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"Auctions, Second Version"

by

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Simple Auctions*

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Abstract

Standard Bayesian models assume agents know and fully exploit prior distributions over types. We are interested in modeling agents who lack detailed knowledge of prior distributions.

In auctions, that agents know priors has two consequences: (i) signals about own valuation come with precise inference about signals received by others; (ii) noisier estimates translate into more weight put on priors.

We revisit classic questions in auction theory, exploring environments in which no such complex inferences are precluded. This is done in a parsimonious model of auctions in which agents are restricted to using simple strategies.

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1 Introduction

I might go to an auction at Sotheby's, see a painting and decide that my value for a painting is \$1100, but have little idea of where my valuation stands in relation to the valuations of other bidders. That is, I might think it is equally likely that my valuation is the highest, second highest, or lowest; in other words, my valuation gives me little guidance in predicting my rank in the valuations. Had my value of the painting been \$1200 rather than \$1100 I might not think it any more likely that my valuation was the highest: whatever made my value higher might well have made other bidders' values higher as well, and I might find it impossible to sort out whether the basis of my higher value was common to all bidders or was idiosyncratic to me.

We are interested in considering auction models that reflect the above difficulty – that of using personal valuation as an instrument in predicting rank.[1]

We propose a model in which a bidder is limited in his ability to exploit the signals that he gets: his behavior is driven by own welfare considerations, *but only to a limited extent.* That is, we depart from a standard auction model in that we put restrictions on the set of rules that the agent considers. Without restriction, optimal bidding would result in behavior that is finely tuned to the particular joint distribution over valuations that the analyst assumes, and personal valuation would become a precise instrument to estimate how one's valuation compares to others.

The particular restriction that we analyze is that bidders apply constant shading to their value estimates: shading reflects a degree of caution, and caution is not adjusted to each particular value realization, but only on average across value estimate realizations. Caution is by assumption the same across all realizations, reflecting the idea that the agent cannot determine whether his value estimate realization is high or not, or that there is no natural reference point to which it can be compared.[2]

[1] Of course, some signals might be good instruments: If I see another participant at the painting auction arriving with a \$4000 suit, that may be a strong indication that my rank is low. Our view however is that personal valuation is often not a useful instrument to estimate rank, and the model we propose will reflect that.

[2] Our approach is similar to that explored in Compte and Postlewaite (2012). In that paper, the agent gets an estimate of some underlying state s, say $x = s + \varepsilon$. Ideally, the agent would like to know his estimation error ε, but we should not expect him to be able

This paper is part of a broader project that attempts to model agents who *lack detailed knowledge of prior distributions*. Following Wilson (1987)'s critique, a number of authors have questioned the standard perspective that agents would have precise knowledge of prior distributions. Attempts to weaken that perspective generally follow the traditional Bayesian route that takes anything that is not known as a random variable, assuming a known prior over priors and enriching the type spaces with additional private signals.[3] Strategy restrictions constitute another route in that they have the effect of limiting agents' ability to finely adjust their behavior to details of the environment that they cannot plausibly know.

An alternative interpretation of the model is that agents are *boundedly rational*. Compared to many traditional bounded rationality models that often assume that agents make evaluation errors in comparing alternatives, we assume that agents do not make errors in comparing bid functions.[4] Rather, they are simply assumed to not compare *all* bid functions.[5]

The model proposed yields a parsimonious model of auctions because an agent's behavior is characterized by a single parameter rather than a function. Standard issues (existence, revenue comparisons, comparative statics) can be addressed within the model. It thus provides insights on these issues that complement standard intuitions.

The model also permits disentangling valuation related information from rank related information, since in equilibrium, valuation is not used by

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to make precise inferences about the size or sign of ε based on the estimate x he forms.

[3]This is the route taken by the robustness in mechanism design literature (Bergemann Morris, 2005), as well as the global game literature (See Morris and Shin, 2006 for a survey). The underlying motivation is to avoid predictions that would be overly sensitive to common knowledge assumptions (as in Rubinstein 1989's email game). The technique consists of "relaxing" common knowledge by adding further private signals. (In the context of auctions, see Fang and Morris, 2006, where the additional signals permit breaking the one-to one relationship between value and beliefs).

These papers, however, assume common knowledge of the distribution over this enriched type space, making the agent's optimization ever more sophisticated.

[4]The usual bounded rationality route consists of assuming one particular choice rule – in general subjective expected utility maximization, based on possibly mistaken evaluations. The source of the errors may be left exogenous (Block and Marchak, 1960), procedural (Osborne and Rubinstein, 1998), or stem from the way beliefs are formed (Geanakoplos, 1989, Gilboa and Schmeidler, 1995, Jehiel, 2005).

[5]In that sense, the route we take is closest to the one that consists of restricting the strategy space to finite automata in repeated games (Aumann, 1981).

agents to make rank related inferences.

More generally, that agents know priors in standard auction models has two consequences: (i) signals received come with a precise inference about signals received by others, implying that optimal shading behavior depends finely on one's inference about others' types; (ii) noisier estimates translate into more weight put on priors: some "regression to the mean" has to take place. A contribution of this paper is to explore environments in which such complex inferences are assumed not made.

Why is this interesting? Apart from providing insights for auctions played by only mildly sophisticated bidders, it permits one to examine the extent to which standard insights are driven by the modelling strategy that assumes known priors.

To illustrate, the second type of inference (ii) implies that in a standard Bayesian model, a more "poorly informed" bidder (i.e., with noisy value estimates) has more concentrated posterior beliefs, hence, from the perspective of other players, a more predictable behavior (unless he uses a mixed strategy). The consequences are numerous: rents are easier to extract from poorly informed agents; ignorance promotes competition (between few symmetric agents) (Ganuza, 2004); in competing with a more informed agent, the less informed agent gets no rents (Milgrom, 1981). It also implies that for an uninformed agent, one benefit of (publicly) getting more precise information is to make behavior less predictable. For sellers, incentives to provide information thus have to balance efficiency gains with the additional rents that less predictable behavior generates (Bergemann and Pesendorfer, 2007).[6]

In many problems however, there may be no obvious reference point that would justify that a more poorly informed agent behaves more predictably. In the absence of such a reference point, or if agents lack detailed knowledge of the joint distribution over value and value estimates, it is likely that the opposite will be true, that is, that from the perspective of other players, behavior of more poorly informed agents is *less* predictable.

Finally, a contribution of the paper is to propose a theory that starts

[6] These remarks apply more generally, to other strategic situations. In Kamenica and Gentzkow (2011) for example, information transmission generates dispersion in the receiver's posteriors, explaining why the sender does not benefit from persuasion when his value function (expressed as a function of the receiver's beliefs) is concave.

from *cognitively* less demanding models, and subsequently investigating the consequences of increased sophistication, adding coarse rank related information to the model for example. We see two benefits: comparative statics with respect to sophistication, and a careful consideration of the cognitive demands that are implicit in standard models. For example, to the analyst, the independent private value model seems relatively simple. Cognitively however, the model is quite demanding: it requires the agent to combine in subtle ways information about rank and dispersion based on each possible value realization he may have.

Section 2 starts by describing the bidder's *decision problem*, and introduces our main modelling assumption (strategy restrictions). Section 3 moves to the analysis of a simple first price auction with perfect value estimates. Existence, revenue comparisons, and comparative statics results are obtained. Section 4 incorporates additional private signals into agents' bidding decisions, showing that providing rank information has an ambiguous effect on sellers' revenue. Section 5 introduces noisy estimates. Section 6 provides a discussion and relates results to the literature. Section 7 concludes. Appendices A and B extend the analysis to several additional standard questions: buyer-seller relationships and comparison of discriminatory and uniform auctions (with unitary demands).[7]

2 The bidder's decision problem

We consider first price auctions and start by describing the *decision problem* that a given agent faces. In any first price auction, the agent chooses a bid, denoted b. That agent wins if and only if b exceeds some price p. Letting v denote his valuation for the object being sold, he obtains $v - p$ if he wins, and 0 otherwise. The agent may be unsure about his valuation v and the threshold price p that will win the object. We can think of the pair (v, p) as a state, and the agent's preferences over his possible alternatives (the bids b) depend on this state; they are characterized by:

$$
\begin{aligned}
u(b, v, p) &= v - b \text{ if } b \geq p \\
&= 0 \text{ otherwise.}
\end{aligned}
$$

[7] The working paper (Compte and Postlewaite, 2010) includes extensions to asymmetric auctions, auctions for bundles and sequential auctions.

While the agent faces uncertainty in that he does not know the state (v, p), he has some imperfect knowledge of it. This is modelled by assuming that he gets *data*, $z \in Z$ correlated with the true state (v, p). The joint distribution over state and data is denoted ω.[8]

The "data" z may take various forms: we define it to be a signal (or set of signals) that the agent may condition his behavior on. In our basic auction model we assume perfect estimates, that is, $z = v$. More generally the data z may consist of a noisy estimate of v. Formally, we define

$$z = v + \varepsilon$$

where ε is a noise term independent of v.[9]

Plausible bidding rules. We define a family of bidding rules that depend on z, parameterized by a scalar γ. When z is a noisy estimate of v, we define rule r_γ as:

$$r_\gamma(z) \equiv z - \gamma.$$

The parameter γ characterizes how *conservative* the agent is (with higher γ connoting more conservative). It may also be interpreted as a level of cautiousness (if the agent fears that $z > v$). Obviously, one could imagine other ways to parameterize conservativeness or cautiousness, and we certainly do not wish to argue in favor of a specific shape. Our main motivation lies not in the shape of the rules considered, but in the assumption that the agent is unable to adjust γ to each particular realization of z, reflecting the idea that z is not (or cannot be) used as an instrument to adjust how conservative or cautious one ought to be.

Each rule r_γ induces an expected utility (or performance)

$$v_\omega(r) = E_\omega u(r(z), s).$$

Throughout most of the paper, we shall assume that the agent finds the optimal rule within the set:

$$R = \{r_\gamma\}_{\gamma \in \mathcal{R}}.$$

[8] When analyzing an auction with several strategic bidders, the distribution ω is endogenous. For now however, we shall keep it exogenous.

[9] In Section 4, we shall consider the case where bidders also receive a signal correlated with rank (e.g.,. the \$4000 suit – see footnote 1), that is

$$z = (v, k)$$

where $k \in \{H, L\}$ is a signal (High or Low) correlated with p.

Optimal bidding rules.

The optimal bidding rule is characterized by a single parameter γ^* that measures the optimal extent of shading. We illustrate with two cases.

(i) Perfect estimates ($z = v$).

Define $\phi(\gamma)$ as the probability of winning when the agent uses r_γ:

$$\phi(\gamma) \equiv \Pr_\omega(v - p > \gamma).$$

The expected payoff that the agent gets when using rule r_γ is:

$$v(r_\gamma) = \gamma\phi(\gamma).$$

The derivation of the optimal bidding rule is thus analogous to a standard monopoly pricing model in which $\phi(\gamma)$ is interpreted as a demand function. Optimal shading γ^* is characterized by the following first order condition:[10]

$$\gamma^* = \frac{\phi(\gamma^*)}{-\phi'(\gamma^*)},$$

Shading is thus larger when the agent has higher chances of winning (that is, if $\phi(\gamma)$ shifts up), or when the distribution over $v - p$ is more dispersed (that is when $\mid \phi' \mid$ shifts down). The latter effect can be interpreted as a Bertrand competition effect: it is stronger when small changes in shading induce large changes in the probability of winning.

(ii) Noisy estimates ($z = v + \varepsilon$).

Now define

$$\phi_\varepsilon(\gamma) \equiv \Pr(v + \varepsilon - p > \gamma) \text{ and } \psi_\varepsilon(\gamma) \equiv E[\varepsilon \mid v - p + \varepsilon = \gamma].$$

$\phi_\varepsilon(\gamma)$ is the probability of winning when the agent uses r_γ, while $\psi_\varepsilon(\gamma)$ is the expectation of the estimation error conditional on using r_γ and winning by a 0-margin. The optimal bidding rule is characterized by a shading level γ^* that solves:[11]

$$\gamma^* = \frac{\phi_\varepsilon(\gamma^*)}{-\phi'_\varepsilon(\gamma^*)} + \psi_\varepsilon(\gamma^*)$$

[10] Assuming that $\gamma + \frac{\phi(\gamma)}{\phi'(\gamma)}$ is increasing, the first order condition uniquely defines γ^*.

[11] Assuming that $\gamma + \frac{\phi_\varepsilon(\gamma)}{\phi'_\varepsilon(\gamma)} - \psi_\varepsilon(y)$ is increasing, this first order condition again uniquely defines γ^*.

In words, optimal shading is now derived from two considerations. First, as in the perfect estimate case, it depends on demand elasticity. In addition, when estimates are noisy, a bidder runs the risk of getting the object only because the error term ε was positive (and high). In other words, *noisy estimates* make a bidder subject to *a winner's curse, or selection bias*, and an optimal reaction to that possibility is cautiousness – or shading more.[12]

Due to noisier estimates, optimal shading may thus increase for two different reasons: more dispersion in estimates (thus weakening the Bertrand competition effect) and a stronger winner's curse effect.

3 A basic auction model.

We now apply our approach to strategic interactions between n bidders. We assume that bidder i's value for an object is of the form

$$v_i = \alpha + \theta_i,$$

where α represents a common component of all bidders' values and θ_i an idiosyncratic component of i's value. This model captures the logic of the painting example: i may know his value v_i but does not know how much of his value is idiosyncratic, that is, he does not observe either α or θ_i. We assume that the vector of idiosyncratic terms is drawn independently of α.

Each bidder faces a decision problem similar to that described in Section 2. For each bidder i, we consider a set of bidding rules R_i, where each rule $r_i \in R_i$ maps the data z_i that the bidder receives into a bid b_i. Throughout the paper, we shall mostly focus on the case where z_i is a possibly noisy estimate of v_i, that is,

$$z_i = v_i + \varepsilon_i,$$

where error terms ε_i are drawn independently of v_i. In Section 4, we shall also consider the case where z_i includes information about rank. For now, we assume as before that

$$r_\gamma(z_i) = z_i - \gamma \text{ and } R_i = \{r_\gamma\}_{\gamma \in \mathcal{R}}$$

[12] Note that the sign of the term $\psi_\varepsilon(\gamma)$ depends on the distribution over $v - p$. With fierce enough competition, prices tends to be high and $\psi_\varepsilon(\gamma)$ positive.

For any vector $\gamma = (\gamma_1, ..., \gamma_n)$, each rule profile $r_\gamma = (r_{\gamma_1}, ..., r_{\gamma_n})$ induces an expected payoff which we denote by $v_i(\gamma_1, ..., \gamma_n)$. An equilibrium is then defined in the usual way.

> **Definition:** *A shading vector* $\gamma^* = (\gamma_1^*, ..., \gamma_n^*)$ *is an equilibrium iff* $v_i(\gamma^*) \geq v_i(\gamma_i, \gamma_{-i}^*)\ \forall \gamma_i$.

Compared to standard auction models, behavior is characterized by a one-dimensional parameter, and equilibrium behavior is relatively simple to derive. It has the same complexity as a problem of price competition with differentiated products, where γ_i is the price set by player i and $v_i(\gamma_i, \gamma_{-i})$ is the profit that results from the price vector (γ_i, γ_{-i}).

Note that our formulation restricts attention to pure strategy equilibria, and existence is not guaranteed. We shall derive sufficient conditions for existence in Section 3.2., and unless otherwise mentioned, assume throughout that existence holds.

We illustrate the approach with the perfect estimate case.

3.1 Perfect estimates

We assume that each bidder observes his valuation without noise, and that the vector of idiosyncratic terms is drawn from a symmetric distribution. We look for a symmetric equilibrium in which all bidders pick the same rule r_{γ^*}.

Define $\phi(y)$ as the probability that bidder i's valuation exceeds all others by at least y:

$$\phi(y) = \Pr(\theta_i - \max_{j \neq i} \theta_j > y).$$

We have

$$v_i(\gamma_i, \gamma^*) = \gamma_i \phi(\gamma_i - \gamma^*).$$

The first order condition for a symmetric equilibrium can thus be written as[13]

$$\gamma \phi'(0) + \phi(0) = 0.$$

By symmetry, $\phi(0) = \frac{1}{n}$, thus implying the following Proposition:

[13] This first order condition is sufficient when $y \to y + \frac{\phi(y)}{\phi'(y)}$ is increasing in y. We shall come back to existence issues in the next Section.

Proposition 1: *In a symmetric equilibrium, we must have:* $\gamma^* = \frac{1}{-\phi'(0)n}$.

In other words, equilibrium shading is driven by the expected chance of winning ($1/n$) and the dispersion of the idiosyncratic terms. Indeed, to interpret $-\phi'(0)$, consider the case where the idiosyncratic elements θ_i are i.i.d., each drawn from a distribution with density f. Let $\theta^{(2)} = \max_{j \neq i} \theta_j$. It is easy to check that

$$-\phi'(0) = Ef(\theta^{(2)}).$$

The coefficient $-1/\phi'(0)$ thus measures the dispersion of second highest valuations. In the special case where the distribution is uniform on the interval $[\underline{\theta}, \overline{\theta}]$, we have: $-\phi'(0) = f(\theta) = 1/(\overline{\theta} - \underline{\theta})$, hence

$$\gamma^* = \frac{\overline{\theta} - \underline{\theta}}{n}.$$

Comments:

1. We assume that bidders look for the optimal bidding rule among a *limited set* of bidding rules of the form $v_i - \gamma$. That bidders use rules of the form $v_i - \gamma$ could be motivated even without restrictions, by assuming that the common component α is drawn from a diffuse prior.[14,15] Our perspective however is not to argue in favor of a specific shape, on the ground that it is optimal or approximately optimal for some distributions. The forces that shape bid functions are likely to be driven by considerations that lie outside a specific auction model. Rather than endogenizing *all* aspects of behavior, we take as given a shape (additive shading) and endogenize just one aspect of behavior (the extent of shading).

[14] Alternatively, if the distribution over α is flat on a large interval, then, even if bidders look for the optimal strategy among all possible bid functions, then, except near the boundary of the distribution over v_i, learning v_i is not informative about θ. Formally, let g be the density function of α, assumed to be flat over $[\underline{\alpha}, \overline{\alpha}]$; then for any $v_i \in [\underline{\alpha}+\overline{x}, \overline{\alpha}-\underline{x}]$ we have:

$$f(\theta \mid \alpha + \theta_i = v_i) = \frac{f(\theta)g(v_i - \theta_i)}{\int_{y_i} f(y_i)g(v_i - \theta_i)d\theta_i} = f(\theta).$$

[15] This diffuse prior model, along with idiosyncratic terms drawn from independant and uniform distributions, has also been proposed as a tractable affiliated value model in Klemperer (1999, Appendix D)

2. The number of bidders participating in the auction affects bidding in two ways: through the chance of winning ($1/n$), and through the dispersion term ($-1/\phi'(0)$). To evaluate the effect of the number of bidders on shading, write $\phi_n(y)$ to indicate the probability that θ_i exceeds $\max_{j\neq i}\theta_j$ by more than y when there are n bidders. Assuming existence (see next Subsection), we denote by γ_n^* the level of equilibrium shading when there are n bidders present. We have the following proposition:

> **Proposition 2.** *Assume* θ_i are i.i.d., drawn from f, *with* f *centered, symmetric around* 0*, and single peaked. Define* $\beta_n = -\phi_n'(0)$*. Then* $\gamma_n^* = \frac{1}{n\beta_n}$ and β_n *is a decreasing sequence. However, as n increases,* $n\beta_n$ *increases without bound.*

Intuitively, when the number of bidders increases, the second highest realization tends to be higher. Since f is single-peaked, the distribution of second highest valuations tends to be more dispersed, and β_n decreases.

3.2 Existence

In standard auctions, existence of an equilibrium with monotone strategies is a difficult issue in general, and providing economic insights as to when existence fails may be difficult. Our approach deals with shading levels rather than shading functions, and interpretation is easier. As with a standard problem of *price competition with differentiated products*, existence depends on the shape of the "demand" function ϕ.

"Local" deviations are taken care of by first order conditions, and there are two types of "large" deviations that may create difficulties: either shading by a much larger amount, with the hope that the chance to win does not vanish; Or shading by a much smaller amount with hope that the chance of winning will be much greater.

A classic condition that guarantees existence is that ϕ is log-concave.[16] Another condition, weaker but still sufficient, is that $y + \frac{\phi}{\phi'}$ is nondecreasing.[17]

$A2$: ϕ is log-concave.

$A2'$: $y + \frac{\phi}{\phi'}$ is increasing.

[16] This means that $Log\phi$ is concave.

[17] Another yet weaker condition is that $y + \frac{\phi(y)}{\phi'(y)} - \frac{\phi(0)}{\phi'(0)}$ crosses 0 only once.

Proposition 3: *Under either A2 or A2′, existence of a pure strategy equilibrium is guaranteed.*

Proof: $A2$ implies $A2'$. $A2'$ implies that the best response is uniquely defined, hence the shading level γ^* derived from first order conditions is an equilibrium. QED

Intuitively, one expects ϕ to be S-shaped, so convex on some range. Convexities are potentially problematic because they may induce incentives for large deviations, either upward or downward. A2 and A2' are conditions that limit the extent to which ϕ is convex, making it sufficient to check for first-order conditions.

A typical case where these conditions (and existence) fail is when the density function f simultaneously exhibits *some concentration* and *fat tails.* Concentration implies a strong Bertrand competition effect, hence little shading (and little profit) in any tentative equilibrium, while fat tails imply that the chances to win remain nonnegligible even when shading substantially. Thus there exists a force toward large shading: you can take a chance on a large benefit, even if it is at the risk of having little chance of winning.

Similarly, these conditions will fail when there is significant *uncertainty about the dispersion* of idiosyncratic terms, as we now illustrate.

Example 1. Consider an auction with two bidders where idiosyncratic terms are either drawn from a distribution with density g_1 (say, this is state $k = 1$) or g_2 (under state $k = 2$), and assume that state $k = 1$ has probability q. Define

$$\phi_k(y) \equiv \Pr_k\{\theta_i - \theta_j > y\}$$

We have:

Proposition 4: *If* $\phi_1'(0) \max_y y\phi_2(y) > \frac{1}{4q(1-q)}$ *then existence of pure strategy equilibria fails.*

In words, existence fails for example when the function $\phi = E\phi_k$ simultaneously exhibits concentration (due to high $\phi_1'(0)$) and large dispersion (high $\max_y y\phi_2(y)$).

Proof: Assume existence holds. The first order condition implies an equilibrium shading level γ^* that satisfies:

$$\gamma^* = -\frac{E\phi_k(0)}{E\phi_k'(0)} = -\frac{q\phi_1(0) + (1-q)\phi_2(0)}{q\phi_1'(0) + (1-q)\phi_2'(0)} \leq \frac{1}{-2q\phi_1'(0)},$$

hence a profit at most equal to $\frac{1}{-4q\phi_1'(0)}$. By picking a large shading level γ that maximizes $y\phi_2(y)$, a player can secure at least $(1-q)\max y\phi_2(y)$. So existence fails when the condition of the Proposition holds. QED

Note that although pure strategy equilibria may fail to exist, relatively simple equilibria in mixed strategies can be constructed. Our working paper (Compte and Postlewaite, 2010) provides an illustration, with bidders randomizing between only two levels of shading.

3.3 Revenue Rankings

We compare below two auction formats: first price and second price auctions. The usual insight concerning private value auctions is that if valuations are affiliated, then a second price auction generates more revenue than a first price auction. As for existence our approach simplifies the analysis, and it proposes an alternative interpretation for the comparison.

In a second price auction, the winner, say player i, gets $y = \theta_i - \max_{j\neq i}\theta_j$ in events where y is non negative. Since y is distributed according to the density $-\phi'(y)$ (by definition of ϕ), a bidder's expected gain, which we denote G^{II}, is therefore:

$$G^{II} = \int_{y\geq 0} -y\phi'(y)dy = \int_{y\geq 0}\phi(y)dy.$$

In a first price auction, a bidder's expected gain, which we denote G^I, is equal to

$$G^I = \gamma^*\phi(0) = \frac{[\phi(0)]^2}{-\phi'(0)}.$$

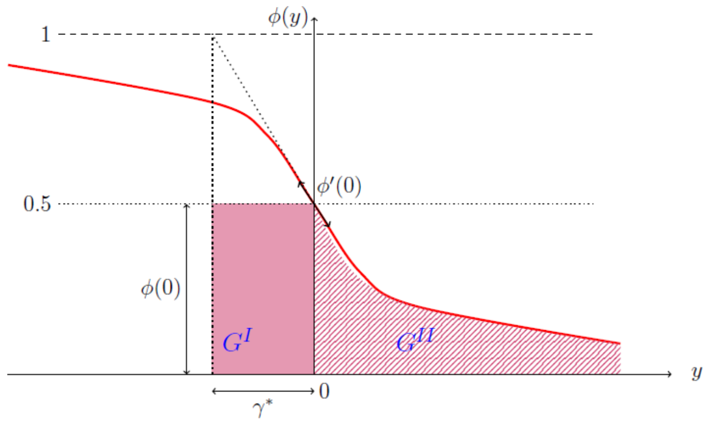
Since the allocation does not change across formats, the seller's revenue is highest when the bidder's expected gain is smallest. So we conclude:

Proposition 5: *The first price auction generates more revenue than the second price if and only if*

$$\mid \phi'(0) \mid \int_{y\geq 0}\phi(y)dy > [\phi(0)]^2 \qquad \text{(P)}$$

The first price is thus preferable when the "demand" function *combines* high concentration (i.e., high $\mid \phi'(0) \mid$, implying strong Bertrand competition effect, hence low gains in the first price auction) and substantial dispersion (implying high rents in the second price auction). Intuitively, in the first price auction, bidders who happen to get high realizations do not know/realize it, so they cannot tailor shading to that event and figure that increased shading would be profitable: this is a source of increased revenues for the seller.

The following figure illustrates graphically the gains G^I and G^{II}.



Comment 1. The conditions (i.e., shapes of ϕ) that make first price generate more revenue are also conditions under which existence may become problematic. In particular, if one imposes stronger than necessary conditions to guarantee existence (say, A2), then condition (P) of Proposition 5 cannot hold and the second price generates more revenue.[18] However, existence and condition (P) are compatible, essentially because $\int_{y\geq 0}\phi(y) > \max_y y\phi(y)$.

Comment 2. As in example 1, uncertainty about the dispersion of θ_i's may create the appropriate combination (strong competition and yet substantial dispersion) making first price preferable. In particular, one can check that in example 1, if $\phi_1'(0)$ is large enough, there is a range of values for q such that existence and condition (P) both hold.

To illustrate further the effect of uncertainty about dispersion, we consider example 2 below.

[18] For the same reason, imposing affiliation in standard auctions (a stronger than needed condition for existence) makes the second price preferable. To see why A2 and condition (P) are not compatible, observe that A2 implies $\phi'(y) \leq \phi(y)\frac{\phi'(0)}{\phi(0)}$, hence $-\phi(0) \leq \frac{\phi'(0)}{\phi(0)}\int_{y\geq 0}\phi(y)dy$.

Example 2: Assume $\theta_i = d\eta_i$ where the η_i are i.i.d., and where $d \in [\underline{d}, \overline{d}]$ is a positive dispersion parameter, drawn independently from the η_i's. Let $\phi_0(y) = \Pr(\eta_i - \max_{j \neq i} \eta_j \geq y)$. Condition (P) becomes:

$$\kappa \mid \phi_0'(0) \mid \int_{y \geq 0} \phi_0(y) dy > [\phi_0(0)]^2 \text{ with } \kappa = E(1/d)Ed.$$

Uncertainty about the dispersion parameter d ensures that $\kappa > 1$, and therefore makes it easier to meet condition (P). Intuitively, when dispersion is likely to be small, there is strong Bertrand competition effect, and in events where dispersion happens to be high, bidders cannot tailor shading to that event: shading remains small whether dispersion is small or large, and this is a source of increased revenues for the seller.

3.4 Releasing information about dispersion

To conclude this Section, we compare the case where bidders have access to information about the dispersion of valuations, to the case where they don't. We model this information as a signal $k \in K$ that may be released (or not) to bidders, signal k having probability q_k. We assume that this information preserves symmetry and define:

$$\phi_k(y) \equiv \Pr(\theta_i - \max_{j \neq i, j \in I} \theta_j > y \mid k).$$

Let γ^* denote the equilibrium shading level when participants do not know k, and γ_k^* the equilibrium shading when they know k. We have:

Proposition 6: $\gamma^* < E\gamma_k^*$.

That is, bidders bid more aggressively on average when they do not know k. Hence for the seller, the *policy* of not releasing information about k generates more revenue.

Proof: From Proposition 1, we have:

$$\gamma_k^* = \frac{\phi_k(0)}{-\phi_k'(0)} \text{ and } \gamma^* = \frac{\sum_k q_k \phi_k(0)}{-\sum_k q_k \phi_k'(0)}. \tag{1}$$

Let $p(k \mid i)$ denote the probability that the state is k given that i wins: $p(k \mid i) = q_k \phi_k(0) / \sum_k q_k \phi_k(0)$. We have:

$$\frac{1}{\gamma^*} = \sum_k p(k \mid i) \frac{1}{\gamma_k^*} > \frac{1}{\sum_k p(k \mid i) \gamma_k^*}, \tag{2}$$

where the inequality follows from $y \longmapsto 1/y$ being a convex function. Given the symmetry assumption, $\phi_k(0) = 1/n$ for all k so $p(k \mid i) = q_k$. QED

This Proposition immediately extends to the case where the *number of participants is stochastic* and where k provides the number or the identity of the participants, to the extent that symmetry is preserved (all players have an equal chance of being a participant).

> **Proposition 7**: *Assume that the set of participants I is stochastic and that k reveals the number of participants $\mid I \mid$. If $\Pr\{i \in I \mid k\}$ is independent of i (hence equal to $\frac{|I|}{n}$), then Proposition 6 holds.*

Proposition 7 thus confirms the results of Matthews (1987) and McAfee and McMillan (1987), and points out that releasing information about the number of participants is analogous to releasing information about dispersion.[19]

Proof: Redefine $\phi_k(y)$ as the ex ante probability of being a participant and winning when the state is k and while shading more than all other participants by exactly y:

$$\phi_k(y) = \Pr\{i \in I \mid k\} \Pr(\theta_i - \max_{j \neq i, j \in I} \theta_j > y \mid k).$$

Equations and inequality (1) and (2) hold unchanged, as does the equality $p(k \mid i) = q_k$ since $\phi_k(0) = \frac{|I|}{n} \frac{1}{|I|} = 1/n$. QED

4 Incorporating information about rank

We motivated our model by arguing that in many contexts, a bidder's own valuation is a poor tool for estimating how his value compares to others'

[19] Note that the statement assumes that existence of a pure strategy equilibrium obtains whether k is observed or not. With a stochastic number of bidders, existence of a pure strategy equilibrium for each realization of n with n known to bidders does not guarantee existence of a pure strategy equilibrium in the uncertain case. The reason is identical to that provided in Proposition 4.

This comment that existence may fail with a stochastic number of bidders actually applies to the standard approach as well, and to our knowledge, it has not been noted. We make this precise in the working paper version.

valuations. We do not suggest, however, that bidders would never get and/or exploit relevant rank related information.

We illustrate below how rank related information can be incorporated in our basic model: we assume that bidders receive signals correlated with rank, and then examine two different signal structures in which bidders receive one of two possible signals, indicating "high rank" or not. We restrict attention to the case of two bidders and show that information about rank can either be pro-competitive (Proposition 8) or anti-competitive (Proposition 9).[20] A pro-competitive effect obtains when the "high rank" signal is more likely whenever own rank is higher (i.e. $v_i > v_j$), while an anti-competitive effect obtains when the "high rank" signal is delivered if and only if own value is *substantially* higher (i.e. $v_i > v_j + \Delta$).

We next discuss the relationship with standard auction models.

4.1 Extension.

Formally, bidder i's data is now defined as:

$$z_i = (v_i, k_i),$$

where $k_i \in K_i$ and K_i is a finite set. A plausible rule for bidder i, denoted r_γ, will now consist of a vector of shading levels, one for each possible private signal k_i. For any $\gamma = (\gamma^k)_{k \in K_i}$, we define

$$r_\gamma(v, k) = v - \gamma^k,$$

and assume that the set of rules R_i consists of all such rules. Equilibrium definition is unchanged.[21]

[20] The insight that additional signals may be anti-competitive (and decrease revenues) has been documented by Fang and Morris (2006), in an independent private value auction where valuation may take two values. The insight that additional signals may be pro-competive (and increase revenues) has been documented by Lansberger et al. (2001) and Fang and Morris (2006), in a standard independent private value model with a continuum of valuations, and two signals correlated with rank.

Propositions 8 and 9 thus confirm these insights, in a setting where bidders do not use valuation realizations to make inferences about rank. The propositions also provide some insight about how the nature of the signal structure translates into more or less competition.

[21] Such signals could also be introduced in the standard model. The technical difficulty is that when signals are not perfectly correlated, bidders then have a two dimensional type, and equilibria are then difficult to characterize.

4.2 A pro-competitive effect.

Consider the case of two bidders who receive private (and possibly noisy) information about their rank.[22] Specifically, assume two possible signals, i.e. $K_i = \{0, 1\}$, and the following technology, whereby for any given valuation vector $v = (v_i)_i$, the signals k_i are drawn independently, according to:

$$\Pr\{k_i = 1 \mid v_i > v_j\} = p \text{ and } \Pr\{k_i = 1 \mid v_i < v_j\} = 1 - p.$$

So when $p = 1/2$, the signal is uninformative, while for $p = 1$, it is perfectly informative of whether i has the higher valuation.

Define (γ_0^*, γ_1^*) as the equilibrium shading levels for each $k_i = 0, 1$. We refer to γ^* as the equilibrium shading level when no signal about rank is available. We have

Proposition 8: Under $A2$, for any $p > 1/2$, we have $\gamma_0^* < \gamma_1^* < \gamma^*$.

The proof is in the Appendix. The intuition is as follows. The private signal creates an asymmetry, with the bidder receiving bad news willing to be more aggressive than the one receiving good news (i.e., $\gamma_0^* < \gamma_1^*$). A bidder who receives $k_i = 1$ might be willing to exploit that signal to bid less aggressively than if no information was available (i.e., $\gamma_1^* > \gamma^*$). Given the particular signal structure assumed however, it turns out that this is not the case. If $p = 1$ for example, the good news $k_i = 1$ just moves the winning probability from $\phi(y)$ to $2\phi(y)$, thus not altering marginal incentives to shade hence best responses. Under A2, best responses are monotonic, that is, more aggressive behavior from one's opponent triggers a more aggressive response, implying that $\gamma_1^* < \gamma^*$.

4.3 An anti-competitive effect.

Proposition 8 highlights that information about rank may be pro-competitive. The signal structure assumed is important however. If a bidder gets good news only when he is far ahead of the other, then in these events, he will be

[22] The case in which bidders receive public and perfect information about rank has been studied by Landsberger et al. (2001). We extend their insights to the case of imperfect private information about rank, but our main motivation here is to illustrate the simplicity of the approach.

less aggressive, and over all, such signal structure may decrease competition. The next Proposition confirms that intuition.

Formally, fix $\Delta > 0$ and consider the following technology, whereby for any valuation vector $v = (v_i)_i$, bidder i observes $k_i = 1$ if and only if $v_i > v_j + \Delta$, and $k_i = 0$ otherwise. We consider the non-degenerate case where $\phi(\Delta) > 0$.[23] We have:

> **Proposition 9:** Under $A2$, if $\gamma^* < \Delta$, then $\gamma_1^* = \gamma_0^* + \Delta$ and $\gamma_0^* = \gamma^*(1 - 2\phi(\Delta))$. The bidder getting $k_i = 1$ wins with probability one, and bidders expected gains are higher than when they do not receive signals.

In words, the bidder who gets good news can afford to bid $\gamma_0^* + \Delta$ and get the object with probability one. When $\gamma^* < \Delta$, he does not want to take the risk of losing the object, so he bids $\gamma_0^* + \Delta$. The bidder who receives bad news is more aggressive ($\gamma_0^* < \gamma^*$). Overall however, the latter effect has less impact on expected gains, and on average, being able to observe these signals generates more expected gains for bidders.

4.4 Discussion

A standard division in studying auctions is whether values are *independent or correlated*. We argue below that from the perspective of a mildly sophisticated agent, a useful dividing line may be *whether or not he gets and exploits rank related signals.*

Standard approaches.

We have represented values as the sum of two *random* components, a common component α and an idiosyncratic component θ_i. Given this representation, the classic *independent private value* environment corresponds to the case where bidders observe both the value v_i *and* the common component α, that is,

$$z_i = (v_i, \alpha).$$

From a purely *technical perspective*, the independent private value model seems relatively simple: given the structure of the model, the difference $v_i - \alpha$

[23] This ensures that the events $k_i = 1$ arise with positive probability

is a useful (and optimal) instrument, correlated with rank and dispersion, and in equilibrium, optimal bidding is of the form $r(v_i, \alpha) = b(v_i - \alpha)$.

The solution hinges on precise knowledge of the common component α, and one may want to weaken this informational assumption. This could be done by assuming that each bidder receives a noisy estimate of α, say

$$\beta_i = \alpha + \xi_i.$$

A player's data would then consist of a vector

$$z_i = (v_i, \beta_i)$$

that combines information about personal value and about the support of the distribution over valuations. The environment would then correspond to a *correlated* private value environment with two-dimensional types.[24]

Either formulation is complex, as each vector realization z_i comes with *implicit* information about the vector z_j received by others, and the analyst must decide how one ought to exploit that information, using fine knowledge of prior distributions.

In addition, from a *bidder's perspective,* both cases seem cognitively demanding. First, even if interested in his rank, a bidder may find impossible to come up with a relevant or reasonably accurate reference point (α) to which his value v_i could be compared. Second, even if he could form an estimate (β_i) of such a reference point, this estimate is likely to be noisy, without the bidder being able to perceive how noisy β_i is. Then, how one should combine the signals v_i and β_i seems a formidable task.

Our perspective.

Our simple auction models are meant to reflect the cognitive difficulties that agents face, starting from mildly sophisticated agents, and then investigating what changes in behavior arise when bidders become more sophisticated. The simplest environment examined has been that in which bidders do not attempt to use rank-related observations (Section 2 and 3). Next we considered more sophisticated bidders, modelling them as receiving and utilizing rank-related information in a crude way (only two signals -

[24] The independent private value case corresponds to the degenerate case with no noise.

$k_i = 0, 1$),[25] and this Section has illustrated different ways in which such information may alter the strategic interaction, either decreasing or increasing competition.

Standard private value models, whether independent value or correlated value, push sophistication even further, as agents ultimately use rank and dispersion related signals, including valuations, in very sophisticated ways. In contrast, the models proposed allow us to *ignore* the possibility that v_i itself could be an instrument in assessing rank and dispersion, thereby allowing us to focus exclusively on the rank related information conveyed by the crude signal k_i.

5 Noisy Estimates

We return in this section to our basic model, assuming that z_i is a noisy estimate of v_i. The noise terms ε_i are assumed centered and drawn from independent and identical distributions. We look for a symmetric equilibrium of the first price auction. Denote by $\phi_\varepsilon(y)$ the probability that z_i exceeds $\max_{j \neq i} z_j$ by more than y, that is,

$$\phi_\varepsilon(y) = \Pr(z_i - \max_{j \neq i} z_j > y).$$

Denote by $v_i(\gamma_i, \gamma)$ the expected payoff that bidder i derives when he shades by γ_i while others shade by γ. Bidder i obtains a payoff equal to $v_i - (v_i + \varepsilon_i - \gamma_i)$ when he wins, so we have:

$$v(\gamma_i, \gamma) = (\gamma_i - E[\varepsilon_i \mid z_i - \max_{j \neq i} z_j > \gamma_i - \gamma])\phi_\varepsilon(\gamma_i - \gamma).$$

Define

$$\psi_\varepsilon(y) = E[\varepsilon_i \mid z_i - \max_{j \neq i} z_j = y].$$

First order conditions immediately yield:

Proposition 10: *In a symmetric equilibrium, bidders shade their bid by*

$$\gamma_\varepsilon^* = \frac{1}{-n\phi'_\varepsilon(0)} + \psi_\varepsilon(0).$$

[25] The model proposed in this section may be interpreted as one in which (v_i, β_i) would be processed according to some mental processing rule, in which the agent thinks his rank is high ($k_i = 1$) or not so high ($k_i = 0$) depending on whether $v_i - \beta_i$ is higher than some threshold h, and consequently the agent has "data" $z_i = (v_i, k)$.

Shading thus has two components. The first term is analogous to that derived in the private value case: $(\frac{1}{n})$ corresponds to the expected probability of winning, and $\frac{1}{-\phi'_\varepsilon(0)}$ captures how bidders take advantage of the dispersion in valuations. The second term $\psi_\varepsilon(0)$ stems from the fact that more optimistic bidders tend to win the auction (i.e., the winner's curse), and rational bidders should correct for that: $\psi_\varepsilon(0)$ captures the expected optimism of the *marginal winner*, i.e., the bidder that wins by a 0-margin.

Note that as before, existence of a pure strategy equilibrium is not guaranteed. As shown in the Appendix, a sufficient condition for existence is:

$$y - \psi_\varepsilon(y) + \frac{\phi_\varepsilon(y)}{\phi'_\varepsilon(y)} \text{ increases in } y.$$

5.1 Comparative statics.

We study below how noisy estimates affect equilibrium gains for sellers and buyers. When estimates are noisier, the dispersion of estimates increases and bidders may take advantage of that by shading their bids more (because $\mid \phi'_\varepsilon(0) \mid$ increases with small noise). Increased shading however does not necessarily translate into greater expected gains for bidders because a bidder only gains $\gamma^*_\varepsilon - \varepsilon_i$ in the event he wins, and conditional on winning a bidder tends to be optimistic (higher ε_i). The first effect (or dispersion effect) benefits bidders because competition is less intense. The second effect (or winner optimism) hurts bidders because the winner loses $E[\varepsilon_i \mid z_i > \max_{j\neq i} z_j]$ while he corrected his bid by only $E[\varepsilon_i \mid z_i = \max_{j\neq i} z_j]$, the expected optimism of the marginal winner.

Proposition 11 shows that when the dispersion of idiosyncratic elements is small, then the first effect dominates. Proposition 12 next provides conditions under which the second effect dominates. Finally, we provide an example that illustrates how additional noise can translate into competition among fewer bidders, i.e. optimistic ones. To fix ideas, we denote by $\overline{\varepsilon}$ the largest error term and by Δ the largest possible realization of $\theta_i - \theta_j$.

First we show:

Proposition 11: If Δ is small enough, then bidders necessarily benefit from estimates being noisy.

Intuitively, when Δ is small, bidders get almost no rent in equilibrium. Noisy estimates ensure that in equilibrium bidders get a payoff bounded away from 0, independently of Δ. A corollary is that the seller is necessarily hurt, because noise necessarily reduces efficiency.

Proof: Consider equilibrium shading γ_ε^*, which cannot be negative (otherwise payoffs would be negative). A bidder may choose to bid $\gamma_\varepsilon^* + \frac{3}{2}\overline{\varepsilon}$, in which case he wins with probability $\phi_\varepsilon(\frac{3}{2}\overline{\varepsilon})$ and gains at least $\frac{3}{2}\overline{\varepsilon} - \overline{\varepsilon}$ (because the error term is at most $\overline{\varepsilon}$), so in equilibrium, he may secure

$$\phi_\varepsilon(\frac{3}{2}\overline{\varepsilon})\frac{\overline{\varepsilon}}{2}.$$

For all $\Delta \leq \frac{\overline{\varepsilon}}{4}$, $\phi_\varepsilon(\frac{3}{2}\overline{\varepsilon}) \geq \Pr(\varepsilon_i - \max_{j\neq i} \varepsilon_j > 7\overline{\varepsilon}/8) > 0$, implying a lower bound on equilibrium payoffs (independent of Δ). **QED**

Next, we show that if the dispersion of idiosyncratic elements θ_i is not small, small noise may benefit the seller and hurt the bidders. We consider the case of two bidders, and denote by h the density over the difference $\theta_i - \theta_j$. We have:

> **Proposition 12:** Assume h is smooth, centered, and that $\mid h''(0) \mid < 2[h(0)]^3$. Then the addition of a small noise benefits the seller and hurts the bidders.

Intuitively, when $\mid h''(0) \mid$ is small, the dispersion effect is small and the winner optimism effect prevails. The proof is in the Appendix.

Example: We consider a simple noise structure and illustrate how pressure towards higher shading can arise. We assume ε can take two values, $\overline{\varepsilon}$ (with probability p) or $\underline{\varepsilon}$, so that bidders are either optimistic or pessimistic. We also assume that each θ_i is drawn from a uniform distribution. Let $\Delta_\varepsilon = \overline{\varepsilon} - \underline{\varepsilon}$ and assume that $\Delta_\varepsilon > \Delta$. In a symmetric equilibrium, bidder i can only win when he is optimistic ($\varepsilon_i = \overline{\varepsilon}$) or when all bidders are pessimistic ($\varepsilon_j = \underline{\varepsilon}$ for all j). Define $\widetilde{n}$ as the random variable that gives the number of bidders who have a chance to win for each realization $(\varepsilon_i)_i$, that is:

$$\widetilde{n} = \#\{i, \varepsilon_i = \max_j \varepsilon_j\}.$$

We have:

> **Proposition 13**: *In a symmetric equilibrium, we have:*
>
> $$\gamma_\varepsilon^* \geq E[\varepsilon \mid \varepsilon_i = \max_j \varepsilon_j] + \Delta E \frac{1}{\widetilde{n}}. \qquad (3)$$

The first term corresponds to the expected "optimism" of the marginal winner, and that term gets close to $\bar{\varepsilon}$ when n increases. The second term describes the dispersion effect. Compared to the case without noise where they would shade by Δ/n, bidders shade more because they are facing less intense competition: because only optimistic bidders may win (except in the event all are pessimistic), a bidder is endogenously facing fewer competitors.

5.2 Comparison with second price

We now consider second price auctions and analyze how noisier estimates affect rents. We find as before that when dispersion (Δ) is small enough, noisier estimates benefit bidders. However Proposition 12 has no equivalent: with two bidders (and centered distributions), noisy estimates always hurt the seller. The reason is that the loser sets the price, and on average, with noise, the loser is pessimistic.

We start by deriving equilibrium shading. In a second price auction, bidder i now gains $v_i - (\max z_j - \gamma) = z_i - \max z_j + \gamma - \varepsilon_i$ when he wins. Letting $h_\varepsilon(y) = -\phi'_\varepsilon(y)$, we have

$$v(\gamma_i, \gamma) = \int_{y \geq \gamma_i - \gamma} y h_\varepsilon(y) dy + (\gamma - E[\varepsilon_i \mid z_i - \max_{j \neq i} z_j > \gamma_i - \gamma])\phi_\varepsilon(\gamma_i - \gamma).$$

First order conditions now imply:

Proposition 14: *In the second price auction, in a symmetric equilibrium, bidders shade by*

$$\gamma_\varepsilon^{SP} = \psi_\varepsilon(0).$$

Bidders thus correct for the winner's curse in the same way as before. Going from second to first price auctions, the change in bidding thus only stems from bidders taking advantage of the dispersion in estimates.

As with the first price auction, noisier estimates may help bidders. Conditional on winning, a bidder obtains:

$$G^{II} \equiv E[\theta_i - \max z_j \mid z_i > \max z_j] + \gamma_\varepsilon^{SP}.$$

With two bidders, $\psi_\varepsilon(0) = 0$ by symmetry, so $\gamma_\varepsilon^{SP} = 0$, and we have:

$$G^{II} = E[\theta_i - \theta_j \mid z_i > z_j] - E[\varepsilon_j \mid z_i > z_j].$$

The first term is the efficiency gain that the winner brings. The second term is an additional rent that the winner obtains because the loser is pessimistic on average. As noise increases, the first term decreases because the allocation is less efficient. The second term however increases. When the idiosyncratic components θ_i become more concentrated, the second term prevails, and the agents get rents that they would not get without noise.

The same is true with many bidders. As the idiosyncratic components θ_i become more concentrated (vanishing Δ), the winner's gain tends to

$$G_0^{II} = E[\max \varepsilon_j \mid \max_j \varepsilon_j = \varepsilon_i] - E[\max \varepsilon_j \mid \max_{j \neq i} \varepsilon_j < \varepsilon_i],$$

which is strictly positive. So dispersion of estimates always increases rents when idiosyncratic components are concentrated. These rents result from the dispersion of estimates.

We conclude with the following proposition:

> **Proposition 15**: *Assume two bidders, with θ_i and ε_i drawn from centered distributions. Then noisy estimates never benefit the seller.*

Proof: With two bidders, each bidder bids his estimate z_i, so the seller obtains $G = E\alpha + E[z_j \mid z_j < z_i]$, so since θ_i and ε_i are centered (around 0), we have $G = E\alpha - \frac{1}{2}E[z_i - z_j \mid z_i > z_j]$. We show below that noise inflates $H \equiv E[z_i - z_j \mid z_i > z_j]$, thereby concluding the proof. We have

$$2H = -\int_{y>0} y\phi'_\varepsilon(y)dy = \int_{y>0} \phi_\varepsilon(y)dy = \int_{y>0}\int_\varepsilon \phi(y+\varepsilon)h(\varepsilon)d\varepsilon dy$$

where h denotes the density over $\varepsilon = \varepsilon_i - \varepsilon_j$. Define $\varphi(\varepsilon) = \int_{y>0} \phi(y+\varepsilon)dy$. φ is convex, implying that $H = \int_\varepsilon \varphi(\varepsilon)h(\varepsilon)d\varepsilon > \varphi(0) = \int_{y>0} \phi(y)dy$, as desired. **QED**

5.3 Discussion

Dispersion rents or Information rents?

We often refer to information rents to describe the gains that an "informed" player gets. A more appropriate qualification might be *"dispersion rents"*: bid dispersion generates rents, and private information in standard models generates rents in so far as it creates bid dispersion. Our model however illustrates that poor information (i.e. noisier estimates) may translate into higher bid dispersion, and that improving an agent's information (i.e. less noisy estimates) may induce lesser bid dispersion, hence smaller rents.

This effect of noise on bid dispersion would not hold in a standard model; the opposite would actually be true. Noisier estimates would translate into less dispersed posteriors (by a regression to the mean effect), and therefore greater competition when symmetry is assumed. The latter conclusion, however, is (in our view) an unfortunate artifact of the standard model, and of the implicit assumption that agents know (or behave as if they knew) all distributions: as noise increases, value estimates decrease in importance and more weight is put on priors.

Common values, interdependence and estimation errors

In modelling auctions, the distinction between private and common values is often seen as a key dividing line. In common or interdependent value auctions, the bids of others reveal information about one's own valuation, and a rational bidder ought to take into account those inferences. An omniscient bidder will indeed find this advice useful. To most bidders however, the precise ways in which preferences are interdependent are likely obscure, and the appropriate inference likely out of reach.

From a less sophisticated bidder's perspective, a more useful dividing line may be whether he is *subject to estimation errors* or not. If he is subject to estimation errors, he should exert caution because he is subject to a selection bias: he is more likely to win when the error is positive.[26] This warning is not specific to auctions. It arises for any decision problem in which an agent compares an alternative that is easy to evaluate (not buying) to one that is more difficult to evaluate (Compte and Postlewaite (2012)).

Now the level of caution depends on context, and indeed, the degree of interdependence then matters. If idiosyncratic components are less dispersed (which can be interpreted as values being more interdependent), the estimation errors carry more weight and caution should increase.[27]

[26] The view that the winner's curse results from a selection bias appears in Capen et al. (1971). Compte (2001) examines the effect of increasing the number of bidders on this selection bias, in the context of the second price auction.

[27] Caution should also increase when the number of bidders increases (Compte 2001).

6 Discussion and Related literature

6.1 Simple rules and strategy restrictions.

Our approach studies auctions where bidders only compare simple rules. Following Rothkopf (1969), a number of authors have analyzed auctions when bidders use multiplicative bidding strategies. More recently, Klemperer (1999), Compte (2001), Compte and Postlewaite (2010), Satterthwaite et al. (2012) have considered the additive structure proposed here.

Auction models proposing simple bidding rules generally view them as arising from standard equilibrium considerations. These models put a special structure on the joint distributions over valuations ensuring that using simple rules is optimal *across all possible rules* (given that others use simple rules as well). This special structure typically assumes that valuations are functions of a common component drawn from a *diffuse prior.* We have proposed a different perspective, based on *direct restrictions* on the strategy space. While both perspectives capture the notion that bidders might have difficulties extracting rank information from their valuations, we propose the path that restricts the strategy space for various reasons:
(i) Our perspective is not to argue in favor of a specific shape, on the ground that it is optimal or approximately optimal for *some* distributions. The forces that shape bid functions are likely to be driven by considerations that lie outside a specific auction model. Rather than endogenizing *all* aspects of behavior, we take as given a shape (additive shading) and endogenize a single aspect of behavior (the extent of shading), with the aim of uncovering economic insights that can be captured in this way.
(ii) We believe that strategy restrictions are a useful tool not only in auctions but also in other strategic environments – dynamic environments for example – for which the diffuse prior assumption would have no equivalent;
(iii) From a positive perspective, we are implicitly attempting to deal with agents having *limited knowledge* of the environment they are facing (say the joint distribution over valuations). The standard way to deal with limited knowledge would be to follow the traditional Bayesian route that takes anything that is not known as a random variable, and then possibly assume that agents receive signals correlated with the realized joint distribution. However, following Heiner (1983), one would expect limited knowledge to be associated with *less* sophistication, not more. Restrictions on the strategy

space incorporate such a bound on sophistication that limited knowledge would seem to call for.

6.2 Existence issues and revenue rankings

Existence. In standard auction models, the concern has been to establish existence of equilibria in monotonic strategies. The main difficulty is that based on his signal/valuation, an omniscient bidder makes sophisticated inferences about the distribution over other bidders' valuations, and it is not guaranteed that his best response will be monotonic in his signal. Affiliation is a condition that ensures that best responses are indeed monotonic in one's signal (when other players use monotonic strategies).[28]

Our analysis demonstrates that existence issues arise even when players are much less sophisticated (in particular, monotonicity of bid functions is not an issue – monotonicity is assumed). The reasons are analogous to those that lead to nonexistence of pure strategy equilibria in models of price competition with differentiated products (Caplin and Nalebuff, 1986). The basic problem is that a bidder may simultaneously have strong incentives to be slightly more aggressive than others (buying a substantial chance of winning at little cost), and strong incentives to take a chance and opt for large shading, betting that the other players have a substantially lower valuation. Whether both strategies turn out to be attractive depends on the shape of the "demand function" ϕ that characterizes the dispersion of idiosyncratic components.

Revenue rankings. Revenue equivalence between first and second price auctions holds for independent private value auctions (Myerson, 1981). Second price auctions generate more revenue when private valuations are affiliated (Milgrom and Weber, 1982). Economic intuition for these comparisons is somewhat difficult to provide. Milgrom and Weber (1982) appeal to a "linkage principle", which itself seems to apply as soon as valuations are positively correlated. However affiliation is stronger than positive correlation, and De Castro (2007) reports simulations where valuations are assumed to be positively correlated and yet first price auctions generate greater revenue.[29]

[28] See, for example, Athey (2001), Lizzeri and Persico (1995), and Reny and Zamir (2004).

[29] Fang and Morris (2006) also question the revenue rankings; by providing agents with

Also, standard revenue comparisons are made assuming bidders can finely extract rank and dispersion information from their valuations and adjust bidding accordingly. Our analysis attempts to understand what drives revenue comparisons when agents are less sophisticated.

In this vein, Proposition 5 provides a simple characterization based on the shape of the demand function ϕ, and it helps build intuition. In essence, first price is better when idiosyncratic elements have a small chance (but not too small) of being sufficiently dispersed. The seller benefits from this situation because "high valuation" bidders cannot take advantage of that dispersion: they cannot extract information about rank and dispersion from their valuation, so effectively, they do not know that they are "high valuation" bidders (if they could they would shade more, thereby decreasing sellers' revenue).

Finally, Proposition 5 also sheds light on revenue ranking in *standard models*: if the common component is drawn from a relatively flat distribution with large support (or a diffuse prior), then even very sophisticated bidders cannot extract information about rank, and the intuition above applies.

Comparative statics.

In standard models, the seller's revenue depends on the fine inferences about own valuation, rank and dispersion that agents make based on their value estimate. Access to other signals or information beyond one's value estimate modifies these inferences, hence also the seller's revenue. When the additional information released is affiliated with the bidders' signals, the effect on revenue is nonnegative (Milgrom and Weber, 1982).

We explored three kinds of comparative statics that separate the effect of further information on dispersion, on rank, and on own valuation. Propositions 6 and 7 show that the policy of disclosing information about the dispersion of valuations (or the number of bidders) while keeping symmetry has an anti-competitive effect, beneficial to bidders (hence detrimental to seller's revenue – since the allocation does not change).

Propositions 8 and 9 show that providing information about rank to bidders may have an ambiguous effect on bidders' gains and seller's revenue. In particular, revenue may decrease despite the fact that information about rank is positively correlated with valuation (which thus provides another

an additional private signal correlated with the other player's valuation, symmetry is broken, inefficiencies naturally arise, but these may be conducive to higher revenues.

illustration of the difference between affiliation and positive correlation).

Finally, we have examined the effects of the quality of estimates on bidders' gains and seller's revenue. When estimates are noisy, shading in the first price auction is driven two factors: the dispersion of estimates, and the selection bias (i.e., the error conditional on winning by a zero margin). Proposition 10 illustrates the two channels by which noisier estimates can increase shading: a weaker Bertrand competition effect (due to higher dispersion of estimate), and a stronger selection bias. When idiosyncratic elements are not too dispersed, noise benefits bidders and hurts the seller. For a given dispersion of idiosyncratic elements however, and depending on the shape of distributions, a small noise may benefit the seller. Nevertheless, and contrasting with Ganuza (2004) who shows that with two bidders ignorance promotes competition, we find that with two bidders, noisier estimates always hurt the seller in the second price auction. The reason for this difference has been explained earlier: standard models implicitly assume that weaker information translates into more concentrated posteriors, hence stronger competition.

7 Conclusion

We have proposed a model in which players only consider a limited set of strategies. This limitation can be interpreted as a bound on rationality, or a bound on the ability to determine what strategies are optimal when the strategy set is large. It can also be viewed as an analyst's device or methodological tool to deal with agents' lack of detailed knowledge over prior distributions, or to deal with agents' inability to use effectively this prior information.

We see various benefits from the approach. Starting from the sophisticated end of the spectrum, it offers a way to check the robustness of insights derived from standard models. It may constitute a useful alternative to the robustness literature, in particular when there are no easy or tractable ways to enrich the types space. It may also shed a different light on known results.

Starting from the other end of the spectrum (in terms of sophistication), it offers a more parsimonious theory of auctions, that can be amended by increasing sophistication, up to a degree that the analyst considers plausible.

Finally, it questions what poor information means. We have taken the view that poorer information means larger discrepancy between value and value estimate. Whether poorer information leads to more dispersed estimates or opinions is probably a matter of context, or at least an empirical question. The existence of a natural reference point to which one's data can be compared is probably a prerequisite to the conclusion that poor information leads to less dispersion. Given the well documented inability for agents to correctly take into account priors in forming beliefs (Tversky and Kahneman, 1974), the existence of such reference point may not be sufficient though.

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Appendix.

A. Buyer/Seller

We consider a seller with a value v_1 for the object to be sold, and a buyer with a value v_2 for the object. As before, we assume that

$$v_i = \alpha + \theta_i.$$

We are interested in comparing three selling mechanisms: take-it-or-leave it offer by the seller, take-it-or-leave it offer by the buyer and the split-the-difference mechanism (Chatterjee and Samuelson (1983)),[30] under the assumption that players are restricted to bidding rules of the form $r(v) = v + \gamma$, again capturing the idea that players cannot disentangle common and private components. The question asked is thus similar to that addressed by Lindsey et al. (1996).

Define

$$\phi(y) = \Pr\{\theta_2 - \theta_1 \geq y\}$$

and let $S(y)$ denote the expected surplus that results when transactions take place if and only if $\theta_2 > \theta_1 + y$, that is

$$S(y) = \int_{x>y} -x\phi'(x)dz.$$

We have:

> **Proposition 16**: *Let* $\gamma^* = \arg\max y\phi(y)$. *Whether the seller or the buyer makes a take-it-or-leave-it offer, or if players adopt the split-the-difference mechanism, the expected surplus is identical and equal to* $S(\gamma^*)$. *The seller prefers to let the buyer make the offer when*
>
> $$\int_{y>\gamma^*} \phi(y)dy > \gamma^*\phi(\gamma^*).$$
>
> *Otherwise he prefers to make the offer. Neither player finds the split-the-difference mechanism most attractive.*

[30] Under the split the difference mechanism (see Chatterjee and Samuelson (1983)), the buyer and seller simultaneously offer respectively prices p_1 and p_2. In the event $p_2 - p_1 \geq 0$, the transaction takes place at price $p = (p_1 + p_2)/2$, otherwise it does not take place.

As for the first and second price auction comparison, the critical issue is the extent to which ϕ has a fat tail. If ϕ has a fat tail, it is preferable to let the other player makes the offer: since "high valuation" buyers cannot learn (by assumption) from their valuation that they have indeed a high valuation, they tend to leave high rents to the seller.

B. Uniform and discriminatory auctions

We now consider k objects for sale with n potential buyers, all interested in buying only *one* unit. As before we assume that $v_i = \alpha + \theta_i$, and we are interested in comparing two selling mechanisms: The uniform auction in which the seller price is set at the $k+1^{th}$ bid, and the discriminatory auction where bidders pay their bids.

Proposition 5, which ranks revenues from first and second price auctions, easily extends. Denote by $\theta_i^{(k)}$ the k^{th} largest idiosyncratic term among bidders other than i, and define

$$\phi(y) = \Pr\{\theta_i - \theta_i^{(k)} \geq y\}$$

Our analysis of the first price auction extends to the discriminatory auction readily with this new definition for ϕ, and we have:

> **Proposition 17**: *The discriminatory auction generates more revenue than the uniform auction if and only if*
>
> $$\mid \phi'(0) \mid \int_{y \geq 0} \phi(y) dy > [\phi(0)]^2$$

The proof is identical to that of Proposition 5, and the intuition is the same. High dispersion (fat tail for ϕ) does not generate high rents for bidders in the discriminatory auction because bidders who happen to get a high realization do not know/realize it, so they cannot tailor shading to that event, realizing that increased shading would be profitable.

C. Proofs

Proof of Proposition 2: We already know that $\beta_n = -\phi'(0)$. By definition, $\phi(y) = \Pr(\theta_i - \max_{j \neq i} \theta_j \geq y) = \int_x f(x)(F(x-y))^{n-1} dx$, thus implying:

$$-\phi'(0) = \int (n-1)(f(x))^2 [F(x)]^{n-2} dx. \tag{4}$$

Integrating by parts the right hand side of (4) and observing that $f'(x) = f'(-x)$ and $F(-x) = 1 - F(x)$, we obtain:

$$\beta_n = f(\overline{\theta}) + \int_{\underline{x}}^{0} f'(x)([1-F(x)]^{n-1} - [F(x)]^{n-1})dx.$$

Now for any given $p < 1/2$, define $\Delta_n = (1-p)^n - p^n$, and $k_n = \Delta_n/(1-2p)$. We have $k_1 = 1$. Since

$$\Delta_{n+1} = (1-p)\Delta_n + (1-2p)p^n = (1-2p)((1-p)k_n + pp^{n-1})$$

This directly implies $k_n > p^{n-1}$ for all n, which further implies that k_{n+1} is an average of k_n and p^{n-1}, hence k_n is a strictly decreasing sequence. Applying this observation to each $p = F(x)$, we obtain that β_n is also a strictly decreasing sequence. To check the last assertion, assuming for example that $f'(\underline{x}) > 0$, it is sufficient to take a Taylor expansion of $F(x)$ close to $\underline{x}$ and observe that $[1-F(x)]^{n-1}$ remains bounded away from 0 on an interval of size $O(n^{-1/2})$. QED

Proof of Proposition 8. In what follows, we look for an equilibrium where both players follows (γ_1, γ_0), and where $z^* \equiv \gamma_1 - \gamma_0$ is strictly positive.[31] Define

$$\bar{\phi}(y) = \Pr\{\theta_i > \theta_j + y \mid \theta_i > \theta_j\} \text{ and } \underline{\phi}(y) = \Pr\{\theta_i > \theta_j + y \mid \theta_i < \theta_j\}.$$

We have $\bar{\phi}(y) = 2\phi(y)$ for $y \geq 0$, $\bar{\phi}(y) = 1$ for $y \leq 0$, and by symmetry $\underline{\phi}(y) = 1 - \bar{\phi}(-y)$. The event $(k_1, k_2) = (1,1)$ has probability $p(1-p)$, and conditional on this event, there is equal chance (1/2) that $\theta_1 > \theta_2$ or $\theta_1 < \theta_2$. The event $(k_1, k_2) = (1,0)$ has probability $(p^2 + (1-p)^2)/2$, and conditional on this event, $\theta_1 > \theta_2$ has probability $p^2/(p^2 + (1-p)^2)$. So for bidder 1, given that player 2 follows (γ_1, γ_0), the value from shading by γ in event $k_i = 1$ is:

$$V^1(\gamma) = \gamma[p^2\bar{\phi}(\gamma-\gamma^0) + p(1-p)\bar{\phi}(\gamma-\gamma^1) + p(1-p)\underline{\phi}(\gamma-\gamma^1) + (1-p)^2\underline{\phi}(\gamma-\gamma^0)].$$

Similarly, the value from bidding γ in event $k_i = 0$ is:

$$V^0(\gamma) = \gamma[p^2\underline{\phi}(\gamma-\gamma^1) + p(1-p)\underline{\phi}(\gamma-\gamma^0) + (1-p)p\bar{\phi}(\gamma-\gamma^0) + (1-p)^2\bar{\phi}(\gamma-\gamma^1))].$$

[31] It can be checked that if $z^* < 0$, then a player who received the weak signal $k_i = 0$ has strict incentives to reduce shading.

Assuming $\gamma_1 > \gamma_0$, first order conditions give:[32]

$$\gamma^1 = \frac{p^2\phi(z) + p(1-p)\phi(0)}{-[p^2\phi'(z) + p(1-p)\phi'(0)]} \text{ and}$$
$$\gamma^0 = \frac{p^2(1-2\phi(z)) + 2(1-p)\phi(0)}{-[2p^2\phi'(z) + 2(1-p)p\phi'(0)]}.$$

The equilibrium difference $z^* = \gamma^1 - \gamma^0$ thus solves:

$$z^* = \frac{4p^2\phi(z^*) - (p^2 + (1-p)^2)}{-[2p^2\phi'(z^*) + 2(1-p)p\phi'(0)]}.$$

So indeed, for any $p > 1/2$, this equation is compatible with z^* being strictly positive. Without information about rank, $\gamma^* = -\frac{\phi(0)}{\phi'(0)}$. Since $\frac{\phi(y)}{-\phi'(y)}$ is decreasing under A2, we get $\gamma^1 < \gamma^*$. QED

Proof of Proposition 9: Let $q \equiv \Pr\{v_i > v_j + \Delta\} = \phi(\Delta)(< 1/2)$. We assume $\Delta \geq \gamma^*$ and look for an equilibrium where $\gamma_0^* \leq \gamma^*$. There are three possible events, $k_1 = 1$, $k_2 = 1$ and $k_1 = k_2 = 0$, with respective probabilities q, q and $1 - 2q$. If i observes $k_i = 1$, player j must have seen $k_j = 0$, so i solves $\max_\gamma(\gamma_0^* + \Delta, \frac{\gamma\phi(\gamma-\gamma_0^*)}{q})$, and under A2, he thus finds optimal to shade by $\gamma_0^* + \Delta$ and wins with probability 1.[33] If player i observes $k_i = 0$, he wins only in events where $k_j = 0$ (so $v_i - v_j \in [-\Delta, \Delta]$) and $v_i - v_j \geq \gamma - \gamma_0^*$, so his expected gain is $\frac{\gamma(\phi(\gamma-\gamma_0^*)-q)}{1-q}$. First order conditions yield $\gamma_0^* = \frac{\phi(0)-\phi(\Delta)}{-\phi'(0)} = \gamma^*(1-2q)$, hence the desired conclusion that $\gamma_0^* \leq \gamma^*$. The allocation is unchanged compared to the case where no rank information is available. The winner shades by $\gamma_0^* + \Delta$ under events $k_1 = 1$ and $k_2 = 1$, and by γ_0^* under event $k_1 = k_2 = 0$. Expected gains are thus equal to

$$\gamma_0^* + 2q\Delta = (1-2q)\gamma^* + 2q\Delta > \gamma^*$$

Proof of Proposition 10: Define $x = z_i - \max_{j\neq i} z_j$ and let $h(\varepsilon_i, x)$ denote the joint distribution over ε_i and x, and let $H(y) = \int_{x\geq y} \varepsilon_i h(\varepsilon_i, x)d\varepsilon_i dx$. Note that $\psi_\varepsilon(y) = \frac{-H'(y)}{-\phi'_\varepsilon(y)}$. We have:

$$v(\gamma_i, \gamma) = \int_{x\geq\gamma_i-\gamma} (\gamma_i - \varepsilon_i)h(\varepsilon_i, x)d\varepsilon_i dx = \gamma_i\phi_\varepsilon(\gamma_i - \gamma) - H(\gamma_i - \gamma).$$

[32] Note that for either $\bar{\phi}$ or $\underline{\phi}$, right and left derivates do not coincide. However, for V^1 and V^0, they do coincide.

[33] This is because under A2, $\gamma\phi(\gamma - \gamma_0^*)$ is maximized for some $\gamma \leq \gamma^*$ when $\gamma_0^* \leq \gamma^*$.

Letting $x = \gamma_i - \gamma$, the first order condition can be written as the equality:

$$\gamma = -x + \frac{\phi_\varepsilon(x)}{-\phi'_\varepsilon(x)} + \psi_\varepsilon(x)$$

holding for $x = 0$, thus yielding the desired conclusion. Existence is guaranteed when the right hand side is monotonic in x. QED

Proof of Proposition 12: With two bidders, by symmetry, $\psi_\varepsilon(0) = 0$. Denote by g the density over $\varepsilon = \varepsilon_j - \varepsilon_i$. The density over $z_i - z_j$ is denoted h_ε. We have: $h_\varepsilon(y) = \int_\varepsilon h(y+\varepsilon)g(\varepsilon)d\varepsilon$. To evaluate how the seller's revenue is affected, we need to see how

$$E[z_i \mid z_i > z_j] - \gamma^*_\varepsilon = \int_{y\geq 0} \phi_\varepsilon(y)dy - \frac{1}{-2\phi'_\varepsilon(0)}$$

changes with noise.[34] The result obtains by taking Taylor expansions, omitting terms of order larger than 2 in ε:

$$\begin{aligned}\int_{y\geq 0}(\phi_\varepsilon(y) - \phi(y))dy &= \int_{y\geq 0}\int_\varepsilon (\phi(y+\varepsilon) - \phi(y))g(\varepsilon)d\varepsilon dy \\ &= \int_{y\geq 0}\phi''(y)dy \int_\varepsilon \frac{\varepsilon^2}{2}g(\varepsilon)d\varepsilon = h(0)\int_\varepsilon \frac{\varepsilon^2}{2}g(\varepsilon)d\varepsilon,\end{aligned}$$

while

$$-(\phi'_\varepsilon(0) - \phi'(0)) = h_\varepsilon(0) - h(0) = \int_\varepsilon (h(\varepsilon) - h(0))g(\varepsilon)d\varepsilon = h''(0)\int_\varepsilon \frac{\varepsilon^2}{2}g(\varepsilon)d\varepsilon$$

Proof of Proposition 13: Let $\rho = (1-p)^n/p$ and denote by A the event $\{\varepsilon_i = \max \varepsilon_j\}$. This event arises either when $\varepsilon_i = \bar{\varepsilon}$ or when all bidders are pessimistic so $\Pr A = p + (1-p)(1-p)^{n-1} = p(1+\rho)$. We have:

$$v_i(\gamma_i, \gamma) = (\sum_{n_0 \geq 1} \Pr(\widetilde{n} = n_0, \varepsilon_i = \bar{\varepsilon})\phi_{n_0}(\gamma_i - \gamma)(\gamma_i - \bar{\varepsilon})) + \Pr(A, \varepsilon_i = \underline{\varepsilon})\phi_n(\gamma_i - \gamma)(\gamma_i - \underline{\varepsilon}).$$

With uniform distributions $\phi'_n(0) = -1/\Delta$ for all $n > 1$. Since $\phi'_1(0) = 0$, and since $\phi_{n_0}(0) = \frac{1}{n_0}$, the first order condition implies:

$$\frac{1}{\Delta}[(\Pr A, \varepsilon_i = \bar{\varepsilon})(\gamma_i - \bar{\varepsilon}) + (1-p)^n(\gamma_i - \underline{\varepsilon})] \geq \sum \frac{1}{n_0}\Pr(A, \widetilde{n} = n_0, \varepsilon_i = \bar{\varepsilon}) + \frac{1}{n}\Pr(A, \varepsilon_i = \underline{\varepsilon}).$$

[34] This is because by symmetry, we have, $E[z_i \mid z_i > z_j] = \frac{1}{2}E[z_i - z_j \mid z_i > z_j] = \int_{y\geq 0} y h_\varepsilon(y)dy = \int_{y\geq 0}\phi_\varepsilon(y)dy$.

The left hand side coincides with $\frac{\Pr A}{\Delta}[\gamma_i - E[\varepsilon_i \mid A]]$. The right hand side corresponds to the expected probability of winning for bidder i, so by symmetry it must equal $\frac{1}{n}$. Since $\frac{1}{n} = (\Pr A)E[\frac{1}{\tilde{n}}]$, we obtain:

$$\gamma_\varepsilon^* \geq E[\varepsilon_i \mid A] + \Delta E[\frac{1}{\tilde{n}}]$$

Proof of Proposition 14: Using $H(y)$ as defined in Proposition 10, and $h(y) = -\phi_\varepsilon'(y)$, we have:

$$v(\gamma_i, \gamma) = \int_{y \geq \gamma_i - \gamma} y h_\varepsilon(y) dx + \gamma \phi_\varepsilon(\gamma_i - \gamma) - H(\gamma_i - \gamma)$$

First order conditions yield $\gamma \phi_\varepsilon'(0) = H'(0)$, hence the desired result. QED

Proof of proposition 16: When the seller makes an offer equal to $p = v_1 + \gamma_1$, the buyer accepts iff $\theta_2 - \theta_1 \geq \gamma_1$, hence the seller obtains an expected payoff equal to

$$G_S = v_1 + \gamma_1 \phi(\gamma_1).$$

When the buyer makes an offer $p = v_2 - \gamma_2$, the seller accepts if $p \geq v_1$, that is, if $\theta_2 - \gamma_2 \geq \theta_1$, hence the buyer obtains an expected payoff equal to

$$G_B = \gamma_2 \phi(\gamma_2).$$

The optimal values of γ_1 and γ_2 are thus the same, and we call this value $\gamma^* = \arg\max \gamma\phi(\gamma)$, and denote by G_S^* and G_B^* the corresponding gains for the seller and the buyer. Note that the expected surplus to be shared is the same whether the seller or the buyer makes the offer, and it is equal to $S(\gamma^*)$. Who makes the offer thus only affects how the expected surplus is shared.

To see how the expected surplus $S(\gamma^*)$ is shared, observe that when the buyer makes the offer, the seller obtains

$$\begin{aligned} R_S &= v_1 + E[\max(\theta_2 - \theta_1 - \gamma^*, 0)] \\ &= v_1 + S(\gamma^*) - \gamma^* \phi(\gamma^*). \end{aligned}$$

So the seller prefers to make the offer when $G_S^* > R_S$, that is, when

$$S(\gamma^*) < 2\gamma^* \phi(\gamma^*).$$

Since $S(y) = \int_{x>y} -x\phi'(x)dx = y\phi(y) + \int_{x>y} \phi(x)dx$, we conclude the proof.

Under the split the difference mechanism the seller chooses γ_1 and offers a price $p_1 = v_1 + \gamma_1$, while the buyer chooses γ_2 and offers a price $p_2 = v_2 - \gamma_2$. The transaction takes place in the event $p_2 - p_1 \geq 0$, that is, in the event $x_2 - x_1 \geq \gamma_1 + \gamma_2$, so the expected gain of the seller can be written as

$$\int_{y \geq \gamma_1 + \gamma_2} \frac{y + \gamma_1 - \gamma_2}{2} \phi(y)dy.$$

Similarly, the expected gain for the buyer can be written

$$\int_{y \geq \gamma_1 + \gamma_2} \frac{y + \gamma_2 - \gamma_1}{2} \phi(y)dy.$$

Let γ^* be as defined earlier. We verify that $\gamma_1^* = \gamma_2^* = \gamma^*/2$ is an equilibrium. Assume 1 chooses $b_1 = \gamma^*/2 + \delta$. Then he obtains a payoff $H(\delta)$:

$$H(\delta) = \frac{1}{2} \int_{y \geq \gamma^* + \delta} (y + \delta)\phi(y)dy = \frac{1}{2}\left(\int_{y \geq \gamma^* + \delta} (y - \gamma^*)\phi(y)dy + G(\gamma^* + \delta)\right).$$

Each of the terms on the right hand side is maximum for $\delta = 0$. So $\gamma_1^* = \gamma_2^* = \gamma^*/2$ is an equilibrium. **QED**