

Penn Institute for Economic Research  
Department of Economics  
University of Pennsylvania  
3718 Locust Walk  
Philadelphia, PA 19104-6297  
[pier@econ.upenn.edu](mailto:pier@econ.upenn.edu)  
<http://economics.sas.upenn.edu/pier>

## *PIER Working Paper 13-042*

“Modeling the Evolution of Expectations and  
Uncertainty in General Equilibrium”

by

Francesco Bianchi and Leonardo Melosi

<http://ssrn.com/abstract=2275515>

# Modeling the Evolution of Expectations and Uncertainty in General Equilibrium

Francesco Bianchi  
Duke University

Leonardo Melosi  
Federal Reserve Bank of Chicago

December, 2012

## Abstract

This paper develops methods to study the evolution of agents' expectations and uncertainty in general equilibrium models. A central insight consists of recognizing that the evolution of agents' beliefs can be captured by defining a set of *regimes* that are characterized by the degree of agents' pessimism, optimism, and uncertainty about future equilibrium outcomes. Once this kind of structure is imposed, it is possible to create a mapping between the evolution of agents' beliefs and *observable* outcomes. Agents in the model are fully rational, conduct Bayesian learning, and *they know that they do not know*. Therefore, agents form expectations taking into account that their beliefs will evolve according to what they observe in the future. The new modeling framework accommodates both gradual and abrupt changes in agents' beliefs and allows an analytical characterization of uncertainty. Shocks to beliefs are shown to have both first-order and second-order effects. To illustrate how to apply the methods, we use a prototypical Real Business Cycle model in which households form beliefs about the likely duration of high-growth and low-growth regimes.

**Keywords:** Markov switching DSGE model, Bayesian econometrics, beliefs, uncertainty, Bayesian learning, rational expectations.

*JEL classification:* D83, E52, E52.

---

We thank Martin Eichenbaum, Eric Leeper, Dan Waggoner, and Tao Zha for very useful comments and discussion. Todd Messer provided excellent research assistance. Francesco Bianchi gratefully acknowledges financial support from the National Science Foundation through grant SES-1227397. Francesco Bianchi, Department of Economics, Duke University, 213 Social Sciences Building, Durham, NC, 27708-0089, USA. Leonardo Melosi, Federal Reserve Bank of Chicago, 230 South La Salle street, Chicago, IL 60604-1413, USA. Email: lmelosi@frbchi.edu. The views in this paper are solely the responsibility of the authors and should not be interpreted as reflecting the views of the Federal Reserve Bank of Chicago or any other person associated with the Federal Reserve System.

# 1 Introduction

A centerpiece of the rational expectations revolution is that macroeconomic outcomes critically depend on how agents' expectations about future events evolve over time. Therefore, correctly modeling the dynamics of the private sector's beliefs is essential to accurately predict economic outcomes. Most general equilibrium models are solved assuming that agents know the exact structure of the whole economy and are certain about the rules governing the future behavior of other players. This is certainly a strong restriction imposed upon the dynamics of beliefs. For instance, the private sector is likely to have limited information about the future path of policy-makers' decisions, dividend payments, economic growth, etc.

In this paper we develop methods to solve models in which forward-looking and *fully rational* agents are subject to waves of pessimism, optimism, and uncertainty that turn out to critically affect macroeconomic outcomes. Such outbursts of pessimism, optimism, and uncertainty may happen abruptly or may gradually unfold over a long period of time in response to the behavior of other agents or to the realizations of economic outcomes. All results are derived within a modeling framework suitable for structural estimation that will allow researchers to bring the models to the data.

The evolution of agents' beliefs is modelled assuming the existence of different states of the world that differ according to the statistical properties of the exogenous shocks or based on the behavior of some of the agents in the model. Such regimes follow a Markov-switching process, which may be correlated with other aspects of the model. For example, the government could be more likely to inflate debt away when the level of spending is high. Agents are assumed to observe economic outcomes, but not the regimes themselves. Agents will then adopt Bayesian learning to infer which regime is in place. This will determine the evolution of agents' beliefs about future economic outcomes.

Our modeling framework goes beyond the assumption of *anticipated utility* that is often used in models characterized by a learning process. Such an assumption implies that agents forecast future events assuming that their beliefs will never change in the future. Instead, agents in our models *know that they do not know*. Therefore, when forming expectations, they take into account that their beliefs will evolve according to what they observe in the future. In our context, it is possible to go beyond the anticipated utility assumption because there are only a finite number of relevant beliefs and they are strictly linked to observable outcomes through the learning mechanism in a way that we can keep track of their evolution.

The proposed model framework is flexible enough to encompass both abrupt and gradual changes in beliefs. For example, augmenting the modeling framework with signals about the regime in place allows one to capture the effect of news on the evolution of the economy or to study the macroeconomic implications of changes in *animal spirits* about future events. At

the same time, through the learning process, we can model situations in which agents' beliefs gradually change in response to the behavior of other agents or the realizations of stochastic events. This sluggish adjustment of public expectations is hard to reproduce through rational expectations models in which the functioning of the whole economy is common knowledge among agents.

We show how to apply these methods using a prototypical Real Business Cycle (RBC) model. In the model, total factor productivity (TFP) growth can assume two values: high or low. For each value of TFP growth, we allow for a long-lasting and a short-lasting regime. Therefore, while agents can observe the current TFP growth rate, they are uncertain about its future values, because they do not know if the current value is likely to last for a short time or for a long time. We consider a wide range of specifications, allowing for smooth transitions or abrupt changes in agents' optimism about future realizations of TFP growth. Each of these different specifications can be easily captured with the appropriate transition matrix governing the evolution of TFP growth. This has the important implication that the dynamics of pessimism, optimism, and uncertainty are consistent in equilibrium. Whenever a short-lasting regime is in fact realized, with the benefit of hindsight, agents' beliefs turn out to over-react to the regime change because agents always take into account the possibility that the economy entered a long-lasting regime. On the other hand, if, in fact, the regime is long-lasting, it takes time for agents' beliefs to line up with the actual realization. This implies that although agents are fully rational, their beliefs are generally misaligned with respect to the actual state of the economy. Such a misalignment is found to substantially influence consumption and capital allocation in the RBC model.

The methods introduced in this paper can be combined with the techniques developed by Bianchi (2012) to obtain an analytical characterization of the evolution of uncertainty in response to changes in agents' beliefs. We expand our analysis of the RBC model to study the case in which agents receive signals about the likely duration of the current regime. In this environment, signals work as shocks to agents' beliefs that have first-order and second-order effects. Uncertainty about macroeconomic events evolves over time as agents' beliefs drift, creating interesting comovements between volatility and real activity. This might shed further light on the link between uncertainty and macroeconomic outcomes with respect to the fascinating study of Bloom (2009).

The methods developed in this paper are based on the idea of expanding the number of regimes to take into account the learning mechanism. The central insight consists of recognizing that the evolution of agents' beliefs can be captured by defining an expanded set of regimes indexed with respect to agents' beliefs themselves. Once this structure has been imposed, the model can be recast as a Markov-switching dynamic stochastic general equilibrium (MS-DSGE) model with perfect information. If regime changes enter *additively* the model can be

solved with standard solution methods such as *gensys* (Sims, 2002) and Blanchard and Kahn (1980), following the approach described in Schorfheide (2005) and Liu, Waggoner, and Zha (2011). If instead regime changes enter *multiplicatively* the model can be solved with any of the methods developed for solving MS-DSGE models, such as Davig and Leeper (2007), Farmer, Waggoner, and Zha (2009), Cho (2012), and Foerster, Rubio-Ramirez, Waggoner, and Zha (2011).

In both cases, the resulting solution is suitable for likelihood-based estimation. This is because even if the final number of regimes is very large, there is a tight link between observable outcomes and the evolution of agents' beliefs. In other words, the transition matrix governing the joint dynamics of the economy and agents' beliefs is highly restricted. For example, Bianchi and Melosi (2012a) apply these methods and Bayesian techniques to estimate a model in which agents are uncertain about the future stance of monetary policy. This paper is therefore related to a growing literature that models parameter instability to capture changes in the evolution of the macroeconomy. This consists of two branches: Schorfheide (2005), Justiniano and Primiceri (2008), Bianchi (forthcoming), Davig and Doh (2008), and Fernandez-Villaverde and Rubio-Ramirez (2008) introduce parameter instability in DSGE models, while Sims and Zha (2006), Primiceri (2005), and Cogley and Sargent (2005) work with structural VARs. Finally, to the extent that we can model situations in which agents' beliefs evolve in response to policy-makers' behavior, our work is also linked to papers that study how inflation expectations respond to policy decisions, such as Mankiw, Reis, and Wolfers (2004), Nimark (2008), Del Negro and Eusepi (2010), and Melosi (2012a,b).

The remainder of the paper is organized as follows. Section 2 introduces the class of models and derives the main results. Section 3 applies the methods to a prototypical RBC model in which agents try to infer the likely value of future TFP growth. Section 4 extends the analysis to allow for signals. Section 5 provides an overview of alternative applications. Section 6 concludes.

## 2 The Model Framework

In this section, we introduce the modeling environment to which our methods are applicable. This framework turns out to be suitable for studying the effects of waves of pessimism, optimism, and uncertainty on macroeconomic dynamics. Various sources of uncertainty can be considered, such as dividends, relative volatility of shocks, or the type of the central bank. For instance, one can use this framework to study how *animal spirits* about the incoming phase of the business cycle evolve over time and, in turn, affect the business cycle itself. Alternatively, one can use this framework to study how long it takes agents to learn about structural changes in the dynamics of fundamentals or in the behavior of policy-makers.

The class of models we focus on has three salient features:

1. A system of equations for the variables of the model:

$$\Gamma_0(\xi_t) S_t = \Gamma_c(\xi_t) + \Gamma_1(\xi_t) S_{t-1} + \Psi(\xi_t) \varepsilon_t + \Pi \eta_t \quad (1)$$

where  $S_t$  is a vector containing all variables of the model known at time  $t$  (including conditional expectations formed at time  $t$ ),  $\eta_t$  is a vector containing the endogenous expectation errors, and the random vector  $\varepsilon_t$  contains the familiar Gaussian shocks. The hidden variable  $\xi_t$  controls the parameter values in place at time,  $\theta(\xi_t)$ , assumes discrete values  $\xi_t \in \{1, \dots, n\}$ , and evolves according to a Markov-switching process with transition matrix  $\mathcal{P}$ .

2. Agents have to forecast the dynamics of the endogenous variables  $S_{t+1}$  on the basis of Model (1) and their information set at time  $t$ ,  $\mathcal{I}_t$ . This includes the history of model variables and shocks, but not the history of regimes,  $\xi^t$ :  $\mathcal{I}_t \equiv \{S^t, \varepsilon^t\}$ .
3. Some regimes are assumed to bring about the same model parameters,  $\theta(\xi_t)$ . Let us group the regimes into  $m$  blocks  $b_j = \{\xi_t \in \{1, \dots, n\} : \theta(\xi_t) = \theta_{b_j}\}$ , for  $j \in \{1, \dots, m\}$ .

Given that agents know the structure of the model (*sub 1*) and can observe the endogenous variables and the shocks (*sub 2*), they can also determine which set of parameters is in place at each point in time. However, while this is enough to establish the history of blocks, agents cannot exactly infer the realized regime  $\xi_t$ , because the regimes within each block share the same parameter values (*sub 3*). It is very important to emphasize that regimes that belong to the same block are not identical in all respects, as they can differ in their stochastic properties such as average persistence and the probability of switching to other regimes. These properties are known to agents that will use them to learn about the regime in place today and to form expectations about the future. Therefore, points 1-3 describe a model in which agents learn about the latent variable  $\xi_t$ . As will be shown below, such a learning process affects the equilibrium law of motion of the economy. However, agents cannot extract any additional information about the underlying regime from observing the resulting law of motion because this reflects their own beliefs.

Henceforth, we will consider a benchmark case in which there are two blocks ( $m = 2$ ) and two regimes within each block. This choice is made in order to keep notation simple. The extension to the case in which  $m > 2$  is straightforward. The probabilities of moving across

regimes are summarized by the transition matrix:

$$\mathcal{P} = \left[ \begin{array}{cc|cc} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ \hline p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{array} \right] \quad (2)$$

in which the probability of switching to regime  $j$  given that we are in regime  $i$  is denoted by  $p_{ij}$ . Without loss of generality, we assume that regimes  $\xi_t = 1$  and  $\xi_t = 2$  belong to Block 1, while regimes  $\xi_t = 3$  and  $\xi_t = 4$  belong to Block 2. We consider only non-trivial blocks that satisfy  $p_{11} + p_{12} + p_{21} + p_{22} \neq 0$  and  $p_{33} + p_{34} + p_{43} + p_{44} \neq 0$ . The excluded cases are trivial as both blocks last only one period. Furthermore, we require that the two regimes that belong to the same block differ either in their persistence or in the probability of moving from one another; that is, we require that either  $p_{11} \neq p_{22}$  or  $p_{12} \neq p_{21}$  and either  $p_{33} \neq p_{44}$  or  $p_{34} \neq p_{43}$ . This condition makes the within-block Bayesian learning non-trivial. Finally, we will impose that  $p_{11} + p_{22} > 0$  and  $p_{33} + p_{44} > 0$ . This last assumption guarantees that within a block at least one of the two regimes can last more than one period. Summarizing, for each block, we will maintain the following *benchmark assumptions* throughout the paper:

**A1** *Non-triviality assumption:*  $p_{11} + p_{12} + p_{21} + p_{22} \neq 0$  and  $p_{33} + p_{34} + p_{43} + p_{44} \neq 0$

**A2** *Non-trivial-learning assumption:* Either  $p_{11} \neq p_{22}$  or  $p_{12} \neq p_{21}$  and either  $p_{33} \neq p_{44}$  or  $p_{34} \neq p_{43}$

**A3** *Non-jumping assumption:*  $p_{11} + p_{22} > 0$  and  $p_{33} + p_{44} > 0$

We will now proceed in two steps. First, in Subsection 2.1 we will characterize the evolution of agents' beliefs within a block for arbitrary prior beliefs. Second, in Subsection 2.2 we will combine these results with different assumptions regarding agents' prior beliefs following a switch across blocks. For each of these cases, we will describe how to recast the model with information frictions as a perfect information rational expectations model obtained by expanding the number of regimes to keep track of agents' beliefs.

## 2.1 Evolution of Beliefs *Within* a Block

In what follows, we will derive the law of motion of agents' beliefs conditional on being in a specific block. The formulas derived below will provide a recursive law of motion for agents' beliefs based on Bayes' theorem. Such recursion applies for any starting values for agents' beliefs. These will be determined by agents' beliefs at the moment the system enters the new block. We will characterize these initial beliefs in the next subsection.

As we have noticed in the previous section, agents can infer the history of the blocks. Therefore, at each point in time, agents know the number of consecutive periods spent in the current block since the last switch. Let us denote the number of consecutive realizations of Block  $i$  at time  $t$  as  $\tau_t^i$ ,  $i \in \{1, 2\}$ . To fix ideas, suppose that the system is in Block 1 at time  $t$ , implying that  $\tau_t^1 > 0$  and  $\tau_t^2 = 0$ . Then, there are only two possible outcomes for the next period. The economy can spend an additional period in Block 1, implying that  $\tau_{t+1}^1 = \tau_t^1 + 1$  and  $\tau_{t+1}^2 = 0$ , or it can move to Block 2, implying  $\tau_{t+1}^1 = 0$  and  $\tau_{t+1}^2 = 1$ . In this subsection, we restrict our attention to the first case.

Using Bayes' theorem and the fact that  $\text{prob}(\xi_{t-1} = 2 | \tau_{t-1}^1) = 1 - \text{prob}(\xi_{t-1} = 1 | \tau_{t-1}^1)$ , the probability of being in Regime 1 given that we have observed  $\tau_t^1$  consecutive realizations of Block 1,  $\text{prob}(\xi_t = 1 | \tau_t^1)$ , is given by:<sup>1</sup>

$$\text{prob}(\xi_t = 1 | \tau_t^1) = \frac{\text{prob}(\xi_{t-1} = 1 | \tau_{t-1}^1) (p_{11} - p_{21}) + p_{21}}{\text{prob}(\xi_{t-1} = 1 | \tau_{t-1}^1) (p_{11} + p_{12} - p_{21} - p_{22}) + p_{21} + p_{22}} \quad (3)$$

where  $\tau_t^1 = \tau_{t-1}^1 + 1$  and for  $\tau_t^1 > 1$ . Notice that for  $\tau_t^1 = 1$ ,  $\text{prob}(\xi_t = 1 | \tau_t^1)$  denotes the initial beliefs that will be discussed in Subsection 2.2. Equation (3) is a rational first-order difference equation that allows us to *recursively* characterize the evolution of agents' beliefs about being in Regime 1 while the system is in Block 1. The probability of being in Regime 3 given that we have observed  $\tau_t^2$  consecutive realizations of Block 2,  $\text{prob}(\xi_t = 3 | \tau_t^2)$ , can be analogously derived:

$$\text{prob}(\xi_t = 3 | \tau_t^2) = \frac{\text{prob}(\xi_{t-1} = 3 | \tau_{t-1}^2) (p_{33} - p_{43}) + p_{43}}{\text{prob}(\xi_{t-1} = 3 | \tau_{t-1}^2) (p_{33} + p_{34} - p_{43} - p_{44}) + p_{43} + p_{44}} \quad (4)$$

where  $\tau_t^2 = \tau_{t-1}^2 + 1$  and for  $\tau_t^2 > 1$ .

The recursive equations (3) and (4) characterize the dynamics of agents' beliefs in both blocks for a given set of prior beliefs. In what follows, we will show under which conditions these beliefs converge. This result will be key to being able to apply the following proposition and recast Model (1)-(2) in terms of a finite dimensional set of regimes indexed with respect to agents' beliefs.

**Proposition 1** *For any  $\varepsilon > 0$  there exists a  $\tau_1^* \in \mathbb{N}$  and  $\tau_2^* \in \mathbb{N}$  such that:*

$$\begin{aligned} \text{prob}(\xi_t = 1 | \tau_1^*) - \text{prob}(\xi_t = 1 | \tau_1^* + 1) &< \varepsilon \\ \text{prob}(\xi_t = 2 | \tau_2^*) - \text{prob}(\xi_t = 2 | \tau_2^* + 1) &< \varepsilon \end{aligned}$$

**Proof.** Easy to see once it has been proved that under the required conditions the rational

---

<sup>1</sup>A detailed derivation of equation (3) is provided in Appendix A.



difference equations (3) and (4) converge. The propositions below prove this point in steps. ■

We will characterize the convergence of  $\text{prob}(\xi_t = 1|\tau_t^1)$  as the number of consecutive periods spent in Block 1  $\tau_t^1$  grows large. We will denote  $\lim_{\tau_t^1 \rightarrow \infty} \text{prob}(\xi_t = 1|\tau_t^1) = x$  using  $\text{prob}(\xi_t = 1|\tau_t^1) \longrightarrow x$  and the characteristic roots of equation (3) with :

$$\tilde{\lambda}_1 \equiv \frac{p_{11} - p_{22} - 2p_{21} - \sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}}}{2(p_{11} + p_{12} - p_{21} - p_{22})} \quad (5)$$

$$\tilde{\lambda}_2 \equiv \frac{p_{11} - p_{22} - 2p_{21} + \sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}}}{2(p_{11} + p_{12} - p_{21} - p_{22})} \quad (6)$$

The following propositions provide the conditions under which the difference equation (3) converges to the stable root  $\tilde{\lambda}_2$ . An analogous pair of roots,  $\tilde{\lambda}_3$  and  $\tilde{\lambda}_4$ , with  $\tilde{\lambda}_4$  being the stable root, can be derived for Block 2. Similarly, all results that follow will also apply to Block 2.

**Proposition 2** *If (i)  $p_{11} + p_{12} - p_{21} - p_{22} \neq 0$ , (ii)  $p_{11}p_{22} \neq p_{21}p_{12}$ , (iii)  $p_{11} \neq p_{22}$  or both  $p_{12} \neq 0$  and  $p_{21} \neq 0$ , and the initial probability is such that  $\text{prob}(\xi_t = 1|\tau_t^1 = 1) \neq \tilde{\lambda}_1$ , then  $\text{prob}(\xi_t = 1|\tau_t^1) \longrightarrow \tilde{\lambda}_2 \in [0, 1]$ . If conditions (i), (ii), and (iii) hold and the initial probability is such that  $\text{prob}(\xi_t = 1|\tau_t^1 = 1) = \tilde{\lambda}_1$ , then  $\text{prob}(\xi_t = 1|\tau_t^1) = \tilde{\lambda}_1$  for any  $\tau_t^1$ .*

**Proof.** See Appendix B. ■

The next proposition relaxes condition (iii) of the above proposition.

**Proposition 3** *If (i)  $p_{11} + p_{12} - p_{21} - p_{22} \neq 0$ , (ii)  $p_{11}p_{22} \neq p_{21}p_{12}$ , (iii)  $p_{11} = p_{22}$  and either  $p_{12} = 0$  or  $p_{21} = 0$ , then  $\text{prob}(\xi_t = 1|\tau_t^1) \rightarrow \tilde{\lambda}_1 = \tilde{\lambda}_2$  and the roots are either equal to zero (if  $p_{21} = 0$ ) or one (if  $p_{12} = 0$ ).*

**Proof.** See Appendix C. ■

If the two regimes have the same persistence ( $p_{11} = p_{22}$ ) and the system has remained in Block 1 for sufficiently long, then agents will eventually believe they are in the regime that is an absorbing state (conditional on staying in the block). The next proposition relaxes condition (ii) of the previous propositions.

**Proposition 4** *If (i)  $p_{11} + p_{12} - p_{21} - p_{22} \neq 0$ , (ii)  $p_{11}p_{22} = p_{21}p_{12}$ , then  $\text{prob}(\xi_t = 1|\tau_t^1) = \frac{p_{11} - p_{21}}{p_{11} + p_{12} - p_{21} - p_{22}}$ .*

**Proof.** See Appendix D. ■

Note that if conditions (i) and (ii) are satisfied,  $\text{prob}(\xi_t = 1|\tau_t^1)$  suddenly converges by jumping to  $(p_{11} - p_{21}) / (p_{11} + p_{12} - p_{21} - p_{22})$  as the system enters Block 1. The recursion (3) can be shown to become a linear difference equation. The solution of this equation is characterized in the following two propositions.

**Proposition 5** *If (i)  $p_{11} + p_{12} - p_{21} - p_{22} = 0$  and (ii)  $p_{11} \neq p_{21}$ , then  $\text{prob}(\xi_t = 1 | \tau_t^1) \rightarrow \frac{p_{21}}{p_{22} - p_{11} + 2p_{21}}$ , with  $\frac{p_{21}}{p_{22} - p_{11} + 2p_{21}} \in [0, 1]$ .*

**Proof.** See Appendix E. ■

**Proposition 6** *If (i)  $p_{11} + p_{12} - p_{21} - p_{22} = 0$ , (ii)  $p_{11} = p_{21}$ , then  $\text{prob}(\xi_t = 1 | \tau_t^1) = \frac{p_{21}}{p_{22} + p_{21}}$ .*

**Proof.** See Appendix F. ■

When  $p_{11} = p_{21}$ , beliefs  $\text{prob}(\xi_t = 1 | \tau_t^1)$  suddenly jump to  $\frac{p_{21}}{p_{22} + p_{21}}$  for any  $\tau_t^1 \geq 1$  (as the system enters Block 1).

To sum up, given the benchmark assumptions  $A1-A3$ , we have shown that equation (3) always converges. Note that Proposition 2 implies that beliefs do not converge to  $\tilde{\lambda}_2$ , if the starting beliefs  $\text{prob}(\xi_t = 1 | \tau_t^1 = 1) = \tilde{\lambda}_1$ . It can be proved that  $\tilde{\lambda}_1 \leq 0$  or  $\tilde{\lambda}_1 \geq 1$ , implying that the only admissible values for probabilities are either zero or one.<sup>2</sup> Therefore, there are only a few limiting cases in which equation (3) does not converge to  $\tilde{\lambda}_2$ . As will become clear later on, it is sufficient to set the probability ratios  $0 < p_{i3} / (p_{i3} + p_{i4}) < 1$  for any  $i \in \{1, 2\}$  to rule out these cases that are not very relevant in practice.

## 2.2 Evolution of Beliefs Across Blocks

In the previous subsection, we characterized the evolution of agents' beliefs conditional on being in a specific block. The formulas derived above apply to any set of initial beliefs. In this subsection, we will pin down agents' beliefs at the moment the economy moves *across* blocks. These beliefs will serve as starting points for the recursions (3) and (4) governing the evolution of beliefs within a block.

Suppose for a moment that before switching to the new block, agents could observe the regime that was in place in the old block. Notice that in this case the transition matrix conveys all the information necessary to pin down agents' prior beliefs about the regime in place within the new block. Specifically, we have that if the economy moves from Block 2 to Block 1, the probability of being in Regime 1 is given by

$$\text{prob}(\xi_t = 1 | \xi_{t-1} = 3, \tau_t^1 = 1) = \frac{p_{31}}{p_{31} + p_{32}},$$

if the economy was under Regime 3 in the previous period, or by

$$\text{prob}(\xi_t = 1 | \xi_{t-1} = 4, \tau_t^1 = 1) = \frac{p_{41}}{p_{41} + p_{42}}$$

---

<sup>2</sup>A proof is provided in Appendix G.

if the economy was under Regime 4 in the previous period. Symmetrically, the probability of being in Regime 3 given that the economy just moved to Block 2 is given by

$$\text{prob}(\xi_t = 3 | \xi_{t-1} = 1, \tau_t^2 = 1) = \frac{p_{13}}{p_{13} + p_{14}},$$

if the economy was under Regime 1 in the previous period, or by

$$\text{prob}(\xi_t = 3 | \xi_{t-1} = 2, \tau_t^2 = 1) = \frac{p_{23}}{p_{23} + p_{24}}$$

if the economy was previously under Regime 2.

However, in the model, agents never observe the regime that is in place. Therefore, their beliefs at the moment the economy moves from one block to the other will be a weighted average of the probabilities outlined above. The weights, in turn, will depend on agents' beliefs at the moment of the switch. In what follows we will focus on three cases:

1. *Static prior beliefs.* In this case, the transition matrix  $\mathcal{P}$  is such that every time the economy enters a new block, agents' beliefs about which regime has been realized do not depend on their beliefs right before the switch. Thus, what has been observed in the past block does not help agents to form expectations in the new block. This restriction has the virtue of delivering a nice closed-form analytical characterization for the dynamics of beliefs.
2. *Dynamic prior beliefs.* In this case, the transition matrix  $\mathcal{P}$  is such that beliefs about which regime is prevailing within a block affect prior beliefs the moment the economy moves to the new block.
3. *Signals.* Exogenous signals  $\varpi_t$  about the current regime are also observed by agents. Signals are assumed to be distributed according to  $p(\varpi_t | \xi_t)$ .

It is worth clarifying that nothing prevents the researcher from combining the three cases described above. For example, static prior beliefs could characterize one block but not another or agents could receive a signal every time the economy enters a new block.

### 2.2.1 The Case of Static Prior Beliefs

In the case of static prior beliefs, every time the system enters a new block, agents' beliefs are the same regardless of the history of past beliefs. It is immediate to show that necessary and

sufficient conditions for this to happen are:

$$prob(\xi_t = 1 | \tau_t^1 = 1) = \frac{p_{31}}{p_{31} + p_{32}} = \frac{p_{41}}{p_{41} + p_{42}} = prob(\xi_t = 1 | \tau_t^1 = 1) \quad (7)$$

$$prob(\xi_t = 3 | \tau_t^2 = 1) = \frac{p_{13}}{p_{13} + p_{14}} = \frac{p_{23}}{p_{23} + p_{24}} = prob(\xi_t = 3 | \tau_t^2 = 1) \quad (8)$$

In other words, when the economy leaves a block, the relative probability of the two regimes in the new block is not affected by the regime that was in place before. Agents are fully rational and know the transition matrix governing the evolution of regimes. Therefore, their beliefs are uniquely pinned down by (7) and (8).

The recursive equations (3) and (4) combined with the initial conditions (7) and (8) *uniquely* characterize the dynamics of agents' beliefs in each block. To see this, notice that for each block, there is a *unique* path for the evolution of agents' beliefs, given that (7) and (8) make agents' beliefs before entering the block irrelevant. Furthermore, Proposition 1 guarantees that there exists a  $\tau_1^* \in \mathbb{N}$  and  $\tau_2^* \in \mathbb{N}$  such that agents' beliefs converge for an arbitrary level of accuracy. Therefore, in the case of static priors the number of consecutive periods spent in a block ( $\tau_t^i$ ) is a sufficient statistic to pin down the dynamics of beliefs in both blocks. Equipped with this important result, we can re-cast Model (1)-(2) in terms of a new set of regimes indexed with respect to the number of consecutive periods spent in a block  $\tau_t^i$ ,  $i \in \{1, 2\}$ :

$$\Gamma_0(\tau_t) S_t = \Gamma_c(\tau_t) + \Gamma_1(\tau_t) S_{t-1} + \Psi(\tau_t) \varepsilon_t + \Pi \eta_t \quad (9)$$

where  $\varepsilon_t \sim N(0, \Sigma_\varepsilon)$  is a vector of exogenous Gaussian shocks,  $\eta_t$  is a vector of endogenous expectation errors, and the  $\tau_1^* + \tau_2^*$  regimes  $\tau_t \equiv (\tau_t^1, \tau_t^2)$  evolve according to the transition matrix

$$\tilde{\mathcal{P}} = \begin{bmatrix} \tilde{\mathcal{P}}_{11} & \tilde{\mathcal{P}}_{12} \\ \tilde{\mathcal{P}}_{21} & \tilde{\mathcal{P}}_{22} \end{bmatrix},$$

where the matrices  $\tilde{\mathcal{P}}_{11}$  and  $\tilde{\mathcal{P}}_{12}$  are given by

$$\tilde{\mathcal{P}}_{11} \equiv \begin{bmatrix} 0 & prob\{\tau_{t+1}^1 = 2 | \tau_t^1 = 1\} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \dots & 0 & prob\{\tau_t^1 = \tau^* | \tau_t^1 = \tau^* - 1\} \\ 0 & 0 & \dots & 0 & prob\{\tau_{t+1}^1 > \tau^* | \tau_t^1 = \tau^*\} \end{bmatrix}$$

$$\tilde{\mathcal{P}}_{12} \equiv \begin{bmatrix} 1 - prob\{\tau_{t+1}^1 = 2 | \tau_t^1 = 1\} & \mathbf{0}_{1 \times (\tau^* - 1)} \\ \vdots & \vdots \\ 1 - prob\{\tau_{t+1}^1 > \tau^* | \tau_t^1 = \tau^*\} & \mathbf{0}_{1 \times (\tau^* - 1)} \end{bmatrix}$$

with the elements of the matrices given by

$$prob \{ \tau_{t+1}^i = \tau_t^i + 1 | \tau_t^i \} = prob \{ \xi_t = 1 | \tau_t^1 \} (p_{11} + p_{12}) + (1 - prob \{ \xi_t = 1 | \tau_t^1 \}) (p_{21} + p_{22}) \quad (10)$$

where  $prob \{ \xi_t = 1 | \tau_t^1 \}$  can be obtained from the recursive equation (3) and equation (7). The matrices  $\tilde{\mathcal{P}}_{21}$  and  $\tilde{\mathcal{P}}_{22}$  can be analogously derived.

Notice that the newly defined set of regimes keeps track of both the parameters in place at each point in time and the evolution of agents' beliefs. Since Model (9) is a Markov-switching DSGE model with perfect information, it can be solved using the techniques developed by Schorfheide (2005), Liu, Waggoner, and Zha (2011), Davig and Leeper (2007), Farmer, Waggoner, and Zha (2009), Cho (2012), and Foerster, Rubio-Ramirez, Waggoner, and Zha (2011). The result is an MS-VAR in the DSGE state vector  $S_t$ :

$$S_t = c \left( \tau_t, \tilde{\mathcal{P}} \right) + T \left( \tau_t, \tilde{\mathcal{P}} \right) S_{t-1} + R \left( \tau_t, \tilde{\mathcal{P}} \right) \varepsilon_t \quad (11)$$

where the law of motion of the economy depends on agents' beliefs as captured by  $\tau_t$ . With the results of Proposition 1 at hand, the solution of Model (9) with a truncated number of regimes  $\tau_t$  approximates the solution of the original model with learning (1). Notice that the accuracy of this approximation can be made arbitrarily precise simply by increasing the number of regimes  $\tau^*$ . Furthermore, it is worth pointing out that in the case of static priors the approximation error stems only from truncating agents' learning process. For all regimes such that  $\tau_t^i < \tau_i^*$  agents' beliefs *exactly* coincide with the analytical values derived using (3) and (4) and conditions (7) and (8).

### 2.2.2 The Case of Dynamic Prior Beliefs

When conditions (7) and (8) do not hold, past beliefs always influence current beliefs. In this case, the number of consecutive periods  $\tau_t$  spent in a block is no longer a sufficient statistic for agents' beliefs. However, as pointed out before, the recursive equations (3) and (4) hold for any prior beliefs. Therefore, these equations still capture the dynamics of beliefs while the system stays in a block. Furthermore, it follows that the sufficient conditions for convergence derived in Subsection 2.1 still apply. Nevertheless, the initial conditions are now different from (7) and (8) as they will depend on beliefs in the past block. Specifically, agents' starting beliefs upon the shift from Block 2 to Block 1 are given by

$$prob \{ \xi_t = 1 | \mathcal{I}_t \} = \frac{prob \{ \xi_{t-1} = 3 | \mathcal{I}_{t-1} \} p_{31} + (1 - prob \{ \xi_{t-1} = 3 | \mathcal{I}_{t-1} \}) p_{41}}{prob \{ \xi_{t-1} = 3 | \mathcal{I}_{t-1} \} (p_{31} + p_{32}) + (1 - prob \{ \xi_{t-1} = 3 | \mathcal{I}_{t-1} \}) (p_{41} + p_{42})} \quad (12)$$

while if the system just entered Block 2, starting beliefs read

$$\text{prob} \{ \xi_t = 3 | \mathcal{I}_t \} = \frac{\text{prob} \{ \xi_{t-1} = 1 | \mathcal{I}_{t-1} \} p_{13} + (1 - \text{prob} \{ \xi_{t-1} = 1 | \mathcal{I}_{t-1} \}) p_{23}}{\text{prob} \{ \xi_{t-1} = 1 | \mathcal{I}_{t-1} \} (p_{13} + p_{14}) + (1 - \text{prob} \{ \xi_{t-1} = 1 | \mathcal{I}_{t-1} \}) (p_{23} + p_{24})} \quad (13)$$

Notice that, using their information set  $\mathcal{I}_t$ , agents can keep track of both the number of consecutive deviations and their starting beliefs. Therefore, in the case of dynamic prior beliefs two variables pin down the dynamics of beliefs over time: how many consecutive periods the system has spent in the current block *and* the initial beliefs agents had when the system entered the current block. We then tackle the problem of solving Model (1) when prior beliefs are dynamic by making a grid for agents' beliefs. Denote the grid for beliefs  $\text{prob} \{ \xi_t = 1 | \mathcal{I}_t \}$  as  $\mathcal{G}_{b_1} = \{ \mathcal{G}_1, \dots, \mathcal{G}_{g_1} \}$  and for beliefs  $\text{prob} \{ \xi_t = 3 | \mathcal{I}_t \}$  as  $\mathcal{G}_{b_2} = \{ \mathcal{G}_{g_1+1}, \dots, \mathcal{G}_{g_1+g_2} \}$  where  $0 \leq \mathcal{G}_i \leq 1$ , all  $1 \leq i \leq g = g_1 + g_2$ . Furthermore, we denote the whole grid as  $\mathcal{G} = \mathcal{G}_{b_1} \cup \mathcal{G}_{b_2}$ . Endowed with such a grid, we can recast the original model in terms of a new set of regimes  $\zeta_t \in \{1, \dots, g_1 + g_2\}$ , any  $t$ . The new regime  $\zeta_t$  captures the knot of the grid  $\mathcal{G}$  that best approximates agents' beliefs; that is, in our notation  $\text{prob} \{ \xi_t = 1 | \mathcal{I}_t \}$  when the system is in Block 1 and  $\text{prob} \{ \xi_t = 3 | \mathcal{I}_t \}$  when the system is in Block 2. The transition probability matrix for these new regimes can be pinned down using the recursions (3) and (4) and the initial conditions (12) and (13). The algorithm below illustrates how exactly to perform this task.

**Algorithm** Initialize the transition matrix  $\widehat{\mathcal{P}}$  for the new regimes  $\zeta_t$ , setting  $\widehat{\mathcal{P}} = \mathbf{0}_{g \times g}$ .

**Step 1** For each of the two blocks, do the following steps (without loss of generality we describe the steps for Block 1):

**Step 1.1** For any grid point  $\mathcal{G}_i \in \mathcal{G}_{b_1}$ ,  $1 \leq i \leq g_1$ , compute

$$\widehat{\mathcal{P}}(i, j) = \text{prob} \{ \xi_{t-1} = 1 | \mathcal{I}_{t-1} \} (p_{11} + p_{12}) + (1 - \text{prob} \{ \xi_{t-1} = 1 | \mathcal{I}_{t-1} \}) (p_{21} + p_{22})$$

where  $\text{prob} \{ \xi_{t-1} = 1 | \mathcal{I}_{t-1} \} = \mathcal{G}_i$  and  $j \leq g_1$  is set so as to  $\min | \text{prob} \{ \xi_t = 1 | \mathcal{I}_t \} - \mathcal{G}_j |$ , where  $\text{prob} \{ \xi_t = 1 | \mathcal{I}_t \}$  is computed using the recursive equation (3) along with the approximation  $\text{prob} \{ \xi_{t-1} = 1 | \mathcal{I}_{t-1} \} = \mathcal{G}_i$ . To ensure the convergence of beliefs, we correct  $j$  as follows: if  $j = i$  and  $\mathcal{G}_i \neq \widetilde{\lambda}_2$ , then set  $j = \min(j + 1, g_1)$  if  $\mathcal{G}_i < \widetilde{\lambda}_2$  or  $j = \max(1, j - 1)$  if  $\mathcal{G}_i > \widetilde{\lambda}_2$ .

**Step 1.2** For any grid point  $\mathcal{G}_i \in \mathcal{G}_{b_1}$ ,  $1 \leq i \leq g_1$ , compute  $\widehat{\mathcal{P}}(i, l) = 1 - \widehat{\mathcal{P}}(i, j)$  with  $l > g_1$  satisfying

$$\min \left| \frac{\text{prob} \{ \xi_{t-1} = 1 | \mathcal{I}_{t-1} \} p_{13} + (1 - \text{prob} \{ \xi_{t-1} = 1 | \mathcal{I}_{t-1} \}) p_{23}}{\text{prob} \{ \xi_{t-1} = 1 | \mathcal{I}_{t-1} \} (p_{13} + p_{14}) + (1 - \text{prob} \{ \xi_{t-1} = 1 | \mathcal{I}_{t-1} \}) (p_{23} + p_{24})} - \mathcal{G}_l \right|$$

where  $prob \{ \xi_{t-1} = 1 | \mathcal{I}_{t-1} \} = \mathcal{G}_i$ .

**Step 2** If no column of  $\widehat{\mathcal{P}}$  has all zero elements, stop. Otherwise, go to Step 3.

**Step 3** Construct the matrix  $T$  as follows. Set  $j = 1$  and  $l = 1$ . While  $j \leq g$ , if  $\sum_{i=1}^g \widehat{\mathcal{P}}(i, j) \neq 0$  then do three things: (1) set  $T(j, l) = 1$ , (2) set  $T(j, v) = 0$  for any  $1 \leq v \leq g$  and  $v \neq l$ , (3) set  $l = l + 1$  and (4) set  $j = j + 1$ ; otherwise (i.e., if  $\sum_{i=1}^g \widehat{\mathcal{P}}(i, j) = 0$ ), set  $j = j + 1$ .

**Step 4** Write the transition equation as  $\widehat{\mathcal{P}}^R = T \cdot \widehat{\mathcal{P}} \cdot T'$ . If no column of  $\widehat{\mathcal{P}}^R$  has all zero elements, set  $\widehat{\mathcal{P}} = \widehat{\mathcal{P}}^R$  and stop. Otherwise, go to step 3.

Step 1.1 determines the regime  $j$  the system will go to if it stays in Block 1 next period and fills up the appropriate element  $(i, j)$  of the transition matrix  $\widehat{\mathcal{P}}$  with the probability of moving to Regime  $j$ . If  $j = i$  and the corresponding diagonal element is not the knot that beliefs converge to, the step makes a correction to prevent this from happening. Otherwise, beliefs could get stuck at a grid point that is not the convergence point, inaccurately representing the dynamic of beliefs.<sup>3</sup> Step 1.2 computes the regime  $l$  the system will go to if it leaves Block 1 and fills up the appropriate element  $(i, l)$  of matrix  $\widehat{\mathcal{P}}$ . Steps 2-4 are not necessary but help to keep the dimension of the grid small, getting rid of regimes that will never be reached. For computational convenience, we always add the convergence points for the two blocks (i.e.,  $\widetilde{\lambda}_2$  in the case of Block 1) to the grid  $\mathcal{G}$ . On many occasions, it is a good idea to make the grid near the convergence knot very fine to improve the precision of the approximation.

Once the transition matrix  $\widehat{\mathcal{P}}$  for the new set of regimes is characterized, the original Model (1) can be recast in terms of the new set of regimes  $\zeta_t$ :

$$\Gamma_0(\zeta_t) S_t = \Gamma_c(\zeta_t) + \Gamma_1(\zeta_t) S_{t-1} + \Psi(\zeta_t) \varepsilon_t + \Pi \eta_t \quad (14)$$

where  $\zeta_t \in \{1, \dots, g_1 + g_2\}$ . Therefore, up to an approximation error that can be made arbitrarily small, the task of solving the model with learning in (1) boils down to solving the perfect-information model (14) using solution algorithms for MS-DSGE models. The resulting law of motion is once again an MS-VAR:

$$S_t = c(\zeta_t, \widetilde{\mathcal{P}}) + T(\zeta_t, \widetilde{\mathcal{P}}) S_{t-1} + R(\zeta_t, \widetilde{\mathcal{P}}) \varepsilon_t \quad (15)$$

---

<sup>3</sup>Similarly, in the case of oscillatory convergence, one has to prevent the algorithm from getting stuck on some grid point that does not represent the convergence point. To avoid complicating the discussion of the paper with too many technical details, we omit this correction.

### 2.2.3 Signals

Let us assume that agents observe signals about the realized regime. To fix notation, denote the signal as  $\varpi_t$  and, for simplicity, assume that it can have only two values, 1 or 2. We denote the probability that the signal is equal to  $q \in \{1, 2\}$ , conditional on the regime being equal to  $h \in \{1, 2, 3, 4\}$  as  $\text{prob}\{\varpi_t = q | \xi_t = h\}$ . The model with signals can be solved by introducing a new system of regimes  $\zeta_t$ , which indexes the grid points corresponding to the probabilities  $\text{prob}\{\xi_t = 1 | \mathcal{I}_t, \varpi^t\}$  and  $\text{prob}\{\xi_t = 3 | \mathcal{I}_t, \varpi^t\}$ , and following the same logic used in the previous subsection.

To fill up the transition matrix  $\widehat{\mathcal{P}}$  for the new set of regimes, one can implement the algorithm detailed in Subsection 2.2.2 with only the little tweak of updating beliefs using the information contained in the observed signal. For instance, we compute the ex-post-probability  $\text{prob}(\xi_t = 1 | \mathcal{I}_t, \varpi^t)$

$$\text{prob}(\xi_t = 1 | \mathcal{I}_t, \varpi^{t-1}, \varpi_t = q) = \frac{\text{prob}(\varpi_t = q | \xi_t = 1) \text{prob}(\xi_t = 1 | \mathcal{I}_t, \varpi^{t-1})}{\sum_{i=1}^2 \text{prob}(\varpi_t = q | \xi_t = i) \text{prob}(\xi_t = i | \mathcal{I}_t, \varpi^{t-1})}, \quad q \in \{1, 2\} \quad (16)$$

where  $\text{prob}(\xi_t = 1 | \mathcal{I}_t, \varpi^{t-1})$  is computed using the recursive equation (3) for a given initial point in the grid  $\mathcal{G}$  that approximates  $\text{prob}(\xi_{t-1} = 1 | \mathcal{I}_{t-1}, \varpi^{t-1})$ . We use the probability computed in equation (16) to determine the appropriate points of the grid  $\mathcal{G}$ , which we denote as  $j_q$ ,  $q \in \{1, 2\}$ . Note that for any given initial belief  $\text{prob}(\xi_{t-1} = 1 | \mathcal{I}_{t-1}, \varpi^{t-1}) \in \mathcal{G}$ , the (ex-post) belief  $\text{prob}(\xi_t = 1 | \mathcal{I}_t, \varpi^{t-1}, \varpi_t = q)$  now pins down the grid points, depending on the realization of the signal  $\varpi_t$ . Once these two points in the grid are determined, we can fill up the transition probability as follows:

$$\widehat{\mathcal{P}}(i, j_q) = \sum_{v=1}^2 \text{prob}\{\xi_t = v | \mathcal{I}_{t-1}, \varpi^{t-1}\} \text{prob}\{\varpi_t = q | \xi_t = v\}, \quad q \in \{1, 2\} \quad (17)$$

where

$$\text{prob}\{\xi_t = v | \mathcal{I}_{t-1}, \varpi^{t-1}\} = \sum_{u=1}^2 \text{prob}\{\xi_{t-1} = u | \mathcal{I}_{t-1}, \varpi^{t-1}\} p_{uv} \quad (18)$$

and we approximate  $\text{prob}\{\xi_{t-1} = 1 | \mathcal{I}_{t-1}, \varpi^{t-1}\} \in \mathcal{G}$ . Note that in the case of signals, each row of the transition matrix  $\widehat{\mathcal{P}}$  has up to four non-zero elements. This completes the derivation of the submatrix  $\widehat{\mathcal{P}}_{11}$ , which governs the evolution of beliefs within Block 1. How to obtain the other submatrices  $\widehat{\mathcal{P}}_{12}$ ,  $\widehat{\mathcal{P}}_{21}$ , and  $\widehat{\mathcal{P}}_{22}$  is detailed in Appendix H.

## 2.3 Agents Know That They Do Not Know

Summarizing, the methods outlined above show that one can recast the Markov-switching DSGE model with learning as a Markov-switching rational expectations system, in which the



regimes are indexed with respect to agents' beliefs. In the case of static priors, the number of consecutive realizations of a block represents a sufficient statistic to index agents' beliefs. In the case of dynamic priors, agents' beliefs are mapped into a grid. In both cases, a new transition matrix that characterizes the joint evolution of agents' beliefs and model parameters is derived.

It is worth emphasizing that this way of recasting the learning process allows us to easily model economies in which *agents know that they do not know*. In other words, agents form expectations taking into account that their beliefs will change in the future according to what they will observe in the economy. This is why the laws of motion (11) and (15) characterizing the behavior of the model depend on the current beliefs and the expanded transition matrix defining the joint evolution of agents' beliefs and model parameters. This represents a substantial difference with the *anticipated utility* approach in which agents form expectations without taking into account that their beliefs about the economy will change over time (e.g., Evans and Honkapohja, 2001; Cogley, Matthes, and Sbordone, 2011). Furthermore, the approach described above differs from the one traditionally used in the learning literature in which agents form expectations according to a reduced-form law of motion that is updated recursively using the discounted least-squares estimator (Eusepi and Preston, 2011). The advantage of adaptive learning is the extreme flexibility given that, at least in principle, no restrictions need to be imposed on the type of parameter instability characterizing the model. However, such flexibility does not come without a cost, given that agents are not really aware of the model they live in, but only of the implied law of motion. Instead, in this paper agents fully understand the model, they are uncertain about the future, and they are aware of the fact that their beliefs will evolve over time.

It is also important to emphasize the extreme tractability of the approach taken in this paper. The solutions (11) and (15) can be easily combined with an observation equation and used in an estimation algorithm. For example, Bianchi and Melosi (2012a) estimates a prototypical New-Keynesian DSGE model, in which agents form beliefs about the likely duration of deviations from active inflation stabilization policies. The estimation of this new class of models is possible for three main reasons. First, even if the final number of regimes can be extremely high, the model imposes very specific restrictions on the allowed regime paths and on the link between observable outcomes and agents' beliefs. This implies that when evaluating the likelihood, a relatively small number of regime paths has to be taken into account. Second, the statistical properties of the different regimes can vary substantially and depend on the probability of moving across regimes. Therefore, identification of the transition matrix is not only given by the frequency with which the different regimes occur, but also by the laws of motion characterizing the different regimes. Finally, the number of extra parameters with respect to a model with perfect information is very low, if not zero, while the resulting dynamics can be substantially enriched. For example, Bianchi and Melosi (forthcoming) show that a period of fiscal distress

can lead to a run-up in inflation that lasts for decades.

From a computational point of view, there might be a concern about the time required to solve the model when the final number of regimes becomes very large. This turns out not to be a problem. If regime changes enter in an additive way, affecting only the matrix  $\Gamma_c$ , the model can be solved with standard solution algorithms such as *gensys* (Sims, 2002) and Blanchard and Kahn (1980) and the high dimensionality of the transition matrix is not a problem. However, in many situations we might want to model regime changes that enter in a multiplicative way. For example, we might want to allow for changes in the Taylor rule parameters. In this case, the matrices  $\Gamma_0$  and  $\Gamma_1$  are also affected and we need to rely on solution methods developed to solve MS-DSGE models. However, according to our experience based on the use of the approach proposed by Farmer, Waggoner, and Zha (2009), even in this case a solution can be obtained in a matter of seconds because the transition matrix governing the evolution of the regimes is very sparse. Therefore, the methods described in this paper provide a promising tools for modeling information frictions, *animal spirits*, and agents' beliefs in a general equilibrium framework suitable for structural estimation.

### 3 Applications

In this section, we introduce a prototypical RBC model to illustrate the properties of the methods detailed above. Central to our discussion will be the evolution of optimism and pessimism and the implications thereof for consumption and saving decisions. The representative household maximizes the sequence of consumption  $c_t$  and capital  $k_t$ :

$$\max_{c_t, k_t} \tilde{E}_0 \sum_{t=0}^{\infty} \beta^t \ln c_t$$

subject to the resource constraint  $c_t + k_t = z_t k_{t-1}^\alpha + (1 - \delta) k_{t-1}$  with  $\alpha < 1$  and  $0 < \delta < 1$ . Let  $\tilde{E}_t(\cdot)$  denote the expectation operator conditional on households' information set at time  $t$ . We assume that total factor productivity (TFP)  $z_t$  follows an exogenous process, such that

$$\ln z_t = \mu(\xi_t) + \ln z_{t-1} + \sigma_z \varepsilon_t \tag{19}$$

where  $\varepsilon_t \stackrel{iid}{\sim} N(0, 1)$  and  $\xi_t$  denotes a discrete Markov process affecting the drift of TFP. This process evolves according to the transition probability matrix  $\mathcal{P}$ . We assume that  $\xi_t$  can take four values; that is,  $\xi_t \in \{1, 2, 3, 4\}$ . These values map into values for the TFP drift  $\mu(\xi_t)$  as follows  $\xi_t \in \{1, 2\} \implies \mu_t(\xi_t) = \mu_H$  and  $\xi_t \in \{3, 4\} \implies \mu_t(\xi_t) = \mu_L$ , where  $\mu_L < \mu_H$ . In Block 1, Regimes 1 and 2 differ in their likely persistence:  $p_{11} < p_{22}$ . The same applies to Regimes 3 and 4 in Block 2:  $p_{33} < p_{44}$ . We call Regimes 1 and 2 *high-growth regimes* and Regimes 3 and

4 *low-growth regimes*. Households are assumed to observe the history of the model variables (i.e.,  $k_t$ ,  $c_t$ , and  $z_t$ ) and that of the TFP shocks (i.e.,  $\varepsilon_t$ ). Therefore, households can establish whether the economy is in the high-growth block or in the low-growth block.

We introduce the stationary variables  $\mu_t \equiv \ln(z_t/z_{t-1})$ ,  $\tilde{c}_t \equiv c_t/z_t^{1/(1-\alpha)}$ ,  $\tilde{k}_t \equiv k_t/z_t^{1/(1-\alpha)}$  and, following Schorfheide (2005) and Liu, Waggoner, and Zha (2011), we define the steady state as the stationary equilibrium in which all shocks are shut down, including the regime shocks to the growth rate of TFP. We then derive a log-linear approximation to the equilibrium equations around the steady-state equilibrium for these stationary variables. The log-linearized Euler equation reads:<sup>4</sup>

$$\hat{c}_t = \tilde{E}_t \hat{c}_{t+1} - (\alpha - 1) \left( 1 + (\delta - 1) \beta \overline{M}^{\frac{1}{\alpha-1}} \right) \hat{k}_t - \left( \frac{1}{\alpha - 1} + \beta \overline{M}^{\frac{1}{\alpha-1}} (\delta - 1) + 1 \right) \tilde{E}_t \hat{\mu}_{t+1} \quad (20)$$

where  $\overline{M} \equiv \exp(\bar{\mu})$ ,  $\bar{\mu} \equiv (\bar{p}_1 + \bar{p}_2) \mu_H + (\bar{p}_3 + \bar{p}_4) \mu_L$  being the ergodic mean of the log growth rate of the economy, and  $\bar{p}_i$  stands for the ergodic probability of being in regime  $i$ ,  $\hat{\mu}_t$ ,  $\hat{c}_t$  and  $\hat{k}_t$  denote log-deviations of the stationary TFP growth, consumption, and capital, respectively, from their steady-state value, and  $\hat{\mu}(\xi_t) \equiv \mu_t(\xi_t) - \bar{\mu}$  is the log-deviation of TFP drift from its ergodic mean  $\bar{\mu}$ . The resource constraint is

$$c_{ss} \hat{c}_t + k_{ss} \hat{k}_t = \left( \frac{\overline{M}^{\frac{\alpha}{\alpha-1}} k_{ss}^{\alpha}}{\alpha - 1} + \frac{1 - \delta}{\alpha - 1} \overline{M}^{\frac{1}{\alpha-1}} k_{ss} \right) \hat{\mu}_t + \left( \overline{M}^{\frac{\alpha}{\alpha-1}} k_{ss}^{\alpha} + (1 - \delta) \overline{M}^{\frac{1}{\alpha-1}} k_{ss} \right) \hat{k}_{t-1} \quad (21)$$

Finally, the log-deviation of the growth rate of TFP from its ergodic level follows

$$\hat{\mu}_t = \hat{\mu}(\xi_t) + \sigma_z \varepsilon_t \quad (22)$$

As is standard for any RBC model, households adjust capital so as to smooth consumption intertemporally. The occurrence of TFP shocks and the succession of low-growth and high-growth regimes challenge households' ability to smooth consumption over time. When the economy is in the high-growth regime, households expect that, with some probability, the economy will enter into the low-growth regime in the future, making it harder to raise future consumption. Therefore, *ceteris paribus* agents raise capital today so as to raise future expected consumption  $\tilde{E}_t \hat{c}_{t+1}$  *vis-a-vis* current consumption  $\hat{c}_t$ . When the economy is in the low-growth regime, agents expect that, with some probability, the economy will enter into the high-growth regime in the future, making it easier to raise future consumption. Therefore, *ceteris paribus* agents reduce capital today so as to raise current consumption  $\hat{c}_t$  *vis-a-vis* expected future consumption  $\tilde{E}_t \hat{c}_{t+1}$ .

---

<sup>4</sup>A detailed derivation of the steady-state equilibrium for the stationary variables and the log-linearized equations is provided in Appendix I.

Clearly, the persistence of the regime in place critically affects consumption and capital decisions. When the current regime is short lasting, households generally adjust capital more aggressively than when it is long lasting, because they deem that a switch in the next period is more likely. In contrast, households do not adjust capital so aggressively if they expect the regime to be very long lasting. When households expect that low growth or high growth has become a structural characteristic of the environment, they understand that consumption cannot be smoothed out over time by simply adjusting capital. Thus, very persistent regimes are mostly characterized by structural changes in the level of consumption.

Given that households have limited information, the log-linearized Model (20)-(22) cannot be solved using the existing techniques that are used to solve Markov-switching models with perfect information. However, we proceed as described in the previous sections, by introducing a new set of regimes that capture the evolution of the representative household's beliefs over time. It is important to notice that in the RBC model described above, regime changes enter additively. In other words, they only affect the vector of constants  $\Gamma_c(\cdot)$  in the canonical forms (9) or (14). In this case, the state space can be augmented with a series of dummy variables as in Schorfheide (2005), Liu, Waggoner, and Zha (2011), and Bianchi, Ilut, and Schneider (2012) and the models under imperfect information can be easily solved using standard solution methods for DSGE model such as *gensys* (Sims, 2002) and Blanchard and Kahn (1980). When regime changes enter multiplicatively, the matrices  $\Gamma_0$  and  $\Gamma_1$  are also affected. In this case, the model can be solved with any of the solution methods that have been developed for MS-DSGE models. Bianchi and Melosi (2012a, 2012b, Forthcoming) consider these cases and solve the model using the algorithm developed by Farmer, Waggoner, and Zha (2009).

In what follows, we adopt a standard calibration of the RBC model. We set capital's share parameter  $\alpha$  to equal 0.33. The discount factor  $\beta$  is equal to 0.9976 and the parameter for the physical depreciation of capital is set to equal 0.0250. The standard deviation of the TFP shock  $\sigma$  is set to 0.007. We set the growth rate of TFP in the high-growth state to equal the annualized rate of 4%:  $\mu_H = .01$ . We assume that under low-growth, the growth rate of TFP is simply zero:  $\mu_L = 0$ . The values for the transition matrix  $\mathcal{P}$  will vary across simulations and will be described case by case.

### 3.1 Static Prior with Symmetric Speed of Learning

Let us assume that the persistence of the short-lasting regimes is the same in the two blocks:  $p_{11} = p_{33} = 0.5$ . Analogously, we set the probabilities of staying in the long-lasting regimes so that  $p_{22} = p_{44} = 0.95$ . For simplicity we assume that regimes belonging to the same block do not communicate with each other; that is,  $p_{12} = p_{21} = p_{34} = p_{43} = 0$ . Furthermore, the transition matrix implies that once a switch to a new block occurs, agents always attach a 95%

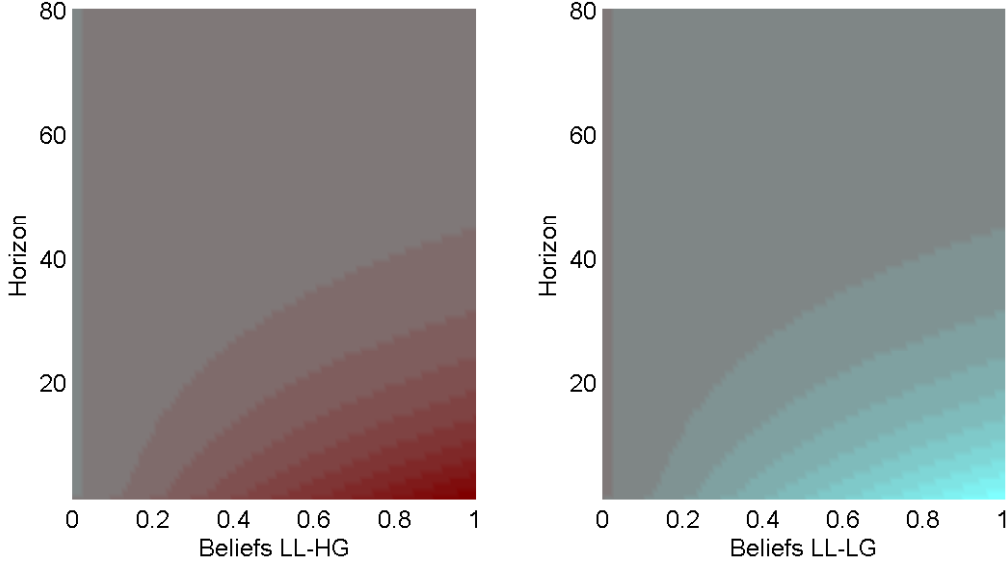


Figure 1: Static Prior Beliefs; Expected Growth Rate of Technology as a Function of Beliefs. The scale of the expected growth rates ranges from -1.6 to 1.6. Lighter blue areas capture expected rates that are lower than the ergodic rate. Darker red areas capture expected rates that are higher than the ergodic rate. Gray areas denote expected rates that are similar to the ergodic rate. *Left graph*: Annualized percentage deviations of the growth rate of technology from its ergodic value at different horizons as a function of the probability that the observed high-growth regime is long lasting (Beliefs LL-HG). *Right graph*: Annualized percentage deviations of the growth rate of technology from its ergodic value at different horizons as a function of the probability that the observed low-growth regime is long lasting (Beliefs LL-LG).

probability to being in the short-lasting regime:

$$\frac{p_{31}}{p_{31} + p_{32}} = \frac{p_{41}}{p_{41} + p_{42}} = 0.95 \quad (23)$$

$$\frac{p_{13}}{p_{13} + p_{14}} = \frac{p_{23}}{p_{23} + p_{24}} = 0.95 \quad (24)$$

Notice that conditions (23)-(24) imply static prior beliefs: agents always enter the high-growth (low-growth) block with the same level of optimism (pessimism). To sum up, we work with the following transition matrix:

$$\mathcal{P} = \left[ \begin{array}{cc|cc} 0.50 & 0 & 0.475 & 0.025 \\ 0 & 0.95 & 0.0475 & 0.0025 \\ \hline 0.475 & 0.025 & 0.50 & 0 \\ 0.0475 & 0.0025 & 0 & 0.95 \end{array} \right]$$

Figure 1 shows agents' expectations about the growth rate of TFP  $\mu_t$  in deviations from its ergodic level  $\bar{\mu}$  at different horizon as agents' beliefs about being in the high-growth long-

lasting regime (left plot) or in the long-lasting low-growth regime (right plot) vary from 0 to 1. These expectations take into account regime uncertainty and can be quickly computed using the methods described in Bianchi (2012). When the economy is in the high-growth block and agents are extremely optimistic about being in the long-lasting regime, they expect a growth rate of TFP that exceeds the ergodic rate by 1.6% for the next few quarters. left graph. See the dark red area in left graph. However, at very long horizons, agents simply expect the growth rate of TFP to return to the annualized ergodic mean of 2%, regardless of how confident they are to be in the long-lasting high-growth regime. When agents mostly expect to be in a short-lasting regime, the expected growth rate of TFP is fairly similar to the ergodic rate at every horizon. Similarly, when the economy is on a low-growth path, the more pessimistic about the duration of this phase agents are, the longer the horizon at which the expected growth rate of TFP stays below the ergodic rate. See the light blue area in the right graph.

To illustrate the consequences of fluctuations in agents' beliefs, we simulate the economy assuming a typical path for the regimes and setting all Gaussian shocks  $\varepsilon_t$  to zero. We assume that capital is initialized at its steady-state value. The results are reported in Figure 2. In each panel, the gray areas denote the periods of low growth. Short-lasting regimes last for their typical duration of 2 quarters. Long-lasting regimes last for their typical duration of 20 quarters. The two right graphs report the evolution of consumption and capital in the model with learning compared to the model with perfect information in which agents can observe the current regime. The panel in the upper-left corner shows the evolution of agents' beliefs about being in the long-lasting high-growth regime and in the long-lasting low-growth regime. The panel in the lower-left corner reports the evolution of expected average TFP growth at 4-, 8-, 20-, and 40- quarter horizons. Notice that this is a convenient measure of agents' optimism/pessimism that takes into account uncertainty about the regime in place today and the possibility of regime changes.

Three features of Figure 2 deserve to be emphasized. First, right after a switch to a new block, agents believe that this switch is most likely to be short lasting. This can be seen in the top left graph when switches to new blocks occur. The reason is that agents are rational and hence are aware that regardless of whether the past regime was short lasting or long lasting, the probability of switching to the short-lasting regime in the new block is always as high as 95%. This stems from the restrictions in (23)-(24), which imply static prior beliefs. Second, whenever a short-lasting regime is in fact realized, with the benefit of hindsight, agents' beliefs turn out to over-react to the regime change because agents *rationally* attach a non-zero probability to being in the long-lasting regime. Third, the probability of being in the long-lasting regime smoothly increases as more realizations of the same block are observed. The top left graph shows that the probability of being in the long-lasting regime rises monotonically with the number of consecutive realizations of a particular growth rate. For instance, from  $t = 117$  to

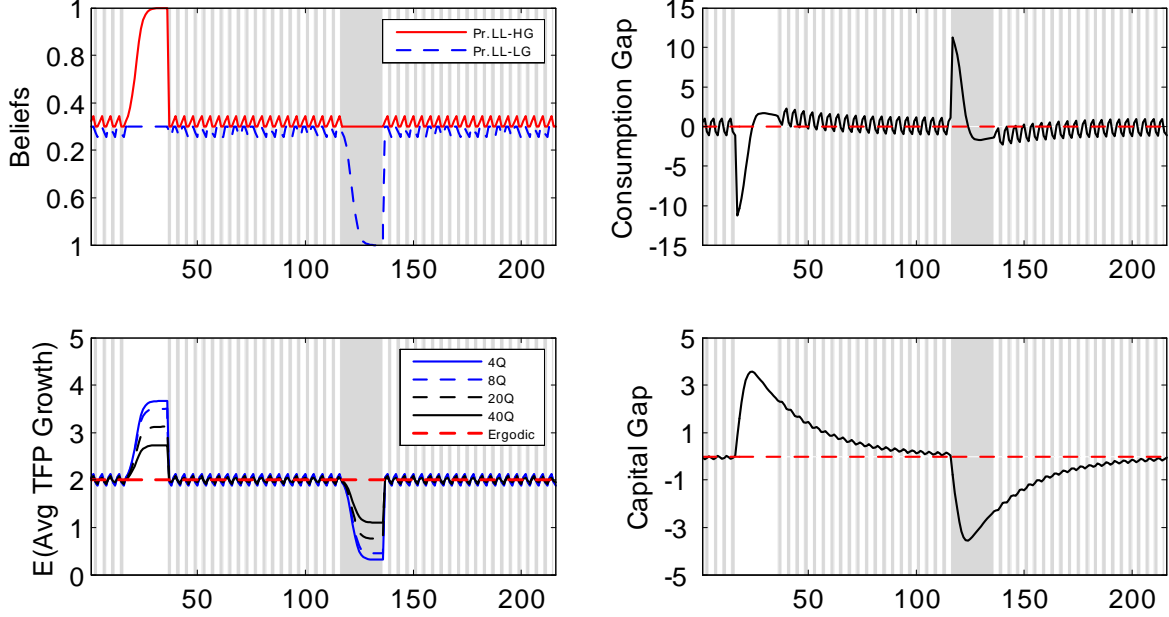


Figure 2: Static Prior Beliefs and Symmetric Speed of Learning. *Top left graph:* Evolution of beliefs of being in the long-lasting high-growth regime (red solid line) and in the long-lasting low-growth regime (blue dashed line). *Top right graph:* Annualized percentage deviations of consumption from the perfect-information benchmark. *Bottom left graph:* Expected average growth rate of technology (annualized percentage) at various horizons. *Bottom right graph:* Annualized percentage deviations of capital from the perfect-information benchmark. In all graphs, gray areas denote periods of low growth.

$t = 136$ , the economy is in a long-lasting low-growth regime. While agents initially attach a small probability to being in the long-lasting regime, they become fully convinced after 12 consecutive periods of low TFP growth.

Furthermore, Figure 2 shows the evolution of optimism and pessimism and the associated dynamics of the *consumption gap* and the *capital gap*, which are defined as the optimal level of consumption and capital in deviations from their corresponding levels under perfect information. When the economy enters the long-lasting low-growth period, imperfectly informed agents are not very pessimistic about the duration of the low-growth regime. This is reflected in their expectations about the average growth rate of TFP that barely moves in the bottom left graph. Given that they expect that the low-growth period will be short lasting, they decide to slow down capital accumulation so as to smooth consumption. In contrast, if agents knew the actual realization of the low-growth regime, they would adjust their stock of capital less aggressively and let consumption fall more. This is why in Figure 2 we observe a *positive consumption gap* and a *negative capital gap* when the economy enters a period of long-lasting low growth.

As the period of low-growth consolidates, imperfectly informed agents update their beliefs until they eventually become convinced that they are in the long-lasting regime. This happens in roughly 12 quarters after the switch. As illustrated in the bottom left graph, such slow-

moving beliefs cause the expected average growth rate of TFP over the next few years to also adjust sluggishly. This eventually determines an adjustment in the path for consumption and the consumption gap slowly fades away. Interestingly, at the end of the long-lasting low-growth period, the consumption gap becomes negative. The reason is that the sluggish evolution of pessimism prompted households to decumulate capital rapidly at the beginning of the period of low growth. The small capital stock depresses consumption as households become pessimistic, leading to a negative consumption gap. A specular pattern characterizes the economy the moment it enters a long-lasting high-growth period at the beginning of the simulation.

As pointed out before, even when the economy repeatedly alternates between short-lasting periods, agents' beliefs are misaligned with the truth. Let us focus on the first 16 quarters during which a sequence of short-lasting regimes are realized. While the economy is in the short-lasting high-growth regime, imperfectly informed households consume more and accumulate less capital than in the case of perfect information. The reason is that imperfectly informed agents attach some non-negligible - albeit small - probability to being in the long-lasting regime. By the same token, when the economy is going through a short-lasting period of low growth, imperfectly informed households consume less and accumulate more capital than under perfect information.

### 3.2 Static Prior with Differential Speed of Learning

Now we study a situation in which optimism and pessimism evolve at differential speed. The speed of learning within a block is affected by the relative persistence of the corresponding two regimes. Generally speaking, if the persistences of the two regimes become more similar, it takes longer for rational agents to figure out which regime is in place. To see this point, let us calibrate the model as in the previous section, using the restrictions in (23)-(24), but now we set the probability of staying in the short-lasting high-growth regime to be  $p_{11} = 0.75 > 0.5$ . The probability of staying in the long-lasting high-growth regime is unchanged ( $p_{22} = 0.95$ ). Figure 3 shows the dynamic of beliefs, the expected average growth rate of TFP at various horizons (4, 8, 20, and 40 quarters), the consumption gap, and the capital gap when the economy goes through the same sequence of regimes as that in Figure 2, with the only difference that now the typical duration of the short-lasting high-growth regime is longer: 4 quarters instead of 2. The typical duration of all other regimes is the same. The crucial point to notice is that in Figure 3 the typical realization of 20 quarters of high growth is not enough to induce households fully make up their mind that the realized regime is of the long-lasting type. Agents attach only an 80% probability of being in the long-lasting regime after having observed 20 consecutive periods of high growth. In contrast, when the economy is going through a period of long-lasting low growth, it takes roughly 12 quarters for households to be fully convinced that they are in the long-lasting regime, exactly as in Figure 2.



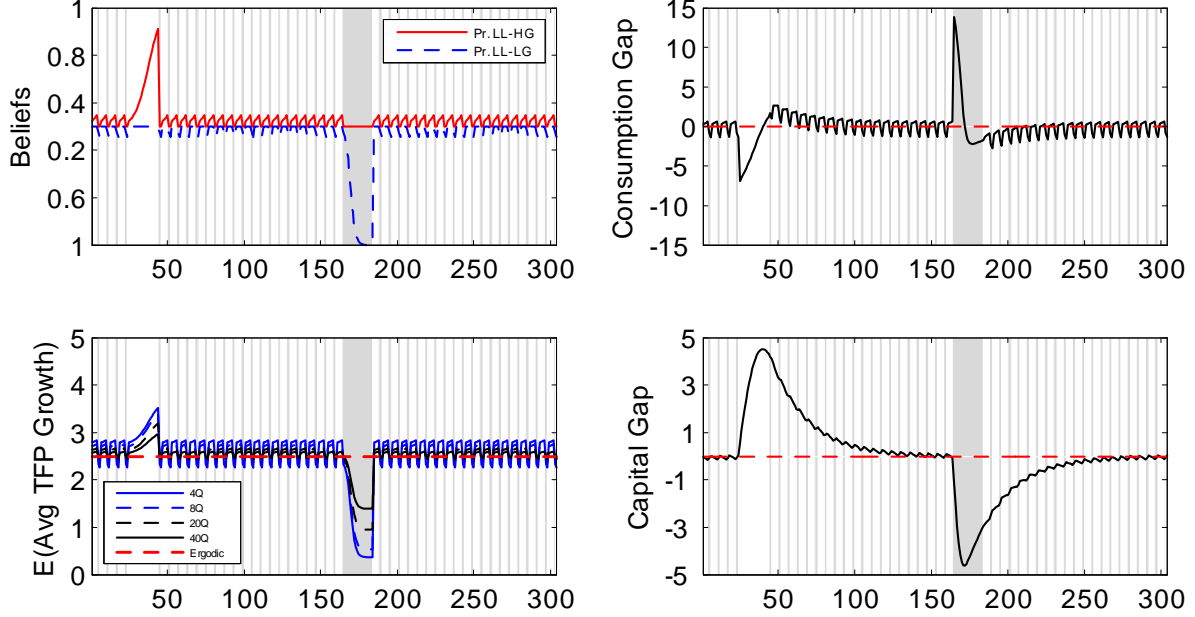


Figure 3: Static Prior Beliefs and Differential Speed of Learning. *Top left graph:* Evolution of beliefs of being in the long-lasting high-growth regime (red solid line) and in the long-lasting low-growth regime (blue dashed line). *Top right graph:* Annualized percentage deviations of consumption from the perfect-information benchmark. *Bottom left graph:* Expected average growth rate of technology (annualized percentage) at various horizons. *Bottom right graph:* Annualized percentage deviations of capital from the perfect-information benchmark. In all graphs, gray areas denote periods of low growth.

Differential speeds of learning have an impact on the dynamics of consumption and capital. During the long-lasting high-growth regime, the misalignment of agents' beliefs is more persistent than in the case of the low-growth regime. This implies a more persistent negative consumption gap because households raise capital to smooth consumption in the future. On the other hand, the consumption gap is less pronounced than that under a symmetric speed of learning. The reason is that the expected duration of the short-lasting high-growth regime is now longer than that in the previous subsection. This is captured by the graph in the lower left corner that shows a larger jump in the expected average growth rate as the economy enters the high-growth block when compared to its counterpart in 2.

### 3.3 Evolution of Optimism and Pessimism

So far, we have assumed that prior beliefs are static. Static prior beliefs imply that previous beliefs do not affect current beliefs once the observed TFP growth rate changes. Let's consider a transition matrix of the following type. We assume that the probability of staying in the short-lasting regimes is  $p_{11} = p_{33} = 0.75$ . We set the probabilities of staying in the long-lasting regimes so that  $p_{22} = p_{44} = 0.95$ . For simplicity we assume that the regimes belonging to the

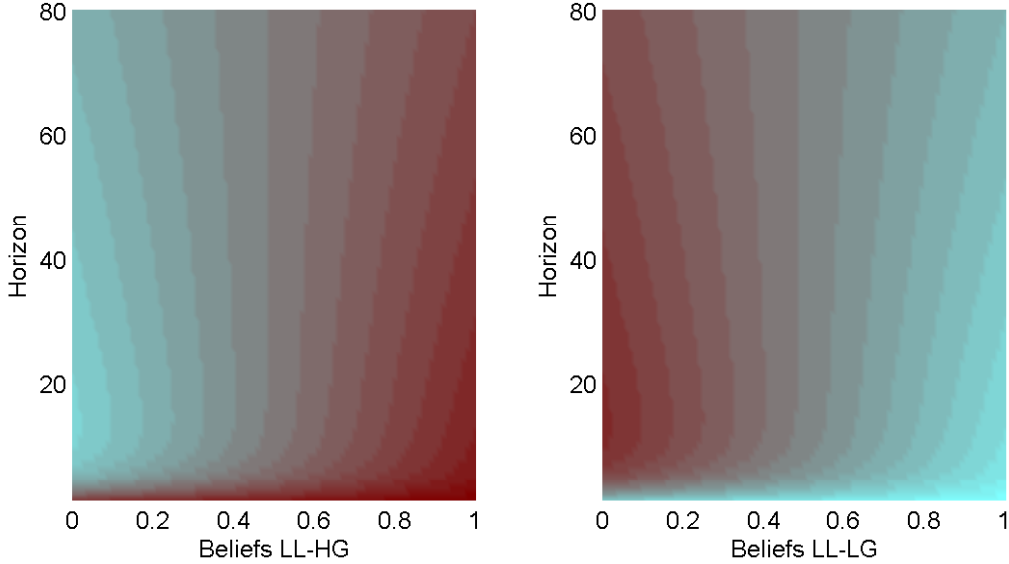


Figure 4: Dynamic Prior Beliefs; Expected Growth Rate of Technology as a Function of Beliefs. The scale of the expected growth rates ranges from -1.6 to 1.6. Lighter blue areas capture expected rates that are lower than the ergodic rate. Darker red areas capture expected rates that are higher than the ergodic rate. Gray areas denote expected rates that are similar to the ergodic rate. *Left graph*: Annualized percentage deviations of the growth rate of technology from its ergodic value at different horizons as a function of the probability that the observed high-growth regime is long lasting (Beliefs LL-HG). *Right graph*: Annualized percentage deviations of the growth rate of technology from its ergodic value at different horizons as a function of the probability that the observed low-growth regime is long lasting (Beliefs LL-LG).

same block do not communicate with each other:  $p_{12} = p_{21} = p_{34} = p_{43} = 0$ . We will relax this restriction later on.

We want to model an economy that goes through two types of *phases* over time: a *high-growth phase* that is mostly characterized by long-lasting high-growth periods with only very short low-growth periods and a *low-growth phase* that is mostly characterized by persistent periods of low-growth and high-growth periods of rather short duration. This can be done by introducing the following restrictions on the parameters of the transition matrix  $\mathcal{P}$ :

$$\frac{p_{31}}{p_{31} + p_{32}} = 0.05 < \frac{p_{41}}{p_{41} + p_{42}} = 0.95 \quad (25)$$

$$\frac{p_{13}}{p_{13} + p_{14}} = 0.05 < \frac{p_{23}}{p_{23} + p_{24}} = 0.95 \quad (26)$$

To sum up, the transition matrix reads:

$$\mathcal{P} = \left[ \begin{array}{cc|cc} 0.75 & 0 & 0.0125 & 0.2375 \\ 0 & 0.95 & 0.0475 & 0.0025 \\ \hline 0.0125 & 0.2375 & 0.75 & 0 \\ 0.0475 & 0.0025 & 0 & 0.95 \end{array} \right]$$

It is important to emphasize that, in this model, the fact that the economy is currently in the high-growth or low-growth regime plays a minor role in affecting the growth rate that agents expect. Most of the action stems from whether agents believe that the economy has been going through a *high-growth phase* or a *low-growth phase*. Figure 4 shows agents' expectations about the growth rate of TFP  $\mu_t$  in deviations from its ergodic level  $\bar{\mu}$  at different horizons (from 1 to 80 quarters) and for various initial levels of probability of being in the long-lasting high-growth regime (left plot) and low-growth regime (right plot). Notice that, unlike in the case of static prior beliefs discussed in Figure 1, high-growth regimes may be associated with a lower medium-horizon expected growth rate of TFP than low-growth regimes. This is because, in this economy, agents know that short-lasting regimes are more likely to be followed by the long-lasting regime of the opposing block. Importantly, the expected growth rate of technology in the long-lasting high-growth (low-growth) regime differs from that in the short-lasting low-growth (high-growth) regime only at a very short horizon.

The upper left graph of Figure 5 reports the evolution of agents' beliefs, consumption, and capital for the case of dynamic prior beliefs. We assume a typical path for the regimes. We initialize agents' beliefs so that agents are confident of being in a high-growth phase.<sup>5</sup> As agents observe 4 quarters of high growth, followed by 20 quarters of low growth, agents start to fear that the economy has switched to the low-growth phase. As a result, households are less optimistic when the economy returns to the high-growth regime. When the second realization of the long-lasting low-growth regime occurs, households become immediately convinced that the long-lasting low-growth regime is in place. Symmetrically, when the economy returns to high TFP growth for the second time, households believe that the high-growth regime will be long-lasting with only a 6% probability. Afterwards, the economy enters the high-growth phase by going through a short-lasting low-growth regime. Households are initially very pessimistic about the persistence of this regime. It takes two realizations of the long-lasting high-growth regimes to make them confident that the economy has shifted to the high-growth phase.

The lower left graph of Figure 5 provides further evidence that households slowly learn about changes in the two paths. Observe that when the economy enters the first long-lasting low-growth period, households mostly believe that they are still in the high-growth phase and

---

<sup>5</sup>This can easily happen if the economy just went through a high-growth phase.

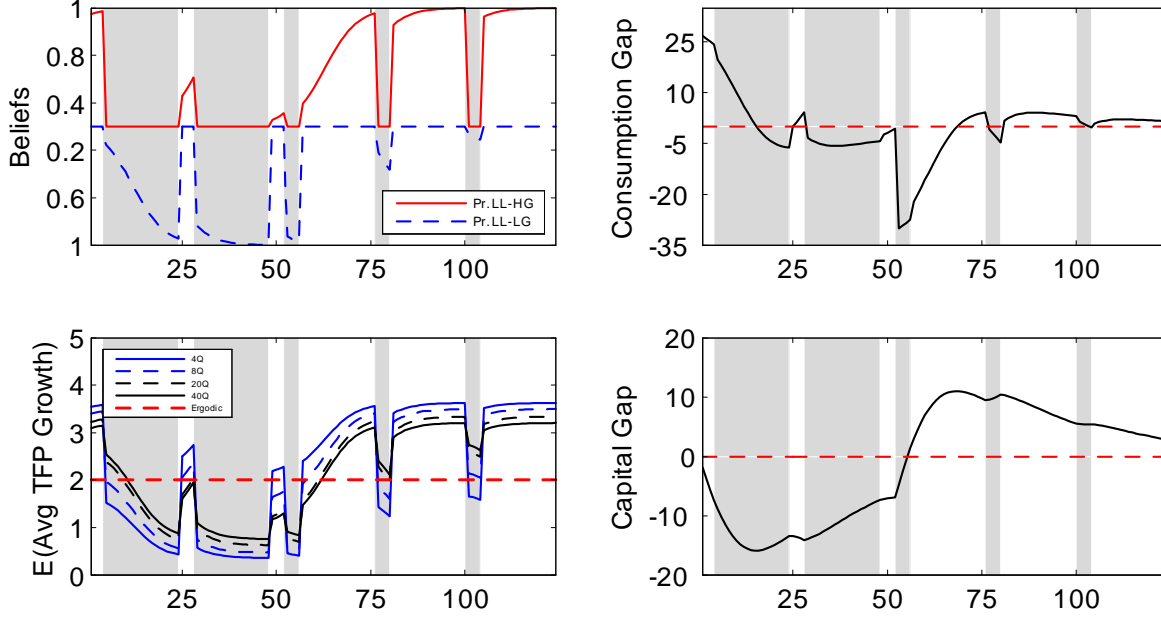


Figure 5: Dynamic Prior Beliefs; Expected Growth Rate of Technology as a Function of Beliefs. *Left graph:* Annualized percentage deviations of the growth rate of technology from its ergodic value at different horizons as a function of the probability that the observed high-growth regime is long lasting (Beliefs LL-HG). *Right graph:* Annualized percentage deviations of the growth rate of technology from its ergodic value at different horizons as a function of the probability that the observed low-growth regime is long lasting (Beliefs LL-LG).

expect an average growth rate of TFP over the next 20 or 40 quarters that is above the ergodic level. The same sluggishness in the expected average growth rate of TFP can be observed as the economy enters the first long-lasting high-growth period. Furthermore, the sluggish dynamics of optimism and pessimism are confirmed by a quick comparison of the expected average growth rate of TFP across short-lasting periods. It is important to notice that the earliest short-lasting high-growth (low-growth) regime is associated with rates that are largely below (above) the ergodic rate at all horizons.

The behavior of consumption and capital during the low-growth and the high-growth phase is analyzed in the righthand graphs of Figure 5. We observe that at the beginning of the first short-lasting high-growth regime, which is associated with high optimism, the consumption gap is positive. The reason is that imperfectly informed households expect this regime to be much longer lasting than what it actually turns out to be. This implies that imperfectly informed households do not raise capital as they would if they knew that the high-growth regime is, in fact, short lasting. This leads to a negative capital gap and a positive consumption gap. When the economy enters the long-lasting low-growth regime for the first time, the consumption gap remains positive as households are not very pessimistic about the persistence of this regime. This happens because households decide to cut capital fairly aggressively in order to sustain current consumption, since they mostly expect this low-growth regime to be short lasting.

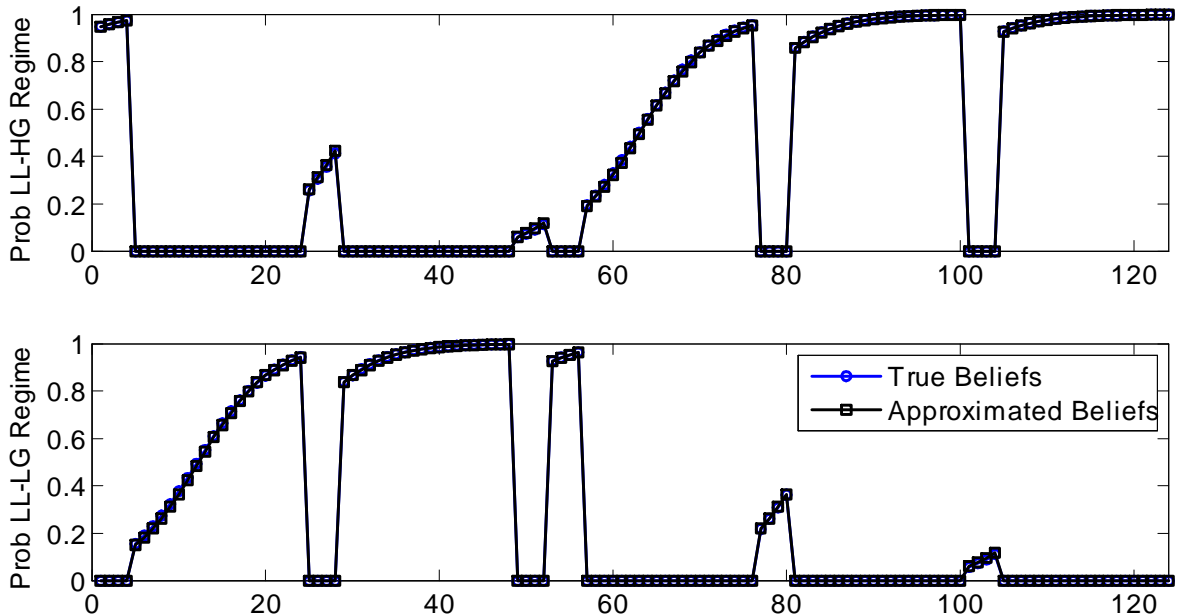


Figure 6: Accuracy. *Top graph:* True and estimated path of agents' beliefs of being in the long-lasting high-growth regime. Lines with squares: approximated beliefs. Lines with circles: true beliefs. *Bottom graph:* True and estimated path of agents' beliefs of being in the long-lasting low-growth regime. Lines with squares: approximated beliefs. Lines with circles: true beliefs.

Households would do otherwise, if they knew that the economy just entered the long-lasting low-growth regime.

During the first long-lasting low-growth spell households update their beliefs until they realize that this regime is most likely long lasting, signifying that the economy must have switched to the low-growth phase. This change in agents' beliefs causes consumption and capital (the latter with some sluggishness) to become similar to the perfect-information benchmark. Interestingly, the consumption gap changes sign and becomes negative at the end of the long-lasting low-growth spell. This is due to the fact that convergence of capital to its perfect-information level is relatively more sluggish than that of consumption. When the second short-lasting high-growth regime occurs, optimism is a bit smaller than in the previous high-growth period, resulting in a more contained hike in the consumption gap. The dynamics of the consumption gap and the capital gap are clearly reversed during the high-growth phase.

Let us focus on the accuracy of our method in tracking the dynamics of beliefs over time. We initially set 400 equally spaced knots in our grid  $\mathcal{G}$ . Furthermore, we add 100 knots to make the grid finer for beliefs near the convergence points for  $prob\{\xi_t = 1|\tau_t^1\}$  and  $prob\{\xi_t = 3|\tau_t^1\}$ , which are zero in both cases. After the refinement of the grid of beliefs introduced in steps 10-11 of Section 2.2.2, we are left with 152 grid points per block. Even if the number of regimes seems enormous, solving the model is fast. It takes 0.97 second to compute the matrix  $\hat{\mathcal{P}}$  and 6.42 seconds to solve the model with *gensys* in Matlab on a 64-bit desktop endowed with an

Intel core processor i7-2600 CPU at 3.40 GHz.<sup>6</sup> Figure 6 shows that the grid of beliefs fits the actual sequence of beliefs very accurately as squares and circles are almost perfectly concentric.

### 3.4 Sneaky Switches to Low-Growth Phase

So far we have studied cases in which transitions between high-growth and low-growth phases are always marked by an *observable switch* in regimes. For instance, a switch to the low-growth phase is characterized by an observable change of TFP growth from high to low. This is because, so far, we have assumed that the probability of switching between regimes belonging to the same block is zero. In this section we relax this assumption.

Let us use the baseline calibration and the same values for the transition matrix  $\mathcal{P}$  as those used in Subsection 3.3, with the only exception being that now the probability of switching to the short-lasting high-growth regime conditional on being in the long-lasting high-growth regime is  $p_{21} = 0.04$ . The probabilities  $p_{23}$  and  $p_{24}$  are re-scaled so that (26) is satisfied. In this context, a switch from the *high-growth phase* to the *low-growth phase* may happen when the economy is in the long-lasting high-growth regime. This means that the system could switch to the low-growth phase while agents do not observe any switch from high growth to low growth. Although the probability that such a *sneaky* switch would happen is quite small ( $p_{21} = 0.04$ ), such a possibility dramatically influences the dynamics of agents' beliefs and allocations.

Figure 7 shows the dynamics of consumption and capital in the case of learning and perfect information using the same sequence of regimes as that analyzed in the previous example.<sup>7</sup> Let us focus on the second half of the periods when the economy enters the high-growth phase. The top left graph of Figure 7 shows that agents' beliefs about being in the long-lasting high-growth regime do not converge to unity even when a large number of high-growth periods occur. This is different from what we observe in Figure 5. Thus, an important implication of introducing sneaky switches to the low-growth phase is that agents will never be fully convinced that they are in the high-growth phase. Furthermore, as short-lasting low-growth regimes occur, agents are relatively more concerned about the possibility of having entered a long-lasting low-growth period. The reason is that agents are aware that a sneaky switch may have occurred during the last high-growth regime.

The rightside graphs of Figure 7 show the consumption and capital gaps with respect to the perfect-information benchmark. As we allow for the possibility of sneaky switches, the high-growth phase is characterized by recurrent negative consumption gaps as the economy is going through short-lasting low-growth regimes. This is different from the case of no sneaky

<sup>6</sup>In the case with static prior beliefs, which was analyzed in Section 3.1, it takes 0.10 second to compute the matrix  $\hat{\mathcal{P}}$  and to solve the model with *gensys*.

<sup>7</sup>To ease the comparison with the previous case with no sneaky switches, the scale of the y-axes is the same as that in Figure 5.

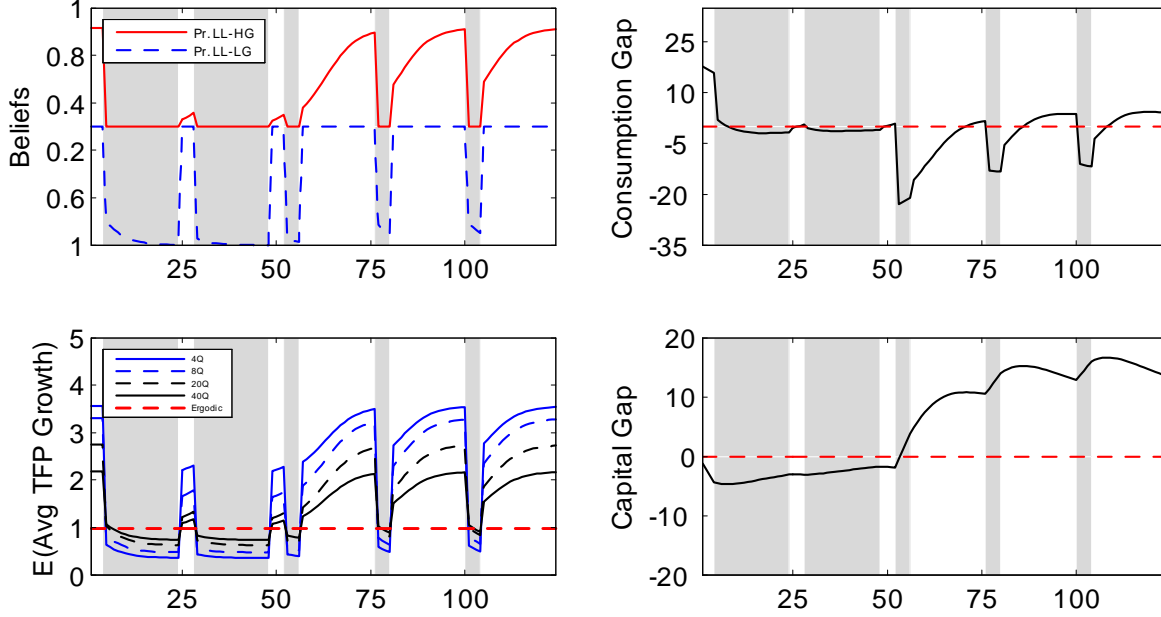


Figure 7: Sneaky Switches. *Top left graph:* Evolution of beliefs of being in the long-lasting high-growth regime (red solid line) and in the long-lasting low-growth regime (blue dashed line). *Top right graph:* Annualized percentage deviations of consumption from the perfect-information benchmark. *Bottom left graph:* Expected average growth rate of technology (annualized percentage) at various horizons. *Bottom right graph:* Annualized percentage deviations of capital from the perfect-information benchmark. In all graphs, gray areas denote periods of low growth.

switches in Figure 5 in which we observe only one large negative consumption gap that gradually disappears as the economy remains in the high-growth phase. The reason is that the possibility of sneaky switches to the low-growth phase prompts households to interpret short-lasting low-growth regimes as long lasting. As a result, households adjust their consumption more than their capital stock. When long-lasting high-growth periods occur, agents are initially not very optimistic, expecting a quite short-lasting high-growth period. As a result, they speed up capital accumulation to achieve consumption smoothing. Quite interestingly and unlike the example in Figure 5, high pessimism during short-lasting low-growth periods causes the capital gap to not exhibit mean reversion during a typical high-growth phase. Therefore, the possibility of sneaky switches induces households to *hoard* capital during high-growth phases. Capital hoarding during high-growth phases is due to households' inability to fully learn when the economy is in the high-growth phase because of the possibility of sneaky switches to the low-growth phase. Finally, in the low-growth phase, households learn faster that the economy is on a low-growth path than in Figure 5. This translates into smaller departures of consumption and capital allocations from the perfect-information benchmark.

### 3.5 Over-pessimism

So far, we have studied economies in which agents' pessimism or optimism monotonically increases as the system remains in the same block. In this section, we study the case in which pessimism over-reacts as the system enters the low-growth block and gradually falls as the economy remains in the low-growth block. Let us consider the same parameterization as in Section 3.3. We make three departures from this case. First, the probability of switching from the long-lasting low-growth regime to the short-lasting one is  $p_{43} = 0.15 > 0$ . Second, the probability of staying in the long-lasting low-growth regime  $p_{44}$  is set to  $0.80 < 0.95$ . Third, the probability of switching from the long-lasting high-growth regime to the short-lasting low-growth regime conditional on the system having switched to the low-growth block is

$$\frac{p_{23}}{p_{23} + p_{24}} = 0.25 \tag{27}$$

To sum up, the transition matrix reads:

$$\mathcal{P} = \left[ \begin{array}{cc|cc} 0.75 & 0 & 0.0125 & 0.2375 \\ 0 & 0.95 & 0.0125 & 0.0375 \\ \hline 0.0125 & 0.2375 & 0.75 & 0 \\ 0.0475 & 0.0025 & 0.15 & 0.80 \end{array} \right]$$

Consider the following simulation in which all regimes occur with their expected durations. First, starting from the highest optimism, the long-lasting high-growth regime is realized for 20 quarters followed by 4 quarters of the short-lasting low-growth regime. After that, the economy switches again to the long-lasting high-growth regime and finally to the long-lasting low-growth regime for 5 quarters. Finally, the system switches to the short-lasting low-growth regime for 4 quarters.

The results are reported in Figure 8. During the first 20 quarters agents are correct in believing that they are in the long-lasting high-growth regime. Therefore, consumption and capital allocations are the same as those in the perfect-information benchmark. Upon the switch to the short-lasting low-growth regime, we observe a hike in agents' pessimism captured by a sharp increase in the probability attached to the long-lasting low-growth regime and a drop in expected TFP growth at all horizons. The blue dashed line in the upper left graph shows that the *direction of learning* goes from the long-lasting regime to the short-lasting regime, and therefore, it is opposite to what is observed in all the previous examples. As the system stays in the low-growth block, agents attach less probability to being in the long-lasting low-growth regime. This is because from the long-lasting low-growth regime the economy can move to the short-lasting low-growth regime. Agents are aware of this, and as time goes by, they take



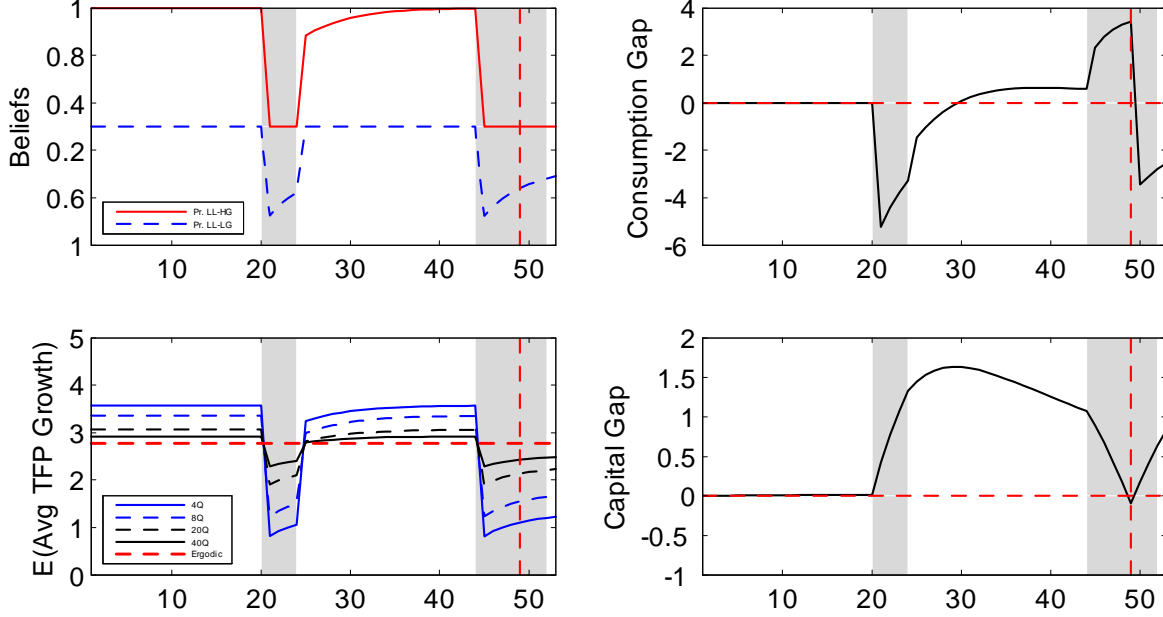


Figure 8: Over-Pessimism. *Top left graph:* Evolution of beliefs of being in the long-lasting high-growth regime (red solid line) and in the long-lasting low-growth regime (blue dashed line). *Top right graph:* Annualized percentage deviations of consumption from the perfect-information benchmark. *Bottom left graph:* Expected average growth rate of technology (annualized percentage) at various horizons. *Bottom right graph:* Annualized percentage deviations of capital from the perfect-information benchmark. In all graphs, gray areas denote periods of low growth. The vertical dashed red lines in all graphs mark the time at which the economy switches from the long-lasting low-growth regime to the short-lasting low-growth regime.

into account the possibility that this event might in fact have occurred. Therefore, pessimism initially *overshoots* and then falls as agents observe more and more periods of low growth. Such *over-pessimism* has profound implications for consumption and capital allocations. The upper right graph shows that the first hike in pessimism causes consumption to fall dramatically with respect to the case of perfect information. This is because of a misalignment of households' expectations. As agents' pessimism goes down, the consumption gap shrinks and the capital gap widens.

When the economy switches to the long-lasting low-growth regime at  $t = 46$ , we observe a second hike in pessimism. Yet, unlike in the previous low-growth period, households are now correct in mostly expecting the current low-growth period to be long lasting. Thus, the consumption and capital gaps do not exhibit any dramatic change upon the switch. In the subsequent periods, households' pessimism gradually declines and only then do the consumption and capital gaps widen. This happens because uninformed agents take into account the possibility that the economy experienced a sneaky switch to the short-lasting low-growth regime, while perfectly informed agents can observe that the long-lasting low-growth regime is still in place. In the last four periods of low growth, the sneaky switch actually happens and it is

marked with a vertical red dashed line. Since this switch is hidden, it does not affect households' beliefs and allocations in our economy, while in the counterfactual perfect information economy it determines a drastic change in consumption and capital allocations that manifests itself with a quick reversal in the dynamics of the consumption and capital gaps.

Summarizing, during low-growth periods the direction of learning is reversed with respect to what we have seen in all previous examples. The longer the low-growth span, the less pessimistic about the future duration of the low-growth period agents become. When the true regime is long lasting, such a pattern of beliefs causes households to slow down capital accumulation more aggressively than under perfect information. On the contrary, when the true regime is short lasting, agents are initially too pessimistic and accumulate an extra stock of capital with respect to what they would do if they knew the stochastic duration of the regime.

### 3.6 Bipolar Beliefs

In this section we provide a limiting case in which agents' beliefs change dramatically in every period. We make the following two key assumptions: *(i)* high-growth regimes are assumed to communicate a lot ( $p_{12} = 0.95$ ,  $p_{21} = 0.65$ ) and *(ii)* the relative persistence of low-growth regimes is markedly different ( $p_{33} = 0.25$  and  $p_{44} = 0.99$ ). Furthermore, the probability of staying in the high-growth regimes is  $p_{11} = 0$  and  $p_{22} = 0.3$ . Low-growth regimes do not communicate ( $p_{34} = p_{43} = 0$ ). We use (25) and (26) for the probabilities linking regimes across blocks. Condition (26), which implies that the persistent high-growth regime is associated with a much smaller probability of switching to the very persistent low-growth regime, and condition *(ii)* together imply that whether forward-looking households believe that they are in the short-lasting or in the long-lasting high-growth regime has large effects on consumption and capital allocations.

Consider a sequence of high-growth regimes that alternates one period of Regime 1 and one period of Regime 2 ten times. The top left graph of Figure 9 reports the dynamics of agents' beliefs about being in the long-lasting high-growth regime and the true probabilities are shown in blue squares. Consistent with assumptions (25) and (26), we initialize agents' beliefs to attach a probability of 0.95 that the economy is in the short-lasting Regime 1 at time 1. Lots of communication between regimes belonging to the same block brings about bipolar dynamics of beliefs during the early quarters spent in the high-growth block. Such a pattern is exacerbated by the fact that Regime 1 has zero persistence. As the system remains in the high-growth block, bipolarism fades away because agents are not receiving any additional information. Therefore, the probability that the high-growth regime has been in place for an even number of consecutive periods keeps growing, breaking the bipolar dynamics.

The rightside graphs highlight the difference between these allocations and the perfect-

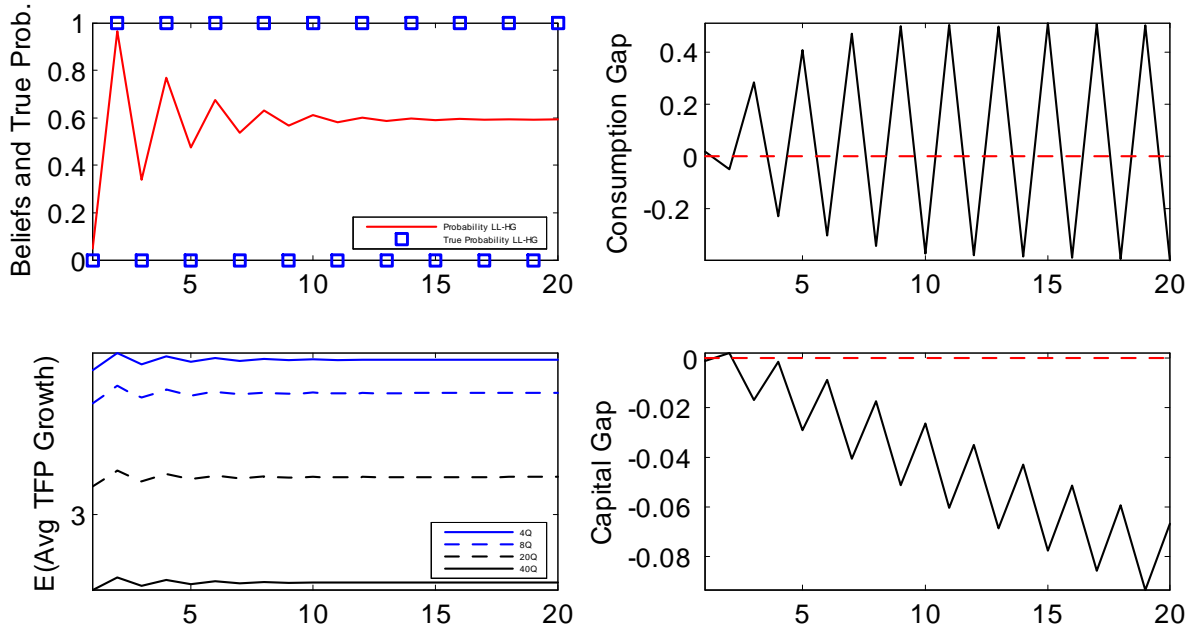


Figure 9: Bipolar Beliefs. *Top left graph:* Evolution of beliefs of being in the long-lasting high-growth regime (red solid line) and the true probability (blue squares) of being in the long-lasting high-growth regime. *Top right graph:* Annualized percentage deviations of consumption from the perfect-information benchmark. *Bottom left graph:* Expected average growth rate of technology (annualized percentage) at various horizons. *Bottom right graph:* Annualized percentage deviations of capital from the perfect-information benchmark.

information benchmark. When the short-lasting Regime 1 is in place, perfectly informed agents know that there is a fairly large probability that the economy will move to the highly persistent low-growth Regime 4 if the system switches blocks. As a result, perfectly informed households slow down the growth rate of consumption and raise that of capital, expecting the worst for the future. Imperfectly informed households follow the same policy but much less aggressively, bringing about a positive consumption gap. When the long-lasting Regime 2 is in place, perfectly informed households know that the most likely low-growth regime to occur in the next period is the short-lasting one. Therefore, perfectly informed households reduce investment and raise the growth rate of consumption. Imperfectly informed households are uncertain about whether the regime is short or long lasting. Thus, they adjust capital less aggressively. This leads to a negative consumption gap. Note that the lower left graph shows that when the system has been in the high-growth block sufficiently long, the expected average growth rate at every horizon does not change over time as beliefs start changing.

## 4 Signals and Implications for Uncertainty

Consider the following transition matrix:

$$\mathcal{P} = \left[ \begin{array}{cc|cc} 0.85 & 0.1 & 0.025 & 0.025 \\ 0.05 & 0.9 & 0.05 & 0 \\ \hline 0 & 0.99 & 0.01 & 0 \\ 0.1 & 0 & 0 & 0.99 \end{array} \right]$$

While the high-growth regimes exhibit similar persistence, the low-growth regimes have markedly different persistence. There is a very small probability that, once the system is in Regime 3, it will stay in the low-growth block next period; most likely it will switch to the long-lasting high-growth regime. Note that the probability of staying in the high-growth block is 0.95 for both the short-lasting and long-lasting high-growth regimes. However, Regime 1 has a larger downside risk with a bigger probability of moving to the long-lasting low-growth Regime 4.

When under the high-growth block, households receive a public signal  $\varpi_t$  about the regime in place. The signal can take two values: 1 or 2. We assume that  $prob\{\varpi_t = 1|\xi_t = 1\} = 0.20$  and  $prob\{\varpi_t = 1|\xi_t = 2\} = 0.80$ , implying that receiving a signal  $\varpi_t = 1$  is more likely when the economy is the short-lasting high-growth regime. Conversely, receiving a signal  $\varpi_t = 2$  is more likely when the economy is in the long-lasting high-growth regime. We study the evolution of allocations and beliefs when Regime 2 is in place for its typical duration of 10 quarters. Households always receive the same signal  $\varpi_t = 2$  during the period except at time  $t = 3$  and  $t = 6$ , when they receive  $\varpi_t = 1$ . Figure 10 shows the dynamics of beliefs and allocations (black dashed line) and compare them with those of an economy in which households always receive  $\varpi_t = 2$  at any time (solid blue line).

Receiving signals  $\varpi_3 = 1$  and  $\varpi_6 = 1$  influences agents' beliefs by reducing their optimism. Note that nothing is really changed in the economy's fundamentals as the economy remains in Regime 2 at all times. Hence, signals play the role of *shocks to beliefs* with the effect of reducing optimism. If households did not receive the signals  $\varpi_3 = 1$  and  $\varpi_6 = 1$ , their beliefs would have not changed (see the black dashed line). Such shocks to beliefs change consumption and capital allocations. The first shock to beliefs reduces consumption by 0.85% and the second one by 0.8% three periods later. Furthermore, Figure 10 shows that shocks to beliefs prompt agents to accumulate more capital. The higher accumulated capital pushes consumption up at the end of the simulated sample when the effects of signals on beliefs has faded away.

Importantly, signals give rise to shocks to beliefs that have second-order effects. In this simulation, bad signals raise the expected probability of switching to the very persistent low-growth Regime 4, determining an increase in the downside risk and uncertainty. The bottom middle panel and the bottom right panel show that the increase in downside risk translates

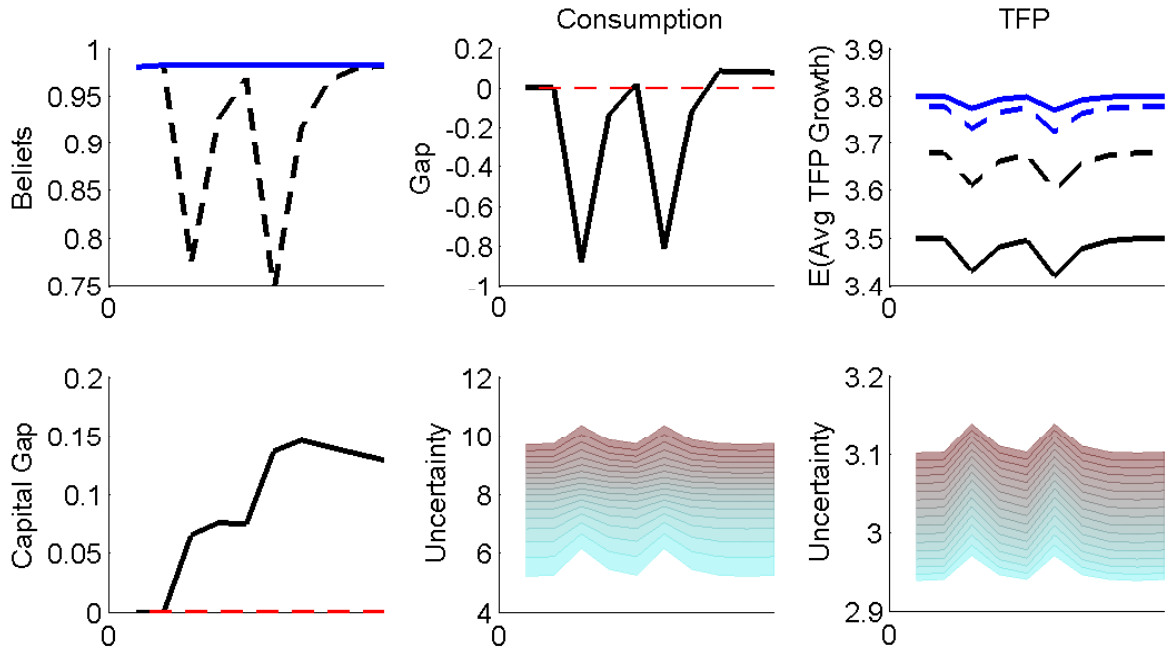


Figure 10: Belief Shocks. *Top left graph:* Evolution of beliefs of being in the long-lasting high-growth regime in the case of shocks to beliefs at time  $t=3$  and  $t=6$  (black dashed line) and in the case of no shock to beliefs (solid blue line). *Top middle graph:* Annualized percentage deviations of consumption from the case of no shock to beliefs. *Top right graph:* Expected average growth rate of technology (annualized percentage points) at various horizons (4 quarters, 8 quarters, 20 quarters, and 40 quarters) for the case of shocks to beliefs at  $t=3$  and  $t=6$ . From top to bottom: the solid blue line denotes the horizon of 4 quarters, the solid black line denotes the horizon 40 quarters. *Bottom left graph:* Annualized percentage deviations of capital from the case of no shock to beliefs. *Bottom middle graph:* Uncertainty about future consumption at horizons from 4 quarters to 20 quarters. Lighter blue areas denote shorter horizons. Darker red areas denote longer horizons. *Bottom right graph:* Uncertainty about future TFP growth at horizons from 4 quarters to 20 quarters.

into a spike in uncertainty about future consumption and future TFP growth. Note that such changes in uncertainty are not supported by any changes in the economy's fundamentals, but rather are due to signals that change the *perceived* probability of switching to Regime 4.

While this is not the first paper to use signals as shocks to beliefs (e.g., Lorenzoni, 2009, Angeletos and La'O, 2010 and Forthcoming) the approach proposed in this paper has the important advantage of keeping the model very tractable. This feature makes our methods suitable for studying shocks to beliefs in likelihood-based estimated large-scale DSGE models (e.g., Christiano, Eichenbaum, and Evans, 2005 and Smets and Wouters, 2007).

## 5 Modeling Beliefs about Policy Makers

This section provides a concise review of two alternative applications of the methods described in this paper.

### 5.1 Modeling Communication and Constrained Discretion

Bianchi and Melosi (2012a) apply the methods introduced in this paper to quantitatively study monetary policy and the role central banks' communication in a prototypical New-Keynesian DSGE model. In the model, monetary policy alternates between periods of active inflation stabilization, in an active regime, and periods during which the emphasis on inflation stabilization is reduced, in a passive regime. Agents in the model are fully rational and able to infer if monetary policy is active or not. However, when the passive regime prevails, they are uncertain about the nature of the observed deviation. In other words, agents are not sure if the central bank is engaging in a short-lasting or long-lasting deviation from active monetary policy.

Agents conduct Bayesian learning in order to infer the likely duration of the deviations from active monetary policy. As agents observe more and more deviations, they become more and more convinced that they are in the long-lasting passive regime and are increasingly pessimistic about a quick return to the active regime. When the model is estimated to the U.S. economy, we find that inflation expectations and agents' uncertainty sluggishly increase as the Federal Reserve keeps deviating from active policy and they accelerate only after 20 quarters of deviations. These features make the model well-suited to explaining the monetary policy framework that has been adopted by the Federal Reserve, which Bernanke and Mishkin (1997) have termed *constrained discretion*. Bernanke (2003) explains that under constrained discretion, the central bank retains some flexibility in the conduct of monetary policy in order to accommodate short-run disturbances. However, such flexibility is constrained to the extent that the central bank should maintain strong credibility for keeping inflation and inflation expectations firmly under control.

Furthermore, Bianchi and Melosi (2012a) build on the methods introduced in this paper to provide a very general framework to model policy-makers' communication. We study the case (*transparency*) in which the central bank *systematically* announces the number of consecutive deviations from active monetary policy. Since the model is purely forward-looking, a sufficient statistic to solve the model with systematic announcements is the number of periods of announced passive policy that lie ahead before switching to the active policy. In other words, the laws of motion associated with a situation in which the central bank announces five periods of consecutive deviations from the active regime are exactly the same as those associated with a situation in which a central bank has carried out five consecutive deviations out of ten announced deviations. Therefore, we can redefine the structure of regimes as follows: Regime

1 is the active regime; Regime 2 is a regime in which only one period of announced passive policy (i.e., the current one) is left before switching to the active regime; Regime 3 is a regime in which two consecutive periods of passive policy will be conducted before switching to the active policy; etc. To avoid the possibility that the dimensionality of the set of regimes blows up to infinity, we truncate the duration of passive deviations to  $\tau_a^*$ . For any  $\varepsilon > 0$  we can find a  $\tau_a^*$  such that the probability of a deviation longer than  $\tau_a^*$  has a probability smaller than  $\varepsilon$ , implying that the approximation can be made arbitrarily accurate.

Endowed with this result, we can study the impact of monetary policy communication on welfare by redefining the structure of regimes in terms of the number of announced deviations yet to be carried out  $\tau_a$  as follows:

$$(\phi_\pi(\tau_t^a = i), \phi_y(\tau_t^a = i)) = \begin{bmatrix} (\phi_\pi^A, \phi_y^A), & \text{if } i = 1 \\ (\phi_\pi^P, \phi_y^P), & \text{if } 1 < i < \tau_a^* \end{bmatrix}$$

where  $\phi_\pi^A$  and  $\phi_y^A$  are the inflation and output gap coefficient in the monetary policy reaction function when monetary policy is active and  $\phi_\pi^P$  and  $\phi_y^P$  when monetary policy is passive. The re-defined set of regimes  $\tau_t^a$  are governed by the  $(\tau_a^* + 1) \times (\tau_a^* + 1)$  transition matrix  $\tilde{\mathcal{P}}^A$ , which is defined as:

$$\tilde{\mathcal{P}}^A = \begin{bmatrix} p_{11} & \tilde{p}_A \\ \mathbf{I}_{\tau_a^*} & \mathbf{0}_{\tau_a^* \times 1} \end{bmatrix}$$

where  $p_{11}$  is the probability of staying in the active regime,  $\mathbf{I}_{\tau_a^*}$  is a  $\tau_a^* \times \tau_a^*$  identity matrix,  $\mathbf{0}_{\tau_a^* \times 1}$  is a  $(\tau_a^* \times 1)$  column vector of zeros. and  $\tilde{p}_A$  is a  $1 \times \tau_a^*$  row vector whose typical  $i$ -th element is the probability of announcing  $i$  consecutive periods of passive monetary policy followed by a switch to the active regime, conditional on being in the active regime. Formally,  $\tilde{p}_A(i) \equiv p_{12}p_{22}^i p_{21} + p_{13}p_{33}^i p_{31}$  for  $1 \leq i \leq \tau_a^* - 1$  and  $\tilde{p}_A(\tau_a^*) = 1 - p_{11} - \sum_{i=1}^{\tau_a^*-1} \tilde{p}_A(i)$ . Note that the matrix  $\tilde{\mathcal{P}}^A$  is a function of the probabilities of the transition matrix  $\mathcal{P}$  for the primitive regimes.

## 5.2 Dormant Fiscal Shocks

Bianchi and Melosi (forthcoming) develop a model in which the current behavior of the fiscal and monetary authorities influences agents' beliefs about the way debt will be stabilized. The standard policy mix consists of a virtuous fiscal authority that moves taxes in response to debt and a central bank that has full control over inflation. When policy-makers deviate from this virtuous policy mix, agents conduct Bayesian learning to infer the likely duration of the deviation. As agents observe more and more deviations, they become increasingly pessimistic about a prompt return to the virtuous regime and inflation starts moving to keep debt on a stable path. Shocks that were *dormant* under the virtuous policy mix now start to manifest themselves

and uncertainty about the macroeconomy starts increasing among agents. These changes are initially imperceptible, but they unfold over decades and accelerate as agents become convinced that the fiscal authority will not raise taxes. Dormant fiscal shocks can account for the run-up of inflation in the '70s and the deflationary pressure of the early 2000s. The paper also shows that the currently low long-term interest rates and inflation expectations might hide the true risk of inflation faced by the US economy.

Bianchi and Melosi (2012b) construct an economy in which policy-makers' reputation for fiscal virtue smoothly fluctuates over time. The monetary and fiscal policy mix alternates between periods of fiscal virtue and periods of fiscal irresponsibility. Under fiscal virtue, a virtuous rule is in place most of the time: the central bank stabilizes inflation and the government strongly adjusts taxes to stabilize public debt. Under fiscal irresponsibility, a fiscally irresponsible rule is in place most of the time: the central bank de-emphasizes inflation stabilization and the fiscal authority is not committed to keeping debt under control. A strong reputation for fiscal virtue is generally desirable because it leads to a stable macroeconomic environment, but when the economy enters the zero lower bound, policy-makers face a trade-off between preserving their reputation and escaping a large recession. Given that policy-makers' behavior is constrained at the zero lower bound, beliefs about the exit strategy are substantially more important than policies implemented during the crisis. Announcing a period of austerity is detrimental in the short run, but it preserves macroeconomic stability in the long run. A severe recession can be avoided by abandoning fiscal virtue, but this results in a sharp increase in macroeconomic instability. Contradictory announcements by the fiscal and monetary authorities can lead to high inflation and large output losses. Finally, the paper shows that high uncertainty is an inherent implication of entering the zero lower bound while deflation is not, because agents are likely to be uncertain about the way policy-makers will deal with the large stock of debt arising from a severe recession.

## 6 Concluding Remarks

This paper has developed methods to solve general equilibrium models in which agents are subject to waves of optimism, pessimism, and uncertainty. Agents in the model are fully rational, understand the structure of the economy, and *know that they do not know*. Therefore, when forming expectations they take into account that their beliefs will evolve in response to realized observable economic outcomes, the behavior of other agents in the model, or both. The central insight consists of creating an expanded number of regimes indexed with respect to agents' beliefs. The resulting law of motion reflects agents' uncertainty and can be expressed in state space form. Therefore, the framework proposed in this paper is suitable for structural estimation.



## References

- ANGELETOS, G.-M., AND J. LA'O (2010): "Noisy Business Cycles," in *NBER Macroeconomics Annual 2009, Volume 24*, pp. 319–378. National Bureau of Economic Research, Inc.
- ANGELETOS, G. M., AND J. LA'O (forthcoming): "Sentiments," *Econometrica*.
- BERNANKE, B. S. (2003): "'Constrained Discretion" and Monetary Policy," Discussion paper, Remarks by Governor Ben S. Bernanke.
- BERNANKE, B. S., AND F. S. MISHKIN (1997): "Inflation Targeting: A New Framework for Monetary Policy?," *Journal of Economic Perspectives*, 11(2), 97–116.
- BIANCHI, F. (2012): "Methods for Markov-Switching Models," Manuscript.
- BIANCHI, F. (forthcoming): "Regime Switches, Agents' Beliefs, and Post-World War II U.S. Macroeconomic Dynamics," *Review of Economic Studies*.
- BIANCHI, F., C. ILUT, AND M. SCHNEIDER (2012): "Business Cycles and Asset Prices: The Role of Volatility Shocks under Ambiguity Aversion," Duke University-Stanford University, working paper.
- BIANCHI, F., AND L. MELOSI (2012a): "Constrained Discretion and Central Bank Transparency," Manuscript.
- (2012b): "Escaping the Great Recession," Manuscript.
- (forthcoming): "Dormant Shocks and Fiscal Virtue," in *NBER Macroeconomics Annual 2013, Volume 28*, pp. 319–378. National Bureau of Economic Research, Inc.
- BLANCHARD, O. J., AND C. M. KAHN (1980): "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48(5), 1305–11.
- BLOOM, N. (2009): "The Impact of Uncertainty Shocks," *Econometrica*, 77(3), 623–685.
- CHO, S. (2012): "Characterizing Markov-Switching Rational Expectations Models," working paper.
- CHRISTIANO, L. J., M. EICHENBAUM, AND C. L. EVANS (2005): "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy," *Journal of Political Economy*, 113(1), 1–45.
- COGLEY, T., C. MATTHES, AND A. M. SBORDONE (2011): "Optimal Disinflation under Learning," Staff Reports 524, Federal Reserve Bank of New York.

- COGLEY, T., AND T. J. SARGENT (2005): “Drift and Volatilities: Monetary Policies and Outcomes in the Post WWII U.S.,” *Review of Economic Dynamics*, 8(2), 262–302.
- DAVIG, T., AND T. DOH (2008): “Monetary Policy Regime Shifts and Inflation Persistence,” Research Working Paper RWP 08-16, Federal Reserve Bank of Kansas City.
- DAVIG, T., AND E. M. LEEPER (2007): “Generalizing the Taylor Principle,” *American Economic Review*, 97(3), 607–635.
- DEL NEGRO, M., AND S. EUSEPI (2010): “Fitting Observed Inflation Expectations,” Staff Reports 476, Federal Reserve Bank of New York.
- EUSEPI, S., AND B. PRESTON (2011): “Expectations, Learning, and Business Cycle Fluctuations,” *American Economic Review*, 101(6), 2844–72.
- EVANS, G. W., AND S. HONKAPOHJA (2001): *Learning and Expectations in Macroeconomics*. Princeton University Press.
- FARMER, R. E., D. F. WAGGONER, AND T. ZHA (2009): “Understanding Markov-switching rational expectations models,” *Journal of Economic Theory*, 144(5), 1849–1867.
- FERNANDEZ-VILLAYERDE, J., AND J. F. RUBIO-RAMIREZ (2008): “How Structural Are Structural Parameters?,” in *NBER Macroeconomics Annual 2007, Volume 22*, pp. 83–137. National Bureau of Economic Research, Inc.
- FOERSTER, A. T., J. RUBIO-RAMIREZ, D. WAGGONER, AND T. ZHA (2011): “Perturbation Methods for Markov-Switching Models,” Mimeo Duke University.
- JUSTINIANO, A., AND G. E. PRIMICERI (2008): “The Time-Varying Volatility of Macroeconomic Fluctuations,” *American Economic Review*, 98(3), 604–41.
- LIU, Z., D. F. WAGGONER, AND T. ZHA (2011): “Sources of macroeconomic fluctuations: A regime-switching DSGE approach,” *Quantitative Economics*, 2(2), 251–301.
- LORENZONI, G. (2009): “A Theory of Demand Shocks,” *American Economic Review*, 99(5), 2050–84.
- MANKIW, N. G., R. REIS, AND J. WOLFERS (2004): “Disagreement about Inflation Expectations,” in *NBER Macroeconomics Annual 2003, Volume 18*, pp. 209–270. National Bureau of Economic Research, Inc.
- MELOSI, L. (2012a): “Estimating Models with Dispersed Information,” Mimeo.

- (2012b): “Signaling Effects of Monetary Policy,” Discussion paper.
- NIMARK, K. (2008): “Dynamic Pricing and Imperfect Common Knowledge,” *Journal of Monetary Economics*, 55(8), 365–382.
- PRIMICERI, G. E. (2005): “Time Varying Structural Vector Autoregressions and Monetary Policy,” *Review of Economic Studies*, 72(3), 821–852.
- SCHORFHEIDE, F. (2005): “Learning and Monetary Policy Shifts,” *Review of Economic Dynamics*, 8(2), 392–419.
- SIMS, C. A. (2002): “Solving Linear Rational Expectations Models,” *Computational Economics*, 20(1-2), 1–20.
- SIMS, C. A., AND T. ZHA (2006): “Were There Regime Switches in U.S. Monetary Policy?,” *American Economic Review*, 96(1), 54–81.
- SMETS, F., AND R. WOUTERS (2007): “Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach,” *American Economic Review*, 97(3), 586–606.

# Appendices

The appendices are organized as follows. Appendix A works out the recursions (3) and (4) that pin down the dynamics of beliefs *within* blocks. Appendices B-F prove Propositions 2-6. Appendix G shows that the unstable root  $\tilde{\lambda}_1$  is either smaller than or equal to zero or higher than or equal to unity. Appendix H details the algorithm to construct the transition matrix  $\hat{\mathcal{P}}$  when agents receive signals. Appendix I characterizes the steady-state equilibrium for stationary variables in the RBC model and obtains the log-linearized equations of this model.

Note that the convergence results, which are proven in Appendices B-G, could be derived by working on the submatrices of each block. However, we have decided to work with the solution of the difference equations (3) and (4) because this approach is familiar to a wider audience.

## A Deriving the Law of Motion for Beliefs

In this appendix, we want to show two propositions.

**Proposition 7** *The rational difference equations (3) and (4) hold true*

**Proof.** Recall that equation (3) describes the dynamics of beliefs within Block 1. Consequently, this equation holds when  $\tau_t^1 > 1$ . The Bayes' theorem can be applied to characterize the probability of being in Regime 1 given that the system is in Block 1 ( $\tau_t^1 > 1$ ):

$$prob(\xi_t = 1 | \tau_t^1) = \frac{p(\tau_t^1 = \tau_{t-1}^1 + 1 | \xi_t = 1) p(\xi_t = 1 | \tau_{t-1}^1)}{\sum_{i=1}^4 p(\tau_t^1 = \tau_{t-1}^1 + 1 | \xi_t = i) p(\xi_t = i | \tau_{t-1}^1)}$$

But if  $\tau_t^1 = \tau_{t-1}^1 + 1$ , then the likelihood is such that

$$p(\tau_t^1 = \tau_{t-1}^1 + 1 | \xi_t = 1) = p(\tau_t^1 = \tau_{t-1}^1 + 1 | \xi_t = 2) > 0$$

and

$$p(\tau_t^1 = \tau_{t-1}^1 + 1 | \xi_t = 3) = p(\tau_t^1 = \tau_{t-1}^1 + 1 | \xi_t = 4) = 0$$

The equality in the first expression reflects the fact that agents cannot distinguish regimes belonging to the same block. The inequality sign in the first expression and the equality sign in the second expression are due to the fact that the system is in Block 1 at time  $t$ , ruling out the possibility that either Regime 3 or Regime 4 is realized. These results allow us to write:

$$prob(\xi_t = 1 | \tau_t^1) = \frac{p(\xi_t = 1 | \tau_{t-1}^1)}{\sum_{i=1}^2 p(\xi_t = i | \tau_{t-1}^1)}$$

Since  $p(\xi_t = i | \tau_{t-1}^1) = \sum_{j=1}^2 p(\xi_{t-1} = j | \tau_{t-1}^1) p_{ji}$ , then

$$prob(\xi_t = 1 | \tau_t^1) = \frac{\sum_{j=1}^2 p(\xi_{t-1} = j | \tau_{t-1}^1) p_{j1}}{\sum_{i=1}^2 \sum_{j=1}^2 p(\xi_{t-1} = j | \tau_{t-1}^1) p_{ji}}$$

Furthermore, note that  $p(\xi_{t-1} = 2 | \tau_{t-1}^1) = 1 - p(\xi_{t-1} = 1 | \tau_{t-1}^1)$  and after straightforward manipulations leads to equation (3). Equation (4) can be proved analogously. ■

## B Proof of Proposition 2

If (i)  $p_{11} + p_{12} - p_{21} - p_{22} \neq 0$ , (ii)  $p_{11}p_{22} \neq p_{21}p_{12}$ , (iii)  $p_{11} \neq p_{22}$  or both  $p_{12} \neq 0$  and  $p_{21} \neq 0$ , and the initial probability is such that  $\text{prob}(\xi_t = 1|\tau_t^1 = 1) \neq \lambda_1$ , then  $\text{prob}(\xi_t = 1|\tau_t^1) \rightarrow \lambda_2 \in [0, 1]$ . If conditions (i), (ii), and (iii) hold and the initial probability is such that  $\text{prob}(\xi_t = 1|\tau_t^1 = 1) = \tilde{\lambda}_1$ , then  $\text{prob}(\xi_t = 1|\tau_t^1) = \tilde{\lambda}_1$  for any  $\tau_t^1$ .

The difference equation (3) can be expressed as

$$\text{prob}(\xi_t = 1|\tau_t^1) = \frac{a \cdot \text{prob}(\xi_{t-1} = 1|\tau_{t-1}^1) + b}{c \cdot \text{prob}(\xi_{t-1} = 1|\tau_{t-1}^1) + d} \quad (28)$$

where

$$\begin{aligned} a &\equiv p_{11} - p_{21}, \quad b \equiv p_{21} \\ c &\equiv p_{11} + p_{12} - p_{21} - p_{22}, \quad d \equiv p_{21} + p_{22} \end{aligned}$$

Condition (i) ensures that the difference equation of interest is rational because it implies  $c > 0$ . In Appendices E and F, we will deal with the case of  $c = 0$ . We then proceed as follows. Denote  $\text{prob}(\xi_t = 1|\tau_t^1) + \frac{d}{c}$  as  $x_t$  and re-write the difference equation above as

$$x_t = \alpha - \frac{\beta}{x_{t-1}} \quad (29)$$

where

$$\begin{aligned} \alpha &\equiv \frac{p_{11} + p_{22}}{p_{11} + p_{12} - p_{21} - p_{22}} \\ \beta &\equiv \frac{p_{11}p_{22} - p_{21}p_{12}}{(p_{11} + p_{12} - p_{21} - p_{22})^2} \end{aligned}$$

Condition (ii) ensures that  $\beta \neq 0$ . The case of  $\beta = 0$  will be studied in Appendix D. The above equation can be reduced to a homogeneous linear difference equation by defining  $x_t = \varphi_t/\varphi_{t-1}$  where:

$$\varphi_t - \alpha\varphi_{t-1} + \beta\varphi_{t-2} = 0 \quad (30)$$

If  $\lambda_1$  and  $\lambda_2$  are the solutions of the characteristic equation, namely  $\frac{1}{2}\alpha \pm \frac{1}{2}\sqrt{\alpha^2 - 4\beta}$ , then the general solution of (30) is

$$\varphi_t = C_1\lambda_1^t + C_2\lambda_2^t, \quad \text{if } \lambda_1 \neq \lambda_2 \quad (31)$$

$$\varphi_t = (C_1 + C_2t)\lambda_1^t, \quad \text{if } \lambda_1 = \lambda_2 \quad (32)$$

The general solution of (29) is then:

$$x_t = \frac{C_1\lambda_1^t + C_2\lambda_2^t}{C_1\lambda_1^{t-1} + C_2\lambda_2^{t-1}} \quad (33)$$

when  $C_2 = 0$ ,  $x_t = \lambda_1$  for all  $t$ . When  $C_1 = 0$ ,  $x_t = \lambda_2$  for all  $t$ . When neither  $C_1$  nor  $C_2$  is zero, then

$$x_t = \lambda_2 \frac{\left(\frac{\lambda_1}{\lambda_2}\right)^{t+1} + C}{\left(\frac{\lambda_1}{\lambda_2}\right)^t + C}, \quad C \neq 0 \quad (34)$$

Note that  $\alpha^2 \geq 4\beta$  is required for the characteristic roots  $\lambda_1$  and  $\lambda_2$  to be real. This condition is

$$\left[ \frac{p_{11} + p_{22}}{p_{11} + p_{12} - p_{21} - p_{22}} \right]^2 \geq 4 \frac{p_{11}p_{22} - p_{21}p_{12}}{(p_{11} + p_{12} - p_{21} - p_{22})^2}$$

and after simplifying

$$p_{11}^2 + p_{22}^2 + 2p_{11}p_{22} \geq 4p_{11}p_{22} - 4p_{21}p_{12}$$

Some straightforward manipulation leads us to

$$(p_{11} - p_{22})^2 \geq -4p_{21}p_{12} \quad (35)$$

From condition (iii), the inequality above is strict and the characteristic roots are unequal. The case in which the characteristic roots are identical is dealt with in Appendix C. Let  $|\lambda_2| > |\lambda_1|$  then  $|\lambda_1/\lambda_2|^t \rightarrow 0$  and (34) implies that  $x_t \rightarrow \lambda_2$  as long as  $x_1 \neq \lambda_1$ . The root with highest absolute value can be seen to be always  $\frac{p_{11}+p_{22}+\sqrt{(p_{11}-p_{22})^2+4p_{21}p_{12}}}{2(p_{11}+p_{12}-p_{21}-p_{22})}$ . Recall that  $x_t \equiv \text{prob}(\xi_t = 1|\tau_t^1) + \frac{d}{c}$ . After some straightforward algebraic manipulations we obtain:

$$\text{prob}(\xi_t = 1|\tau_t^1) \rightarrow \tilde{\lambda}_2 = \frac{p_{11} - p_{22} - 2p_{21} + \sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}}}{2(p_{11} + p_{12} - p_{21} - p_{22})}$$

where  $\tilde{\lambda}_2$  is the stable root for the variable of interest  $\text{prob}(\xi_t = 1|\tau_t^1)$ . The unstable root for  $\text{prob}(\xi_t = 1|\tau_t^1)$  can be easily seen to be:

$$\tilde{\lambda}_1 = \frac{p_{11} - p_{22} - 2p_{21} - \sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}}}{2(p_{11} + p_{12} - p_{21} - p_{22})}$$

We only need to show that  $\tilde{\lambda}_2 \in [0, 1]$ . We want to show that

$$\frac{p_{11} - p_{22} - 2p_{21} + \sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}}}{2(p_{11} + p_{12} - p_{21} - p_{22})} \geq 0$$

If  $p_{11} + p_{12} - p_{21} - p_{22} > 0$  and  $p_{11} - p_{22} - 2p_{21} \geq 0$ , then the statement is clearly true. When  $p_{11} + p_{12} - p_{21} - p_{22} > 0$  and  $p_{11} - p_{22} - 2p_{21} < 0$ , then

$$\sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}} \geq -(p_{11} - p_{22} - 2p_{21})$$

Since the right-hand side is positive we can square both sides of this equation:

$$\begin{aligned} (p_{11} - p_{22})^2 + 4p_{21}p_{12} &\geq (p_{11} - p_{22} - 2p_{21})^2 \\ 4p_{21}p_{12} &\geq 4p_{21}^2 - 4(p_{11} - p_{22})p_{21} \end{aligned}$$

If  $p_{21} = 0$ , the statement is true. If  $p_{21} > 0$

$$p_{12} - p_{21} + (p_{11} - p_{22}) \geq 0$$

which is true.

If  $p_{11} + p_{12} - p_{21} - p_{22} < 0$ , then  $p_{11} - p_{22} - 2p_{21} < 0$ . We need to show that

$$p_{11} - p_{22} - 2p_{21} \leq -\sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}}$$

Since both sides of the inequality are negative, then

$$(p_{11} - p_{22} - 2p_{21})^2 \geq (p_{11} - p_{22})^2 + 4p_{21}p_{12}$$

and after manipulating:

$$-4(p_{11} - p_{22})p_{21} + 4p_{21}^2 \geq 4p_{21}p_{12}$$

If  $p_{21} = 0$ , the inequality is obviously verified. If  $p_{21} > 0$ , then

$$0 \geq (p_{11} - p_{22}) + p_{12} - p_{21}$$

which is true.

We want to show that

$$\frac{p_{11} - p_{22} - 2p_{21} + \sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}}}{2(p_{11} + p_{12} - p_{21} - p_{22})} \leq 1$$

If  $p_{11} + p_{12} - p_{21} - p_{22} > 0$ , then after some manipulations

$$\sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}} \leq p_{11} + 2p_{12} - p_{22}$$

Note that  $p_{11} + 2p_{12} - p_{22} > p_{11} + p_{12} - p_{21} - p_{22} > 0$ . Hence, taking the square on both sides of the inequality yields:

$$(p_{11} - p_{22})^2 + 4p_{21}p_{12} \leq (p_{11} + 2p_{12} - p_{22})^2$$

and finally

$$4p_{21}p_{12} \leq 4p_{12}^2 + 4(p_{11} - p_{22})p_{12}$$

If  $p_{12} = 0$ , this is true. If  $p_{12} > 0$ , then

$$p_{21} \leq p_{12} + (p_{11} - p_{22})$$

which is true.

If  $p_{11} + p_{12} - p_{21} - p_{22} < 0$ , then after some manipulations

$$\sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}} \geq p_{11} + 2p_{12} - p_{22}$$

If  $p_{11} + 2p_{12} - p_{22} < 0$ , this inequality is obviously true. If  $p_{11} + 2p_{12} - p_{22} \geq 0$ , then

$$(p_{11} - p_{22})^2 + 4p_{21}p_{12} \geq (p_{11} + 2p_{12} - p_{22})^2$$

and then

$$4p_{21}p_{12} \geq 4p_{12}^2 + 4(p_{11} - p_{22})p_{12}$$

If  $p_{12} = 0$ , this is true. If  $p_{12} > 0$ , then

$$p_{21} \geq p_{12} + p_{11} - p_{22}$$

which is true.

## C Proof of Proposition 3

We want to show that if (i)  $p_{11} + p_{12} - p_{21} - p_{22} \neq 0$ , (ii)  $p_{11}p_{22} \neq p_{21}p_{12}$ , (iii)  $p_{11} = p_{22}$  and either  $p_{12} = 0$  or  $p_{21} = 0$ , then  $\text{prob}(\xi_t = 1|\tau_t^1) \rightarrow \tilde{\lambda}_1 = \tilde{\lambda}_2$  and the roots are either equal to zero (if  $p_{21} = 0$ ) or one (if  $p_{12} = 0$ ). This result follows from observing that condition (iii) implies that condition (35) delivers coincident characteristic roots  $\tilde{\lambda}_1$  and  $\tilde{\lambda}_2$ ; that is,

$$\tilde{\lambda}_1 = \tilde{\lambda}_2 = \frac{p_{21}}{p_{21} - p_{12}}$$

If  $p_{12} = 0$ , then  $\text{prob}(\xi_t = 1|\tau_t^1) \rightarrow \tilde{\lambda}_1 = \tilde{\lambda}_2 = 1$ . If  $p_{21} = 0$ , then  $\text{prob}(\xi_t = 1|\tau_t^1) \rightarrow \tilde{\lambda}_1 = \tilde{\lambda}_2 = 0$ .

## D Proof of Proposition 4

We want to show that if (i)  $p_{11} + p_{12} - p_{21} - p_{22} \neq 0$ , (ii)  $p_{11}p_{22} = p_{21}p_{12}$ , then  $\text{prob}(\xi_t = 1|\tau_t^1) = \frac{p_{11} - p_{21}}{p_{11} + p_{12} - p_{21} - p_{22}}$ . Condition (ii) implies  $\beta = 0$  in equation (29) and hence (using the notation introduced in Appendix B)

$$x_t = \alpha \equiv \frac{p_{11} + p_{22}}{p_{11} + p_{12} - p_{21} - p_{22}}$$

From Appendix B, recall that  $x_t = \text{prob}\{\xi_t = 1|\tau_t^1\} + d/c$ , then it follows that

$$\text{prob}\{\xi_t = 1|\tau_t^1\} = \frac{p_{11} - p_{21}}{p_{11} + p_{12} - p_{21} - p_{22}}.$$

## E Proof of Proposition 5

We want to show that if (i)  $p_{11} + p_{12} - p_{21} - p_{22} = 0$  and (ii)  $p_{11} \neq p_{21}$ , then  $\text{prob}(\xi_t = 1|\tau_t^1) \rightarrow \frac{p_{21}}{p_{22} - p_{11} + 2p_{21}}$ , with  $\frac{p_{21}}{p_{22} - p_{11} + 2p_{21}} \in [0, 1]$ . If  $p_{11} + p_{12} - p_{21} - p_{22} = 0$ , then  $c = 0$  in the difference equation (28), which hence boils down to the first-order *linear* difference equation below:

$$\text{prob}(\xi_t = 1|\tau_t^1) = \frac{a}{d} \cdot \text{prob}(\xi_{t-1} = 1|\tau_{t-1}^1) + \frac{b}{d} \quad (36)$$

where  $a = p_{11} - p_{21}$ ,  $b = p_{21}$ ,  $d = p_{21} + p_{22}$ . Stability is ensured by  $|\frac{a}{d}| = \left| \frac{p_{11} - p_{21}}{p_{21} + p_{22}} \right| < 1$ . First note that the *background assumption* A1 combined with condition (i) implies that  $d \neq 0$  and hence the ratio  $|\frac{a}{d}|$  is well-defined. Condition (ii) rules out the possibility that the ratio  $|\frac{a}{d}|$  is zero. We consider this case in Appendix F.

The condition  $p_{11} + p_{12} - p_{21} - p_{22} = 0$  allows us to re-write the stability condition  $|\frac{a}{d}| = \left| \frac{p_{11} - p_{21}}{p_{21} + p_{22}} \right|$  as  $\left| \frac{p_{11} - p_{21}}{p_{11} + p_{12}} \right|$ . Hence, showing that  $p_{12} + p_{21} > 0$  implies stability. Recall that the *background assumption* A2 requires that either  $p_{11} \neq p_{22}$  or  $p_{12} \neq p_{21}$ . If the latter condition is satisfied, then  $p_{12} + p_{21} > 0$  trivially follows. If the latter condition is not satisfied, then it must be that  $p_{11} \neq p_{22}$ , which, combined with condition (i), implies that  $p_{12} + p_{21} > 0$ .

It is easy to see that the difference equation (36) implies that  $\text{prob}(\xi_t = 1|\tau_t^1) \rightarrow \frac{b}{d} (1 - \frac{a}{d})^{-1}$ , that is,

$$\text{prob}(\xi_t = 1|\tau_t^1) \rightarrow \frac{p_{21}}{p_{21} + p_{22}} \left( 1 - \frac{p_{11} - p_{21}}{p_{21} + p_{22}} \right)^{-1}$$



After easy algebraic manipulations

$$\text{prob}(\xi_t = 1 | \tau_t^1) \rightarrow \frac{p_{21}}{p_{22} - p_{11} + 2p_{21}}.$$

Note that

$$0 \leq \frac{p_{21}}{p_{22} - p_{11} + 2p_{21}} \leq 1$$

To see that, recall that in this case,  $p_{11} + p_{12} - p_{21} - p_{22} = 0$ , implying that  $p_{22} - p_{11} = p_{12} - p_{21}$ . Substituting this result into the inequalities above yields

$$0 \leq \frac{p_{21}}{p_{12} + p_{21}} \leq 1$$

which is clearly verified.

## F Proof of Proposition 6

We want to show that if (i)  $p_{11} + p_{12} - p_{21} - p_{22} = 0$ , (ii)  $p_{11} = p_{21}$ , then  $\text{prob}(\xi_t = 1 | \tau_t^1) = \frac{p_{21}}{p_{22} + p_{21}}$ . Condition (i) implies that  $c = 0$  in the difference equation (28), which hence boils down to the first-order linear difference equation below:

$$\text{prob}(\xi_t = 1 | \tau_t^1) = \frac{a}{d} \cdot \text{prob}(\xi_{t-1} = 1 | \tau_{t-1}^1) + \frac{b}{d} \quad (37)$$

where  $a = p_{11} - p_{21}$ ,  $b = p_{21}$ ,  $d = p_{21} + p_{22}$ . Condition (ii) implies that  $a = 0$  and hence  $\text{prob}(\xi_t = 1 | \tau_t^1) = b/d = p_{21}/(p_{21} + p_{22})$ .

## G Admissible Region for the Unstable Root

Recall that

$$\tilde{\lambda}_1 \equiv \frac{p_{11} - p_{22} - 2p_{21} - \sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}}}{2(p_{11} + p_{12} - p_{21} - p_{22})}$$

We want to show that  $0 \leq \tilde{\lambda}_1 \leq 1$ . This claim is implied by the following two Lemmas.

**Lemma 8** *If  $p_{11} + p_{12} - p_{21} - p_{22} > 0$ , then  $\tilde{\lambda}_1 \leq 0$ .*

**Proof.** We want to show that

$$\frac{p_{11} - p_{22} - 2p_{21} - \sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}}}{2(p_{11} + p_{12} - p_{21} - p_{22})} \leq 0$$

If  $p_{11} + p_{12} - p_{21} - p_{22} > 0$ , then the above implies

$$p_{11} - p_{22} - 2p_{21} \leq \sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}}$$

Note that the *background assumption* A3 excludes that  $p_{11} - p_{22} - 2p_{21} = 0$ . Hence there are two possible cases left: (a) if  $p_{11} - p_{22} - 2p_{21} < 0$ , then the above is true; (b) if  $p_{11} - p_{22} - 2p_{21} > 0$ , then we can take the square on both sides of the above equation to get

$$(p_{11} - p_{22} - 2p_{21})^2 \leq (p_{11} - p_{22})^2 + 4p_{21}p_{12}$$

Straightforward manipulations lead to

$$p_{21}^2 - p_{11}p_{21} + p_{22}p_{21} \leq p_{21}p_{12}$$

If  $p_{21} = 0$ , then the above is true. Otherwise, we can divide both sides of the above inequality by  $p_{21}$  to get

$$p_{11} + p_{12} - p_{21} - p_{22} \geq 0$$

that is obviously true because  $p_{11} + p_{12} - p_{21} - p_{22} > 0$ . ■

**Lemma 9** *If  $p_{11} + p_{12} - p_{21} - p_{22} < 0$ , then  $\tilde{\lambda}_1 \geq 1$ .*

**Proof.** We want to show that

$$\frac{p_{11} - p_{22} - 2p_{21} - \sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}}}{2(p_{11} + p_{12} - p_{21} - p_{22})} \geq 1$$

Since  $p_{11} + p_{12} - p_{21} - p_{22} < 0$ , the above implies

$$p_{11} - p_{22} - 2p_{21} - \sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}} \leq 2(p_{11} + p_{12} - p_{21} - p_{22})$$

and after simplifying

$$-\sqrt{(p_{11} - p_{22})^2 + 4p_{21}p_{12}} \leq p_{11} - p_{22} + 2p_{12}$$

Note that the *background assumption A3* excludes that  $p_{11} - p_{22} + 2p_{21} = 0$ . If  $p_{11} - p_{22} + 2p_{12} > 0$ , the above is obviously true. If  $p_{11} - p_{22} + 2p_{12} < 0$ , then taking the square on both sides

$$(p_{11} - p_{22})^2 + 4p_{21}p_{12} \geq (p_{11} - p_{22} + 2p_{12})^2$$

After some manipulations:

$$p_{12} + p_{11} - p_{12} - p_{22} \leq 0$$

that is obviously true because  $p_{11} + p_{12} - p_{21} - p_{22} < 0$ . ■

## H Algorithm for the Case with Signals

**Algorithm** Set  $i = 1$  and initialize the matrix  $\hat{\mathcal{P}} = \mathbf{0}_{g \times g}$

**Step 1** Find  $j_1 \leq g_1$  and  $j_2 \leq g_1$  so as to min  $|\text{prob}\{\xi_t = 1 | \mathcal{I}_t, \varpi^{t-1}, \varpi_t = q\} - \mathcal{G}_{j_q}|$  with  $q \in \{1, 2\}$  where

$$\text{prob}(\xi_t = 1 | \mathcal{I}_t, \varpi^{t-1}, \varpi_t = q) = \frac{\text{prob}(\varpi_t = q | \xi_t = 1) \text{prob}(\xi_t = 1 | \mathcal{I}_t, \varpi^{t-1})}{\sum_{j=1}^2 \text{prob}(\varpi_t = q | \xi_t = j) \text{prob}(\xi_t = j | \mathcal{I}_t, \varpi^{t-1})}, \quad q \in \{1, 2\} \quad (38)$$

and agents' beliefs about being in Regime 1 before observing the signal read:

$$\text{prob}(\xi_t = 1 | \mathcal{I}_t, \varpi^{t-1}) = \frac{\text{prob}(\xi_{t-1} = 1 | \mathcal{I}_{t-1}, \varpi^{t-1}) (p_{11} - p_{21}) + p_{21}}{\text{prob}(\xi_{t-1} = 1 | \mathcal{I}_{t-1}, \varpi^{t-1}) (p_{11} + p_{12} - p_{21} - p_{22}) + p_{21} + p_{22}} \quad (39)$$

using the approximation  $\text{prob}\{\xi_{t-1} = 1 | \mathcal{I}_{t-1}, \varpi^{t-1}\} = \mathcal{G}_i$ . To ensure convergence of beliefs, we correct  $j_1$  and  $j_2$  as follows. If  $j_q = i$  and  $\mathcal{G}_i \neq \tilde{\lambda}_2$  ( $q \in \{1, 2\}$ ), then set  $j_q = j_q + 1$  if  $\mathcal{G}_i < \tilde{\lambda}_2$  and  $j_q = \max(1, j_q - 1)$  if  $\mathcal{G}_i > \tilde{\lambda}_2$ .

**Step 2** Setting  $prob(\xi_{t-1} = 1|\mathcal{I}_{t-1}, \varpi^{t-1}) = \mathcal{G}_i$ , the (ex-ante) transition probability can be computed as:

$$\widehat{\mathcal{P}}(i, j_q) = \sum_{v=1}^2 prob\{\xi_t = v|\mathcal{I}_{t-1}, \varpi^{t-1}\} prob\{\varpi_t = q|\xi_t = v\}, \quad q \in \{1, 2\} \quad (40)$$

where

$$prob\{\xi_t = v|\mathcal{I}_{t-1}, \varpi^{t-1}\} = \sum_{u=1}^2 prob\{\xi_{t-1} = u|\mathcal{I}_{t-1}, \varpi^{t-1}\} p_{uv} \quad (41)$$

**Step 3** Find  $j_1 > g_1$  and  $j_2 > g_1$  so as to min  $|prob\{\xi_t = 3|\mathcal{I}_t, \varpi^{t-1}, \varpi_t = q\} - \mathcal{G}_{j_q}|$  with  $q \in \{1, 2\}$ , where

$$prob(\xi_t = 3|\mathcal{I}_t, \varpi^{t-1}, \varpi_t = q) = \frac{prob(\varpi_t = q|\xi_t = 3) prob(\xi_t = 3|\mathcal{I}_t, \varpi^{t-1})}{\sum_{j=3}^4 prob(\varpi_t = q|\xi_t = j) prob(\xi_t = j|\mathcal{I}_t, \varpi^{t-1})}, \quad q \in \{1, 2\}$$

and the beliefs about being in Regime 3 upon the shift to Block 2 (before having observed the signal  $\varpi_t$ ) are given by:

$$\begin{aligned} prob\{\xi_t = 3|\mathcal{I}_t, \varpi^{t-1}\} &= \frac{\sum_{j \in b_1} prob\{\xi_{t-1} = j|\mathcal{I}_{t-1}, \varpi^{t-1}\} p_{j3}}{\sum_{i \in b_2} \sum_{j \in b_1} prob\{\xi_{t-1} = j|\mathcal{I}_{t-1}, \varpi^{t-1}\} p_{ji}} \\ &= \frac{prob\{\xi_{t-1} = 1|\mathcal{I}_{t-1}, \varpi^{t-1}\} p_{13} + (1 - prob\{\xi_{t-1} = 1|\mathcal{I}_{t-1}, \varpi^{t-1}\}) p_{23}}{prob\{\xi_{t-1} = 1|\mathcal{I}_{t-1}, \varpi^{t-1}\} (p_{13} + p_{14}) + (1 - prob\{\xi_{t-1} = 1|\mathcal{I}_{t-1}, \varpi^{t-1}\}) (p_{23} + p_{24})} \end{aligned}$$

using the approximation that  $prob(\xi_{t-1} = 1|\mathcal{I}_{t-1}, \varpi^{t-1}) = \mathcal{G}_i$ . Setting  $prob(\xi_{t-1} = 1|\mathcal{I}_{t-1}, \varpi^{t-1}) = \mathcal{G}_i$ , the (ex-ante) transition probabilities as

$$\widehat{\mathcal{P}}(i, j_q) = \widehat{\mathcal{P}}(i, j_q) + \sum_{v=3}^4 \left( \sum_{u=1}^2 prob\{\xi_{t-1} = u|\mathcal{I}_{t-1}, \varpi^{t-1}\} p_{uv} \right) prob\{\varpi_t = q|\xi_t = v\}, \quad q \in \{1, 2\} \quad (42)$$

**Step 4** If  $i = g_1$  then set  $i = i + 1$  and go to step 6; otherwise, set  $i = i + 1$  and go to step 1.

**Step 5** Find  $j_1 > g_1$  and  $j_2 > g_1$  so as to min  $|prob\{\xi_t = 3|\mathcal{I}_t, \varpi^{t-1}, \varpi_t = q\} - \mathcal{G}_{j_q}|$  with  $q \in \{1, 2\}$  where

$$prob(\xi_t = 3|\mathcal{I}_t, \varpi^{t-1}, \varpi_t = q) = \frac{prob(\varpi_t = q|\xi_t = 3) prob(\xi_t = 3|\mathcal{I}_t, \varpi^{t-1})}{\sum_{j=3}^4 prob(\varpi_t = q|\xi_t = j) prob(\xi_t = j|\mathcal{I}_t, \varpi^{t-1})}, \quad q \in \{1, 2\}$$

and agents' beliefs about being in Regime 3 before observing the signal read:

$$prob(\xi_t = 3|\mathcal{I}_t, \varpi^{t-1}) = \frac{prob(\xi_{t-1} = 3|\mathcal{I}_{t-1}, \varpi^{t-1}) (p_{33} - p_{43}) + p_{43}}{prob(\xi_{t-1} = 3|\mathcal{I}_{t-1}, \varpi^{t-1}) (p_{33} + p_{34} - p_{43} - p_{44}) + p_{43} + p_{44}} \quad (43)$$

using the approximation  $prob\{\xi_{t-1} = 3|\mathcal{I}_{t-1}, \varpi^{t-1}\} = \mathcal{G}_i$ . To ensure convergence of beliefs, we correct  $j_1$  and  $j_2$  as follows. If  $j_q = i$  and  $\mathcal{G}_i \neq \tilde{\lambda}_4$  ( $q \in \{1, 2\}$ ), then set  $j_q = \min(j_q + 1, g)$  if  $\mathcal{G}_i < \tilde{\lambda}_4$  and  $j_q = j_q - 1$  if  $\mathcal{G}_i > \tilde{\lambda}_4$ .

**Step 6** Setting  $prob(\xi_{t-1} = 3|\mathcal{I}_{t-1}, \varpi^{t-1}) = \mathcal{G}_i$ , the (ex-ante) transition probability can be computed as:

$$\widehat{\mathcal{P}}(i, j_q) = \widehat{\mathcal{P}}(i, j_q) + \sum_{v=3}^4 \left( \sum_{u=3}^4 prob\{\xi_{t-1} = u|\mathcal{I}_{t-1}, \varpi^{t-1}\} p_{uv} \right) prob\{\varpi_t = q|\xi_t = v\}, \quad q \in \{1, 2\} \quad (44)$$

**Step 7** Find  $j_1 \leq g_1$  and  $j_2 \leq g_1$  so as to min  $|\text{prob}\{\xi_t = 1|\mathcal{I}_t, \varpi^{t-1}, \varpi_t = q\} - \mathcal{G}_{j_q}|$  with  $q \in \{1, 2\}$ , where

$$\text{prob}(\xi_t = 1|\mathcal{I}_t, \varpi^{t-1}, \varpi_t = q) = \frac{\text{prob}(\varpi_t = q|\xi_t = 1)\text{prob}(\xi_t = 1|\mathcal{I}_t, \varpi^{t-1})}{\sum_{j=1}^2 \text{prob}(\varpi_t = q|\xi_t = j)\text{prob}(\xi_t = j|\mathcal{I}_t, \varpi^{t-1})}, \quad q \in \{1, 2\}$$

and the beliefs about being in Regime 1 upon the shift to Block 1 (before having observed the signal  $\varpi_t$ ) are given by:

$$\begin{aligned} \text{prob}\{\xi_t = 1|\mathcal{I}_t, \varpi^{t-1}\} &= \frac{\sum_{j \in b_2} \text{prob}\{\xi_{t-1} = j|\mathcal{I}_{t-1}, \varpi^{t-1}\} p_{j1}}{\sum_{i \in b_1} \sum_{j \in b_2} \text{prob}\{\xi_{t-1} = j|\mathcal{I}_{t-1}, \varpi^{t-1}\} p_{ji}} \\ &= \frac{\text{prob}\{\xi_{t-1} = 3|\mathcal{I}_{t-1}, \varpi^{t-1}\} p_{31} + (1 - \text{prob}\{\xi_{t-1} = 3|\mathcal{I}_{t-1}, \varpi^{t-1}\}) p_{41}}{\text{prob}\{\xi_{t-1} = 3|\mathcal{I}_{t-1}, \varpi^{t-1}\} (p_{31} + p_{32}) + (1 - \text{prob}\{\xi_{t-1} = 3|\mathcal{I}_{t-1}, \varpi^{t-1}\}) (p_{41} + p_{42})} \end{aligned}$$

using the approximation that  $\text{prob}(\xi_{t-1} = 3|\mathcal{I}_{t-1}, \varpi^{t-1}) = \mathcal{G}_{g_1+i}$ . Setting  $\text{prob}(\xi_{t-1} = 3|\mathcal{I}_{t-1}, \varpi^{t-1}) = \mathcal{G}_i$ , the (ex-ante) transition probability can be computed as:

$$\widehat{\mathcal{P}}(i, j_q) = \widehat{\mathcal{P}}(i, j_q) + \sum_{v=1}^2 \left( \sum_{u=3}^4 \text{prob}\{\xi_{t-1} = u|\mathcal{I}_{t-1}, \varpi^{t-1}\} p_{uv} \right) \text{prob}\{\varpi_t = q|\xi_t = v\}, \quad q \in \{1, 2\} \quad (45)$$

**Step 8** If  $i = g$ , then go to step 9; otherwise, set  $i = i + 1$  and go to step 5.

**Step 9** If no column of  $\widehat{\mathcal{P}}$  has all zero elements, then stop. Otherwise, go to step 10.

**Step 10** Construct the matrix  $T$  as follows. Set  $j = 1$  and  $l = 1$ . While  $j \leq g$ , if  $\sum_{i=1}^g \widehat{\mathcal{P}}(i, j) \neq 0$  then do three things: (1) set  $T(j, l) = 1$ , (2) set  $T(j, v) = 0$  for any  $1 \leq v \leq g$  and  $v \neq l$ , (3) set  $l = l + 1$  and (4) set  $j = j + 1$ ; otherwise (i.e., if  $\sum_{i=1}^g \widehat{\mathcal{P}}(i, j) = 0$ ), set  $j = j + 1$ .

**Step 11** Write the transition equation as  $\widehat{\mathcal{P}}^R = T \cdot \widehat{\mathcal{P}} \cdot T'$ . If no column of  $\widehat{\mathcal{P}}^R$  has all zero elements, set  $\widehat{\mathcal{P}} = \widehat{\mathcal{P}}^R$  and stop. Otherwise, go to step 10.

## I Log-Linearization of the RBC Model

Solving the problem of the representative household in Section 3 leads to:

$$c_t^{-1} = \beta \widetilde{E}_t c_{t+1}^{-1} [\alpha z_{t+1} k_t^{\alpha-1} + 1 - \delta] \quad (46)$$

$$c_t + k_t = z_t k_{t-1}^{\alpha} + (1 - \delta) k_{t-1} \quad (47)$$

The stochastic process of TFP (19) and equations (46)-(47) imply that consumption and capital are non-stationary. Denote the stationary variables  $\widetilde{c}_t \equiv c_t/z_t^{(1-\alpha)^{-1}}$ ,  $\widetilde{k}_t \equiv k_t/z_t^{(1-\alpha)^{-1}}$ ,  $\mu_t \equiv \ln(z_t/z_{t-1})$ , and  $M_t \equiv z_t/z_{t-1}$  as the gross growth rate of TFP. The stationary version of the model reads:

$$\widetilde{c}_t^{-1} = \beta \widetilde{E}_t \widetilde{c}_{t+1}^{-1} M_{t+1}^{\frac{1}{\alpha-1}} [\alpha M_{t+1} \widetilde{k}_t^{\alpha-1} + 1 - \delta] \quad (48)$$

$$\widetilde{c}_t + \widetilde{k}_t = M \widetilde{k}_{t-1}^{\alpha} + (1 - \delta) M_t^{\frac{1}{\alpha-1}} \widetilde{k}_{t-1} \quad (49)$$

Following Schorfheide (2005) and Liu, Waggoner, and Zha (2011), we define a steady-state equilibrium for the stationary consumption  $\widetilde{c}_t$  and capital  $\widetilde{k}_t$  when  $\varepsilon_t = 0$  all  $t$  and the growth rate of TFP

is at its ergodic value  $\bar{\mu}$ . The steady-state equilibrium level of consumption  $c_{ss}$  and capital  $k_{ss}$  is:

$$k_{ss} = \left[ \frac{1}{\alpha \bar{M}} \left( \frac{\bar{M}^{\frac{1}{1-\alpha}}}{\beta} - 1 + \delta \right) \right]^{\frac{1}{\alpha-1}} \quad (50)$$

$$c_{ss} = \bar{M}^{\frac{\alpha}{\alpha-1}} k_{ss}^{\alpha} + \left[ (1-\delta) \bar{M}^{\frac{1}{\alpha-1}} - 1 \right] k_{ss} \quad (51)$$

where  $\bar{M} \equiv \exp(\bar{\mu})$ ,  $\bar{\mu} \equiv (\bar{p}_1 + \bar{p}_2) \mu_H + (\bar{p}_3 + \bar{p}_4) \mu_L$  is the ergodic mean of the log growth rate of the economy, and  $\bar{p}_i$  stands for the ergodic probability of being in Regime  $i$ .

Taking the log-linear approximation of equations (48)-(49) around the steady-state equilibrium (50)-(51) leads to

$$\hat{c}_t = \tilde{E}_t \hat{c}_{t+1} - (\alpha - 1) \left( 1 + (\delta - 1) \beta \bar{M}^{\frac{1}{\alpha-1}} \right) \hat{k}_t - \left( \frac{1}{\alpha - 1} + \beta \bar{M}^{\frac{1}{\alpha-1}} (\delta - 1) + 1 \right) \tilde{E}_t \hat{\mu}_{t+1}$$

where we use the fact that  $\beta \bar{M}^{\frac{1}{\alpha-1}} (\alpha \bar{M} k_{ss}^{\alpha-1} + 1 - \delta) = 1$  from equation (50) and  $\hat{\mu}_t \equiv \mu_t - \bar{\mu}$  is the log-deviation of the growth rate of TFP from its ergodic mean  $\bar{\mu}$ .  $\hat{c}_t$  and  $\hat{k}_t$  denote log-deviations of the stationary consumption and capital, respectively, from their steady-state value, and  $\hat{\mu}_t(\xi_t) \equiv \mu_t(\xi_t) - \bar{\mu}$  is the log-deviation of the TFP drift from its ergodic mean  $\bar{\mu}$ . The resource constraint is

$$c_{ss} \hat{c}_t + k_{ss} \hat{k}_t = \left( \bar{M}^{\frac{\alpha}{\alpha-1}} k_{ss}^{\alpha} \frac{\alpha}{\alpha-1} + \frac{1-\delta}{\alpha-1} \bar{M}^{\frac{1}{\alpha-1}} k_{ss} \right) \hat{\mu}_t + \left( \bar{M}^{\frac{\alpha}{\alpha-1}} k_{ss}^{\alpha} \alpha + (1-\delta) \bar{M}^{\frac{1}{\alpha-1}} k_{ss} \right) \hat{k}_{t-1}$$

and the log-deviations of the growth rate of TFP from its ergodic level follows

$$\hat{\mu}_t = \hat{\mu}_t(\xi_t) + \sigma_z \varepsilon_t. \quad (52)$$