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"Implementation with Interdependent Valuations"
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Implementation with Interdependent Valuations*

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Abstract

It is well-known that the ability of the Vickrey-Clarke-Groves (VCG) mechanism to implement efficient outcomes for private value choice problems does not extend to interdependent value problems. When an agent's type affects other agents' utilities, it may not be incentive compatible for him to truthfully reveal his type when faced with CGV payments. We show that when agents are informationally small, there exist small modifications to CGV that restore incentive compatibility. We further show that truthful revelation is an approximate ex post equilibrium. Lastly, we show that in replicated settings aggregate payments sufficient to induce truthful revelation go to zero.

Keywords: Auctions, Incentive Compatibility, Mechanism Design, Interdependent Values, Ex Post Incentive Compatibility

JEL Classifications: C70, D44, D60, D82

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1. Introduction

There is a large literature aimed at characterizing the social choice functions that can be implemented in Bayes Nash equilibria. This literature typically takes agents' information as exogenous and fixed throughout the analysis. For some problems this may be appropriate, but the assumption is problematic for others. A typical analysis, relying on the revelation principle, maximizes some objective function subject to an incentive compatibility constraint requiring that truthful revelation be a Bayes-Nash equilibrium. It is often the case that truthful revelation is not ex post incentive compatible, that is, for a given agent, there are some profiles of the other agents' types for which the agent may be better off by misreporting his type than by truthfully revealing it. Truthful revelation, of course, may still be a Bayes equilibrium, because agents announce their types without knowing other agents' types: choices must be made on the basis of their beliefs about other agents' types. The assumption that agents' information is exogenous can lead to a difficulty: if truthful revelation is not ex post incentive compatible, then agents have incentives to learn other agents' types. To the extent that an agent can, at some cost, learn something about the types of other agents, then agents' beliefs at the stage at which agents actually participate in the mechanism must be treated as endogenous: if an agent can engage in preplay activities that provide him with some information about other agents' types, then that agent's beliefs when he actually plays the game are the outcome of the preplay activity.

A planner who designs a mechanism for which truthful revelation is ex post incentive compatible can legitimately ignore agents' incentives to engage in espionage to discover other agents' types, and consequently, ex post incentive compatibility is desirable. The Vickrey-Clarke-Groves- mechanism (hereafter VCG)[1] for private values environments is a classic example of a mechanism for which truthful revelation is ex post incentive compatible. For this mechanism, each agent submits his or her valuation. The mechanism selects the outcome that maximizes the sum of the agents' submitted valuations, and prescribes a transfer to each agent. These transfers can be constructed in such a way that it is a dominant strategy for each agent to reveal his valuation truthfully. Cremer and McLean (1985) (hereafter CM) consider a similar problem in which agents have private information, but interdependent valuations; that is, each agent's valuation can depend on other agents' information. They consider the mechanism design problem in which the aim is to maximize the revenue obtained from auctioning an

[1] See Clarke (1971), Groves (1973) and Vickrey (1961).

object. They analyze revelation games in which agents announce their types, and construct special transfers different from those in the VCG mechanism. Because each agent's valuation depends on other agents' announced types, truthful revelation will not generally be a dominant strategy in the CM mechanism. They show, however, that under certain conditions[2] truthful revelation will be ex post incentive compatible, i.e., the truth is an ex-post Nash equilibrium.

There has recently been renewed interest in mechanisms for which truthful revelation is ex post incentive compatible. Dasgupta and Maskin (2000), Perry and Reny (2002) and Ausubel (1999) (among others) have used the solution concept in designing auction mechanisms that assure an efficient outcome. Chung and Ely (2001) and Bergemann and Morris (2003) analyze the solution concept more generally. These papers (and Cremer and McLean), however, essentially restrict attention to the case in which an agent's private information is one dimensional[3], a serious restriction for many problems. Consider, for example, a problem in which an oil field is to be auctioned, and each agent may have private information about the quantity of the oil in the field, the chemical characteristics of the oil, the capacity of his refinery to handle the oil and the demand for the refined products in his market, all of which affect this agent's valuation (and potentially other agents' valuations as well). While the assumption that information is single dimensional is restrictive, Jehiel *et al.* (2006) show that for general mechanism design problems with interdependent values and multidimensional signals, for nearly all valuation functions, truthful revelation will be an ex post equilibrium only for trivial outcome functions.

Thus, except for extreme cases, we can hope for ex post equilibria for interdependent value problems only in the case of single dimensional information that . But even in the single dimensional case, there are difficulties. Most work on mechanism design in problems with asymmetric information begins with utilities of the form $u_i(c; t_i, t_{-i})$, where c is a possible outcome, t_i represents agent i's private information and t_{-i} is a vector representing other agents' private information. In the standard interpretation, u_i is a reduced form utility function that defines the utility of agent i for the outcome c under the particular circumstances likely to obtain given the agent's information. In the oil field problem above, for example, an agent's utility for the oil may depend on (among other things) the amount and chemical composition of the oil and the future demand for oil products, and the

[2]The conditions are discussed in section 3.

[3]Formally, what is necessary is that agents' types are ordered in a particular way that typically fails in multidimensional information settings.

information of other agents will affect i's (expected) value for the field insofar as i's beliefs about the quantity and composition of the oil and the demand for oil products are affected by their information. In this paper, we begin from more primitive data in which i has a utility function $v_i(c, \theta; t_i)$ where θ is a complete description of the state of nature and t_i represents his private information. For the oil example, θ would include those things that affect i's value for the oil – the amount and composition of the oil, the demand for oil, etc. The relationship between agents' private information and the state is given by a probability distribution P over $\Theta \times T$. This formulation emphasizes the fact that the information possessed by other agents will affect agent i precisely to the extent that the information of others provides information about the state of nature.

The reduced form utility function that is normally the starting point for mechanism design analysis can be calculated from this more primitive structure: $u(c, t) \equiv \Sigma_\theta v_i(c, \theta; t) P(\theta|t)$. Most work that employs ex post incentive compatibility makes additional assumptions regarding the reduced form utility functions u_i. It is typically assumed that each agent's types are ordered, and that agents' valuations are monotonic in any agent's type. Further, it is assumed that the utility function of each individual agent satisfies a classic single-crossing property and that, across agents, their utilities are linked by an "interagent crossing property." This latter property requires that a change in an agent's type from one type to a higher type causes his valuation to increase at least as much as any other agent's valuation. We show that the conditions on the primitive data of the problem that would ensure that the reduced form utility functions satisfy these crossing properties are stringent; the reduced form utility functions associated with very natural single dimensional information problems can fail to satisfy these properties.

In summary, while ex post incentive compatibility is desirable, nontrivial mechanisms for which truthful revelation is ex post incentive compatible fail to exist for a large set of important problems. We introduce in this paper a notion of *weak $\varepsilon-$ex post incentive compatibility:* a mechanism is weakly $\varepsilon-$ex post incentive compatible if truthful revelation is ex post incentive compatible with conditional probability at least $1 - \varepsilon$. If truthful revelation is weakly ε-ex post incentive compatible for a mechanism, then the incentive that agents have to collect information about other agents is bounded by ε times the maximal gain from espionage. If espionage is costly, a mechanism designer can be relatively comfortable in taking agents' beliefs as exogenous when ε is sufficiently small. We show that the existence of mechanisms for which there are weakly ε-incentive compatible equilibria is related to the concept of informational size introduced in McLean and

Postlewaite (2002, 2004). When agents have private information, the posterior probability distribution on the set of states of the nature Θ will vary depending on a given agent's type. Roughly, an agent's informational size corresponds to the maximal expected change in the posterior on Θ as his type varies, fixing other agents' types. We show that for any ε, there exists a δ such that, if each agent's informational size is less than δ, then there exists an efficient mechanism for which truthful revelation is a weak ε-ex post incentive compatible equilibrium.

The weakly ε-ex post incentive compatible mechanism that is used in the proof of the result elicits agents' private information and employs payments to agents that depend on their own announcement and the announcements of others. The payments employed are nonnegative and are small when agents are informationally small. When there are many agents, each will typically be informationally small, and hence, the payment needed to elicit truthful revelation of any agent's private information will be small. But the accumulation of a large number of small payments is potentially large. We show, however, that for a replica problem in which the number of agents goes to infinity, agents' informational size goes to zero exponentially and the aggregate payments needed to elicit the private information necessary to ensure efficient outcomes goes to zero.

We describe the model in the next section and provide a brief history of ex post incentive compatibility in Section 3. In Section 4 we introduce a generalized VCG mechanism and, in Lemma 1, we identify a property of the generalized VCG mechanism that is fundamental to all of our results. In section 5, we present the relationship between ex -post incentive compatibility and nonexclusive information and extend these observations to the relationship between approximate ex post incentive compatibility and small informational size in Section 6. In Section 7, we demonstrate that, when agents have sufficiently small informational size the generalized VCG mechanism can be modified by adding small positive transfers so as to induce truthful revelation as an approximate ex post incentive compatible and an exact Bayes-Nash equilibrium. This result shows that the additional transfers required to effect exact interim incentive compatibility are small when agents are informationally small but does not address the size of the sum of these transfers. In Section 8, we show that, in a "conditionally independent" informational framework, this sum becomes small as the number of agents increases. The paper concludes with the discussion Section 9.

2. The Model

Let $\Theta = \{\theta_1, .., \theta_m\}$ represent the finite set of states of nature and let T_i denote the finite set of types of player i. Let C denote the finite set of social alternatives. Agent $i's$ payoff is represented by a nonnegative valued function $v_i : C \times \Theta \times T_i \to \mathbb{R}_+$. We will assume that there exists $c_0 \in C$ such that $v_i(c_0, \theta, t_i) = 0$ for all $(\theta, t_i) \in \Theta \times T_i$ and that there exists $M > 0$ such that $v_i(\cdot, \cdot, \cdot) \leq M$ for each i. Since v_i takes on only nonnegative values, c_0 is the "uniformly worst" outcome for all agents. We will say that v_i satisfies the *pure common value property* if v_i depends only on $(c, \theta) \in C \times \Theta$ and the *pure private value property* if v_i depends only on $(c, t_i) \in C \times T_i$. Our notion of common value is more general than that typically found in the literature in that we do not require that all agents have the same value for a given decision. According to our definition of pure common value, an agent's "fundamental" valuation depends only on the state θ, and not on any private information he may have.

Let $(\widetilde{\theta}, \widetilde{t}_1, \widetilde{t}_2, ..., \widetilde{t}_n)$ be an (n+1)-dimensional random vector taking values in $\Theta \times T$ $(T \equiv T_1 \times \cdots \times T_n)$ with associated distribution P where

$$P(\theta, t_1, .., t_n) = Prob\{\widetilde{\theta} = \theta, \widetilde{t}_1 = t_1, ..., \widetilde{t}_n = t_n\}.$$

We will make the following full support assumptions regarding the marginal distributions: $P(\theta) =$Prob$\{\widetilde{\theta} = \theta\} > 0$ for each $\theta \in \Theta$ and $P(t_i) =$Prob$\{\widetilde{t}_i = t_i\} > 0$ for each $t_i \in T_i$. If K is a finite set, let $\Delta(K)$ denote the set of probability measures on K. The set of probability measures in $\Delta(\Theta \times T)$ satisfying the full support conditions will be denoted $\Delta^*_{\Theta \times T}$. If $P \in \Delta^*_{\Theta \times T}$, let $T^* := \{t \in T | P(t) > 0.\}$ (The set T^* depends on P but we will suppress this dependence to keep the notation lighter.)

In many problems with differential information, it is standard to assume that agents have utility functions $u_i : C \times T \to R_+$ that depend on other agents' types. It is worthwhile noting that, while our formulation takes on a different form, it is equivalent. Given a problem as formulated in this paper, we can define $u_i(c, t_{-i}, t_i) = \sum_{\theta \in \Theta} [v_i(c, \theta, t_i) P(\theta | t_{-i}, t_i)]$. Alternatively, given utility functions $u_i : C \times T \to \mathbb{R}_+$, we can define $\Theta \equiv T$ and define $v_i(c, t, t_i') = u_i(c, t_{-i}, t_i')$. Our formulation will be useful in that it highlights the nature of the interdependence: agents care about other agents' types to the extent that they provide additional information about the state θ. Because of the separation of an agent's fundamental valuation function from other agents' information, this formulation allows an analysis of the effects of changing the information structure while keeping an agent's fundamental valuation function fixed.

A *social choice problem* is a collection $(v_1, .., v_n, P)$ where $P \in \Delta^*_{\Theta\times T}$. An *outcome function* is a mapping $q : T \to C$ that specifies an outcome in C for each profile of announced types. We will assume that $q(t) = c_0$ if $t \notin T^*$, where c_0 can be interpreted as a status quo point. A *mechanism* is a collection $(q, x_1, .., x_n)$ (written simply as $(q, (x_i))$ where $q : T \to C$ is an outcome function and the functions $x_i : T \to \mathbb{R}$ are transfer functions. For any profile of types $t \in T^*$, let

$$\hat{v}_i(c; t) = \hat{v}_i(c; t_{-i}, t_i) = \sum_{\theta\in\Theta} v_i(c, \theta, t_i)P(\theta|t_{-i}, t_i).$$

Although $\hat{v}$ depends on P, we suppress this dependence for notational simplicity as well. Finally, we make the simple but useful observation that the pure private value model is mathematically identical to a model in which $|\Theta| = 1$ where $|K|$ denotes the cardinality of a finite set K.

Finally, we make the following important notational convention. Throughout the paper, $||\cdot||_2$ will denote the 2-norm and, for notational simplicity, $||\cdot||$ will denote the 1-norm. The real vector spaces on which these norms are defined will be clear from the context.

Definition: Let $(v_1, .., v_n, P)$ be a social choice problem. A mechanism $(q, (x_i))$ is:

ex post incentive compatible if truthful revelation is an *ex post Nash equilibrium*: for all $i \in N$, all $t_i, t_i' \in T_i$ and all $t_{-i} \in T_{-i}$ such that $(t_{-i}, t_i) \in T^*$,

$$\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + x_i(t_{-i}, t_i) \geq \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + x_i(t_{-i}, t_i').$$

strongly ex post incentive compatible if truthful revelation is an *ex post dominant strategy* equilibrium: for all $i \in N$, all $t_i, t_i' \in T_i$, all $\sigma_{-i} \in T_{-i}$ and all $t_{-i} \in T_{-i}$ such that $(t_{-i}, t_i) \in T^*$,

$$\hat{v}_i(q(\sigma_{-i}, t_i); t_{-i}, t_i) + x_i(\sigma_{-i}, t_i) \geq \hat{v}_i(q(\sigma_{-i}, t_i'); t_{-i}, t_i) + x_i(\sigma_{-i}, t_i').$$

interim incentive compatible if truthful revelation is a *Bayes-Nash equilibrium*: for each $i \in N$ and all $t_i, t_i' \in T_i$

$$\sum_{\substack{t_{-i}\in T_{-i}\\ :(t_{-i},t_i)\in T^*}} \left[\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + x_i(t_{-i}, t_i)\right] P(t_{-i}|t_i)$$
$$\geq \sum_{\substack{t_{-i}\in T_{-i}\\ :(t_{-i},t_i)\in T^*}} \left[\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + x_i(t_{-i}, t_i')\right] P(t_{-i}|t_i)$$

ex post individually rational if

$$\hat{v}_i(q(t);t) + x_i(t) \geq 0 \text{ for all } i \text{ and all } t \in T^*.$$

feasible if for each $t \in T^*$,

$$\sum_{j \in N} x_j(t) \leq 0.$$

balanced if for each $t \in T^*$,

$$\sum_{j \in N} x_j(t) = 0.$$

outcome efficient if for each $t \in T^*$,

$$q(t) \in \arg\max_{c \in C} \sum_{j \in N} \hat{v}_j(c;t).$$

Clearly, strong ex-post IC implies ex post IC and ex post IC implies interim IC. If, for all i, $\hat{v}_i(c;t)$ does not depend on t_{-i}, then the notions of ex post dominant strategy equilibrium and ex post Nash equilibrium coincide. In this private value setting, the two definitions actually reduce to the usual notion of dominant strategy equilibrium. There is, of course, a definition of dominant strategy equilibrium that is appropriate for the actual Bayesian game. This (interim) equilibrium concept is weaker than ex post dominant strategy equilibrium and stronger than Bayes-Nash equilibrium, but is not logically nested with respect to ex-post Nash equilibrium. For a discussion of the relationship between ex post dominant strategy equilibrium, dominant strategy equilibrium, ex post Nash equilibrium and Bayes-Nash equilibrium, see Cremer and McLean (1985) (henceforth, CM (1985)).

3. Monotonicity and Implementation with Interdependent Values

As mentioned in the introduction, the typical modeling approach to mechanism design and implementation in quasilinear settings with interdependent valuations begins with a collection of functions $u_i : C \times T \to \mathbb{R}$ as the primitive objects of study. In the typical problem, the elements of each T_i are totally ordered and two "crossing" properties (see below) are imposed. For concreteness, consider an auction for a single indivisible object when agents' valuations for the object exhibit interdependency. In this problem, one is typically interested in constructing an ex post efficient auction that awards the object to the bidder who values it most. While not usually stated explicitly, the typical approach to the ex post efficient auction problem actually consists of two distinct parts. In the first step, an "interagent crossing condition" (e.g., assumption 3A in CM (1985) is used to show that ex post efficient allocation of the object is monotone with respect to agents' types: the probability that agent i is awarded the object when he is of type t_i is not more than the probability that agent i is awarded the object when he is of type $t'_i > t_i$. The second step consists of showing that monotonicity of the allocation rule established in step 1, together with the usual "single crossing property", allows one to explicitly construct an ex post efficient auction mechanism for which truthful reporting is an ex post Nash equilibrium.

To our knowledge, the earliest construction of an ex post IC mechanism in the interdependent framework that identifies the role of monotonicity in this two step approach to ex post efficient implementation appears in CM (1985). In their setup, $T_i = \{1, 2, ..., m_i\}$ and $C \subseteq \mathbb{R}^n$ and they define a class of mechanisms as follows.

Definition 1: Let $q : T \to C$ be an outcome function. An *E(xtraction)-mechanism* is any mechanism $(q, (x_i))$ satisfying

$$x_i(t_{-i}, t_i) = x_i(t_{-i}, 1) - \sum_{\sigma_i=2}^{t_i} [u_i(q_i(t_{-i}, \sigma_i), t_{-i}, \sigma_i) - u_i(q_i(t_{-i}, \sigma_i - 1), t_{-i}, \sigma_i)]$$

whenever $t_{-i} \in T_{-i}$ and $t_i \in T_i \backslash \{1\}$.

There are many E- mechanisms, depending on the specification of $x_i(t_{-i}, 1)$ for each i and $t_{-i} \in T_{-i}$. In their 1985 paper, CM define such mechanisms and use them (in conjunction with a full rank condition) to derive their full extraction results. If $t_i \mapsto q_i(t_{-i}, t_i)$ is monotonic for each i and t_{-i} and if each u_i satisfies the classic single crossing property, then an E-mechanism will implement q as an ex post Nash equilibrium (this is Lemma 2 in CM (1985).)

If one is interested in implementing a specific outcome function (e.g., an ex post efficient outcome function), then one must make further assumptions that guarantee that q satisfies the monotonicity condition. This is step 1 in our previous discussion and this is the point at which the interagent crossing property comes into play. We will illustrate this in the special case of a single object auction with interdependent valuations studied in CM (1985). In this case, a single object is to be allocated to one of n bidders. If i receives the object, his value is the nonnegative number $w_i(t)$. In this framework, $q(t) = (q_1(t), .., q_n(t))$ where each $q_i(t) \geq 0$ and $q_1(t) + \cdots + q_n(t) \leq 1$ and

$$u_i(q_i(t_{-i}, t'_i); t_{-i}, t_i) + x_i(t_{-i}, t'_i) = q_i(t_{-i}, t'_i)w_i(t_{-i}, t_i) + x_i(t_{-i}, t'_i)$$

and outcome efficiency means that

$$\sum_{i \in N} q_i(t)w_i(t) = \max_{i \in N}\{w_i(t)\}.$$

The next result is extracted from Corollary 2 in CM (1985).

Theorem 1: Suppose that
(i) for each $i \in N, t_{-i} \in T_{-i}, t_i \in T_i \backslash \{m_i\}$

$$w_i(t_{-i}, t_i) < w_i(t_{-i}, t_i + 1)$$

(ii) For all $i, j \in N, t_{-i} \in T_{-i}, t_i \in T_i \backslash \{m_i\}$

$$w_i(t_{-i}, t_i + 1) - w_j(t_{-i}, t_i + 1) \geq w_i(t_{-i}, t_i) - w_j(t_{-i}, t_i)$$

Then there exists an outcome efficient, ex post IR, ex post IC auction mechanism.

Condition (i) is the single crossing property (called Assumptions $2'$ in CM (1985) which, in the auction case, reduces to the simple assumption that an agent's valuation for the object is increasing in his own type. Condition (ii) is the inter-agent crossing property (called Assumptions $3A'$ in CM (1985) that guarantees that i's probability of winning (i.e., $q_i(t_{-i}, t_i)$) is nondecreasing in i's type t_i.

In this paper, we do not take the $u_i : C \times T \to \mathbb{R}$ as the primitive objects of study. Instead, we derive the reduced form $\hat{v}_i : C \times T \to \mathbb{R}$ from the function $v_i : C \times \Theta \times T_i \to \mathbb{R}_+$ and the conditional distributions $P_\Theta(\cdot|t)$. In a single object auction framework (such as that studied in McLean and Postlewaite (2004)), this reduced form payoff for bidder i is defined by the reduced form valuation function

$$\hat{w}_i(t) = \sum_\theta w_i(\theta, t_i)P_\Theta(\theta|t).$$

In this natural special case, the applicability of a result such as Theorem 1 may require quite restrictive assumptions regarding the underlying data. For example, suppose that $w_i(\theta, t_i) = \alpha_i\theta + \beta_i$ for each i where $\alpha_i > 0$. Then

$$\hat{w}_i(t) = \alpha_i \left[\sum_\theta \theta P_\Theta(\theta|t)\right] + \beta_i := \alpha_i\overline{\theta}(t) + \beta_i$$

and conditions (i) and (ii) of Theorem 1 can only be satisfied if $\alpha_i = \alpha_j$. To see this, note that (ii) is satisfied only if

$$(\alpha_i - \alpha_j)\left[\overline{\theta}(t_{-i}, t_i + 1) - \overline{\theta}(t_{-i}, t_i)\right] \geq 0$$

and

$$(\alpha_j - \alpha_i)\left[\overline{\theta}(t_{-j}, t_j + 1) - \overline{\theta}(t_{-j}, t_j)\right] \geq 0$$

for each i and j. Consequently, (i) implies that $\alpha_i = \alpha_j$.

These observations are not restricted to the case in which each T_i is a totally ordered finite set; it is straightforward to show that the same implication holds for the case that agents' type sets are intervals.

We emphasize again that, in this paper, we do not take the $u_i : C \times T \to \mathbb{R}$ as the primitive objects of study. We do not investigate the assumptions that v_i and $P_\Theta(\cdot|t)$ would need to satisfy in order to apply the monotonicity technique to the reduced form $\hat{v}_i$. Instead, we take a complementary approach and make certain assumptions regarding the distribution $P \in \Delta^*_{\Theta\times T}$ but no assumptions regarding the primitive valuation function v_i.

4. The Generalized VCG Mechanism and Ex post Incentive Compatibility

Let q be an outcome function and define transfers as follows:

$$\begin{aligned}\alpha_i^q(t) &= \sum_{j\in N\backslash i} \hat{v}_j(q(t); t) - \max_{c\in C}\left[\sum_{j\in N\backslash i} \hat{v}_j(c; t)\right] \text{ if } t \in T^* \\ &= 0 \text{ if } t \notin T^*\end{aligned}$$

Note that $\alpha_i^q(t) \leq 0$ for each i and t. The resulting mechanism $(q, (\alpha_i^q))$ is the *generalized VCG mechanism with interdependent valuations* (GVCG for short.)

(Ausubel (1999) and Chung and Ely (2002) use the term generalized Vickrey mechanisms, but for different classes of mechanisms.) It is straightforward to show that the GVCG mechanism is ex post individually rational and feasible. If $\hat{v}_i$ depends only on t_i (as in the pure private value case case where $|\Theta| = 1$ or, more generally, when $\tilde{\theta}$ and $\tilde{t}$ are stochastically independent), then the GVCG mechanism reduces to the classical VCG mechanism for private value problems and it is well known that, in this case, the VCG mechanism satisfies strong ex post IC. In general, however, the GVCG mechanism will *not even* satisfy interim IC. However, we will show that the GVCG mechanism is ex post IC when P satisfies a property called nonexclusive information (Postlewaite and Schmeidler (1986).

Before proceeding to the main result for nonexclusive information, let us review the logic of the VCG mechanism in the case of pure private values. In that case, we obtain (abusing notation slightly),

$$\begin{aligned}\alpha_i^q(t) &= \sum_{j\in N\backslash i} \hat{v}_j(q(t);t_j) - \max_{c\in C}\left[\sum_{j\in N\backslash i} \hat{v}_j(c;t_j)\right] \text{ if } t\in T^* \\ &= 0 \text{ if } t\notin T^*\end{aligned}$$

In computing $\max_{c\in C}\left[\sum_{j\in N\backslash i}\hat{v}_j(c;t_j)\right]$, we maximize the total payoff of the players in $N\backslash i$ and, as a consequence of the pure private values assumption, only utilize the information of the agents in $N\backslash i$. Hence, the value of the optimum only depends on t_{-i}. In the interdependent case, this computation can be extended in two ways. First, we could maximize the total payoff of the players in $N\backslash i$ using the information of all agents. The associated transfer is then equal to

$$\sum_{j\in N\backslash i} \hat{v}_j(q(t);t) - \max_{c\in C}\sum_{j\in N\backslash i}\left[\sum_{\theta\in\Theta} v_j(c,\theta,t_j)P(\theta|t_{-i},t_i)\right].$$

Alternatively, we could maximize the total payoff of the players in $N\backslash i$ using only the information of the agents in $N\backslash i$. The associated transfer is then equal to

$$\sum_{j\in N\backslash i} \hat{v}_j(q(t);t) - \max_{c\in C}\sum_{j\in N\backslash i}\left[\sum_{\theta\in\Theta} v_j(c,\theta,t_j)P(\theta|t_{-i})\right].$$

In the first payment scheme, agent i pays the cost that he imposes on other agents assuming that they have access to his information even though he is not present.

In the second scheme, agent i pays the cost that he imposes on other agents assuming that the other agents do not have access to his information. In the pure private values model, these two approaches yield the same transfer scheme.

These payment schemes induce different games in the case of interdependent values. We are interested in the first of the payment schemes that uses agent i's information when calculating the cost that he imposes on other agents. One can think of the designer's problem as encompassing two stages. In the first stage, the designer elicits the agents' information to determine the posterior probability distribution over the states and makes that probability distribution available to the agents. The second stage consists of a VCG mechanism where the agents' values are computed with respect to the probability distribution from the first stage. If the designer has elicited truthful revelation in the first stage, the problem in the second stage is a private values problem, and truthful revelation is a dominant strategy. The interdependence of agents matters only for the first stage; our method is to show how the designer can extract the information needed to compute the probability distribution over the states, following which the problem becomes a private value problem. In this private value problem, the first payment scheme mimics the standard VCG mechanism. This intuition can be formalized using the two stage implementation game presented in Section 7.3 below.

We next identify a special "gain-bounded" property of the GVCG mechanism that is key to our results. (All proofs are relegated to the appendix.)

Lemma A: Suppose that $q : T \to C$ is outcome efficient for the problem $(v_1, .., v_n, P)$. If $(t_{-i}, t_i), (t_{-i}, t'_i) \in T^*$, then

$$\begin{aligned} &\hat{v}_i(q(t_{-i}, t'_i); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t'_i) - \ \hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i) \\ &\leq 2M(n-1)||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t'_i)|| \end{aligned}$$

In the case of the GVCG mechanism, Lemma A provides an upper bound on the "ex post gain" to agent i when i's true type is t_i but i announces t'_i and others announce truthfully. An important implication of Lemma A is that an agent's gain by misreporting his type is essentially bounded by the degree to which his type affects the posterior probability distribution on Θ; we return to this below.

5. Ex post Incentive Compatibility and Nonexclusive Information

If v_i does not depend on θ, then (letting $|\Theta| = 1$), we recover Vickrey's classic dominant strategy result for the VCG mechanism in the pure private values case as a special case of Lemma A. We can use lemma A to extend the classic private values result to a special class of problems with interdependent valuations in which ex post Nash equilibrium replaces dominant strategy equilibrium. These are the problems in which P exhibits *nonexclusive information*.

Definition 2: A measure $P \in \Delta^*_{\Theta \times T}$ satisfies *nonexclusive information* (NEI) if

$$t \in T^* \Rightarrow P_\Theta(\cdot|t) = P_\Theta(\cdot|t_{-i}) \text{ for all } i \in N$$

or, equivalently, if

$$[(t_{-i}, t_i) \in T^* \text{ and } (t_{-i}, t'_i) \in T^*] \Rightarrow P_\Theta(\cdot|t_{-i}, t_i) = P_\Theta(\cdot|t_{-i}, t'_i) \text{ for all } i \in N.$$

As an immediate application of Lemma A, we have the following result.

Proposition 1: Let $\{v_1, .., v_n\}$ be a collection of payoff functions. If $P \in \Delta^*_{\Theta \times T}$ exhibits nonexclusive information and if $q : T \to C$ is outcome efficient for the problem $(v_1, .., v_n, P)$, then the GVCG mechanism (q, α^q_i) is ex post IC and ex post IR.

If agents have "zero informational size" – that is, if P exhibits nonexclusive information – then $||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t'_i)|| = 0$ if $(t_{-i}, t_i), (t_{-i}, t'_i) \in T^*$. Hence, truth is an ex post Nash equilibrium as a consequence of Proposition 1. Note that the private values problem in which v_i does not depend on θ is the special case of NEI where $|\Theta| = 1$. Since ex post Nash equilibrium coincides with dominant strategy equilibrium in the private values case, we conclude that Proposition 1 implies Vickrey's classic dominant strategy result for the VCG mechanism in the pure private values case.

It is important to point out that every "reduced form" valuation function $t \mapsto u_i(c; t)$ is expressible as

$$u_i(c; t) = \sum_{\theta \in \Theta} v_i(c, \theta, t_i) P(\theta | t_{-i}, t_i)$$

by defining $\Theta = \Pi_{i\in N} T_i$, $P(\theta|t) = 1$ if and only if $\theta = t$ and

$$v_i(c, \theta, t_i) = u_i(c; \theta).$$

For this formulation of Θ however, the distribution $P \in \Delta^*_{\Theta\times T}$ exhibits NEI if and only if $P(t_i|t_{-i}) = 1$ whenever $P(t_{-i}, t_i) > 0$. Consequently, the vector t_{-i} determines t_i when the NEI assumption holds and the support $T^* \neq T$.

Nonexclusive information, while subsuming the private values model, is a strong assumption. Our goal in this paper is to identify conditions under which we can modify the GVCG payments so that the new mechanism is interim IC. We begin by presenting a continuity result that is motivated by Proposition 1. If we (informally) think of NEI as meaning that an agent has no effect on the posterior distribution on Θ in the presence of the information of other agents, then we can interpret Proposition 1 as follows: if each agent has "no information effect" on the posterior on Θ, then the GVCG is "exactly ex post incentive compatible". We will prove the following continuity result: if each agent has a "small information effect" on the posterior on Θ, then the GVCG is "approximately" ex post incentive compatible. Of course, this result requires that the notions of "small informational effect" and "approximate ex post incentive compatibility" be formalized and to accomplish this, we introduce the notions of informational size and ε- ex post Nash equilibrium in the next section.

6. Approximate Ex Post Incentive Compatibility and Small Informational Size

6.1. Informational Size

If $t \in T^*$, recall that $P_\Theta(\cdot|t) \in \Delta_\Theta$ denotes the induced conditional probability measure on Θ. A natural notion of an agent's informational size is one that measures the degree to which he can alter the best estimate of the state θ when other agents are announcing truthfully. In our setup, that estimate is the conditional probability distribution on Θ given a profile of types t. Any profile of agents' types $t = (t_{-i}, t_i) \in T^*$ induces a conditional distribution on Θ and, if agent i unilaterally changes his announced type from t_i to t'_i, this conditional distribution will (in general) change. We consider agent i to be informationally small if, for each t_i, there is a "small" probability that he can induce a "large" change in the induced conditional distribution on Θ by changing his announced type from t_i to some other t'_i. This is formalized in the following definition.

Definition 3: Let

$I^i_\varepsilon(t'_i, t_i) = \{t_{-i} \in T_{-i} | (t_{-i}, t_i) \in T^*, (t_{-i}, t'_i) \in T^* \text{ and } ||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t'_i)|| > \varepsilon\}$

The *informational size* of agent i is defined as

$$\nu^P_i = \max_{t_i \in T_i} \max_{t'_i \in T_i} \min\{\varepsilon \geq 0 | \text{ Prob}\{\tilde{t}_{-i} \in I^i_\varepsilon(t'_i, t_i) | \tilde{t}_i = t_i\} \leq \varepsilon\}.$$

Loosely speaking, we will say that agent i is *informationally small* with respect to P if his informational size ν^P_i is small. If agent i receives signal t_i but reports $t'_i \neq t_i$, the effect of this misreport is a change in the conditional distribution on Θ from $P_\Theta(\cdot|t_{-i}, t_i)$ to $P_\Theta(\cdot|t_{-i}, t'_i)$. If $t_{-i} \in I_\varepsilon(t'_i, t_i)$, then this change is "large" in the sense that $||P_\Theta(\cdot|\hat{t}_{-i}, t_i) - P_\Theta(\cdot|\hat{t}_{-i}, t'_i)|| > \varepsilon$. Therefore, $\text{Prob}\{\tilde{t}_{-i} \in I_\varepsilon(t'_i, t_i) | \tilde{t}_i = t_i\}$ is the probability that i can have a "large" influence on the conditional distribution on Θ by reporting t'_i instead of t_i when his observed signal is t_i. An agent is informationally small if for each of his possible types t_i, he assigns small probability to the event that he can have a "large" influence on the distribution $P_\Theta(\cdot|t_{-i}, t_i)$, given his observed type. Informational size is closely related to the notion of nonexclusive information: if all agents have zero informational size, then P must satisfy NEI. In fact, we have the following easily demonstrated result: $P \in \Delta^*_{\Theta \times T}$ satisfies NEI if and only if $\nu^P_i = 0$ for each $i \in N$. If $T^* = T$, then ν^P is the *Ky Fan distance* between the r.v.s $P_\Theta(\cdot|\tilde{t}_{-i}, t_i)$ and$P_\Theta(\cdot|\tilde{t}_{-i}, t_i)$ with respect to the probability measure $P_{T_{-i}}(\cdot|t_i)$ (see, e.g., Dudley (2002), Section 9.2).[4]

6.2. Approximate Ex Post Incentive Compatibility

Definition 4: Let $\varepsilon \geq 0$. A mechanism $(q, (x_i))$ is *weakly* $\varepsilon-$ ex post incentive compatible if for all i and all $t_i, t'_i \in T_i$,

$$\begin{aligned} \Pr ob\{(\tilde{t}_{-i}, t_i) &\in T^* \text{ and } \hat{v}_i(q(\tilde{t}_{-i}, t'_i); \tilde{t}_{-i}, t_i) + x_i(\tilde{t}_{-i}, t'_i)) \\ &> \hat{v}_i(q(\tilde{t}_{-i}, t_i); \tilde{t}_{-i}, t_i) + x_i(\tilde{t}_{-i}, t_i) + \varepsilon | \tilde{t}_i = t_i\} \leq \varepsilon. \end{aligned}$$

Note that $(q, (x_i))$ is a weakly $0-$ ex post incentive compatible mechanism if and only if $(q, (x_i))$ is an ex post incentive compatible mechanism.

Other notions of approximate incentive compatibility and informational size are possible and these are discussed in Section 9.2 below.

[4] If X and Y are random variables defined on a probability space $(\Omega, \mathcal{F}, \mu)$ taking values in a metric space (S, d), then the Ky Fan metric is defined as $\min[\varepsilon \geq 0 : \mu\{d(X, Y) > \varepsilon\} \leq \varepsilon]$. If $T^* = T$, then ν^P is the Ky Fan distance between the r.v.s $X = P_\Theta(\cdot|\tilde{t}_{-i}, t_i)$ and $Y = P_\Theta(\cdot|\tilde{t}_{-i}, t_i)$ with respect to the probability measure $\mu = P_{T_{-i}}(\cdot|t_i)$.

6.3. The Result

Proposition 2: Suppose that $q : T \rightarrow C$ is outcome efficient for the problem $(v_1, .., v_n, P)$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that, if $\nu_i^P < \delta$ for each agent i, then the GVCG mechanism $(q, (\alpha_i^q))$ is weakly $\varepsilon-$ ex post incentive compatible.

To explain Proposition 2, note that Lemma A provides an upper bound on the "ex post gain" to agent i when i's true type is t_i but i announces t_i' and others announce truthfully. If agent i is informationally small, then (informally) we can deduce that

$$\Pr ob\{||P_\Theta(\cdot|\tilde{t}_{-i}, t_i) - P_\Theta(\cdot|\tilde{t}_{-i}, t_i')|| \approx 0|\tilde{t}_i = t_i\} \approx 1$$

so truth is an approximate ex post equilibrium for the GVCG in the sense that

$$\Pr ob\{(\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i)) - (\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i')) \underset{\approx}{>} 0|\tilde{t}_i = t_i\} \approx 1.$$

Consequently, we obtain the following continuity result embodied in Proposition 2: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that truth will be a weak $\varepsilon-$ex post Nash equilibrium whenever $\nu_i^P < \delta$ for each i.

Lemma A has a second important consequence: if agent i is informationally small, then truth is an approximate Bayes-Nash equilibrium in the GVCG mechanism so the mechanism is approximately interim incentive compatible. More precisely, we can deduce from Lemma A that the interim expected gain from misreporting one's type is essentially bounded from above by one's informational size. If we want the mechanism to be exactly interim incentive compatible, then we must alter the mechanism (specifically, construct an augmented GVCG mechanism) in order to provide the correct incentives for truthful behavior. We turn to this next.

7. Bayesian Incentive Compatibility and Augmented Mechanisms

Proposition 2, leaves two important questions unanswered. First, we would like to identify conditions under which agents are informationally small so that an outcome efficient social choice function is ε-ex post implementable for small ε. It is reasonable to conjecture that this will be the case, inter alia, when there are many agents, and we provide a precise analysis of this case in Section 8 below. The second question concerns the possibility of modifying a mechanism via the introduction of small transfers so that the resulting modified mechanism is *exactly*, rather than *approximately*, interim incentive compatible when agents are informationally small. Given Proposition 2, the existence of such a mechanism is at least plausible since an agent's ex post gain from lying, i.e., his ex post informational rent, is small with high probability when the agent is informationally small. Consequently, his expected informational rent conditional on his type is small and truth will be an approximate Bayes-Nash equilibrium when agents are informationally small. In this section, we provide conditions under which a modified GVCG mechanism is approximately ex post incentive compatible and (exactly) Bayesian incentive compatible and the sum of the agents' ex post transfers is bounded by a number close to 0 when agents are informationally small.

7.1. Variability of Agents' Beliefs

Whether an agent i can be given incentives to reveal his information will depend on the magnitude of the difference between $P_{T_{-i}}(\cdot|t_i)$ and $P_{T_{-i}}(\cdot|t_i')$, the conditional distributions on T_{-i} given different types t_i and t_i' for agent i. If $P \in \Delta_{\Theta\times T}$, let $P_{T_{-i}}(\cdot|t_i) \in \Delta_{T_{=i}}$ be the conditional distribution on T_{-i} given that i receives signal t_i and define

$$\Lambda_i^P = \min_{t_i\in T_i} \min_{t_i'\in T_i\setminus\{t_i\}} \left\| \frac{P_{T_{-i}}(\cdot|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2} - \frac{P_{T_{-i}}(\cdot|t_i')}{||P_{T_{-i}}(\cdot|t_i')||_2} \right\|_2^2$$

where $||\cdot||_2$ denotes the 2-norm on $\mathbb{R}^{|T_{-i}|}$. This is the measure of the "variability" of the conditional distribution $P_{T_{-i}}(\cdot|t_i)$ as a function of t_i.

As mentioned in the introduction, our work is related to that of Cremer and McLean (1985, 1989). Those papers and subsequent work by McAfee and Reny (1992) demonstrated how one can use correlation to fully extract the surplus in certain mechanism design problems. The key ingredient there is the assumption that the collection of conditional distributions $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i\in T_i}$ is a linearly independent set for each i. This of course, implies that $P_{T_{-i}}(\cdot|t_i) \neq P_{T_{-i}}(\cdot|t_i')$ if $t_i \neq t_i'$ and, therefore, that $\Lambda_i^P > 0$. While linear independence implies that $\Lambda_i^P > 0$, the actual (positive) size of Λ_i^P is not relevant in the Cremer-McLean constructions, and full extraction will be possible. In the present work, we do not require that the collection $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i\in T_i}$ be linearly independent (or satisfy the weaker cone condition in Cremer and McLean (1988)). However, the "closeness" of the members of $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i \in T_i}$ is an important issue. It can be shown that for each i, there exists a collection of numbers $\varsigma_i(t)$ satisfying $0 \le \zeta_i(t) \le 1$ and

$$\sum_{t_{-i} \in T_{-i}} [\varsigma_i(t_{-i}, t_i) - \varsigma_i(t_{-i}, t'_i)] \, P_{T_{-i}}(t_{-i}|t_i) > 0$$

for each $t_i, t'_i \in T_i$ if and only if $\Lambda_i^P > 0$. The elements of the collection $\{\varsigma_i(t)\}_{i \in I, t \in T}$ can be thought of as "incentive payments" to the agents to reveal their information. The above inequality assures that, if the posteriors $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i \in T_i}$ are all distinct, then the incentive compatibility inequalities above are strict. However, the expression on the left hand side decreases as $\Lambda^P \to 0$. Hence, the difference in the expected reward from a truthful report and from a false report will be very small if the conditional posteriors are very close to each other. Our results require that informational size be small relative to the variation in these posteriors.

7.2. The Result

Let $(z_i)_{i \in N}$ be an n-tuple of functions $z_i : T \to \Re_+$ each of which assigns to each $t \in T$ a nonnegative number, interpreted as a "reward" to agent i. If $(q, x_1, ..x_n)$ is a mechanism, then the associated *augmented* mechanism is defined as $(q, x_1 + z_1, .., x_n + z_n)$ and will be written simply as $(q, (x_i + z_i))$.

Theorem 2: Let $(v_1, .., v_n)$ be a collection of payoff functions.

(i) Suppose that $P \in \Delta^*_{\Theta \times T}$ satisfies $\Lambda_i^P > 0$ for each i and suppose that $q : T \to C$ is outcome efficient for the problem $(v_1, .., v_n, P)$. Then there exists an augmented GVCG mechanism $(q, \alpha_i^q + z_i)$ for the social choice problem problem $(v_1, .., v_n, P)$ satisfying ex post IR and interim IC.

(ii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta^*_{\Theta \times T}$ satisfies

$$\max_i \nu_i^P \le \delta \min_i \Lambda_i^P,$$

and whenever $q : T \to C$ is outcome efficient for the problem $\{v_1, .., v_n, P\}$, there exists an augmented GVCG mechanism $(q, (\alpha_i^q + z_i))$ with $0 \le z_i(t) \le \varepsilon$ for every i and t satisfying ex post IR, interim IC and weak $\varepsilon-$ex post IC. Consequently, $\sum_i (\alpha_i^q + z_i) \le n\varepsilon$.

7.3. A Two Stage implementation Game

Informally, Theorem 2 can be explained in the following way. If a problem is a pure private value problem, then the VCG mechanisms will implement efficient outcomes. In the presence of interdependent values, these mechanisms are no longer incentive compatible. With interdependent values, a given agent's utility depends on other agents' types, insofar as their types are correlated with the state θ. If there is correlation in the components of the agents' information that are related to θ, then those components can be truthfully elicited via payments to the agents that are of the magnitude of their informational sizes; it is these payments that "augment" the GVCG transfers. Once the part of an agent's information that affects the probability distribution over the states is obtained, the problem becomes a private value problem, and VCG-type payments can be used to extract the residual private information that agents may have, that is, their private values.

This informal description can be formalized as a two stage game. Fix a social choice problem $(v_1, .., v_n, P)$ and assume that $T = T^*$ to simplify the presentation. Next, suppose that $q : T \to C$ is an outcome efficient social choice function for $(v_1, .., v_n, P)$. In particular,

$$q(t) \in \arg\max_{c \in C} \sum_{i \in N} \hat{v}_i(c; t) = \arg\max_{c \in C} \sum_{i \in N} \left[\sum_{\theta \in \Theta} v_i(c, \theta, t_i) P_\Theta(\theta | t) \right]$$

for each $t \in T$ and recall that, given q, the GVCG transfers are defined as follows:

$$\alpha_i^q(t) = \sum_{j \in N \backslash i} \hat{v}_j(q(t); t) - \max_{c \in C} \left[\sum_{j \in N \backslash i} \hat{v}_j(c; t) \right] \text{ if } t \in T.$$

For each $\pi \in \Delta(\Theta)$, let

$$w_i(c, \pi, t_i) = \sum_{\theta \in \Theta} v_i(c, \theta, t_i) \pi(\theta).$$

Throughout this section, we will use the following notational convention:

$$\rho(t_{-i}, t_i) = P_\Theta(\cdot | t_{-i}, t_i) \text{ and } \rho_\theta(t_{-i}, t_i) = P_\Theta(\theta | t_{-i}, t_i).$$

With this convention, note that for $t = (t_{-i} t_i)$ we have

$$w_i(c, \rho(t), t_i) = \sum_{\theta \in \Theta} v_i(c, \theta, t_i) \rho_\theta(t) = \sum_{\theta \in \Theta} v_i(c, \theta, t_i) P_\Theta(\theta | t) = \hat{v}_i(c; t).$$

Next, define for each $\pi \in \Delta(\Theta)$ an outcome function $\hat{q}(\cdot|\pi) : T \to C$ where

$$\hat{q}(t|\pi) \in \arg\max_{c\in C} \sum_{j\in N} w_j(c, \pi, t_j).$$

and $\hat{q}(t|\rho(t))$ is chosen so that

$$\hat{q}(t|\rho(t)) = q(t) \text{ for each } t \in T.$$

Finally, define transfers as follows:

$$\hat{x}_i(t|\pi) = \sum_{j\in N\backslash i} w_j(\hat{q}(t|\pi), \pi, t_j) - \max_{c\in C}\left[\sum_{j\in N\backslash i} w_j(c, \pi, t_j)\right]$$

and note that $\hat{q}(t|\rho(t)) = q(t)$ for each $t \in T$ implies that

$$\hat{x}_i(t|\rho(t)) = \alpha_i^q(t).$$

The mechanism $(\hat{q}, \hat{x})$ is the simple VCG mechanism for the private value problem in which i's valuation is $w_i(c, \pi, t_i)$ and π is treated as a parameter. Consequently, it is a dominant strategy to honestly report one's type, i.e.,

$$t_i \in \arg\max_{s_i\in T_i} w_i(\hat{q}(t_{-i}, s_i|\pi), \pi, t_i) + \hat{x}_i(t_{-i}, s_i|\pi)$$

for all $t_i \in T_i, t_{-i} \in T_{-i}$ and $\pi \in \Delta(\Theta)$.

We now define an extensive form game that formalizes the two stage process that lies behind the intuition for the augmented GVCG mechanism. Let $(z_i)_{i\in N}$ be an n-tuple of functions $z_i : T \to \mathbb{R}_+$ each of which assigns to each $t \in T$ a nonnegative number $z_i(t)$ interpreted as a "reward" to agent i.

Stage 1: Each agent i learns his type t_i and makes a (not necessarily honest) report $r_i \in T_i$ of his signal to the mechanism designer. If $(r_1, .., r_n)$ is the profile of stage 1 reports, then agent i receives the nonnegative payment $z_i(r_1, .., r_n)$ and the game moves to stage 2.

Stage 2: If $(r_1, .., r_n) = r \in T$ is the announced type profile in stage 1, the mechanism designer publicly posts the probability measure $\rho(r) = P_\Theta(\cdot|r)$. Agents observe this posting (but not the reported profile r) and make a second (not necessarily honest) report to the mechanism designer. If $(s_1, .., s_n) = s \in T$

is the second stage profile of reports, then the mechanism designer chooses the social alternative $\hat{q}(s|\rho(r))$, each agent i receives the transfer $\hat{x}_i(s|\rho(r))$ and the game ends.

To fix ideas, note that player i's ex post payoff when i's type is t_i, the players report $(r_1, .., r_n) = r$ in stage 1 and $(s_1, .., s_n) = s$ in stage 2 is given by

$$w_i(\hat{q}(s|\rho(r)), \rho(r), t_i) + \hat{x}_i(s|\rho(r)) + z_i(r).$$

Let

$$\Pi := \{\rho(t)|t \in T\} = \{P_\Theta(\cdot|t)|t \in T\}$$

denote the (finite) set of conditional measures on Θ. A strategy for agent i in this game is a pair (α_i, β_i) where $\alpha_i : T_i \to T_i$ specifies a type dependent report $\alpha_i(t_i) \in T_i$ in stage 1 and $\beta_i : T_i \times \Pi \times T_i \to T_i$ specifies a second stage report $\beta_i(r_i, \pi, t_i) \in T_i$ as a function of i's first stage report $r_i \in T_i$, the posted distribution $\pi \in \Pi$, and $i's$ type $t_i \in T_i$.

In stage 1, players are asked to reveal their private information in order to compute the posterior distribution on the state space Θ. Stage 2 is a simple implementation problem with private values in which the mechanism designer chooses a socially optimal action for the posterior computed in stage 1 and players are assessed their (private value) VCG transfers. If agents are truthful in both stages of the game, then for type profile t, the ex post payoff to agent i corresponding to type profile t is

$$\begin{aligned} w_i(\hat{q}(t|\rho(t)), \rho(t), t_i) + \hat{x}_i(t|\rho(t)) + z_i(t) &= w_i(q(t), t_i) + \alpha_i(t) + z_i(t) \\ &= \hat{v}_i(q(t), t) + \alpha_i(t) + z_i(t). \end{aligned}$$

Consequently, we are interested in a Perfect Bayesian Equilibrium (PBE) assessment for the two stage implementation game consisting of a strategy profile $(\alpha_i, \beta_i)_{i\in N}$ and a system of second stage beliefs in which players truthfully report their private information at each stage.

Definition: A strategy (α_i, β_i) for player i is *truthful* for i if $\alpha_i(t_i) = t_i$ for all $t_i \in T_i$ and $\beta_i(t_i, \pi, t_i) = t_i$ for all $\pi \in \Pi$ and $t \in T$. A strategy profile $(\alpha_i, \beta_i)_{i\in N}$ is *truthful* if (α_i, β_i) is truthful for each player i.

Formally, a system of beliefs for player i is a collection of probability measures on $\Theta \times T_{-i}$ indexed by $T_i \times \Pi \times T_i$, i.e., a collection of the form

$$\{\mu_i(\cdot|r_i, \pi, t_i) \in \Delta(\Theta \times T_{-i} \times T_{-i}) : (r_i, \pi, t_i) \in T_i \times \Pi \times T_i\}.$$

with the following interpretation: when player i of type t_i reports r_i in Stage 1 and observes the posted distribution π, then player i assigns probability mass $\mu_i(\theta, r_{-i}, t_i | r_i, \pi, t_i)$ to the event that other players have true types t_{-i} and have reported r_{-i} and that the state of nature is θ. As usual, an *assessment* is a pair $\{(\alpha_i, \beta_i)_{i\in N}\ , (\mu_i)_{i\in N}\}$ consisting of a strategy profile and a system of beliefs for each player.

Definition 5: An assessment $\{(\alpha_i, \beta_i)_{i\in N}\ , (\mu_i)_{i\in N}\}$ is an *incentive compatible Perfect Bayesian equilibrium (ICPBE) assessment* if $\{(\alpha_i, \beta_i)_{i\in N}\ , (\mu_i)_{i\in N}\}$ is a Perfect Bayesian equilibrium assessment and the profile $(\alpha_i, \beta_i)_{i\in N}$ is truthful.

We can prove the following two results.

Proposition 3: Suppose that $P \in \Delta^*_{\Theta\times T}$ satisfies $\Lambda_i^P > 0$ for each i. Then there exist $(z_i)_{i\in N}$ such that the associated two stage game has a ICPBE.

Proposition 4: For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta^*_{\Theta\times T}$ satisfies

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P,$$

there exist (z_i) with $0 \leq z_i(t) \leq \varepsilon$ for every i such that the associated two stage game has a ICPBE.

It may seem strange that, in the two stage mechanism above, the agents are announcing their types twice. Since the mechanism "knows" the agents' types after the first stage, why ask them to report a second time? There are several reasons to structure the problem as we have. For many problems an agent's type may consist of different parts, some of which affect the beliefs about the state θ and some of which do not. For the oil auction example, an agent may have geological information that affects his beliefs regarding the amount of oil in the tract, and may also have information regarding the amount of oil in his current inventory. When an agent's type can be decomposed into distinct parts in this way, only the part relevant to θ need be announced in the first stage and only the part that concerns the agent's inventory need be announced in the second stage.

Having two separate announcements also allows us to distinguish an agent's rent associated with his information regarding θ from the rent associated with information not related to θ. It's useful to distinguish these, since, as we show below, the first will often asymptotically vanish as the number of agents gets large while the second typically does not. In addition, the first of the two rents is independent of other agents' preferences while the second is not.

8. Asymptotic Results

Informally, an agent is informationally small when the probability that he can affect the posterior distribution on Θ is small. One would expect, in general, that agents will be informationally small in the presence of many agents. For example, if agents receive conditionally independent signals regarding the state θ, then the announcement of one of many agents is unlikely to significantly alter the posterior distribution on Θ. Hence, it is reasonable to conjecture that (under suitable assumptions) an agent's informational size goes to zero in a sequence of models with an increasing number of agents. Consequently, the required rewards z_i that induce truthful behavior will also go to zero as the number of agents grows. We will show below that this is in fact the case. Of greater interest, however, is the behavior of the *aggregate* reward necessary to induce truthful revelation. The argument sketched above only suggests that each individual's z_i becomes small as the number of agents goes to infinity, but does not address the asymptotic behavior of the sum of the z_i's. Roughly speaking, the size of the z_i that is necessary to induce agent i to reveal truthfully is of the order of magnitude of his informational size. Hence, the issue concerns the speed with which agents' informational size goes to zero as the number of agents increases. We will demonstrate below that, under reasonably general conditions, agents' informational size goes to zero at an exponential rate and that the total reward $\sum_{i \in N} z_i$ goes to zero as the number of agents increases.

8.1. Notation and Definitions:

We will assume that all agents have the same finite signal set $T_i = A$. Let $J_r = \{1, 2, ..., r\}$. For each $i \in J_r$, let $v_i^r : C \times \Theta \times A \to \Re_+$ denote the payoff to agent i. For any positive integer r, let $T^r = A \times \cdots \times A$ denote the r-fold Cartesian product and let $t^r = (t_1^r, .., t_r^r)$ denote a generic element of T^r.

Definition 6: A sequence of probability measures $\{P^r\}_{r=1}^{\infty}$ with $P^r \in \Delta(\Theta \times T^r)$ is a *conditionally independent sequence* if there exists $P \in \Delta(\Theta \times A)$ such that

(a) $P(\theta, t) > 0$ for all $(\theta, t) \in \Theta \times A$ and for every $\theta, \hat{\theta}$ with $\theta \neq \hat{\theta}$, there exists a $t \in A$ such that $P(t|\theta) \neq P(t|\hat{\theta})$.

(b) For each r and each $(\theta, t_1, .., t_r) \in \Theta \times T^r$,

$$P^r(t_1^r, .., t_r^r|\theta) = \text{Prob}\{\widetilde{t}_1^r = t_1, \widetilde{t}_2^r = t_2, ..., \widetilde{t}_r^r = t_r|\tilde{\theta} = \theta\} = \prod_{i=1}^{r} P(t_i|\theta).$$

Because of the symmetry in the objects defining a conditionally independent sequence, it follows that, for fixed r, the informational size of each $i \in J_r$ is the same. In the remainder of this section we will drop the subscript i and will write ν^{P^r} for the value of the informational size of agents in J_r.

Lemma B: Suppose that $\{P^r\}_{r=1}^{\infty}$ is a conditionally independent sequence. For every $\varepsilon > 0$ and every positive integer k, there exists an $\hat{r}$ such that

$$r^k \nu^{P^r} \leq \varepsilon$$

whenever $r > \hat{r}$.

The proof is provided in the appendix and is an application of a classic large deviations result due to Hoeffding (1960). With this lemma, we can prove the following asymptotic result.

Theorem 3: Suppose that $\{P^r\}_{r=1}^{\infty}$ is a conditionally independent sequence. Let M and ε be positive numbers. Let $\{(v_1^r, .., v_r^r)\}_{r\geq 1}$ be a sequence of payoff function profiles and for each r, let $\{q^{P^r}(r), \alpha_1^{P^r}(r), .., \alpha_r^{P^r}(r)\}$ denote the GVCG mechanism for the SCP $(v_1^r, .., v_r^r, P^r)$. Suppose that $0 \leq v_i^r(\cdot, \cdot, \cdot) \leq M$ for all r and $i \in J_r$ and that P satisfies the following condition: for each pair t, t' in A with $t \neq t'$, there exists as $s \in A$ such that

$$\sum_{\theta} P(s|\theta)P(\theta|t) \neq \sum_{\theta} P(s|\theta)P(\theta|t').$$

Then for every $\varepsilon > 0$, there exists an $\hat{r}$ such that for all $r > \hat{r}$, there exists an augmented GVCG mechanism $(q^r, \alpha_1^r + z_1^r, .., \alpha_r^r + z_r^r)$ for the social choice problem $(v_1^r, .., v_r^r, P^r)$ satisfying ex post IR, interim IC and weak $\varepsilon-$ex post IC. Furthermore, for each $i \in J_r$ and each $t^r \in T^r$, $z_i^r(t^r) \geq 0$ and $\sum_{i\in J_r}^{r} z_i^r(t^r) \leq \varepsilon$.

8.2. An Auction Application

The significance of Theorem 3 can be illustrated in the case of a Vickrey auction with interdependent valuations as studied in McLean and Postlewaite (2004). For simplicity, suppose that $T^* = T$. If i receives the object, his value is the nonnegative number $w_i(\theta, t_i)$ and his "reduced form" value is

$$\hat{w}_i(t) = \sum_{\theta} w_i(\theta, t_i) P_\Theta(\theta|t) \text{ for each } t \in T.$$

In this framework, $q(t) = (q_1(t), .., q_n(t))$ where each $q_i(t) \geq 0$ and $q_1(t) + \cdots + q_n(t) \leq 1$ and

$$\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + x_i(t_{-i}, t_i') = q_i(t_{-i}, t_i')\hat{w}_i(t_{-i}, t_i) + x_i(t_{-i}, t_i').$$

Finally, outcome efficiency means that

$$\sum_{i \in N} q_i(t) w_i(t) = \max_{i \in N}\{w_i(t)\}.$$

Let $w^*(t) := \max_i \hat{w}_i(t)$ and let $I(t) := \{i \in N | \hat{w}_i(t) = w^*(t)\}$. If

$$\begin{aligned} q_i^*(t) &= \frac{1}{|I(t)|} \text{ if } i \in I(t) \\ &= 0 \text{ if } i \notin I(t) \end{aligned}$$

then q^* is outcome efficient. Defining $w_{-i}^*(t) := \max_{j:j \neq i}\{w_j(t)\}$, it is easy to verify that the GVCG transfers associated with q^* are given by

$$\begin{aligned} \alpha_i^*(t) &= -\frac{w_{-i}^*(t)}{|I(t)|} \text{ if } i \in I(t) \\ &= 0 \text{ if } i \notin I(t). \end{aligned}$$

If the GVCG mechanism $(q^*, (\alpha_i^*))$ were ex post IC (as in the pure private value case or, more generally, the case of nonexclusive information), then the auctioneer's ex post revenue would be exactly

$$-\sum_{i=1}^{n} \alpha_i^*(t) = \sum_{i \in I(t)} \frac{w_{-i}^*(t)}{|I(t)|}.$$

In an augmented mechanism $(q^*, (\alpha_i^* + z_i))$, the auctioneer's ex post revenue is

$$-\sum_{i=1}^{n} \alpha_i^*(t) = \sum_{i \in I(t)} \frac{w_{-i}^*(t)}{|I(t)|} - \sum_{i=1}^{n} z_i(t)$$

so the auctioneer's ex post revenue is reduced by the total of the reward payments necessary to elicit truthful revelation of types. In a large conditionally independent model of an auction, we know that the rewards z_i can be constructed so that the augmented mechanism $(q^*, (\alpha_i^* + z_i))$ is ex post IR, interim IC and approximately ex post IC. Furthermore, the sum $\sum_{i=1}^{n} z_i(t)$ is converging to zero as the number of bidders grows. Consequently, the auctioneer's ex post revenue will be close to the auctioneer's ex post revenue from the unaugmented GVCG auction in the presence of many bidders.

9. Discussion

9.1. Relation to the literature

Jehiel, Meyer-ter-Vehn, Moldovanu and Zame (2006) showed that , generically, the only mechanisms for which truthful revelation of types is an ex post equilibrium in the presence of interdependent valuations and multidimensional signals must be constant. That is, the revealed information cannot affect the mechanism outcome. In order to explain the connection between their model and results of this paper, we need to reformulate our basic model to accommodate infinite signal sets. Suppose that Θ is finite but now, as in Jehiel et al., suppose that agent i's type t_i is drawn from $T_i = [0,1]^{k_i}$, $k_i > 1$.

Let $T = \Pi_{i \in N} T_i$, let $P \in \Delta(\Theta \times T)$ be a probability measure whose marginal on T has full support and suppose that the conditional probability function

$$t \in T \mapsto P(\cdot|t)$$

exists and satisfies the smoothness hypotheses imposed in Jehiel et al. For any profile of types $t \in T^*$, we again define

$$\hat{v}_i(c;t) = \hat{v}_i(c;t) = \sum_{\theta \in \Theta} v_i(c, \theta, t_i) P(\theta|t)$$

so that the map

$$t \in T \mapsto \hat{v}_i(c;t)$$

also satisfies the smoothness hypotheses imposed in Jehiel et al.

In this model with a continuum of multidimensional types, Proposition 2 holds verbatim: for every $\varepsilon > 0$, there exists $\delta > 0$ such that, if $\nu_i^P < \delta$ for each agent i, then the GVCG mechanism $(q, (\alpha_i^q))$ is weakly $\varepsilon-$ ex post incentive compatible. That is, there are interesting nonconstant mechanisms for which truthful revelation of types is an approximate ex post equilibrium.

9.2. Alternative Notions of Informational Size and approximate Ex Post Nash Equilibrium

Roughly speaking, when an agent has informational size ε, then that agent's (conditional) probability that he can change the posterior distribution on Θ by more than ε is at most ε. One might consider an alternative definition of informational size whereby an agent's informational size is ε if *with probability one* he cannot change the posterior distribution on Θ by more than ε.

Definition 7: The *strict informational size* of agent i is defined as

$$\sigma_i^P = \max_{t_i \in T_i} \max_{t_i' \in T_i} \max_{t_{-i} \in T_{-i}} \{||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t_i')|| : (t_{-i}, t_i), (t_{-i}, t_i') \in T^*\}.$$

We will refer to an agent as strictly informationally small if his strict informational size is small. From the definitions, it follows that $\nu_i^P \leq \sigma_i^P$. For economic problems with a small number of agents, it is often the case that every agent is informationally small but no agent is strictly informationally small. For example, consider a problem with two equiprobable states, θ_1 and θ_2, and three agents, each of whom receives a noisy signal about the state θ. With very accurate signals, each agent's signal is the true state θ with high probability. In this case, it is easy to verify that any agent who unilaterally misreports his signal will, *with high probability*, have only a small effect on the posterior distribution and, consequently, agents are informationally small. However, it is also easy to see that agents will not be *strictly* informationally small. When the agents' signals are very accurate, then all agents' signals will correspond to the true state θ with high probability. However, the probability that two agents, say agent 1 and agent 2, receive different signals is positive. In this case, agent 3's announcement will have a large effect on the posterior distribution: whether he announces θ_1 or θ_2, one of the other two agents' announcements will match his announcement and one will not. Consequently, agent 3 cannot be strictly informationally small in this case.

The discussion above illustrates the advantage of analyses that employ the weaker notion of informational size rather than strict informational size: a large and interesting class of problems is covered by the former notion that will not be covered by the latter. There is, of course, a cost: theorems employing the weaker hypothesis will have weaker consequences. If a mechanism satisfies our notion of *weak* $\varepsilon-$ex post IC, then with (conditional) probability at most ε, a change in an agent's reported type (given other agents' types) will increase his utility by more than ε. This, of course allows for the possibility that a change could lead to a large increase in his utility for some (low probability) profiles of other agents' types. The small probability of large utility gains is connected to the fact that, with small probability, an agent's report will have a large effect on the posterior distribution. In interdependent type mechanisms, an agent's transfer depends on other agents' valuations, and those valuations depend on the posterior distribution on Θ; large changes in the posterior distribution can translate into large changes in utility.

The above discussion suggests a stronger notion of approximate ex post incentive compatibility:

Definition 8: Let $\varepsilon \geq 0$. A mechanism is ε- ex post incentive compatible if truthful revelation is an $\varepsilon-$ex post Nash equilibrium*:* if for all i, all $t_i, t_i' \in T_i$ and all $t_{-i} \in T_{-i}$ such that $(t_{-i}, t_i) \in T^*$,

$$(\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + x_i(t_{-i}, t_i')) - (\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + x_i(t_{-i}, t_i)) \leq \varepsilon.$$

That is, a mechanism is ε- ex post incentive compatible if, with conditional probability one, no agent can increase his utility by more than ε regardless of other agents' types. Ex post incentive compatibility is stronger than ε- ex post incentive compatibility, and ε- ex post incentive compatibility is stronger than weak ε- ex post incentive compatibility.

Recall that Proposition 1 provides the following continuity result for the GVCG mechanism: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that truth will be an weak $\varepsilon-$ex post Nash equilibrium whenever $\nu_i^P < \delta$ for each i. Strict informational size is related to ε- ex post incentive compatibility in the same way: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that truth will be an $\varepsilon-$ex post Nash equilibrium whenever $\sigma_i^P < \delta$ for each i.

9.3. Pure Common Value Problems and GVCG Transfers

For pure common value problems, there is (by definition) no residual private information, so it might seem that the VCG-type payments can be dispensed with. However, simply dropping the GVCG payments introduces a problem. In the description of the intuition of the proof of Theorem 1, we pointed out that the part of an agent's information that affects the utility of other agents can be extracted by augmenting the VCG payments. If agent i has true type t_i but announces t'_i when other agents announce t_{-i}, then the ex post payoff to agent i in the unaugmented GVCG mechanism is

$$U_i(t'_i|t_{-i}, t_i) := \hat{v}_i(q(t_{-i}, t'_i); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t'_i).$$

As a consequence of Lemma A, we know that the gain from a lie (i.e., $U_i(t'_i|t_{-i}, t_i) - U_i(t_i|t_{-i}, t_i)$) is small if $||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t'_i)||$ is small. If we simply drop the GVCG transfers, then the gain to lying (i.e., $\hat{v}_i(q(t_{-i}, t'_i); t_{-i}, t_i) - \hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i)$) will typically no longer be small when $||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t'_i)||$ is small. Consequently, it will no longer be true that small informational size assures that an agent's information can be extracted with small payments. There are two important properties of the GVCG payments in our framework: they are used to elicit agents' private information and, in addition, they assure that an agent's ex post payoff behaves nicely with respect to the posterior distribution on Θ.

For pure private value problems, Green and Laffont (1979) show that the VCG payments are essentially unique. It may be the case that when there is a nontrivial private value component to agents' information, transfers that embody VCG payments are necessary, but for pure common value problems that is not the case. For pure common value problems with positive variability, there exist transfer schemes that have no relation to the GVCG mechanism. What is necessary is that the transfer payments accomplish what the GVCG payments accomplish: they must ensure that small changes in the posterior distribution on Θ do not translate into a large utility gain. This requires a "continuity" assumption on the mapping from posterior distributions on Θ into agents' utilities and we address this in the next section.

9.4. Gain-Bounded Mechanisms

In a typical implementation or mechanism design problem, one computes the mechanism for each instance of the data that defines the social choice problem. Therefore, in most cases of interest, the mechanism is parametrized by the valuation functions and probability structure that define the social choice problem. If we fix a profile $(v_1, .., v_n)$ of payoff functions, then we can analyze the parametric dependence of the mechanism on the probability distribution P and this dependence can be modelled as a mapping that associates a mechanism with each $P \in \Delta^*_{\Theta\times T}$. We will denote this mapping $P \mapsto (q^P, x_1^P, .., x_n^P)$. For example, the mapping naturally associated with the GVCG mechanism is defined by

$$q^P(t) \in \arg\max_{c\in C} \sum_{j\in N}\sum_{\theta\in\Theta} v_i(c,\theta,t_i)P(\theta|t_{-i},t_i) \text{ if } t \in T^*$$

$$q^P(t) = c_0 \text{ if } t \notin T^*$$

and

$$x_i^P(t) = \sum_{j\in N\setminus i}\sum_{\theta\in\Theta} v_i(q^P(t),\theta,t_i)P(\theta|t_{-i},t_i) - \max_{c\in C}\left[\sum_{j\in N\setminus i}\sum_{\theta\in\Theta} v_i(c,\theta,t_i)P(\theta|t_{-i},t_i)\right] \text{ if } t \in T^*$$
$$= 0 \text{ if } t \notin T^*.$$

Definition 9: Let $(v_1, .., v_n)$ be a profile of payoff functions. For each $P \in \Delta^*_{\Theta\times T}$, let $(q^P, x_1^P, .., x_n^P)$ be a mechanism for the social choice problem $(v_1, .., v_n, P)$. We will say that the mapping $P \mapsto (q^P, x_1^P, .., x_n^P)$ is *gain-bounded with respect to conditional probabilities*, or simply *gain-bounded*, if there exists a $K > 0$ such that for all $P \in \Delta^*_{\Theta\times T}$,

$$\hat{v}_i(q^P(t_{-i},t_i'); t_{-i},t_i) + x_i^P(t_{-i},t_i') - \hat{v}_i(q^P(t_{-i},t_i); t_{-i},t_i) + x_i^P(t_{-i},t_i)$$
$$\leq K||P_\Theta(\cdot|t_{-i},t_i) - P_\Theta(\cdot|t_{-i},t_i')||$$

whenever $(t_{-i},t_i), (t_{-i},t_i') \in T^*$.

Note that, in Definition 9 above, the social choice function need not be outcome efficient. Lemma A shows that the GVCG mechanism is gain-bounded with $(K = 2M(n-1))$ and this is the essential property of the GVCG mechanism that drives Propositions 1 and 2 and Theorem 2. In fact, using the same proof, an important extension of Theorem 2 holds for any gain-bounded mechanism.

Theorem 4: Let $(v_1, .., v_n)$ be a collection of payoff functions and suppose that $P \mapsto (q^P, x_1^P, .., x_n^P)$ is gain-bounded.

(i) If $\Lambda_i^P > 0$ for each i, then there exists an augmented mechanism $(q^P, (x_i^P + z_i^P))$ for the social choice problem problem $(v_1, .., v_n, P)$ satisfying ex post IR and interim IC.

(ii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta^*_{\Theta\times T}$ satisfies

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P,$$

there exists an augmented mechanism $(q^P, (x_i^P + z_i^P))$ with $0 \leq z_i^P(t) \leq \varepsilon$ for every i and t satisfying ex post IR, interim IC and weak $\varepsilon-$ex post IC.

We now present an example of a balanced gain-bounded mechanism for pure common value models which is quite different from the GVCG mechanism. Let $(v_1, .., v_n)$ be a collection of payoff functions. For each $P \in \Delta^*_{\Theta\times T}$ suppose that $q^P : T \to C$ is a social choice function for the problem $(v_1, .., v_n, P)$ and define transfer payments associated with q^P as follows:

$$\beta_i^P(t) = \frac{1}{n}\sum_j \hat{v}_j(q^P(t), t) - \hat{v}_i(q^P(t), t).$$

In this simple scheme, agent i receives money if his individual payoff is less than the average payoff and he pays out money if his individual payoff is greater than the average payoff. Furthermore, note that

$$\sum_i \beta_i^P(t) = 0$$

so that the mechanism $(q^P, (\beta_i^P))$ is balanced for each $P \in \Delta^*_{\Theta\times T}$.

If q^P is outcome efficient for the problem $(v_1, .., v_n, P)$, then the associated mechanism with transfer payments $(\beta_i^P)_{i\in N}$ is gain-bounded in pure common value problems (though not for general problems).

Theorem 5: Let $(v_1, .., v_n)$ be a collection of payoff functions satisfying the pure common value assumption. For each $P \in \Delta^*_{\Theta\times T}$ suppose that $q^P : T \to C$ is outcome efficient for the problem $(v_1, .., v_n, P)$ and let (β_i^P) be the transfer payments associated with q^P as defined as above.

(i) The mapping $P \mapsto (q^P, \beta_1^P, .., \beta_n^P)$ is gain-bounded.

(ii) If $\Lambda_i^P > 0$ for each i, then there exists an augmented mechanism $(q^P, \beta_i^P + z_i^P\}_{i\in N}$ for the social choice problem problem $(v_1, .., v_n, P)$ satisfying ex post IR and interim IC.

(iii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta^*_{\Theta \times T}$ satisfies

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P,$$

there exists an augmented mechanism $(q^P, \beta_i^P + z_i^P)$ for the social choice problem problem $(v_1, .., v_n, P)$ with $0 \leq z_i^P(t) \leq \varepsilon$ for every i and t satisfying ex post IR, interim IC and weak $\varepsilon-$ex post IC.

The augmented mechanism of Theorem 5(iii) is not balanced in general, but we do know that $0 \leq \sum_i (\beta_i^P + z_i^P) = \sum_i z_i^P \leq n\varepsilon$. If $n\varepsilon$ is small then this mechanism is "nearly" balanced.

9.5. Extending the Model

A result more general than Theorem 2 is possible and we discuss this now. In many problems of interest, an agent's type has several components, only some of which are correlated with the state of nature θ. For example, consider the bidding model studied in McLean and Postlewaite (2004). There, $t_i = (a_i, s_i) \in A_i \times S_i := T_i$ is the type of a bidder on an oil tract in which θ represents the amount of oil in the tract, a_i represents the bidder's private extraction cost and s_i represents a signal (resulting from, e.g., geological tests) that is correlated with θ. Let P^1 (resp. P^2) denote the marginal of P on $\Theta \times S_1 \times \cdots \times S_n$ (resp. $A_1 \times \cdots \times A_n$.) If

$$P(\theta, t_1, .., t_n) = P(\theta, a_1, s_1, , .., a_n, s_n) = P^1(\theta, s_1, , .., s_n) P^2(a_1, ..a_n)$$

so that only a bidder's signal is informationally relevant with respect to θ, then

$$P_\Theta(\cdot|t_1, .., t_n) = P_\Theta^2(\cdot|s_1, , .., s_n)$$

implying that $\Lambda_i^P = 0$ for each i. However, it is the relationship between informational size and variation in beliefs computed with respect to P^1 that matters. If each $\nu_i^{P^1}$ is small enough relative to $\Lambda_i^{P^1}$, the conclusion of Theorem 1 will still hold. In particular, if each $\nu_i^{P^1}$ is small enough relative to $\Lambda_i^{P^1}$, then we can still find an augmented mechanism $(q, (\alpha_i^q + z_i))$ in which each z_i is small and depends *only* on the announced signal profile $(s_1, ., s_n)$.

Using precisely the same argument as that for Theorem 2, we have the following generalization.

Theorem 6: Let $(v_1, .., v_n)$ be a collection of payoff functions. Suppose that

$T_i = A_i \times S_i$ for each i and suppose that

$$P(\theta, t_1, .., t_n) = P(\theta, a_1, s_1, , .., a_n, s_n) = P^1(\theta, s_1, , .., s_n)P^2(a_1, ..a_n)$$

for each $(\theta, t_1, .., t_n) \in \Theta \times T_1 \times \cdots \times T_n$.

(i) Suppose that $P \in \Delta^*_{\Theta \times T}$ satisfies $\Lambda_i^{P^1} > 0$ for each i and suppose that $q : T \to C$ is outcome efficient for the problem $(v_1, .., v_n, P)$. Then there exists an augmented GVCG mechanism $(q, \alpha_i^q + z_i)$ for the social choice problem problem $(v_1, .., v_n, P)$ satisfying ex post IR and interim IC.

(ii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta^*_{\Theta \times T}$ satisfies

$$\max_i \nu_i^{P^1} \leq \delta \min_i \Lambda_i^{P^1},$$

and whenever $q : T \to C$ is outcome efficient for the problem $(v_1, .., v_n, P)$, there exists an augmented GVCG mechanism $(q, (\alpha_i^q + z_i))$ with $0 \leq z_i(t) \leq \varepsilon$ for every i and t satisfying ex post IR, interim IC and weak $\varepsilon-$ex post IC. Consequently, $\sum_i (\alpha_i^q + z_i) \leq n\varepsilon$.

10. Proofs:

We begin with a simple result regarding Lipschitz continuity of the optimal value function.

Lemma 1: For each $S \subseteq N$ and for each $p \in \Delta(\Theta)$, let

$$F_S(p) = \max_{\hat{c} \in C} \sum_{\theta \in \Theta} \sum_{i \in S} v_i(\hat{c}, \theta, t_i) p(\theta).$$

Then for each $p, p' \in \Delta(\Theta)$,

$$|F_S(p) - F_S(p')| \leq |S| M ||p - p'||.$$

Proof: Choose $S \subseteq N$ and $p, p' \in \Delta(\Theta)$. Choose c and c' so that

$$\begin{aligned}
\sum_{\theta \in \Theta} \sum_{i \in S} v_i(c, \theta, t_i) p(\theta) &= \max_{\hat{c} \in C} \sum_{\theta \in \Theta} \sum_{i \in S} v_i(\hat{c}, \theta, t_i) p(\theta) \\
\sum_{\theta \in \Theta} \sum_{i \in S} v_i(c', \theta, t_i) p(\theta) &= \max_{\hat{c} \in C} \sum_{\theta \in \Theta} \sum_{i \in S} v_i(\hat{c}, \theta, t_i) p'(\theta)
\end{aligned}$$

Then,

$$\begin{aligned}
F_S(p) - F_S(p') &= \sum_{\theta\in\Theta}\sum_{i\in S} v_i(c,\theta,t_i)\,[p(\theta) - p'(\theta)] + \sum_{\theta\in\Theta}\sum_{i\in S}[v_i(c,\theta,t_i) - v_i(c',\theta,t_i)]\,p'(\theta) \\
&\leq \sum_{\theta\in\Theta}\sum_{i\in S} v_i(c,\theta,t_i)\,[p(\theta) - p'(\theta)] \\
&\leq |S|M||p - p'||.
\end{aligned}$$

Reversing the roles of p and p' yields the result.

10.1. Proof of Lemma A

Choose $(t_{-i}, t_i), (t_{-i}, t'_i) \in T^*$. Then

$$\begin{aligned}
\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + \alpha_i(t_{-i},t_i) = \hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) &+ \sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i);t_{-i},t_i) \\
&- \max_{c\in C}\left[\sum_{j\in N\setminus i}\hat{v}_j(c;t_{-i},t_i)\right]
\end{aligned}$$

and

$$\begin{aligned}
\hat{v}_i(q(t_{-i},t'_i);t_{-i},t_i) + \alpha_i(t_{-i},t'_i) = \hat{v}_i(q(t_{-i},t'_i);t_{-i},t_i) &+ \sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t'_i);t_{-i},t_i) \\
&- \sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t'_i);t_{-i},t_i) \\
&+ \sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t'_i);t_{-i},t'_i) - \max_{c\in C}\left[\sum_{j\in N\setminus i}\hat{v}_j(c;t_{-i},t'_i)\right]
\end{aligned}$$

Since

$$\begin{aligned}
&\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + \sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i);t_{-i},t_i) \\
\geq\ &\hat{v}_i(q(t_{-i},t'_i);t_{-i},t_i) + \sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t'_i);t_{-i},t_i)
\end{aligned}$$

we conclude that

$$
\begin{aligned}
&(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i(t_{-i},t_i))-(\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\alpha_i(t_{-i},t_i'))\\
&\geq \max_{c\in C}\left[\sum_{j\in N\backslash i}\hat{v}_j(c;t_{-i},t_i')\right]-\max_{c\in C}\left[\sum_{j\in N\backslash i}\hat{v}_j(c;t_{-i},t_i)\right]\\
&\quad -\sum_{j\in N\backslash i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i')+\sum_{j\in N\backslash i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i)
\end{aligned}
$$

Lemma 1 implies that

$$
\max_{c\in C}\left[\sum_{j\in N\backslash i}\hat{v}_j(c;t_{-i},t_i')\right]-\max_{c\in C}\left[\sum_{j\in N\backslash i}\hat{v}_j(c;t_{-i},t_i)\right]\geq -(n-1)M||P_\Theta(\cdot|t_{-i},t_i)-P_\Theta(\cdot|t_{-i},t_i')||.
$$

so the result follows from the observation that

$$
\left|\sum_{j\in N\backslash i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i)-\sum_{j\in N\backslash i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i')\right|\leq (n-1)M||P_\Theta(\cdot|t_{-i},t_i)-P_\Theta(\cdot|t_{-i},t_i')||.
$$

10.2. Proof of Proposition 2

Suppose that $(t_{-i},t_i)\in T^*$ and define

$$
U_i(t_i'|t_{-i},t_i)=\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\alpha_i^q(t_{-i},t_i').
$$

If $(t_{-i},t_i')\notin T^*$, then

$$
\begin{aligned}
U_i(t_i'|t_{-i},t_i)-U_i(t_i|t_{-i},t_i) &= (\hat{v}_i(c_0;t_{-i},t_i)+0)-(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i^q(t_{-i},t_i))\\
&= -(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i^q(t_{-i},t_i))\\
&\leq 0
\end{aligned}
$$

and we conclude that

$$
U_i(t_i'|t_{-i},t_i)-U_i(t_i|t_{-i},t_i)>2M(n-1)\nu_i^P \text{ implies that } (t_{-i},t_i')\in T^*.
$$

Applying Lemma A, we observe that

$$
\begin{aligned}
\{t_{-i}|(t_{-i},t_i) &\in T^* \text{ and } U_i(t_i'|t_{-i},t_i)-U_i(t_i|t_{-i},t_i)>2M(n-1)\nu_i^P\}\\
&= \{t_{-i}|(t_{-i},t_i)\in T^*,\ (t_{-i},t_i')\in T^*,\\
&\qquad \text{and } U_i(t_i'|\tilde{t}_{-i},t_i)-U_i(t_i|\tilde{t}_{-i},t_i)>2M(n-1)\nu_i^P\}\\
&\subseteq \{t_{-i}\in T_{-i}|\ (t_{-i},t_i)\in T^*,(t_{-i},t_i')\in T^*,||P_\Theta(\cdot|t_{-i},t_i)-P_\Theta(\cdot|t_{-i}t_i')||>\hat{\nu}_i^P\}.
\end{aligned}
$$

If $\varepsilon > 0$, then choosing $0 < \delta < \min\{\frac{\varepsilon}{2M(n-1)}, \varepsilon\}$ and $\hat{\nu}_i^P < \delta$ yields

$$\begin{aligned}
\Pr ob\{(\tilde{t}_{-i}, t_i) &\in T^* \text{ and } U_i(t_i'|\tilde{t}_{-i}, t_i) - U_i(t_i|\tilde{t}_{-i}, t_i) > \varepsilon\} \\
&\leq \Pr ob\{(\tilde{t}_{-i}, t_i) \in T^* \text{ and } U_i(t_i'|\tilde{t}_{-i}, t_i) - U_i(t_i|\tilde{t}_{-i}, t_i) > 2M(n-1)\nu_i^P\} \\
&\leq \Pr ob\{\ (\tilde{t}_{-i}, t_i) \in T^*, (\tilde{t}_{-i}, t_i') \in T^*, ||P_\Theta(\cdot|\tilde{t}_{-i}, t_i) - P_\Theta(\cdot|\tilde{t}_{-i}t_i')|| > \hat{\nu}_i^P\} \\
&\leq \hat{\nu}_i^P \\
&\leq \varepsilon
\end{aligned}$$

and the proof is complete.

10.3. Proof of Theorem 2

We prove part (ii) first. Choose $\varepsilon > 0$. Recall that $0 \leq v_i(\cdot, \cdot, \cdot) \leq M$ for each i and let $|T|$ denote the cardinality of T. Choose δ so that

$$0 < \delta < \min\left\{\frac{\varepsilon}{4M(n+1)\sqrt{|T|}}, \frac{\varepsilon}{4}\right\}$$

Suppose that $P \in \Delta^*_{\Theta \times T}$ satisfies

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P.$$

Define $\hat{\nu}^P = \max_i \nu_i^P$and $\Lambda^P = \min_i \Lambda_i^P$. Therefore $\hat{\nu}^P \leq \delta \Lambda^P$.

Now we define an augmented GVCG mechanism. For each $t \in T$, define

$$z_i(t_{-i}, t_i) = \varepsilon \frac{P_{T_{-i}}(t_{-i}|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2}.$$

Since $0 \leq \frac{P_{T_{-i}}(t_{-i}|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2} \leq 1$, it follows that

$$0 \leq z_i(t_{-i}, t_i) \leq \varepsilon$$

for all i, t_{-i} and t_i.For each $(t_{-i}, t_i) \in T^*$, define

$$U_i(t_i'|t_{-i}, t_i) = \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i').$$

The augmented VCG mechanism $\{q, \alpha_i^q + z_i\}_{i \in N}$ is clearly ex post efficient. Individual rationality follows from the observations that

$$\hat{v}_i(q(t); t) + \alpha_i^q(t) \geq 0$$

and

$$z_i(t) \geq 0.$$

Claim 1: For i and for each $t_i, t'_i \in T_i$,

$$\sum_{t_{-i}:(t_{-i},t_i)\in T^*} (z_i(t_{-i}|t_i) - z_i(t_{-i}|t'_i))\, P(t_{-i}|t_i) = \sum_{t_{-i}} (z_i(t_{-i}|t_i) - z_i(t_{-i}|t'_i))\, P(t_{-i}|t_i) \geq \frac{\varepsilon}{2\sqrt{|T|}}\Lambda_i^P$$

Proof of Claim 1:

$$\begin{aligned}\sum_{t_{-i}} (z_i(t_{-i}|t_i) - z_i(t_{-i}|t'_i))\, P(t_{-i}|t_i) = &\sum_{t_{-i}} \varepsilon \left[\frac{P_{T_{-i}}(t_{-i}|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2} - \frac{P_{T_{-i}}(t_{-i}|t'_i)}{||P_{T_{-i}}(\cdot|t'_i)||_2}\right] P(t_{-i}|t_i) \\ &= \frac{\varepsilon||P_{T_{-i}}(\cdot|t_i)||_2}{2}\left\|\frac{P_{T_{-i}}(\cdot|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2} - \frac{P_{T_{-i}}(\cdot|t'_i)}{||P_{T_{-i}}(\cdot|t'_i)||_2}\right\|^2 \\ &\geq \frac{\varepsilon}{2\sqrt{|T|}}\Lambda_i^P.\end{aligned}$$

This completes the proof of Claim 1.

Claim 2: For each i and for each $t_i, t'_i \in T_i$,

$$\begin{aligned}\sum_{t_{-i}:(t_{-i},t_i)\in T^*} &[U_i(t_i|t_{-i}, t_i) - U_i(t'_i|t_{-i}, t_i)]\, P(t_{-i}|t_i) \\ &\geq -(n+1)2M\hat{\nu}^P\end{aligned}$$

Proof of Claim 2: Define

$$A_i(t'_i, t_i) = \{t_{-i} \in T_{-i}|\ (t_{-i}, t_i) \in T^*, (t_{-i}, t'_i) \in T^*, ||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}t'_i)|| > \hat{\nu}^P\}$$

and

$$B_i(t'_i, t_i) = \{t_{-i} \in T_{-i}|\ (t_{-i}, t_i) \in T^*, (t_{-i}, t'_i) \in T^*, ||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}t'_i)|| \leq \hat{\nu}^P\}$$

and

$$C_i(t'_i, t_i) = \{t_{-i} \in T_{-i}|\ (t_{-i}, t_i) \in T^*, (t_{-i}t'_i) \notin T^*\}$$

Since $\nu_i^P \leq \hat{\nu}^P$, we conclude that

$$\text{Prob}\{\tilde{t}_{-i} \in A_i(t'_i, t_i)|\tilde{t}_i = t_i\} \leq \nu_i^P \leq \hat{\nu}^P.$$

Next, note that

$$0 \leq \hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i) \leq \hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) \leq M$$

for all i, t_i and t_{-i}. Therefore,

$$\begin{aligned}
|U_i(t_i'|t_{-i}, t_i)| &= |\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) - \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i') \\
&\quad + \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i') + \alpha_i^q(t_{-i}, t_i')| \\
&\leq |\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) - \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i')| \\
&\quad + |\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i') + \alpha_i^q(t_{-i}, t_i')| \\
&\leq 3M
\end{aligned}$$

for all i, t_i, t_i' and t_{-i}. Applying the definitions and Lemma A, it follows that

$$\sum_{t_{-i} \in A_i(t_i', t_i)} [U_i(t_i|t_{-i}, t_i) - U_i(t_i'|t_{-i}, t_i)]\, P(t_{-i}|t_i) \geq -4M \sum_{t_{-i} \in A_i(t_i', t_i)} P(t_{-i}|t_i) \geq -4M\hat{\nu}^P.$$

In addition,

$$\begin{aligned}
&\sum_{t_{-i} \in B_i(t_i', t_i)} [U_i(t_i|t_{-i}, t_i) - U_i(t_i'|t_{-i}, t_i)]\, P(t_{-i}|t_i) \\
&\geq -2M(n-1) \sum_{t_{-i} \in B_i(t_i', t_i)} ||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i} t_i')|| P(t_{-i}|t_i) \\
&\geq -2M(n-1)\hat{\nu}^P
\end{aligned}$$

and, finally,

$$\begin{aligned}
&\sum_{t_{-i} \in C_i(t_i', t_i)} [U_i(t_i|t_{-i}, t_i) - U_i(t_i'|t_{-i}, t_i)]\, P(t_{-i}|t_i) \\
&= \sum_{t_{-i} \in C_i(t_i', t_i)} [(\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i)) - (\hat{v}_i(c_0; t_{-i}, t_i) + 0)]\, P(t_{-i}|t_i) \\
&= \sum_{t_{-i} \in C_i(t_i', t_i)} (\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i))\, P(t_{-i}|t_i) \\
&\geq 0.
\end{aligned}$$

Combining these observations completes the proof of the claim 2.

Applying Claims 1 and 2, it follows that

$$\begin{aligned}
&\sum_{t_{-i}:(t_{-i},t_i)\in T^*} \left[U_i(t_i|t_{-i},t_i) - U_i(t_i'|t_{-i},t_i)\right] P(t_{-i}|t_i) \\
&+ \sum_{t_{-i}:(t_{-i},t_i)\in T^*} \left(z_i(t_{-i},t_i) - z_i(t_{-i},t_i')\right) P(t_{-i}|t_i) \\
&\geq \frac{\varepsilon}{2\sqrt{|T|}}\Lambda_i^P - (n+1)2M\hat{\nu}^P \\
&\geq 0.
\end{aligned}$$

and the mechanism is interim incentive compatible. If $(t_{-i},t_i) \in T^*$ but $t_{-i} \notin A_i(t_i',t_i)$, then $\Lambda^P \leq 2$ implies that

$$U_i(t_i'|t_{-i},t_i) - U_i(t_i|t_{-i},t_i) \leq 2M(n-1)\nu^P \leq 2M(n-1)\frac{\varepsilon}{4M(n+1)\sqrt{|T|}}\Lambda^P \leq \varepsilon.$$

In addition, $\Lambda^P \leq 2$ implies that $\nu^P \leq \frac{\varepsilon}{4}\Lambda^P \leq \varepsilon$. Therefore,

$$\begin{aligned}
\Pr ob\{(\tilde{t}_{-i},t_i) &\in T^* \text{ and} \\
\hat{v}_i(q(\tilde{t}_{-i},t_i');\tilde{t}_{-i},t_i) + x_i(\tilde{t}_{-i},t_i')) &\leq \hat{v}_i(q(\tilde{t}_{-i},t_i);\tilde{t}_{-i},t_i) + x_i(\tilde{t}_{-i},t_i) + \varepsilon|\tilde{t}_i = t_i\} \\
&\geq Prob\{\tilde{t}_{-i} \notin A_i(t_i',t_i)|\tilde{t}_i = t_i\} \\
&= 1 - Prob\{\tilde{t}_{-i} \in A_i(t_i',t_i)|\tilde{t}_i = t_i\} \\
&\geq 1 - \nu^P \\
&\geq 1 - \varepsilon
\end{aligned}$$

and it follows that the mechanism is weakly ε-ex post incentive compatible. This completes the proof of part (ii).

Part (i) follows from the computations in part (ii). We have shown that, for any positive number κ, there exists an augmented GVCG mechanism $\{q, \alpha_i^q + z_i\}_{i\in N}$ satisfying

$$\begin{aligned}
&\sum_{t_{-i}:(t_{-i},t_i)\in T^*} \left[(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + \alpha_i^q(t_{-i},t_i)) - (\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + \alpha_i^q(t_{-i},t_i'))\right] P(t_{-i}|t_i) \\
&\geq \frac{\kappa}{2\sqrt{|T|}}\Lambda_i^P - (n+1)2M\hat{\nu}^P
\end{aligned}$$

for each i and each t_i, t_i'. If $\Lambda_i^P > 0$ for each i, then κ can be chosen large enough so that incentive compatibility is satisfied. This completes the proof of part (i).

10.4. Proofs of Propositions 3 and 4

We begin with the following Lemma.

Lemma C: For each $t \in T$ and all $\pi, \pi' \in \Delta(\Theta)$,

$$\left[\sum_{\theta\in\Theta} v_i(\hat{q}(t|\pi'), \theta, t_i)\pi'(\theta) + \hat{x}_i(t|\pi')\right] - \left[\sum_{\theta\in\Theta} v_i(\hat{q}(t|\pi), \theta, t_i)\pi(\theta) + \hat{x}_i(t|\pi)\right] \leq (2n-1)M||\pi - \pi'||.$$

Proof:

$$\begin{aligned}
& \left[\sum_{\theta\in\Theta} v_i(\hat{q}(t|\pi), \theta, t_i)\pi(\theta) + \hat{x}_i(t|\pi)\right] - \left[\sum_{\theta\in\Theta} v_i(\hat{q}(t|\pi'), \theta, t_i)\pi'(\theta) + \hat{x}_i(t|\pi')\right] \\
= \; & \sum_k \sum_{\theta\in\Theta} v_k(\hat{q}(t|\pi), \theta, t_i)\pi(\theta) - \max_{c\in C}\left[\sum_{j\in N\setminus i} \sum_{\theta\in\Theta} v_j(c, , \theta, t_j)\pi(\theta)\right] \\
& - \sum_k \sum_{\theta\in\Theta} v_k(\hat{q}(t|\pi'), \theta, t_i)\pi(\theta) + \sum_k \sum_{\theta\in\Theta} v_k(\hat{q}(t|\pi'), \theta, t_i)\pi(\theta) \\
& - \sum_k \sum_{\theta\in\Theta} v_k(\hat{q}(t|\pi'), \theta, t_i)\pi'(\theta) + \max_{c\in C}\left[\sum_{j\in N\setminus i} \sum_{\theta\in\Theta} v_j(c, , \theta, t_j)\pi'(\theta)\right] \\
\geq \; & \max_{c\in C}\left[\sum_{j\in N\setminus i} \sum_{\theta\in\Theta} v_j(c, , \theta, t_j)\pi'(\theta)\right] - \max_{c\in C}\left[\sum_{j\in N\setminus i} \sum_{\theta\in\Theta} v_j(c, , \theta, t_j)\pi(\theta)\right] \\
& + \sum_k \sum_{\theta\in\Theta} v_k(\hat{q}(t|\pi'), \theta, t_i)[\pi(\theta) - \pi'(\theta)] \\
\geq \; & -(n-1)M||\pi - \pi'|| - nM||\pi - \pi'||
\end{aligned}$$

where the final inequality follows from Lemma A.

Proof of Proposition 4:

Step 1: Next, let $\alpha_j(t_j) = t_j$ for each j and for each $(r_i, \pi, t_i) \in T_i \times \Pi \times T_i$, let

$$\beta_i(r_i, \pi, t_i) \in \arg\max_{s_i\in T_i} \sum_{t_{-i}\in T_{-i}} \sum_{\theta\in\Theta} \left[v_i(\hat{q}(t_{-i}, s_i|\pi), \theta, t_i) + \hat{x}_i(t_{-i}, s_i|\pi)\right] \mu_i(\theta, t_{-i}|r_i, \pi, t_i).$$

where

$$\begin{aligned}
\mu_i(\theta, t_{-i}|r_i, \pi, t_i) &= \frac{\rho_\theta(t_{-i}, t_i)P(t_{-i}|t_i)}{\sum_{\hat{t}_{-i}:\rho(\hat{t}_{-i}, r_i)=\pi} P(\hat{t}_{-i}|t_i)} \text{ if } \rho(t_{-i}, r_i) = \pi \\
&= 0 \text{ otherwise}
\end{aligned}$$

To show that (α_i, β_i) is truthful, we must show that $\beta_i(t_i, \pi, t_i) = t_i$, i.e., that

$$t_i \in \arg\max_{s_i \in T_i} \sum_{t_{-i} \in T_{-i}} \sum_{\theta \in \Theta} \left[v_i(\hat{q}(t_{-i}, s_i|\pi), \theta, t_i) + \hat{x}_i(t_{-i}, s_i|\pi)\right] \mu_i(\theta, t_{-i}|t_i, \pi, t_i).$$

for each t_i and each $\pi \in \Pi$. To see this, note that

$$\begin{aligned}
& \sum_{t_{-i} \in T_{-i}} \sum_{\theta \in \Theta} \left[v_i(\hat{q}(t_{-i}, s_i|\pi), \theta, t_i) + \hat{x}_i(t_{-i}, s_i|\pi)\right] \mu_i(\theta, t_{-i}|t_i, \pi, t_i) \\
= & \sum_{\substack{t_{-i} \in T_{-i} \\ :\rho(t_{-i}, t_i) = \pi}} \sum_{\theta \in \Theta} \left[v_i(\hat{q}(t_{-i}, s_i|\pi), \theta, t_i) + \hat{x}_i(t_{-i}, s_i|\pi)\right] \left[\frac{\rho_\theta(t_{-i}, t_i) P(t_{-i}|t_i)}{\sum_{\hat{t}_{-i}:\rho(\hat{t}_{-i}, t_i) = \pi} P(\hat{t}_{-i}|t_i)}\right] \\
= & \sum_{\substack{t_{-i} \in T_{-i} \\ :\rho(t_{-i}, t_i) = \pi}} \left[\sum_{\theta \in \Theta} v_i(\hat{q}(t_{-i}, s_i|\pi), \theta, t_i) \rho_\theta(t_{-i}, t_i) + \hat{x}_i(t_{-i}, s_i|\pi)\right] \left[\frac{P(t_{-i}, t_i)}{\sum_{\hat{t}_{-i}:\rho(\hat{t}_{-i}, t_i) = \pi} P(\hat{t}_{-i}, t_i)}\right] \\
= & \sum_{\substack{t_{-i} \in T_{-i} \\ :\rho(t_{-i}, t_i) = \pi}} \left[\sum_{\theta \in \Theta} v_i(\hat{q}(t_{-i}, s_i|\pi), \theta, t_i) \pi(\theta) + \hat{x}_i(t_{-i}, s_i|\pi)\right] \left[\frac{P(t_{-i}, t_i)}{\sum_{\hat{t}_{-i}:\rho(\hat{t}_{-i}, t_i) = \pi} P(\hat{t}_{-i}, t_i)}\right] \\
\leq & \sum_{\substack{t_{-i} \in T_{-i} \\ :\rho(t_{-i}, t_i) = \pi}} \left[\sum_{\theta \in \Theta} v_i(\hat{q}(t_{-i}, t_i|\pi), \theta, t_i) \pi(\theta) + \hat{x}_i(t_{-i}, t_i|\pi)\right] \left[\frac{P(t_{-i}, t_i)}{\sum_{\hat{t}_{-i}:\rho(\hat{t}_{-i}, t_i) = \pi} P(\hat{t}_{-i}, t_i)}\right] \\
= & \sum_{t_{-i} \in T_{-i}} \sum_{\theta \in \Theta} \left[v_i(\hat{q}(t_{-i}, t_i|\pi), \theta, t_i) + \hat{x}_i(t_{-i}, t_i|\pi)\right] \mu_i(\theta, t_{-i}|t_i, \pi, t_i)
\end{aligned}$$

and the proof is complete.

Step 2: Next, we construct beliefs that are consistent with the strategy profile (α, β). Suppose that player is of true type t_i, the other players have true type profile t_{-i}, player i reports r_i in stage 1 and reports s_i in stage 2. Given the definition of α_j, it follows that each player different from i reports truthfully in stage 1 implying that t_{-i} is reported in stage 1 by players different from i. Since i has reported r_i, *all* players will observe the posted distribution $\rho(t_{-i}, r_i)$ at stage 2. Upon observing $\rho(t_{-i}, r_i)$ and having reported truthfully in stage 1, it follows from the definition of β_j that each player $j \neq i$ reports $\beta_j(t_j, \rho(t_{-i}, r_i), t_j) = t_j$ in stage 2. Since $\alpha_j(t_j) = t_j$ for each j and $P(t_j) > 0$ for each j, it follows that player i of type t_i who has chosen report r_i in stage 1 and who observes $\pi \in \Pi$ at stage 2 will assign positive probability

$$\sum_{\hat{t}_{-i}:\rho(\hat{t}_{-i},r_i)=\pi} P(\hat{t}_{-i}|t_i) > 0$$

to the event

$$\{\hat{t}_{-i} \in T_{-i} : \rho(\hat{t}_{-i}, r_i) = \pi\}.$$

It follows that i's updated beliefs regarding (θ, r_{-i}, t_{-i}) are given by

$$\begin{aligned} \mu_i(\theta, r_{-i}, t_{-i}|r_i, \pi, t_i) &= \frac{\rho_\theta(t_{-i}, t_i)P(t_{-i}|t_i)}{\sum_{\hat{t}_{-i}:\rho(\hat{t}_{-i},r_i)=\pi} P(\hat{t}_{-i}|t_i)} \text{ if } t_{-i} = r_{-i} \text{ and } \rho(t_{-i}, r_i) = \pi \\ &= 0 \text{ otherwise} \end{aligned}$$

Note that the mild abuse of notation is justified by the observation that

$$\sum_{r_{-i}\in T_{-i}} \mu_i(\theta, r_{-i}, t_{-i}|r_i, \pi, t_i) = \mu_i(\theta, t_{-i}|t_i, \pi, t_i)$$

We complete the proof that (α, β, μ) is an ICPBE in the two steps.

Step 3: Deviations at second stage information sets are unprofitable.

The second stage expected payoff to player i given the beliefs μ_i defined above is

$$\begin{aligned} &\sum_{t_{-i}\in T_{-i}} \sum_{r_{-i}\in T_{-i}} \sum_{\theta\in\Theta} \big[v_i(\hat{q}((\beta_j(r_j, \pi, t_j))_{j\neq i}, s_i|\pi), \theta, t_i) \\ &\qquad + \hat{x}_i((\beta_j(r_j, \pi, t_j))_{j\neq i}, s_i|\pi)\big]\, \mu_i(\theta, r_{-i}, t_{-i}|r_i, \pi, t_i) \\ &\quad = \sum_{t_{-i}\in T_{-i}} \sum_{\theta\in\Theta} \left[v_i(\hat{q}(t_{-i}, s_i|\pi), \theta, t_i) + \hat{x}_i(t_{-i}, s_i|\pi)\right] \mu_i(\theta, t_{-i}|r_i, \pi, t_i) \end{aligned}$$

so the definition of β_i implies that

$$\begin{aligned} &\beta_i(r_i, \pi, t_i) \in \\ &\quad \arg\max_{s_i\in T_i} \sum_{t_{-i}\in T_{-i}} \sum_{r_{-i}\in T_{-i}} \sum_{\theta\in\Theta} \big[v_i(\hat{q}((\beta_j(r_j, \pi, t_j))_{j\neq i}, s_i|\pi), \theta, t_i) \\ &\qquad\qquad + \hat{x}_i((\beta_j(r_j, \pi, t_j))_{j\neq i}, s_i|\pi)\big]\, \mu_i(\theta, r_{-i}, t_{-i}|r_i, \pi, t_i) \end{aligned}$$

Step 4: Coordinated deviations at stages 1 and 2 are unprofitable.

To show that coordinated deviations are unprofitable for player i, we assume that other players use $(\alpha_{-i}, \beta_{-i})$ and we must show that, for any $r_i \in T_i$ and $\hat{\beta}_i : T_i \times \Pi \times T_i \to T_i$, we have

$$\begin{aligned}
&\sum_{t_{-i} \in T_{-i}} \left(\sum_{\theta \in \Theta} [v_i(\hat{q}(t_{-i}, t_i | \rho(t_{-i}, t_i)), \theta, t_i) \rho_\theta(t_{-i}, t_i) \right. \\
&\left. + \hat{x}_i(t_{-i}, t_i | \rho(t_{-i}, t_i))] + z_i(t_{-i}, t_i) \right) P(t_{-i} | t_i) \\
\geq &\sum_{t_{-i} \in T_{-i}} \left(\sum_{\theta \in \Theta} \left[v_i(\hat{q}(t_{-i}, \hat{\beta}_i(r_i, \rho(t_{-i}, r_i), t_i) | \rho(t_{-i}, r_i)), \theta, t_i) \rho_\theta(t_{-i}, r_i) \right. \right. \\
&\left. \left. + \hat{x}_i(t_{-i}, \hat{\beta}_i(r_i, \rho(t_{-i}, r_i), t_i) | \rho(t_{-i}, r_i)) \right] + z_i(t_{-i}, r_i) \right) P(t_{-i} | t_i)
\end{aligned}$$

for each $t_i \in T_i$.Note that for each t_{-i} and each r_i,

$$\begin{aligned}
&\max_{s_i \in T_i} \left[\sum_{\theta \in \Theta} [v_i(\hat{q}(t_{-i}, s_i | \rho(t_{-i}, r_i)), \theta, t_i) \rho_\theta(t_{-i}, r_i) + \hat{x}_i(t_{-i}, s_i | \rho(t_{-i}, r_i))] \right] \\
= &\sum_{\theta \in \Theta} [v_i(\hat{q}(t_{-i}, t_i | \rho(t_{-i}, r_i)), \theta, t_i) \rho_\theta(t_{-i}, r_i) + \hat{x}_i(t_{-i}, t_i | \rho(t_{-i}, r_i))]
\end{aligned}$$

so it suffices to show that

$$\begin{aligned}
&\sum_{t_{-i} \in T_{-i}} \left(\sum_{\theta \in \Theta} v_i(\hat{q}(t_{-i}, t_i | \rho(t_{-i}, t_i)), \theta, t_i) \rho_\theta(t_{-i}, t_i) + \hat{x}_i(t_{-i}, t_i | \rho(t'_{-i}, t_i)) + z_i(t_{-i}, t_i) \right) P(t_{-i} | t_i) \\
\geq &\sum_{t_{-i} \in T_{-i}} \left(\sum_{\theta \in \Theta} v_i(\hat{q}(t_{-i}, t_i | \rho(t_{-i}, r_i)), \theta, t_i) \rho_\theta(t_{-i}, r_i) + \hat{x}_i(t | \rho(t_{-i}, r_i)) + z_i(t_{-i}, r_i) \right) P(t_{-i} | t_i)
\end{aligned}$$

From Lemma C, it follows that

$$\begin{aligned}
&\left(\sum_{\theta \in \Theta} v_i(\hat{q}(t_{-i}, t_i | \rho(t_{-i}, r_i)), \theta, t_i) \rho_\theta(t_{-i}, r_i) + \hat{x}_i(t | \rho(t_{-i}, r_i)) \right) \\
&- \left(\sum_{\theta \in \Theta} v_i(\hat{q}(t_{-i}, t_i | \rho(t_{-i}, t_i)), \theta, t_i) \rho_\theta(t_{-i}, t_i) + \hat{x}_i(t_{-i}, t_i | \rho(t'_{-i}, t_i)) \right) \\
\leq &\ (2n-1) M ||\rho(t_{-i}, r_i) - \rho(t_{-i}, t_i)||
\end{aligned}$$

Therefore,

$$\sum_{t_{-i}\in T_{-i}}\left(\sum_{\theta\in\Theta} v_i(\hat{q}(t_{-i},t_i|\rho(t_{-i},r_i)),\theta,t_i)\rho_\theta(t_{-i},r_i)+\hat{x}_i(t|\rho(t_{-i},r_i))+z_i(t_{-i},r_i)\right)P(t_{-i}|t_i)$$
$$-\sum_{t_{-i}\in T_{-i}}\left(\sum_{\theta\in\Theta} v_i(\hat{q}(t_{-i},t_i|\rho(t_{-i},t_i)),\theta,t_i)\rho_\theta(t_{-i},t_i)+\hat{x}_i(t_{-i},t_i|\rho(t'_{-i},t_i))+z_i(t_{-i},t_i)\right)P(t_{-i}|t_i)$$
$$\leq \sum_{t_{-i}\in T_{-i}}(2n-1)M||\rho(t_{-i},r_i)-\rho(t_{-i},t_i)||P(t_{-i}|t_i)+\sum_{t_{-i}\in T_{-i}}\left[z_i(t_{-i},r_i)-z_i(t_{-i},t_i)\right]P(t_{-i}|t_i)$$

Choosing

$$z_i(t_{-i},t_i)=\varepsilon\frac{P_{T_{-i}}(t_{-i}|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2}.$$

it follows that

$$0\leq z_i(t_{-i},t_i)\leq\varepsilon$$

since $0\leq\frac{P_{T_{-i}}(t_{-i}|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2}\leq 1$, for all i, t_{-i} and t_i. Therefore,

$$\sum_{t'_{-i}\in T_{-i}}[z_i(t_{-i},r_i)-z_i(t_{-i},t_i)]P(t_{-i}|t_i)\leq-\frac{\varepsilon}{2\sqrt{|T|}}\Lambda_i^P$$

Choosing

$$0<\delta<\frac{\varepsilon}{2(2n-1)M\sqrt{|T|}}$$

yields the result.

Proof of Proposition 3: Using the same argument as that for part (i) of Theorem 2, Proposition 3 follows immedediately from the proof of Proposition 4 above

10.5. Proof of Lemma B

For each $\theta\in\Theta$, let $P(\cdot|\theta)$ denote the conditional measure on A given $\theta\in\Theta$ and for each r, let $P_\Theta(\cdot|t^r)$ denote the conditional measure on Θ given $t^r\in T^r$. For each $\alpha\in A$, let $f_\alpha(t^r)=\#\{i\in J_r|t_i^r=\alpha\}$ and define $f(t^r)=(f_\alpha(t^r))_{\alpha\in A}$.

Step 1: If $s \in A$ and $t^r_{-i} \in T^{r-1}$, then

$$\left|\frac{f_\alpha(t^r_{-i}, s)}{r} - \frac{f_\alpha(t^r_{-i})}{r-1}\right| = \left|\frac{f_\alpha(t^r_{-i})+1}{r} - \frac{f_\alpha(t^r_{-i})}{r-1}\right| = \frac{r-1-f_\alpha(t^r_{-i})}{r(r-1)} \leq \frac{1}{r} \text{ if } \alpha = s$$

$$\left|\frac{f_\alpha(t^r_{-i}, s)}{r} - \frac{f_\alpha(t^r_{-i})}{r-1}\right| = \left|\frac{f_\alpha(t^r_{-i})}{r} - \frac{f_\alpha(t^r_{-i})}{r-1}\right| = f_\alpha(t^r_{-i})\frac{1}{r(r-1)} \leq \frac{1}{r} \text{ if } \alpha \neq s$$

implying that

$$\left\|\frac{f_\alpha(t^r_{-i}, s)}{r} - \frac{f_\alpha(t^r_{-i})}{r-1}\right\| \leq \frac{|A|}{r}.$$

Step 2: For each θ, let

$$\mu(\theta) := \max_{\hat{\theta}\neq\theta} \prod_{\alpha\in A} \left[\frac{P(\alpha|\hat{\theta})}{P(\alpha|\theta)}\right]^{P(\alpha|\theta)}$$

and let $R = \max_\theta \mu(\theta)$. Let $\chi_\theta \in \Delta(\Theta)$ denote the Dirac measure with $\chi_\theta(\theta) = 1$ and let $\beta := \min_{\theta\in\Theta} P(\theta)$. There exists a $\delta > 0$ such that, for each $\theta \in \Theta$ and each r,

$$||\frac{f(t^r)}{r} - P(\cdot|\theta)|| < \delta \Rightarrow ||\chi_\theta - P_\Theta(\cdot|t^r)|| \leq \frac{2R^{r/2}}{\beta}$$

To see this, fix θ and note that Assumption (a) in the definition of conditionally independent sequence and the strict concavity of the function $ln(\cdot)$ imply that $\mu(\theta) < 1$. Again by computing the logarithm, there exists a $\delta_\theta > 0$ such that

$$\prod_{\alpha\in A} \left[\frac{P(\alpha|\hat{\theta})}{P(\alpha|\theta)}\right]^{\frac{f_\alpha(t^r)}{r} - P(\alpha|\theta)} \leq \frac{1}{\sqrt{\mu(\theta)}}$$

whenever $\hat{\theta} \neq \theta$ and $||\frac{f(t^r)}{r} - P(\cdot|\theta)|| < \delta_\theta$. Letting $R = \max_\theta \mu(\theta)$ and $\delta = \min \delta_\theta$, we conclude that for each $\theta \in \Theta$, $||\frac{f(t^r)}{r} - P(\cdot|\theta)|| < \delta$ implies that

$$\frac{P_\Theta(\hat{\theta}|t^r)P(\theta)}{P_\Theta(\theta|t^r)P(\hat{\theta})} = \left[\prod_{\alpha\in A}\left[\frac{P(\alpha|\hat{\theta})}{P(\alpha|\theta)}\right]^{P(\alpha|\theta)} \prod_{\alpha\in A}\left[\frac{P(\alpha|\hat{\theta})}{P(\alpha|\theta)}\right]^{\frac{f_\alpha(t^r)}{r} - P(\alpha|\theta)}\right]^r \leq \left[\mu(\theta)\frac{1}{\sqrt{\mu(\theta)}}\right]^r \leq R^{r/2}$$

whenever $\hat{\theta} \neq \theta$. Therefore, $||\frac{f(t^r)}{r} - P(\cdot|\theta)|| < \delta$ implies that

$$||\chi_\theta - P_\Theta(\cdot|t^r)|| = 2\sum_{\hat{\theta}\neq\theta} P_\Theta(\hat{\theta}|t^r) \leq 2\sum_{\hat{\theta}\neq\theta} \frac{P(\hat{\theta})}{P(\theta)} P_\Theta(\theta|t^r) R^{r/2} \leq \frac{2R^{r/2}}{\beta}$$

Step 3: To complete the argument, choose θ and $t_i, t'_i \in A$ and note that for all sufficiently large r we have

$$\begin{aligned}
||P_\Theta(\cdot|t^r_{-i}, t_i) - P_\Theta(\cdot|t^r_{-i}, t'_i)|| &> \frac{4R^{r/2}}{\beta} \\
&\Rightarrow \exists s \in A : ||\chi_\theta - P_\Theta(\cdot|t^r_{-i}, s)|| > \frac{2R^{r/2}}{\beta} \\
&\Rightarrow \exists s \in A : ||\frac{f(t^r_{-i}, s)}{r} - P(\cdot|\theta)|| \geq \delta \\
&\Rightarrow \exists s \in A : ||\frac{f(t^r_{-i})}{r-1} - P(\cdot|\theta)|| + ||\frac{f(t^r_{-i}, s)}{r} - \frac{f(t^r_{-i})}{r-1}|| \geq \delta \\
&\Rightarrow ||\frac{f(t^r_{-i})}{r-1} - P(\cdot|\theta)|| \geq \delta - \frac{|A|}{r} \\
&\Rightarrow ||\frac{f(t^r_{-i})}{r-1} - P(\cdot|\theta)|| \geq \delta/2.
\end{aligned}$$

Applying Hoeffding's inequality (see Hoeffding (1963)), it follows that for all sufficiently large r we have

$$\Pr ob\{||\frac{f(\tilde{t}^r_{-i})}{r-1} - P(\cdot|\theta)|| \geq \delta/2|\tilde{\theta} = \theta\} \leq 2|A| \exp(\frac{-(r-1)\delta^2}{2}).$$

Therefore,

$$\begin{aligned}
\Pr ob\{||P_\Theta(\cdot|\tilde{t}^r_{-i}, t_i) - P_\Theta(\cdot|\tilde{t}^r_{-i}, t'_i)|| > \frac{4R^{r/2}}{\beta}|\tilde{t}_i = t_i\} \\
&\leq \Pr ob\{||\frac{f(\tilde{t}^r_{-i})}{r-1} - P(\cdot|\theta)|| \geq \delta/2|\tilde{t}_i = t_i\} \\
&= \sum_\theta \Pr ob\{||\frac{f(\tilde{t}^r_{-i})}{r-1} - P(\cdot|\theta)|| \geq \delta/2|\tilde{\theta} = \theta\} P(\theta|t_i) \\
&\leq 2|A| \exp(\frac{-(r-1)\delta^2}{2})
\end{aligned}$$

Hence, for all r sufficiently large,

$$\nu_i^{P^r} \leq \max\{\frac{4R^{r/2}}{\beta}, 2|A| \exp(\frac{-(r-1)\delta^2}{2})\}.$$

10.6. Proof of Theorem 3

The proof is essentially identical to that of Theorem 2. First, note that $(T^r)^* = T^r$. For notational ease, we will write q_i, α_i, and z_i instead of q_i^r, α_i^r and z_i^r and T, t, t_{-i} and t_i instead of T^r, t^r, t_{-i}^r and t_i^r. Choose $\varepsilon > 0$. Let M be the bound defined in the statement of the Theorem. For each $(t_1, ..t_r) \in T$, For each $\alpha, \beta \in A$, let

$$Q(\beta|\alpha) = \sum_{\theta} P(\beta|\theta)P(\theta|\alpha)$$

so that

$$||Q(\cdot|\alpha)||_2 = \left[\sum_{\beta \in A} Q(\beta|\alpha)^2\right]^{\frac{1}{2}}$$

For each i and $(t_1, ..t_r) \in T$, , define

$$\begin{aligned} z_i^r(t_{-i}, t_i) &= \frac{\varepsilon}{r} \frac{Q(t_{i+1}|t_i)}{||Q(\cdot|t_i)||_2} \text{ if } i = 1, .., r-1 \\ &= \frac{\varepsilon}{r} \frac{Q(t_1|t_r)}{||Q(\cdot|t_r)||_2} \text{ if } i = r \end{aligned}$$

Therefore,

$$0 \leq z_i^r(t_{-i}, t_i) \leq \frac{\varepsilon}{r}$$

for all i, t_{-i} and t_i. Individual rationality of the augmented mechanism follows from the observations that

$$\hat{v}_i(q(t); t) + x_i(t) \geq 0$$

and

$$z_i(t) \geq 0.$$

If $\alpha, \alpha' \in A$ with $\alpha \neq \alpha'$, then, by assumption, $Q(\cdot|\alpha) \neq Q(\cdot|\alpha')$ and, therefore,

$$\Lambda^* := \min_{\alpha \in A} \min_{\alpha' \in A \setminus \{\alpha\}} \left\| \frac{Q(\cdot|\alpha)}{||Q(\cdot|\alpha)||_2} - \frac{Q(\cdot|\alpha')}{||Q(\cdot|\alpha')||_2} \right\|_2^2 > 0.$$

Finally, note that

$$\sum_{s \in A} \left[\frac{Q(s|t_i)}{||Q(\cdot|t_i)||_2} - \frac{Q(s|t_i')}{||Q(\cdot|t_i)||_2}\right] Q(s|t_i) = \frac{||Q(\cdot|t_i)||_2}{2} \Lambda^* \geq \frac{\Lambda^*}{2\sqrt{|A|}}.$$

Claim 1: Let $|A|$ denote the cardinality of A. Then

$$\sum_{t_{-i}} \left(z_i^r(t_{-i}|t_i) - z_i^r(t_{-i}|t_i')\right) P^r(t_{-i}|t_i) \geq \frac{\varepsilon}{2r\sqrt{|A|}}\Lambda^*$$

Proof of Claim 1: If $1 \leq i \leq r-1$, then

$$\begin{aligned}
\sum_{t_{-i}} \left(z_i^r(t_{-i}|t_i) - z_i^r(t_{-i}|t_i')\right) P^r(t_{-i}|t_i) &= \sum_{t_{i+1}} \sum_{t_{-\{i,i+1\}}} \frac{\varepsilon}{r} \left[\frac{Q(t_{i+1}|t_i)}{||Q(\cdot|t_i)||_2} - \frac{Q(t_{i+1}|t_i')}{||Q(\cdot|t_i)||_2}\right] P^r(t_{-i}|t_i) \\
&= \sum_{t_{i+1}} \frac{\varepsilon}{r} \left[\frac{Q(t_{i+1}|t_i)}{||Q(\cdot|t_i)||_2} - \frac{Q(t_{i+1}|t_i')}{||Q(\cdot|t_i)||_2}\right] Q(t_{i+1}|t_i) \\
&\geq \frac{\varepsilon}{2r\sqrt{|A|}}\Lambda^*.
\end{aligned}$$

A similar computation is applied when $i = r$ and this completes the proof of Claim 1.

Claim 2:

$$\sum_{t_i} \left[(\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + x_i(t_{-i}, t_i)) - (\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + x_i(t_{-i}, t_i'))\right] P^r(t_{-i}|t_i) \geq -2M(r+1)$$

Proof of Claim 2: For each $(t_{-i}, t_i) \in T$, define

$$U_i(t_i'|t_{-i}, t_i) = \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i(t_{-i}, t_i').$$

As in the proof of Theorem 2, define

$$A_i(t_i', t_i) = \{t_{-i} \in T_{-i}|\ ||P_\Theta^r(\cdot|t_{-i}, t_i) - P_\Theta^r(\cdot|t_{-i}t_i')|| > \hat{\nu}^{P^r}\}$$

and

$$B_i(t_i', t_i) = \{t_{-i} \in T_{-i}|\ ||P_\Theta^r(\cdot|t_{-i}, t_i) - P_\Theta^r(\cdot|t_{-i}t_i')|| \leq \hat{\nu}^{P^r}\}.$$

Using the arguments of Theorem 2, we conclude that

$$\text{Prob}\{\tilde{t}_{-i} \in A_i(t_i', t_i)|\tilde{t}_i = t_i\} \leq \nu^{P^r},$$

$$0 \leq \hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i(t_{-i}, t_i) \leq \hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) \leq M$$

and

$$|\hat{v}_i^r(q(t_{-i}, t_i'); t_{-i}, t_i) + x_i(t_{-i}, t_i')| \leq 3M|$$

for all i, t_i, t_i' and t_{-i}. Again using the arguments of Theorem 2, it follows that

$$\sum_{t_{-i} \in A_i(t_i', t_i)} [U_i(t_i|t_{-i}, t_i) - U_i(t_i'|t_{-i}, t_i)] \, P^r(t_{-i}|t_i) \geq -4M\hat{\nu}^P.$$

and that

$$\sum_{t_{-i} \in B_i(t_i', t_i)} [U_i(t_i|t_{-i}, t_i) - U_i(t_i'|t_{-i}, t_i)] \, P^r(t_{-i}|t_i) \geq -2M(r-1)\nu^{P^r}$$

Combining these observations completes the proof of the claim 2.

Applying Lemma B and Claims 1 and 2, it follows that, for sufficiently large r,

$$\begin{aligned}
&\sum_{t_{-i}} (U_i(t_i|t_{-i}, t_i) + z_i(t_{-i}, t_i)) \, P^r(t_{-i}|t_i) - \sum_{t_{-i}} (U_i(t_i'|t_{-i}, t_i) + z_i(t_{-i}, t_i')) \, P^r(t_{-i}|t_i) \\
= &\sum_{t_{-i}} [U_i(t_i|t_{-i}, t_i) - U_i(t_i'|t_{-i}, t_i)] \, P^r(t_{-i}|t_i) + \sum_{t_{-i}} (z_i(t_{-i}, t_i) - z_i(t_{-i}, t_i')) \, P^r(t_{-i}|t_i) \\
\geq &\frac{\varepsilon}{2r\sqrt{|A|}}\Lambda^* - 2M(r-1)\nu^{P^r} \\
= &\frac{1}{r}\left[\frac{\varepsilon}{2\sqrt{|A|}}\Lambda^* - 2Mr(r-1)\nu^{P^r}\right] \\
\geq &\, 0.
\end{aligned}$$

and the proof of interim IC is complete.

11. Bibliography

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