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by

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Coasian Bargaining with An Arriving Outside Option

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1 Introduction

In a recent paper, Board and Pycia (2013) consider a Coasian bargaining model where a seller of a good chooses a sequence of prices over time, and a buyer has an outside option available in every period. They show that there is a unique equilibrium in which the seller posts the monopoly price in every period, and the buyer either immediately takes the offer or exercises his outside option. Their result implies that even when the seller can change the price frequently, the initial price does not converge to the lowest valuation of the buyer, so the Coase conjecture fails. Later in the paper, they apply their result to a model of sequential search where the buyer can sample a new trade opportunity in each period and argue that their result therefore provides a justification of the "no-haggling" assumption in Wolinsky's (1986) search model where the buyer can sample a new seller in each period.

In this short paper, we perturb Board and Pycia (2013) by assuming the buyer's outside option arrives at a stochastic time and examine the robustness of Board and Pycia's (2013) prediction. In many environments, especially in markets with search frictions, it is reasonable to assume that the buyer's outside option arrives randomly. Before the arrival of the outside option, the buyer can keep bargaining with the current seller. In the labor search literature, one typically assumes that the worker can search on-the-job. (See Burdett and Mortensen, 1998) In the consumer search literature, one typically assumes that the consumer has perfect recall. (See Stahl,1989 and Wolinsky, 1986) In principle, the consumer can always search "on-the-bargaining". Or more broadly, our model can be viewed as a simplified version of a model where the value of the buyer's outside option changes stochastically over time.

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We model a bargaining game between a seller and a buyer over an indivisible good. The buyer's valuation of the good is either high or low. In each period, the seller announces a price and the buyer decides whether to buy the good. If the buyer rejects the seller's offer, an outside option arrives with positive probability. Once the outside option arrives, the buyer can exercise it in any period.

We consider cases where the arrival of an outside option is observable to the seller and where the arrival is unobservable. In both cases, we show that for any arrival rate of the outside option (1) the seller makes multiple rounds of offers on the path of play, and (2) a generalized Coase conjecture holds: when the seller lacks commitment power, the initial price is arbitrarily close to the low-valuation buyer's reservation price, which is the difference between his valuation of the good and his expected payoff by waiting for the outside option. Notice that when the arrival rate is sufficiently large, the outside option arrives in "the twinkling of an eye." In Proposition 2 of Board and Pycia (2013), they show that the failure of the Coase conjecture requires that the buyer have a nonzero outside option. Our exercise implies that their result also requires that the buyer have the outside option available at any time.

A new incentive issue arises in the model with an unobservable outside option regarding the buyer's decision to opt out. The high-valuation buyer has a strong incentive to trade with the current seller, so he has a weaker incentive to exercise the outside option (when it is available) than the low-valuation buyer. Hence, if the seller observes that the buyer stays in bargaining, then he believes that the buyer's valuation is high with higher probability. Hence the buyer must take into account the effect of additional information regarding his opting-out behavior. We show that the standard Coasian equilibrium may not exist due to the presence of such an additional incentive issue. However, when the seller can make offer arbitrarily frequently, an equilibrium exhibiting the Coase conjecture exists.

We present the model in section 2 and study the observable and unobservable outside option cases in section 3 and 4 respectively. In section 5, we study the limit property of both model by allowing the seller to make offers frequently.

2 Model

Time is discrete and the length of each period is $\Delta > 0$, so $t = 0, \Delta, 2\Delta, ...$ We consider a bargaining game between a seller and a buyer. The seller owns an indivisible good that has zero value to himself. In each period the seller makes a price offer p. The buyer's type θ is either high (H) or low (L), which is her private information. Type-θ buyer's valuation of the good is v_θ, where $v_H > v_L$. Let $q(t) \in (0, 1)$ be the posterior belief of the seller in period t that the buyer is the low type. If the buyer accepts the seller's offer, she obtains a payoff $v_\theta - p$, the seller obtains p, and the game ends. If the buyer declines the offer, and if the buyer has yet to receive an outside option, an outside option arrives with probability $\lambda = 1 - e^{-\mu\Delta}$. Once the outside option arrives it is available in any period. If the buyer has an outside option available, she chooses whether to exercise the option. If the buyer opts out, the game ends and the buyer and the seller obtain payoffs ω and 0, respectively.

Since the buyer can always ignore all offers and wait for the outside option, the lower bound of her equilibrium payoff is given by

$$\lambda\omega + \delta(1-\lambda)\omega + \ldots = \frac{\lambda}{1-\delta(1-\lambda)}\omega \equiv W^*$$

where $\delta = e^{-r\Delta}$ is the buyer and seller's common discount factor. To avoid a trivial case, we assume that $v_H > v_L > W^*$.[1]

The timing in each period is summarized as follows: At the beginning of period t, the seller makes an offer $p(t)$. The buyer decides whether to accept the offer. If she declines the offer, the outside option may arrive. If the outside option is available, the buyer can decide to take it immediately or wait. The game continues to the next period only if the buyer decides to wait.

A price history until time t is the sequence of previous rejected price offers $\{p(\tau)\}_{\tau=0}^{t}$. At time t, the index function $o(t)$ equals 0 if the outside option has not arrived yet, and $o(t) = 1$ otherwise. In this paper, we consider a model where the arrival of the outside option is observable and a model where the arrival is unboservable. In the first model, both the price history and the arrival of the outside option is the public history $\{p(\tau), o(\tau)\}_{\tau=0}^{t}$, while in the second model, the public history only summarizes the price sequence $\{p(\tau)\}_{\tau=0}^{t}$ and the arrival of the outside option is the buyer's private information.

A *perfect Bayesian equilibrium* (PBE) is a strategy profile: a public-history-contingent pricing rule of the seller, and the buyer's acceptance and exercise decisions, and updated beliefs about the buyer's values and the arrival of the outside option such that: actions are optimal given beliefs; beliefs are derived from actions according to Bayes' rule whenever possible. We say a PBE is a *Coasian equilibrium* if (1) it is a weak-Markov perfect equilibrium, and (2) on the path of play, the price $p(t)$ declines over time and belief $q(t)$ rises over time. In this paper, we will mainly focus on Coasian equilibria unless otherwise mentioned.

[1]Otherwise, the low type would not accept any positive price offer, and the seller's screening problem is trivial.

3 Observable Outside Option

In this section, we assume that the arrival of the buyer's outside option is public information. The first observation is that in any equilibrium, both types of buyers opt out immediately if the outside option arrives. The reason is as follows. When the outside option is available, the seller's price cannot be lower than $\underline{p} = v_L - \omega$, so the low type's payoff is bounded above by ω. Since waiting is costly, once the outside option is available in period t, the low type will exercise it immediately to obtain ω instead of waiting. In anticipation of the low type's choice, the seller's optimal price will be bounded below by $v_H - \omega$ once the outside option is available. A similar argument implies that the high type will exercise the outside option as soon as possible.

Therefore, we can construct the equilibrium as follows. When q_t is large enough, the seller is almost convinced that the buyer's valuation is low. To avoid the cost of delay and the risk of losing the buyer, he charges a price $v_L - W^*$ and the buyer accepts it regardless of her type. When $q(t)$ is small, in equilibrium, only the high type buyer accepts the price offer with certain probability. Hence, the high type's indifference condition implies that the price sequence p_t must satisfy the following recursive equation:

$$v_H - p(t) = \lambda\omega + (1-\lambda)\,\delta\,(v_H - p(t+\Delta)). \tag{1}$$

In each period, the high type accepts the offer $p(t)$ with probability $\beta(t)$, and the low type declines any offer with $p > v_L - W^*$. Both types of buyers exercise the outside option once it arrives. Over time, if the game is still being played, the belief $q(t)$ rises and reaches the cutoff q^* in finitely many periods, the seller posts an offer $v_L - W^*$, and the game ends. By the standard technique in the Coase conjecture literature, we can show that the above equilibrium is the unique PBE generically.[2] The result is summarized in the following proposition:

Lemma 1. *A Coasian equilibrium exists, and generically it is the unique PBE.*

The proof is a straightforward extension of that in the standard Coasian bargaining model; hence, it is omitted here. We provide some limit properties of the equilibrium as follows.

Proposition 1. *In the equilibrium,*

1. *for any $\delta \in (0,1)$, as λ goes to 1, if $q(0) \leq \frac{v_H - v_L}{v_H - \omega}$, the initial price $p(0)$ converges to $v_H - \omega$, and it converges to $v_L - \omega$ if $q(0) \leq \frac{v_H - v_L}{v_H - \omega}$.*

2. *for any $\delta \in (0,1)$, as λ goes to 0, the initial price goes to that in the standard Coasian bargaining model.*

[2] See Fudenberg, Levine and Tirole (1985) and Gul, Sonnenschein and Wilson (1986).

Proof. In the Appendix. □

When $\lambda = 1$, the model is a two-type version of Board and Pycia (2013). In the unique equilibrium, the seller posts a constant price in each period. If $q(0) < \frac{v_H - v_L}{v_H - \omega}$, the price is $v_H - \omega$, the high type takes the offer and the low type exercises the outside option immediately. If $q(0) \geq \frac{v_H - v_L}{v_H - \omega}$, the price is $v_L - \omega$, and both the high type and the low type accept the offer immediately. Proposition 1 says that when the arrival of the outside option is observable, the equilibrium correspondence is continuous at $\lambda = 1$. On the other hand, as λ goes to zero, the equilibrium prediction converges to that in the standard Coasian bargaining model.

In the equilibrium, the seller makes multiple serious offers to screen the buyer. Notice that this result depends on the specification of the timing: in each period, the seller makes an offer before the arrival of the outside option. If we assume an alternative timing: the possible arrival of the outside option is at the beginning of each period. Once the arrival (or not) outcome is realized, the seller makes the offer. Then, the buyer decides to take the offer or exercise the outside option. Under such a specification, the seller may have an incentive to wait for the arrival of the outside option. Once it arrives, the seller simply replicate the strategy in Board and Pycia (2013) by charging a constant price which is either $v_L - \omega$ or $v_H - \omega$. By doing so, the seller can expect a discounted monopoly profit $\frac{\lambda}{1-\delta(1-\lambda)}\pi$, where $\pi = max\{v_L - \omega, q(0)(v_H - \omega)\}$. Notice that π is the commitment payoff of the seller when the outside option is available. Hence, when λ and δ is large enough, the seller has the incentive to wait for the arrival of the outside option as a commitment device. In equilibrium, only one serious offer is made when the outside option is available. When the arrival of the outside option is not observable by the seller, such a strategy is invalid.

4 Unobservable Outside Option

Now we consider the unobservable outside option model. In this scenario, the buyer's incentive to exercise the outside option is different from that in the previous model. The reason is as follows. First, the price is never lower than $v_L - \omega$, so the low type exercises the outside option whenever it is available. As a result, in equilibrium, the seller serves the low type only if the outside option has not arrived yet, and therefore, the price lower bound is $v_L - W^*$, which is the price that the low type is indifferent between accepting and rejecting to wait for the arrival of the outside option. If a Coasian equilibrium exists, the seller screens the buyer over time, and the price declines to $v_L - W^*$ in finitely many periods. As a high type, the payoff to wait for such a price is $\delta^\tau (v_H - v_L + W^*)$, where τ is the remaining period before the seller charges $v_L - W^*$. When τ is large, both the high type and the low type will exercise the outside option once it is available, so the seller's intertemporal belief updating is exactly the same as the one in the case of the observable outside option case. When τ is small, the cost of delay is negligible, so waiting for the low price is better than exercising the outside option immediately. Consequently, if the buyer enters next period, he could be a low type without an available outside option, or a high type who ignores his available outside option; therefore, the seller can update his belief $q(t)$ from two facts over time: (1) the offer is rejected, so the buyer is a high type with higher probability, and (2) no one exercises the outside option, so the buyer is a low type with higher probability. As a result, we conjecture that the equilibrium has two phases.

- In phase I, the price is high, the high type randomizes between taking the offer or not, and both types exercise the outside option once it is available. Over time, the belief $q(t)$ rises and the price $p(t)$ declines.
- In phase II, the price is low, the high type randomizes between taking the offer or not, but only the low type exercises the outside option once it is available.

We construct the Coasian equilibrium as follows. Let $p_0 = v_L - W^*$ be the last price the seller charges to end the game. Then there exists q_1 where the the seller is indifferent between offering p_0 and offering some $p_1 > p_0$ to induce two-period screening. To pin down p_1, we need the high type's indifference condition:

$$v_H - p_1 = \delta(v_H - p_0),$$

or $p_1 = (1-\delta)v_H + \delta p_0$. Now consider the seller's incentive. At $q = q_1$, if the seller offers p_1, then high-type buyer accepts it for sure, and low-type buyer opts out if the option arrives, and the remaining low-type buyer accepts p_0 in the next period. Hence, the seller's expected payoff is

$$(1-q_1)p_1 + q_1\delta(1-\lambda)p_0$$

which must be the same as p_0, the payoff from immediate agreement. Therefore, q_1 is given by

$$q_1 = \frac{p_1 - p_0}{p_1 - \delta(1-\lambda)p_0.}$$

Furthermore, we can construct $\{p_k, q_k\}_{k=2}^{\infty}$ recursively as follows.

- Given p_{k-1}, calculate $\tilde{p} = (1-\delta)v_H + \delta p_{k-1}$.
- If $\omega \leq \delta(v_H - p_{k-1})$, then $p_k = \tilde{p}$. In this case, we are still in Phase II, and the high-type buyer does not opt out after the rejection.

- If $\omega > \delta(v_H - p_{k-1})$, then $p_k = (1-\delta(1-\lambda))v_H + \delta(1-\lambda)p_{k-1} - \lambda\omega$. In this case, we are in Phase I, and the high-type buyer opts out if the option is available. The indifference condition takes into account the payoff from outside option. It is clear from the equation above that once $p_k > v_H - \omega$, then $p_{k'} > v_H - \omega$ for all $k' > k$.

- For each price, the high type accepts the offer p_k with probability $\beta_k \in (0,1)$ and rejects it with the complementary probability. Such a probability β_k can pin down the belief sequence $\{q_k\}$. If the belief sequence is decreasing, then for any prior $q(0)$, after constructing finitely many steps, we have $q_K < q(0)$ so the equilibrium construction is finished.

Notice that when δ is small, $\omega > \delta(v_H - p_k)$ for any $p_k \geq v_L - W^*$, and phase II does not exist. As a result, once the outside option arrives, the buyer exercises it regardless of his type, and therefore the equilibrium is essentially identical to that in the case of the observed outside option. To focus on the more interesting case, in the rest of this paper, we assume δ is large. We will show that, such Coasian equilibria may not always exist in our model.

Proposition 2. *Fix a large discount factor $\delta < 1$. There exists a cutoff $\lambda_\delta < 1$ such that there is a Coasian equilibrium if $\lambda \in [0, \lambda_\delta)$. Moreover, $\lim_{\delta\to 1}\lambda_\delta < 1$.*

Proof. In the Appendix. □

In contrast to the standard Coasian bargaining model, Proposition 2 points out that a Coasian equilibrium may fail to exist. To see the intuition, consider the last three periods of the game. Suppose that $\omega \leq \delta(v_H - p_1)$, so we are in phase II when $k = 2$. The high type's indifference condition yields $p_2 = (1-\delta^2)v_H + \delta^2 p_0$. By charging p_2, the seller's value is

$$(1-q_2)\,\beta(q_2, q_1)\,p_2 + [(1-q_2)(1-\beta) + q_2(1-\lambda)]\,\delta V(q_1)$$

where $V(q_1)$ is the seller's continuation value. Since the seller is indifferent between charging p_0 and p_1 given the belief q_1, $V(q_1) = p_0$. On the other hand, by charging p_1, the high type will take the offer for sure, and the low type will wait and take the next period offer p_0 if no outside option is available, so the seller's value is

$$(1-q_2)\,p_1 + q_2(1-\lambda)\,\delta p_0.$$

Since the seller is indifferent between charging p_2 and p_1, some simple algebra implies that $\beta = \frac{v_H}{v_H + \delta(v_H - v_L + W^*)}$. Since only the low type exercises the outside option, the seller's belief likelihood updating is given by

$$\frac{q_1}{1-q_1} = \frac{q_2}{1-q_2}\frac{1-\lambda}{1-\beta}.$$

When $\lambda > \beta$, $q_1 < q_2$ which is inconsistent with the hypothesis! What is more, since $\lim_{\delta \to 1} \beta = \frac{v_H}{2v_H - v_L + W^*} < 1$, for large λ, Coasian equilibria fail to exist even when the discount factor δ is arbitrarily close to 1. Hence, in order to have a Coasian equilibrium, the arrival probability λ must be small enough.

Remark 1. *The results are robust to the specification of the timing since the arrival of the outside option is unobservable. Hence, in any equilibrium, the seller makes multiple offers.*

The result in proposition 2 can easily apply to the model where the buyer has a time-varying outside option. Suppose $\omega = \{\omega_L, \omega_H\}$ and $v_H > v_L > \omega_H > \omega_L$. Initially, $\omega(0) = \omega_H$ with probability ϑ. In each period, the outside option may switch from ω_θ to $\omega_{\theta'}$ with a probability $\lambda_{\theta\theta'}$ where $\theta, \theta' \in \{L, H\}$. In our baseline model, we assume that $\omega_L = 0$, $\lambda_{LH} = \lambda$ and $\lambda_{HL} = 0$. However, nothing will change if we assume $\omega_L > 0$ and $\lambda_{HL} > 0$. When the outside option is unobservable, a buyer with a low current outside option may wait for its change. In a market with search frictions, the change in the outside option can be interpreted as the arrival (disappearance) of a better trade opportunity.

If the Coasian equilibrium fails to exist when λ is large, what is the equilibrium? In a slightly different environment, Hwang (2013) shows that there is an equilibrium when λ is large.[3] In the equilibrium, which he calls a *deadlock equilibrium*, the uninformed seller fails to learn the type of the buyer and continues to make the same randomized pricing through the bargaining process. As a result, the equilibrium behavior produces an outcome path that resembles an outcome of the bargaining deadlock and its resolution. The equilibrium with deadlock has inefficient outcomes such as a real-time delay and a breakdown in negotiations. For an intermediate value of λ, the Coasian equilibrium and the deadlock equilibrium may coexist. Moreover, as λ converges to zero, the Coasian equilibrium converges to the unique sequential equilibrium of Fudenberg, Levine, and Tirole (1985). On the other hand, as λ converges to one, the deadlock equilibrium converges to the monopoly pricing equilibrium of Board and Pycia (2013).

5 Frequent Offers

In this section, we consider the equilibrium behavior when the seller can make offers arbitrarily frequently. We fix the discount rate r and the arrival rate of the outside option μ, then take the length of each period Δ to zero. Note that for any value of r and μ, as $\Delta \to 0$, $\delta = e^{rt}$ and

[3] Hwang (2013) considers bargaining in which a seller is informed about the quality of the good. A buyer makes an offer in each period, and the seller receives an outside option after each rejection. So his model is mathematically equivalent to ours, although the roles of the seller and the buyer are reversed.

$\lambda = 1 - e^{-\mu\Delta}$ converge to one and zero, respectively. The following proposition says that there is an equilibrium with Coasian dynamics in the limit of frequent offers.

Proposition 3. *(1) Suppose the outside option is observable. Then as $\Delta \to 0$, the initial price in the unique PBE converges to $v_L - \frac{\mu}{\mu+r}\omega$.*

(2) Suppose the outside option is unobservable. Then as $\Delta \to 0$, there exists a PBE where the initial price converges to $v_L - \frac{\mu}{\mu+r}\omega$.

Proof. In the Appendix. □

Proposition 3 points out that for any fixed arrival rate of the outside option, the introductory price converges to the buyer's lowest possible reservation price as the seller's commitment power disappears. Note that in both cases, $q_2 \to 0 < q(0)$ and $W^* \to \frac{\mu}{\mu+r}\omega$ as $\Delta \to 0$.

Note that if the arrival rate is high, the outside option arrives almost immediately. However, it is not profitable for the seller to charge a higher price, which is in sharp contrast to Board and Pycia (2013). Our result implies that the "no haggling" result in Board and Pycia (2013) depends on the fact that the buyer has an outside option at any time with probability one. As previously mentioned, in many environments including markets with search frictions, it is more plausible to assume that the buyer receives the outside option (new trade opportunity) randomly. When the seller has limited commitment power, he can make new offers after a very short time period. In such a time period, the probability of the arrival of the outside option is very small. Hence, to justify the "no haggling" assumption in search models, one may have to assume that the seller has some commitment power.

Appendix

Proof of Proposition 1

Similar to the proof of Proposition 2, the equilibrium is characterized by a sequence of cutoff beliefs $\{q_k\}$ and a sequence of prices $\{p_k\}$ where $k \in \mathbb{N}$. Both types exercise the outside option once it arrives. Before the arrival of the outside option, the seller screens the buyer over time. When the seller's belief is $q(t) = q_1$, he is indifferent between charging p_0 and p_1 where

$$\begin{aligned} p_0 &= v_L - W^* \\ p_1 &= v_H - \lambda\omega - (1-\lambda)\,\delta\,(v_H - v_L + W^*)\,, \\ q_1 &= \frac{p_1 - p_0}{p_1 - \delta(1-\lambda)p_0}. \end{aligned}$$

If $q(t) > q_1$, the seller charges p_0 and the game ends. If $q(t) \leq q_1$, the seller screens the high type by charging a higher price. When $q(t) = q_k$, the seller charges p_k. The low type accepts the offer only if the price $p(t) \leq v_L - W^*$, while the high type accepts price p_k with probability β_k. The price sequence is pinned down by

$$v_H - p_{k+1} = \lambda\omega + (1-\lambda)\,\delta\,(v_H - p_k)\,. \tag{2}$$

The belief updating is pinned down by equation (11), and the high type's strategy β is consistent with the belief updating equation:

$$\frac{q_{k-1}}{1-q_{k-1}} = \frac{q_k}{1-q_k}\frac{1}{1-\beta_k}.$$

Owing to the arrival of the outside option, the cutoff belief q_1 and the equilibrium price sequence are different from those in standard Coasian bargaining model.

As $\lambda \to 1$, $W^* \to \omega$, $q_1 \to \frac{v_H - v_L}{v_H - \omega}$, and

$$p_1 \to \begin{cases} v_H - \omega & \text{if } q(t) \geq q_1, \\ v_L - \omega & \text{otherwise.} \end{cases}$$

By equation (2), $p_k \to p_{k+1}$. When $q(t) > q_1$, for any k, $p_k \to v_h - \omega$ as $\lambda \to 1$. and therefore $p(0) \to v_h - \omega$.

As $\lambda \to 0$, both q_1 and the recursive equation of the equilibrium price go to those in the standard Coasian model, so the equilibrium converges to the equilibrium in the standard Coasian bargaining model. ∎

Proof of Proposition 2

Constuction of Sequences of Prices and Cutoff Beliefs In order to construct the Coasian equilibrium, we characterize a sequence of equilibrium prices p_k and cutoff beliefs q_k. First, recursive equations for p_k are given as follows:

- given p_{k-1}, calculate $\tilde{p} = (1-\delta)v_H + \delta p_{k-1}$.
- if $\omega \leq \delta\,(v_H - p_{k-1})$, then $p_k = \tilde{p}$.
  - In this case, we are still in Phase II, and the high-type buyer does not opt out after the rejection.
- if $\omega > \delta\,(v_H - p_{k-1})$, then $p_k = (1-\delta(1-\lambda))v_H + \delta(1-\lambda)p_{k-1} - \lambda\omega$.

– We are in Phase I, and high-type buyer opts out if the option is available. The indifference condition takes into account the payoff from the outside option.

– It is clear from the equation above that once $p_k > v_H - \omega$, then $p_{k'} > v_H - \omega$ for all $k' > k$.

Given p_k, q_k is given by the seller's indifference condition. At $q = q_k$, the seller is indifferent between offering p_k(higher price but more delay) and p_{k-1}(less delay but lower price). The indifference condition depends on whether we are in Phase I or II. Before we do the analysis, let us define several notations.

- Let $\beta(q, q')$ be the high-type buyer's acceptance probability which induces posterior from q to q', given that both types of buyers take the outside option. That is, $\beta(q, q')$ satisfies
$$\frac{q'}{1-q'} = \frac{q}{1-q}\frac{1}{1-\beta(q,q')}.$$
So $\beta(q, q') = 1 - \frac{q}{1-q}\frac{1-q'}{q'}$.

- Let $\beta'(q, q')$ be the high-type buyer's acceptance probability, which induces posterior from q to q', given that only the low-type buyer takes the outside option. That is, $\beta'(q, q')$ satisfies
$$\frac{q'}{1-q'} = \frac{q}{1-q}\frac{1-\lambda}{1-\beta'(q,q')}.$$
So $\beta(q, q') = 1 - \frac{q}{1-q}\frac{1-q'}{q'}(1-\lambda)$.

- Note that $(1-q)\beta(q, q') = 1 - \frac{q}{q'}$ and $(1-q)\beta'(q, q') = 1 - \frac{q}{q'}(1-\lambda(1-q'))$.

Let $V(q)$ be the seller's payoff in the Coasian equilibrium when the belief is q.

1. If $p_k \leq v_H - \omega$ (Phase II): if the seller offers p_k, then the high-type buyer accepts with probability $\beta'(q_k, q_{k-1})$, and only the low-type buyer takes the outside option. On the other hand, if he offers p_{k-1}, the high-type buyer accepts with higher probability $\beta'(q_k, q_{k-2})$. Hence the indifference condition is given by
$$\begin{aligned} V(q_k) &= (1-q_k)\beta'(q_k, q_{k-1})p_k + (1-(1-q_k)\beta'(q_k, q_{k-1}) - q_k\lambda)\delta V(q_{k-1}) \\ &= (1-q_k)\beta(q_k, q_{k-1})p_{k-1} + (1-(1-q_k)\beta(q_k, q_{k-1}))V(q_{k-1}). \end{aligned}$$
Simplifying, we have
$$V(q_k) = (1-\gamma_k(1-\lambda+\lambda q_{k-1}))p_k + \gamma_k(1-\lambda)\delta V(q_{k-1}) \tag{3}$$
$$= (1-\gamma_k)p_{k-1} + \gamma_k V(q_{k-1}), \tag{4}$$

where $\gamma_k = \frac{q_k}{q_{k-1}}$. From the indifference condition, we have

$$(1-\delta(1-\lambda))\gamma_k V(q_{k-1}) = (1-\gamma_k(1-\lambda+\lambda q_{k-1}))(1-\delta)v_H+(\delta\lambda\gamma_k(1-q_{k-1})-(1-\delta)(1-\gamma_k))p_{k-1}. \tag{5}$$

Putting (5) into (4), we have

$$(1-\delta(1-\lambda))V(q_k) = (1-\gamma_k(1-\lambda+\lambda q_{k-1}))(1-\delta)v_H + (1-q_k)\lambda\delta p_{k-1}. \tag{6}$$

Putting (6) again into (5) and simplifying, we have

$$\begin{aligned}
\gamma_k &= \frac{(1-\delta)(v_H - p_{k-1})}{(1+(1-\gamma_{k-1})(1-\lambda))(1-\delta)v_H + (1-q_{k-1})\lambda\delta p_{k-2} - ((1-\delta)+\delta\lambda(1-q_{k-1}))p_{k-1}} \\
&= \frac{v_H - p_{k-1}}{(1+(1-\gamma_{k-1})(1-\lambda))v_H - p_{k-1} - (1-q_{k-1})\lambda\delta(v_H - p_{k-2})} \\
&= \frac{v_H - p_{k-1}}{v_H - p_{k-1} + (1-\gamma_{k-1})v_H - \lambda\left[(1-\gamma_{k-1})\, v_H + (1-q_{k-1})\delta(v_H - p_{k-2})\right]}.
\end{aligned}$$

To be consistent with the hypothesis, we must have $\gamma_k < 1$ for all k, which requires that

$$\lambda < \lambda_k \equiv \frac{(1-\gamma_{k-1})v_H}{(1-\gamma_{k-1})\, v_H + (1-q_{k-1})\delta(v_H - p_{k-2})},$$

for all k.

2. If $p_k > v_H - \omega$ (Phase I): the seller is indifferent between offering p_k, which the high-type buyer accepts with probability $\beta(q_k, q_{k-1})$, and offering p_{k-1}, which the high-type buyer accepts with probability $\beta(q_k, q_{k-2})$. In both cases, all types of buyers take the outside option. Hence, the indifference condition is given by

$$\begin{aligned}
V(q_k) &= (1-q_k)\beta(q_k, q_{k-1})p_k + (1-(1-q_k)\beta(q_k, q_{k-1}) - \lambda)\delta V(q_{k-1}) \\
&= (1-q_k)\beta(q_k, q_{k-1})p_{k-1} + (1-(1-q_k)\beta(q_k, q_{k-1}))V(q_{k-1}).
\end{aligned}$$

Simplifying, we have

$$\begin{aligned}
V(q_k) &= (1-\gamma_k)p_k + \gamma_k(1-\lambda)\delta V(q_{k-1}) \qquad (7)\\
&= (1-\gamma_k)p_{k-1} + \gamma_k V(q_{k-1}). \qquad (8)
\end{aligned}$$

From the indifference condition, we have

$$\gamma_k V(q_{k-1}) = (1-\gamma_k)((v_H - p_{k-1}) - W^*). \tag{9}$$

Putting (9) into (8), we have

$$V(q_k) = (1 - \gamma_k)(v_H - W^*). \tag{10}$$

Putting (10) again into (9) and simplifying, we have

$$\gamma_k = \frac{(v_H - W^*) - p_{k-1}}{(2 - \gamma_{k-1})(v_H - W^*) - p_{k-1}}. \tag{11}$$

Since $\gamma_{k-1} < 1$, $\gamma_k < 1$.

Equilibrium Profile Using $\{p_k\}$ and $\{q_k\}$, we can describe the equilibrium behavior of the Coasian equilibrium. Let K be an integer such that a decreasing sequence of $\{q_k\}$ goes below the prior $q(0)$ for the first time. That is,

$$K = \min\{k : q_k \leq q(0) \text{ and } \gamma_j < 1 \text{ for all } j \leq k\}.$$

where $\gamma_k = \frac{q_k}{q_{k-1}}$. Note that if $\gamma_{k'} \geq 1$ for some k', then there is no such K for any prior $q(0) < q_{k'}$. Consider a generic case where $q_K < q(0)$. Then the equilibrium behavior of the Coasian equilibrium is as follows:

- On the equilibrium path, the seller offers a price

$$p(t) = \begin{cases} p_{K-1} & \text{if } q(t) \in [q(0), q_{K-1}], \\ p_j & \text{if } q(t) \in (q_{j+1}, q_j] \text{ for } j = 1, \dots, K-2, \\ p_0 & \text{if } q(t) \in (q_1, 1]. \end{cases}$$

- The high type accepts $p(t)$ with probability

$$\sigma(t) = \begin{cases} 0 & \text{if } p(t) > p_{K-1}, \\ \max\{\beta(q(t), q_j), 0\} & \text{if } p(t) \in (p_j, p_{j+1}] \text{ for } j = 1, \dots, K-2 \text{ and } p(t) > v_H - w, \\ \max\{\beta'(q(t), q_j), 0\} & \text{if } p(t) \in (p_j, p_{j+1}] \text{ for } j = 1, \dots, K-2 \text{ and } p(t) \leq v_H - w, \\ 1 & \text{if } p(t) \leq p_1. \end{cases}$$

- The high type opts out at period t if and only if $p(t) > v_H - w$.

Off the Path of Play. Since the buyer's deviation is unobservable, it does not change the continuation play. Once a seller deviates by charging a "wrong" price, players are off the path of play. As long as the price is higher than $v_L - W^*$, the low type will decline it and the high type will mix between accepting it and waiting. To sustain the high type's indifference condition, the seller must randomize between different (equilibrium) prices in the next period. The probability distribution of this randomization depends on the deviation price.

Note that the equilibrium profile does not need to specify the behavior when the posterior belief is less than the prior. This is because given the equilibrium behavior, the seller's belief is no less than the prior after any history.

Proof of Proposition 2 Define $\hat{k}$ to be a minimum integer such that γ_k is greater than 1. Note that $\hat{k}$ does not exists if γ_k is less than 1 for all k. Then the Coasian equilibrium exists if and only if the prior belief is greater than $q_{\hat{k}}$.

Fix a prior $q(0)$. In order to prove Proposition 2, it suffices to show that $q_{\hat{k}} < q(0)$ for sufficiently small λ. We prove the claim by showing that for sufficiently small λ, γ_k is bounded away from 1 for any k. For $k = 2$, we already know that when λ is small enough, there exists a $\varepsilon > 0$, such that $\gamma_2 < 1 - \varepsilon$. Now we use an induction argument to show that $\gamma_k < 1 - \varepsilon$ for any $k \in \mathbb{N}$.

$$\begin{aligned}
\gamma_k &= \frac{v_H - p_{k-1}}{v_H - p_{k-1} + (1-\gamma_{k-1})v_H - \lambda\left[(1-\gamma_{k-1})\,v_H + (1-q_{k-1})\delta(v_H - p_{k-2})\right]} \\
&\leq \frac{v_H - p_{k-1}}{v_H - p_{k-1} + (1-\gamma_{k-1})v_H\,(1-\lambda) - \lambda\delta v_H} \\
&= \frac{1}{1 + \frac{(1-\gamma_{k-1})v_H(1-\lambda) - \lambda\delta v_H}{v_H - p_{k-1}}}
\end{aligned}$$

By the hypothesis, $1 - \gamma_{k-1} > \varepsilon$. Let $\lambda = \varepsilon$. When ε is small, $\frac{1}{1+\frac{(1-\gamma_{k-1})v_H(1-\lambda)-\lambda\delta v_H}{v_H - p_{k-1}}} \simeq 1 - \frac{(1-\gamma_{k-1})v_H(1-\lambda)-\lambda\delta v_H}{v_H-p_{k-1}}$. We need to show that $\frac{(1-\gamma_{k-1})v_H(1-\lambda)-\lambda\delta v_H}{v_H-p_{k-1}} < \varepsilon$. Since $p_{k-1} \geq v_L - W^*$, we have $\frac{(1-\gamma_{k-1})v_H(1-\lambda)-\lambda\delta v_H}{v_H-p_{k-1}} < \frac{v_H}{v_H-v_L+W^*}\left[(1-\varepsilon)\,\varepsilon - \varepsilon\delta\right] = \frac{v_H}{v_H-v_L+W^*}\left[\varepsilon\,(1-\delta) - \varepsilon^2\right]$. Since $\frac{v_H}{v_H-v_L+W^*}\,(1-\delta) < 1$, we have $\frac{(1-\gamma_{k-1})v_H(1-\lambda)-\lambda\delta v_H}{v_H-p_{k-1}} < \varepsilon$. Hence, we have the desired result. ■

Proof of Proposition 3

Fix any r and μ. It suffices to show that there exists $\bar{\Delta} > 0$ such that for any $\Delta < \bar{\Delta}$, there exists $\alpha > 0$ such that $\gamma_k < 1 - \alpha$ for all k. The rest of the proof is identical to the proof in the standard Coase conjecture literature, so we only provide a heuristic argument here to illustrate the idea.

- For $k = 1$, in both the observable and the unobservable outside option model, we have

$$\begin{aligned}\gamma_1 = q_1 &= \frac{p_1 - p_0}{p_1 - \delta(1-\lambda)p_0} \\ &= \frac{v_H - p_0}{v_H - p_0 + \frac{1-\delta(1-\lambda)}{1-\delta}p_0} \\ &\to \frac{v_H - p_0}{v_H + \frac{\mu}{r}p_0} < 1,\end{aligned}$$

  as Δ goes to zero.

- For any $k \in \mathbb{N}$, $p_k \to p_0$ as $\Delta \to 0$ in both the observable and the unobservable outside option model.

- When the outside option is unobservable, for $k > 1$, in Phase II,

$$\begin{aligned}\gamma_k &= \frac{v_H - p_{k-1}}{v_H - p_{k-1} + (1-\gamma_{k-1})v_H - \lambda\left[(1-\gamma_{k-1})\, v_H + (1-q_{k-1})\delta(v_H - p_{k-2})\right]} \\ &\to \frac{v_H - p_0}{(2-\gamma_{k-1})v_H - p_0},\end{aligned}$$

  as Δ goes to zero. Since the function $f(x) = \frac{v_H - p_0}{(2-x)v_H - p_0}$ is convex and has fixed points of 1 and $1 - \frac{p_0}{v_H}$, if $\gamma_k \in (1 - \frac{p_0}{v_H}, 1)$ then $\gamma_{k+1} < \gamma_k$.

- Similarly, when the outside option is unobservable, for $k > 1$, in Phase I,

$$\gamma_k \to \frac{v_H - W^{**} - p_0}{(2-\gamma_{k-1})(v_H - W^{**}) - p_0},$$

  as Δ goes to zero, where $W^{**} = \lim_{\Delta\to} W^* = \frac{\mu}{\mu+r}\omega$.

- When the outside option is observable, for $k > 1$, the belief updating is identical to that in phase I of the unobservable outside option model.

Hence, for any $q(0)$, as $\Delta \to 0$, there exists a finite K such that $q(0) > q_k$ when $k > K$. So the proof is complete. ■

References

[1] Board, S., and Pycia, M. 2013, "Outside Option and the Failure of Coase Conjecture" *American Economic Review*, forthcoming.

[2] Burdett, K., and Mortensen, D., 1998, "Wage Differentials, Employer Size, and Unemployment" *International Economic Review*, Vol. 39, pp. 257-273.

[3] Fudenberg, D., Levine, D., and Tirole, J. 1985, "Infinite-Horizon Models of Bargaining with One-Sided Incomplete Information" in *Game-Theoretic Models of Bargaining*, ed. Alvin E. Roth, 7398. Cambridge, UK: Cambridge University Press.

[4] Gul, F., Sonnenschein, H., and Wilson, R. 1986, "Foundations of Dynamic Monopoly and the Coase Conjecture" *Journal of Economic Theory*, Vol. 39, pp. 155 - 190.

[5] Hwang, I. 2013, "A Theory of Bargaining Deadlock" *unpublished manuscript.*

[6] Wolinsky, A., 1986, "True Monopolistic Competition as a Result of Imperfect Information" *The Quarterly Journal of Economics*, Vol. 101, pp. 493-512.

[7] Stahl, D., 1989, "Oligopolistic Pricing with Sequential Consumer Search " *American Economic Review*, Vol. 79, pp. 700-712.