.*

Proof: The main idea of the proof is to have at each step at most five outcomes: x, μ, z , and up to two outcomes between x and μ . In a typical move either μ or one of the outcomes between x and μ , denote it w , is split “as far as possible,” which means:

1. If $w \in (x, \frac{x+\mu}{2}]$, then split its probability between x and $w + (w-x) = 2w-x$. Observe that $x < 2w-x \leq \mu$.

2. If $w \in [\frac{x+z}{2}, \mu]$, then split its probability between z and $w - (z - w) = 2w - z$. Observe that $x \leq 2w - z < \mu$.
3. If $w \in (\frac{x+\mu}{2}, \frac{x+z}{2})$, then split its probability between μ and $w - (\mu - w) = 2w - \mu$. Observe that $x < 2w - \mu < \mu$.

If $r = \frac{1}{2}$, that is, if $\mu = \frac{x+z}{2}$ then the sequence terminates after the first split. We will therefore assume that $r < \frac{1}{2}$. Observe that the this procedures never split the probabilities of x and z hence these probabilities form increasing sequences. We identify and analyze three cases: *a.* For every i the support of X_i is $\{x, y_i, z\}$. *b.* There is $k > 1$ such that the support of X_k is $\{x, \mu, z\}$. *c.* Case *b* does not happen, but there is $k > 1$ such that the support of X_k is $\{x, w_k, \mu, z\}$. We also show that if for all $i > 1$, μ is not in the support of X_i , then case *a* prevails.

a. The simplest case is when for every i the support of X_i has three outcomes at most, $x < y_i < z$. By construction, the probability of y_i is $\frac{1}{2^i}$, hence X_i puts $1 - \frac{1}{2^i}$ probability on x and z . In the limit these converge to a lottery over x and z only, and since for every i , $E[X_i] = \mu$, this limit must be Y . The two examples of Section 3 show that this procedure may or may not terminate after a finite number of steps.

b. Suppose now that even though at a certain step the obtained lottery has more than three outcomes, it is nevertheless the case that after k splits we reach a lottery of the form $X_k = (x, p_k; \mu, q_k; z, 1 - p_k - q_k)$. For example, let $X = (17, 1)$ and $Y = (24, \frac{17}{24}; 0, \frac{7}{24})$. The first five splits are

$$\begin{aligned}
X &= (17, 1) \rightarrow (10, \frac{1}{2}; 24, \frac{1}{2}) \rightarrow (3, \frac{1}{4}; 17, \frac{1}{4}; 24, \frac{1}{2}) \rightarrow \\
&\quad (0, \frac{1}{8}; 6, \frac{1}{8}; 17, \frac{1}{4}; 24, \frac{1}{2}) \rightarrow (0, \frac{3}{16}; 12, \frac{1}{16}; 17, \frac{1}{4}; 24, \frac{1}{2}) \rightarrow \\
&\quad (0, \frac{7}{32}; 17, \frac{1}{4}; 24, \frac{17}{32})
\end{aligned} \tag{5}$$

By construction $k \geq 2$ and $q_k \leq \frac{1}{4}$. Repeating these k steps j times will yield the lottery $X_{jk} = (x, p_{jk}; \mu, q_k^j; z, 1 - p_{jk} - q_k^j) \rightarrow Y$ as $q_k^j \rightarrow 0$ and as the expected value of all lotteries is μ , $p_{jk} \uparrow r$ and $1 - p_{jk} - q_k^j \uparrow 1 - r$.

c. If at each stage X_i puts no probability on μ then we are in case *a*. The reason is that as splits of type 3 do not happen, in each stage the probability of the outcome between x and z is split between a new such outcome and either x or z , and the number of different outcomes is still no more than three. Suppose therefore that at each stage X_i puts positive probability on

at least one outcome w strictly between x and μ (although these outcomes w may change from one lottery X_i to another) and at some stage X_i puts (again) positive probability on μ . Let $k \geq 2$ be the first split that puts positive probability on μ . We consider two cases.

c_1 . $k = 2$: In the first step, the probability of μ is divided between z and $2\mu - z$ and in the second step the probability of $2\mu - z$ is split and half of it is shifted back to μ (see for example the second split in eq. (5) above). In other words, the first split is of type 2 while the second is of type 3. By the description of the latter,

$$\frac{x + \mu}{2} < 2\mu - z < \frac{x + z}{2} \iff \frac{2}{3} < \frac{\mu - x}{z - x} < \frac{3}{4} \quad (6)$$

The other one quarter of the original probability of μ is shifted from $2\mu - z$ to

$$2\mu - z - (\mu - [2\mu - z]) = 3\mu - 2z \leq \frac{x + \mu}{2} \iff 4(z - x) \geq 5(\mu - x)$$

Which is satisfied by eq. (6). Therefore, in the next step a split of type 1 will be used, and one eighth of the original probability of μ will be shifted away from $2\mu - z$ to x . In other words, in three steps $\frac{5}{8}$ of the original probability of μ is shifted to x and z , one quarter of it is back at μ , and one eighth of it is now on an outcome $w_1 < \mu$. \diamond

c_2 . $k \geq 3$: For example, $X = (29, 1)$ and $Y = (48, \frac{29}{48}; 0, \frac{19}{48})$. Then

$$\begin{aligned} X &= (29, 1) \rightarrow (10, \frac{1}{2}; 48, \frac{1}{2}) \rightarrow (0, \frac{1}{4}; 20, \frac{1}{4}; 48, \frac{1}{2}) \rightarrow \\ &(0, \frac{1}{4}; 11, \frac{1}{8}; 29, \frac{1}{8}; 48, \frac{1}{2}) \rightarrow \dots \end{aligned} \quad (7)$$

After k splits $\frac{1}{2^k}$ of the original probability of μ is shifted back to μ and $\frac{1}{2^k}$ is shifted to another outcome $w_1 < \mu$. The rest of the original probability is split (not necessarily equally) between x and z . \diamond

Let $\ell = \max\{k, 3\}$. We now construct inductively a sequence of cycles, where the length of cycle j is $\ell + j - 1$. Such a cycle will end with the probability distributed over $x < w_j < \mu < z$. Denote the probability of μ by p_j and that of w_j by q_j . We show that $p_j + q_j \rightarrow 0$. The probabilities of x and z are such that the expected value is kept at μ , and as $p_j + q_j \rightarrow 0$, it will follow that the probabilities of x and z go up to r and $1 - r$, respectively.

In the example of eq. (7), $\ell = 3$, the length of the first cycle (where $j = 1$) is 3, and $w_1 = 11$.

Suppose that we've finished the first j cycles. Cycle $j + 1$ starts with splitting the p_j probability of μ to $\{x, w_1, \mu, z\}$ as in the first cycle. One of the outcomes along this sequence may be w_j , but we will continue to split only the “new” probability of this outcome (and will not yet touch the probability q_j of w_j). At the end of this part of the new cycle, the probability is distributed over x, w_1, w_j, μ , and z . At least half of p_j , the earlier probability of μ , is shifted to $\{x, z\}$, and the probabilities of both these outcomes did not decrease. Continuing the example of eq. (7), the first part of the second cycle (where $j = 1$) is

$$\begin{aligned} (0, \frac{1}{4}; 11, \frac{1}{8}; 29, \frac{1}{8}; 48, \frac{1}{2}) &\rightarrow (0, \frac{1}{4}; 10, \frac{1}{16}; 11, \frac{1}{8}; 48, \frac{9}{16}) \rightarrow \\ (0, \frac{9}{32}; 11, \frac{1}{8}; 20, \frac{1}{32}; 48, \frac{9}{16}) &\rightarrow (0, \frac{9}{32}; 11, \frac{9}{64}; 29, \frac{1}{64}; 48, \frac{9}{16}) \end{aligned}$$

The second part of cycle $j + 1$ begins with $j - 1$ splits starting with w_1 . At the end of these steps, the probability is spread over x, w_j, μ , and z . Split the probability of w_j between an element of $\{x, \mu, z\}$ and w_{j+1} which is not in this set to get p_{j+1} and q_{j+1} . In the above example, as $j = 1$ there is only one split at this stage to

$$(0, \frac{45}{128}; 22, \frac{9}{128}; 29, \frac{1}{64}; 48, \frac{9}{16})$$

And $w_2 = 22$. The first part of the third cycle ($j = 2$) leads to

$$(0, \frac{91}{256}; 11, \frac{1}{512}; 22, \frac{9}{128}; 29, \frac{1}{512}; 48, \frac{73}{128})$$

The second part of this cycle has two splits. Of $w_1 = 11$ into 0 and 22, and then of $w_2 = 22$ into $\mu = 29$ and $w_3 = 15$.

$$\rightarrow (0, \frac{365}{1024}; 22, \frac{73}{1024}; 29, \frac{1}{512}; 48, \frac{73}{128}) \rightarrow (0, \frac{365}{1024}; 15, \frac{73}{2048}; 29, \frac{77}{2048}; 48, \frac{73}{128})$$

We now show that for every j ,

$$p_{j+2} + q_{j+2} \leq \frac{3}{4}(p_j + q_j) \tag{8}$$

We first observe that for every j , $p_{j+1} + q_{j+1} < p_j + q_j$. This is due to the fact that the rest of the probability is spread over x and z , the probability of z must increase (because of the initial split in the probability of μ), and the probabilities of x and z cannot go down.

When moving from (p_j, q_j) to (p_{j+2}, q_{j+2}) , half of p_j is switched to z . Later on, half of q_j is switched either to x or z , or to μ , in which case half of it (that is, one quarter of q_j) will be switched to z on the move from p_{j+1} to p_{j+2} . This proves inequality (8), hence the lemma. \square

Lemma 2 *Let $X = (x_1, p_1; \dots; x_n, p_n)$ and $Y = (y_1, q_1; \dots; y_m, q_m)$ where $x_1 \leq \dots \leq x_n$ and $y_1 \leq \dots \leq y_m$ be two lotteries such that X dominates Y by second-order stochastic dominance. Then there is a sequence of lotteries X_i such that $X_1 = X$, $X_i \rightarrow Y$, X_{i+1} is obtained from X_i by a symmetric (not necessarily always left or always right) split of one of the outcomes of X_i , all the outcomes of X_i are between y_1 and y_m , and the probabilities the lotteries X_i put on y_1 and y_m go up to q_1 and q_m , respectively.*

Proof: From Rothschild and Stiglitz [31, p. 236] we know that we can present Y as $(y_{11}, q_{11}; \dots; y_{nn}, q_{nn})$ such that $\sum_j q_{kj} = p_k$ and $\sum_j q_{kj} y_{kj} / p_k = x_k$, $k = 1, \dots, n$.

Let $Z = (z_1, r_1; \dots; z_\ell, r_\ell)$ such that $z_1 < \dots < z_\ell$ and $E[Z] = z$. Let $Z_0 = (z, 1)$. One can move from Z_0 to Z in at most ℓ steps, where at each step some of the probability of z is split into two outcomes of Z without affecting the expected value of the lottery, in the following way. If

$$\frac{r_1 z_1 + r_\ell z_\ell}{r_1 + r_\ell} \geq z \tag{9}$$

then move r_1 probability to z_1 and $r'_\ell \leq r_\ell$ to z_ℓ such that $r_1 z_1 + r'_\ell z_\ell = z(r_1 + r'_\ell)$. However, if the sign of the inequality in (9) is reversed, then move r_ℓ probability to z_ℓ and $r'_1 \leq r_1$ probability to z_1 such that $r'_1 z_1 + r_\ell z_\ell = z(r'_1 + r_\ell)$. Either way the move shifted all the required probability from z to one of the outcomes of Z without changing the expected value of the lottery.

Consequently, one can move from X to Y in ℓ^2 steps, where at each step some probability of an outcomes of X is split between two outcomes of Y . By Lemma 1, each such split can be obtained as the limit of symmetric splits (recall that we do not require in the current lemma that the symmetric splits will be left or right splits). That all the outcomes of the obtained lotteries are between y_1 and y_m , and that the probabilities these put on y_1 and y_m go up to q_1 and q_m follow by Lemma 1. \square

Lemma 3 *Let $Y = (y_1, p_1; \dots; y_n, p_n)$, $y_1 \leq \dots \leq y_n$, with expected value μ be skewed to the left such that $F_Y(\mu) \geq \frac{1}{2}$. Then there is a sequence of*

lotteries X_i with expected value μ such that $X_1 = (\mu, 1)$, $X_i \rightarrow Y$, and X_{i+1} is obtained from X_i by a left symmetric split. Moreover, if r_i and r'_i are the probabilities of y_1 and y_n in X_i , then $r_i \uparrow p_1$ and $r'_i \uparrow p_n$.

Proof: Suppose wlg that $y_{j^*} = \mu$ (of course, it may be that $p_{j^*} = 0$). Since $F_Y(y_{j^*}) \geq \frac{1}{2}$, it follows that $t := \sum_{j=j^*+1}^n p_j \leq \frac{1}{2}$. As Y is skewed to the left, $y_n - \mu \leq \mu - y_1$, hence $2\mu - y_n \geq y_1$. Let $m = n - j^*$ be the number of outcomes of Y that are strictly above the expected value μ . Move from X_1 to $X_m = (2\mu - y_n, p_n; \dots; 2\mu - y_{j^*+1}, p_{j^*+1}; y_{j^*}, 1 - 2t; y_{j^*+1}, p_{j^*+1}; \dots; y_n, p_n)$ by repeatedly splitting probabilities away from μ . All these splits are symmetric, hence left symmetric splits.

Next we show that Y is a mean preserving spread of X_m . Obviously, $E[X_m] = E[Y] = \mu$. Integrating by parts, we have for $x \geq \mu$

$$\begin{aligned} y_n - \mu &= \int_{y_1}^{y_n} F_Y(z) dz = \int_{y_1}^x F_Y(z) dz + \int_x^{y_n} F_Y(z) dz \\ y_n - \mu &= \int_{y_1}^{y_n} F_{X_m}(z) dz = \int_{y_1}^x F_{X_m}(z) dz + \int_x^{y_n} F_{X_m}(z) dz \end{aligned}$$

Since F_Y and F_{X_m} coincide for $z \geq \mu$, we have, for $x \geq \mu$, $\int_{y_1}^x F_{X_m}(z) dz = \int_{y_1}^x F_Y(z) dz$ and in particular, $\int_{y_1}^x F_{X_m}(z) dz \leq \int_{y_1}^x F_Y(z) dz$.

For $x < \mu$ it follows by the assumption that Y is skewed to the left and by the construction of X_m as a symmetric lottery around μ that

$$\begin{aligned} \int_{y_1}^x F_{X_m}(z) dz &= \int_{2\mu-x}^{2\mu-y_1} [1 - F_{X_m}(z)] dz = \\ &= \int_{2\mu-x}^{2\mu-y_1} [1 - F_Y(z)] dz \leq \int_{y_1}^x F_Y(z) dz \end{aligned}$$

Since to the right of μ , X_m and Y coincide, we can view the left side of Y as a mean preserving spread of the left side of X_m . By Lemma 2 the left side of Y is the limit of symmetric mean preserving spreads of the left side of X_m . Moreover, all these splits take place between y_1 and μ and are therefore left symmetric splits. By Lemma 2 it also follows that $r_i \uparrow p_1$ and $r'_i \uparrow p_n$. \square

We now show that Lemma 3 holds without the restriction $F_Y(\mu) \geq \frac{1}{2}$.

Lemma 4 *Let Y with expected value μ be skewed to the left. Then there is a sequence of lotteries X_i with expected value μ such that $X_1 = (\mu, 1)$, $X_i \rightarrow Y$, and X_{i+1} is obtained from X_i by a left symmetric split.*

Proof: The first step in the proof of Lemma 3 was to create a symmetric distribution around μ such that its upper tail (above μ) agrees with F_Y . Obviously this can be done only if $F_Y(\mu) \geq \frac{1}{2}$, which is no longer assumed. Instead, we apply the proof of Lemma 3 successively to mixtures of F_Y and δ_μ , the distribution that yields μ with probability one.

Suppose that $F_Y(\mu) = \lambda < \frac{1}{2}$. Let $\gamma = 1/2(1 - \lambda)$ and define Z to be the lottery obtained from the distribution $\gamma F_Y + (1 - \gamma)\delta_\mu$. Observe that

$$F_Z(\mu) = \gamma F_Y(\mu) + (1 - \gamma)\delta_\mu(\mu) = \frac{\lambda}{2(1 - \lambda)} + \frac{1 - 2\lambda}{2(1 - \lambda)} = \frac{1}{2}$$

It follows that the lotteries Z and $(\mu, 1)$ satisfy the conditions of Lemma 3, and therefore there is sequence of lotteries X_i with expected value μ such that $X_1 = (\mu, 1)$, $X_i \rightarrow Z$, and X_{i+1} is obtained from X_i by a left symmetric split. This is done in two stages. First we create a symmetric distribution around μ that agrees with Z above μ (denote the number of splits needed in this stage by t), and then we manipulate the part of the distribution which is weakly to the left of μ by taking successive symmetric splits (which are all left symmetric splits when related to μ) to get nearer and nearer to the second-order stochastically dominated left side of Z as in Lemma 2. Observe that the highest outcome of this part of the distribution Z is μ , and its probability is $1 - \gamma$. By Lemma 2, for every $k \geq 1$ there is ℓ_k such that after ℓ_k splits of this second phase the probability of μ will be at least $r_k = (1 - \gamma)(1 - \frac{1}{k+1})$ and $\|X_{t+\ell_k} - Z\| < \frac{1}{k}$.

The first cycle will end after $t + \ell_1$ splits with the distribution F_{Z_1} . Observe that the probabilities of the outcomes to the right of μ in Z_1 are those of the lottery Y multiplied by γ . The first part of second cycle will be the same as the first cycle, applied to the r_k conditional probability of μ . At the end of this part we'll get the lottery Z'_1 which is the same as Z_1 , conditional on the probability of μ . We now continue the second cycle by splitting the combination of Z_1 and Z'_1 for the total of $t + \ell_1 + \ell_2$ steps. As we continue to add such cycles inductively we get closer and closer to Y , hence the lemma. \square

Next we show that part 1 of the theorem can be achieved by using bounded spreads. The first steps in the proof of Lemma 3 involve shifting probabilities from μ to all the outcomes of Y to the right of μ , and these outcomes are not more than $\max y_i - \mu$ away from μ . All other shifts are symmetric shifts involving only outcomes to the left of μ . The next lemma

shows that such shifts can be achieved as the limit of symmetric bounded shifts.

Lemma 5 *Let $Z = (z - \alpha, \frac{1}{2}; z + \alpha, \frac{1}{2})$ and let $\varepsilon > 0$. Then there is a sequence of lotteries Z_i such that $Z_0 = (z, 1)$, $Z_i \rightarrow Z$, and Z_{i+1} is obtained from Z_i by a symmetric (not necessarily left or right) split of size smaller than ε .*

Proof: The claim is interesting only when $\varepsilon < \alpha$. Fix n such that $\varepsilon > \frac{\alpha}{n}$. We show that the lemma can be proved by choosing the size of the splits to be $\frac{\alpha}{n}$. Consider the $2n + 1$ points $z_k = z + \frac{k}{n}$, $k = -n, \dots, n$ and construct the sequence $\{Z_i\}$ where $Z_i = (z - \alpha, p_{i,-n}; z - \frac{n-1}{n}\alpha, p_{i,-n+1}; \dots; z + \alpha, p_{i,n})$ as follows.

THE INDEX i IS ODD: Let z_j be the highest outcome in $\{z, \dots, z + \frac{n-1}{n}\alpha\}$ with the highest probability in Z_{i-1} . Formally, j satisfies:

- $0 \leq j \leq n - 1$
- $p_{i-1,j} \geq p_{i-1,k}$ for all k
- If for some $j' \in \{0, \dots, n - 1\}$, $p_{i-1,j'} \geq p_{i-1,k}$ for all k , then $j \geq j'$.

Split the probability of z_j between $z_j - \frac{\alpha}{n}$ and $z_j + \frac{\alpha}{n}$ (i.e., between z_{j-1} and z_{j+1}). That is, $p_{i,j-1} = p_{i-1,j-1} + \frac{1}{2}p_{i-1,j}$, $p_{i,j+1} = p_{i-1,j+1} + \frac{1}{2}p_{i-1,j}$, $p_{i,j} = 0$, and for all $k \neq j - 1, j, j + 1$, $p_{i,k} = p_{i-1,k}$.

THE INDEX i IS EVEN: In this step we create the mirror split of the one done in the previous step. Formally, If j of the previous stage is zero, do nothing. Otherwise, split the probability of z_{-j} between $z_{-j} - \frac{\alpha}{n}$ and $z_{-j} + \frac{\alpha}{n}$. That is, $p_{i,-j-1} = p_{i-1,-j-1} + \frac{1}{2}p_{i-1,-j}$, $p_{i,-j+1} = p_{i-1,-j+1} + \frac{1}{2}p_{i-1,-j}$, $p_{i,-j} = 0$, and for all $k \neq -j - 1, -j, -j + 1$, $p_{i,k} = p_{i-1,k}$.

After each pair of these steps, the probability distribution is symmetric around z . Also, the sequences $\{p_{i,-n}\}_i$ and $\{p_{i,n}\}$ are non decreasing. Being bounded by $\frac{1}{2}$, they converge to a limit L . Our aim is to show that $L = \frac{1}{2}$. Suppose not. Then at each step the highest probability of $\{p_{i-1,-n+1}, \dots, p_{i-1,n-1}\}$ must be at least $\ell := (1 - 2L)/(2n - 1) > 0$. The variance of Z_i is bounded from above by the variance of $(\mu - \alpha, \frac{1}{2}; \mu + \alpha, \frac{1}{2})$, which is α^2 . Splitting p probability from z to $z - \frac{\alpha}{n}$ and $z + \frac{\alpha}{n}$ will increase the variance by $p(\frac{\alpha}{n})^2$. Likewise, for $k \neq -n, 0, n$, splitting p probability

from $z + \frac{k\alpha}{n}$ to $z + \frac{(k+1)\alpha}{n}$ and $z - \frac{(k-1)\alpha}{n}$ will increase the variance by $\frac{p}{2}\left(\frac{\alpha}{n}\right)^2$. Therefore, for positive even i we have

$$\sigma^2(Z_i) - \sigma^2(Z_{i-2}) \geq \frac{1 - 2L}{2n - 1} \left(\frac{\alpha}{n}\right)^2$$

If $L < \frac{1}{2}$, then after enough steps the variance of Z_i will exceed α^2 , a contradiction. \square

That we can do part 1 of the theorem for all lotteries Y follows by the fact that a countable set of countable sequences is countable.

Finally, the following lemma proves part 2 of the theorem.

Lemma 6 *Any sequence of left symmetric split starting at δ_μ converges (in the L^1 topology) to a skewed to the left distribution with expected value μ .*

Proof: That such sequences converge follows from the fact that a symmetric split will increase the variance of the distribution, but as all distributions are over the bounded $[\underline{x}, \bar{x}]$ segment of \mathfrak{R} , the variances of the distributions increase to a limit. Replacing (x, p) with $(x - \alpha, \frac{p}{2}; x + \alpha, \frac{p}{2})$ increases the variance of the distribution by

$$\frac{p}{2}(x - \alpha - \mu)^2 + \frac{p}{2}(x + \alpha - \mu)^2 - p(x - \mu)^2 = p\alpha^2$$

and therefore the distance between two successive distributions in the sequences is bounded by $\bar{x} - \underline{x}$ times the change in the variance. The sum of the changes in the variances is bounded, as is therefore the sum of distances between successive distributions, hence Cauchy criterion is satisfied and the sequence converges.

Next we prove that the limit is a skewed to the left distribution with expected value μ . Let F be the distribution of $X = (x_1, p_1; \dots; x_n, p_n)$ with expected value μ be skewed to the left. Suppose wlg that $x_1 \leq \mu$, and break it symmetrically to obtain $X' = (x_1 - \alpha, \frac{p_1}{2}; x_1 + \alpha, \frac{p_1}{2}; x_2, p_2; \dots; x_n, p_n)$ with the distribution F' . Note that $E[X'] = \mu$. Consider the following two cases.

Case 1: $x_1 + \alpha \leq \mu$. Then for all δ , $\eta_2(F, \delta) = \eta_2(F', \delta)$. For δ such that $\mu - \delta \leq x_1 - \alpha$ or such that $x_1 + \alpha \leq \mu - \delta$, $\eta_1(F', \delta) = \eta_1(F, \delta) \geq \eta_2(F, \delta) = \eta_2(F', \delta)$. For δ such that $x_1 - \alpha < \mu - \delta \leq x_1$, $\eta_1(F', \delta) = \eta_1(F, \delta) + [(\mu - \delta) - (x_1 - \alpha)]\frac{p_1}{2} > \eta_1(F, \delta) \geq \eta_2(F, \delta) = \eta_2(F', \delta)$. Finally, for δ such that

$x_1 < \mu - \delta < x_1 + \alpha (\leq \mu)$, $\eta_1(F', \delta) = \eta_1(F, \delta) + [(x_1 + \alpha) - (\mu - \delta)] \frac{\rho_1}{2} > \eta_1(F, \delta) \geq \eta_2(F, \delta) = \eta_2(F', \delta)$.

Case 2: $x_1 + \alpha > \mu$. Then for all δ such that $\mu + \delta \geq x_1 + \alpha$, $\eta_2(F, \delta) = \eta_2(F', \delta)$. For δ such that $\mu - \delta \leq x_1 - \alpha$, $\eta_1(F', \delta) = \eta_1(F, \delta) \geq \eta_2(F, \delta) = \eta_2(F', \delta)$. For δ such that $x_1 - \alpha < \mu - \delta \leq x_1$, $\eta_1(F', \delta) = \eta_1(F, \delta) + [(\mu - \delta) - (x_1 - \alpha)] \frac{\rho_1}{2} \geq \eta_2(F, \delta) + [(\mu - \delta) - (x_1 - \alpha)] \frac{\rho_1}{2} \geq \eta_2(F, \delta) + \max\{0, (x_1 + \alpha) - (\mu + \delta)\} \frac{\rho_1}{2} = \eta_2(F', \delta)$. Finally, for δ such that $\mu - \delta > x_1$, $\eta_1(F', \delta) = \eta_1(F, \delta) + [(x_1 + \alpha) - (\mu - \delta)] \frac{\rho_1}{2} \geq \eta_2(F, \delta) + \max\{0, (x_1 + \alpha) - (\mu + \delta)\} \frac{\rho_1}{2} = \eta_2(F', \delta)$.

If $X_n \rightarrow Y$, all have the same expected value and for all n , X_n is skewed to the left, then so is Y . ■

Appendix B: Other Proofs

Proof of Claim 1: To simplify notation, assume wlg that F is defined over $[0, \bar{x}]$, hence $F(0) = 0$ and $F(\bar{x}) = 1$. Define $F_0 = \delta_\mu$, and for $n = 1, \dots$ define F_n in the following way. Divide the segment $[0, \mu]$ into n equal segments, and define for $\frac{i\mu}{n} \leq x < \frac{(i+1)\mu}{n}$, $i = 0, \dots, n-1$,

$$F_n(x) = \frac{n}{\mu} \int_{\frac{i\mu}{n}}^{\frac{(i+1)\mu}{n}} F(x) dx$$

In other words, F and F_n have the same areas under their graphs on each of these n segments. To define F_n on $[\mu, \bar{x}]$, let $q = F(\mu)$ and let y_i be the minimal number such that $F(y_i) \geq q + \frac{(1-q)i}{n}$, $i = 0, \dots, n$. Define $\mu < x_1 \leq \dots \leq x_n$ such that

$$\frac{(1-q)x_i}{n} = \frac{(1-q)y_{i-1}}{n} + \int_{y_{i-1}}^{y_i} \left[q + \frac{(1-q)i}{n} - F(x) \right] dx$$

Typical segments of F_n are depicted in Fig. 4.

We first show that $F_n \rightarrow F$. On $[0, \mu]$ we obtain

$$\begin{aligned} \int_0^\mu |F_n(x) - F(x)| dx &\leq \\ &\frac{\mu}{n} \sum_{i=0}^{n-1} \left[F\left(\frac{(i+1)\mu}{n}\right) - F\left(\frac{i\mu}{n}\right) \right] = \frac{\mu}{n} F(\mu) \rightarrow 0 \end{aligned}$$

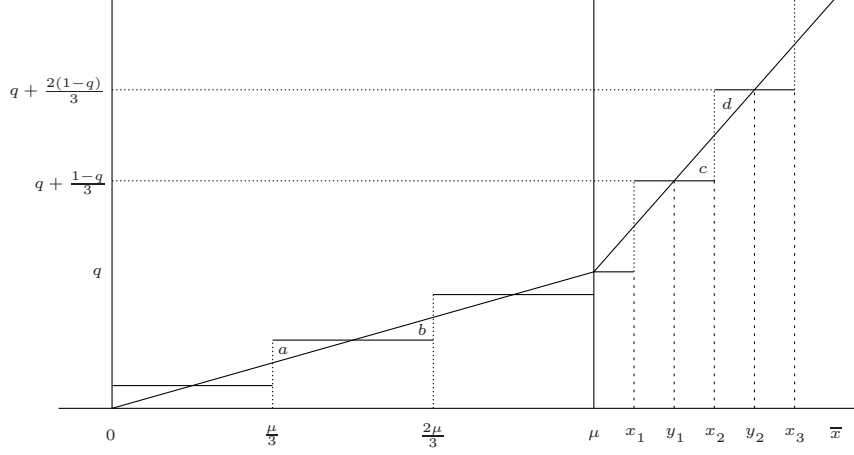


Figure 4: F_3 (bars) and F (diagonal lines). $a = b$, $c = d$

While on $[\mu, \bar{x}]$ we get

$$\int_{\mu}^{\bar{x}} |F_n(x) - F(x)| dx \leq \frac{1-q}{n} \sum_{i=0}^{n-1} [y_i - y_{i-1}] = \frac{(1-q)(\bar{x} - \mu)}{n} \rightarrow 0$$

Let $\delta > 0$. We show now that for $y = \mu - \delta$

$$\eta_1(F_n, \delta) = \int_0^y F_n(x) dx \geq \int_0^y F(x) dx = \eta_1(F, \delta) \quad (10)$$

By definition, for $y = \frac{i\mu}{n}$, $i = 1, \dots, n$, $\int_0^{\frac{i\mu}{n}} F_n(x) dx = \int_0^y F(x) dx$. Inequality (10) follows from the fact that on the segment $[\frac{i\mu}{n}, \frac{(i+1)\mu}{n}]$, $F_n - F$ is a decreasing function. On the other hand, for $y = \mu + \delta$,

$$\eta_2(F_n, \delta) = \int_y^{\bar{x}} [1 - F_n(x)] dx \leq \int_y^{\bar{x}} [1 - F(x)] dx = \eta_2(F, \delta) \quad (11)$$

Here, by definition, for y_i , $i = 0, \dots, n-1$, $\int_{y_i}^{y_{i+1}} [1 - F_n(x)] dx = \int_{y_i}^{y_{i+1}} [1 - F(x)] dx$. Inequality (11) follows from the fact that on the segments $[y_i, x_{i+1}]$

and $[x_{i+1}, y_{i+1})$, $F_n - F$ is a decreasing function. Inequalities (10) and (11), together with the fact that F is skewed to the left imply for all δ ,

$$\eta_1(F_n, \delta) \geq \eta_1(F, \delta) \geq \eta_2(F, \delta) \geq \eta_2(F_n, \delta)$$

Hence F_n is skewed to the left. \blacksquare

Proof of Claim 2: Let the lottery Y be obtained from the lottery Z by a left symmetric split. Denote by x their common mean and assume that $\bar{m}(Z) \geq x$. As Y is obtained from Z by splitting one of its outcomes $z_i \leq x \leq \bar{m}(Z)$, this split can only increase the mass on the distribution above x , thus (weakly) increasing its median.

By Theorem 1, X is the limit of a sequence of left symmetric splits starting with $(x, 1)$, hence the claim. \blacksquare

Proof of Claim 3: Let the lottery Y be obtained from the lottery Z by a left symmetric split and denote by x their common mean. For example, the outcome $z_i \leq x$ with probability p_i of Z is split into $z_i - \alpha$ and $z_i + \alpha$, each with probability $\frac{p_i}{2}$. Denote the distributions of Y and Z by F and G . Since for $t < 0$ and odd n , t^n is a concave function, it follows that if $z_i + \alpha \leq x$, then

$$\begin{aligned} & \int_{\underline{x}}^{\bar{x}} (t-x)^n dF(t) - \int_{\underline{x}}^{\bar{x}} (t-x)^n dG(t) = \\ & \frac{p_i}{2} [(z_i - \alpha - x)^n + (z_i + \alpha - x)^n] - p_i (z_i - x)^n \leq 0 \end{aligned} \quad (12)$$

If $z_i + \alpha > x$ we need to manipulate eq. (12) a little further. Let $\xi = z_i - x$ and obtain

$$\begin{aligned} & \frac{p_i}{2} [(z_i - \alpha - x)^n + (z_i + \alpha - x)^n] - p_i (z_i - x)^n = \\ & \frac{p_i}{2} [(\xi - \alpha)^n + (\xi + \alpha)^n] - p_i \xi^n = \\ & \frac{p_i}{2} \xi^n + \frac{p_i}{2} \sum_{j=1}^{\frac{n-1}{2}} \binom{n}{2j-1} \xi^{2j-1} \alpha^{n-2j+1} - \frac{p_i}{2} \sum_{j=0}^{\frac{n-1}{2}} \binom{n}{2j} \xi^{2j} \alpha^{n-2j} + \\ & \frac{p_i}{2} \xi^n + \frac{p_i}{2} \sum_{j=1}^{\frac{n-1}{2}} \binom{n}{2j-1} \xi^{2j-1} \alpha^{n-2j+1} + \frac{p_i}{2} \sum_{j=0}^{\frac{n-1}{2}} \binom{n}{2j} \xi^{2j} \alpha^{n-2j} - p_i \xi^n = \\ & p_i \sum_{j=1}^{\frac{n-1}{2}} \binom{n}{2j-1} \xi^{2j-1} \alpha^{n-2j+1} \leq 0 \end{aligned}$$

Where the last inequality follows by the fact that $\xi \leq 0$. Since X with expected value μ is skewed to the left it follows by Theorem 1 that it can be obtained as the limit of a sequence of left symmetric splits. At δ_μ (the distribution of $(\mu, 1)$), $\int_{\underline{x}}^{\bar{x}} (y - \mu)^n d\delta_\mu = 0$. The claim follows by the fact (proved above) that each left symmetric split reduces the value of the integral. ■

Proof of Theorem 2: Fix $x > y$, and let c_p be the certainty equivalent of the lottery $(x, p; y, 1 - p)$. The two-stage lottery $\langle p - \alpha, \varepsilon; p, 1 - 2\varepsilon; p + \alpha, \varepsilon \rangle$ translates in the recursive model into the lottery $(c_{p-\alpha}, \varepsilon; c_p, 1 - 2\varepsilon; c_{p+\alpha}, \varepsilon)$. With a slight abuse of notation, let δ_p be the degenerate distribution yielding c_p with probability 1. Since the decision maker always rejects symmetric noise, it follows that the local utility u_{δ_p} satisfies

$$u_{\delta_p}(\delta_p) \geq \frac{1}{2}u_{\delta_p}(\delta_{p-\alpha}) + \frac{1}{2}u_{\delta_p}(\delta_{p+\alpha})$$

By Hypothesis II, for every $r \geq p$,

$$u_{\delta_r}(\delta_p) \geq \frac{1}{2}u_{\delta_r}(\delta_{p-\alpha}) + \frac{1}{2}u_{\delta_r}(\delta_{p+\alpha}) \quad (13)$$

Consider first the lottery over the probabilities in $(x, p; y, 1 - p)$ given by $Q = \langle p_1, q_1; \dots; p_m, q_m \rangle$ where $\sum q_i p_i = p$ (we deal with the distributions with non-finite support at the end of the proof). If Q is skewed to the left, then it follows by Theorem 1 that there is a sequence of lotteries $Q_i = \langle p_{i,1}, q_{i,1}, \dots, p_{i,n_i}, q_{i,n_i} \rangle \rightarrow Q$ such that $Q_1 = \langle p, 1 \rangle$ and for all i , Q_{i+1} is obtained from Q_i by a left symmetric split. Suppose $p_{i,j}$ is split into $p_{i,j} - \alpha$ and $p_{i,j} + \alpha$. By eq. (13), as $p > p_{i,j}$,

$$\begin{aligned} \mathbb{E}[u_{\delta_p}(Q_i)] &= \\ q_{i,j} u_{\delta_p}(\delta_{p_{i,j}}) &+ \sum_{m \neq j} q_{i,m} u_{\delta_p}(\delta_{p_{i,m}}) \geq \\ \frac{1}{2} q_{i,j} u_{\delta_p}(\delta_{p_{i,j} - \alpha}) &+ \frac{1}{2} q_{i,j} u_{\delta_p}(\delta_{p_{i,j} + \alpha}) + \sum_{m \neq j} q_{i,m} u_{\delta_p}(\delta_{p_{i,m}}) = \\ \mathbb{E}[u_{\delta_p}(Q_{i+1})] \end{aligned}$$

As $Q_i \rightarrow Q$, and as for all i , $u_{\delta_p}(\delta_p) \geq u_{\delta_p}(Q_i)$, it follows by continuity that $u_{\delta_p}(\delta_p) \geq \mathbb{E}[u_{\delta_p}(Q)]$. By Fréchet Differentiability

$$\left. \frac{\partial}{\partial \varepsilon} V(\varepsilon Q + (1 - \varepsilon)\delta_p) \right|_{\varepsilon=0} \leq 0$$

Quasi-concavity now implies that $V(\delta_p) \geq V(Q)$. Finally, as preferences are continuous, it follows by Claim 1 that the theorem holds for all Q , even if its support is not finite. \blacksquare

Proof of Claim 4: The rank-dependent (3) value of a lottery of the form $(x, p; y, 1 - p)$ with $x > y$ is $u(x)f(p) + u(y)[1 - f(p)]$. As u is unique up to positive affine transformations, we can assume that $u(x) = 1$ and $u(y) = 0$. The certainty equivalent of $(x, p; y, 1 - p)$ for $x > y$ is therefore $u^{-1}(f(p))$. We thus obtain that the recursive utility value of the noise $\langle p_h, q; p_\ell, 1 - q \rangle$ where $p_h > p_\ell$ is

$$f(p_h)f(q) + f(p_\ell)[1 - f(q)] \quad (14)$$

while the value of the lottery with the average probability $p = qp_h + (1 - q)p_\ell$ is $f(p)$.

By eq. (14), the value of the symmetric noise $\langle p + \alpha, \frac{1}{2}; p - \alpha, \frac{1}{2} \rangle$ is $g(\alpha) := f(p + \alpha)f(\frac{1}{2}) + f(p - \alpha)[1 - f(\frac{1}{2})]$. As f is convex (recall that we assume risk aversion), $f(\frac{1}{2}) < \frac{1}{2}$, hence $g'(0) < 0$. Also, if f is convex then so is g , hence the decision maker rejects all symmetric binary noise iff $\langle p, 1 \rangle \succeq \langle 2p, \frac{1}{2}; 0, \frac{1}{2} \rangle$ for $p \leq \frac{1}{2}$ and $\langle p, 1 \rangle \succeq \langle 1, \frac{1}{2}; 2p - 1, \frac{1}{2} \rangle$ for $p > \frac{1}{2}$. By eq. (14), the function f must satisfy

$$\forall p \leq \frac{1}{2}, \frac{f(p)}{f(2p)} \geq f\left(\frac{1}{2}\right) \quad (15)$$

and

$$\forall p > \frac{1}{2}, f(p) \geq f\left(\frac{1}{2}\right) + f(2p - 1)[1 - f\left(\frac{1}{2}\right)] \quad (16)$$

Let n be the first derivative such that $f^{(n)}(0) \neq 0$. By l'Hospital's rule we obtain that

$$\lim_{p \rightarrow 0} \frac{f(p)}{f(2p)} \geq f\left(\frac{1}{2}\right) \implies \frac{1}{2^n} \geq f\left(\frac{1}{2}\right) \quad (17)$$

For $p = \frac{1}{2}$, $\frac{f(p)}{f(2p)} = f\left(\frac{1}{2}\right)$, hence a sufficient condition for inequality (15) is that eq. (17) is satisfied and that $f(p)/f(2p)$ is a concave function, both guaranteed by the assumptions of the claim.

For inequality (16), observe that for $p \in \{\frac{1}{2}, 1\}$, $h(p) := f(p) - f\left(\frac{1}{2}\right) - f(2p - 1)[1 - f\left(\frac{1}{2}\right)] = 0$. Also, $h''(p) = f''(p) - 4f''(2p - 1)[1 - f\left(\frac{1}{2}\right)] <$

$f''(p) - 2f''(2p-1)$. The claim assumed that $f''' < 0$ (implying, in particular, $f''(0) > 0$). Then since $p \geq 2p-1$, it follows that h is concave hence non-negative on $[\frac{1}{2}, 1]$ and inequality (16) is satisfied. This completes the proof of the first part of the claim, that the decision maker rejects symmetric noise.

Consider now the skewed to the right noise $\langle kp, \frac{1}{k}; 0, 1 - \frac{1}{k} \rangle$, where $p \leq \frac{1}{k}$. The decision maker prefers this noise to $\langle p, 1 \rangle$ if

$$f(\frac{1}{k}) > \frac{f(p)}{f(kp)} \tag{18}$$

We show first that $f'(0) = 0$. Suppose instead that $f'(0) > 0$. Then $\lim_{p \rightarrow 0} f(p)/f(kp) = \frac{1}{k}$, but the convexity of f implies that we cannot have $f(\frac{1}{k}) > \frac{1}{k}$, as $f(1) = 1$, in contradiction to inequality (18). It thus follows that $f'(0) = 0$ and since (as noted above) $f''(0) > 0$, we obtain that $\lim_{p \rightarrow 0} f(p)/f(kp) = \frac{1}{k^2}$. If for a sufficiently large k we have $f(\frac{1}{k}) > \frac{1}{k^2}$, then there are skewed to the right binary noises which the decision maker will find attractive.

The equivalent skewed to the left noise is $\langle 1, 1 - \frac{1}{k}; 1 - kp, \frac{1}{k} \rangle$ with the expected value $1 - p$. The decision maker prefers $1 - p$ to this noise if $f(1 - p) > f(1 - \frac{1}{k}) + f(1 - kp)[1 - f(1 - \frac{1}{k})]$. The two sides are equal for $p = 0$, and the derivative of the left side at $p = 0$ is larger than that of the right side. Indeed,

$$\begin{aligned} -f'(1 - p)|_{p=0} &> -kf'(1 - kp)[1 - f(1 - \frac{1}{k})]|_{p=0} \iff \\ f(1 - \frac{1}{k}) &< 1 - \frac{1}{k} \end{aligned}$$

which follows by the convexity of f . ■

Appendix C: Examples

Example 1 Consider the rank-dependent model (3) (Quiggin [30]) with a concave utility function u , and where the probability transformation function $f : [0, 1] \rightarrow [0, 1]$ is given by

$$f(p) = \begin{cases} 1.8p^2 & p \leq \frac{1}{3} \\ 1.2p - 0.2 & p > \frac{1}{3} \end{cases}$$

Of course $f(0) = 0$, $f(1) = 1$, f is continuous, strictly increasing, and differentiable. It is also weakly convex, hence risk averse (see [6]). By Fact 1

of section 4.2, the decision maker rejects any symmetric noise of the form $\langle p - \alpha, \frac{1}{2}; p + \alpha, \frac{1}{2} \rangle$ for a sufficiently small α , which may depend on p .

Next we show that the decision maker accepts all skewed-to-the-right noise of the form $\langle p, q; 0, 1 - q \rangle$ around pq , provided $q \leq \frac{1}{3}$. Indeed, for $q \leq \frac{1}{3}$ and $p \leq \frac{1}{3}$,

$$\langle p, q; 0, 1 - a \rangle \succ \langle pq, 1 \rangle \iff f(p)f(q) \geq f(pq) \iff 1.8^2 p^2 q^2 > 1.8(pq)^2$$

While for $q \leq \frac{1}{3}$ and $p > \frac{1}{3}$,

$$\begin{aligned} \langle p, q; 0, 1 - q \rangle \succ \langle pq, 1 \rangle &\iff \\ (1.2p - 0.2)1.8q^2 > 1.8p^2q^2 &\iff \\ p^2 - 1.2p + 0.2 < 0 &\iff p \in (0.2, 1) \end{aligned}$$

On the other hand, for $p > \bar{p} > \frac{1}{3}$ and $q > \frac{1}{2}$, the decision maker prefers the known probability $pq + \bar{p}(1 - q)$ to the noise $\langle p, q; \bar{p}, 1 - q \rangle$:

$$\begin{aligned} \langle p, q; \bar{p}, 1 - q \rangle \prec \langle pq + \bar{p}(1 - q), 1 \rangle &\iff \\ (1.2p - 0.2)(1.2q - 0.2) + 1.2(1.2\bar{p} - 0.2)(1 - q) < & \\ 1.2(pq + \bar{p}(1 - q)) - 0.2 &\iff \\ pq + \bar{p} < \bar{p}q + p &\iff \\ (p - \bar{p})(1 - q) > 0 &\iff p > \bar{p} \end{aligned}$$

Finally, we show that for every $\bar{p} \leq \frac{1}{3}$ there exists $p^* > \bar{p}$ such that for q satisfying $pq + \bar{p}(1 - q) > \frac{1}{3}$ and for all $p > p^*$, the decision maker prefers the known probability $pq + \bar{p}(1 - q)$ to the noise $\langle p, q; \bar{p}, 1 - q \rangle$:

$$\begin{aligned} \langle p, q; \bar{p}, 1 - q \rangle \prec \langle pq + \bar{p}(1 - q), 1 \rangle &\iff \\ (1.2p - 0.2)(1.2q - 0.2) + 1.8 \cdot 1.2\bar{p}^2(1 - q) < & \\ 1.2(pq + \bar{p}(1 - q)) - 0.2 &\iff \\ 9\bar{p}^2 - 5\bar{p} + 1 < p & \end{aligned}$$

Clearly $9\bar{p}^2 - 5\bar{p} < 0$ iff $\bar{p} < \frac{5}{9}$, hence it is possible to find $p^* < 1$ satisfying our requirements. ■

Example 2 This is an example for smooth preferences that reject symmetric noise as well as skewed-to-the-left noise around high probabilities, but sometimes prefer skewed-to-the-right noise around lower probabilities.

The polynomial utility of order n is an extension of the quadratic model (Chew, Epstein, and Segal [5]).

$$V(F) = \int \cdots \int \varphi(x_1, \dots, x_n) dF(x_1) \dots dF(x_n)$$

where φ is symmetric and strictly monotonic. We assume wlg that $\varphi(0, \dots, 0) = 0$. For binary lotteries we obtain

$$V(x, p; y, 1 - p) = \sum_{i=0}^n \binom{n}{i} p^{n-i} (1-p)^i \varphi(\underbrace{x, \dots, x}_{n-i}, \underbrace{y, \dots, y}_i)$$

Let

$$x(i) = (\underbrace{x, \dots, x}_{n-i}, \underbrace{0, \dots, 0}_i)$$

Denote by x_p the certainty equivalent of the lottery $(x, p; 0, 1 - p)$, satisfying $(x_p, 1) \sim (1, p; 0, 1 - p)$. It follows that

$$\varphi(x_p, \dots, x_p) = \sum_{i=0}^n \binom{n}{i} p^{n-i} (1-p)^i \varphi(1(i)) \quad (19)$$

We want to compare $A = (1, pq; 0, 1 - pq)$ with the noise $B = ((1, p; 0, 1 - p), q; 0, 1 - q)$. For $q = \frac{1}{2}$, B is symmetric noise around A . For $q > \frac{1}{2}$ ($q < \frac{1}{2}$), the noise represented by B is skewed to the left (right). According to the recursive model, $B \sim C = (x_p, q; 0, 1 - q)$. Hence (recall that $\varphi(1(n)) = \varphi(0, \dots, 0) = 0$)

$$\begin{aligned} A \succeq B &\iff \\ \sum_{i=0}^n \binom{n}{i} (pq)^{n-i} (1-pq)^i \varphi(1(i)) &\geq \sum_{i=0}^n \binom{n}{i} q^{n-i} (1-q)^i \varphi(x_p(i)) = \\ q^n \sum_{j=0}^n \binom{n}{j} p^{n-j} (1-p)^j \varphi(1(j)) &+ \sum_{i=1}^n \binom{n}{i} q^{n-i} (1-q)^i \varphi(x_p(i)) \iff \\ \sum_{i=1}^{n-1} \binom{n}{i} (pq)^{n-i} (1-pq)^i \varphi(1(i)) &- \\ q^n \sum_{j=1}^{n-1} \binom{n}{j} p^{n-j} (1-p)^j \varphi(1(j)) &- \sum_{i=1}^{n-1} \binom{n}{i} q^{n-i} (1-q)^i \varphi(x_p(i)) \geq 0 \end{aligned} \quad (20)$$

Observe that for $q \in \{0, 1\}$, the two sides of the above inequality are equal. As both sides are polynomial of order n in q , it follows that for $n = 1$ (expected utility) both sides are always the same (hence noise makes no difference), and for $n = 2$ (quadratic utility) the sign of the inequality cannot change between 0 and 1.

For $q = \frac{1}{2}$ we want to have $A \succ B$, while for sufficiently small $q > 0$ (and positive p) we want to have $B \succ A$. Substitute $q = \frac{1}{2}$ in eq. (20) and obtain the requirement

$$\sum_{i=1}^{n-1} \binom{n}{i} \frac{1}{2^{n-i}} p^{n-i} \left(1 - \frac{p}{2}\right)^i \varphi(1(i)) - \frac{1}{2^n} \sum_{j=1}^{n-1} \binom{n}{j} p^{n-j} (1-p)^j \varphi(1(j)) - \frac{1}{2^n} \sum_{i=1}^{n-1} \binom{n}{i} \varphi(x_p(i)) > 0 \quad (21)$$

The requirement that for small q we have $B \succ A$ is satisfied if the derivative of eq. (20) with respect to q at zero is negative, that is, if

$$p\varphi(1, 0, \dots, 0) < \varphi(x_p, 0, \dots, 0) \quad (22)$$

Let $n = 3$ and denote $\zeta = \varphi(1, 1, 1)$, $\theta = \varphi(1, 1, 0)$, and $\kappa = \varphi(1, 0, 0)$. Eqs. (19), (21) (after multiplying by 8), and (22) become

$$\varphi(x_p, x_p, x_p) = \zeta p^3 + 3\theta p^2(1-p) + 3\kappa p(1-p)^2 \quad (23)$$

$$\theta p^2 + \kappa(3p - 2p^2) - \varphi(x_p, x_p, 0) - \varphi(x_p, 0, 0) > 0 \quad (24)$$

$$\kappa p < \varphi(x_p, 0, 0) \quad (25)$$

Let $\kappa = \frac{1}{2}$, $\theta = \frac{3}{4}$, and $\varphi(x, x, x) = \sqrt{x}$ (hence $\zeta = 1$). For $p = \frac{1}{2}$, Eq. (23) implies $x_{0.5} = \frac{361}{1024}$. Eq. (25) and monotonicity of φ require

$$\frac{19}{32} = \varphi\left(\frac{361}{1024}, \frac{361}{1024}, \frac{361}{1024}\right) > \varphi\left(\frac{361}{1024}, \frac{361}{1024}, 0\right) > \varphi\left(\frac{361}{1024}, 0, 0\right) > \frac{1}{4}$$

which, since $\theta p^2 + \kappa(3p - 2p^2) = \frac{11}{16}$, can be accommodated by eq. (24). ■

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