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"Dynamic Education Signaling with Dropout"

by

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Dynamic Education Signaling with Dropout*

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Abstract

We present a dynamic signaling model where wasteful education takes place over several periods of time. Workers pay an education cost per unit of time and cannot commit to a fixed education length. Workers face an exogenous dropout risk before graduation. Since low-productivity workers' cost is high, pooling with early dropouts helps them to avoid a high education cost. In equilibrium, low-productivity workers choose to endogenously drop out over time, so the productivity of workers in college increases along the education process. We find that (1) wasteful education signals exist even when job offers are privately made and the length of the period is small, (2) the maximum education length is decreasing in the prior about a worker being highly productive, and (3) the joint dynamics of returns to education and the dropout rate are characterized, which is consistent with previous empirical evidence.

Keywords: Dynamic Education Signaling, Dropout

JEL Classification Codes: D83, J31

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1 Introduction

In his seminal paper, Spence (1973) argues that people may rationally engage in unproductive education to reveal their ability. Cho and Kreps (1987) provide a game-theoretic analysis of the Spence model. In their model, a worker, whose productivity is either high or low, makes a one-shot education choice to which he commits. Education is assumed to be more costly for a worker with low productivity. When a worker finishes his education, he goes on the job market and firms simultaneously make him offers. Many perfect Bayesian equilibria exist in their model, but only the least costly separating equilibrium, found by Riley (1979), satisfies the intuitive criterion. In this equilibrium, workers fully separate: the low-type worker chooses zero education and obtains a wage equal to his productivity, and the high-type worker obtains a wage equal to his productivity by choosing the least costly separating education duration such that a low-type worker has no incentive to mimic him.

Even though this canonical signaling model can justify the presence of unproductive education, it does not capture some features that are tied to the dynamic nature of the education process. Indeed, receiving education is a time-consuming activity where decisions are sequentially made over time. Due to the lack of commitment, one of the decisions is to remain in the education process or leave it, that is, to drop out.[1] If a separating equilibrium is supposed to be played, when a student arrives on the first day of school, the separation has already happened, and firms believe the student has high productivity. Hence, the student should drop out immediately. Cho and Kreps (1987) avoid this challenge by assuming that a worker can commit to his decision about education duration. In practice, it is hard to see where the commitment power comes from. This suggests that we should think about students' dropout decision in an environment without commitment.

To capture dropping out behavior and analyze its impact on students' education choice and education returns, we develop a dynamic education signaling model in which students make an education choice in each period, and both an exogenous dropout shock and an endogenous dropout choice are considered. Particularly, in every period, a worker may have to drop out and go on the job market with some exogenous probability. We interpret this exogenous dropout process as random shocks faced by students, and the shocks are driven by exogenous problems such as financial constraints, family reasons and the arrival of utility shocks. Since whether and when the

[1] Empirical evidence suggests that the dropout rate in US colleges is not small. In a survey paper, Bound and Turner (2011) reports that only about half of those who begin first-level degree programs actually obtain their degrees. A study from the Bill & Melinda Gates Foundation (2009) shows that students drop out of college for many reasons. For example, 52% of dropouts mentioned that "I just couldn't afford the tuition and fees,"71% mentioned that "I needed to go to work and make money," and so on.

student will be forced to drop out is not known with certainty by the student at the beginning of his education, one can expect, under these features, that students who drop out do not have any offer when they leave college. This establishes a timing that distinguishes our work from the literature on dynamic signal with preemptive offers: in our model, the informed party, the student, moves first (going to the job market or not) and, consequently, conditional on being in the job market, the uninformed agents, firms, make him offers.[2]

Since a high-productivity worker leaves education with positive probability, a low-productivity worker may have incentives to mimic him by voluntarily quitting school in order to save future education costs. Nevertheless, if in some period a low-productivity worker dropped out with probability one while the high-productivity worker stayed with positive probability, the next period's beliefs about the worker being a high-productivity worker would jump to one. If the corresponding jump in wage was large enough, the low-productivity worker would have incentives not to drop out in the current period, leading to a contradiction. On the other hand, if low-productivity workers did not endogenously drop out at some period, then learning would be slow, which makes education less attractive for them, incentivizing dropout in the current period. We show that, in our benchmark model, in equilibrium, low-productivity workers will mix between dropping out and staying in almost all periods, in order to balance these two forces. Hence, as the first main result of our model, the joint dynamics of education signaling and the dropout rate are characterized. We show that wasteful education signaling appears as an equilibrium phenomenon. Furthermore, to ensure the low-type worker's randomization, the wage increment in each period must equal the marginal cost of education for the low-type worker. As a result, in the two-type model, the return to education is linear. As we will show in extensions, the return to education can be concave in a multiple-type model.

In addition, the model proposed generates an implication on the relation between maximum equilibrium education duration and the prior about a worker being highly productive. As pointed out by Mailath, Okuno-Funjiwara and Postlewaite (1993), one unsatisfactory property of the prediction of the canonical signaling model is that, in the unique equilibrium that passes the intuitive criterion, the education duration is strictly positive (significantly different from that in the symmetric information case, which is zero) and it does not depend on the initial prior of a worker's type. They ask: "But is it reasonable to believe that the outcome in a game with a 1 in 1,000,000 chance of a worker of low ability will differ significantly from that in a game with no chance of such a worker? If not, this discontinuity with respect to the probability distribution over the two types (of workers) is disturbing." In our model, the maximum equilibrium education duration is

[2] This is not the first paper to consider dynamic commitment issues in a signaling model (we discuss Weiss (1983) and Admati and Perry (1987) and related literature in Section 5.1).

decreasing in the prior. In particular, when the prior goes to one, no wasteful education appears in any perfect Bayesian equilibrium. The intuition is as follows. Since the worker exogenously drops out with a positive rate, in any equilibrium, early dropouts are on the path of play, and the posterior about workers' type conditional on the dropout is specified by Bayes' rule. When the prior about a worker being high type is close to one, the lower bound of the posterior about workers' type conditional on dropping out is also close to one; thus, the maximum marginal benefit by waiting one more period is very small, but the cost is non-trivial. Hence, neither the low-type nor the high-type worker has incentives to take one more period of education, and the game ends immediately at no education. Consequently, the equilibrium of the asymmetric information game converge to that of the symmetric information game, in which the worker is a high type with probability one, as the information asymmetry vanishes. This is sharply different from the equilibrium education duration prediction in the canonical signaling model.

To check the robustness of our equilibrium prediction, we extend our two-type model to a finitely-many-type one and show that the properties of the equilibria are similar. In particular, in each period, there is a marginal type who is indifferent between dropping out and receiving more education. Workers whose cost is higher than this marginal type's strictly prefer to drop out, while those whose cost is lower strictly prefer to receive more education. Over time, the cutoff type's cost decreases, and therefore, the return to education is concave, which is consistent with previous empirical findings.

We also relax the assumption of a homogeneous exogenous dropout rate. When the exogenous dropout rate of the high-type worker is greater than that of the low-type worker, the set of equilibria is identical to that in the benchmark model. When the exogenous dropout rate of the high-type worker is smaller than that of the low-type worker, equilibria may be different from those in the benchmark model. We show that the difference vanishes when the difference between the exogenous dropout rates is small.

The rest of this paper is organized as follows. In the next section we present the model with a type-independent dropout rate and characterize the set of equilibria. We consider a multiple-type version of our model in Section 3. In Section 4 we turn to a model with a type-dependent dropout rate. In Section 5, we review the related literature and conclude. All omitted proofs are in the Appendix.

2 Model

Time is discrete, $t = 0, 1, 2, \ldots$. There is one worker who has a type $\theta \in \{H, L\}$, which is his private information with a common prior $p_0 = \Pr(\theta = H) \in (0, 1)$. The productivity of type θ

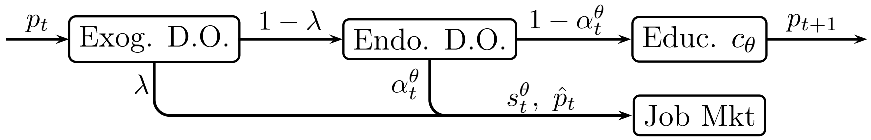

Figure 1: Schematic representation of the timing of the model (D.O. denotes dropout).

worker (henceforth, the θ-worker) is Y_θ. We normalize $Y_H = 1$ and $Y_L = 0$. In period 0 the worker decides whether to go to school or not. In the rest of the periods, if the worker continues going to school, he pays a type-contingent cost per unit of time, c_θ, where $0 < c_H < c_L$ and $c_H < 1$. The worker, regardless of his type, in each period is subject to an exogenous shock that results in the student being forced to drop out of school with probability $\lambda \in (0, 1)$, regardless of his type. The exogenous dropout is interpreted as financial or utility shocks. In addition to this exogenous dropout, the worker may decide to endogenously drop out and go on the job market voluntarily.

The timing is summarized as follows. First, nature determines the type of the worker, choosing H with probability p_0. If the worker is still in school, in period t: (1) the worker exogenously drops out with probability λ and, if he does not exogenously drop out, decides whether to endogenously drop out or not. (2) If the worker decides not to drop out, he pays the education cost and goes to the next period. If the worker drops out, he goes on the job market.[3] (3) Two short-lived firms enter the job market and simultaneously make private job offers to the worker who has dropped out. (4) The worker can choose to take either offer or a zero-value outside option. Figure 1 schematically represents the timing of the model.

The utility of the θ-worker who received t periods of education and accepts a wage of w is $U(w, t) = w - c_\theta t$. The profit of a firm that employs a θ-worker at a wage w is given by $Y_\theta - w$. When a firm hires no worker, its profit is zero.

A dropout (behavior) strategy for the θ-worker is $\alpha^\theta : \{0, 1, ...\} \to [0, 1]$, the probability that type θ worker chooses to drop out at t conditional on reaching its decision point. We use $s_t^\theta \equiv \lambda + (1-\lambda)\alpha_t^\theta$ to denote the total probability of dropping out in period t. Finally, S_t^θ denotes the probability of reaching t, that is

$$S_t^\theta \equiv \prod_{\tau=0}^{t-1} (1 - s_\tau^\theta).$$

For each strategy profile, let $T^\theta \equiv \min\{t | S_{t+1}^\theta = 0\} \in \{0\} \cup \mathbb{N} \cup \infty$, which is the maximum number of education periods the type θ worker may receive under the given strategy profile.

[3]Note, in contrast to Swinkels (1999), firms cannot make offers to a worker in school. This will play an important role in the existence of equilibria with education, which will be discussed below.

Define p_t to be the beliefs about a worker who reached period t, and $\hat{p}_t$ be the beliefs about a worker who dropped out at t. When a worker goes on the job market in period t, two firms Bertrand-compete given their updated belief $\hat{p}_t$. We denote the sequence of wage offers by w. So, they both will offer $w_t = \hat{p}_t$. On the path of play, firms have correct beliefs about the dropout's type, $\hat{p}_t$; thus, they obtain zero expected profit. The worker will take the offer with the higher wage if it is positive. The solution concept we employ is a perfect Bayesian equilibrium:

Definition 1. *A **perfect Bayesian equilibrium (PBE)** is a strategy profile* $\{(\alpha^\theta)_{\theta=L,H}, w\}$ *and two belief sequences p and $\hat{p}$ such that:*

1. *the θ-worker chooses α^θ to maximize his expected payoff given w,*
2. *if a worker drops out with education t, firms offer $w_t = \hat{p}_t$,*
3. *when it is well defined, $\hat{p}_t$ satisfies the Bayes' rule*

$$\hat{p}_t = \frac{p_t s_t^H}{p_t s_t^H + (1 - p_t) s_t^L} \ , \tag{1}$$

and,

4. *when it is well defined, p_t is updated following Bayes' rule*

$$p_{t+1} = \frac{p_t(1 - s_t^H)}{p_t(1 - s_t^H) + (1 - p_t)(1 - s_t^L)} \ . \tag{2}$$

The value function of the θ-worker in period t is

$$V_t^\theta = \lambda \hat{p}_t + (1 - \lambda) W_t^\theta \ ,$$

where $\hat{p}_t$ is his payoff when he exogenously drops out, and $W_t^\theta \equiv \max\{\hat{p}_t, V_{t+1}^\theta - c_\theta\}$ is his continuation value in the complementary event. The worker will decide to endogenously drop out when $\hat{p}_t > V_{t+1}^\theta - c_\theta$, stay in school when $\hat{p}_t < V_{t+1}^\theta - c_\theta$, and potentially randomize when $\hat{p}_t = V_{t+1}^\theta - c_\theta$.

Lemma 1. *In any equilibrium, in all periods $t < T^L$,*

1. *there is positive voluntary dropout by the L-worker, that is $\alpha_t^L > 0$, and*
2. *there is no voluntary dropout by the H-worker, that is $\alpha_t^H = 0$.*

Proof. The proof is in the appendix on page 20. □

Since, by Lemma 1, L-workers are randomizing in any PBE in every period before T^L, for all periods $t < T^L$,

$$\hat{p}_{t+1} - \hat{p}_t = c_L, \tag{3}$$

so the low-type worker is always indifferent between dropping out and staying in school except (possibly) in his last possible period T^L. This fact implies that the wage must linearly increase before T^L. In other words, except (possibly) in the last period, education has constant returns over time. As we will show, this implication depends on the two-type assumption. When there are more than two types, the returns to education are concave.

Lemma 2. *In any equilibrium, $T^H \in \{T^L, T^L + 1\}$.*

Proof. The proof is in the appendix on page 20. □

The intuition behind the previous lemma is as follows. On the one hand, if $T^H > T^L + 1$, firms are convinced that the type of the worker is H (so $p_{T^L+1} = 1$). Since there is (exogenous) dropping out in period $T^L + 1$, then $\hat{p}_{T^L+1} = 1$. Therefore, there is no reason for the H-worker to receive any extra education after $T^L + 1$. On the other hand, Lemma 1 tells us that the H-worker does not endogenously drop out unless L-workers completely drop out, so $T^H \geq T^L$.

Theorem 1. *Set*

$$p_0^1 \equiv \frac{1 - c_H}{1 - (1 - \lambda)c_H} \ . \tag{4}$$

Then, if $p_0 > p_0^1$, the only equilibrium outcome is pooling at no education.

Proof. The proof is in the appendix on page 20. □

The intuition behind Theorem 1 is as follows. Given a p_0, the lowest possible wage is obtained when $s_0^L = 1$ (the L-worker drops out for sure at $t = 0$) and $s_0^H = \lambda$ (the H-worker does not voluntarily drop out). In this case, the wage in period 0 is $\frac{\lambda p_0}{\lambda p_0 + 1 - \lambda}$. Since the maximum wage next period is 1, an upper bound on the gain from not dropping out at period 0 is $1 - \frac{\lambda p_0}{\lambda p_0 + 1 - \lambda}$. If p_0 is close to 1, the maximum gain gets close to 0. Nevertheless, the marginal cost for type $\theta \in \{L, H\}$ is $c_\theta > 0$. Hence, when p_0 is close to 1, receiving education is not attractive for both types, and the game ends immediately with a pooling equilibrium at no education.

In the standard signaling model by Cho and Kreps (1987), wasteful signaling can be supported even when the prior about the type being high (p_0) is very close to 1. The reason is that, off the path of play, a belief threat may be imposed by the firms, so early dropouts are punished with low wages. In our model, since $\lambda > 0$, there is positive dropout in any period before T^H. Hence, there cannot be belief threats off the path of play for dropouts in all periods before T^H.

When $p_0 = 1$, i.e., when there is common knowledge that the worker is high type, the unique equilibrium both in our model and in the standard model exhibits no wasteful education signaling. However, in the standard model, if information is asymmetric between firms and the worker ($p_0 < 1$), the set of equilibria can support non-trivial wasteful signaling, which is very different from the equilibrium education choice in the symmetric information game. What is more, for any $p_0 \in (0, 1)$, by imposing some refinement concept, for example, the intuitive criterion or D1, the equilibrium prediction of the standard signaling model is the Riley outcome, in which a non-trivial wasteful signal is sent. In other words, a discontinuity appears as the information asymmetry, measured by $1 - p_0$, vanishes. Mailath, Okuno-Fujiwara and Postlewaite (1993) provide a belief-based solution concept, undefeated equilibrium, to address this problem. In our model, without imposing any refinement concept, there is no signaling waste when $p_0 \to 1$ in any equilibrium. Consequently, the equilibrium education length converges to that in the symmetric information model as p_0 goes to 1.

The following theorem characterizes possible education lengths in the set of all equilibria:

Theorem 2. *Let* $T^* \equiv \lceil \frac{1-c_H}{c_L} \rceil$.[4] *There exists a partition of* $[0, 1]$ *characterized by* $\{p_0^k\}_{k=0}^{T^*+1}$, *with* $p_0^0 = 1$, $p_0^k > p_0^{k+1}$ *for all* k *and* $p_0^{T^*+1} = 0$, *such that for all* $0 \leq k \leq T^*$ *and* $0 \leq T \leq k$, *if* $p_0 \in (p_0^{k+1}, p_0^k]$ *then*

1. *there exists an equilibrium with* T *periods of education and*
2. *there is no equilibrium with more than* k *periods of education.*

Proof. The proof is in the appendix on page 21. □

Theorem 2 shows that the maximum possible equilibrium education duration is finite. Furthermore, the maximum duration is non-increasing in the prior p_0 and goes to zero as p_0 goes to 1. Finally, given p_0, we have at least one equilibrium for each duration lower than the maximal duration at this p_0.

In Theorem 1 we already discussed the case where p_0 is close to 1. Now, consider the case in which p_0 is not close to 1. As we have shown in Lemma 1, the low-type endogenously drops out with positive probability and the high-type does not voluntarily drop out; thus $s_t^L > s_t^H$, which means that p_t is pushed up over time. The low-type indifference condition (3) implies that $\hat{p}_t$ is linear before T^L. These two observations imply that p_t and $\hat{p}_t$ will be high enough (close to 1) after finitely many periods. The smaller the prior p_0, the more periods of education can be supported in an equilibrium. This suggests that the upper bound of the education duration supported by

[4] $\lceil x \rceil$ denotes the smallest integer no lower than x.

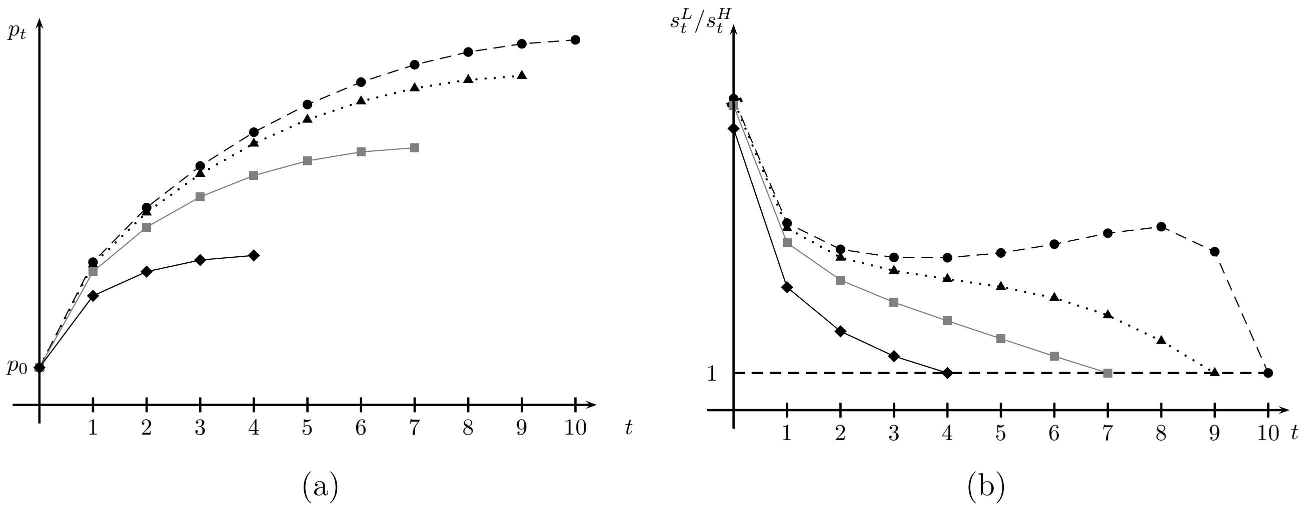

Figure 2: (a) p_t for different equilibria. (b) s_t^L/s_t^H in different equilibria. The dots with the same color correspond to the same equilibrium, and they are linked with a straight line for visual clarity. The parameter values are $c_H = 0.032$, $c_L = 0.097$, $\lambda = 0.1$ and $p_0 = 0.1$.

an equilibrium is non-increasing in p_0. In Figure 2, we plot some equilibrium belief sequences p_t and dropout rate ratio sequences s_t^L/s_t^H. In each equilibrium, the belief p_t goes up over time, the high-type worker's dropout rate $s_t^H = \lambda$ for all $t < T^H$ and $s_t^H = 1$ at $t = T^H$, and the low-type worker's dropout rate may not be monotone.

Let's finally make some comparative statics with respect to the parameter λ. Figure 3 plots $\{p_0^k\}_{k=1}^{T^*}$ for different values of λ. As we see, when $\lambda \to 0$, p_0^k for all k collapses to 1. This implies that, when λ is low, for almost all priors the maximum length of an equilibrium is T^*. This is consistent with the canonical signaling model, where $\lambda = 0$. In the other limit, when $\lambda \to 1$, $p_0^k - p_0^{k+1} = c_L$ for all $k > 1$. This is a consequence of the fact that when λ is close to 1, so are s^L and s^H. Therefore, as we see in (1), $\hat{p}_t$ is close to p_t for all t. Since $\hat{p}_t$ increases linearly in any equilibrium, this imposes a nearly linear evolution for p_t and therefore also to p_0^k .

2.1 Refinement

Without imposing any refinement, multiple equilibria exist for most p_0. The reason we do not have equilibrium uniqueness is the arbitrariness of belief after T^H off the path of play, which is the same as that in Cho and Kreps (1987). Hence, we still have belief threats that push duration down. Nevertheless, due to the positive exogenous dropout rate, we do not have belief threats during the periods where education takes place. Therefore, in our model, education cannot be

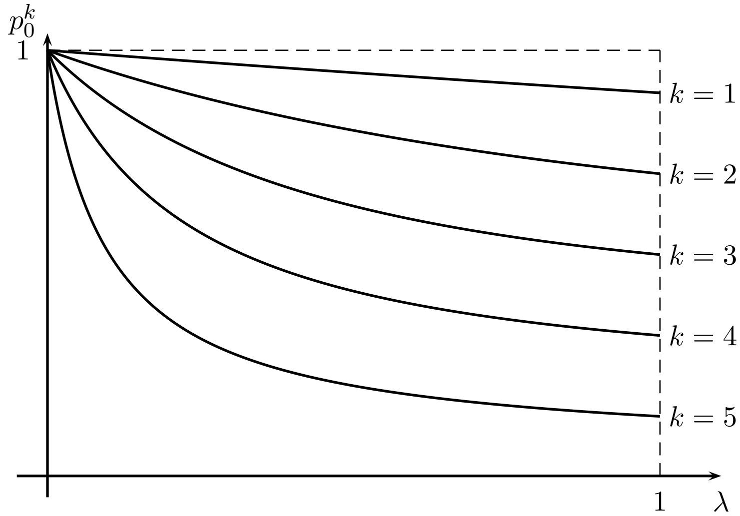

Figure 3: $\{p_0^k\}$ as a function of λ.

longer than the lowest individually irrational education duration of the L-worker (Riley outcome).

By imposing an appropriate criterion on beliefs off the path of play, for example, D1 defined by Banks and Sobel (1987) or NWBR defined by Kohlberg and Mertens (1986), one can shrink the equilibrium set. The spirit of these refinements requires that, off the path of play, firms put a positive probability only on that type is most likely to deviate. In our model, since the marginal cost of education of the high-type worker is strictly smaller than that of the low-type worker, any sequence of wage off the path of play (after T^H) that induces a deviation of the low-type worker deviate must induce a deviation of the high-type worker. As a result, off the path of play, firms put a positive belief only on the high-type worker, i.e., $p_t = \hat{p}_t = 1$ for any $t > T^H$. Given this belief sequence off the path of play, we will say a PBE is eliminated by NWBR if $\hat{p}_{T^H} < 1 - c_H$, since otherwise the high-type worker would have incentives to stay in college one more period. These concepts are not enough to select a unique equilibrium, similarly to Nöldeke and van Damme (1990). The key reason of the multiplicity is that, in our model, the education choice is an integer instead of a real number. Consider the following case as an example.

Example 1. *Suppose $p_0 \in (1 - c_H, p_1^0)$. It is easy to show that there is a PBE in which $s_0^H = \lambda$ and $s_0^L = 1$. Since, in this equilibrium, $p_1 = \hat{p}_1 = 1$ is on the path of play, it survives the elimination of NWBR. However, there is another PBE consisting on pooling at no education, that is, $s_0^H = s_0^L = 1$, so $p_0 = \hat{p}_0 > 1 - c_H$. Hence, pooling at no education also survives the elimination of NWBR.*

Nevertheless, as shown below, when the length of the interval is small, the D1 criterion is essentially unique, in the sense that the outcomes of all equilibria satisfying D1 become arbitrarily close.

2.2 Frequent Dropout Decision

In our model, since $\lambda > 0$, both the high-type and low-type workers drop out before T^H with positive probability, and therefore, there is no fully separating equilibrium. For any positive c_H, c_L, and $\lambda > 0$ such that $c_H < c_L$, when $p_0 \leq p_0^1$ there exist equilibria with wasteful signaling. This is in sharp contrast to the result in Swinkels (1999), who studies a dynamic education signaling model with preemptive offers. In Swinkels (1999), the only equilibrium is pooling at no education when the worker can adjust his education choice very frequently. In other words, wasteful education cannot appear in any equilibrium if the length of the interval is short enough.

In our model, we assume that the time length between two consecutive periods is 1. Nevertheless, our main results do not qualitatively change when this length is $\Delta \in (0, 1)$ instead of 1, the education cost and dropout probability in each period are given by $c_\theta \equiv \tilde{c}_\theta \Delta$ and $\lambda \equiv \tilde{\lambda}\Delta$, respectively.[5] Specifically, $T^*_\Delta \equiv \lceil \frac{1-\Delta\tilde{c}_H}{\Delta\tilde{c}_L} \rceil$ increases as Δ decreases, but the real time ΔT^*_Δ is finite and bounded away from zero, and is asymptotically equal to $\frac{1}{\tilde{c}_L}$. The following lemma establishes that the maximum length of an equilibrium is a non-trivial function of p_0 when Δ gets small:

Lemma 3. *Consider any strictly decreasing sequence $\Delta_n \to 0$, and a corresponding sequence of models with $\lambda_n \equiv \tilde{\lambda}\Delta_n$, $c_{L,n} \equiv \tilde{c}_L\Delta$ and $c_{H,n} \equiv \tilde{c}_H\Delta_n$, for some $\tilde{\lambda}, \tilde{c}_L, \tilde{c}_H \in \mathbb{R}_{++}$, with $\tilde{c}_L > \tilde{c}_H$. Fix a $p_0 \in (0, 1)$, and let $\kappa(p_0; \Delta_n)$ be the maximum real time length of an equilibrium when the length of the period is Δ_n. Then, $\kappa(p_0) \equiv \lim_{n\to\infty} \kappa(p_0; \Delta_n)$ exists, belongs to $(0, \frac{1}{c_L})$ and is strictly decreasing in p_0.*

Proof. The proof is in the appendix on page 26. □

Therefore, for a fixed $p_0 \in (0, 1)$ and small $\Delta > 0$, there are equilibria with education duration $\kappa(p_0) + O(\Delta)$. The contrast between this result and Swinkles' illustrates the critical role of timing in the two models. In both models, offers are privately made. However, when firms can make preemptive offers, they can attract the worker in school and end the game immediately. In this case, when the time interval between two consecutive periods is small, firms can post an appropriate wage w to (1) attract both types, and (2) obtain a non-negative profit. In our model, firms cannot directly disturb the worker's signaling process by making an in-school offer, and therefore, semi-separating equilibria can survive.

[5] This limit corresponds to interpreting $\tilde{c}_\theta$ to be the flow cost for each $\theta \in \{L, H\}$, and interpreting $\tilde{\lambda}$ as the rate at which students are exogenously forced to drop out.

As noted before, D1 selects PBE where $p_T \in [1 - \Delta\tilde{c}_L, 1]$. As Δ goes to zero, the last period equilibrium belief converges to 1. The following lemma establishes that, when Δ is small, only in equilibria with a real education length close to $\kappa(p_0)$ (the maximum length at p_0) the last period equilibrium belief is close to 1.

Lemma 4. *Consider an equilibrium sequence as in Lemma 3. Fix $p_0 \in (0,1)$ and any $\tau \in (0, \kappa(p_0))$. Let $p_{T_n}(\Delta_n)$ be the maximum last period beliefs of an equilibrium with $T_n \equiv \lceil \tau/\Delta_n \rceil$ periods of education for Δ_n. Then, $\lim_{n\to\infty} p_{T_n}(\Delta_n)$ exists and is strictly lower than 1. If, instead, $\tau = \kappa(p_0))$, then $\lim_{n\to\infty} p_{T_n}(\Delta_n) = 1$.*

Proof. The proof is in the appendix on page 27. □

Therefore, Lemma 4 implies that equilibria satisfying D1 have a real duration of $\kappa(p_0)+O(\Delta_n)$. Indeed, otherwise the last period's beliefs are bounded away from 1, and hence lower than $1-\Delta\tilde{c}_L$. In the proof of Theorem 2 (see Lemma 11) we explicitly construct for each p_0 equilibria with the last period's beliefs belong to $[1 - \Delta\tilde{c}_L, 1]$. So, for each p_0 and small $\Delta > 0$, there are equilibria satisfying D1, and their duration is close to $\kappa(p_0)$.

Finally, consider again the case where the dropout rate is small, that is, when $\tilde{\lambda}$ is small. From the equation that $\kappa(p_0)$ satisfies (equation (10) in the proof of Lemma 3), it is easy to see that $\lim_{\tilde{\lambda}\to 0} \kappa(p_0) = \frac{1}{\tilde{c}_L}$ for all $p_0 \in (0,1)$. Indeed, as we see in Figure 4, as $\tilde{\lambda}$ gets small, $\tau(p_0)$ converges to $\frac{1}{\tilde{c}_L}$ for all $p_0 \in (0,1)$. Hence, the length of an equilibrium satisfying D1 gets close to $\frac{1}{\tilde{c}_L}$ when the interval gets short and $\tilde{\lambda}$ gets small. This is consistent with the finding of Cho and Kreps (1987) that the only equilibrium that satisfies D1 is the least costly separating equilibrium, found by Riley (1979), that requires an education length equal to $\frac{1}{\tilde{c}_L}$.

3 Multiple Types

Now we consider the $N > 2$ types case in which $\theta \in \{1, 2, 3, ..., N\}$ with a prior p_0^θ, where $\sum_{\theta=1}^{N} p_0^\theta = 1$. Type θ worker has a cost of waiting c^θ, $c^\theta > c^{\theta+1}$. The productivity of θ is Y^θ, $Y^\theta < Y^{\theta+1}$. All types exogenously drop out with probability λ.

The equilibrium concept is the same as in Definition 1 but adapted to the fact that now we have many types. Note that firms' offers depend only on the expected productivity and not on other moments of the productivity distribution. This fact helps us to keep our definition simple:

Definition 2. *A **perfect Bayesian equilibrium (PBE)** is a strategy profile $\{(\alpha^\theta)_{\theta=1,\dots,N}, w\}$, beliefs sequences p^θ and $\hat{p}^\theta$ for all $\theta \in \{1, ..., N\}$ such that:*

1. *θ-worker chooses α^θ to maximize her expected payoff given w,*

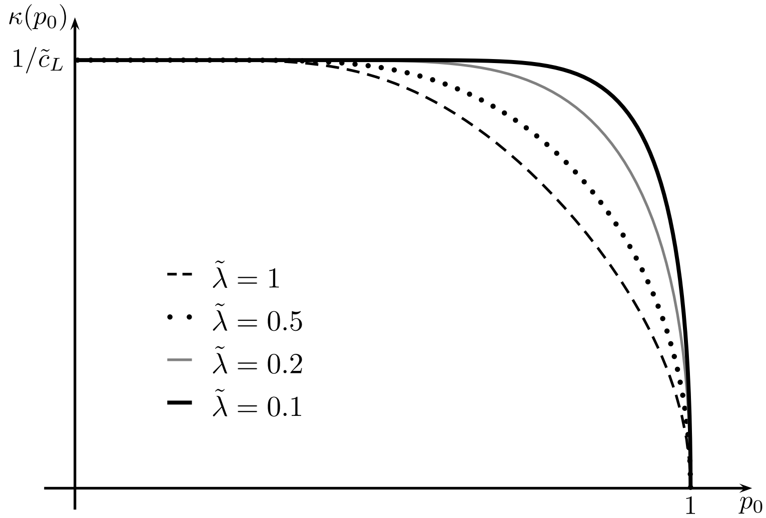

Figure 4: $\kappa(p_0)$ as a function of p_0, for different values of $\tilde{\lambda}$.

2. *if a worker drops out with education t, firms offer $w_t = \sum_{\theta=1}^{N} \hat{p}_t^\theta Y^\theta$*
3. *when it is well defined, $\hat{p}_t^\theta$ satisfies*

$$\hat{p}_t^\theta = \frac{p_t^\theta s_t^\theta}{\sum_{\theta'=1}^{N} p_t^{\theta'} s_t^{\theta'}} \ , \tag{5}$$

   and

4. *when it is well defined, p_t^θ is updated according to the Bayes' rule*

$$p_{t+1}^\theta = \frac{p_t^\theta (1 - s_t^\theta)}{\sum_{\theta'=1}^{N} p_t^{\theta'} (1 - s_t^{\theta'})} \ . \tag{6}$$

Let T^θ be the last time the θ-worker is in school. The following theorem shows that our insight into the binary-type model can be easily extended to a multiple-types model.

Theorem 3. *Under the previous assumptions, in any equilibrium:*

1. *in each period t, there is at most one type, indifferent to dropping out,*
2. *more productive types stay longer in education, $T^\theta \leq T^{\theta+1}$,*
3. *there is positive voluntary dropout in all periods, and*
4. *the expected productivity of dropouts, $\hat{Y}_t \equiv \sum_{\theta=1}^{N} \hat{p}_t^\theta Y^\theta$, is concave in t.*

Proof. The proof can be found in the appendix on page 27. □

Most features in the two-type model are preserved. However, note that under many types we have decreasing returns to education instead of linear ones, since lower types are skimmed out before higher types in equilibria. This pattern of decreasing returns to education is consistent with many empirical studies, for example, Frazis (2002), Habermalz (2003), Heckman *et al.* (2008) and Manoli (2008). The equilibrium construction in multiple-type models is almost identical to that in the two-type model, and thus is omitted.

4 Type-Dependent Exogenous Dropout Rate

In our model, the exogenous dropout rate is independent of the worker's type. This assumption may seem counterintuitive. Indeed, our intuition tells us that low-productivity workers should have a higher probability of dropping out than high-productivity workers, which seems to conflict with our assumption. Our intuition is based on the observed dropping-out behavior, which is driven both by workers' choices (that are related to their productivities) and by exogenous shocks (that may not be related to their productivities).

In our model, even though the exogenous dropout rate is the same across types, the implied observed dropout rate of low-type workers is higher than that of high-type workers. Since both the posterior about a worker being high-type and the endogenous dropout rate of low-type workers change over time, the observable dropout rate changes over time. Hence, assuming a homogeneous exogenous dropout rate is enough to generate an endogenous partial separation and interesting dropout rate dynamics.

Nevertheless, it is useful to know how robust our results are if we relax the assumption of an equal exogenous dropout rate. In this section, we consider a model in which a worker's dropout rate is correlated with his productivity. It turns out that our predictions in Section 2 are robust. There are three relevant cases: (1) $\lambda_H > \lambda_L \geq 0$, (2) $\lambda_L > \lambda_H > 0$, and (3) $\lambda_L \geq \lambda_H = 0$.

4.1 $\lambda_H > \lambda_L \geq 0$ Case

The first case we consider is $\lambda_H > \lambda_L \geq 0$, that is, the high-type worker exogenously drops out at a higher rate than the low-type worker. The following lemma implies that the equilibrium set in this case coincides with the base model when $\lambda = \lambda_H$:

Lemma 5. *Assume $\lambda_H > \lambda_L \geq 0$. Then, $(\alpha^L, \alpha^H, w, p, \hat{p})$ is a PBE if and only if it is also a PBE in the benchmark model with $\lambda = \lambda_H$.*

Proof. The proof can be found in the appendix on page 29. □

The intuition behind this lemma is that, in our original model, by Lemma 1, the endogenous dropout rate of the low-type worker is positive in all periods before (maybe) the last. So, the constraint $s_t^L \geq \lambda$ was never binding in equilibrium. Therefore, all equilibria from the base model for $\lambda = \lambda_H$ are also equilibria for the case $\lambda_H > \lambda_L \geq 0$. On the other hand, for any equilibrium in the case where $\lambda_H > \lambda_L$, let $\tilde{\alpha}_t^L$ denote the low type's strategy. It must be true that $\tilde{\alpha}_t^L \geq \lambda_H - \lambda_L$. Define $\hat{\alpha}_t^L = \tilde{\alpha}_t^L - (\lambda_H - \lambda_L) \geq 0$. One can easily verify that $\hat{\alpha}_t^L$ can be supported in a PBE of the game with a symmetric exogenous dropout rate, $\lambda = \lambda_H$.

4.2 $\lambda_L > \lambda_H > 0$ Case

As we can see in Figure 5, s^L may be non-monotone. In particular, there are some equilibria where it is initially decreasing and then increasing and finally it goes down again. Now, s^L is restricted to be no lower than $\lambda_L > \lambda_H$. We may guess that this constraint will be potentially binding in two connected regions, one for large $\hat{p}$ and the other for intermediate values. In any equilibrium, when this constraint is binding, both types strictly prefer to wait. Different from the benchmark model, the equilibrium belief p_t still goes up since $\lambda_L > \lambda_H$. After some periods, the constraint may become not binding anymore, and the low-type worker starts to play a mixed strategy again. However, the neat equilibrium characterization in the benchmark model can not survive for some parameters. Fortunately, the following theorem shows that the equilibrium characterization in the benchmark model still works when λ_L is not significantly larger than λ_H.

Theorem 4. *For any given set of parameters (λ, c_L, c_H, p_0) there exist $\varepsilon > 0$ such that if $\lambda_H = \lambda$ and $\lambda_L = (\lambda, \lambda + \varepsilon]$ then the set of PBE is the same.*

Proof. The proof can be found in the appendix on page 29. □

4.3 $\lambda_L \geq \lambda_H = 0$ Case

In this case, there is no exogenous drop out by the *H*-worker. Consider first $\lambda_L = 0$. In this case our model is equivalent to Cho and Kreps (1987), only corrected by the fact that the education choice is restricted to be discrete. The reason is that the worker decides his education without interacting with the firms. Once the decision to drop out has been made, the worker cannot change the market's belief about his type. Furthermore, early dropping out may be off the path of play, so beliefs can be arbitrarily assigned in those events. Therefore, the equilibrium predictions of both models share the same characteristics.

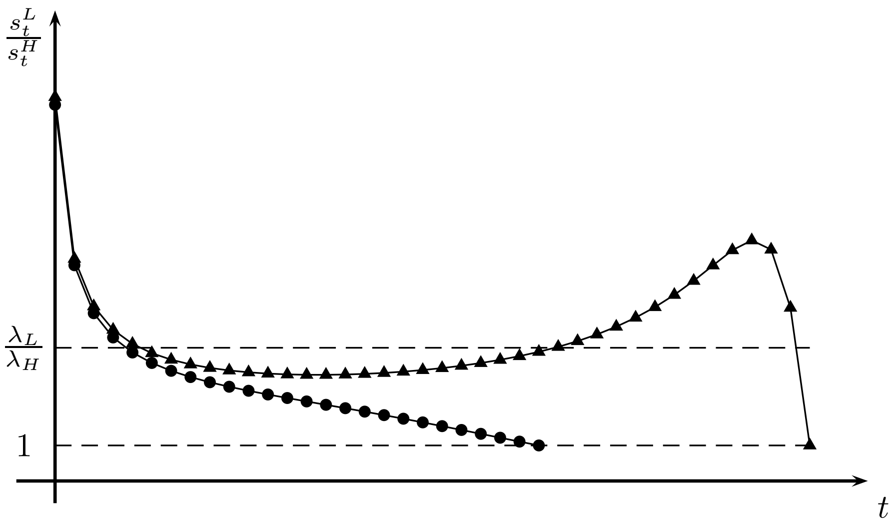

Figure 5: Endogenous dropout rate of the low-type worker

Intuitively, when $\lambda_L > 0$, nothing essential changes. The reason is that the belief threats off the path of play when $\lambda_L = 0$ are replaced by potentially exogenous dropping out by the L-worker, so now deviations to early dropout are still punished.

Note that our main mechanism in the benchmark model is not present here. Indeed, in our benchmark model, as is proven in Lemma 1, the L-worker uses the fact that the H-worker exogenously drops out to mimic him in order to save a high cost of education. Since the H-worker exogenously drops out, early dropout cannot be punished too much, constraining the belief threats by the firms. This is no longer true when $\lambda_H = 0$, so the set of equilibria is qualitatively different from the $\lambda_H > 0$ case.

5 Related Literature and Concluding Remarks

5.1 Related Literature

This is not the first paper to consider dynamic commitment issues in a signaling model. Weiss (1983) and Admati and Perry (1987) (henceforth WAP) pointed out the critical role of commitment in the story of education signaling: when a student arrives on the first day of school, the separation has already happened, and WAP raise the question: "why don't firms hire individuals immediately after they have sorted themselves?" Cho and Kreps (1987) avoid this challenge by directly assuming that a worker can commit to his decision about education duration. Nöldeke and van Damme (1990) formulate an explicitly dynamic game-theoretic version of the Spence model and try to answer WAP's question. In their model, long-lived firms simultaneously make *public offers* to the worker in each period, and the worker decides to accept an offer or continue receiving education. They focus on equilibria that satisfy the *never a weak best response* (NWBR) requirement provided by Kohlberg and Mertens (1986), and they find that equilibria outcome converges to the Riley outcome when the time interval between two education decision points goes to zero. Nonetheless, Swinkels (1999) argues that Nöldeke and Van Damme's result crucially depends on the fact that job offers are publicly made. Hence, he considers a model where two short-lived firms enter and simultaneously make *private offers* to the worker in each period before the worker decides on whether to receive further education and he provides a second answer to WAP's question. Swinkels finds that, when the interval between consecutive offers goes to zero, the unique sequential equilibrium in this game is a pooling one at no education.

Our analysis sheds light on the degree to which Swinkels' result depends on the presence of *preemptive offers*. In Cho and Kreps (1987), the Riley outcome can always be supported since, off the path of play, a belief threat can be imposed to punish early dropouts with low wages. However, in Swinkels (1999), firms can directly interfere with workers' education by making private preemptive offers. Consequently, within a period, the game between current period firms and the worker is a screening game instead of a signaling one, and, as a by-product, the belief threat in Cho and Kreps (1987) cannot be used. It is not clear whether the result in Swinkels (1999) results from the non-existence of a belief threat in Cho and Kreps (1987) or other factors induced by his screening-style timing. To eliminate the effect of a belief threat in Cho and Kreps (1987), we assume that a worker in school faces an exogenous dropout risk, even though he does not choose to do so, and firms can make private offers to a worker only when he has dropped out of school and is available on the market. By doing so, we restore the existence of unproductive education signaling even though offers are privately made and the time interval between two consecutive offers is short enough. Note, our model should be interpreted as a complement rather than a substitute for Swinkels (1999). In practice, both in-school offers and in-market offers take place in many industries. Our goal is to understand how the interaction between the decision sequence and the privacy of offers affects the equilibrium dynamics and the existence of wasteful signaling.

There is a growing literature that considers the impact of extra signaling in Swinkels' model. Kremer and Skrzypacz (2007) introduce an extra (noisy) signal, which is observed at a given time, into Swinkels' model. They interpret the extra signal as students' grades. By doing this, they restore the wasteful signaling and show the presence of a degree premium. Daley and Green (2011) follow Kremer and Skrzypacz (2007) and consider a dynamic lemons market model where the seller's (worker) type is gradually revealed to buyers (firms). In particular, they assume that the news follows a type-contingent diffusion process: the high type has a greater drift rate than the low type. They show that the game ends in one of two ways: either enough good news arrives, restoring confidence and a transaction happens with a high wage, or enough bad news arrives, making the worker more pessimistic so that he accepts a low-wage offer. Our paper is different from those in the following aspects. (1) Kremer and Skrzypacz (2007) follow Swinkels' timing and introduce an extra signal at the deadline of the education. Daley and Green (2011) consider a continuous time model where a type-contingent diffusion process, as a sequence of extra signals, is realized over time. In contrast, we do not introduce any extra signal in addition to education length, and we adopt a different decision sequence. (2) In Daley and Green (2011), the extra signal for informed agents with different types is generated by different processes. In our benchmark model, both high-type and low-type workers are forced to drop out at the same exogenous rate, and the difference in the total dropout rate is determined by the equilibrium instead of by the exogenous assumption. Dilme (2012) considers a similar model introducing moral hazard in the signaling process, obtaining similar results.

In addition, in our model, since an exogenous dropout rate exists, education length is a noisy signal of the worker's type. Some papers also consider signaling models with noisy signals. Matthews and Mirman (1983) consider a noisy signaling model and show that the equilibrium can be unique, which depends on prior beliefs. Bar-Isaac (2003) investigates learning and reputation in a dynamic signaling model where a privately informed monopolist faces a type-contingent but random demand, which can be treated as a noisy signal, and decides whether to sell in each period.

Last, our model is also related to the dynamic adverse selection literature. Janssen and Roy (2002) study a dynamic lemons market problem and show that each equilibrium involves a sequence of increasing prices and qualities traded over time. Trade is delayed and therefore inefficient, but all goods are sold out in finitely many periods. In their model, the time-on-the-market of a good is used to signal the quality of the good. Horner and Vieille (2009) study a dynamic bargaining game in which a single seller faces a sequence of buyers and show that the observability of previously rejected prices can cause a bargaining impasse. Kim (2011) examines the roles of different pieces of information about sellers' past behavior in a dynamic decentralized lemons market. He suggests that market efficiency is not monotone in the amount of information available to buyers but depends crucially on what information is available under what market conditions. Camargo and Lester (2011) investigate a dynamic decentralized lemons market with one-time entry. They demonstrate how prices and the composition of assets evolve over time given an initial fraction of lemons. They find that the patterns of trade depend systematically on the initial fraction of lemons, which is similar to the structure of our result. However, they focus on the dynamics of trade and price.

5.2 Concluding Remarks

This paper presents a new dynamic signaling model where wasteful education takes place over several periods of time. Workers pay an education cost per unit of time and can not commit to a pre-fixed education length. We adopt a timing that is different from that in the traditional literature and introduce an exogenous dropout rate. By doing this, we make three contributions to the literature. First, we highlight the importance of timing in the dynamic signaling model without assuming perfect commitment power. If a job offer is private but not preemptive, the education signaling will not disappear in a model where education is not productive. The difference between our equilibrium prediction and that in Swinkels (1999) highlights the importance of the assumption of timing. In our model, the result is robust to the commitment assumption. Hence, we reconcile the basic idea of Spence (1973) and WAP's challenge. Second, in our equilibrium, the maximum length of education is decreasing in the prior about the worker being productive, and therefore, the equilibrium correspondence is lower hemicontinuous with respect to the information asymmetry perturbation. Third, our model provides rich empirical implications: the joint dynamics of education returns and the dropout rate can be derived.

Even though we present our model in an education signaling environment, our insight is also useful to understand some other environments where sending signals is not only costly but also time-consuming. For example, consider a firm owner trying to sell his firm. In order to signal the type of the firm, the owner may wait some time. The opportunity cost of waiting is likely to be low if the quality of the firm is good. Dropout may be reinterpreted as liquidity shocks that force the owner to sell the firm early. Another example is given by central banks defending themselves from currency attacks. In this case, the cost of defending may depend on the fundamentals of the economy, known only by the central bank. As time passes, the posterior about the economy being healthy increases, so the size of the attacks decreases and the attacks eventually vanish. The exogenous shocks may be given by random events in the international markets, such as a devaluation of the foreign currency used to defend attacks.

In our model, there is no exogenous constraint on the education length. In practice, there is an upper bound on it. A possible extension of our paper is to consider the deadline effect on the set of equilibria. Also, as we show, the empirical implementation of our model is interesting and it may help to distinguish a dynamic signaling model from a human capital model empirically. In particular, one can consider a dynamic education choice model where a worker's productivity depends on both his privately observed ability and on his accumulation of human capital in school. Fang (2006) estimates a static education choice model with both human capital accumulation and a signaling mechanism and claims that the signaling effect is at most about one-third of the actual college wage premium. By using data on workers' education and earnings, one can estimate our model to fit the dynamic process of education returns and the dropout rate, instead of a premium, and decompose the time-varying education returns into human capital effects and signaling effects. We leave these questions for further research.

A Appendix

A.1 The Proof of Lemma 1

Let's first prove a preliminary result:

Lemma 6. *(The L-worker does not beat the market) For all PBE and t,* $V_t^L \leq p_t$*.*

Proof of Lemma 6. Fix a PBE. Let τ be the time at which the game ends. Then,

$$p_t V_t^H + (1 - p_t) V_t^L \leq \mathbb{E}_t[w_\tau | \tau \geq t],$$

and

$$\mathbb{E}_t[w_\tau | \tau \geq t] = \sum_{\tau=t}^{\infty} \Pr(\tau, t)\hat{p}_\tau = \sum_{\tau=t}^{\infty} \Pr(\tau, t) \frac{s_\tau^H p_t \Pr^H(\tau, t)}{\Pr(\tau, t)} = p_t \sum_{\tau=t}^{\infty} s_\tau^H \Pr^H(\tau, t) = p_t \ .$$

where $\Pr(\tau, t)$ denotes the conditional probability in period t that the game ends in period τ, and $\Pr^H(\tau, t) = s_\tau^H \prod_{t'=t}^{\tau-1} (1 - s_{t'}^H)$ is further conditioning on the dropout being type H. The last equality holds because the high type has strictly positive dropout rate and therefore he drops out in finite time with probability one. Since $V_t^H \geq V_t^L$ (the H-worker can mimic the L-worker at a cheaper price) the result holds. □

Suppose there is no endogenous dropout by the L-worker in period t, then $p_{t+1} \leq p_t \leq \hat{p}_t$. But, $\hat{p}_t \leq W_t^L = V_{t+1}^L - c_L$ due to the fact that the L-worker does not voluntarily drop out. By Lemma 6, $V_{t+1}^L \leq p_{t+1} \leq \hat{p}_t$; thus $\hat{p}_t \leq \hat{p}_t - c_L$, which is a clear contradiction. So (1) is true. Therefore (2) is also true, since $W_t^H \geq V_{t+1}^H - c_H \geq \hat{p}_{t+1} - c_H$ by definition of W_t^H and V_t^H, and $\hat{p}_{t+1} - c_H = \hat{p}_t + c_L - c_H > \hat{p}_t$ by the indifferent condition of the L-worker. *Q.E.D.*

A.2 The Proof of Lemma 2

Assume first $T^H > T^L + 1$. In this case, $p_{T^L+1} = 1$. Using equation (1) we know $\hat{p}_{T^L+1} = 1$. Since the payoff of the worker is bounded by 1, and waiting to next period is costly, the worker is better off dropping out at $T^L + 1$. This is a contradiction.

Lemma 1 implies that $S_{T^L}^H > 0$, and therefore $T^H \geq T^L$. *Q.E.D.*

A.3 The Proof of Theorem 1

The wage in period $t = 1$ is bounded above by 1. This implies that for the H-worker to be (weakly) willing to receive one period education, it must be the case that $w_0 \leq 1 - c_H$. This implies that

$$1 - c_H \geq \hat{p}_0 = \frac{p_0 s_0^H}{p_0 s_0^H + (1-p_0) s_0^L} \geq \frac{p_0 \lambda}{p_0 \lambda + 1 - p_0} \ .$$

Solving for p_0 under the equality, we get that the threshold for the existence of an equilibrium with non zero education satisfies equation (4). *Q.E.D.*

A.4 The Proof of Theorem 2

The proof of Theorem 2 is divided in several steps. To make the proof clear to the reader, we clarify that we will be following this **road map**:

1. We begin defining and proving some properties of the "pull-back functions", which will be used to construct equilibria in the rest of the proof (lemmas 7 and 8).
2. In subsection A.4.1 we define some putative values for p_0^k, denoted $\tilde{p}_0^k$, and we prove by induction that, if $p_0 \in (\tilde{p}_0^{k+1}, \tilde{p}_0^k]$, then there is no equilibrium with more than k periods of education.
3. Then, in subsection A.4.2 we show that, if $p_0 \in (\tilde{p}_0^{k+1}, \tilde{p}_0^k]$, there exists an equilibrium where the L-worker is indifferent on dropping out or not for all periods except (maybe) the last for all $T \in \{0, ..., k-1\}$.
4. Finally, in subsection A.4.3 we show that, if $p_0 \in (\tilde{p}_0^{k+1}, \tilde{p}_0^k]$, there exists an equilibrium with length k. Therefore, $p_0^k = \tilde{p}_0^k$.

We begin this proof by stating and proving two results that will simplify the rest of the proof and the proofs of other results in our paper. The first one defined and states two properties of the "**pull-back functions**" $\mu_\tau(\cdot,\cdot)$ and $\hat{\mu}_\tau(\cdot)$:

Lemma 7. *For any $\tau \in \mathbb{N}$, let $\mu_\tau : [0,1]^2 \to [0,1]$ and $\hat{\mu}_\tau : [0,1] \to \mathbb{R}$ be the functions defined by*

$$\mu_\tau(p,\hat{p}) \equiv \frac{\mu_{\tau-1}(p,\hat{p})\hat{\mu}_\tau(\hat{p})}{\hat{\mu}_\tau(\hat{p})(1-\lambda) + \mu_{\tau-1}(p,\hat{p})\lambda} \ , \tag{7}$$

$$\hat{\mu}_\tau(\hat{p}) \equiv \hat{p} - \tau c_L \ , \tag{8}$$

and $\mu_0(p,\hat{p}) \equiv p$ and $\hat{\mu}_0(\hat{p}) = \hat{p}$. Then, for any $\tau > 0$, $\mu_\tau(\cdot,\cdot)$ is continuous and strictly increasing in both arguments.

Proof of Lemma 7. It is obvious when $\tau = 1$, and it holds when $\tau > 1$ by induction argument. □

The meaning of the pull-back functions is the following. Assume that, for a given equilibrium, the L-worker is indifferent on dropping out or not in period $t > 0$. Then, for all $\tau \in \{0, ..., t\}$, we have $p_{t-\tau} = \mu_\tau(p_t, \hat{p}_t)$ and $\hat{p}_{t-\tau} = \hat{\mu}_\tau(\hat{p}_t)$. So, since by Lemma 1 the L-worker is indifferent on dropping out or not in all periods except their last period, the pull-back functions give us the values of the beliefs sequences p and $\hat{p}$ for all periods prior to a given period. They are obtained using equations (1) and (2). The following lemma formalizes this intuition:

Lemma 8. *For any equilibrium with $T > 1$ periods of education and any $T > \tau \geq \tau' \geq 0$ we have*

$$p_{\tau'} = \mu_{\tau-\tau'}(p_\tau, \hat{p}_\tau) \quad \text{and} \quad \hat{p}_{\tau'} = \hat{\mu}_{\tau-\tau'}(\hat{p}_\tau) \ .$$

Proof of Lemma 8. Note that, by Lemma 1, in all periods $t < T - 1$, the L-worker is indifferent on dropping out or not and $s_t^H = \lambda$. This implies that if $t < T - 1$, $\hat{p}_{t-1} = \hat{p}_t - c_L$. Combining equations (1) and (2) with $s_t^H = \lambda$, we have

$$p_t \equiv \frac{p_{t+1}\hat{p}_t}{\hat{p}_t(1-\lambda) + p_{t+1}\lambda} = \frac{p_{t+1}(\hat{p}_{t+1} - c_L)}{(\hat{p}_{t+1} - c_L)(1-\lambda) + p_{t+1}\lambda} = \mu_1(p_{t+1}, \hat{p}_{t+1}) \ .$$

Using this formula recursively and the fact that $\mu_\tau(p, \hat{p}) = \mu_{\tau-1}(\mu_1(p, \hat{p}), \hat{\mu}_1(\hat{p}))$ we obtain the desired result. □

A.4.1 Constructing the Upper Bound on the Length

Define the sequence $\tilde{p}_0^k \equiv \mu_{k-1}(p_0^1, 1 - c_H)$, where p_0^1 is defined in (4). Our goal is to show that $\tilde{p}_0^k$ has the same properties that p_0^k (stated in the statement of the theorem), so $p_0^k = \tilde{p}_0^k$. We are going to prove first, by induction, that if $p_0 \in (\tilde{p}_0^{k+1}, \tilde{p}_0^k]$ then there is no equilibrium with more than k periods of education:

Step 1 (induction hypothesis): If $p_0 \in (\tilde{p}_0^{k+1}, \tilde{p}_0^k]$ there is no equilibrium with more than k periods of education. If an equilibrium has k periods of education, then $\hat{p}_0 \leq \hat{p}_0^k \equiv \hat{\mu}_{k-1}(1 - c_H)$.[6]

Step 2 (proof for $k = 0$ periods of education): By Theorem 1 there is no equilibrium with education for $p_0 > p_0^1$. Also, in the same proof, it is shown that all equilibria in this region, $\hat{p}_0 = p_0 \geq p_0^1 > 1 - c_H = \hat{\mu}_0(1 - c_H)$.

Step 3 (proof for $k = 1$ period of education): Assume that p_0 is such that there is an equilibrium with 1 period of education. Then, $\hat{p}_0 \leq \hat{p}_0^1 \equiv 1 - c_H$ (at least the H-worker has to be willing to wait). Using Bayes' update (equations (1) and (2)) we can express $\hat{p}_0 \equiv \hat{p}_0(p_0, s_0^L, s_0^H)$

[6] The second induction hypothesis is included in order to make the argument simple in the induction argument (step 4).

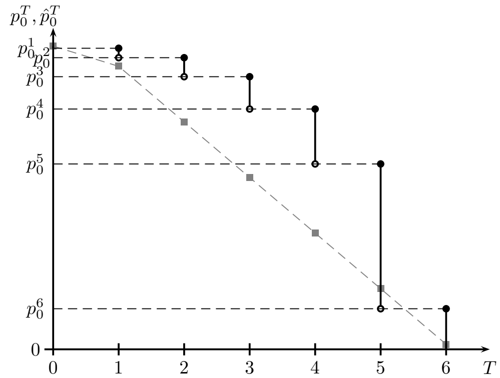

Figure 6: $\tilde{p}_0^T$ (black filled dots), $\hat{p}_0^T$ (gray square dots) and maximum length of equilibria (black).

and $p_1 = p_1(p_0, s_0^L, s_0^H)$. Therefore, using these equations, we can write p_0 in terms of $\hat{p}_0$, p_1 and s_0^H in the following way:

$$p_0 = p_{-1}(p_1, \hat{p}_0, s_0^H) \equiv \frac{p_1 \hat{p}_0}{\hat{p}_0(1 - s_0^H) + p_1 s_0^H} \ .$$

The RHS of the previous expression is maximized when $s_0^H = \lambda$. Therefore, if an equilibrium ends with length of two periods, the initial prior is at most $p_0^1 \equiv \frac{1-c_H}{1-c_H(1-\lambda)}$.

Step 4 (induction argument for $k > 1$): Assume the induction hypothesis is true for $k-1$, for $k > 1$. We need to verify whether it is true for k.

Assume that p_0 is such that there exists some equilibrium with k periods of education. Denote the beliefs sequences for this equilibrium p and $\hat{p}$. Note that, by the induction hypothesis, $p_1 \leq p_0^{k-1}$ and $\hat{p}_1 \leq \hat{p}_0^{k-1}$, since the continuation play after 1 is itself an equilibrium with initial prior p_1. Since $k > 2$, by Lemma 1, the H-worker is strictly willing to wait in period 0, so $s_0^H = \lambda$, and the L-worker randomizes in period 0. Then, $\hat{p}_0 = \hat{p}_1 - c_L \leq \hat{p}_0^{k-1} - c_L = \hat{p}_0^k$. Therefore, by Lemma 8, $p_0 = \mu_1(p_1, \hat{p}_1)$, and that this is increasing in both arguments. So, the maximum value it can take is $\tilde{p}_0^k \equiv \mu_1(\tilde{p}_0^{k-1}, \hat{p}_0^{k-1})$.

Step 5 (T^* is the limit): Note that T^* is such that

$$\hat{p}_0^{T^*+1} \leq 0 < \hat{p}_0^{T^*} \ .$$

Then, since $\hat{p}_0^{T^*+1} \leq 0$, there is no equilibrium longer than T^* periods of education.

A graphical intuition of the proof can be found in Figure 6. It graphically represents both $\tilde{p}_0^T$ and $\hat{p}_0^T$ used in the proof.

A.4.2 Constructing L-equilibria

Now, we prove a result related to the set of equilibria where the L-worker is indifferent in all periods, which is similar to Theorem 2 itself. For each $p_0 \in (0,1)$, we use $\tilde{T}^L(p_0)$ to denote the maximum number of education periods of an equilibrium where the L-worker is indifferent to drop out in all periods except (maybe) the last. We name these equilibria L-equilibria. The following lemma shows that, for any $p_0 \in (0,1)$, there is a finite integer k such that, for each $T = 0, 1, ..., k$ there is an L-equilibrium that lasts for T periods of education, and no L-equilibrium with length more than k.

Lemma 9. *Let's define $T^{**} \equiv \lceil \frac{1-c_L}{c_L} \rceil$, $p_0^{L,k} \equiv \mu_k(1,1)$ for $k = 0, ..., T^{**}$ and $p_0^{L,T^{**}+1} \equiv 0$. Then, if $p_0 \in (p_0^{L,k+1}, p_0^{L,k}]$ for some $k = 0, ..., T^{**}$, we have $\tilde{T}^L(p_0) = k$. Furthermore, for each $T \leq \tilde{T}^L(p_0)$, there is a unique L-equilibrium with T periods of education.*

Proof of Lemma 9. Fix some $p_0 \in (0,1)$. If $p_0 > \mu_k(1,1)$ for some $k \leq T^{**}$ there is no L-equilibrium with k periods of education. Indeed, if there was one (ending at $p_k = \hat{p}_k$), then $p_0 = \mu_k(p_k, p_k)$. But since $\mu_k(p_k, p_k)$ is strictly increasing in p_k and $p_0 > \mu_T(1,1)$, then $p_0 > \mu_k(p,p)$ for all $p \in [0,1]$. This is clearly a contradiciton. Note also that, in an L-equilibrium with T periods of education, $\hat{p}_T - \hat{p}_0 = Tc_L \leq 1$. Since $(T^{**}+1)c_L > 1$, we have $\tilde{T}(p_0) < T^{**} + 1$,.

Fix $k < T^{**}$, $p_0 \in (p_0^{L,k+1}, p_0^{L,k}]$ and $T \leq k$. Note that $\mu_T(p,p)$ is continuous and strictly increasing when $p > Tc_L$ for any $T \leq T^{**}$ and $\lim_{p \searrow Tc_L} \mu_T(p,p) = 0$.[7] So, since $p_0 \leq \mu_k(1,1) \leq \mu_T(1,1)$, there exists a unique $p_T \in (Tc_L, 1)$ such that $p_0 = \mu_T(p_T, p_T)$. Furthermore, there is an equilibrium with length T with $p_t = \mu_{T-t}(p_T, p_T)$ and $\hat{p}_t = \hat{\mu}_{T-t}(p_T)$. The argument for $k = T^{**}$ is analogous. □

Lemma 10. *For any $k \leq T^{**}$, we have $p_0^{L,k} \in (\tilde{p}_0^{k+1}, \tilde{p}_0^k)$.*

Proof of Lemma 10. Note first that

$$\overbrace{\frac{1-c_H}{1-(1-\lambda)c_H}}^{p_0^1} > \overbrace{\frac{1-c_L}{1-(1-\lambda)c_L}}^{p_0^{1,L}} = \mu_1(1,1) > \mu_1(p_0^1, 1-c_H)$$

By definition, for $k > 1$, $p_0^k = \mu_{k-1}(p_0^1, 1-c_H) = \mu_1(p_0^{k-1}, \hat{\mu}_{k-2}(1-c_H))$ and $p_0^{k,L} = \mu_{k-1}(p_0^{1,L}, 1-c_L) = \mu_1(p_0^{k-1}, \hat{\mu}_{k-2}(1-c_L))$. Also, note that $\hat{\mu}_k(1-c_H) > \hat{\mu}_k(1-c_L) > \hat{\mu}_{k+1}(1-c_H)$. Therefore, since $\mu_1(\cdot,\cdot)$ is strictly increasing in both arguments, we have $p_0^{L,k} \in (\tilde{p}_0^{k+1}, \tilde{p}_0^k)$. □

[7] Note that, if $T \leq T^{*}*$ then $Tc_L < 1$, and, by definition, $\hat{\mu}_T(Tc_L) = 0$. Using the definition of $\mu_\tau(\cdot,\cdot)$, we have that $\mu_T(c_L T, c_L T) = 0$.

A.4.3 Constructing H-equilibria

Lemma 9 implies that for any $p_0 \in (0,1)$, an L-equilibrium lasting for at most k periods can be constructed, where k satisfies that $p_0 \in (p_0^{L,k+1}, p_0^{L,k}]$. However, Lemma 10 shows that $p_0^{L,k} < \tilde{p}_0^k$. For $p_0 \in (p_0^{L,k}, \tilde{p}_0^k]$, there is no L-equilibrium lasting for k periods. The question is now whether any other equilibrium which lasts for k periods in this last region. Lemma 11 shows that the answer to this question is yes.

An equilibrium which lasts for $T > 0$ periods of education is an H-equilibrium if and only if, in equilibrium, the L-worker strictly prefers dropping out in period $T-1$. In other words, in an H-equilibrium $p_T = 1$. Note each equilibrium is either L-equilibrium or H-equilibrium, and never both.

Lemma 11. *If $p_0 \in (p_0^{L,k}, \tilde{p}_0^k]$, there exists an H-equilibrium of length k, for $k \in \{1, ..., T^{**}\}$. If $p_0 \in (\tilde{p}_0^{k+1}, p_0^{L,k}]$, there exists an L-equilibrium of length k, for $k \in \{1, ..., T^* - 1\}$.*

Proof of Lemma 11. For $p_0 \in (\tilde{p}_0^{k+1}, p_0^{L,k}]$ the proof of the previous lemma tells us that there exists an L-equilibrium of length k. To prove the case $p_0 \in (p_0^{L,k}, \tilde{p}_0^k]$, we define the function $\hat{p} : (p_0^{1,L}, p_0^1] \to (1-c_L, 1-c_H]$ as follows

$$\hat{p}(p) \equiv \frac{\lambda p}{\lambda p + 1 - p} \; .$$

Then for all $p_0 \in (p_0^{L,k}, p_0^k]$ there exists a unique $f(p_0) \in (p_0^{1,L}, p_0^1]$ such that $p_0 \equiv \mu_{k-1}(f(p_0), \hat{p}(f(p_0)))$. Indeed, we have that $\lim_{p \searrow p_0^{1,L}} \hat{p}(p) = 1 - c_L$ and $\hat{p}(p_0^1) = 1 - c_H$. So, we have

$$\lim_{p \searrow p_0^{L,1}} \mu_{k-1}(p, \hat{p}(p)) = p_1^{L,k} \quad \text{and} \quad \mu_{k-1}(p_0^1, \hat{p}(p_0^1)) = \tilde{p}_0^k \; .$$

Since $\hat{p}(\cdot)$ is continuous and strictly increasing, $\mu_{k-1}(\cdot,\cdot)$ is continuous in both arguments and strictly increasing, then there exists such $f(p_0)$, and is unique.

Let's construct one equilibrium with k education periods when $p_0 \in (p_0^{L,k}, \tilde{p}_0^k]$, for $k \leq T^* - 1$. Our claim is that it can be defined by $p_k = \hat{p}_k = 1$, $p_t = \mu_{t-1}(f(p_0), \hat{p}(f(p_0)))$ and $\hat{p}_t = \hat{p}(f(p_0)) - c_L(k-t-1)$, for $t \in \{0, ..., k-1\}$. To prove that we show that the corresponding strategies are well defined. Note that, if the L-worker is indifferent in period 0, we have

$$s_t^L = \frac{1}{1 + \frac{(1-\lambda)(1-p_t)\hat{p}_t}{\lambda p_t(1-\hat{p}_t)}} = \frac{\lambda}{1 - \frac{(1-\lambda)(p_t - \hat{p}_t)}{p_t(1-\hat{p}_t)}} \; .$$

The first equality shows that $s_t^L < 1$. The second equality shows that, if $p_t^1 > \hat{p}_t$, then $s_t^L > \lambda$, what is equivalent to $p_0^2 < p_0^1$, which is true as long as $\hat{p}_t > 0$. Since, when $k < T^*$, $\hat{p}_0 = \hat{p}(f(p_0)) - c_L(k-1) > 0$, the result holds in this case.

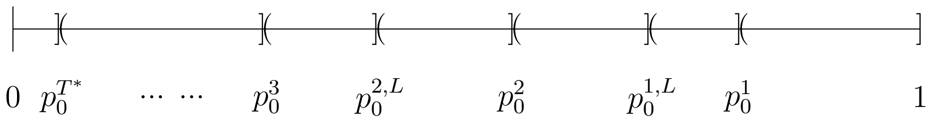

Figure 7: partition construction

Finally, there are two possible cases. If $T^{**} = T^*$, we know from the previous lemma that there exists an L-equilibrium in with length T^{**} in $(0, p_0^{L,T^*})$. If $T^{**} = T^* - 1$ then there exists some $p \in (p_0^{1,L}, p_0^1]$ such that $\hat{p}(p) = T^{**}c_L$. Indeed, in this case $1 \leq T^* c_L < 1 - c_H + c_L$, so $T^{**}c_L \in (1 - c_H, 1 - c_L]$. Therefore, we can use the same argument as for $p_0 \in (p_0^{L,k}, \tilde{p}_0^k]$, for $k \leq T^* - 1$. The idea of the partition construction can be summarized in Figure 7. □

Finally, note that the set $\{\tilde{p}_0^k\}_{k=0}^{T^*+1}$ is such that $\tilde{p}_0^k > \tilde{p}_0^{k+1}$ for all k. Furthermore, for all $0 \leq k \leq T^*$ and $0 \leq T \leq k$, if $p_0 \in (\tilde{p}_0^{k+1}, \tilde{p}_0^k]$ then there exists an equilibrium with T periods of education and no equilibrium with length larger than k. So, $p_0^k \equiv \tilde{p}_0^k$, for $k = 0, ..., T^* + 1$, satisfy the statement of Theorem 2, and therefore its proof is complete. *Q.E.D.*

A.5 The Proof of Lemma 3

We will do the proof first fixing the maximum real time and solving for the corresponding p_0, and then showing that for all p_0 there exists a unique limit for the maximum real time. Fix $\bar{\kappa} \in (0, \frac{1}{\tilde{c}_L})$. In order to save notation, consider a strictly decreasing sequence Δ_n such that $\frac{\bar{\kappa}}{\Delta_n} \in \mathbb{N}$ for all $n \in \mathbb{N}$. Using the Bayes' rule, we have the following equation relating $p_0^{\bar{\kappa}/\Delta_n, L}$ and $p_0^{\bar{\kappa}/\Delta_n - 1, L}$:[8]

$$\frac{1}{p_0^{\bar{\kappa}/\Delta_n, L}} = \frac{\tilde{\lambda}\Delta_n}{1 - \tilde{c}_L \bar{\kappa}} + \frac{1 - \tilde{\lambda}\Delta_n}{p_0^{\bar{\kappa}/\Delta_n - 1, L}} = \sum_{m=0}^{\bar{\kappa}/\Delta_n} \frac{\tilde{\lambda}\Delta_n (1 - \tilde{\lambda}\Delta_n)^m}{1 - \tilde{c}_L(\bar{\kappa} - m\Delta_n)} + (1 - \tilde{\lambda}\Delta_n)^{\bar{\kappa}/\Delta_n} . \tag{9}$$

When Δ_n is small, each term of the sum can be approximated as follows

$$\frac{\tilde{\lambda}\Delta_n (1 - \tilde{\lambda}\Delta_n)^m}{1 - \tilde{c}_L(\bar{\kappa} - m\Delta_n)} = \frac{\tilde{\lambda} e^{-\tilde{\lambda}s}}{1 - \tilde{c}_L(\bar{\kappa} - s)} \Delta_n + O(\Delta_n^2)$$

where $s \equiv m\Delta_n$. The last term of the RHS of equation (9) is approximated by $e^{-\tilde{\lambda}\bar{\kappa}} + O(\Delta_n)$. Since this each term in the sum is a bounded function (note that s ranges from 0 to $\bar{\kappa}$) multiplied by Δ_n, at the limit $\Delta_n \searrow 0$ the sum converges to the integral, so we have

$$\frac{1}{\tilde{p}_0(\bar{\kappa})} \equiv \lim_{n \to \infty} \frac{1}{p_0^{\bar{\kappa}/\Delta_n, L}} = e^{-\tilde{\lambda}\bar{\kappa}} + \int_0^{\bar{\kappa}} \frac{e^{-\tilde{\lambda}s}\tilde{\lambda}}{1 - \tilde{c}_L s} ds .$$

[8] We use $p_0^{\bar{\kappa},L}$ defined in Lemma 9 instead of $p_0^{\bar{\kappa},L}$ for simplicity. Lemma 10 and the fact that $p_0^{\bar{\kappa}/\Delta_n,L} - p_0^{t/\Delta_n - 1, L} = O(\Delta_n)$ guarantees that $p_0^{\bar{\kappa}/\Delta_n,L}$ and $p_0^{\bar{\kappa}/\Delta_n,L}$ will be asymptotically equal.

Note that the RHS of the previous expression is equal to 1 when $\bar{\kappa} = 0$. Differentiating it with respect to $\bar{\kappa}$ we find

$$\frac{d}{d\bar{\kappa}}\frac{1}{\tilde{p}_0(\bar{\kappa})} = -\tilde{\lambda}e^{-\tilde{\lambda}\bar{\kappa}} + \frac{e^{-\tilde{\lambda}\bar{\kappa}}\tilde{\lambda}}{1-\tilde{c}_L\bar{\kappa}} = \frac{e^{-\tilde{\lambda}\bar{\kappa}}\tilde{\lambda}\tilde{c}_L\bar{\kappa}}{1-\tilde{c}_L\bar{\kappa}} \geq 0 \ .$$

Therefore, $p_0^{\bar{\kappa}} \in (0,1)$ when $\bar{\kappa} \in (0, \frac{1}{\tilde{c}_L})$.

Note that, for each $p_0 \in (0,1)$, there exist a unique $\bar{\kappa}$ such that $\tilde{p}_0(\bar{\kappa}) = p_0$. Indeed, $\lim_{\bar{\kappa}\to 0}\tilde{p}_0(\bar{\kappa}) = 1$, $\lim_{\bar{\kappa}\to 1/\tilde{c}_L}\tilde{p}_0(\bar{\kappa}) = 0$ and $\tilde{p}_0(\cdot)$ is strictly increasing in $(0, \frac{1}{\tilde{c}_L})$. Therefore, for each p_0 there exists a unique $\kappa(p_0) \equiv \tilde{p}_0^{-1}(p_0)$ such that satisfies the conditions of the lemma. It is given by the solution of

$$\frac{1}{p_0} = e^{-\tilde{\lambda}\kappa(p_0)} + \int_0^{\kappa(p_0)} \frac{e^{-\tilde{\lambda}s}\tilde{\lambda}}{1-\tilde{c}_L s}ds \ . \tag{10}$$

Q.E.D.

A.6 The Proof of Lemma 4

Proceeding similarly as in the proof of Lemma 3, we have that

$$\frac{1}{p_0} = \frac{e^{-\tilde{\lambda}\tau}}{p_{T_n}(\Delta_n)} + \int_0^{\tau} \frac{e^{-\tilde{\lambda}s}\tilde{\lambda}}{p_{T_n}(\Delta_n) - \tilde{c}_L s}ds + O(\Delta_n) \ .$$

Note that the RHS of the previous equation is decreasing in $p_{T_n}(\Delta_n)$. Furthermore, the RHS is lower than $\frac{1}{p_0}$ when $p_{T_n}(\Delta_n) = 1$, since it would be equal if $\tau = \kappa(p_0)$, but, by assumption, $\tau < \kappa(p_0)$. Also, when $p_{T_n}(\Delta_n) = p_0$, the RHS is larger than p_0. Indeed, it would be equal to p_0 if $\tau = 0$ but, $\tau > 0$ and, as it is shown in the proof of Lemma 3, the RHS is increasing in τ. Therefore, there exists a unique limit of $p_{T_n}(\Delta_n)$, and is strictly lower than 1. *Q.E.D.*

A.7 The Proof of Theorem 3

1. Assume that, in period t, there are two types $\theta_1, \theta_2 \in \Theta$, with $c^{\theta_1} < c^{\theta_2}$, both indifferent on dropping out or not. Let τ_1 and τ_2 denote, respectively, the stopping times of the continuation strategies that make players indifferent on dropping out or not.[9] Then, we have

$$\hat{Y}_t = \mathbb{E}[w_{\tau_{\theta_2}} - c^{\theta_2}\tau_{\theta_2}] \geq \mathbb{E}[w_{\tau_{\theta_1}} - c^{\theta_2}\tau_{\theta_1}] > \mathbb{E}[w_{\tau_{\theta_1}} - c^{\theta_1}\tau_{\theta_1}] = \hat{Y}_t \ .$$

[9]For this proof, for a given strategy, it is convenient to use the random variable τ, which gives the duration of the game.

The first (weak) inequality is from the optimality of the θ_2-worker. The strong inequality is because $\mathbb{E}[\tau_{\theta_1}] > 0$ and $c^{\theta_1} < c^{\theta_2}$. The equalities come from the fact that i-workers, with $i \in \{1, 2\}$ are indifferent on taking on dropping out (and getting $\hat{Y}_t$) or staying and following τ_i. Therefore, we have a contradiction.

2. Assume otherwise, so there exist $\theta_1, \theta_2 \in \Theta$ such that $\theta_1 < \theta_2$ and $T^{\theta_1} > T^{\theta_2}$. Let τ_{θ_1} be the stopping time of the continuation strategy after T^{θ_2}, given by the strategy of θ_1. Then, note that

$$\hat{Y}_{T^{\theta_2}} \geq \mathbb{E}[w_{\tau_{\theta_1}} - c^{\theta_2}\tau_{\theta_1}] > \mathbb{E}[w_{\tau_{\theta_1}} - c^{\theta_1}\tau_{\theta_1}] \geq \hat{Y}_{T^{\theta_2}} \ .$$

This is clearly a contradiction. The first inequality comes from the optimality of the θ_2-worker choosing to drop out at T^{θ_2} (since they could deviate to mimic the θ_1-worker). The second inequality is given by the fact that since $\theta_1 < \theta_2$, $c^{\theta_2} < c^{\theta_1}$ and since $T^{\theta_1} > T^{\theta_2}$, $\mathbb{E}[\tau_{\theta_1}] > 0$. The last inequality comes from the optimality of the θ_1-worker choosing to drop out at $T^{\theta_1} > T^{\theta_2}$ (since they could deviate to mimic the θ_2-worker).

3. Define $\Theta_t = \{\theta | T^\theta \geq t\}$ and $\theta_t = \min\{\Theta_t\}$. We proceed as in the proof of Lemma 6. Now we have

$$\begin{aligned} \mathbb{E}_t[w_\tau | \tau \geq t] &= \sum_{\tau=t}^{\infty} \Pr(\tau, t)\hat{Y}_\tau = \sum_{\tau=t}^{\infty} \Pr(\tau, t) \frac{\sum_\theta Y^\theta s_\tau^\theta p_t^\theta \Pr{}^\theta(\tau, t)}{\Pr(\tau, t)} \\ &= \sum_\theta p_t^\theta Y^\theta \sum_{\tau=t}^{\infty} s_\tau^\theta \Pr{}^\theta(\tau, t) = \sum_\theta p_t^\theta Y^\theta = Y_t \ , \end{aligned}$$

where $\Pr(\tau, t)$ and $\Pr^\theta(\tau, t) = s_\tau^\theta \prod_{t'=t}^{\tau-1}(1 - s_{t'}^\theta)$ are defined as in the proof of Lemma 6.

Note that, by the previous result,

$$\sum_{\theta=\theta_t}^{N} p_t^\theta V_t^\theta = \mathbb{E}_t[w_\tau | \tau \geq t] - \sum_{\theta=\theta_t}^{N} p_t^\theta c^\theta \tau^\theta(t) < \mathbb{E}_t[w_\tau | \tau \geq t] \ ,$$

where $\tau^\theta(t)$ is the stopping time for the θ-worker conditional on reaching t. Since $V_t^\theta \leq V_t^{\theta+1}$ (since the $(\theta+1)$-worker can mimic the θ-worker at a lower cost), and $\sum_{\theta=\theta_t}^{N} p_t^\theta = 1$ we have that $V_t^{\theta_t} < Y_t$.

Assume that in period t there is no voluntary dropout. In this case, $\hat{Y}_t = Y_t$. Since we just showed $V_{\theta_t} < Y_t$, the θ_t-worker is willing to drop out, which is a contradiction.

4. Note that, by part 3 of this theorem, we have that $\hat{Y}_{t+1} - c^{\theta_t} \leq \hat{Y}_t$. Furthermore, $\hat{Y}_{t+1} - c^{\theta_{t+1}} \geq \hat{Y}_t$. This implies that $\hat{Y}_{t+1} - \hat{Y}_{t+1} \in [c^{\theta_{t+1}}, c^{\theta_t}]$. Since c^θ is decreasing in θ and, by part 2 of this theorem, the θ_t-worker is (weakly) increasing in t, $\hat{Y}_t$ is concave in t.

Q.E.D.

A.8 The Proof of Lemma 5

We first prove that Lemma 1 (that holds when $\lambda_H = \lambda_L$) is still valid when $\lambda_H \geq \lambda_L$. Consider T as the maximum periods lower than T^L where $s_t^L \leq s_t^H$. In this case

$$p_{T+1} \leq p_T \leq \hat{p}_T \ .$$

Furthermore, since the L-worker is voluntarily dropping out at time $T+1$, this implies $\hat{p}_T \leq \hat{p}_{T+1} - c_L$. Nevertheless, since $s_{T+1}^L \geq s_{T+1}^H$, we have $\hat{p}_{T+1} \geq p_{T+1}$, which is a contradiction, since

$$\hat{p}_{T+1} \leq p_{T+1} \leq p_T \leq \hat{p}_T \leq \hat{p}_{T+1} - c_L \ .$$

So, when $\lambda_H \geq \lambda_L$, it is still true that $s_t^L > s_t^H$ in all periods of all equilibria before T^L. Therefore, the relaxation of the constraint $\lambda_L = \lambda_H = \lambda$ to $\lambda_L \leq \lambda_H = \lambda$ does not introduce new equilibria. Trivially, it does not destroy any equilibria, since in the model $\lambda_L = \lambda_H = \lambda$, in all equilibria, $s_t^L > \lambda$ for all equilibria and period $t \leq T^L$. *Q.E.D.*

A.9 The Proof of Theorem 4

Note that Lemma 6 still holds (the H-worker can imitate the strategy of the L-worker). Now we try to prove a result analogous to Lemma 1. Assume that the L-worker is not voluntarily dropping out in period t, so his dropout rate is $\lambda + \varepsilon$. First assume that the dropout rate of the H-worker is larger than $\lambda + \varepsilon$. In this case, we can apply the exact same argument as in the proof of Lemma 1, so we obtain again a contradiction. Assume now that $s^H \in [\lambda, \lambda + \varepsilon)$. In this case $p_{t+1} = p_t + O(\varepsilon)$ and $\hat{p}_t = p_t + O(\varepsilon)$, so $\hat{p}_t - p_{t+1} = O(\varepsilon)$. Then, using the same logic as in the proof of Lemma 1 we have

$$\hat{p}_t \leq W_t^L \leq V_{t+1}^L - c_L \leq p_{t+1} - c_L \ .$$

Therefore, $\hat{p}_t - p_{t+1} \leq -c_L$. But this is inconsistent with $\hat{p}_t - p_{t+1} = O(\varepsilon)$. That proves that, if $\varepsilon > 0$ is small enough, the model with $\lambda_H = \lambda$ and $\lambda_L = \lambda + \varepsilon$ does not have more equilibria than for the case $\varepsilon = 0$.

Let's prove the reverse. Assume that there exists a sequence $\{\varepsilon_n > 0\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} \varepsilon_n = 0$ and, for each n, there exists an equilibrium in our original model and t_n reached with positive probability on the path of play under this equilibrium such that $s_{t_n}^L \in [\lambda, \lambda + \varepsilon_n)$. This implies

$p_{t_n+1} = p_{t_n} + O(\varepsilon_n)$ and $\hat{p}_{t_n} = p_{t_n} + O(\varepsilon_n)$, so $\hat{p}_{t_n} - p_{t_n+1} = O(\varepsilon_n)$.[10] So,

$$\hat{p}_{t_n} = W^L_{t_n} = V^L_{t_n+1} - c_L \leq p_{t_n+1} - c_L \ .$$

This, again, is a contradiction. *Q.E.D.*

[10] Using some abuse of notation, p_{t_n} and $\hat{p}_{t_n}$ denote the corresponding posteriors in the n-th equilibrium of the sequence.

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