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"Optimal Taxation in a Limited Commitment Economy"

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Optimal Taxation in a Limited Commitment Economy

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Abstract

This paper studies optimal Ramsey taxation when risk sharing in private insurance markets is imperfect due to limited enforcement. In a limited commitment economy, there are externalities associated with capital and labor because individuals do not take into account that their labor and saving decisions affect aggregate supply, wages and thus the value of autarky. Due to these externalities, the Ramsey government has an additional goal, which is to internalize the externalities of labor and capital to improve risk sharing, in addition to its usual goal - minimizing distortions when financing government expenditures. These two goals drive capital and labor taxes in opposite directions. By balancing these conflicting goals, the steady-state optimal capital income taxes are levied only to remove the negative externality of the capital, and optimal labor income taxes are set to meet the budgetary needs of the government in the long run, despite positive externalities of labor.

Keywords: Ramsey Taxation, Limited Enforcement
JEL Classification Codes: D52, E62, H21, H23

1 Introduction

What should be the optimal fiscal policy when private insurance against idiosyncratic income risk is not perfect? The canonical Ramsey tax literature cannot answer this question because it adopts a representative agent model framework which, in the presence of idiosyncratic risk, requires perfect insurance markets. Thus, it focuses only on optimal ways to finance a given stream of government expenditures by minimizing distortions on labor supply and capital accumulation of households. Recently there has been a growing interest in optimal taxation in models with imperfect risk sharing against idiosyncratic uncertainty. In the Bewley-Aiyagari class of models, where risk sharing is limited because of incomplete asset markets, there have been several studies that analyze optimal taxation considering both insurance and distortions of labor and capital (for theoretical results, see Aiyagari (1995), Dávila, Hong, Krusell, and Ríos-Rull (forthcoming); for quantitative results, see: Conesa and Krueger (2006), Conesa, Kitao, and Krueger (2009)). In limited commitment models, where imperfect risk sharing arises endogenously because of limited enforcement, however, studies on optimal taxation have focused only on risk-sharing effects based on an endowment economy or a production economy with exogenous labor supply (Krueger and Perri, 2011; Chien and Lee, 2010). No studies have investigated optimal taxation under consideration of both risk-sharing and distortion effects in the latter type of imperfect risk-sharing models.

This paper studies optimal Ramsey taxation with heterogeneous agents when risk sharing in private insurance markets is not perfect due to limited enforcement of contracts (from now on, we will call this environment a limited commitment economy), considering both the risk-sharing and distortion effects of taxes. In a limited commitment economy, there is a complete set of contingent insurance contracts. Private risk-sharing, however, is limited because private insurance contracts are not fully enforceable. These contracts can only be sustained through the threat of exclusion from participating in future insurance markets upon default. Thus, the private insurance that can be traded is limited by the extent to which the contract can keep households from defaulting. Since both the value of staying in the contract and the cost of default are affected by tax policy, risk sharing in this economy endogenously responds to taxes set by the government. Thus, the government needs to consider how tax policies endogenously change private risk sharing.

In a limited commitment economy with production, private risk sharing is not efficient even if we restrict our attention to constrained efficiency[1] because of externalities associated

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[1]Private risk-sharing in a limited commitment economy with endowment income is constrained efficient,

with capital and labor. As Abraham and Carceles-Poveda (2006), Chien and Lee (2010) and others have pointed out, an increase in aggregate capital raises the autarky value by increasing the wage in autarky. This is the external cost of capital, since it has adverse effects on private risk sharing, but individuals do not take into account the effects on wage in autarky when making their capital accumulation decision. In the same manner, there is an external benefit of aggregate labor because its increase lowers the wage in autarky and this effect is not internalized by households' decisions. Due to these externalities of labor and capital, the Ramsey government in a limited commitment environment has two goals when it sets taxes on labor and capital. The first goal is the standard one - minimizing distortions when financing exogenous government expenditures. The second goal, which is new in the Ramsey optimal taxation literature, is to internalize the externalities of labor and capital to improve risk sharing in a limited commitment economy.

These two goals of the Ramsey government conflict with each other in the following sense. In the canonical Ramsey taxation literature, when the Ramsey government does not have risk-sharing concerns, it is well known that taxes on capital should be set to zero in the steady-state and all the distortions due to the government's budgetary needs will be put on the labor income taxes in the long run. On the other hand, if the government does not have to finance its expenditures using distortionary taxes and only takes into account the impact of taxes on private risk sharing, it wants to tax capital and subsidize labor to internalize the negative externality of capital and the positive externality of labor. These observations raise a crucial question: what is the optimal mix of labor income taxes and capital income taxes when the Ramsey government tries to achieve these two conflicting goals simultaneously? The main contribution of this paper is to provide an answer to this question. we show that in a limited commitment economy, long-run optimal capital income taxes are levied only to remove the externality of capital, not to finance government expenditures, even though the level of the optimal capital tax need to be normalized by the opportunity cost of capital disinvestment from the Ramsey government's perspective and thus might depend on factors affecting government revenues (shadow cost of distortion and elasticities). We argue that all the tax distortions due to the budgetary needs of the government are still put on labor income taxes in the long run, despite the external benefit of labor.

One implication of this result is that limited commitment provides a rationale for capital income taxes. The positive capital income taxes make individuals internalize the effects of capital investment on the tightening of enforcement constraints, improving private risk

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where the constraints include the enforcement constraints, but is not fully efficient either.

sharing. This positive capital income tax, however, does not contradict the zero capital taxation of Chamley-Judd (Chamley, 1986; Judd, 1985) and can be interpreted as a version of the Chamley-Judd's result. Depending on the existence of a capital externality, the result is either an exactly zero capital tax or a generalized version of zero capital taxation. If the wage in autarky does not depend on equilibrium capital state, then there is no capital externality and thus the optimal capital income tax will be exactly zero in the steady state. If the wage in autarky is endogenously determined by responding to equilibrium capital, the externality of capital makes a positive capital income tax optimal, which is still in line with the Chamley-Judd logic in the following sense. The non-zero capital tax arises not because of the government's budgetary needs at all but only to internalize the externality.

A secondary contribution of this paper is a method of characterizing an equilibrium with limited commitment, a 'Kehoe-Levine (K-L) equilibrium', referring to Kehoe and Levine (1993). *Ex-post* heterogeneity of agents due to idiosyncratic shocks makes the characterization of an equilibrium complicated. Adopting Werning (2007)'s methodology to a limited commitment model, however, we provide a concise characterization of a K-L equilibrium with aggregate allocations and Pareto-Negish weights, so that we can use a primal approach. Based on this characterization, the Ramsey problem boils down to a simple programming problem of choosing aggregate allocations, as in the canonical Ramsey literature, but with additional constraints.

To clearly understand the role of capital income taxes and labor income taxes in achieving the two goals of government, we first consider a benchmark model when lump-sum taxes are allowed. In this case, the government can achieve the budgetary needs without inducing tax distortions. Thus, with only one goal, which is internalizing externalities, optimal capital taxes are exactly equal to the external cost of capital, and labor income taxes are negative, implying a subsidy. On the other hand, if lump-sum taxes are excluded and the wage in autarky does not depend on equilibrium capital and labor, then the optimal tax property reverts back to that of the canonical Ramsey problem - a zero capital tax, to minimize the distortions of taxation. If lump-sum taxes are excluded and the wage in autarky does depend on equilibrium capital and labor, then the government considers both goals and the main results above hold. In the steady-state, optimal capital taxes are levied only to remove the externality of capital, and all the taxation distortions due to the government's budgetary needs will be placed on labor income taxes, despite the external benefit of labor. For separable isoelastic preference, this result holds even outside the steady-state.

As a robustness check, we investigate whether the optimal tax structure with a linear tax

system still applies if we allow non-linear progressive labor income taxes. We show that for a specific functional form of progressive labor income taxation, the optimal structure of the capital income tax and the labor income tax is qualitatively equivalent to that of a linear tax system.

Our paper is related to the literature on models with limited commitment. Earlier works by Kehoe and Levine (1993, 2001), Kocherlakota (1996), Alvarez and Jermann (2000) analyze optimal risk-sharing contracts that are constrained by limited enforcement. Several studies have analyzed the effects of taxation with this limited commitment model. Krueger and Perri (2011) study the impact of the progressive income taxation on private risk sharing. They analyze taxation in an endowment economy and focus only on the risk-sharing effects of the progressivity of taxes, while our paper models a production economy and considers both the risk-sharing and the distortion effects of taxes. Chien and Lee (2010) decentralize a constrained efficient allocation using capital taxes, but their economy also abstracts from endogenous labor supply and focuses only on the risk-sharing effects, while our paper models capital accumulation and labor supply together. Richer structure of our paper enables us to take into account both risk sharing effects and distortions of labor and capital in the design of optimal taxes. Abraham and Carceles-Poveda (2006) was the first paper to show the inefficiency of a competitive equilibrium in a limited commitment economy with capital accumulation, but they decentralized the constrained efficiency by putting an upper limit on capital accumulation instead of using taxes.

Our paper is also related to the large literature on optimal Ramsey taxation. One of the best known results of the Ramsey literature is the zero capital tax result of Chamley (1986) and Judd (1985). Atkeson, Chari, and Kehoe (1999) summarize how it is robust in a wide class of models. This zero capital tax result has been validated in a variety of other settings, including a model with human capital (Jones, Manuelli, and Rossi, 1997), neoclassical growth models with aggregate risk (Zhu, 1992; Chari, Christiano, and Kehoe, 1994), and an overlapping generation model, provided that labor taxes can be age-conditioned (Garriga, 2001; Erosa and Gervais, 2002). On the contrary, there has been some attempts to find fundamental reasons why the capital income tax is positive in the Ramsey literature. Aiyagari (1995) argues that if idiosyncratic risk is not insurable because of incomplete insurance markets and borrowing constraints, the optimal capital income tax is positive due to precautionary saving. Recently, Acemoglu, Golosov, and Tsyvinski (2011) show that a political economy can give a rationale for optimal positive capital tax, since it relaxes the political economy constraints if politicians are more impatient than citizens. Our paper contributes to both streams of the literature. First, we provide another fundamental reason for a positive capital

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income tax - the externality in a limited commitment economy. If there is no externality of capital, however, despite limited commitment, the optimal capital income tax is zero as in the celebrated Chamley-Judd result.

Finally, this paper is very closely related to the literature on the heterogeneous-agent Ramsey taxation. Methodologically, our paper is mostly related to Werning (2007), who analyzes Ramsey taxation with redistributional concerns. The heterogeneity Werning (2007) studies is *ex-ante* and permanent heterogeneity in productivity, while this paper introduces *ex-post* heterogeneity due to idiosyncratic shocks. Moreover, Werning focuses on the redistributional role of distortionary labor income taxes when lump-sum taxes are allowed, whereas this paper focuses on the optimal structure of the capital tax and labor tax when the government has two conflicting goals. Earlier, Aiyagari (1995) analyzed Ramsey taxation in the Bewley-Aiyagari class of incomplete market model with heterogeneous agents. Different from Aiyagari (1995), our paper has complete asset markets, but risk-sharing is imperfect because of the limited commitment. The different market structure in this paper leads to an optimal taxation different from that of Aiyagari (1995). Dávila, Hong, Krusell, and Ríos-Rull (forthcoming) also analyze optimal capital taxes in an incomplete market. Their analysis is different from Ramsey literature since there is no government expenditure and thus no need to use distortionary taxes to finance it. The constrained efficiency concept in their paper, however, is in line with our paper in the sense that government does not try to alter the market structure and just affects prices.

This paper is organized as follows. Section 2 presents the economy. Section 3 sets up the Ramsey problem using a primal approach, and Section 4 analyzes the properties of optimal Ramsey taxation. Section 5 investigates the robustness of the results to the permission of non-linear labor income taxes, and Section 6 concludes.

2 The economy

The economy is populated by a continuum of households with idiosyncratic productivity shocks. Productivity shocks, θ_t, are drawn every period from $\Theta = \{\theta_1, \cdots, \theta_N\}$. The history of productivity shocks is denoted as $\theta^t = (\theta_0, \cdots, \theta_t)$. The probability of θ^t is $\pi(\theta^t)$. We assume that there is no *ex-ante* heterogeneity in initial productivity realizations (with same θ_0) among agents in period 0.

The preference of an infinitely lived household is represented by the expected lifetime utility function,

$$\sum_{t=0}^{\infty} \sum_{\theta^t} \beta^t \pi(\theta^t) \left[u(c_t(\theta^t)) - v(l_t(\theta^t)) \right],$$

where $c_t(\theta^t)$ denotes consumption and $l_t(\theta^t)$ denotes labor. The concavity of the lifetime utility is guaranteed by assuming $u' > 0, u'' < 0, v' > 0$, and $v'' > 0$. Standard inada conditions on preferences are imposed; $\lim_{c \to 0} u'(c) = \infty$, $\lim_{c \to \infty} u'(c) = 0$, $\lim_{l \to 0} v'(l) = 0$, and there exists $\bar{l} < \infty$ such that $\lim_{l \to \bar{l}} v'(l) = \infty$. The labor supply of the household is endogenous due to a disutility of labor. The agent who has history θ_t, will supply $l_t(\theta^t)$, resulting in $\theta_t l_t(\theta^t)$ efficiency units of labor supply.

The production function of the representative firm is $F(K_t, L_t)$, where K_t is aggregate capital and L_t is aggregate labor in the economy. We assume that F is strictly increasing and concave in both of its arguments, continuously differentiable, and exhibits constant returns to scale. The firm is producing output by renting labor and capital from households. To guarantee positive aggregate capital, we will assume that $F(0, L) = 0$, for all L. Also, there exists $\bar{K} < \infty$ such that $F(\bar{K}, \bar{L}) < \bar{K}$, ensuring that the steady-state level of output is finite. The depreciation rate for capital is denoted by δ.

Assuming a law of large numbers, aggregate allocations are given as follows.

$$K_t = \sum_{\theta^{t-1}} \pi(\theta^{t-1}) k_t(\theta^{t-1}), \quad L_t = \sum_{\theta^t} \pi(\theta^t) \theta_t l_t(\theta^t)$$

As in the canonical Ramsey literature, we assume that the government can finance its exogenous stream of expenditures using debt and taxes on labor income and capital income. It is also assumed that the government can fully commit to a sequence of taxes. Both capital income and labor income taxes are linear, with tax rates $\tau_{k,t}$, $\tau_{l,t}$, respectively. Later, we relax this assumption by allowing non-linear labor income taxes and check the robustness of our results. Lump-sum taxes are not permitted. It will be allowed only in the benchmark case for purposes of comparison.

Asset markets are assumed to be complete. Households can purchase Arrow-Debreu state-contingent consumption claims at period 0. Risk sharing in the private insurance markets is limited, however, because of limited commitment. A household has the option to renege on a risk-sharing contract, at any time, at any history. Because households have this option, the amount of a contingent consumption claim of a particular history that can be purchased in the private insurance market at period 0 is limited by the extent to which the contract can keep households from defaulting. Formally, households will face enforcement constraints

that guarantee that a household would never be better off reverting permanently to autarky. After default, all assets and capital are seized and the household will be excluded from the asset and capital trading markets. Thus, after default, the household will live in financial autarky with only labor income. Then the period utility of autarky is defined by

$$U^{aut}(\theta^t; w_t, \tau_{l,t}) = \max_{\tilde{c}_t, \tilde{l}_t} \left[u(\tilde{c}_t) - v(\tilde{l}_t) \right]$$

$$s.t. \quad \tilde{c}_t \leq (1 - \tau_{l,t}) w_t \theta_t \tilde{l}_t$$

We assume that the wage in autarky is equal to the wage in equilibrium throughout most of the paper, but this assumption will be relaxed when we analyze the properties of optimal taxation and thus we will look at three different cases. We also assume that the government cannot discriminate between tax rates in and outside autarky. The enforcement constraint at θ^t is:

$$\sum_{s=t}^{\infty} \sum_{\theta^s | \theta^t} \beta^{s-t} \pi(\theta^s | \theta^t) \left[u(c_s(\theta^s)) - v(l_s(\theta^s)) \right] \geq \sum_{s=t}^{\infty} \sum_{\theta^s | \theta^t} \beta^{s-t} \pi(\theta^s | \theta^t) U^{aut}(\theta^s; w_s, \tau_{l,s}),$$

where $c_s(\theta^s)$ specifies the consumption allocation to agents in the private insurance contract who experience history θ^s. Since there is no private information, given that the enforcement constraints are imposed, households never have an incentive to default, thus there will be no default in equilibrium.

In the appendix, we present an analysis under the alternative assumption on autarky that the government cannot tax households that are in autarky. If we take this alternative assumption, the following Ramsey taxation analysis applies exactly with little modification. The interpretation of no taxing in autarky, however, might be problematic because the wage in autarky still depends on aggregate capital and labor in the economy.[2] Not only that, being able to tax labor income in autarky could have an additional benefit in this economy, which is to mitigate the external cost of capital. Thus, we assume that the government can tax labor income in autarky. There is, however, one advantage of taking the alternative assumption - no taxing in autarky. We can compare the constrained efficient allocation with that of a Ramsey equilibrium since construction of the planner's problem is straightforward with the alternative assumption. This is because the enforcement constraints in the planner's problem can be easily expressed only with allocations when the autarky values do not depend on taxes.[3] Thus, in the appendix, we provide an analysis based on the assumption of no taxation in autarky.

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[2]Thus, we should not think of living in autarky as working in a completely informal sector.

[3]Construction of a social planner's problem (for constrained efficiency) with the assumption of taxing

2.1 Kehoe-Levine equilibrium with taxes

As we mentioned in the introduction, we will call a competitive equilibrium with enforcement constraints a "Kehoe-Levine (K-L) equilibrium", referring to Kehoe and Levine (1993). We start with the household problem in a K-L equilibrium. We will denote by $Q(\theta^t)$ the price of a contract at period 0 that specifies delivery of one unit of a consumption good at period t to a person who has experienced a history of productivity shocks θ^t. In period 0, agents are identical by assumption. So, we normalize the price of the consumption good at period 0 to 1.

The household problem, given a linear tax $\{\tau_l, \tau_k\}$ is:

$$\max_{\{c_t, l_t, k_{t+1}\}} \quad \sum_{t=0}^{\infty} \sum_{\theta^t} \beta^t \pi(\theta^t) \left[u(c_t(\theta^t)) - v(l_t(\theta^t)) \right] \tag{1}$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} \sum_{\theta^t} Q(\theta^t) \left[c_t(\theta^t) + k_{t+1}(\theta^t) - (1 - \tau_{l,t}) w_t \theta_t l_t(\theta^t) - (1 + r_t(1 - \tau_{k,t})) k_t(\theta^{t-1}) \right]$$

$$\leq (1 + r_0^b) B_0$$

$$\sum_{s=t}^{\infty} \sum_{\theta^s} \beta^{s-t} \pi(\theta^s | \theta^t) \left[u(c_s(\theta^s)) - v(l_s(\theta^s)) \right] \geq \sum_{s=t}^{\infty} \sum_{\theta^s} \beta^{s-t} \pi(\theta^s | \theta^t) U^{aut}(\theta^s; \tau_{l,t})$$

This is a standard household problem in an Arrow-Debreu equilibrium except that it has additional enforcement constraints restricting the consumption possibility set.[4] Now we can define the K-L equilibrium.

Definition 1. *Given $K_0, B_0,$ and $\{G_t, \tau_{l,t}, \tau_{k,t}\}$, a **K-L equilibrium** is allocations $\{c_t^*(\theta^t),$ $l_t^*(\theta^t), k_{t+1}^*(\theta^t)\}$ and prices $\{Q^*(\theta^t), r_t^*, w_t^*\}$ if*

1. *Given $\{Q^*(\theta^t), r_t^*, w_t^*, \tau_{l,t}, \tau_{k,t}\}$, $\{c_t^*(\theta^t), l_t^*(\theta^t), k_{t+1}^*(\theta^t)\}$ solves the household problem with enforcement constraints.*

2. *$(K_t, L_t) \in \arg\max F(K_t, L_t) - w_t^* L_t - (r_t^* + \delta) K_t$*

3. *Government's Budget Balance*
   $\sum_{t=0}^{\infty} \sum_{\theta^t} Q_t(\theta^t) [\tau_{l,t} w_t L_t + \tau_{k,t} r_t K_t - G_t] = (1 + r_0^b) B_0$

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in autarky is tricky, because the planner will tax labor income as much as he can to make the enforcement constraints nonbinding, reverting to a complete market. The planner's ability of completing markets is against the notion of constrained efficiency.

[4]Here, we included a sequence of capital allocations in the budget constraint to explicitly show the effect of capital tax on households' decision. Simple manipulations with no arbitrage condition allow us to rewrite the budget constraint without capital terms on the left-hand side and the initial capital term, $[1+r_0(1-\tau_{k,0})]k_0$, on the right-hand side, as in a budget constraint of the standard Arrow-Debreu equilibrium.

4. Market Clear

$i \ \sum_{\theta^t} \pi(\theta^t) c_t^*(\theta^t) + K_{t+1} + G_t = F(K_t, L_t) + (1 - \delta) K_t$

$ii \ K_t = \sum_{\theta^{t-1}} \pi(\theta^{t-1}) k_t^*(\theta^{t-1})$

$iii \ L_t = \sum_{\theta^t} \pi(\theta^t) \theta_t l_t^*(\theta^t)$

3   The Ramsey problem

The original Ramsey government problem is to pick the set of tax rates that generate the K-L equilibrium allocation that achieves maximum social welfare. We adopt an equivalent way of formulating the Ramsey problem, a primal approach, which is to make the government choose an allocation directly (rather than a set of tax rates), subject to a series of constraints guaranteeing that the allocation can be decentralized as a K-L equilibrium. To construct these constraints, we start by characterizing a K-L equilibrium with aggregate allocation and Pareto-Negish weights. We closely follow the treatment of Werning (2007) for this characterization.

We can characterize a K-L equilibrium with only aggregate allocations and stochastic Pareto-Negish weights instead of individual allocations. In a K-L equilibrium, given an aggregate allocation, individual allocations will be assigned based on histories of binding enforcement constraints because enforcement constraints are the only source of imperfect risk sharing. Since we can find stochastic Pareto-Negish weights that summarize all the information on binding enforcement constraints, aggregate allocations and Pareto-Negish weights will be sufficient to characterize a K-L equilibrium allocation. Below, we will show how we can construct such Pareto-Negish weights.

This characterization involving only aggregate allocation is useful because what matters for the Ramsey government to achieve its goal is only aggregate allocations. The goal of the Ramsey government is to maximize welfare subject to an allocation being a K-L equilibrium. To achieve this goal, government wants to 1. minimize distortions when financing the government expenditures, and 2. internalize the externalities of capital and labor. For both objectives, only aggregate allocations matter. For the first objective, the distortions due to taxes are confined to aggregate allocations because given an aggregate allocation, individual allocations will be assigned efficiently. Of course, the notion of efficiency is constrained efficiency. For the second objective, only aggregate labor and capital allocations matter because the only channel of the externality is autarky wages, which are determined

by aggregate labor and capital. Thus, to analyze the properties of optimal Ramsey taxation, the characterization with aggregate allocations is useful.[5] In the following subsection, we will restate the household problem as an equivalent planner's static problem given an aggregate allocation and Pareto-Negish weights reflecting the efficiency of individual allocations for a fixed aggregate allocation, which will then be useful for characterizing a K-L equilibrium.

3.1 Planner's static problem and the fictitious representative agent

With linear taxes, individual allocations of a K-L equilibrium are constrained efficient for a fixed aggregate allocation. We can see this efficiency from the equal marginal rates of substitution of consumption and labor across agents. That is, distortions due to taxation will only be confined to determination of aggregate allocations. Given an aggregate allocation, we can use a planner's static problem to solve for individual consumption and labor allocations, reflecting efficiency.

For a given aggregate allocation (C_t, L_t) and Pareto-Negish weights $\{M_t(\theta^t)\}$, a planner's static problem is defined as

$$U^m(C_t, L_t; M) = \max_{c_t(\theta^t), l_t(\theta^t)} \sum_{\theta^t} \pi(\theta^t) M_t(\theta^t) \left[u(c_t(\theta^t)) - v(l_t(\theta^t)) \right] \tag{2}$$

$$s.t. \quad \sum_{\theta^t} \pi(\theta^t) c_t(\theta^t) = C_t \tag{3}$$

$$\sum_{\theta^t} \pi(\theta^t) \theta_t l_t(\theta^t) = L_t. \tag{4}$$

We will denote the solution to the above problem by

$$h(\theta^t, C_t, L_t; M) = \left(h^c(\theta^t, C_t, L_t; M), h^l(\theta^t, C_t, L_t; M) \right). \tag{5}$$

The planner distributes consumption and labor across households to maximize social welfare subject to aggregate feasibility. This planner's static problem is exactly the same as that of Werning (2007)[6] except that the planner will face stochastic Pareto-Negish weights while the planner in Werning (2007) faces fixed weights.

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[5]Of course, to calculate the exact level of taxes, we need to know the Pareto-Negish weights, which will be defined and characterized below.

[6]Werning used the term, 'market weights' instead of Pareto-Negish weights. The weights in that paper are not stochastic because asset markets in his economy depend only on an innate heterogeneity in productivity that is invariant over time.

We use Pareto-Negish weights $\{M_t(\theta^t)\}$ to derive a consumption-sharing rule for individuals in a K-L equilibrium. By setting Pareto-Negish weights of one period to the sum of the history of all multipliers on enforcement constraints up to that period, all promises made in the past can be summarized into Pareto-Negish weights. Then, the relative size of Pareto-Negish weights determines the consumption share of each individual, having the following pattern: every time the enforcement constraint of an agent is binding, his consumption weight is increased permanently so that the private insurance contract can keep him from deviating. Then, given aggregate allocations and proper Pareto-Negish weights, we can solve the planner's static problem to get individual allocations of a K-L equilibrium. In subsection 3.3, we will explicitly show how we can construct Pareto-Negish weights $\{M_t(\theta^t)\}$ using Lagrange multipliers of the enforcement constraints.

Following Werning (2007), we will call the value function of the planner's static problem a utility of a fictitious representative agent. The reason for this name is that we can treat this economy as one where there is only a representative agent who has this utility, $U^m(C_t, L_t; M)$. Notice that the degree of risk sharing affects this utility since it depends on $\{M_t(\theta^t)\}$. This notation will make the characterization of a K-L equilibrium very compact. Using these definitions and notations, we now can characterize a K-L equilibrium.

3.2   Simple characterization of a K-L equilibrium

The next proposition shows how we can characterize the set of aggregate allocations and Pareto-Negish weights that can be supported as a K-L equilibrium. That is, we derive conditions that the aggregate allocations and Pareto-Negish weights have to satisfy so that individual allocations associated with these aggregate allocations and Pareto-Negish weights can be decentralized as a K-L equilibrium. Then, imposing these constraints on the Ramsey government's problem ensures that any aggregate allocations and Pareto-Negish weights chosen by the government can be supported as a K-L equilibrium.

For this proposition, we need an additional assumption on period utility function and allocations.

Assumption 2. *There exists constants $\zeta_1, \zeta_2 < \infty$ such that for all t, θ^t,*

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\begin{aligned}
\left| u(c_t(\theta^t)) \right| &\leq \zeta_1 u'(c_t(\theta^t)) c_t(\theta^t) \\
\left| v(l_t(\theta^t)) \right| &\leq \zeta_2 v'(l_t(\theta^t)) \theta_t l_t(\theta^t)
\end{aligned}
$$

Assumption 2 is needed to ensure the validity of the Lagrangian method to solve the household problem when we construct a K-L equilibrium. In the presence of infinite sequences of enforcement constraints, first-order conditions of the Lagrangian might not be sufficient conditions of a household's optimality even if the objective function is concave and a constraint set is convex, because the infinite sum in the Lagrangian might not converge. Assumption 2 guarantees that the infinite sum in the Lagrangian converges to a finite limit, validating the Lagrangian method.[7] Notice that Assumption 2 is a joint requirement on the allocation and the utility functions. If both consumption and labor are uniformly bounded away from zero, this assumption is satisfied automatically.[8]

Proposition 3. *Given initial conditions, K_0, B_0, and an initial capital tax rate $\tau_{k,0}$, an aggregate allocation $\{C_t, K_t, L_t\}$ can be supported as a K-L equilibrium if and only if there exist Pareto-Negish weights $\{M_t(\theta^t)\}$ so that the following conditions (i) - (v) hold.*

(i) **Resource constraint:** $C_t + K_{t+1} + G_t = F(K_t, L_t) + (1 - \delta)K_t \quad$ *for all* t $\quad$ (6)

(ii) **Implementability constraint:**

$$\sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left[U_c^m(C_t, L_t; M) h^c(\theta^t, C_t, L_t; M) + U_L^m(C_t, L_t; M) \theta_t h^l(\theta^t, C_t, L_t; M) \right]$$

$$= U_c^m(C_0, L_0; M) \left\{ \left[1 + (F_K(K_0, L_0) - \delta)(1 - \tau_{k,0}) \right] (K_0 + B_0) \right\} \quad (7)$$

where $h^c(\cdot)$, $h^l(\cdot)$ are the solution of planner's static problem (2).

(iii) **Enforcement constraint:** *for all* θ^t

$$\sum_{s=t}^{\infty} \sum_{\theta^s} \beta^{s-t} \pi(\theta^s|\theta^t) \left[u(h^c(\theta^s, C_s, L_s; M)) - v(h^l(\theta^s, C_s, L_s; M)) \right]$$

$$\geq \sum_{s=t}^{\infty} \sum_{\theta^s} \beta^{s-t} \pi(\theta^s|\theta^t) U^{aut}(\theta^s, L_s, K_s; \tau_{l,s}) \quad (8)$$

$$where \quad U^{aut}(\theta^s, L_s, K_s; \tau_{l,s}) = \max_{\tilde{c}_s, \tilde{l}_s} \left[u(\tilde{c}_s) - v(\tilde{l}_s) \right]$$

$$s.t. \quad \tilde{c}_s \leq F_L(K_s, L_s)(1 - \tau_{l,s})\theta_s \tilde{l}_s$$

$$for\ labor\ income\ tax\ \{\tau_{l,t}\}\ s.t.\ (1 - \tau_{l,t}) = -\frac{U_L^m(C_t, L_t; M)}{F_L(K_t, L_t) U_c^m(C_t, L_t; M)}$$

(iv) **Monotonicity:** $M_{t+1}(\theta^t, \theta_{t+1}) \geq M_t(\theta^t) \geq 1 \quad$ *for all* θ^t, θ_{t+1}, *and* $M_0(\theta_0) = 1$ (9)

(v) **High MRS:** *If the enforcement constraint at (θ^t, θ_{t+1}) is not binding,*

$$M_t(\theta^t) = M_{t+1}(\theta^t, \theta_{t+1})$$

---

[7]Assumption 2 is a modified version of condition (c) of proposition 4.6. of Alvarez and Jermann (2000). Alvarez and Jermann (2000) made the same assumption to guarantee that each term of summation associated with enforcement constraints converges to a finite limit in the Lagrangian. This assumption is needed for the "if" part of proposition 3. The "only if" part of this proposition still holds without assumption 2.

[8]For a further sufficient condition, see Alvarez and Jermann (2000)

Given an aggregate allocation, individual allocations can then be computed using equation (5).[9,10]

In the next subsection, we will prove the "only if" part of the proposition. In the Appendix, we will prove the "if" part of the proposition.

Proposition 3 says that the set of aggregate allocations and Pareto-Negish weights $\{C_t, L_t, K_t, \{M_t(\theta^t)\}\}$ that satisfy *(i)-(v)* can be implemented as a K-L equilibrium. That is, there will be some prices and taxes that support the individual allocation associated with an aggregate allocation $\{C_t, L_t, K_t\}$ and Pareto-Negish weights $\{M_t(\theta^t)\}$ - which is the solution of planner's static problem defined in the previous subsection - as a K-L equilibrium. Construction of prices and taxes that decentralize allocation is given below. Notice that conditions *(iii),(iv)*, and *(v)* exactly capture the risk-sharing rule of a K-L equilibrium. The relative size of $\{M_t(\theta^t)\}$ across history $\{\theta^t\}$ at period t will determine the consumption share of the agent who has history θ^t. At time 0, the initial Pareto-Negish weight is equal to 1 across all agents because there is no *ex-ante* heterogeneity. If it were a standard Arrow-Debreu model without enforcement constraints, then this weight would be fixed. In the presence of enforcement constraints, however, this weight increases over time whenever the enforcement constraint is binding. Then, the consumption share of agents with non-binding constraints will drift downward because these agents' Pareto-Negish weights stay constant, while those of others increase. On the other hand, the consumption share of agents whose constraint is binding will jump up to guarantee that these agents do not renege on the contract.

3.3   Proof of the "only if" part of proposition 3

In this subsection, we prove the "only if" part of proposition 3. Deriving conditions in proposition 3 is important in constructing the Ramsey problem because they characterize a

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[9]The enforcement constraint is specified for the specific tax rate in autarky, which is designated in the proposition because of the assumption that the Ramsey government cannot discriminate between tax rates in and outside autarky. The labor income tax rate in autarky is implied by aggregate allocations and Pareto-Negish weights. Thus, the characterization of a K-L equilibrium is given only with aggregate allocations and Pareto-Negish weights.

[10]We are borrowing the name of the last condition, "High MRS", from Alvarez and Jermann (2000). The name comes from the fact that the MRS between today's consumption and tomorrow's consumption of the non-binding agent is the highest. Holding Pareto-Negish weights constant guarantees the highest MRS for the agents with non-binding constraints.

K-L equilibrium with a system of usable equalities and inequalities that can be directly incorporated into the government maximization problem. Imposing these conditions in the Ramsey government problem ensures that any aggregate allocations and Pareto-Negish weights picked up by the government can be implemented as a K-L equilibrium.

The key step of this proof is deriving the implementability constraint. As in the canonical Ramsey problem, an implementability constraint is a household budget constraint whose prices and taxes are replaced by allocations reflecting the connection between tax revenues and the household's marginal rates of substitution. Different from the canonical Ramsey problem, however, individual allocations and aggregate allocations are not the same. Individual allocations in an implementability constraint should be replaced by some function of aggregate allocations and Pareto-Negish weights, so that they are exactly equal to the individual allocations of a K-L equilibrium. We closely follow the treatment of Werning (2007) for this, but different from Werning (2007), agents in a K-L equilibrium are ex-post heterogeneous because of idiosyncratic shocks that are drawn every period. By setting Pareto-Negish weights $\{M_t(\theta^t)\}$ to the original utilitarian planner's weight (which is 1) plus the sum of history of multipliers on the enforcement constraints up to each period t, consumption weights derived from Pareto-Negish weights will be exactly those of a K-L equilibrium. We now give the proof.

First, we need to rewrite the household problem. We write down the Lagrangian of the household problem (1) by attaching Lagrange multipliers λ^{KL}, $\{\beta^t\pi(\theta^t)\mu^{KL}(\theta^t)\}$ to the budget constraint and enforcement constraints, respectively. Then, following Marcet and Marimon (2011), we can construct cumulative Lagrange multipliers for this problem as follows.

$$M_t(\theta^t) \equiv 1 + \sum_{\theta^s \preceq \theta^t} \mu^{KL}(\theta^s), \text{ for all } t, \theta^t$$

Now we can re-express the Lagrangian of the household problem using these cumulative Lagrange multipliers:

$$L = \max_{\{c_t, l_t, k_{t+1}\}} \sum_{t=0}^{\infty} \sum_{\theta^t} \beta^t \pi(\theta^t) \begin{bmatrix} M_t(\theta^t) \left[u(c_t(\theta^t)) - v(l_t(\theta^t)) \right] \\ - \{ M_t(\theta^t) - 1 \} U^{aut}(\theta^t) \end{bmatrix}$$
$$+ \lambda^{KL} \left\{ \sum_t \sum_{\theta^t} Q(\theta^t) \begin{bmatrix} w_t(1 - \tau_{l,t})\theta_t l_t(\theta^t) + (1 + r_t(1 - \tau_{k,t}))k_t(\theta^{t-1}) \\ - c_t(\theta^t) - k_{t+1}(\theta^t) \end{bmatrix} - (1 + r_0^b)B_0 \right\}$$

The first-order conditions of the household problem are

$$\textit{Intratemporal Condition:} \quad w_t(1 - \tau_{l,t}) = \frac{v'(l_t(\theta^t))}{\theta_t u'(c_t(\theta^t))} \tag{10}$$

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Intertemporal Condition (No Arbitrage):

$$1 = [1 + r_{t+1}(1 - \tau_{k,t+1})] \sum_{\theta^{t+1}|\theta^t} \frac{Q(\theta^{t+1})}{Q(\theta^t)} \tag{11}$$

$$= [1 + r_{t+1}(1 - \tau_{k,t+1})] \sum_{\theta^{t+1}|\theta^t} \frac{\beta \pi(\theta^{t+1}|\theta^t) M_{t+1}(\theta^{t+1}) u'(c_{t+1}(\theta^{t+1}))}{M_t(\theta^t) u'(c_t(\theta^t))}$$

Let $\{C_t, L_t, K_{t+1}|\tau_l, \tau_k\}$ be the equilibrium aggregate allocations given tax rates (τ_l, τ_k) and let $\{M_t(\theta^t)|M_t(\theta^t) = 1 + \sum_{\theta^s \preceq \theta^t} \mu^{KL}(\theta^t), \text{ for all } \theta^t\}$ be the equilibrium Pareto-Negish weights. Then, individual allocations of a K-L equilibrium solve the planner's static problem (2) we defined in the previous section. That is, the solution of (2) is equated with the individual allocation of a K-L equilibrium, $\left(h^c(\theta^t, C_t, L_t; M), h^l(\theta^t, C_t, L_t; M) \right) = (c_t(\theta^t), l_t(\theta^t))$.[11]

Note that by the envelope theorem,

$$U_c^m(C_t, L_t; M) = \lambda_t^m = M_t(\theta^t) u'(c_t(\theta^t)) \tag{12}$$

$$U_L^m(C_t, L_t; M) = \mu_t^m = -\frac{1}{\theta_t} M_t(\theta^t) v'(l_t(\theta^t)), \tag{13}$$

where λ_t^m and μ_t^m are Lagrange multipliers on (3) and (4), respectively.

Then, after-tax prices in the equilibrium can be expressed as if the economy were populated by a fictitious representative agent with utility function, $U^m(C, L; M)$.

$$Q(\theta^t) = \beta^t \pi(\theta^t) \frac{M_t(\theta^t) u'(c_t(\theta^t))}{u'(c_0(\theta_0))} = \beta^t \pi(\theta^t) \frac{U_c^m(C_t, L_t; M)}{U_c^m(C_0, L_0; M)} \tag{14}$$

$$w_t(1 - \tau_{l,t}) = \frac{v'(l_t(\theta^t))}{\theta_t u'(c_t(\theta^t))} = -\frac{U_L^m(C_t, L_t; M)}{U_c^m(C_t, L_t; M)} \tag{15}$$

Before-tax prices can be computed from the firm's problem in a K-L equilibrium.

$$r_t = F_K(K_t, L_t) - \delta \tag{16}$$

$$w_t = F_L(K_t, L_t) \tag{17}$$

By substituting out all prices, taxes and individual allocations in the budget constraint of the household with the function of aggregate allocations we derived, we can get an implementability constraint.

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[11]Since only the relative size of the weights $\{M_t(\theta^t)\}$ across $\theta^t \in \Theta^t$ matters for the planner's static problem, any constant times $\{M_t(\theta^t)\}_{\theta^t \in \Theta^t}$ derives the same individual allocations for a fixed aggregate allocation. However, by imposing condition (iv) and (v) of proposition 3, we pin down Pareto-Negish weights $\{M_t(\theta^t)\}$ so that $\{M_t(\theta^t)\}$ are equal to the cumulative multipliers of the enforcement constraints. This characterization of $\{M_t(\theta^t)\}$ makes the construction of a K-L equilibrium the most convenient in the proof of the "if" part of the proposition.

$$\sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left[U_c^m(C_t, L_t; M) h^c(\theta^t, C_t, L_t; M) + U_L^m(C_t, L_t; M) \theta_t h^l(\theta^t, C_t, L_t; M) \right]$$

$$= U_c^m(C_0, L_0; M) \left\{ \left[1 + (F_K(K_0, L_0) - \delta)(1 - \tau_{k,0}) \right] (K_0 + B_0) \right\}$$

The remaining part of the proof of the "only if " part of proposition 3 is obvious. The resource constraint and enforcement constraints come from the definition of a K-L equilibrium as stated. Monotonicity conditions and high MRS conditions are obviously satisfied since $\{M_t(\theta^t)\}$ are the sum of Lagrange multipliers across histories.

In the appendix, we prove the "if" part of proposition 3 - an individual allocation associated with an aggregate allocation and Pareto-Negish weights that satisfy five conditions can be decentralized at a K-L equilibrium.

This characterization of an equilibrium is similar to that of the canonical Ramsey taxation literature except that we are using a fictitious representative agent's utility whose efficiency is constrained by Pareto-Negish weights $\{M_t(\theta^t)\}$ and we need three additional conditions on $\{M_t(\theta^t)\}$ so that $\{M_t(\theta^t)\}$ exactly reflect the limits of risk sharing due to limited enforcement. As we show in the proof of the "if" part of proposition 3 (in the appendix), $\{M_t(\theta^t)\}$ that satisfies these three additional conditions (iii), (iv), and (v) of proposition 3 is equal to the cumulative Lagrange multipliers of the enforcement constraints of the household problem in a K-L equilibrium.

In summary, proposition 3 shows how we can characterize the set of aggregate allocations that can be implemented as a K-L equilibrium with five conditions. The Ramsey government then maximizes social welfare over the set restricted by these conditions. Finally, the equilibrium individual allocation that is associated with the aggregate allocation and Pareto-Negish weights chosen by the government can be derived by solving the planner's static problem (2) in subsection 3.1, and equilibrium prices and taxes can be constructed using the first-order conditions of households and firms.[12]

3.4 Solving the Ramsey problem

We now formulate the Ramsey problem using the characterization of a K-L equilibrium in proposition 3. The Ramsey government chooses aggregate allocations and Pareto-Negish

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[12]As in the canonical Ramsey literature, we assume that if there are multiple K-L equilibria associated with given tax rates (τ_k, τ_l), the government can choose the equilibrium that yields the highest utility.

weights among the implementable set of equilibrium allocations, E^{KL}, defined as

$$E^{KL} = \left\{ \left. \left\{ C_t, K_{t+1}, L_t, \{M_t(\theta^t)\} \right\} \right| \begin{array}{l} \text{satisfy conditions } (i), (ii), (iii), (iv), \text{ and } (v) \\ \text{of proposition 3} \end{array} \right\}$$

Ramsey Problem (RP)

$$\max_{\{C_t, L_t, K_t\{M_t(\theta^t)\}\} \in E^{KL}} \quad \sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left[u(h^c(\theta^t, C_t, L_t; M)) - v(h^l(\theta^t, C_t, L_t; M)) \right]$$

We want to reformulate this Ramsey problem as a simple programming problem, but the last condition - (v) the high MRS condition - is not easy to directly incorporate into such a programming problem because it is a constraint that is imposed only when the enforcement constraint is not binding. We can drop this condition, however, because it turns out that the high MRS condition is not a binding constraint in the maximization problem. The intuition of this result is as follows. To maximize social welfare, the intertemporal marginal rates of substitution should be equalized across agents whenever the enforcement constraint allows it. Otherwise, there will be a Pareto improving allocation assignment without violating any constraint. Formally, we can show this by the following steps. First, we construct a relaxed Ramsey problem (RRP) where condition (v) - high MRS - is dropped. Second, we will show that the optimal solution of the RRP satisfies condition (v). It then follows that the solution of the RRP solves the original Ramsey problem (RP).

Relaxed Ramsey Problem (RRP)

[1] Given $\{\tau_{l,t}\}$,

$$\max_{C_t, L_t, K_{t+1}, M_t(\theta^t)} \quad \sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left[u(h^c(\theta^t, C_t, L_t; M)) - v(h^l(\theta^t, C_t, L_t; M)) \right]$$

$$s.t. \quad C_t + K_{t+1} + G_t \leq F(K_t, L_t) + (1 - \delta)K_t \tag{18}$$

$$\sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left\{ \begin{array}{l} U_c^m(C_t, L_t; M)h^c(\theta^t, C_t, L_t; M) \\ + U_L^m(C_t, L_t; M)\theta_t h^l(\theta^t, C_t, L_t; M) \end{array} \right\}$$

$$= U_c^m(C_0, L_0; M) \left\{ [1 + (F_K(K_0, L_0) - \delta)(1 - \tau_{k,0})](K_0 + B_0) \right\} \tag{19}$$

$$\sum_{s=t}^{\infty} \sum_{\theta^s} \beta^{s-t} \pi(\theta^s | \theta^t) \left\{ u(h^c(\theta^s, C_s, L_s; M)) - v(h^l(\theta^s, C_s, L_s; M)) \right\}$$

$$\geq \sum_{s=t}^{\infty} \sum_{\theta^s} \beta^{s-t} \pi(\theta^s | \theta^t) U^{aut}(\theta^s, L_s, K_s; \tau_{l,s}) \tag{20}$$

$$M_t(\theta^t) \geq M_{t-1}(\theta^{t-1}) \tag{21}$$

[2] $1 - \tau_{l,t} = \dfrac{v'(h^l(\theta^t))}{u'(h^c(\theta^t))F_L(K_t, L_t)\theta_t}$

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The implementability constraint (19) and the monotonicity constraints (21) in the RRP do not contain prices and taxes because we already substituted them out using optimality conditions of the household. On the other hand, the enforcement constraints still depend on the labor income taxes $\{\tau_{l,t}\}$ in autarky. However, the Ramsey problem does not maximize over this labor income tax rate of autarky. The Ramsey government chooses an aggregate allocation and Pareto-Negish weights to maximize the expected lifetime utility given labor income taxes $\{\tau_{l,t}\}$ in autarky, and then, labor income tax rates will be pinned down by condition [2] of the RRP, which comes from the assumption that government cannot discriminate between labor income tax rate in and outside autarky. That is, we substitute out tax rates in autarky after maximizing over allocations, not before maximization. The reason why we did not substitute out tax rates in autarky before maximizing is because of the externality. If the Ramsey government solves the maximization problem after substituting out the tax rates in autarky by condition [2], then it is solving a problem in which households internalize the autarky effects, which contradicts the assumption that households behave competitively in a K-L equilibrium. Households take prices and the value of autarky as given when they decide on allocations. Technically, when we solve for allocations defined in the RRP, we will derive first-order conditions of the maximization problem given $\{\tau_{l,t}\}$, first. Only then, we will substitute out the labor income tax rates in the first-order conditions using the condition [2] of the RRP. Finally, we can solve for allocations and Pareto-Negish weights using the first-order conditions.

The following proposition shows that the solution of the RRP solves the original Ramsey problem (RP).

Proposition 4. *Suppose that an aggregate allocation and Pareto-Negish weights $\left\{C_t^R, L_t^R, K_t^R, \{M_t^R(\theta^t)\}\right\}$ solve the RRP. Then $\left\{C_t^R, L_t^R, K_t^R, \{M_t^R(\theta^t)\}\right\}$ satisfy constraints (v) of proposition 3.*

Proof See the Appendix. ∎

Thus, we can analyze the optimal taxation by solving the RRP. We start analyzing the Ramsey equilibrium allocation by composing a Lagrangian for the RRP. We attach Lagrange multipliers γ_t, λ, $\{\beta^t \pi(\theta^t) \mu(\theta^t)\}$, and $\{\phi(\theta^t)\}$ to the constraints (18), (19), (20), and (21), respectively. By collecting terms on $\left(u(h^c(\theta^t)) - v(h^l(\theta^t))\right)$, we cumulate the Lagrange multipliers of enforcement constraints over history. By defining a cumulative multiplier as follows

$$\xi_t(\theta^t) \equiv 1 + \sum_{\theta^s \preceq \theta^t} \mu_s(\theta^s),$$

we can re-express the Lagrangian of the RRP.

$$
\max_{C_t, L_t, K_{t+1}, M_t(\theta^t)} \quad \sum_t \beta^t W(C_t, L_t; M, \xi, \lambda) - \sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \{\xi_t(\theta^t) - 1\} U^{aut}(\theta^t, L_t, K_t; \tau_{l,t})
$$
$$
+ \sum_t \gamma_t \left[F(K_t, L_t) + (1 - \delta) K_t - C_t - K_{t+1} - G_t \right]
$$
$$
- \lambda U_c^m(C_0, L_0; M) \left\{ \left[1 + (F_K(K_0, L_0) - \delta)(1 - \tau_{k,0}) \right] (K_0 + B_0) \right\}
$$
$$
+ \sum_t \sum_{\theta^t} \phi(\theta^t) [M_t(\theta^t) - M_{t-1}(\theta^{t-1})]
$$

where $M = \{M_t(\theta^t)\}$, $\xi = \{\xi_t(\theta^t)\}$ and the pseudo-utility function $W(C_t, L_t; M, \xi, \lambda)$ is defined by

$$
W(C_t, L_t; M, \xi, \lambda) = \sum_{\theta^t} \pi(\theta^t) \begin{bmatrix} \xi_t(\theta^t) \left[u(h^c(\theta^t, C_t, L_t; M)) - v(h^l(\theta^t, C_t, L_t; M)) \right] \\ + \lambda U_c^m(C_t, L_t; M) h^c(\theta^t, C_t, L_t; M) \\ + \lambda U_L^m(C_t, L_t; M) \theta_t h^l(\theta^t, C_t, L_t; M) \end{bmatrix}.
$$

We will call $\xi_t(\theta^t)$ a Pareto weight of the Ramsey government as we call $M_t(\theta^t)$ a Pareto-Negish weight of a K-L equilibrium. The next lemma shows that these weights are equivalent in the optimal solution.

Lemma 5. *Let $\{M_t^R(\theta^t)\}$ be the Ramsey equilibrium Pareto-Negish weights that solve the RRP and let $\{\xi_t(\theta^t)\}$ be the cumulative multipliers of the enforcement constraints in the RRP, which are defined as $\xi_t(\theta^t) = 1 + \sum_{\theta^s \preceq \theta^t} \mu_s(\theta^s)$, $\forall \theta^t$. Then, for all θ^t, $M_t^R(\theta^t) = \xi_t(\theta^t)$.*

Proof Since both the relaxed Ramsey problem and the household problem of a K-L equilibrium have the same objective function and face exactly the same enforcement constraints, the Lagrange multipliers of enforcement constraints are the same for both problems. Since $\xi_t(\theta^t)$ and $M_t(\theta^t)$ are the cumulative multipliers of each problem, respectively, they are equal to each other. ∎

Throughout the paper, the optimal taxation analysis exploits only the first-order necessary conditions.[13]

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[13]It is well known that the set of allocations that satisfy the implementability constraint is not necessarily convex. Thus, first-order necessary conditions of the Ramsey problem might not be sufficient, as in the canonical Ramsey literature. Moreover, by adding limited commitment, enforcement constraints might make the constraint set "even more non-convex". For the theoretical results we derive in this paper, the necessity of first-order conditions is sufficient. Since the first-order conditions are necessary for the optimality, the properties of taxes we derive using the first-order conditions are satisfied by the optimal tax schedule. However, in order to explicitly compute the optimal tax system using the first-order conditions, we would also require their sufficiency. In our companion paper, Park (2012), we take up this issue by means of numerical examples.

The first-order conditions of the RRP imply that, for $t \geq 1$,

$$\cdot \;\; W_c(C_t, L_t; M, \xi, \lambda) F_L(K_t, L_t) - \sum_{\theta^t} \pi(\theta^t) \{\xi_t(\theta^t) - 1\} \frac{\partial U^{aut}(\theta^t, L_t, K_t; \tau_{l,t})}{\partial L_t} = -W_L(C_t, L_t; M, \xi, \lambda)$$

(22)

$$\cdot \;\; W_c(C_t, L_t; M, \mu, \lambda) + \sum_{\theta^{t+1}} \beta \pi(\theta^{t+1}) \{\xi_{t+1}(\theta^{t+1}) - 1\} \frac{\partial U^{aut}(\theta^{t+1}, L_{t+1}, K_{t+1}; \tau_{l,t+1})}{\partial K_{t+1}}$$
$$= \beta W_c(C_{t+1}, L_{t+1}; M, \xi, \lambda)[F_K(K_{t+1}, L_{t+1}) + 1 - \delta]$$

(23)

The first-order conditions of the RRP have additional terms internalizing the autarky effects of aggregate labor and capital, compared to the canonical Ramsey problem. In this section, we explain the effects of these externalities based on the assumption that the wage in autarky is equal to the wage in equilibrium, an assumption that will be relaxed in the next section. If the wage in autarky is the same as the wage in an equilibrium, it is equal to the marginal productivity of labors. Then, for a production function that has the property $F_{LK} > 0$ (for example, a Cobb-Douglas function), we can easily see that an increase in aggregate labor decreases the wage in autarky and an increase in aggregate capital increases the wage in autarky. The intratemporal condition (22) has an additional marginal benefit term, $\sum_{\theta^t} \pi(\theta^t) \{\xi_t(\theta^t) - 1\} \frac{\partial U^{aut}(\theta^t)}{\partial L_t}$, which captures the positive externality of labor, in the sense that a one-unit increase in labor lowers the wage in autarky, resulting in a decrease in the autarky value, which relaxes enforcement constraints. The intertemporal condition (23) has an additional marginal cost term $\sum_{\theta^{t+1}} \beta \pi(\theta^{t+1}) \{\xi_{t+1}(\theta^{t+1}) - 1\} \frac{\partial U^{aut}(\theta^{t+1})}{K_{t+1}}$, which captures the negative externality of capital. A one-unit increase in capital raises the wage in autarky, leading to an increase in the autarky value, which tightens enforcement constraints. These autarky wage effects are called externalities because households do not internalize these autarky price effects, as is apparent from the households' intratemporal and intertemporal conditions (10), (11).

Notice that the first-order conditions of the Ramsey problem involve the pseudo-utility function, which is the combination of the period utility function, the implementability constraint and enforcement constraints. To exploit the first-order conditions for the taxation analysis, it is useful to know how marginal pseudo-utility is related to the marginal utility of the fictitious representative agent.

Lemma 6. *The marginal pseudo utility with respect to C_t and L_t can be expressed as:*

$$W_c(C_t, L_t; M, \xi, \lambda) = U_c^m(t) \left[1 + \lambda \left(1 + \frac{U_{cc}^m(t)}{U_c^m(t)} C_t\right)\right]$$

$$W_L(C_t, L_t; M, \xi, \lambda) = U_L^m(t) \left[1 + \lambda \left(1 + \frac{U_{LL}^m(t)}{U_L^m(t)} L_t\right)\right]$$

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Proof See the Appendix. ∎

The marginal pseudo-utilities in Lemma 6 are represented in familiar forms as in the canonical Ramsey literature. The only difference is that here we are using the utility of the fictitious representative agent and general equilibrium elasticities, $\frac{U_{cc}^m(t)}{U_c^m(t)} C_t$, $\frac{U_{LL}^m(t)}{U_L^m(t)} L_t$, which are also expressed in terms of the utility of a fictitious representative agent.

4 Properties of optimal taxation

As we discussed above, the goal of the Ramsey government is to maximize welfare subject to an allocation being a K-L equilibrium. To achieve this goal, government wants to 1. minimize distortions when financing the government expenditures, and 2. internalize the externalities of capital and labor. In order to see how these two objectives of the government conflict, we start by analyzing the case with lump-sum taxes. This will serve as a benchmark for optimal Ramsey taxation without lump-sum taxes.

4.1 Benchmark: Allowing lump-sum taxation

In this subsection only, we assume that in every period the government can levy lump-sum taxes T_t in addition to linear labor income taxes and capital income taxes. The only reason we analyze the case allowing lump-sum taxes is to determine in which direction the second goal of the Ramsey government is driving the labor income tax and the capital income tax. When lump-sum taxes are allowed, the Ramsey government does not have to use distortionary taxes to finance government expenditures. Thus, the government will have only one goal, namely to internalize externalities.

The properties of optimal taxation when lump-sum taxes are allowed are summarized in the following proposition. Notice that we only allow the government to use lump-sum taxes to finance government expenditures. In other words, lump-sum taxes cannot be used to relax the enforcement constraints. Technically, this restriction can be satisfied by assuming that the government does not levy lump-sum taxes in autarky. This assumption might seem unnatural. Our purpose in analyzing this benchmark, however, is simply to determine in which direction the second goal of the government drives the tax rates. Thus, we assume the absence of lump-sum taxes in autarky just to secure this end and shed light on the results in the next subsection. Still, the government cannot discriminate between labor income taxes in and outside autarky.

Proposition 7. *With lump-sum taxes, the optimal tax system satisfies for all t,*

$$\tau_{l,t} < 0$$

$$\tau_{k,t+t} = \frac{1}{U_c^m(t+1)[F_K(t+1) - \delta]} \sum_{\theta^{t+1}} \pi(\theta^{t+1})\{\xi_{t+1}(\theta^{t+1}) - 1\}\frac{\partial U^{aut}(\theta^{t+1}; \tau_{l,t+1})}{\partial K_{t+1}} > 0$$

Proof See the Appendix. ∎

The optimal taxes derived in proposition 7 are in line with intuition. The only concern of this government is to internalize the externalities because the government can always choose lump-sum taxes $\{T_t\}$ such that the implementability constraint is not binding anymore. Since there is a positive externality of aggregate labor, the government will subsidize labor ($\tau_{l,t} < 0$), and since there is a negative externality of aggregate capital, the government will tax capital ($\tau_{k,t} > 0$).[14]

4.2 Absence of lump-sum taxes

From now on, we rule out lump-sum taxes ($T_t = 0$). The government will have to use distortionary taxes to finance government expenditures. We now need to make an assumption on an initial tax rate, as in the canonical Ramsey taxation literature.

Assumption 8. $\tau_{k,0} \leq \bar{\tau}_k$ *and* $\bar{\tau}_k$ *is sufficiently small that this constraint is binding*

As in the canonical Ramsey literature, this assumption is needed because if there is no restriction on an initial capital taxation, the government will levy taxes on initial capital so high that it can finance the government expenditures without additional distortionary taxation.

We will analyze optimal taxation for three possible cases that differ in the extent to which externalities with respect to labor and capital are present.

· **Case 1 (No externality):** w_t^{aut} does not depend on L_t and K_t $(\frac{\partial w_t^{aut}}{\partial L_t} = 0, \frac{\partial w_t^{aut}}{\partial K_t} = 0)$

· **Case 2 (Externality of labor):** $w_t^{aut} = G(L_t)$,   where $\frac{\partial w_t^{aut}}{\partial L_t} < 0, \frac{\partial w_t^{aut}}{\partial K_t} = 0$

· **Case 3 (Externality of both capital and labor):**

$$w_t^{aut} = F_L(K_t, L_t), \quad \text{where } \frac{\partial w_t^{aut}}{\partial L_t} < 0, \frac{\partial w_t^{aut}}{\partial K_t} > 0$$

---

[14]The way we incorporate the lump-sum taxes is the same as in Werning (2007). Different from Werning, however, we assume that the government is using utilitarian weights by giving weight 1 to every agent *ex-ante*, while Werning permits arbitrary nonutilitarian Pareto weights.

4.2.1 Case 1: No externality of K and L

In this case, the wage in autarky does not depend on equilibrium labor and capital. For example, the production function with functional form $F(K, L) = \alpha L + f(K)$ has this property. In this case, since there is no externality in the economy,[15] the government has only one goal: to minimize distortions when financing government expenditures. We should emphasize that this is exactly the opposite of the benchmark case with lump-sum taxes in terms of the government's goal. Thus, we can compare in which direction the two different goals of the government drive labor income taxes and capital income taxes.

The next proposition describes the steady-state optimal taxes in the absence of any externality. The steady-state is defined to be an allocation in which aggregate variables remain constant and the distribution of individual allocations is time invariant.

Proposition 9 (No externality case). *If $\frac{\partial w_t^{aut}}{\partial L_t} = 0$, $\frac{\partial w_t^{aut}}{\partial K_t} = 0$ for all t, then the optimal tax system in the steady-state satisfies*

$$
\begin{aligned}
\tau_k &= 0 \\
\tau_l &= 1 - \frac{1 + \lambda \left[1 + \frac{U_{cc}^m}{U_c^m} C\right]}{1 + \lambda \left[1 + \frac{U_{LL}^m}{U_L^m} L\right]} \in (0, 1)
\end{aligned}
$$

Proof See the Appendix. ■

This is a version of the Chamley-Judd result. Even with heterogeneous agents and limited commitment, in the absence any externality of K and L, the capital income tax will be zero in the long run. Thus, limited commitment itself does not change the main result of the Ramsey taxation literature. What matters for the structure of the steady-state labor and capital tax is whether there is an externality in a limited commitment economy.

By comparing this result with proposition 7 in the benchmark, we can see that the two objectives of the government push capital and labor income taxes in opposite directions. In order to minimize distortions, in the steady-state, capital income taxes should be set to zero and the government should use the labor income taxes to finance government expenditures

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[15]When the wage in autarky does not depend on equilibrium labor and capital, a K-L equilibrium with a production economy is constrained efficient, as in an endowment economy. The only difference between the production economy and the endowment economy in this case is that when agents choose allocations at period 0, the contract specifies $\{c_t(\theta^t), l_t(\theta^t), k_{t+1}(\theta^{t+1})\}$, while the contract in the endowment economy specifies only $\{c_t(\theta^t)\}$. $\{l_t(\theta^t), k_{t+1}(\theta^{t+1})\}$ in the contract of the production economy will be determined in consideration of both risk sharing and the total output that can be shared.

in the long run. In order to internalize the externailty, however, we showed that the labor tax should be negative and the capital tax should be positive in the benchmark case with lump-sum taxes (proposition 7). Thus, the two goals conflict when the government is setting taxes in the steady-state. It is therefore interesting to analyze optimal taxes when the government has to consider both objectives together, case 2 and case 3.

4.2.2 Case 2: Externality of L only

Now, we assume that the wage in autarky depends only on aggregate labor. An increase in labor decreases the wage in autarky due to the diminishing marginal productivity of labor, and this leads to a decrease in the autarky value, relaxing enforcement constraints. Since households do not consider these autarky wage effects when making labor supply decision, there is a positive externality of labor. In this case, government has two goals: 1. minimizing distortion when financing government expenditures, and 2. internalizing the external benefit of labor.

Proposition 10. *If $\frac{\partial w_t^{aut}}{\partial L_t} < 0$, $\frac{\partial w_t^{aut}}{\partial K_t} = 0$ for all t, then the optimal tax system in the steady-state satisfies*

$$\tau_k = 0$$

$$1 - \tau_l = \frac{1 + \lambda\left(1 + \frac{U_{cc}^m}{U_c^m}C\right)}{1 + \lambda\left(1 + \frac{U_{LL}^m}{U_L^m}L\right) - (1 - \tau_l)\frac{\Delta}{U_L^m}}$$

$$where \quad \frac{\Delta}{U_L^m} = -\sum_{\theta^t} \pi(\theta^t)\frac{\theta_t}{v'(l(\theta^t))}\frac{\partial U^{aut}(\theta^t)}{\partial w(1 - \tau_l)}\frac{\partial w}{\partial L} > 0$$

Proof See the Appendix. ∎

This proposition shows that if there is no externality of capital, the steady-state optimal capital tax is zero regardless of the externality of labor. Also, notice that even though there is an external benefit of labor, all distortions due to the budgetary needs of the government are still allocated to labor income taxes in the long run.

A natural question that arises from this result is: Is the optimal labor income tax lower than that of case 1, reflecting the external benefit of labor? The answer to this question is not obvious. The external benefit of labor will be the driving force of lowering the labor income tax, but the Ramsey government still needs to finance the same amount of government expenditures using only labor income taxes and government assets accumulated along the transition to the steady state. Thus, if the lower labor income tax due to the positive

externality of labor leads to lower government revenue and a lower level of government assets at the onset of the steady-state, the government needs to increase the steady-state labor income tax rate. Formally, holding λ, $\frac{U^m_{cc}}{U^m_c}C$, $\frac{U^m_{LL}}{U^m_L}L$ constant, the optimal labor income tax rate of case 2 will be lower than that of case 1 because $\Delta < 0$ ($\frac{\Delta}{U^m_L} > 0$), reflecting the external benefit of labor. If a decreased labor income tax rate reduces the government's revenue, however, the implementability constraint will be more binding and this will increase the Lagrange multiplier, λ. Due to the adjustment of λ, the labor income tax rate will increase. If a lower tax rate, τ_l and a higher labor supply, L, can achieve the same present value of the government's revenues, then the labor income tax rate of case 2 will be lower than that of case 1, since λ will be constant. Thus, a conclusive comparison of the labor income tax in case 1 and case 2 is not available in the absence of a full (quantitative) characterization of the optimal tax code.

4.2.3   Case 3: externality of both K and L

Finally we assume that the wage in autarky depends on both equilibrium aggregate labor and capital. An increase in labor decreases the wage in autarky, leading to a decrease in autarky value, while an increase in capital increases the wage in autarky, raising the autarky value. Households do not internalize these aurtarky wage effects when they decide on capital and labor. That is, there is a positive externality of labor and a negative externality of capital. Thus, the government has two goals: 1. minimizing distortions when financing government expenditures, and 2. internalizing both externalities of capital and labor. The next proposition characterizes optimal taxes in this case.

Proposition 11. *If $\frac{\partial w^{aut}_t}{\partial L_t} < 0$, $\frac{\partial w^{aut}_t}{\partial K_t} > 0$ for all t, then the optimal tax system in the steady-state satisfies*

$$\tau_k = \frac{1}{W_c \cdot [F_K - \delta]} \sum_{\theta^t} \pi(\theta^t) \left\{ \xi_t(\theta^t) - 1 \right\} \frac{\partial U^{aut}(\theta^t)}{\partial K} > 0$$

$$1 - \tau_l = \frac{1 + \lambda \left(1 + \frac{U^m_{cc}}{U^m_c}C\right)}{1 + \lambda \left(1 + \frac{U^m_{LL}}{U^m_L}L\right) - (1 - \tau_l)\frac{\Delta}{U^m_L}}$$

$$where \quad \frac{\Delta}{U^m_L} = -\sum_{\theta^t} \pi(\theta^t) \frac{\theta_t}{v'(l(\theta^t))} \frac{\partial U^{aut}(\theta^t)}{\partial w(1 - \tau_l)} \frac{\partial w}{\partial L} > 0$$

Proof   See the Appendix.   ∎

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This proposition shows that optimal capital taxes are no longer zero, even in the steady-state. It is exactly the external cost of capital normalized by $W_c \cdot (F_k - \delta)$ in the steady-state that leads to positive capital income taxes.

This positive capital income tax could be interpreted in two ways. One interpretation is that in a limited commitment model, there is a fundamental reason to tax capital income - the externality of capital, which is distinguished from the canonical Ramsey literature.

Another interpretation is that this positive capital income tax result can be considered a generalized version of the Chamley-Judd result. This is because capital income taxes are levied only to remove the negative externality of capital, but not to finance government expenditures.

Notice that by normalizing with W_c, the capital income tax depends on the general equilibrium elasticity $\frac{U_{cc}}{U_c}C$, and distortion cost λ, because $W_c = U_c^m \left[1 + \lambda \left(1 + \frac{U_c c^m}{U_c^m}C\right)\right]$ from Lemma 6. Since these terms are relevant to raising government revenues, one might think positive capital taxes are levied not only to internalize the externality of capital but also to share the cost of distortions due to the revenue burden with labor income taxes. The dependence on an elasticity and a shadow cost, however, arises just because they affect the opportunity cost of capital disinvestment which is the proper normalization for the Ramsey government. From the perspective of the Ramsey government, the opportunity cost of capital disinvestment is not only forgone utility but also forgone government surplus. Thus, the normalization term includes an elasticity and a shadow cost. Still, the only reason for levying capital income taxes in this economy is to internalize the externaltiy.

Even though the purpose of the capital income tax is to internalize the externality, the revenue from the capital income tax will be used for government expenditures because we do not allow government transfers and we assume that any tax revenue is used for government expenditures as is commonly assumed in the canonical Ramsey literature. If government expenditures are big enough so that the revenue from the capital income tax is not enough to meet the budgetary needs of the government, the remaining budgetary needs will be financed by the labor income taxes and accumulated government assets in the long run. Thus, the level of the steady-state labor income tax will be determined by λ, which implies how binding the government budget constraint (or equivalently, implementability constraint) is. Thus, the labor income tax is responsible for the remaining government expenditures, despite the external benefit of labor.

4.3 Outside the steady-state: the case of separable isoelastic utility

In this subsection, we explore the properties of optimal Ramsey taxation in the case of separable isoelastic utility. With this utility function, we can derive optimal taxes even outside the steady state. Separable isoelastic utility has the following functional form.

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad v(l) = \alpha \frac{l^\gamma}{\gamma} \tag{24}$$

A useful property of this utility function is that $U^m(t)$ and $W(t)$ inherit the separable isoelastic property, so that general equilibrium elasticities are constant over time. (In the appendix, we derive the fictitious representative agent's utility and pseudo utility explicitly.) Thus, when the preferences are isoelastic, the steady-state optimal structure of taxes with general utility functions even applies outside the steady-state. The next proposition describes the optimal tax system with this utility function.

Proposition 12. *If the utility function of the household has the form in (24), then for all $t \geq 1$, the optimal tax system has the following properties.*

- *case 1 :* $\quad \tau_{k,t+1} = 0, \quad \tau_{l,t} = 1 - \dfrac{1+(1-\sigma)\lambda}{1+\gamma\lambda}$

- *case 2 :* $\quad \tau_{k,t+1} = 0, \quad \tau_{l,t} = 1 - \dfrac{1+(1-\sigma)\lambda}{1+\gamma\lambda + (1-\tau_{l,t})\frac{\Delta_t}{U_L^m(t)}}$

- *case 3 :* $\quad \begin{aligned} \tau_{k,t+1} &= \frac{1}{U_c^m(t+1)[1+(1-\sigma)\lambda][F_K(t+1)-\delta]} \sum_{\theta^{t+1}} \pi(\theta^{t+1}) \left\{ \xi_{t+1}(\theta^{t+1}) - 1 \right\} \frac{\partial U^{aut}(\theta^{t+1})}{\partial K_{t+1}} \\ \tau_{l,t} &= 1 - \frac{1+(1-\sigma)\lambda}{1+\gamma\lambda + (1-\tau_{l,t})\frac{\Delta_t}{U_L^m(t)}} \end{aligned}$

This proposition can be interpreted exactly the same way as in the canonical Ramsey tax literature. With a general utility function, the consumption goods' income and price elasticities change over time as the allocation changes over time, and the optimal taxes depend on these changes in elasticities to spread distortions equally across time. With isoelastic utility, however, general equilibrium elasticities are constant, $\frac{U_{cc}^m(t)}{U_c^m(t)}C_t = -\sigma$, $\frac{U_{LL}^m(t)}{U_L^m(t)}L_t = \gamma - 1$, for all t. Hence, the implied tax rates on consumption goods at different dates should be the same even outside the steady-state, which leads to a generalized zero capital income tax result even outside the steady-state.

4.4 Comparison to the incomplete markets case

In this subsection, we relate the results in this paper to the results of studies on optimal taxation in the Bewley-Aiyagari class of incomplete market model. Private risk sharing is

not perfect in both the limited commitment model and the incomplete market model. The reason for imperfect risk sharing, however, is different, and this difference leads to diverging optimal tax results between the two classes of models.

Dávila, Hong, Krusell, and Ríos-Rull (forthcoming) study constrained efficient allocation and their implementation in the incomplete market model with uninsurable idiosyncratic shocks and borrowing constraints. The concept of constrained efficiency in their paper is in line with our paper in the sense that it is the efficiency without altering the market structure and without forcing any transfers between consumers.[16] They show that there is a pecuniary externality of capital in incomplete markets. More specifically, there are two effects of capital accumulation that are not considered when agents make their saving decision. First, an increase in capital leads to a higher wage, which increases the amount of risk the agent is exposed to by scaling up the share of labor income that is stochastic. So, there can be negative externality of capital. The second effect of increasing capital is that the lower interest rate helps the poor (households with low wealth) and hurts the rich (households with high wealth), which is welfare improving in the view of the utilitarian planner. This is positive externality of capital. Depending on calibration, one of these two externalities will dominate the other. Thus, the optimal capital income tax can be either positive or negative, as the government tries to internalize the externalities of capital.[17]

In a limited commitment model, however, there is a different type of externality, which leads to a positive capital income tax. This externality in a limited commitment economy works through autarky prices, whereas the pecuniary externality in the incomplete markets works through equilibrium prices. The pecuniary externality through equilibrium prices does not cause a welfare loss in a limited commitment model because of the complete asset market structure, but it matters for welfare in an economy with incomplete markets because

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[16]The Ramsey government in our paper, however, has additional constraint due to the need to finance government expenditures in addition to the imperfect insurance market structure owing to the limited commitment.

[17]Aiyagari (1995) argues that in the presence of idiosyncratic risk, incomplete market with borrowing constraint always gives the rationale for the positive capital income tax, which is different from Dávila, Hong, Krusell, and Ríos-Rull (forthcoming). This result, however, is crucially dependent on the fact that the government optimally chooses the level of government expenditure that enters the household's utility function. Since the Euler equation has to hold for government expenditures, the government tries to make the pre-tax return to capital equal to the time discount rate. In an equilibrium with incomplete market, however, after-tax interest rate is always lesser than the time discount rate because there is capital overaccumulation due to the precautionary motive. This shows that optimal capital income tax in Aiyagari (1995)'s economy is positive even in the long run.

the change in relative prices induces a change in the feasible consumption set. Thus, the externality of capital in a limited commitment model always derives a positive optimal tax, which is different from incomplete market case.

5   Non-linear labor income taxes

In this section, we analyze whether the result we derived in the previous section with linear labor income taxes applies when we permit non-linear progressive labor income taxes. Capital income taxes are still assumed to be linear. A progressive labor income tax might provide more insurance, relaxing the trade-off between efficiency and insurance. Thus, we want to check whether the optimal tax structure with linear taxes still holds in the presence of progressive labor income taxation.

We want to stress that the purpose here is not to analyze optimal tax progressivity. The question we want to answer in this section is: Does the optimal capital and labor income tax structure still apply when labor income taxes are permitted to be non-linear? The answer is that the optimal tax structure between the capital income tax and labor income tax with linear labor income taxes still holds when we permit non-linear labor income taxes, for an adequately restricted classes of tax functions.

We have to admit that we can check the robustness of this result only for some functional forms of progressive taxes because arbitrarily non-linear taxes might not allow the characterization of equilibrium with only aggregate allocations by eliminating the connection between tax revenues and the agent's marginal rates of substitution. We consider two different functional forms of progressive taxation that allow us to use a primal approach.

5.1   Progressive taxation with a specific function 1

In this subsection, we assume that the labor income tax is a proportional linear income tax with a fixed deduction. The functional form of the labor income tax is:

$$T_{l,t}(y_t) = \tau_{l,t} y_t - d, \quad d \geq 0. \ {}^{18}$$

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[18]Under this tax system, $T_{l,t}(y_t) < 0$ is allowed. This is, however, for simplicity. If we assume $T_{l,t}(y_t) = \max\{0, \tau_{l,t} y_t - d\}$ to avoid a negative labor income tax, the main results below still apply, but the notations we need for derivation get quite complicated.

Here the deduction level 'd' represents the progressivity of the labor income tax system because marginal labor income taxes are proportional and deductions are constant. A higher d implies a larger degree of redistribution from agents with high labor income to agents with low labor income.

With this functional form of taxes, we can follow exactly the same steps that we did with the linear tax case to characterize a K-L equilibrium. Since marginal rates of substitution are equated across agents with this specific progressive tax system, we can construct the fictitious representative agent's utility function in exactly the same way as in the linear tax case and express all the prices with only aggregate allocations and Pareto-Negish weights. The only change in the characterization of a K-L equilibrium is the existence of 'd' in the implementability constraint.

$$\sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left[U_c^m(t) h^c(\theta^t) + U_L^m(t) h^l(\theta^t) - U_c^m(t) d \right] = U_c^m(0) \left\{ 1 + (F_K(0) - \delta)(1 - \tau_{k,0})(B_0 + K_0) \right\}$$

Thus, the relaxed Ramsey problem is also constructed in the same way with pseudo-utility that has an additional term, $-U_c^m(t)d$. Then, given some deduction level 'd,' all the first-order conditions with respect to allocations have exactly the same form and the main result with the linear labor income taxes still holds. The steady-state optimal capital taxes are equal to the external cost of capital normalized by $W_c \cdot (F_K - \delta)$, and the optimal labor taxes are responsible for all the remaining government expenditures, despite the external benefit of labor. That is, with this specific functional form and given progressivity, the optimal structure of the labor income tax and the capital income tax will be exactly the same as in the linear income tax case.

5.2 Progressive taxation with a specific function 2

In this subsection, we assume another functional form of progressive labor income taxes. Following Bénabou (2002), we assume the following functional form.

$$T_{l,t}(y_t; \tau_{l,t}, \tau_p) = y_t - \tau_{l,t} y_t^{1-\tau_p} \tag{25}$$

where τ_p captures the progressivity of the labor income tax system. If $\tau_p = 1$, then the government achieves full redistribution. If $0 < \tau_p < 1$, the labor income tax system is progressive. When $\tau_p = 0$, the labor income tax system goes back to the proportional linear tax with a tax rate $1 - \tau_t$. Finally, if $\tau_p < 0$, the labor income tax system is regressive. According to Heathcote, Storesletten, and Violante (2012), this functional form of taxation

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is simple, but it is sufficiently flexible to offer a reasonable approximation to the actual US tax and transfer system.

5.2.1   Characterization of a K-L equilibrium

With the functional form of labor income taxes that we assume, we cannot directly apply the method of characterizing a K-L equilibrium that we used for the linear tax case because marginal rates of substitution will be different across agents in this case. We need to modify the planner's static problem so that its solution can be equated with a K-L equilibrium allocation. Also, we need to assume a functional form of utility from the beginning, since a modification of the planner's static problem will be different for different utility forms. Here, we will assume an isoelastic preference as defined in section 4.3.

Even though marginal rates of substitution are not equated across agents, we can find modified terms of marginal rates of substitution that are equated across agents using first-order conditions:

$$\cdot \quad \frac{v'(l_t(\theta^t))l_t(\theta^t)^{\tau_p}}{\theta_t^{1-\tau_p}u'(c_t(\theta^t))} = \tau_{l,t}(1-\tau_p)w_t^{1-\tau_p}$$

$$\cdot \quad \frac{\beta^t\pi(\theta^t)M_t(\theta^t)u'(c_t(\theta^t))}{u'(c_0(\theta_0))} = Q(\theta^t)$$

Now, we can define a modified planner's static problem for the separable isoelastic utility. The following modified static problem is constructed so that a K-L equilibrium allocation solves the planner's static problem given an aggregate K-L equilibrium allocation (C_t, L_t), Pareto-Negish weights $\{M_t(\theta^t)\}$, and a progressivity of labor income tax, τ_p.

$$U^m(C_t, L_t, \tau_p; M) \quad = \quad \max_{c_t(\theta^t), l_t(\theta^t)} \sum_{\theta^t} \pi(\theta^t)M_t(\theta^t)\left[u(c_t(\theta^t)) - \frac{\gamma}{\tau_p+\gamma}\theta_t^{\tau_p}l_t(\theta^t)^{\tau_p}v(l_t(\theta^t))\right]$$

$$s.t. \quad (\lambda_t^m) \quad \sum_{\theta^t}\pi(\theta^t)c_t(\theta^t) = C_t$$

$$(\mu_t^m) \quad \sum_{\theta^t}\pi(\theta^t)\theta_t l_t(\theta^t) = L_t$$

Using the envelope condition, we can express prices only with the aggregate allocations, Pareto-Negish weights, and τ_p.

$$\cdot \quad \tau_{l,t}(1-\tau_p)w_t^{1-\tau_p} = \frac{M_t(\theta^t)v'(l_t(\theta^t))l_t(\theta^t)^{\tau_p}}{M_t(\theta^t)u'(c_t(\theta^t))\theta_t^{1-\tau_p}} = -\frac{U_L^m(C_t, L_t, \tau_p; M)}{U_c^m(C_t, L_t, \tau_p; M)}$$

$$\cdot \quad Q(\theta^t) = \frac{\beta^t\pi(\theta^t)M_t(\theta^t)u'(c_t(\theta^t))}{u'(c_0(\theta_0))} = \frac{\beta^t\pi(\theta^t)U_c^m(C_t, L_t, \tau_p; M)}{U_c^m(C_0, L_0, \tau_p; M)}$$

By substituting out these prices in the budget constraint of the household, we obtain an implementability constraint[19] of the form:

$$\sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left[\begin{array}{l} U_c^m(C_t, L_t, \tau_p; M) h^c(\theta^t, C_t, L_t, \tau_p; M) \\ + \frac{1}{1-\tau_p} U_L^m(C_t, L_t, \tau_p; M) \theta_t^{1-\tau_p} \{h^l(\theta^t, C_t, L_t, \tau_p; M)\}^{1-\tau_p} \end{array} \right]$$
$$= U_c^m(C_0, L_0, \tau_l; M) \left\{ \left[1 + (F_K(K_0, L_0) - \delta)(1 - \tau_{k,0}) \right] (K_0 + B_0) \right\}$$

Among five conditions that characterize a K-L equilibrium, only the implementability condition is changed as we change the tax system from linear to non-linear. So we can use the same relaxed Ramsey problem with the modified implementability constraint to analyze optimal taxes. See the Appendix C for a complete description of the relaxed Ramsey problem.

For a fixed τ_p, the Ramsey government still has two conflicting goals: 1. minimizing distortions when financing government expenditures, and 2. internalizing the externality to improve risk sharing. The next proposition discusses the optimal structure of the labor income tax and the capital income tax.

Proposition 13. *Assume an isoelastic utility of the form (24). If the tax system takes the functional form specified in (25), and $\frac{\partial w_t^{aut}}{\partial L_t} < 0$, $\frac{\partial w_t^{aut}}{\partial K_t} > 0$ for all t, then an optimal tax system for $t \geq 1$ satisfies*

$$\tau_{k,t+1} = \frac{1}{W_{c,t+1}(F_{K,t+1} - \delta)} \sum_{\theta^{t+1}} \pi(\theta^{t+1}) \left\{ \xi_{t+1}(\theta^{t+1}) - 1 \right\} \frac{\partial U^{aut}(\theta^{t+1}; \tau_{l,t+1}, \tau_p)}{\partial K_{t+1}}$$

$$\tau_{l,t} = \left[-\frac{U_L^m(t)}{U_c^m(t)} \right] \left[-\frac{W_c(t)}{W_L(t) - \Delta_t(\tau_{l,t})} \right]^{1-\tau_p}$$

$$where \quad \Delta_t(\tau_{l,t}) = \sum_{\theta^t} \pi(\theta^t) \left\{ \sum_{\theta^s \preceq \theta^t} \mu(\theta^s) \right\} \frac{\partial U^{aut}(\theta^t; \tau_{l,t})}{\partial L_t}$$

With this specific functional form of a non-linear tax system and isoelastic preference, proposition 13 shows that capital taxes are levied only to remove the externality of capital, and labor income taxes are responsible for all distortions due to remaining budgetary needs. In particular, if we let $\tau_p = 0$, we can see that the result reduces to the previous finding with linear labor income taxes.

In summary, we here demonstrated that the optimal taxation results - the optimal structure between optimal and labor income tax with linear taxation - are robust to some class of progressive taxation functions.

6 Conclusion

In this paper, we have studied optimal Ramsey taxation in a limited commitment model. The goal of the Ramsey government in this economy is to maximize welfare subject to an allocation being a K-L equilibrium. To achieve this goal, the government faces two conflicting objectives: 1. minimizing distortions when financing government expenditures, and 2. internalizing the externality of labor and capital on the value of autarky and thus the degree of risk-sharing. Balancing these two objectives, the steady-state optimal capital taxes are levied only to remove the external cost of capital, even though its level could depend on the elasticities and the shadow cost of distortion because of normalization. All the remaining budgetary needs of the government will be financed using labor income taxes in the long run, even though labor has a positive externality. This main result can be extended to non-linear labor income taxes if we take specific functional forms of labor income taxes.

Therefore, our analysis shows that there is a good reason to tax capital income in a limited commitment economy - the externality of capital. Our result, however, can also be interpreted as a version of the famous zero capital tax result of Chamley-Judd. If there is no capital externality in a limited commitment economy, then the result reverts to Chamley-Judd's result. If there is an external cost of capital, then the result is a generalization of the Chamley-Judd's result because capital taxes are levied only to remove the externality.

This result, however, implies that when thinking about the real world implications of optimal Ramsey tax analysis, the market structure matters crucially for the optimal level of capital income taxes and labor income taxes. Thus, a policy prescription should be based on a thorough investigation of the relevant financial market structure and its friction associated.

We see this paper as the first attempt to consider the risk-sharing effects endogenous to a tax code in the Ramsey literature. If private insurance markets are not perfect, a tax system implemented by the government affects private risk sharing. Thus, the government has to consider how the tax rate endogenously changes risk sharing in private markets when it optimally sets a tax system to finance government expenditures. This paper provides an analysis of optimal Ramsey taxation focusing on one source of imperfect private risk sharing,

limited commitment. Analyzing optimal Ramsey taxation with other sources of imperfect risk sharing would be important and complementary future work.

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Appendix

A Alternative definition of autarky: No tax in autarky

In this subsection, we analyze optimal taxation under the alternative assumption on au-
tarky - no taxation in autarky. If we interpret living in autarky as working in an informal
sector after default, then the assumption that households in autarky can evade taxes is rea-
sonable. This interpretation, however, might be problematic because we assume the wage
in autarky is equal to the marginal productivity of labor in an equilibrium. Despite this in-
terpretation problem, it is still worth to analyze taxation under this alternative assumption,
because we can construct a social planner's problem in a straightforward way and can com-
pare the social planner's problem with Ramsey problem. When we assume we can tax labor
income in autarky as in the main text, on the other hand, construction of the social planner's
problem (for constrained efficiency) and comparison with Ramsey problem is tricky.[20] Thus,
we used the case of allowing lump-sum taxes as the benchmark in the main text instead of
the social planner's problem. The social planner's problem with the alternative assumption
- no taxing in autarky - is defined as follows.

Social Planner's Problem (Constrained Efficiency) [21]

$$
\max_{\{c_t(\theta^t), l_t(\theta^t), K_{t+1}\}} \quad \sum_{t=1}^{\infty} \sum_{\theta^t} \beta^t \pi(\theta^t) \left[u(c_t(\theta^t)) - v(l_t(\theta^t)) \right]
$$

$$
s.t. \ (\lambda_t^{SP}) \qquad \sum_{\theta^t} \pi(\theta^t) c_t(\theta^t) + G_t \leq F(K_t, \sum_{\theta^t} \pi(\theta^t) \theta_t l_t(\theta^t)) - K_{t+1} + (1-\delta) K_t \quad \forall t
$$

$$
(\beta^t \pi(\theta^t) \mu_t^{SP}(\theta^t)) \quad \sum_{s=t}^{\infty} \sum_{\theta^s | \theta^t} \beta^{s-t} \pi(\theta^s | \theta^t) \left[u(c_s(\theta^s)) - v(l_s(\theta^s)) \right]
$$

$$
\geq \sum_{s=t}^{\infty} \sum_{\theta^s | \theta^t} \beta^{s-t} \pi(\theta^s | \theta^t) U^{aut}(\theta^s, L_s, K_s) \ \forall t, \forall \theta^t
$$

$$
where \qquad U^{aut}(\theta^s, L_s, K_s) = \max_{\tilde{c}_s, \tilde{l}_s} \left[u(\tilde{c}) - v(\tilde{l}) \right]
$$

$$
s.t. \quad \tilde{c}_s \leq F_L(K_s, L_s) \theta_s \tilde{l}_s
$$

---

[20]If we construct a planner's problem as we do in this appendix, we implicitly assume that the planner
does not levy taxes in autarky. However, the Ramsey problem in the main text permits taxing in autarky,
even though it has the restriction - no discrimination between in and outside autarky. So, this gives the
possibility that the Ramsey government might do better than the social planner.

[21]This constrained efficiency is identical to that of Abraham & Carceles-Poveda(2006) and Chien & Lee
(2010)

The first-order conditions of social planner's problem are as follows.

Intratemporal condition

$$v'(l_t(\theta^t)) = u'(c_t(\theta^t))F_L(K_t, L_t)\theta_t - \frac{\sum_{\theta^s \preceq \theta^t} \mu_s^{SP}(\theta^s)}{1 + \sum_{\theta^s \preceq \theta^t} \mu_s^{SP}(\theta^s)} \frac{\partial U^{aut}(\theta^t)}{\partial L_t}\pi(\theta^t)\theta_t$$

Intertemporal condition

$$\left\{1 + \sum_{\theta^s \preceq \theta^t} \mu_s^{SP}(\theta^s)\right\} u'(c_t(\theta^t)) = \left\{1 + \sum_{\theta^s \preceq \theta^{t+1}} \mu_s^{SP}(\theta^s)\right\} \beta[F_K(t+1) + 1 - \delta]u'(c_{t+1}(\theta^{t+1}))$$
$$-\beta\sum_{\theta^{t+1}} \pi(\theta^{t+1})\left\{\sum_{\theta^s \preceq \theta^t} \mu_s^{SP}(\theta^s)\right\} \frac{\partial U^{aut}(\theta^{t+1})}{\partial K_{t+1}}$$

Here, we show one possible way to implement the allocation that solves the social planner's problem as a Kehoe-Levine equilibrium with taxes and subsidies. We can implement this constrained efficiency using the following history independent linear capital income taxes and history dependent linear labor income subsidies.

$$\tau_{k,t+1} = \underbrace{\frac{1}{\{1 + \sum_{\theta^s \preceq \theta^{t+1}} \mu_s^{SP}(\theta^s)\}u'(c_{t+1}(\theta^{t+1}))[F_K(t+1) - \delta]} \sum_{\theta^{t+1}} \pi(\theta^{t+1})\left\{\sum_{\theta^s \preceq \theta^{t+1}} \mu_s^{SP}(\theta^s)\right\} \frac{\partial U^{aut}(\theta^{t+1})}{\partial K_{t+1}}}_{\text{=constant across } \theta^{t+1}} > 0$$

$$\tau_{l,t}(\theta^t) = \frac{1}{u'(c_t(\theta^t))F_L(K_t, L_t)} \frac{\sum_{\theta^s \preceq \theta^t} \mu_s^{SP}(\theta^s)}{\{1 + \sum_{\theta^s \preceq \theta^t} \mu_s^{SP}(\theta^s)\}} \frac{\partial U^{aut}(\theta^T)}{\partial L_t}\pi(\theta^t) < 0$$

Like the Ramsey government who can use lump-sum taxes (benchmark case) in the main text, implementation of the constrained efficiency deals with only one goal - internalizing the externalities. Thus, the signs of taxes on capital and labor income are the same with those of the benchmark in the main text because of exactly the same reasons. Capital income taxes are positive due to the external cost of capital, labor income taxes are negative (subsidy) due to the external benefit of labor. This social planner's problem can be the benchmark as we used the case with lump-sum taxes as the benchmark in the main text.

Now we show the optimal Ramsey taxation in the steady-state for possible three cases. These can be derived by following exactly the same steps we derived the optimal taxes in the main text.

$$\begin{cases} \textbf{case 1 (No externality): } \tau_l = 1 - \frac{1 + \lambda\left[1 + \frac{U_{cc}^m}{U_c^m}C\right]}{1 + \lambda\left[1 + \frac{U_{LL}^m}{U_L^m}L_t\right]} \stackrel{(isoelastic)}{=} 1 - \frac{1 + (1-\sigma)\lambda}{1 + \gamma\lambda}, \quad \tau_k = 0 \\[3em] \textbf{case 2 (Externality of } L\textbf{): } \tau_l = 1 - \frac{1 + \lambda\left[1 + \frac{U_{cc}^m}{U_c^m}C\right]}{1 + \lambda\left[1 + \frac{U_{LL}^m}{U_L^m}L_t\right] - \frac{\Delta}{U_L^m}} \stackrel{(isoelastic)}{=} 1 - \frac{1 + (1-\sigma)\lambda}{1 + \gamma\lambda - \frac{\Delta}{U_L^m}}, \quad \tau_k = 0 \\[3em] \textbf{case 3 (Externality of } K \textbf{ and } L\textbf{):} \\[1em] \quad\quad \tau_l = 1 - \frac{1 + \lambda\left[1 + \frac{U_{cc}^m}{U_c^m}C\right]}{1 + \lambda\left[1 + \frac{U_{LL}^m}{U_L^m}L_t\right] - \frac{\Delta}{U_L^m}} \stackrel{(isoelastic)}{=} 1 - \frac{1 + (1-\sigma)\lambda}{1 + \gamma\lambda - \frac{\Delta}{U_L^m}}, \quad \tau_k = \frac{\chi}{W_C \cdot [F_K - \delta]} \end{cases}$$

where

$$\Delta = \sum_{\theta^t} \pi(\theta^t) \left\{ \xi_t(\theta^t) - 1 \right\} \frac{\partial U^{aut}(\theta^t)}{\partial L}$$
$$\chi = \sum_{\theta^t} \pi(\theta^t) \left\{ \xi_t(\theta^t) - 1 \right\} \frac{\partial U^{aut}(\theta^t)}{\partial K}$$

Here, $U^{aut}(\theta^t)$'s do not depend on taxes, by assumption. Other than that, the result is qualitatively the same with the optimal taxes in the main text. So, all the analysis and comparison among the benchmark and three cases we did in the main text will apply.

B Separable isoelastic utility

In this section, we show the useful properties of the separable isoelastic utility which were exploited in the main text. Assume that $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ and $v(l) = \alpha \frac{l^\gamma}{\gamma}$.

With this preference, individual consumption and labor are proportional to aggregates.

$$
\begin{aligned}
c_t(\theta^t) \quad &= h^c(\theta^t, C_t, L_t; M) = \omega_c(\theta^t) C_t \\
\theta_t l_t(\theta^t) \quad &= \theta_t h^l(\theta^t, C_t, L_t; M) = \omega_L(\theta^t) L_t \\
where \quad &\begin{cases} \omega_c(\theta^t) = \dfrac{M_t(\theta^t)^{\frac{1}{\sigma}}}{\sum_{\theta^t} \pi(\theta^t) M_t(\theta^t)^{\frac{1}{\sigma}}} \\[4mm] \omega_L(\theta^t) = \dfrac{M_t(\theta^t)^{-\frac{1}{\gamma-1}} \theta_t^{\frac{\gamma}{\gamma-1}}}{\sum_{\theta^t} \pi(\theta^t) M_t(\theta^t)^{-\frac{1}{\gamma-1}} \theta_t^{\frac{\gamma}{\gamma-1}}} \end{cases}
\end{aligned}
$$

Moreover, U^m and W inherit the separable and isoelastic form of the utility function.

$$
\begin{aligned}
U^m(C_t, L_t; M) \quad &= \Phi_{u,t}^m u(C_t) - \Phi_{v,t}^m v(L_t) \\
where \quad &\begin{cases} \Phi_{u,t}^m = \left[\sum_{\theta^t} \pi(\theta^t) M_t(\theta^t)^{\frac{1}{\sigma}} \right]^\sigma \\[3mm] \Phi_{v,t}^m = \left[\sum_{\theta^t} \pi(\theta^t) M_t(\theta^t)^{-\frac{1}{\gamma-1}} \theta_t^{\frac{\gamma}{\gamma-1}} \right]^{1-\gamma} \end{cases} \\
W(C_t, L_t; M, \mu, \sigma) \quad &= \Phi_{u,t}^W u(C_t) - \Phi_{v,t}^W v(L_t) \\
where \quad &\begin{cases} \Phi_{u,t}^W = \Phi_{u,t}^m \sum_{\theta^t} \pi(\theta^t) \omega_c(\theta^t) \left[\frac{\xi_t(\theta^t)}{M_t(\theta^t)} + (1-\sigma)\lambda \right] = \Phi_{u,t}^m [1 + (1-\sigma)\lambda] \\[3mm] \Phi_{v,t}^W = \Phi_{v,t}^m \sum_{\theta^t} \pi(\theta^t) \omega_L(\theta^t) \left[\frac{\xi_t(\theta^t)}{M_t(\theta^t)} + \gamma\lambda \right] = \Phi_{v,t}^m [1 + \gamma\lambda] \end{cases}
\end{aligned}
$$

The last equality come form $\xi_t(\theta^t) = M_t(\theta^t), \quad \forall \theta^t.$

C Relaxed Ramsey problem for a specific progressive tax function 2

In this section, we provide a complete description of the RRP with the specific progressive tax function we discussed in the section 5.2 of the main text.

Relaxed Ramsey problem (RRP)

(1) Given labor income tax, $\{\tau_{l,t}\}, \tau_p$,

$$\max_{C_t, L_t, K_{t+1}, M_t(\theta^t)} \quad \sum_t \beta^t W(C_t, L_t; M, \xi, \lambda) - \sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left\{ \xi_t(\theta^t) - 1 \right\} U^{aut}(\theta^t, L_t, K_t; \tau_{l,t}, \tau_p)$$

$$+ \sum_t \gamma_t \left[F(K_t, L_t) + (1-\delta)K_t - C_t - K_{t+1} - G_t \right]$$

$$- \lambda U_c^m(C_0, L_0; M) \left\{ \left[1 + (F_K(K_0, L_0) - \delta)(1 - \tau_{k,0}) \right] (K_0 + B_0) \right\}$$

$$+ \sum_t \sum_{\theta^t} \phi(\theta^t)[M_t(\theta^t) - M_{t-1}(\theta^{t-1})]$$

where

$$W(C_t, L_t, \tau_p; M, \xi, \lambda) = \sum_{\theta^t} \pi(\theta^t) \begin{bmatrix} \xi_t(\theta^t) \begin{bmatrix} u(h^c(\theta^t, C_t, L_t; M, \tau_p)) \\ -v(h^l(\theta^t, C_t, L_t; M, \tau_p)) \end{bmatrix} \\ + \lambda U_c^m(C_t, L_t; M, \tau_p)h^c(\theta^t, C_t, L_t; M, \tau_p) \\ + \frac{\lambda}{1-\tau_p} U_L^m(C_t, L_t; M, \tau_p)\theta_t^{1-\tau_p}\{h^l(\theta^t, C_t, L_t; M, \tau_p)\}^{1-\tau_p} \end{bmatrix}$$

$$U^{aut}(\theta^t, L_t, K_t; \tau_{l,t}, \tau_p) = \max_{\tilde{c}_t, \tilde{l}_t} \left[u(\tilde{c}_t) - v(\tilde{l}_t) \right]$$

$$s.t. \qquad \tilde{c}_t \leq \tau_{l,t} \left[F_L(K_t, L_t)\theta_t \tilde{l}_t \right]^{1-\tau_p}$$

(2) $\tau_{l,t} F_L(K_t, L_t)^{1-\tau_p} = -\frac{U_L^m(t)}{U_c^m(t)}$

Analyzing progressivity is not the purpose of this problem. We assume that there will be some level of progressivity, τ_p, which is determined by the government and it does not have to be the optimal progressivity. For a fixed τ_p, we can discuss the optimal structure of the labor income tax and the capital income tax.

The impact of progressivity on social welfare is mentioned only as an aside. An increase in progressivity reduces the volatility of the labor income process, leading to a lower volatility of autarky income. This changes the agent's incentive to default on private insurance contracts, and can cause either a crowding-in or a crowding-out of private risk sharing as in Krueger and Perri (2011). Analyzing these impacts of progressivity is beyond this paper's scope. We analyze only the taxation that achieves a constrained Pareto efficient equilibrium for a given

progressivity (equivalently, for a given volatility of labor income process), as Krueger and Perri (2011) analyze constrained efficiency for each progressivity (volatility).

D   Proofs of results in main text

D.1   Proof of "if" part of proposition 3

Proof   By construction.

Suppose $\{C_t, L_t, K_t, \{M_t(\theta^t)\}\}$ satisfy $(i), (ii), (iii), (iv)$ and (v). Recall that fictitious representative agent utility was defined by, for all t,

$$U^m(C_t, L_t; M) = \max_{c_t(\theta^t), l_t(\theta^t)} \sum_{\theta^t} \pi(\theta^t) M_t(\theta^t) \left[u(c_t(\theta^t)) - v(l_t(\theta^t)) \right]$$

$$s.t. \quad (\lambda_t^m) \quad \sum_{\theta^t} \pi(\theta^t) c_t(\theta^t) = C_t \tag{26}$$

$$(\mu_t^m) \quad \sum_{\theta^t} \pi(\theta^t) \theta_t l_t(\theta^t) = L_t. \tag{27}$$

We denote the solution of this static problem as $\left(h^c(\theta^t, C_t, L_t; M), h^l(\theta^t, C_t, L_t; M) \right)$. Then, individual allocations of a K-L equilibrium are constructed as follows.

$$c_t(\theta^t) = h^c(\theta^t, C_t, L_t; M), \quad l_t(\theta^t) = h^l(\theta^t, C_t, L_t; M), \quad \text{for all } t$$

With the allocations and Pareto-Negish weights, we define prices and taxes as follows.

$$r_t = F_K(K_t, L_t) - \delta$$

$$w_t = F_L(K_t, L_t)$$

$$Q(\theta^t) = \frac{\beta \pi(\theta^t) U_c^m(C_t, L_t; M)}{U_c^m(C_0, L_0; M)} = \frac{\beta^t \pi(\theta^t) M_t(\theta^t) u'(h^c(\theta^t, C_t, L_t; M))}{M_0(\theta_0) u'(h^c(\theta_0, C_0, L_0; M))}$$

$$(1 - \tau_{k,t+1}) = \frac{\frac{U_c^m(C_t, L_t; M)}{\beta U_c^m(C_{t+1}, L_{t+1}; M)} - 1}{F_K(K_{t+1}, L_{t+1}) - \delta} = \frac{\frac{M_t(\theta^t) u'(h^c(\theta^t, C_t, L_t; M))}{\beta M_{t+1}(\theta^{t+1}) u'(h^c(\theta^{t+1}, C_{t+1}, L_{t+1}; M))} - 1}{F_K(K_{t+1}, L_{t+1} - \delta)}$$

$$(1 - \tau_{l,t}) = -\frac{U_L^m(C_t, L_t; M)}{F_L(K_t, L_t) U_c^m(C_t, L_t; M))} = -\frac{M_t(\theta^t) \frac{1}{\theta_t} v'(h^l(\theta^t, C_t, L_t; M))}{F_L(K_t, L_t) M_t(\theta^t) u'(h^c(\theta^t, C_t, L_t; M))}$$

$$r_t^b = r_t(1 - \tau_{k,t})$$

By the definition of $\{r_t, w_t\}$, firm's optimality is satisfied.

Resource constraints hold by assumption (i). And by constraints of the static problem (26) and (27), goods market and labor market clear.

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After dividing LHS and RHS of the implementability constraint by $U_c^m(C_0, L_0; M)$, we substitute out following terms in the LHS

$$\frac{\beta^t \pi(\theta^t) U_c^m(C_t, L_t; M)}{U_c^m(C_0, L_0; M)} = Q(\theta^t)$$

$$\frac{\beta^t \pi(\theta^t) U_L^m(C_t, L_t; M)}{U_c^m(C_0, L_0; M)} = -\frac{\beta^t \pi(\theta^t) U_c^m(C_t, L_t; M)(1 - \tau_{l,t} w_t)}{U_c^m(C_0, L_0; M)} = -Q(\theta^t)(1 - \tau_{l,t}) w_t$$

and substitute out $F_K(K_0, L_0) - \delta$ in the RHS by r_0. Then, we get a budget constraint (B.C.) of households. Enforcement constraints (E.C.) of the household hold by assumption (iii).

It remains only to verify that $\{c_t(\theta^t), l_t(\theta^t)\}$ is optimal for the household given $(w, r, Q, \tau_k, \tau_l)$. It will suffice to find the nonnegative multipliers associated with the B.C. and E.C. and verify that they are a saddle. First, we construct the nonnegative Lagrange multipliers associated with B.C. and E.C.

The Lagrange multiplier of B.C., λ^{KL} is defined by

$$\lambda^{KL} = M_0(\theta_0) u'(h^c(\theta_0, C_0, L_0; M)) = U_c^m(C_0, L_0; M).$$

The Lagrange multiplier of E.C. at time 0 is set to zero, because enforcement constraints at period 0 are not binding for all agents due to the absence of *ex-ante* uncertainty.

$$\mu^{KL}(\theta_0) = 0$$

For $t > 0$, the multiplier of E.C. at $\theta^t \in \Theta^t$, $\mu^{KL}(\theta^t)$ is defined recursively by

$$\mu^{KL}(\theta^{t-1}, \theta_t) = M_t(\theta^{t-1}, \theta_t) - M_{t-1}(\theta^{t-1}). \tag{28}$$

Then, we get the following equivalence.

$$\left\{ 1 + \sum_{\theta^s \preceq \theta^t} \mu_s^{KL}(\theta^s) \right\} = M_t(\theta^t) \quad \forall t, \forall \theta^t \tag{29}$$

Finally, we need to verify that these multipliers, together with allocations are indeed a saddle. First, by the assumption (iv), $\mu_t^{KL}(\theta^t) \geq 0$ are satisfied for all θ^t and by the assumption (v), $\mu^{KL}(\theta^{t+1}) = 0$ if enforcement constraint for θ^{t+1} is not binding. By construction of the multipliers, it is straightforward that multipliers minimize Lagrangian problem of the household.

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Second, we will show that $\{c_t(\theta^t), l_t(\theta^t)\}$ maximizes $L(\cdot, \lambda^{KL}, \{\mu^{KL}\})$. We first check convergence of each component of sums in the Lagrangian, as a technical requirement. We can check that by following.

$$\left| \sum_t \sum_{\theta^t} \pi(\theta^t) u(c_t(\theta^t)) + \sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \mu^{KL}(\theta^t) \sum_{s=t}^{\infty} \sum_{\theta^s} \beta^{s-t} \pi(\theta^s|\theta^t) u(c_s(\theta^s)) \right|$$

$$= \left| \sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left\{ 1 + \sum_{\theta^s \preceq \theta^t} \mu^{KL}(\theta^t) \right\} u(c(\theta^t)) \right|$$

$$\leq \sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left\{ 1 + \sum_{\theta^s \preceq \theta^t} \mu^{KL}(\theta^t) \right\} \zeta_1 u'(c(\theta^t)) c(\theta^t)$$

$$< \infty,$$

where the second inequality comes from the assumption 2 and the third inequality come from implementability constraint.

Similarly,

$$\left| \sum_t \sum_{\theta^t} \pi(\theta^t) v(l_t(\theta^t)) + \sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \mu^{KL}(\theta^t) \sum_{s=t}^{\infty} \sum_{\theta^s} \beta^{s-t} \pi(\theta^s|\theta^t) v(l_s(\theta^s)) \right|$$

$$= \left| \sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left\{ 1 + \sum_{\theta^s \preceq \theta^t} \mu^{KL}(\theta^t) \right\} v(l(\theta^t)) \right|$$

$$\leq \sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left\{ 1 + \sum_{\theta^s \preceq \theta^t} \mu^{KL}(\theta^t) \right\} \zeta_2 v'(l(\theta^t)) \theta_t l(\theta^t)$$

$$< \infty.$$

By the argument about the convergence of each component of sums, showing optimality of $\{c_t(\theta^t), l_t(\theta^t), k_{t+1}(\theta^t))\}$ given multipliers is equivalent to verify that [22]

$$\sum_t \sum \theta^t \beta^t \pi(\theta^t) \left[u(c_t(\theta^t)) - v(l_t(\theta^t)) \right] + \sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \mu^{KL}(\theta^t) \sum_{s=t}^{\infty} \sum_{\theta^s} \beta^{s-t} \pi(\theta^s|\theta^t) \left[u(c_s(\theta^s)) - v(l_s(\theta^s)) \right]$$

$$+ \lambda^{KL} \sum_t \sum_{\theta^t} Q(\theta^t) \left[w_t(1 - \tau_{l,t}) \theta_t l_t(\theta^t) - c_t(\theta^t) \right]$$

$$\geq \sum_t \sum \theta^t \beta^t \pi(\theta^t) \left[u(\hat{c}_t(\theta^t)) - v(\hat{l}_t(\theta^t)) \right] + \sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \mu^{KL}(\theta^t) \sum_{s=t}^{\infty} \sum_{\theta^s} \beta^{s-t} \pi(\theta^s|\theta^t) \left[u(\hat{c}_s(\theta^s)) - v(\hat{l}_s(\theta^s)) \right]$$

$$+ \lambda^{KL} \sum_t \sum_{\theta^t} Q(\theta^t) \left[w_t(1 - \tau_{l,t}) \theta_t \hat{l}_t(\theta^t) - \hat{c}_t(\theta^t) \right],$$

for all $\left\{ \hat{c}_t(\theta^t), \hat{l}_t(\theta^t), \hat{k}_t(\theta^t) \right\}$

---

[22]Notice that a lifetime budget constraint of a household is written without sequences of capital allocation. This is after simple manipulation of budget constraint of (1) with no arbitrage condition ($\sum_{\theta^{t+1}} Q(\theta^{t+1})[1 + r_{t+1}(1 - \tau_{k,t+1})] = Q(\theta^t)$).

Equivalently, we want to show following.

$$\sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left[\left\{ 1 + \sum_{\theta^s \preceq \theta^t} \mu^{KL}(\theta^s) \right\} \left(u(c_t(\theta^t)) - v(l_t(\theta^t)) \right) \right] + \lambda \sum_t \sum_{\theta^t} Q(\theta^t) \left[w_t(1 - \tau_{l,t}) \theta_t l_t(\theta^t) - c_t(\theta^t) \right]$$

$$\geq \quad \sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left[\left\{ 1 + \sum_{\theta^s \preceq \theta^t} \mu^{KL}(\theta^s) \right\} \left(u(\hat{c}_t(\theta^t)) - v(\hat{l}_t(\theta^t)) \right) \right] + \lambda \sum_t \sum_{\theta^t} Q(\theta^t) \left[w_t(1 - \tau_{l,t}) \theta_t \hat{l}_t(\theta^t) - \hat{c}_t(\theta^t) \right]$$

By concavity of u and convexity of v, we have

$$u(\hat{c}_t(\theta^t)) \quad \leq \quad u(c_t(\theta^t)) + u'(c_t(\theta^t)) \left[\hat{c}_t(\theta^t) - c_t(\theta^T) \right] \tag{30}$$

$$-v(\hat{l}(\theta^t)) \quad \geq \quad -v(l_t(\theta^t)) - v'(l_t(\theta^t)) \left[\hat{l}_t(\theta^t) - l_t(\theta^t) \right] \tag{31}$$

Also, by the definition of $Q(\theta^t)$, λ^{KL} and $(1 - \tau_{l,t})$ we get

$$\beta^t \pi(\theta^t) M_t(\theta^t) u'(h^c(\theta^t, C_t, L_t; M)) \quad = \quad Q(\theta^t) M_0(\theta_0) u'(h^c(\theta_0, C_0, L_0; M))$$

$$= \quad \lambda^{KL} Q(\theta^t)$$

$$\beta^t \pi(\theta^t) M_t(\theta^t) \frac{1}{\theta_t} v'(h^l(\theta^t, C_t, L_t; M)) \quad = \quad w_t(1 - \tau_{l,t}) \beta^t \pi(\theta^t) M_t(\theta^t) u'(h^c(\theta^t, C_t, L_t; M))$$

$$= \quad \lambda^{KL} Q(\theta^t) w_t(1 - \tau_{l,t})$$

And by (28), we get first-order conditions of the household problem.

$$\left\{ 1 + \sum_{\theta^s \preceq \theta^t} \mu_s^{KL}(\theta^s) \right\} \beta^t \pi(\theta^t) u'(h^c(\theta^t, C_t, L_t; M)) \quad = \quad \lambda^{KL} Q(\theta^t) \tag{32}$$

$$\left\{ 1 + \sum_{\theta^s \preceq \theta^t} \mu_s^{KL}(\theta^s) \right\} \beta^t \pi(\theta^t) \frac{1}{\theta_t} v'(h^l(\theta^t, C_t, L_t; M)) \quad = \quad \lambda^{KL} Q(\theta^t) w_t(1 - \tau_{l,t}) \tag{33}$$

Then, using (30), (31), (32), and (33), we obtain the desired inequality.

$$\sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left[\left\{ 1 + \sum_{\theta^s \preceq \theta^t} \mu^{KL}(\theta^s) \right\} \left(u(\hat{c}_t(\theta^t)) - v(\hat{l}_t(\theta^t)) \right) \right] + \lambda \sum_t \sum_{\theta^t} Q(\theta^t) \left[w_t(1 - \tau_{l,t}) \theta_t \hat{l}_t(\theta^t) - \hat{c}_t(\theta^t) \right]$$

$$\leq \quad \sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left[\left\{ 1 + \sum_{\theta^s \preceq \theta^t} \mu^{KL}(\theta^s) \right\} \left(u(c_t(\theta^t)) - v(l_t(\theta^t)) \right) \right]$$

$$+ \sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left[\left\{ 1 + \sum_{\theta^s \preceq \theta^t} \mu^{KL}(\theta^s) \right\} \left\{ u'(c_t(\theta^t)) \left[\hat{c}_t(\theta^t) - c_t(\theta^t) \right] - v'(l_t(\theta^t)) \left[\hat{l}_t(\theta^t) - l_t(\theta^t) \right] \right\} \right]$$

$$+ \lambda \sum_t \sum_{\theta^t} Q(\theta^t) \left[w_t(1 - \tau_{l,t}) \theta_t \hat{l}_t(\theta^t) - \hat{c}_t(\theta^t) \right]$$

$$= \quad \sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left[\left\{ 1 + \sum_{\theta^s \preceq \theta^t} \mu^{KL}(\theta^s) \right\} \left(u(c_t(\theta^t)) - v(l_t(\theta^t)) \right) \right] + \lambda \sum_t \sum_{\theta^t} Q(\theta^t) \left[w_t(1 - \tau_{l,t}) \theta_t l_t(\theta^t) - c_t(\theta^t) \right]$$

$$\blacksquare$$

D.2 Proof of proposition 4

Proof Suppose not, then there exists $(\hat{\theta}^t, \hat{\theta}_{t+1})$ at which the enforcement constraint is not binding and

$$M_{t+1}^R(\hat{\theta}^t, \hat{\theta}_{t+1}) > M_t^R(\hat{\theta}^t). \tag{34}$$

From the first-order conditions of the planner's static problem and the envelope theorem at t and $t+1$, we get

$$\frac{\beta\pi(\hat{\theta}^t, \hat{\theta}_{t+1})M_{t+1}^R(\hat{\theta}^t, \hat{\theta}_{t+1})u'(h^c(\hat{\theta}^t, \hat{\theta}_{t+1}))}{\pi(\hat{\theta}^t)M_t^R(\hat{\theta}^t)u'(h^c(\hat{\theta}^t))} = \frac{\beta\pi(\hat{\theta}^t, \hat{\theta}_{t+1})\lambda_{t+1}^m}{\pi(\hat{\theta}^t)\lambda_t^m} = \frac{\beta\pi(\hat{\theta}^t, \hat{\theta}_{t+1})U_c^m(C_{t+1}^R, L_{t+1}^R; M^R)}{\pi(\hat{\theta}^t)U_c^m(C_t^R, L_t^R; M^R)},$$

$$\frac{\beta\pi(\hat{\theta}^t, \hat{\theta}_{t+1})M_{t+1}^R(\hat{\theta}^t, \hat{\theta}_{t+1})\frac{1}{\hat{\theta}^{t+1}}v'(h^l(\hat{\theta}^t, \hat{\theta}_{t+1}))}{\pi(\hat{\theta}^t)M_t^R(\hat{\theta}^t)\frac{1}{\hat{\theta}^t}v'(h^l(\hat{\theta}^t))} = \frac{\beta\pi(\hat{\theta}^t, \hat{\theta}_{t+1})\mu_{t+1}^m}{\pi(\hat{\theta}^t)\mu_t^m} = \frac{\beta\pi(\hat{\theta}^t, \hat{\theta}_{t+1})U_L^m(C_{t+1}^R, L_{t+1}^R; M^R)}{\pi(\hat{\theta}^t)U_L^m(C_t^R, L_t^R; M^R)}.$$

Let's denote

$$q^c(\hat{\theta}^t, \hat{\theta}_{t+1}) = \frac{\beta\pi(\hat{\theta}^t, \hat{\theta}_{t+1})U_c^m(C_{t+1}^R, L_{t+1}^R; M^R)}{\pi(\hat{\theta}^t)U_c^m(C_t^R, L_t^R; M^R)}$$

$$q^l(\hat{\theta}^t, \hat{\theta}_{t+1}) = \frac{\beta\pi(\hat{\theta}^t, \hat{\theta}_{t+1})U_L^m(C_{t+1}^R, L_{t+1}^R; M^R)}{\pi(\hat{\theta}^t)U_L^m(C_t^R, L_t^R; M^R)}$$

Notice that $q^c(\hat{\theta}^t, \hat{\theta}_{t+1})$ $(q^l(\hat{\theta}^t, \hat{\theta}_{t+1}))$ is the ratio of Ramsey planner's shadow price of consumption (labor) between $\hat{\theta}^t$ and $\hat{\theta}_{t+1}$.

Using (34), we get

$$u'(h^c(\hat{\theta}^t)) > \frac{1}{q^c(\hat{\theta}^t, \hat{\theta}_{t+1})}\beta\pi(\hat{\theta}^{t+1}|\hat{\theta}^t)u'(h^c(\hat{\theta}^t, \hat{\theta}_{t+1})) \tag{35}$$

$$\frac{1}{\hat{\theta}_t}v'(h^l(\hat{\theta}^t)) > \frac{1}{q^l(\hat{\theta}^t, \hat{\theta}_{t+1})}\beta\pi(\hat{\theta}^{t+1}|\hat{\theta}^t)\frac{1}{\hat{\theta}_{t+1}}v'(h^l(\hat{\theta}^t, \hat{\theta}_{t+1})) \tag{36}$$

Now we will construct a (ϵ_1, ϵ_2)-variation $(\epsilon_1, \epsilon_2 > 0)$ of the Pareto-Negish weights, $\{\tilde{M}_t(\theta^t)\}$ and that of the aggregate allocation, $\{\tilde{C}_t, \tilde{L}_t, \tilde{K}_t\}$ as follows.

$$\tilde{M}_s(\theta^s) = \begin{cases} M_s^R(\theta^s) - \epsilon_1 & \text{if } \theta^s = (\hat{\theta}^t, \hat{\theta}_{t+1}) \\ M_s^R(\theta^s) + \epsilon_2 & \text{if } \theta^s = \hat{\theta}^t \\ M_s^R(\theta^s) & o.w. \end{cases}$$

$$\begin{cases} \tilde{C}_t = C_t^R + \pi(\hat{\theta}^t)\left(h^c(\hat{\theta}^t, \tilde{C}_t; \tilde{M}) - h^c(\hat{\theta}^t, C_t^R; M^R)\right) \\ \tilde{C}_{t+1} = C_{t+1}^R + \pi(\hat{\theta}^{t+1})\left(h^c(\hat{\theta}^{t+1}, \tilde{C}_{t+1}; \tilde{M}) - h^c(\hat{\theta}^{t+1}, C_{t+1}^R; M^R)\right) \\ \tilde{C}_s = C_s^R & \text{if } s \notin \{t, t+1\} \end{cases}$$

$$\begin{cases} \tilde{L}_t = L_t^R + \pi(\hat{\theta}^t)\hat{\theta}_t\left(h^l(\hat{\theta}^t, \tilde{L}_t; \tilde{M}) - h^l(\hat{\theta}^t, L_t^R; M^R)\right) \\ \tilde{L}_{t+1} = L_{t+1}^R + \pi(\hat{\theta}^{t+1})\hat{\theta}_{t+1}\left(h^l(\hat{\theta}^{t+1}, \tilde{L}_{t+1}; \tilde{M}) - h^l(\hat{\theta}^{t+1}, L_{t+1}^R; M^R)\right) \\ \tilde{L}_s = L_s^R & \text{if } s \notin \{t, t+1\} \end{cases}$$

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And construct $\{\tilde{K}_{s+1}\}$ so that it satisfies resource constraints for all s. $(\tilde{K}_{s+1} = F(\tilde{K}_s, \tilde{L}_s) + (1-\delta)\tilde{K}_s - \tilde{C}_s)$

Notice that

$$\left(h^c(\theta^s, \tilde{C}_s, \tilde{L}_s; \tilde{M}), h^l(\theta^s, \tilde{C}_s, \tilde{L}_s; \tilde{M}) \right) = \left(h^c(\theta^s, C_s^R, L_s^R; M^R), h^l(\theta^s, C_s^R, L_s^R; M^R) \right) \quad \forall \theta^s \notin \{\hat{\theta}^t, (\hat{\theta}^t, \hat{\theta}_{t+1})\}.$$

That is, we changed the Pareto-Negish weights and aggregate allocation so that they only change individual allocations for history $\hat{\theta}^t$ and $(\hat{\theta}^t, \hat{\theta}_{t+1})$. Also notice that $\{U_c^m(s), U_L^m(s)\}$ are kept constant for all s after variation. $\left(\because U_c^m(s) = \tilde{M}(\tilde{\theta}^s) u'(h^c(\tilde{\theta}^s)), \quad U_L^m(s) = \tilde{M}(\tilde{\theta}^s) \frac{1}{\tilde{\theta}_s} v'(h^l(\tilde{\theta}^s)) \right.$ for all $\tilde{\theta}^s$ and we know that Pareto-Negish weights and individual allocation are not changed for all $\tilde{\theta}^s \notin \{\hat{\theta}^t, \hat{\theta}^{t+1}\}$ $\big)$

Since $(\epsilon_1, \epsilon_2) > 0$, we know that
(using $\tilde{M}_s(\theta^s) u'(h^c(\theta^s)) = \lambda_s^m \quad and \quad \frac{1}{\theta_s} \tilde{M}_s(\theta^s) v'(h^l(\theta^s)) = \mu_s^m \quad \forall \theta^s \in \Theta^s$)

$$h^c(\hat{\theta}^t, \tilde{C}_t, \tilde{L}_t; \tilde{M}) \; > \; h^c(\hat{\theta}^t, C_t^R, L_t^R; M^R) \tag{37}$$

$$h^l(\hat{\theta}^t, \tilde{C}_t, \tilde{L}_t; \tilde{M}) \; < \; h^l(\hat{\theta}^t, C_t^R, L_t^R; M^R) \tag{38}$$

$$h^c(\hat{\theta}^{t+1}, \tilde{C}_{t+1}, \tilde{L}_{t+1}; \tilde{M}) \; < \; h^c(\hat{\theta}^{t+1}, C_{t+1}^R, L_{t+1}^R; M^R) \tag{39}$$

$$h^l(\hat{\theta}^{t+1}, \tilde{C}_{t+1}, \tilde{L}_{t+1}; \tilde{M}) \; > \; h^l(\hat{\theta}^{t+1}, C_{t+1}^R, L_{t+1}^R; M^R) \tag{40}$$

Then, t-period utility of $\hat{\theta}^t$ and $t+1$-period utility of history $\hat{\theta}^{t+1}$ will decrease. Then, by (35),(36), the continuation value at $\hat{\theta}^t$ will increase.

By picking up small $(\epsilon_1, \epsilon_2) > 0$, the enforcement constraints and the monotonicity will be satisfied for all history.

In addition, for given ϵ_1, we can always find ϵ_2 so that implementability constraint holds. The reason we can find such ϵ_2 is following. For given $\epsilon_1 > 0$, we will find $\epsilon_2 > 0$ that satisfy following equation.

$$\beta^t \pi(\hat{\theta}^t) \underbrace{\left[\begin{array}{c} U_c^m(C_t^R, L_t^R; M^R) \left(h^c(\hat{\theta}^t, \tilde{C}_t, \tilde{L}_t; \tilde{M}) - h^c(\hat{\theta}^t, C_t^R, L_t^R; M^R) \right) \\ + U_L^m(C_t^R, L_t^R; M^R) \hat{\theta}_t \left(h^l(\hat{\theta}^t, \tilde{C}_t, \tilde{L}_t; \tilde{M}) - h^l(\hat{\theta}^t, C_t^R, L_t^R; M^R) \right) \end{array} \right]}_{\equiv \Delta_t(\epsilon_2) > 0}$$

$$+ \beta^{t+1} \pi(\hat{\theta}^{t+1}) \underbrace{\left[\begin{array}{c} U_c^m(C_{t+1}^R, L_{t+1}^R; M^R) \left(h^c(\hat{\theta}^{t+1}, \tilde{C}_{t+1}, \tilde{L}_{t+1}; \tilde{M}) - h^c(\hat{\theta}^{t+1}, C_{t+1}^R, L_{t+1}^R; M^R) \right) \\ + U_L^m(C_{t+1}^R, L_{t+1}^R; M^R) \hat{\theta}_t \left(h^l(\hat{\theta}^{t+1}, \tilde{C}_{t+1}, \tilde{L}_t; \tilde{M}) - h^l(\hat{\theta}^{t+1}, C_{t+1}^R, L_{t+1}^R; M^R) \right) \end{array} \right]}_{\equiv \Delta_{t+1}(\epsilon_1) < 0} = 0$$

$$\tag{41}$$

(41) guarantees that implementability constraint is satisfied because our $(\epsilon_1, \epsilon_2)-$variation did not change U_c^m, U_L^m and all other individual allocations for $\theta^s \notin \{\hat{\theta}^t, \hat{\theta}^{t+1}\}$. There always exists $\epsilon_2 > 0$ that makes (41) satisfied for given $\epsilon_1 > 0$, because given $\epsilon_1 > 0$, $\Delta_{t+1}(\epsilon_1) < 0$ and $\Delta_t(\epsilon_2)$ has property: $\Delta_t'(\cdot) > 0$ and $\Delta_t(0) = 0$. Then we can see that $\left\{ \tilde{C}_t, \tilde{L}_t, \tilde{K}_t, \{\tilde{M}_t(\theta^t)\} \right\}$ satisfies all constraints of RRP but increases expected lifetime utility (the objective function of RRP) because of (35),(36), and (37)-(40), which contradicts to optimality of $\left\{ C_t^R, L_t^R, K_t^R, \{M_t^R(\theta^t)\} \right\}$. ∎

D.3 Proof of lemma 6

Proof

$$
\begin{aligned}
W_c(C_t, L_t; M, \xi, \lambda) &= \sum_{\theta^t} \pi(\theta^t) \left[\xi_t(\theta^t) u'(h^c(\theta^t)) \frac{\partial h^c(\theta^t)}{\partial C_t} + \lambda \left\{ U_{cc}^m(t) h^c(\theta^t) + U_c^m(t) \frac{\partial h^c(\theta^t)}{\partial C_t} \right\} \right] \\
&= \sum_{\theta^t} \pi(\theta^t) \left[U_c^m(t) \frac{\partial h^c(\theta^t)}{\partial C_t} + \lambda \left\{ U_{cc}^m(t) h^c(\theta^t) + U_c^m(t) \frac{\partial h^c(\theta^t)}{\partial C_t} \right\} \right] \\
&= U_c^m(t) \left[1 + \lambda \left(1 + \frac{U_{cc}^m(t)}{U_c^m(t)} C_t \right) \right]
\end{aligned}
$$

The second equality is coming from Lemma 5 and (12) and third equality is because the sum of changes of individual allocations is equal to the change of aggregate allocation.

$$
\begin{aligned}
W_L(C_t, L_t; M, \xi, \lambda) &= \sum_{\theta^t} \pi(\theta^t) \left[-\xi_t(\theta^t) v'(h^l(\theta^t)) \frac{\partial h^l(\theta^t)}{\partial L_t} + \lambda \left\{ U_{LL}^m(t) \theta_t h^l(\theta^t) + U_L^m(t) \theta_t \frac{\partial h^l(\theta^t)}{\partial L_t} \right\} \right] \\
&= \sum_{\theta^t} \pi(\theta^t) \left[U_L^m(t) \theta_t \frac{\partial h^l(\theta^t)}{\partial L_t} + \lambda \left\{ U_{LL}^m(t) \theta_t h^l(\theta^t) + U_L^m(t) \theta_t \frac{\partial h^l(\theta^t)}{\partial L_t} \right\} \right] \\
&= U_L^m(t) \left[1 + \lambda \left(1 + \frac{U_{LL}^m(t)}{U_L^m(t)} L_t \right) \right]
\end{aligned}
$$

Again, second equality is coming from Lemma 5 and (13) and third equality is because the sum of changes of individual allocations is equal to the change of aggregate allocation. ∎

D.4 Proof of proposition 7

Proof When allowing lump-sum taxes, implementability constraint is expressed as follows.

$$\sum_t \sum_{\theta^t} \beta^t \pi(\theta^t) \left[U_c^m(C_t, L_t; M) h^c(\theta^t, C_t, L_t; M) + U_L^m(C_t, L_t; M) \theta_t h^l(\theta^t, C_t, L_t; M) \right]$$

$$= U_c^m(C_0, L_0; M) \left\{ \left[1 + (F_K(K_0, L_0) - \delta)(1 - \tau_{k,0}) \right] (K_0 + b_0) - T \right\}$$

$$where \quad T = \sum_t \sum_{\theta^t} Q(\theta^t) T_t$$

Among the five conditions characterizing a K-L equilibrium, only the implementability constraint will be changed by the introduction of lump-sum taxes. In the Ramsey problem, the Ramsey government will also maximize over T.

The first-order condition with respect to T is $U_c^m(0)\lambda = 0$, implying $\lambda = 0$. This is because the government can always choose T such that an implementability constraint is not binding anymore. Then, Lemma 6 implies

$$W_c(C_t, L_t; M, \xi, \lambda) = U_c^m(C_t, L_t; M), \quad W_L(C_t, L_t; M, \xi, \lambda) = U_L^m(C_t, L_t; M).$$

Using these equalities, the first-order conditions of the Ramsey problem (22),(23), and the first-order conditions of a K-L equilibrium, (14), (15), we can derive following labor income taxes and capital income taxes. For all t,

$$1 - \tau_{l,t} = \frac{1}{1 - (1 - \tau_{l,t})\frac{\Delta_t}{U_L^m(t)}} \tag{42}$$

$$where \quad \Delta_t = \sum_{\theta^t} \pi(\theta^t)\{\xi_t(\theta^t) - 1\}\frac{\partial U^{aut}(\theta^t; \tau_{l,t})}{\partial(1 - \tau_{l,t})w_t}\frac{\partial w_t}{\partial L_t} < 0$$

$$\tau_{k,t+1} = \frac{1}{U_c^m(t+1)[F_K(t+1) - \delta]} \sum_{\theta^{t+1}} \pi(\theta^{t+1})\{\xi_{t+1}(\theta^{t+1}) - 1\}\frac{\partial U^{aut}(\theta^{t+1}; \tau_{l,t+1})}{\partial K_{t+1}} > 0$$

We know $\tau_{l,t} < 1$, for all t, since otherwise $L_t = 0$, for all t. We can also prove $\tau_{l,t} < 0$, for all t, by contradiction. Suppose $\tau_{l,t} \geq 0$. Then, the left hand side of (42) is $0 \leq 1 - \tau_{l,t} \leq 1$. Then, because $\Delta_t < 0$, $U_L^m(t) < 0$ and $0 \leq (1 - \tau_{l,t}) \leq 1$, the right hand side of (42) is either greater than 1 or negative, which leads to a contradiction. ∎

D.5 Proof of proposition 9

Proof The first-order conditions of the RRP are:

$$\cdot \quad W_L(C_t, L_t; M, \xi, \lambda) = -W_c(C_t, L_t; M, \xi, \lambda) F_L(K_t, L_t) \tag{43}$$

$$\cdot \quad W_c(C_t, L_t; M, \xi, \lambda) = \beta W_c(C_{t+1}, L_{t+1}; M, \xi, \lambda)\{F_K(K_{t+1}, L_{t+1}) + 1 - \delta\} \tag{44}$$

which are very familiar from the canonical Ramsey problem.

First, we will derive optimal capital income taxes. From (11), (14) of a K-L equilibrium, intertemporal condition of a K-L equilibrium is :

$$U_c^m(C_t, L_t; M) = \beta U_c^m(C_{t+1}, L_{t+1}; M)[1 + r_{t+1}(1 - \tau_{k,t+1})] \qquad (45)$$

Then, τ_k is characterized by making (44) and (45) compatible. By rearranging the equation, we get

$$\frac{\beta W_c(t+1)}{W_c(t)}[1 + F_K(t+1) - \delta] = \frac{\beta U_c^m(t+1)}{U_c^m(t)}[1 + (F_K(t+1) - \delta)(1 - \tau_{k,t+1})]. \qquad (46)$$

(46) shows that if $\frac{W_c(t)}{U_c^m(t)}$ is constant over time, capital taxes should be zero. Thus, we only need to show $\frac{W_c(t)}{U_c^m(t)}$ is constant in the steady-state. By lemma 6, it suffices to show that $\frac{U_{cc}^m(t)}{U_c^m(t)}C_t$ is constant in the steady-state. Since $\{C_t, L_t\}$ are constant and distributions of $\{c_t(\theta^t), l_t(\theta^t)\}$ are invariant in the steady-state, there exist time invariant Pareto-Negish weights $\{\tilde{M}_t(\theta^t)\}$ whose distribution is equivalent to $\{\alpha_t M_t(\theta^t)\}$ for some constant, $\alpha_t > 0$ and whose mean is normalized to 1 ($\sum_{\theta^t} \pi(\theta^t)\tilde{M}_t(\theta^t) = 1$). Then, for every t, $U^m(C_t, L_t; M) = \alpha_t \cdot U^m(C_t, L_t; \tilde{M})$, for some $\alpha_t > 0$, which implies $\frac{U_{cc}^m(t)}{U_c^m(t)}C_t$ is constant in the steady-state. Thus, optimal capital taxes are zero in the steady-state.

Next, we will derive the optimal labor income taxes. From the (15), the intratemporal condition of a K-L equilibrium is :

$$w(1 - \tau_l) = -\frac{U_L^m(C, L; M)}{U_c^m(C, L; M)} \qquad (47)$$

By equating this (47) and (43), we get

$$
\begin{aligned}
\tau_l^* &= 1 - \frac{W_c(C, L; M, \xi, \lambda)}{U_c^m(C, L; M)} \frac{U_L^m(C, L; M)}{W_L(C, L; M, \xi, \lambda)}) \\
&= 1 - \frac{1 + \lambda\left[1 + \frac{U_{cc}^m}{U_c^m}C_t\right]}{1 + \lambda\left[1 + \frac{U_{LL}^m}{U_L^m}L_t\right]}
\end{aligned}
$$

$\tau_l^* \in (0, 1)$ can be shown by the following reasoning. First of all, $\lambda > 0$ since implementability constraint is binding. From (12),(13), we know that the signs of the first-order derivatives of the utility of a fictitious representative agent are $U_c^m(t) > 0$, $U_L^m(t) < 0$ for all t. The signs of second-order derivatives of a utility of fictitious representative agent are

$$U_{cc}^m(t) = M_t(\theta^t)u''(c_t(\theta^t))\frac{\partial c_t(\theta^t)}{\partial C_t} < 0$$

$$U_{LL}^m(t) = -\frac{1}{\theta_t}M_t(\theta^t)v''(l_t(\theta^t))\frac{\partial l_t(\theta^t)}{\partial L_t} < 0$$

Thus, we know $\frac{U_{cc}^m}{U_c^m}C < 0$ and $\frac{U_{LL}^m}{U_L^m}L > 0$. This implies that $1 + \lambda\left[1 + \frac{U_{cc}^m}{U_c^m}C\right] < 1 + \lambda\left[1 + \frac{U_{LL}^m}{U_L^m}L\right]$. The first-order condition of RRP with respect to C_t is $\beta^t W_c(t) = \gamma_t$ where γ_t is the Lagrange multiplier of the resource constraint, implying $\gamma_t > 0$. From lemma 5, $\frac{W_c}{U_c^m} = \left[1 + \lambda\left(1 + \frac{U_{cc}^m}{U_c^m}\right)\right] > 0$. Thus, we get $0 < \tau_l < 1$. $\blacksquare$

D.6 Proof of proposition 10

Proof First-order conditions of the RRP are

$$\cdot \quad W_L(C_t, L_t; M, \xi, \lambda) - \bar{\Delta}_t = -W_c(C_t, L_t; M, \xi, \lambda)F_L(K_t, L_t) \tag{48}$$

$$where \quad \bar{\Delta}_t = \sum_{\theta^t}\pi(\theta^t)\left\{\xi_t(\theta^t) - 1\right\}\frac{\partial U^{aut}(\theta^t; \tau_{l,t})}{\partial L_t}$$

$$\cdot \quad W_c(C_t, L_t; M, \xi, \lambda) = \beta W_c(C_{t+1}, L_{t+1}; M, \xi, \lambda)\{F_K(K_{t+1}, L_{t+1}) + 1 - \delta\} \tag{49}$$

In case of optimal capital taxes, following the same procedure as in case 1, $\tau_{k,t} = 0$ in the steady-state.

Now we derive optimal labor income taxes. By equating the intratemporal condition of a K-L equilibrium (15) and that of the Ramsey government (48), we can derive the following expression for the labor income taxes.

$$1 - \tau_{l,t} = \frac{W_c(C_t, L_t; M, \xi, \lambda)}{U_c^m(C_t, L_t; M)}\frac{U_L^m(C_t, L_t; M))}{W_L(C_t, L_t; M, \xi, \lambda) - (1 - \tau_{l,t})\Delta_t}$$

$$where \quad (1 - \tau_{l,t})\Delta_t = (1 - \tau_{l,t})\underbrace{\sum_{\theta^t}\pi(\theta^t)\left\{\xi_t(\theta^t) - 1\right\}\frac{\partial U^{aut}(\theta^t)}{\partial w^t(1 - \tau_{l,t})}\frac{\partial w^t}{\partial L_t}}_{=\Delta_t} < 0$$

$$= \frac{1 + \lambda\left(1 + \frac{U_{cc}^m(t)}{U_c^m(t)}C_t\right)}{1 + \lambda\left(1 + \frac{U_{LL}^m(t)}{U_L^m(t)}L_t\right) - (1 - \tau_{l,t})\frac{\Delta_t}{U_L^m(t)}}$$

$$where$$

$$\lim_{t\to\infty}\frac{\Delta_t}{U_L^m(t)} = \lim_{t\to\infty}-\sum_{\theta^t}\pi(\theta^t)\frac{\theta_t}{v'(l(\theta^t))}\left\{\frac{\xi_t(\theta^t)}{M_t(\theta^t)} - \frac{1}{M_t(\theta^t)}\right\}\frac{\partial U^{aut}(\theta^t)}{\partial w_t(1 - \tau_{l,t})}\frac{\partial w_t}{\partial L_t}$$

$$= -\sum_{\theta^t}\pi(\theta^t)\frac{\theta_t}{v'(l(\theta^t))}\frac{\partial U^{aut}(\theta^t)}{\partial w(1 - \tau_l)}\frac{\partial w}{\partial L}$$

The last equality holds because $\xi_t(\theta^t) = M_t(\theta^t)$ by lemma 5, and $\lim_{t \to \infty} M_t(\theta^t) = \infty$ ∎

D.7 Proof of proposition 11

Proof First-order conditions of the relaxed Ramsey problem are :

$$\cdot \quad W_L(C_t, L_t; M, \xi, \lambda) - \bar{\Delta}_t = -W_c(C_t, L_t; M, \xi, \lambda) F_L(K_t, L_t) \tag{50}$$

$$where \quad \bar{\Delta}_t = \sum_{\theta^t} \pi(\theta^t) \left\{ \xi_t(\theta^t) - 1 \right\} \frac{\partial U^{aut}(\theta^t; \tau_{l,t})}{\partial L_t}$$

$$\cdot \quad W_c(C_t, L_t; M, \xi, \lambda) = \beta W_c(C_{t+1}, L_{t+1}; M, \xi, \lambda)\{F_K(K_{t+1}, L_{t+1}) + 1 - \delta\} - \beta \bar{\chi}_{t+1} \tag{51}$$

$$where \quad \bar{\chi}_{t+1} = \sum_{\theta^{t+1}} \pi(\theta^{t+1}) \left\{ \xi_{t+1}(\theta^{t+1}) - 1 \right\} \frac{\partial U^{aut}(\theta^{t+1}; \tau_{l,t+1})}{\partial K_{t+1}}$$

First, we derive optimal capital income taxes. By equating the intertemporal condition of a K-L equilibrium (45) and that of the Ramsey government (51), we get

$$\frac{\beta W_c(t+1)}{W_c(t)} [F_K(t+1) + 1 - \delta] - \frac{\beta \bar{\chi}_{t+1}}{W_c(t)}$$

$$= \frac{\beta U_c^m(t+1)}{U_c^m(t)} [F_K(t+1) + 1 - \delta] - \tau_{k,t+1}(F_k(t+1) - \delta) \frac{\beta U_c^m(t+1)}{U_c^m(t)}$$

As we showed in the proof of proposition 9, in the steady-state, $\frac{U_c^m(t+1)}{U_c^m(t)} = \frac{W_c(t+1)}{W_c(t)}$, for all t. Thus, we get

$$\tau_{k,t+1} = \frac{1}{W_c(t+1)[F_K(t+1) - \delta]} \sum_{\theta^{t+1}} \pi(\theta^{t+1}) \left\{ \xi_{t+1}(\theta^{t+1}) - 1 \right\} \frac{\partial U^{aut}(\theta^{t+1})}{\partial K_{t+1}}$$

$\frac{\partial U^{aut}(\theta^{t+1})}{\partial K_{t+1}} = \frac{\partial U^{aut}(\theta^{t+1})}{\partial w_{t+1}} \frac{\partial w_{t+1}}{\partial K_{t+1}} > 0$. Also, $\{\xi_{t+1}(\theta^{t+1}) - 1\} > 0$ in the long run, since $\xi_{t+1}(\theta^{t+1})$ is the cumulative Lagrange multipliers of the enforcement constraints and $\xi_0(\theta_0) = 1$. Also, the normalization term, $W_c(t+1) > 0$ is always positive at the optimum. We can see that from the first-order condition of the Ramsey problem with respect to C_t,

$$\beta^t W_c(t) = \gamma_t,$$

where γ_t is the Lagrange multiplier for resource constraint. Since $\gamma_t > 0$, for all t, we get $W_c(t) > 0$, for all t. By lemma 5, we get $1 + \lambda(1 + \frac{U_{cc}^m(t+1)}{U_c^m(t+1)} C_{t+1}) > 0$. Thus, the optimal capital income taxes are positive in the long run.

Optimal labor income taxes are derived by the same procedure as in the proof of proposition 10. ∎