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"On the Continuous Equilibria of Affiliated-Value, All-Pay Auctions with Private Budget Constraints"

by

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Abstract

We consider all-pay auctions in the presence of interdependent, affiliated valuations and private budget constraints. For the sealed-bid, all-pay auction we characterize a symmetric equilibrium in continuous strategies for the case of N bidders and we investigate its properties. Budget constraints encourage more aggressive bidding among participants with large endowments and intermediate valuations. We extend our results to the war of attrition where we show that budget constraints lead to a uniform amplification of equilibrium bids among bidders with sufficient endowments. An example shows that with both interdependent valuations and private budget constraints, a revenue ranking between the two mechanisms is generally not possible.

Keywords: All-Pay Auction, War of Attrition, Budget Constraints, Common Values, Private Values, Affiliation, Contests
JEL: D44

Suppose firms are lobbying for a lucrative government contract. Clearly, the contract's value to each firm has an idiosyncratic component since the firms likely have different operating costs. On the other hand, each firm also has a privately known limit on how much it is able or willing to spend on the lobbying game. Perhaps the management of one firm is approving of small restaurant meals with officials but expenditures or bribes beyond some threshold are morally too much to stomach. A competitor, in contrast, may be less hampered in its lobbying strategy. How does the lobbying game unfold when competitors differ in their valuation for the prize and in their ability or capacity to compete for it? Would some firms spend *more* on lobbying believing that their competitors have to navigate within some private constraints on actions?

In this essay we consider the class of situations like the above by analyzing all-pay auctions. In an all-pay auction, the highest bidder is the winner of the item for sale; however, all bidders incur

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a payment equal to their bid. As a stylized model of a lobbying contest, the all-pay auction has a long tradition in political economy (Baye *et al.*, 1993).

Despite the frequent application of the all-pay auction to contests, most analyses fail to capture the exogenous but private limits on actions that are relevant in many situations. We introduce these constraints into the all-pay auction and we identify sufficient conditions for the existence of an equilibrium in continuous strategies. Our analysis isolates a general amplification of bids submitted by bidders with intermediate valuations. An extension of our model to the war of attrition (the "second-price all-pay auction") shows the generality of the bid amplification phenomenon and allows for a comparison of the revenue potential of the two mechanisms. Generally, no revenue ranking exists between the two formats in the presence of both budget constraints and affiliated, interdependent values.

Although our model is phrased in the language of auctions (players are called "bidders," etc.), it applies to any situation where resources are irreversibly expended in pursuit of a goal or a prize. The goal or prize can have a value that has both private and common components. The private constraints on bids or effort that we introduce are often quite natural elements of the situation. For example, Hickman (2011) employs a version of the all-pay auction to model students competing for college admissions. It is clear that the return from a college degree varies across students due to personal preferences and ability; it is also natural to assume that idiosyncratic shocks, such as health status, family background, or school location, place an exogenous, heterogenous, and private cap on the "effort" that a student can exert in the college admissions game. These shocks may be orthogonal to the expected return from the degree per se.

As another example consider a patent race between competing firms. Such competition is naturally modeled as either an all-pay auction or as a war of attrition (Leininger, 1991). The expected value of the invention and the budget available to the research division will determine the effort devoted to the race. However, information asymmetries or agency concerns can create a wedge between the available budget and the research division's assessment of the project's value. Moreover, a firm as a whole likely faces a hard, short-run physical resource constraint that will cap its feasible effort level. It is natural to suppose that this resource limit is also private information. The interaction between expected rewards and the heterogenous resource constraints will shape how firms engage in this competition.

While we are motivated by the range of applications of all-pay auctions in modeling social and economic situations, our study also fills a gap in the small but growing literature on auctions with private budget constraints. Our analysis begins with the work of Krishna & Morgan (1997) who study the all-pay auction and the war of attrition with interdependent and affiliated valuations. To this setting we introduce private budget constraints distributed continuously on an interval. Our environment parallels the setting of Fang & Parreiras (2002) and Kotowski (2012) who study the second-price and the first-price auction with private budget constraints respectively. Both of these studies build on Che & Gale (1998), which is the seminal paper in this strand of literature.

In light of this literature, our study contributes along several dimensions. First, by focusing on all-pay mechanisms we put under scrutiny an important allocation mechanism in resource-constrained environments. Many authors examining optimal auctions with budget-constrained participants have resorted to mechanisms that feature "all-pay" payment schemes (Maskin, 2000; Pai & Vohra, 2011). Our analysis therefore complements this literature but we do not attempt the mechanism design exercise here.

Second, by developing our model in a more general setting than traditionally employed we are able to identify additional features of the environment that affect the existence of a well-behaved and (relatively) tractable equilibrium. Previous auction studies lodged in the affiliated and interdependent-value paradigm, such as Fang & Parreiras (2002) and Kotowski (2012), have focused exclusively on the two-bidder case. While some of the intuition from the two-bidder case is relevant generally, the case of two bidders masks much of the nuance that we identify. For example, in the all-pay auction we document how changes in the number of bidders alone directly affects the existence of an equilibrium within the class of strategies traditionally considered by this literature.

Although we relax many assumptions, we do not study the all-pay auction's equilibrium in its fullest generality. Indeed, from the onset we focus on the existence of an equilibrium that is continuous, symmetric, and monotone. We view this restricted scope to be a pragmatic but reasonable choice. From a technical perspective, this restriction allows us to define equilibrium behavior as a solution to a differential equation. This methodology offers a window on the mechanism's economic properties and gives precise and testable predictions concerning bidder behavior. We believe that the set of cases covered is rich and it offers insights that would carry over to a discontinuous (but symmetric and monotone) equilibrium. Undoubtably, continuous equilibria would receive the bulk of attention in applications due to their relative tractability.

The remainder of the paper is organized as follows. Section 1 introduces the model and section 2 studies the symmetric equilibrium in the all-pay auction. We then consider the equilibrium's comparative static properties with focus on changes in the distribution of budgets, changes in the number of bidders, and changes in the public information surrounding the contest. The final section considers this model's second-price analogue, the war of attrition. We explore the symmetric equilibria of this model and we discuss the scope for a revenue ranking. Proofs and supporting lemmas are in the appendix. An online appendix collects additional results and extensions of the main analysis.

1 The Environment

Let $\mathcal{N} = \{1, \ldots, N\}$ be the set of bidders. Each bidder $i \in \mathcal{N}$ has a two-dimensional private type $(s_i, w_i) \in [0, 1] \times [\underline{w}, \bar{w}]$. While (s_i, w_i) is bidder i's private information, the auction's ambient environment—types' prior distributions (defined below), the number of bidders, the auction rules, etc.—are assumed to be common knowledge. In the main text we assume that $0 < \underline{w} < \bar{w}$. The case of $\underline{w} = 0$ is addressed in the online appendix and is qualitatively similar to our main analysis.

A bidder's realized *value-signal*, s_i, is her private information about the item for purchase. For example, in a patent race it would be an estimate of the invention's value. In a political lobbying contest, it may correspond to some private information about the consequences of proposed legislation. Let $\mathbf{s} = (s_1, \ldots, s_N)$ be a profile of realized value-signals.[1] We use capital letters—S_i, etc.—to refer to signals as random variables.

A bidder's realized *budget*, w_i, is a bound above which she cannot bid. We consider a budget to be a hard constraint on expenditures. A budget may correspond to a bidder's cash holdings, her credit limit, or some other private limit on actions. Such limits may be both financial or physical, depending on the application of interest. Alternatively, budget constraints can be modeled as "soft" constraints acting through an increasing cost of bidding. For brevity, we do not explore this extension. Zheng (2001), among others, is an application of this specification of budget constraints.

Bidder i's valuation for the item can be described by a random variable: $V_i = u(S_i, S_{-i})$. We assume that $u\colon [0,1] \times [0,1]^{N-1} \to [0,1]$ is strictly increasing in the first argument and nondecreasing and permutation-symmetric in the last $N-1$ arguments. As standard, we suppose u is continuously differentiable and normalized such that $u(0, \ldots, 0) = 0$ and $u(1, \ldots, 1) = 1$.

A bidding strategy is a (measurable) function $\beta_i\colon [0,1] \times [\underline{w}, \bar{w}] \to \mathbb{R}_+$. Throughout, we adopt Bayesian-Nash equilibrium as our solution concept. An equilibrium is symmetric if all bidders follow the same bidding strategy. We focus on symmetric equilibria and we henceforth suppress player subscripts in our notation whenever possible.

We always assume that bidders are risk neutral. The introduction of risk aversion or more general preferences into this model introduces complications analogous to those seen in the first-price auction. As shown by Kotowski (2012), the interaction of a bidder's private budget with her risk preferences can introduces countervailing incentives rendering the existence of a monotone equilibrium a more involved question.

Two assumptions concerning the distribution of bidders' types define our baseline environment and we maintain them throughout our analysis. Our initial set-up is standard and subsequent assumptions, which are specific to the auction format considered, will impose additional structure.

Assumption A1. *Value-signals have a joint density $h(s_1, \cdots, s_N)$ which is continuous and strictly positive. Moreover, $h(s_1, \cdots, s_N)$ is invariant to permutations of $(s_1, \ldots, s_N)$ and $h(s_1, \cdots, s_N)$ is log-supermodular. Thus, value-signals are affiliated.*

Affiliated signals are a standard assumption introduced to the auction literature by Milgrom & Weber (1982).[2] Except for budget constraints, and the specific conditions introduced below, our model

[1] We use standard notation: $\mathbf{s}_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$, $\mathbf{s} = (s_i, \mathbf{s}_{-i})$, etc.

[2] We refer the reader to this paper for a summary of the properties of affiliated random variables.

closely parallels their classic environment. Although affiliation is more general than independence, it is nonetheless a restrictive statistical property (de Castro, 2010).

Concerning the distribution of players' budgets, we require budgets to be determined independently of value-signals and to be identically distributed.

Assumption A2. *Players' budgets are independently and identically distributed according to the cumulative distribution function* $G(w)$. $G(w)$ *has full support on* $[\underline{w}, \bar{w}]$ *and admits a strictly positive and continuous density* $g(w)$.

While the independence condition is strong, without it the model is not tractable. It is standard in studies of auctions with budget constraints when there is some affiliation in players' value-signals. Our model naturally allows for $\bar{w} = \infty$, but for brevity we present our main discussion assuming $\bar{w} < \infty$. A priori it is clear that the values of $\underline{w}$ and $\bar{w}$ will play an important role in our analysis. In many situations it is natural to assume that such values (if they are different from zero or infinity, respectively) may become common knowledge. For example the parameters of the support of G may become known via posted bonds, (not modeled) participant selection, or obligatory financial disclosures as may occur in the context of political lobbying. In future work we intend to explore the effects of endogenous disclosure of $\underline{w}$ or $\bar{w}$ but for now we take them as given and common knowledge.

2 A Symmetric, Continuous Equilibrium in the All-Pay Auction

The rules of the all-pay auction are well-known. Each bidder i will simultaneously submit a bid b_i. A bid must be feasible given the bidder's budget: $b_i \leq w_i$. If bidder i submits the highest bid she wins the game and her payoff under the realized signal profile $\mathbf{s} = (s_i, \mathbf{s}_{-i})$ is $u(s_i, \mathbf{s}_{-i}) - b_i$; otherwise, it is $-b_i$. Ties among high bidders are resolved by a uniform randomization to designate the winner.

We endeavor to identify a symmetric equilibrium where all bidders follow a strategy of the form

$$\beta(s, w) = \min \{b(s), w\} \tag{1}$$

where $b(s)$ is strictly increasing when less than $\bar{w}$, continuous, and (piecewise) differentiable. Our focus on equilibria meeting these criteria is consistent with previous studies of auctions with private budget constraints. Che & Gale (1998), Fang & Parreiras (2002, 2003), and Kotowski (2012) examine equilibria that reside in this class of strategies.

To motivate the sufficient conditions for equilibrium existence that we propose below, we begin with an heuristic discussion. Suppose for the moment that there is a symmetric equilibrium $\beta(s, w) = \min \{b(s), w\}$ and consider bidder i of type (s, w). If this player bids $b(x) \leq w$ her bid will defeat two categories of opponents assuming all other bidders are following the strategy $\beta(s, w)$.

First it defeats all opponents who have a value-signal $s < x$. Second, it defeats all opponents who have a budget $w < b(x)$ irrespective of value-signal. Since $\beta(s, w)$ is strictly increasing, ties are probability zero events. To account for these two possible cases and to succinctly express a bidder i's expected payoffs from the bid $b(x)$ requires some new notation. We let S be the value-signal observed by bidder i and we relabel the value-signals of the other bidders as $Y_1, \ldots, Y_{N-1}$. Let $\bar{Y}_k = \max(Y_1, \ldots, Y_k)$. Let $f_k(y|s)$ be the density of $\bar{Y}_k|S = s$. We introduce the following terms:

$$v_k(s, y) = \mathbb{E}[u(s, Y_1, \ldots, Y_{N-1})|S = s, \bar{Y}_k = y] \tag{2}$$

$$z_k(x|s) = \int_0^x v_k(s, y) f_k(y|s) dy \tag{3}$$

$$\gamma_k(b) = \binom{N-1}{k} G(b)^{N-1-k}(1 - G(b))^k \tag{4}$$

Adopting the convention that

$$z_0(x|s) = v_0(s, y) = \mathbb{E}[u(s, Y_1, \ldots, Y_{N-1})|S = s],$$

we can write the expected payoff of bidder i from the bid $b(x)$ as

$$U(b(x)|s, w) = \sum_{k=0}^{N-1} \gamma_k(b(x)) z_k(x|s) - b(x). \tag{5}$$

The binomial terms account for the combinations of opponents who are defeated by $b(x)$ due to having a low value-signal or a low budget.[3] We do not need to keep track of the precise identities of these bidders due to the symmetry assumptions on their preferences and on the information structure. Introducing asymmetries would necessitate a more detailed accounting of the different cases. The final term in (5) is the bidder's payment which she makes irrespective of the auction's outcome.

If $b(s) < w$ is indeed this player's equilibrium best response, a local first-order optimality condition must be satisfied. Specifically,

$$\frac{d}{dx} U(b(x)|s, w)\bigg|_{x=s} = 0. \tag{6}$$

Adopting the notation

$$z'_k(x|s) \equiv \frac{\partial}{\partial x} z_k(x|s) = \begin{cases} 0 & \text{if } k = 0 \\ v_k(s|x) f_k(x|s) & \text{if } k \neq 0 \end{cases}$$

[3]The online appendix presents a detailed derivation of this expression for expected utility.

we can evaluate (6) to derive:

$$b'(s) = \frac{\sum_{k=0}^{N-1} \gamma_k(b(s)) z_k'(s|s)}{1 - \sum_{k=0}^{N-1} \gamma_k'(b(s)) z_k(s|s)}. \tag{7}$$

Our subsequent discussion will identify conditions that ensure (7) has a solution which will be consistent with equilibrium bidding.

Two initial observations are worthwhile. First, when $b(s) < \underline{w}$, and it is understood that $G(b(s)) = 0$, equation (7) reduces to

$$b'(s) = v_{N-1}(s,s) f_{N-1}(s|s),$$

which is the differential equation identified by Krishna & Morgan (1997) as characterizing bidding behavior in the all-pay auction absent budget constraints. Therefore, our sufficient conditions must suitably generalize their assumptions. Second, when $b(s) > \underline{w}$, (7) accounts for the change in marginal incentives faced by unconstrained bidders. Slight bid increases not only defeat opponents with slightly higher valuations but they also defeat all opponents with sufficiently low budgets regardless of their valuation. This second effect ameliorates the well-known winner's curse phenomenon in interdependent-value settings.

Regrettably the derivation of (7) was heuristic and we made many implicit assumptions. Specifically, we need to ensure that the solution to (7) satisfying an appropriate boundary condition is strictly increasing. For example, if the denominator of (7) is ever negative, then $b'(s) < 0$ contradicting our original hypothesis that $b(s)$ is increasing. Furthermore, we must also ensure that first-order conditions are sufficient to determine a bidder's optimal bid, which in general may not be true.

To identify sufficient conditions when (7) does characterize equilibrium bidding we introduce two additional assumptions. The first assumption will limit the degree of affiliation among bidder's value-signals. The second assumption will place a restriction on the joint distribution of value-signals and budgets. Both assumptions speak to the complicated interaction among conflicting incentives faced by bidders in the all-pay auction. We elaborate on these assumptions below.

We begin our analysis by introducing the function

$$\alpha(s) = \int_0^s v_{N-1}(y,y) f_{N-1}(y|y) dy. \tag{8}$$

Krishna & Morgan (1997) show that under suitable conditions $\alpha(s)$ defines the equilibrium bidding strategy in the all-pay auction without private budget constraints. Let $\bar{\alpha} = \alpha(1)$, and assume $\underline{w} < \bar{\alpha}$. Otherwise, budget constraints effectively do not bind and $\alpha(s)$ would define a symmetric equilibrium in our model.

The first assumption generalizes the sufficient condition proposed by Krishna & Morgan (1997) supporting $\alpha(s)$ as the equilibrium strategy in the all-pay auction without budget constraints.

Assumption A3. *Let* $\phi(x, w|s) = \sum_{k=1}^{N-1} \gamma_k(w) v_k(s, x) f_k(x|s)$*. For all* (x, w)*,* $\phi(x, w|\cdot)\colon [0, 1] \to \mathbb{R}$ *is nondecreasing.*[4]

Intuitively, Assumption A3 limits the degree of correlation between value-signals relative to the impact of a player's own signal on her valuation. The assumption always holds if signals are independent but it can hold in other cases as well. For example it is satisfied when there are two bidders, $u(s_i, s_j) = (s_i + s_j)/2$ and $h(s_i, s_j) \propto 1 + s_i s_j$.

Whereas Assumption A3 places a restriction on the correlation among value-signals, we additionally require an assumption structuring the joint distribution of value-signals and budgets. Assumption A4 presents this restriction. We defer interpreting this assumption until after presenting our main result and an example illustrating the identified equilibrium. We define the value $\tilde{s} < 1$ as the unique solution to $\alpha(\tilde{s}) = \underline{w}$.

Assumption A4. *Let* $\xi(x, w|s) = 1 - \sum_{k=0}^{N-1} \gamma_k'(w) z_k(x|s)$*. Then the following conditions hold:*

1. *For every* $s \geq \tilde{s}$*, there exists* w_s*,* $\underline{w} \leq w_s < \bar{w}$ *such that* $w < w_s \implies \xi(s, w|s) < 0$ *and* $w > w_s \implies \xi(s, w|s) > 0$*.*
2. *There exists* $\epsilon > 0$ *such that for all* $s \in (\tilde{s} - \epsilon, \tilde{s} + \epsilon)$*,* $\xi(s, \underline{w}|s) > 0$*.*
3. *When* $x \geq \tilde{s}$*,* $\xi(x, w|\cdot)\colon [0, 1] \to \mathbb{R}$ *is non-increasing.*

Although Assumption A4 may appear as a strictly technical assumption, it has an economic interpretation which we discuss below. We thus have the following theorem:

Theorem 1. *Suppose Assumptions A1–A4 are satisfied. Then there exists a continuous, symmetric equilibrium in the all-pay auction where all bidders follow the strategy* $\beta(s, w) = \min\{b(s), w\}$*. For all* $s < \tilde{s}$*,* $b(s) = \alpha(s) = \int_0^s v_{N-1}(y, y) f_{N-1}(y|y) dy$*. For all* $s \geq \tilde{s}$*,* $b(s)$ *is a strictly increasing solution to the differential equation*

$$b'(s) = \frac{\sum_{k=0}^{N-1} \gamma_k(b(s)) z_k'(s|s)}{1 - \sum_{k=0}^{N-1} \gamma_k'(b(s)) z_k(s|s)}$$

satisfying the initial condition $b(\tilde{s}) = \underline{w}$*.*

Remark 1. In the online appendix we show how Theorem 1 obtains allowing for alternative assumptions. For example, if $\underline{w} = 0$, an alternative argument is necessary to identify the appropriate solution for $b(s)$. Another case which we consider is when $G(w)$ is not smooth for all $[\underline{w}, \bar{w}]$. All that is required to obtain an equilibrium in our class is that $G(w)$ is sufficiently smooth in the relevant

[4]When $w = \underline{w}$, $\phi(x, \underline{w}|s) = v_{N-1}(s, x) f_{N-1}(x|s)$. The sufficient condition identified by Krishna & Morgan (1997) in their model of the all-pay auction is that $v_{N-1}(\cdot, x) f_{N-1}(x|\cdot)$ is nondecreasing.

range of bids—specifically, when $w \leq \bar{\alpha}$. The need to avoid circular reasoning, however, renders establishing the bound $b(s) \leq \bar{\alpha}$ a more involved argument.

Remark 2. Both Assumptions A3 and A4(3) are satisfied if $u(\cdot, \mathbf{s}_{-i})h(\mathbf{s}_{-i}|\cdot)\colon [0,1] \to \mathbb{R}$ is nondecreasing. See Assumption AX.3 and Lemma X.1 in the online appendix. We introduced A3 and A4(3) separately to emphasize the parallel between the all-pay auction and the war of attrition, which we analyze below.

The following example highlights several features of the equilibrium of the all-pay auction.

Example 1. Suppose $N = 2$ and that value-signals are given by $S_i \overset{i.i.d.}{\sim} U[0,1]$ while budgets $W_i \overset{i.i.d.}{\sim} U[\frac{2}{25}, \frac{3}{4}]$. Let $u(s_i, s_j) = (s_i + s_j)/2$.

It is readily verified that $b(s) = s^2/2$ for $s < \tilde{s} = 2/5$. Of course, $\alpha(s) = s^2/2$ is also the equilibrium strategy in this model absent budget constraints. For $s > 2/5$, $b(s)$ is the solution to the differential equation

$$b'(s) = \frac{25(3 - 4b(s))s}{25s(3s-2) + 42}$$

satisfying the boundary condition $b\left(\frac{2}{5}\right) = 2/25$. The resulting equilibrium strategy is

$$\beta(s, w) = \begin{cases} \frac{s^2}{2} & s \leq \frac{2}{5} \\ \min\{b(s), w\} & s > \frac{2}{5} \end{cases}$$

Figure 1 plots the functions $b(s)$ and $\alpha(s) = \frac{s^2}{2}$.[5] The introduction of budget constraints rendered $b(s)$ concave for $s > \tilde{s}$ while $\alpha(s)$ is convex. Immediately to the right of $\tilde{s} = 2/5$, $b(s) > \alpha(s)$; therefore, some types of bidders with intermediate value-signals bid more following the introduction of budget constraints. The following corollary demonstrates that such an amplification is indeed a common feature of equilibrium bidding in such environments.

Corollary 1. *Under the conditions of Theorem 1,* $\lim_{s \to \tilde{s}^+} b'(s) > \lim_{s \to \tilde{s}^-} b'(s) = \lim_{s \to \tilde{s}^-} \alpha'(s)$.

The encouragement of more aggressive bidding by bidders with relatively large budgets and intermediate valuations is due to a change in the marginal incentives that bidders experience in the presence of budget constraints. The prospect of defeating additional opponents who are budget constrained increases the marginal return of a higher bid; therefore, some types of bidders respond to this incentive with more aggressive bidding.

Discussion and Interpretation

To interpret the sufficient conditions behind Theorem 1 it is useful to examine in detail the role of Assumption A4. Assumption A4(1) asserts that the function $\xi(s, \cdot|s)$ satisfies a single crossing

[5] In all examples, graphs of numerical solutions are obtained using the Runge-Kutta method.

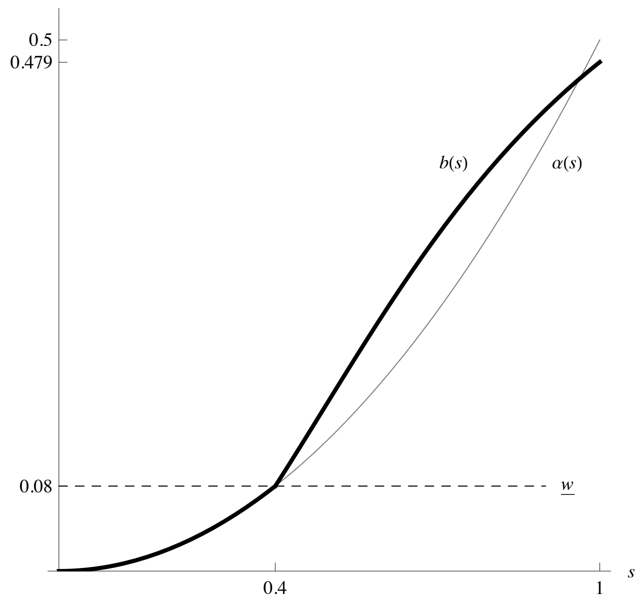

Figure 1: The functions $b(s)$ and $\alpha(s)$ in the characterization of equilibrium bidding in Example 1.

condition and is strictly positive for w sufficiently large. Thus, the assumption ensures that the righthand side of the differential equation (7) is eventually strictly positive. While this is Assumption A4's technical role, it also has an economic interpretation that we outline below.

Writing the condition $\xi(s, w|s) > 0$ explicitly (see Lemma A.1) gives

$$g(w)(N-1)\left[\sum_{k=0}^{N-2}\binom{N-2}{k}G(w)^{N-2-k}(1-G(w))^k(z_k(s|s)-z_{k+1}(s|s))\right] < 1. \tag{9}$$

To simplify further suppose values are private and value-signals are independent draws from a common distribution with c.d.f. $H(s)$. In this case:

$$z_k(s|s) - z_{k+1}(s|s) = u(s)H(s)^k(1-H(s)).$$

$\xi(s, w|s) > 0$ now becomes

$$\begin{aligned} g(w)(N-1)u(s)(1-H(s))\left(G(w)+H(s)-G(w)H(s)\right)^{N-2} &< 1 \\ \iff u(s)\frac{d}{dw}\left[G(w)+H(s)-G(w)H(s)\right]^{N-1} &< 1. \end{aligned}$$

The term $[G(w)+H(s)-G(w)H(s)]^{N-1}$ is the probability of all other bidders having a value-signal less than s or a budget less than w. If $b = \beta(s, w)$—as equilibrium bidding assumes—this would be the probability with which bidder i wins the auction with a bid of b. We can therefore regard Assumption A4 as imposing a limit on the rate of change in the probability of winning owing only to defeating opponents who have a smaller budget. If this probability increases too rapidly at some point $\hat{w}$—for instance, due to an "atom"[6] in the distribution of budgets—then as $b(s)$ crosses $\hat{w}$, $b'(s)$ becomes undefined or negative and the continuous strategy we are considering can no longer be an equilibrium. At such bid levels, a bidder would have an incentive to drastically increase her bid to take advantage of others' budget constraints.

In an interdependent-value setting, the preceding intuition continues to apply. However, it must be extended to incorporate the winner's curse. Defeating low-budget opponents is generally "good news" concerning the expected value of the item. Therefore in its fullest form, (9) additionally incorporates a weighted average controlling for these marginal effects on an opponent-by-opponent basis.

Since the sufficient conditions in Assumption A4 may be difficult to verify in practice, a simple (but exceptionally conservative) alternative is that

$$g(w)(N-1)\mathbb{E}[u(1, Y_1, \ldots, Y_{N-1})|S=1] < 1 \tag{10}$$

The sufficiency of this condition is easy to confirm since

$$\begin{aligned}\sum_{k=0}^{N-1} \gamma_k'(w) z_k(x|s) &= g(w)(N-1)\left[\sum_{k=0}^{N-2} \binom{N-2}{k} G(w)^{N-2-k}(1-G(w))^k (z_k(s|s) - z_{k+1}(s|s))\right] \\ &\le g(w)(N-1)\left[\sum_{k=0}^{N-2} \binom{N-2}{k} G(w)^{N-2-k}(1-G(w))^k\right] z_0(s|s) \\ &\le g(w)(N-1)\mathbb{E}[u(1, Y_1, \ldots, Y_{N-1})|S=1].\end{aligned}$$

Effectively, (10) places a uniform limit on the preponderance of budget constraints in the relevant range of bids. Of course, this strict limit is not necessary for equilibrium existence. Example 1 presented above does not satisfy (10).

Necessity A natural question to pose is to what extent our assumptions are necessary to support a continuous symmetric equilibrium. First, any assumptions concerning the differentiability of relevant functions are needed to ensure that the differential approach we adopt is possible. We consider such conditions to be economically innocuous. It is therefore more apt to examine the extent to which Assumption A4 is necessary since it is the most unusual of the proposed conditions.

First suppose that $\xi(s, \underline{w}|s) < 0$ in a neighborhood of $\tilde{s}$. In this situation, the solution $b(s)$ cannot be extended continuously to bids in the range above $\underline{w}$. All solutions to the differential

[6] We are assuming atom-less distributions of budgets, but the intuition in the extreme case of an atom in $G(w)$ is illuminating. Of course, there exist examples of a similar character when $G(w)$ admits a continuous density.

equation (7) will be decreasing in a neighborhood immediately above $\underline{w}$. In this regard, Assumption A4(2) cannot be relaxed while ensuring an equilibrium in continuous strategies.

From a formal point of view Assumption A4(1) is not necessary for the existence of the equilibrium that we identify. From a practical perspective we view it as necessary. It is the weakest assumption that *guarantees* increasing solutions to (7) on the domain $[\tilde{s}, 1]$ without referring to the solution of (7) itself, which we view as too far removed from model primitives to be economically meaningful. At minimum, A4(1) enjoys an economic interpretation, which we view as plausible. Weaker statements in lieu of Assumption A4(1) would allow $\xi(s, \cdot|s)$ to fail its single crossing condition provided the failure did not substantively affect the desired solution to (7). We present one such alternative statement in the online appendix.

Comparative Statics in the All-Pay Auction

To place the equilibrium in context and to foster intuition for its properties we investigate several comparative statics. Throughout we focus on the effect of changes of the environment on changes in individual bidder behavior. Given the indirect characterization of the equilibrium bidding strategy, we cannot offer detailed conclusions concerning aggregate auction performance, such as expected revenue. In this regard we do not differ from previous studies of auctions with budget constraints. None has yet arrived at a concise description of aggregate statistics allowing for affiliated and interdependent values. Only Fang & Parreiras (2003) are able to document the failure of the linkage principle through an extended example.

Changes in the Distribution of Budgets Consider a change in the environment that makes budget constraints more lax. For example, the distribution of budgets may vary exogenously with broader economic or social conditions. In principle, this relaxation can lead to two competing effects. On one hand, when budget constraints are relaxed, bidders may be encouraged to bid more—constraints on competition have been removed and on the margin a bidder must bid more to influence the auction outcome. The countervailing force, however, draws on the amelioration of the winner's curse associated with budget constraints. Conditional on winning, the item is of relatively higher value when budget constraints bind since there is a good chance of having defeated a budget-constrained opponent. Relaxing budget constraints would dampen this effect which would tend to pull bids down. In the context of the second-price auction, Fang & Parreiras (2002) conclude that the latter effect can dominate.

In the all-pay auction, however, there does not exist a standard and general ordering of bidder's strategies as we change G. This is true even under very restrictive stochastic orders. To appreciate this conclusion, suppose $N = 2$ and fix a distribution of budgets G on $[\underline{w}, \bar{w}]$. Suppose $u(\cdot, s_j)h(s_j|\cdot)$ is nondecreasing. As shown in the online appendix, the equilibrium bidding strategy in this case will be bounded above by $\bar{\alpha}$. Consider the family of distribution functions $G_a(w) \equiv G(w)^a$ for $a \geq 1$. If

$a' > a$, then $G_{a'}$ likelihood-ratio dominates G_a. Intuitively, higher values of a imply more relaxed budget constraints. Denote by $\beta_a(s, w) = \min\{b_a(s), w\}$ an equilibrium strategy parameterized by a and meeting the conditions identified in our analysis. Suppose for $a = 1$, the auction admits an equilibrium β_1. Then for all a sufficiently large $1 - g(w)aG(\bar{\alpha})^{a-1} > 0$ for all $w \in [\underline{w}, \bar{\alpha}]$ since $g(w)$ is bounded. Therefore, for a sufficiently large, β_a will define an equilibrium when budgets are distributed according to G_a. By examining the main differential equations defining $b_a(s)$ as $a \to \infty$, we see that

$$\frac{(1 - G(b)^a)v_1(s, s)f_1(s|s)}{1 - ag(b)G(b)^{a-1}\int_s^1 v_1(s, y)f_1(y|s)dy} \to v_1(s, s)f_1(s|s)$$

uniformly for all s and $b \leq \bar{\alpha}$. Therefore $b_a(s) \to \int_0^s v_1(y, y)f_1(y|y)dy$, as one would expect.

Recall however that for each a, $b_a(\tilde{s} + \epsilon) > \alpha(\tilde{s} + \epsilon)$ while $b_a(1) < \bar{\alpha}$. Therefore a bidder's strategy adjustment is not monotonic across types and in general $b_a(\cdot)$ is neither greater nor less than $b_{a'}(\cdot)$ for $a' \neq a$. Thus, the same qualitative ordering that exists for the second-price auction does not carry over to the case of the all-pay auction.

Changes in the Bidder Population How will changes in the bidder population affect the auction's equilibrium? While original studies of auctions with budget constraints, such as Che & Gale (1998), allowed for variation in the number of bidders, comparative statics exploring the sensitivity of equilibrium to changes in N were not pursued systematically. Studies by Fang & Parreiras (2002) and Kotowski (2012) of the second-price and first-price auction did not extend the model beyond two bidders. The main conclusion from our study is that the existence of an equilibrium in our class is very sensitive to the number of bidders in the auction. This holds for even independent, private-value environments.

Fix an auction environment with private values and suppose there is an equilibrium of the form $\beta(s, w) = \min\{b(s), w\}$ for some $N \geq 2$. Changing N can lead to two salient violations of Assumption A4. First, due to a change in N at the (new) critical value $\tilde{s}$, the (new) expression (9) is such that $\xi(\tilde{s}, \underline{w}|\tilde{s}) < 0$, which violates Assumption A4(2). Second, even if A4(2) is satisfied, following a change in the number of bidders $\xi(s, w|s)$ may instead violate the single-crossing crossing condition of Assumption A4(1). The violation can preclude the existence of a strictly increasing solution to (7) for all $s \geq \tilde{s}$. We illustrate both failures with an example.

Example 2. Suppose there are N bidders with private values, $u(s_i, \mathbf{s}_{-i}) = s_i$. Value-signals are distributed uniformly and independently on the unit interval. Budgets are distributed independently according to the distribution $G(w) = 1 - \exp(-4(w - \underline{w}))$ with support $[\underline{w}, \infty)$. Choose $\underline{w} = 0.1$. As a function of N we can express $b(s)$ for bids below $\underline{w}$ as

$$b_N(s) = \frac{N-1}{N}s^N.$$

The associated critical value is $\tilde{s}_N = \sqrt[N]{\frac{Nw}{N-1}}$. Similarly, for each $N \geq 2$, we can calculate $\xi(s, w|s)$ to be

$$\xi_N(s, w|s) = 1 + 4(N-1)(s-1)se^{\frac{2}{5}-4w}\left((s-1)e^{\frac{2}{5}-4w} + 1\right)^{N-2}.$$

Suppose $N = 2$, then $\xi_2(s, w|s) = 4(s-1)se^{\frac{2}{5}-4w} + 1$ which is strictly positive for all $(s, w) \in [0, 1] \times [\underline{w}, \infty)$ except at the point $(s, w) = \left(\frac{1}{2}, \frac{1}{10}\right)$ where it is zero. Since $\tilde{s}_2 = \frac{1}{\sqrt{5}} \approx 0.447$, Assumption A4 is satisfied and a continuous equilibrium of the form $\min\{b(s), w\}$ exists.

Keeping the environment otherwise the same, suppose $N = 3$. Now $\tilde{s}_3 = \frac{\sqrt[3]{\frac{3}{5}}}{2^{2/3}} \approx 0.531$. At this value, $\xi_3(\tilde{s}_3, \underline{w}|\tilde{s}_3) = \frac{11}{5} - 2\left(\frac{6}{5}\right)^{2/3} < 0$. This is a volition of Assumption A4(2) and a continuous extension of $b(s)$ at $\tilde{s}$ into the range above $\underline{w}$ is not possible. We note that A4(1) is otherwise satisfied.

Finally, suppose $N = 10$. In practical terms this would be a setting with a large number of bidders. Now, $\tilde{s}_{10} = \frac{1}{\sqrt[5]{3}} \approx 0.803$ and $\xi_{10}(\tilde{s}_{10}, \underline{w}|\tilde{s}_{10}) = 5 - 4\sqrt[5]{3} \approx 0.017 > 0$. Thus, Assumption A4(2) is met. However, Assumption A4(1) fails. We illustrate this failure with Figure 2. The figure shows the function $b(s)$ in this case along with its solution satisfying the boundary condition $b(\tilde{s}_{10}) = \underline{w}$.[7] This extension of $b(s)$ above $\underline{w}$ necessarily needs to traverse a region, illustrated in gray, where $\xi_{10}(s, w|s) < 0$. Therefore, there does not exist a strictly increasing solution $b(s)$ for all $s > \tilde{s}_{10}$ which satisfies $b(\tilde{s}_{10}) = \underline{w}$.

The main implication stemming from Example 2 concerns the possibilities and opportunities for inference in auction environments where bidders may be budget constrained. While there does not exist a good theory of inference and identification in auctions with budget constraints (and it is far beyond the scope of this study to develop one), changes in N are a common source of variation exploited in empirical auction studies.[8] Fully exploiting this variation in auctions with budget constraints may be problematic (or at best challenging) due to the qualitative differences of equilibrium bidding as the environment changes with N. For example, for some values of N (depending on the distribution of budgets and valuations), one would not be able to employ first-order conditions to fully characterize a bidder's optimal bid. Much more research is required to develop precise conclusions and restrictions accounting for such concerns.

Public Signals Suppose prior to bidding players observe the realization of some public signal S_0, which we assume is affiliated with bidders' value-signals. We may suppose that a bidder's payoff depends on the value of this signal, i.e. $V_i = u(S_0, S_i, S_{-i})$. For example, this signal may be some piece of information released non-strategically by the auctioneer. We begin by distinguishing two (extreme) types of public signals that the bidders may observe.

[7]Precisely, we plot the solution for the inverse of $b(s)$ to accommodate the points where $b'(s) = \infty$.

[8]See Athey & Haile (2007) for a recent survey of identification in auction models. See Bajari & Hortaçsu (2005) for an implementation.

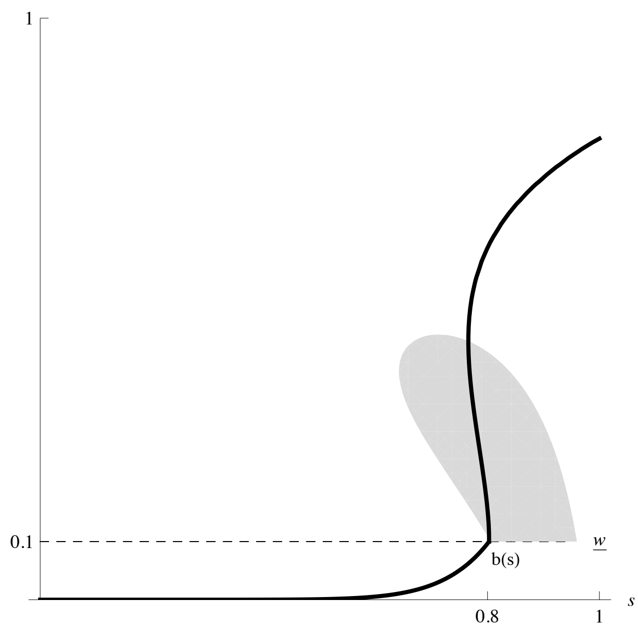

Figure 2: Failure of Assumption A4. The gray region denotes the set $\{(s,w)\colon \xi(s,w|s) < 0\}$. Elsewhere, $\xi(s,w|s) \geq 0$. $\xi(s,\cdot|s)$ crosses zero multiple times at values of s slightly less than 0.8.

Definition 1. A signal S_0 is said to be *value-relevant* if for a.e. $(s_i, \mathbf{s}_{-i})$, $u(s_0, s_i, \mathbf{s}_{-i})$ is strictly increasing in s_0. S_0 is said to be *value-irrelevant* if for all $(s_i, \mathbf{s}_{-i})$, $u(s_0, s_i, \mathbf{s}_{-i})$ is constant in s_0.

Definition 2. A signal S_0 is said to be *information-relevant* if for all $\bar{s}_0 \neq \underline{s}_0 \implies h(\cdot|s_i, \bar{s}_0) \neq h(\cdot|s_i, \underline{s}_0)$ for all s_i.

A signal that is value-relevant conveys information about the value of the item directly; its realized value is effectively a parameter of the bidder's utility function. An information-relevant signal is correlated with other bidders' private information. Therefore, it conveys additional information about others' signals beyond the information contained already in S_i. While nothing precludes a signal from being both value- and information-relevant—indeed, we consider this to be the norm—we will focus only on extreme cases where public signals are either value- or information-relevant, but not both. This dichotomy allows us to characterize the competing effects of information in the all-pay auction. Signals that are purely value-relevant encourage bidders to respond in the intuitive manner—"good news" will encourage uniformly more aggressive bidding. In contrast, high realizations of signals that are solely information-relevant are a discouragement. Some types of bidders place lower bids.

Theorem 2. *Suppose the conditions of Theorem 1 are satisfied. Let $\bar{s}_0 > \underline{s}_0$ be realizations of a public signal S_0 observable to all bidders. Let $\bar{\beta}(s,w)$ ($\underline{\beta}(s,w)$) be the equilibrium strategy in the all-pay auction when the public signal is high (low).*

1. *If the public signal is value-relevant but* $h(\cdot|s_i, \bar{s}_0) = h(\cdot|s_i, \underline{s}_0)$ *for all* s_i*, then* $\bar{\beta}(s,w) \geq \underline{\beta}(s,w)$.
2. *If the public signal is value-irrelevant but* $h(\cdot|s_i, \bar{s}_0) \neq h(\cdot|s_i, \underline{s}_0)$ *for all* s_i*, then there exists an* $\hat{s} > 0$ *such that for all* $0 < s < \hat{s}$, $\bar{\beta}(s,w) \leq \underline{\beta}(s,w)$.

Consider first the case of purely value-relevant information. Noting the preceding discussion, and viewing s_0 as a parameter entering u it is clear that our equilibrium characterization remains the same with statements conditional on s_0 replacing the unconditional statements. An implicit assumption, of course, is that changes in s_0 are sufficiently small to ensure that we maintain an equilibrium of the form $\min\{b(s), w\}$. The associated comparative static is intuitive.

In turning to information-relevant signals, we observe a different reaction. This conclusion is independent of the presence of budget constraints per se but is instead a general feature of the all-pay auction. The intuition is straightforward. Conditional on observing a high public signal $\bar{s}_0$ bidder i can infer that her opponent likely has a high signal and will in consequence bid high. A high bid by the opponent decreases the probability with which bidder i wins the auction, discouraging her from bidding aggressively (recall, in an all-pay auction she must pay her bid irrespective of the outcome). In contrast, if the public signal also has a direct effect on a bidder's value for the item, the resulting boost in expected payoff may be enough to counteract this discouragement effect.

3 The War of Attrition

Given that the first-price, second-price, and all-pay auctions have symmetric equilibria of the form $\beta(s,w) = \min\{b(s), w\}$, a natural conjecture is that the war of attrition—the second-price, all-pay auction—also has an equilibrium in this class. In this section we extend our baseline model to accommodate this auction format as well. Many of the qualitative features of the all-pay auction's equilibrium find natural analogues in the war of attrition. The major distinction is that under a very mild technical condition the war of attrition features a uniform amplification of bids following the introduction of budget constraints among high-budget bidders. In the all-pay auction, such an amplification was present generally only for a subset of types with intermediate value-signals (see Example 1). The section concludes by noting the prospects for a revenue ranking between the all-pay auction and the war of attrition. Generally, such a ranking is not possible if both budget constraints and affiliated interdependent values are present.

We maintain our assumptions concerning the environment from Section 1. Again, bidders will simultaneously submit bids and the highest bidder will be deemed the winner. The winning bidder will make a payment equal to the second-highest bid. All losing bidders continue to incur a cost equal to their bid. Our static treatment of the war of attrition mirrors the treatment in Krishna & Morgan (1997). Therefore, we do not model the war of attrition as an extensive game where bidders sequentially submit additional (incremental) bids. Leininger (1991) and Dekel *et al.* (2006)

consider such models with budget limits and perfect information. Extending our analysis in this direction would introduce many interesting complications such as the role of jump bidding in signaling valuations and budget levels.

As the derivation of the equilibrium strategy in the war of attrition parallels that from the all-pay auction, we abbreviate our discussion accordingly. First, recall that under suitable assumptions Krishna & Morgan (1997) show that the war of attrition without budget constraints has a equilibrium strategy of

$$\omega(s) = \int_0^s \frac{v_{N-1}(y,y) f_{N-1}(y|y)}{1 - F_{N-1}(y|y)} dy. \tag{11}$$

Let $\tilde{\sigma}$ be defined as the unique value where $\underline{w} = \omega(\tilde{\sigma})$. Since $\lim_{s \to 1} \omega(s) = \infty$ and noting the similarity with the all-pay auction, we will identify an equilibrium in the war of attrition with budget constrains which assumes the form $\beta(s,w) = \min\{b(s), w\}$ where

$$b(s) = \begin{cases} \omega(w) & \text{if } s < \tilde{\sigma} \\ \hat{b}(s) & \text{if } s \in [\tilde{\sigma}, \hat{\sigma}] \\ \bar{w} & \text{if } s > \hat{\sigma} \end{cases}. \tag{12}$$

$\hat{b}(s)$ will be defined as a solution to a differential equation while at $\hat{\sigma}$, $\lim_{s \to \hat{\sigma}^-} \hat{b}(s) = \bar{w}$. To derive an expression for $\hat{b}(s)$ we first define

$$F_k(x|s) = \underbrace{\int_0^1 \cdots \int_0^1}_{N-1-k} \underbrace{\int_0^x \cdots \int_0^x}_{k} h(y_1, \ldots, y_{N-1}|s) dy_1 \cdots dy_{N-1}$$

and we let

$$\hat{H}(x, \hat{b}(x)|s) = \sum_{k=0}^{N-1} \gamma_k(\hat{b}(x)) F_k(x|s).$$

If all other bidders are following a bidding strategy as defined in (12), $\hat{H}(x, \hat{b}(x)|s)$ is the probability that all bidders $j \neq i$ are of a type (s_j, w_j) such that $\beta(s_j, w_j) < \hat{b}(x)$. We can thus write the expected utility to bidder i of type (s, w) from the bid $\hat{b}(x)$, as

$$U(\hat{b}(x)|s,w) = \sum_{k=0}^{N-1} \gamma_k(\hat{b}(x)) z_k(x|s) - (1 - \hat{H}(x, \hat{b}(x)|s)) b(x)$$
$$- \int_0^{\tilde{\sigma}} \omega(y) f_{N-1}(y|s) dy - \int_{\tilde{\sigma}}^x \hat{b}(y) \left. \frac{d}{dz} \hat{H}(z, \hat{b}(z)|s) \right|_{z=y} dy$$

The first term is the expected benefit of winning the auction. The second term is the payment the bidder must make if she loses the auction. This is her own bid. The third and fourth terms account for the payment she makes when she wins the auction. Computing $\frac{d}{dx} U(b(x)|s,w)\big|_{x=s} = 0$ leads to the following differential equation:

$$\hat{b}'(s) = \frac{\sum_{k=0}^{N-1} \gamma_k(\hat{b}(s)) z_k'(s|s)}{1 - \hat{H}(s, \hat{b}(s)|s) - \sum_{k=0}^{N-1} \gamma_k'(\hat{b}(s)) z_k(s|s)}. \tag{13}$$

Again, we propose two assumptions—alternatives for Assumptions A3 and A4—which will be sufficient to ensure that (12), where $\hat{b}(s)$ is defined by (13), is indeed an equilibrium strategy profile. The first assumption again places a limit on the relative degree of affiliation and generalizes a condition proposed by Krishna & Morgan (1997). The second assumption is the analogue of (A4) and also structures the joint distribution of value-signals and budgets.

Assumption A5. *Let*

$$\Phi(x, w|s) = \sum_{k=1}^{N-1} \frac{\gamma_k(w)(1 - F_k(x|s))}{\sum_{k'=1}^{N-1} \gamma_{k'}(w)(1 - F_{k'}(x|s))} \left(\frac{v_k(s, x) f_k(x|s)}{(1 - F_k(x|s))} \right) \tag{14}$$

$\Phi(x, w|\cdot)\colon [0, 1] \to \mathbb{R}$ *is nondecreasing.*

Like Assumption A3, Assumption A5 is a restriction on the degree of affiliation among value-signals. It always holds if value-signals are independent. For $w < \underline{w}$, (14) reduces to $\frac{f_{N-1}(x|\cdot)}{1-F_{N-1}(x|\cdot)}$ being nondecreasing. A similar simplification occurs when $N = 2$. As with Assumption A3, we stated A5 as a weighted average emphasizing the interaction between the number of bidders and the limit on affiliation that needs to hold for subsets of signals.

Assumption A6. *Let*

$$\Xi(x, w|s) = 1 - \frac{\sum_{k=0}^{N-1} \gamma_k'(w) z_k(x|s)}{\sum_{k=1}^{N-1} \gamma_k(w)(1 - F_k(x|s))} \tag{15}$$

$\Xi(x, w|s)$ *satisfies the following properties:*

1. *For every $s \geq \tilde{\sigma}$, there exists w_s, $\underline{w} \leq w_s < \bar{w}$ such that $w < w_s \implies \Xi(s, w|s) < 0$ and $w \in (w_s, \bar{w}) \implies \Xi(s, w|s) > 0$.*

2. *There exists $\epsilon > 0$ such that for all $s \in (\tilde{\sigma} - \epsilon, \tilde{\sigma} + \epsilon)$, $\Xi(s, \underline{w}|s) > 0$.*

3. *When $x \geq \tilde{\sigma}$, $\Xi(x, w|\cdot)\colon [0, 1] \to \mathbb{R}$ is non-increasing.*

The conditions in Assumption A6 are direct adaptations of the conditions presented in Assumption A4. Their interpretation and roles are also analogous. Some simplifications of Assumption A6 are possible in special cases of interest. For example, Assumption A6(3) is satisfied automatically if value-signals are independent. A similar conclusion holds if there are only two bidders. In this case Assumption A1 implies Assumption A6(3). To see this, note that when $N = 2$,

$$\begin{aligned}\Xi(x, w|s) &= 1 - \frac{g(w)}{1 - G(w)} \frac{\int_x^1 v_1(s, y) f_1(y|s) dy}{1 - F_1(x|s)} \\ &= 1 - \frac{g(w)}{1 - G(w)} \mathbb{E}[u(s, Y_1)|S = s, Y_1 \geq x].\end{aligned}$$

Since (S, Y_1) are affiliated, $\mathbb{E}[u(s, Y_1)|S = s, Y_1 \geq x]$ is nondecreasing in s. Therefore, $\Xi(x, w|\cdot)$ is non-increasing as needed.

Turning to the equilibrium in the war of attrition, Theorem 3 collects the preceding assumptions and offers sufficient conditions for an equilibrium of the form $\min\{b(s), w\}$. The equilibrium strategy resembles the equilibrium of the all-pay auction. Low value-signal bidders will follow the usual no-budget-constraints equilibrium strategy. Only for bids above $\underline{w}$ will the change in marginal incentives introduced by budget constraints modify equilibrium behavior. Unlike the all-pay auction, bidders with sufficiently large value-signals will desire to expend an arbitrarily large (but feasible) amount in equilibrium. If $\bar{w} = \infty$, then the equilibrium strategy is unbounded, like in the case of no budget constraints.

Theorem 3. *Suppose Assumptions A1, A2, A5, and A6 hold. Then there exists a symmetric equilibrium in the war of attrition where all bidders follow the strategy $\beta(s, w) = \min\{b(s), w\}$. The function $b(s)\colon [0, 1) \to [0, \bar{w}]$ is defined as:*

$$b(s) = \begin{cases} \omega(s) & \text{if } s < \tilde{\sigma} \\ \hat{b}(s) & \text{if } s \in [\tilde{\sigma}, \hat{\sigma}] \\ \bar{w} & \text{if } s > \hat{\sigma} \end{cases}.$$

where $\omega(s)$ is defined in (11)*, $\tilde{\sigma}$ solves $\underline{w} = \omega(\tilde{\sigma})$, $\hat{b}(s)$ is the solution to the differential equation* (13) *satisfying the boundary condition $\hat{b}(\tilde{\sigma}) = \underline{w}$ and $\hat{\sigma}$ is such that $\lim_{s \to \hat{\sigma}^-} \hat{b}(s) = \bar{w}$.*

The following example illustrates an equilibrium strategy in the war of attrition. In many examples, closed-form expressions for equilibrium strategies are available.

Example 3. Suppose $N = 2$ and that value-signals $S_i \overset{i.i.d.}{\sim} U[0, 1]$. Let $u(s_i, s_j) = (s_i + s_j)/2$ and suppose budgets follow the cumulative distribution $G(w) = 1 - e^{-(w - \underline{w})}$ on $[\underline{w}, \infty)$. Choose $\underline{w} = -\frac{7}{20} + \log\left(\frac{20}{13}\right) \approx 0.081$. With these parameters, our equilibrium strategy in the war of attrition is $\beta(s, w) = \min\{b(s), w\}$ where

$$b(s) = \begin{cases} -s - \log(1 - s) & \text{if } s \leq \frac{7}{20} \\ \int_{\frac{7}{20}}^{s} \frac{4y}{3(y-1)^2} dy + \underline{w} & \text{if } s > \frac{7}{20} \end{cases}$$

For $s > 7/20$, we can integrate the above expression to see that

$$b(s) = \frac{1040(s-1)\log(1-s) + 1820(s-1)\log\left(\frac{20}{13}\right) - 1873s + 833}{780(s-1)}.$$

For comparison, Figure 3 presents the functions $b(s)$ and $\omega(s) = -s - \log(1-s)$.

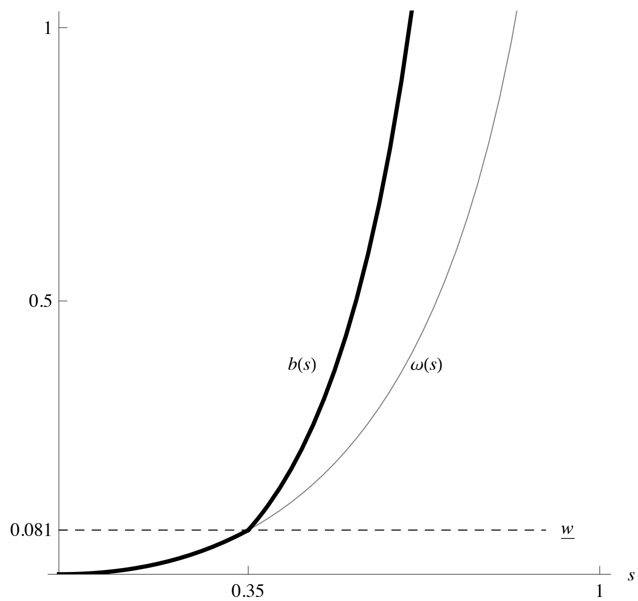

Figure 3: The functions $b(s)$, solid black, and $-s - \log(1-s)$, gray, in the characterization of equilibrium bidding in Example 3. Both functions are not bounded.

As seen in Example 3, bidders with a value-signal of only 0.65 desire to commit to a bid greater than 1, which is the maximum possible value of the available prize. Such "overbidding" is a particular feature of the war of attrition (Albano, 2001). The effect of budget constraints is to amplify this phenomenon further. The following corollary formalizes this observation.

Corollary 2. *Under the conditions of Theorem 3:*

1. $\lim_{s\to\tilde{\sigma}^+} b'(s) > \lim_{s\to\tilde{\sigma}^-} b'(s) = \lim_{s\to\tilde{\sigma}^-} \omega'(s)$.
2. *If* $\frac{f_k(s|s)}{1-F_k(s|s)} \geq \frac{f_{N-1}(s|s)}{1-F_{N-1}(s|s)}$ *for all* k,[9] *then for all* s, $b(s) \geq \omega(s)$.

The equilibrium in the war of attrition exhibits similar comparative statics to the all-pay auction. Again, the equilibrium strategy identified here will converge to the equilibrium in an environment

[9]For example, this condition is satisfied when $S_i \overset{i.i.d.}{\sim} U[0,1]$.

without budget constraints if the constraints are relaxed. Additionally, the same bidder-level comparative statics apply concerning information revelation. The distinction between value-relevant and information-relevant public signals continues to be important in appreciating a bidder's equilibrium reaction to public information.

Theorem 4. *Suppose the conditions of Theorem 3 are satisfied. Then the conclusions of Theorem 2 continue to apply in the context of the war of attrition.*

3.1 Comparing of the All-Pay Auction and the War of Attrition

We conclude our investigation with a brief comparison of the two auction formats. Naturally, we restrict attention to environments where the all-pay auction has an equilibrium of the form $\beta_\alpha(s,w) = \min\{b_\alpha(s), w\}$ and the war of attrition has an equilibrium of the form $\beta_\omega(s,w) = \min\{b_\omega(s), w\}$. Both β_α and β_ω are assumed to exhibit the characteristics identified in our preceding analysis. Our first comparison considers an ordering of the bidding strategies.

Theorem 5. $\beta_\omega(s,w) \geq \beta_\alpha(s,w)$ *for all* (s,w).

Noting Theorem 5, we can employ the arguments in Che & Gale (1998) and also outlined in Krishna (2002) to conclude that the all-pay auction will be more efficient on average than the war of attrition when preferences are reflective of the ordering of bidder's value-signals.

We close with a discussion of revenue comparisons between the two formats. There does not exist a general revenue ranking between the war of attrition and the all pay auction in the presence of budget constraints and affiliated valuations. One can draw this conclusion by documenting the results in extreme cases. First, suppose that budget constraints are very lax. For example, suppose budgets are distributed according to the exponential distribution with a mean that is very large. Since valuations are bounded the equilibrium bids submitted in both formats are essentially those submitted in the case of a no-budget constraints situation. In this case, it is known that in the presence of value interdependence the war of attrition will revenue-dominate the all-pay auction (Krishna & Morgan, 1997).

When budget constraints are more meaningful, and they constrain bidders with non-vanishing probability, the all-pay auction can generate more revenue. Consider the following case. Suppose there are two bidders and value signals are distributed independently according to the uniform distribution. Suppose budget constraints follow the exponential distribution $G(w) = 1 - e^{-(w-\underline{w})}$ on $[\underline{w}, \infty)$. Choose $\underline{w} = \log(\frac{10}{3}) - \frac{7}{10} \approx 0.5039$. Finally, assume bidders have private values: $u_i(s_i, s_j) = s_i$.

In this situation, budget constraints are (just) irrelevant in the case of the all-pay auction. The equilibrium strategy is $\beta_\alpha(s,w) = \min\{b_\alpha(s), w\}$ where $b_\alpha(s) = \frac{s^2}{2}$. The expected revenue in the all-pay auction is $R_\alpha = \frac{1}{3}$.

Since the bidding strategy in the war of attrition is not bounded, the introduced budget constraints will directly affect the equilibrium strategy. It is straightforward to show that the equilibrium bidding strategy is $\beta_\omega(s,w) = \min\{b_\omega(s), w\}$ where

$$b_\omega(s) = \begin{cases} -s - \log(1-s) & \text{if } s < \frac{7}{10} \\ \int_{\frac{7}{10}}^{s} \frac{y}{(y-1)^2} dy + \log(\frac{10}{3}) - \frac{7}{10} & \text{if } s \geq \frac{7}{10} \end{cases}$$

When $s > \frac{7}{10}$, we can write $b_\omega(s)$ in closed form as

$$b_\omega(s) = \frac{s\left(\log\left(\frac{1000}{27}\right) - 10\right) + 7 + \log(27) - 3\log(10)}{3(s-1)} + \log\left(\frac{10}{3} - \frac{10}{3}s\right) - \frac{7}{10}.$$

A direct calculation for the revenue in this case (see the online appendix) gives

$$R_\omega = \frac{1}{1500}\left(527 - 270e^{\frac{20}{3}} \int_{\frac{20}{3}}^{\infty} \frac{e^{-x}}{x} dx\right).$$

The terms in R_ω are straightforward to approximate accurately to conclude that $R_\omega < 0.328$. In this example the impact on revenue following the introduction of budget constraints is very slight for two reasons. First, only a fraction of bidders in the war of attrition are somehow directly impacted by the budget constraint. Additionally, those who are impacted adjust their bidding upward (Corollary 2) which partially ameliorates the revenue decline. This adjustment is not sufficient to preclude a strict revenue decline.

While tractability has guided our discussion of revenues towards comparisons of extreme scenarios, its conclusions apply more generally. It is clear that we can modify our final example by perturbing the distribution of budgets slightly such that it has full support on $[0,\infty)$ without changing the conclusion. Similarly, one can perturb the distribution of value-signals such that they are strictly but "slightly" affiliated while maintaining the strict difference in expected revenues. For example, consider the distribution $h(s_1, s_2) \propto K + s_i s_j$ on $[0,1]^2$ and let K be very large. Finally, one can introduce strict value interdependence by endowing bidders with the preferences $u(s_i, s_j) = (1-\epsilon)s_i + \epsilon s_j$. As we have shown, the boundary of the revenue dominance of one auction format over the other will lie somewhere in between the two extreme cases considered. We hope to explore this boundary further, along with its implications for auction and contest design, in future research.

A Proofs from Section 2 (All-Pay Auction)

Lemma A.1. *Let $G \equiv G(w)$, $g \equiv g(w)$, $z_k \equiv z_k(x|s)$, and $\gamma_k \equiv \gamma_k(w)$. Then,* $\sum_{k=0}^{N-1} \gamma_k' z_k = \sum_{k=0}^{N-2} g(N-1)\binom{N-2}{k} G^{N-2-k}(1-G)^k(z_k - z_{k+1})$.

Proof. Differentiating $\gamma_k(w)$ gives

$$\gamma_k' = \binom{N-1}{k}(N-1-k)G^{N-2-k}g(1-G)^k - \binom{N-1}{k}G^{N-1-k}gk(1-G)^{k-1}.$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{N-1} \gamma_k' z_k = &\sum_{k=0}^{N-2} \binom{N-1}{k}(N-1-k)G^{N-2-k}g(1-G)^k z_k \\ &- \sum_{k=1}^{N-1} \binom{N-1}{k} kG^{N-1-k}g(1-G)^{k-1} z_k.\end{aligned}$$

For $k \leq N-2$, $\binom{N-1}{k}(N-1-k) = \frac{(N-1)!}{(N-1-k)!k!}(N-1-k) = (N-1)\frac{(N-2)!}{(N-2-k)!k!} = (N-1)\binom{N-2}{k}$ and for $k \geq 1$, $\binom{N-1}{k}k = \frac{(N-1)!}{(N-1-k)!(k-1)!} = (N-1)\frac{(N-2)!}{(N-2-(k-1))!(k-1)!} = (N-1)\binom{N-2}{k-1}$. Shifting the index of summation we see that

$$(N-1)g\sum_{k=1}^{N-1}\binom{N-2}{k-1}G^{N-1-k}(1-G)^{k-1}z_k = (N-1)g\sum_{k=0}^{N-2}\binom{N-2}{k}G^{N-2-k}(1-G)^k z_{k+1}.$$

Hence,

$$\begin{aligned}\sum_{k=0}^{N-1} \gamma_k' z_k = &\, g(N-1)\sum_{k=0}^{N-2}\binom{N-2}{k}G^{N-2-k}g(1-G)^k z_k \\ &- g(N-1)\sum_{k=0}^{N-2}\binom{N-2}{k}G^{N-2-k}(1-G)^k z_{k+1} \\ = &\sum_{k=1}^{N-1} g(N-1)\binom{N-2}{k-1}G^{N-1-k}(1-G)^{k-1}(z_{k-1} - z_k) \\ = &\sum_{k=0}^{N-2} g(N-1)\binom{N-2}{k}G^{N-2-k}(1-G)^k(z_k - z_{k+1})\end{aligned}$$

which is the desired conclusion. □

Lemma A.2. *Let $k \geq 1$ and adopt the notation $\mathbf{y} = (y_1, \ldots, y_{N-1})$. Then for all $k \geq 1$,*

1. $z_k(x|s) = \int_0^x v_k(s,x) f_k(x|s) dx = \underbrace{\int_0^1 \cdots \int_0^1}_{N-1-k} \underbrace{\int_0^x \cdots \int_0^x}_{k} u(s,\mathbf{y}) h(\mathbf{y}|s) dy_1 \cdots dy_{N-1}.$

2. $z_k(x|s) - z_{k+1}(x|s) \geq 0$. *Therefore,* $\sum_{k=0}^{N-1} \gamma_k'(w) z_k(x|s) \geq 0$.

Proof. To prove part 1 we work from the definition of $z_k(x|s)$:

$$\begin{aligned} z_k(x|s) &= \int_0^x v_k(s,y) f_k(y|s) dy \\ &= \int_0^x \mathbb{E}[u(S,Y_1,\ldots Y_N)|S=s,\bar{Y}_k=y] f_k(y|s) dy \\ &= \Pr[\bar{Y}_k \leq x|S=s]\mathbb{E}[u(S,Y_1,\ldots Y_N)|S=s,\bar{Y}_k \leq x] \\ &= \underbrace{\int_0^1 \cdots \int_0^1}_{N-1-k} \underbrace{\int_0^x \cdots \int_0^x}_{k} u(s,\mathbf{y}) h(\mathbf{y}|s) dy_1 \cdots dy_{N-1}. \end{aligned}$$

Part 2 is an immediate consequence of the above since (noting symmetry)

$$z_k(x|s) - z_{k+1}(x|s) = \int_x^1 \left(\underbrace{\int_0^1 \cdots \int_0^1}_{N-1-(k+1)} \underbrace{\int_0^x \cdots \int_0^x}_{k} u(s,\mathbf{y}) h(\mathbf{y}|s) dy_1 \cdots dy_{N-2} \right) dy_{N-1}.$$

Whenever $0 < x < 1$, the preceding expression is strictly positive. □

Lemma A.3. *The differential equation*

$$b'(s) = \frac{\sum_{k=0}^{N-1} \gamma_k(b(s)) z_k'(s|s)}{1 - \sum_{k=0}^{N-1} \gamma_k'(b(s)) z_k(s|s)} \tag{16}$$

has a strictly increasing solution $b(s)\colon [\tilde{s},1] \to [\underline{w},\bar{w}]$ *satisfying the boundary condition* $b(\tilde{s}) = \underline{w}$.

Proof. Since all of the terms in (16) are continuous, this lemma follows from a simple application of standard results in the theory of ordinary differential equations. It is readily verified that $b'(s)$ as defined is strictly positive for all $s \in [\tilde{s},1]$ due to Assumption A4. The basic intuition is presented in Figure 4. $\{(s,b)\colon \xi(s,b|s) = 0\}$, denoted by dashed curves, is a set of points where $b'(s)$ is not defined and solutions to the differential equation (16) approach it vertically. In grey regions, all solutions to (16) are (strictly) downward sloping while in the white region $b'(s) > 0$.

We need to however verify that the solution meeting the boundary condition $b(\tilde{s}) = \underline{w}$ is indeed defined for all $s \in [\tilde{s},1]$. The sole alternative is that for some $\tilde{s}' < 1$, $\lim_{s\to\tilde{s}'-} b(s) = \bar{w}$. This possibility is ruled out, however, by noting that the function $\tilde{b}(s) = \bar{w}$ is a solution to (16) satisfying the boundary condition $\tilde{b}(\tilde{s}) = \bar{w}$. Thus, $b(s)$ is bounded above by $\bar{w}$ and indeed the solution $b(s)$ has a maximal domain of $[\tilde{s},1]$ as required. □

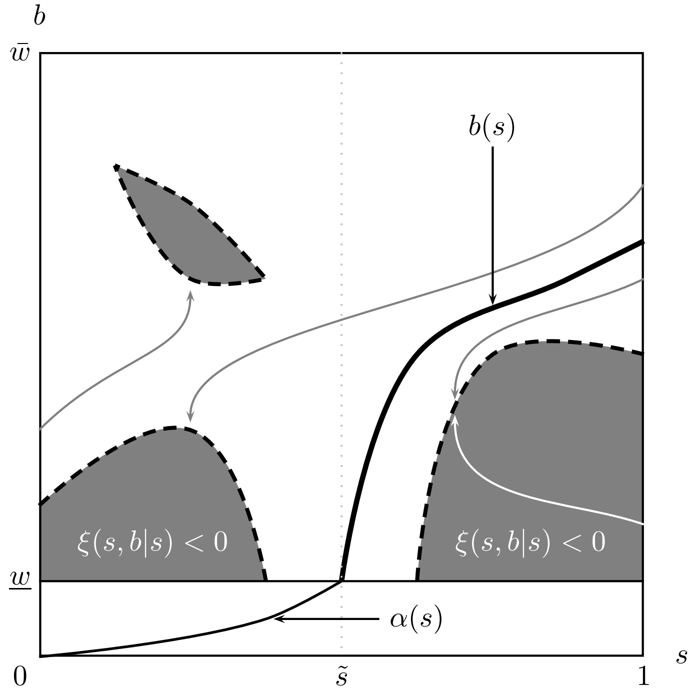

Figure 4: Definition of $b(s)$ in the all-pay auction. From Assumption A4, it follows that $b(s)$ is contained in the white region where all solutions to the differential equation (16) are strictly increasing.

Remark A.1. An alternative argument, contained in an earlier working paper of this study, analyzed instead the solution of the differential equation $q(b) = \psi(b, q(b))$ where

$$\psi(b, q(b)) = \frac{1 - \sum_{k=0}^{N-1} \gamma_k'(b) z_k(q(b)|q(b))}{\sum_{k=0}^{N-1} \gamma_k(b) z_k'(q(b)|q(b))}, \qquad q(\underline{w}) = \tilde{s}.$$

$b(s)$ is then defined as the inverse of $q(b)$. Analyzing the inverse of the main differential equation in auction models has a long history (see for example Lebrun (1999), Maskin & Riley (2003), among others) and assuages concerns related to the set of points where $\xi(s, b|s) = 0$.

Proof of Theorem 1. Let $U(b(x)|s, w)$ be the expected utility of bidder i when he has value-signal s and places the bid $b(x) \leq w$. Noting the definition of $b(x)$, it is easily seen to be absolutely continuous. Moreover, since $G(\cdot)$ is continuously differentiable, $U(b(x)|s, w) = \sum_{k=0}^{N-1} \gamma_k(b(x)) z_k(x|s) - b(x)$

is absolutely continuous[10] and in particular we can write

$$U(b(x)|s,w) = U(b(\tilde{s})|s,w) + \int_{\tilde{s}}^{x} \frac{d}{dt}U(b(t)|s,w)\bigg|_{t=y} dy. \tag{17}$$

We proceed to verify that no type of bidder wishes to deviate to an alternative (feasible) bid.

1. Consider a bidder with value-signal $s < \tilde{s}$. When following the strategy $\beta(s,w)$, this bidder places the bid $b(s) = \alpha(s)$. From Krishna & Morgan (1997), we know that this bidder will not have a profitable deviation to any bid $b(x)$, $x \in [0, \tilde{s}]$. In particular, $U(\beta(s,w)|s,w) \geq U(b(\tilde{s})|s,w)$.

   Suppose instead that this bidder contemplates bidding $b(x)$ for some $x > \tilde{s}$. The expected payoff from this bid is given by (17). It is sufficient to verify that when $s < x$, $\frac{d}{dt}U(b(t)|s,w) \leq 0$. Using Assumptions A3 and A4, we can see that

$$\begin{aligned}
&\frac{d}{dt}U(b(t)|s,w) \\
&= \sum_{k=0}^{N-1} \gamma_k'(b(t))z_k(t|s)b'(t) + \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|s) - b'(t) \\
&= \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|s) - \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|t) \left[\frac{1-\sum_k \gamma_k'(b(t))z_k(t|s)}{1-\sum_k \gamma_k'(b(t))z_k(t|t)}\right] \\
&\leq \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|t) - \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|t) \left[\frac{1-\sum_k \gamma_k'(b(t))z_k(t|t)}{1-\sum_k \gamma_k'(b(t))z_k(t|t)}\right] = 0
\end{aligned}$$

   Hence, deviating to a bid $b(x)$ for $x > s$ is not profitable.

2. Consider instead a bidder with a value-signal $s \geq \tilde{s}$. An argument exactly parallel to the preceding case confirms that a bid $b(x)$, $x > s$, will not be profitable. Suppose instead this bidder bids $b(x) \leq w$ such that $x \in [\tilde{s}, s)$. As above, we have

$$\frac{d}{dt}U(b(t)|s,w) = \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|s) - \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|t) \left[\frac{1-\sum_k \gamma_k'(b(t))z_k(t|s)}{1-\sum_k \gamma_k'(b(t))z_k(t|t)}\right].$$

   By Assumption A3, $\sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|s) \geq \sum_{k=0}^{N-1} \gamma_k(b(t))z_k'(t|t)$. There are two cases:

   (a) If $1 - \sum_k \gamma_k'(b(t))z_k(t|s) < 0$, then we conclude immediately that $\frac{d}{dt}U(b(t)|s,w) \geq 0$.

   (b) Suppose $1 - \sum_k \gamma_k'(b(t))z_k(t|s) \geq 0$. Then by Assumption A4, $1 - \sum_k \gamma_k'(b(t))z_k(t|s) \leq 1 - \sum_k \gamma_k'(b(t))z_k(t|t)$ and so $\frac{d}{dt}U(b(t)|s,w) \geq 0$.

[10] Generally, the composition of absolutely continuous functions need not be absolutely continuous. However, since $G(\cdot)$ is continuously differentiable (hence, Lipschitz) this conclusion holds.

Therefore, there is no profitable deviation to a bid $b(x)$ when $x \in [\tilde{s}, s)$.

Finally, consider a deviation to a bid of $b(x)$ for $x < \tilde{s}$. It is sufficient to show that $\frac{d}{dt}U(b(t)|s,w) \geq 0$ for all $t < \tilde{s}$. Again, Assumption A3 gives

$$\begin{aligned}\frac{d}{dt}U(b(t)|s,w) &= v_{N-1}(s,t)f_{N-1}(t|s) - v_{N-1}(t,t)f_{N-1}(t|t)\\ &\geq v_{N-1}(t,t)f_{N-1}(t|t) - v_{N-1}(t,t)f_{N-1}(t|t) = 0.\end{aligned}$$

The above analysis is exhaustive of all the cases; thus, $\beta(s,w)$ is a symmetric equilibrium strategy.
□

Proof of Corollary 1. By Assumption A4, $1 - \sum_{k=0}^{N-1} \gamma_k'(\underline{w})z_k(\tilde{s}|\tilde{s}) > 0$. However, from Lemmas A.1 and A.2, $\sum_{k=0}^{N-1} \gamma_k'(\underline{w})z_k(\tilde{s}|\tilde{s}) > 0$. Therefore,

$$0 < 1 - \sum_{k=0}^{N-1} \gamma_k'(\underline{w})z_k(\tilde{s}|\tilde{s}) < 1.$$

Also, $\sum_{k=0}^{N-1} \gamma_k(b(s))z_k'(s|s) = \sum_{k=1}^{N-1} \gamma_k(b(s))v_k(s,s)f_k(s|s)$. Since $\gamma_k(\underline{w}) = 0$ for all $k \neq N-1$ and $\gamma_{N-1}(\underline{w}) = 1$, we can take limits to conclude

$$\begin{aligned}\lim_{s\to\tilde{s}^+} b'(s) &= \lim_{s\to\tilde{s}^+} \frac{\sum_{k=0}^{N-1} \gamma_k(b(s))z_k'(s|s)}{1 - \sum_{k=0}^{N-1} \gamma_k'(b(s))z_k(s|s)}\\ &= \frac{\sum_{k=1}^{N-1} \gamma_k(\underline{w})v_k(\tilde{s},\tilde{s})f_k(\tilde{s}|\tilde{s})}{1 - \sum_{k=0}^{N-1} \gamma_k'(\underline{w})z_k(\tilde{s}|\tilde{s})}\\ &= \frac{v_{N-1}(\tilde{s},\tilde{s})f_{N-1}(\tilde{s}|\tilde{s})}{1 - \sum_{k=0}^{N-1} \gamma_k'(\underline{w})z_k(\tilde{s}|\tilde{s})}\\ &> v_{N-1}(\tilde{s},\tilde{s})f_{N-1}(\tilde{s}|\tilde{s}) = \lim_{s\to\tilde{s}^-} b'(s) = \lim_{s\to\tilde{s}^-} \alpha'(s)\end{aligned}$$

□

The following lemma is used in the proof of Theorems 2 and 4 below.

Lemma A.4. *Suppose (X, Y, Z) are affiliated random variables with a strictly positive, bounded, continuous density $f(x, y, z)$ defined on $[0, 1]^3$. Define*

$$f(x|y,z) = \frac{f(x,y,z)}{\int_0^1 f(x,y,z)dx}.$$

Let $z' > z$ and suppose that $f(\cdot|y,z) \neq f(\cdot|y,z')$ for all y. Then there exists $0 < \hat{y} < 1$ such that:

(a) $f(x|x,z') < f(x|x,z)$ for all $x < \hat{y}$.

(b) $\frac{f(x|x,z')}{1-F(x|x,z')} < \frac{f(x|x,z)}{1-F(x|x,z)}$ *for all* $x < \hat{y}$.

Proof. Let $z' > z$ and fix y. Since $(y, z') \geq (y, z)$, by the properties of affiliated random variables (see Milgrom & Weber (1982) or Krishna (2002)) the function

$$\frac{f(\cdot|y,z')}{f(\cdot|y,z)} \colon [0,1] \to \mathbb{R}$$

is nondecreasing. It is also continuous and strictly positive.

Suppose $f(0|y,z') \geq f(0|y,z)$. Then $f(x|y,z') \geq f(x|y,z)$ for all $x \in [0,1]$. Since $f(\cdot|y,z) \neq f(\cdot|y,z')$ there exist an open set $\mathcal{X} \subset [0,1]$ such that for all $x \in \mathcal{X}$, $f(x|y,z) > f(x|y,z')$. But this implies $1 = \int_0^1 f(x|y,z)dx < \int_0^1 f(x|y,z')dx$ which is a contradiction. Therefore $f(0|y,z') < f(0|y,z)$. Specifically, the above conclusion holds when $y = 0$: $f(0|0,z') < f(0|0,z)$. Noting that $f(\cdot|\cdot,z')$ and $f(\cdot|\cdot,z)$ are continuous functions, there exists $\hat{y} > 0$ such that for all $0 < x < \hat{y}$, $f(x|x,z') < f(x|x,z)$ as desired.

To derive the second conclusion, let $x < \hat{y}$. Then for all $\tilde{x} \leq x$,

$$1 \geq \frac{f(x|x,z')}{f(x|x,z)} \geq \frac{f(\tilde{x}|x,z')}{f(\tilde{x}|x,z)}.$$

Thus, $F(x|x,z') = \int_0^x f(\tilde{x}|x,z')d\tilde{x} \leq \int_0^x f(\tilde{x}|x,z)d\tilde{x} = F(x|x,z)$. Hence,

$$\frac{1}{1-F(x|x,z')} \leq \frac{1}{1-F(x|x,z)}.$$

Combining this observation with the first conclusion gives the second result. □

Proof of Theorem 2. Let $\bar{\beta}(s,w) = \min\{\bar{b}(s), w\}$ and $\underline{\beta}(s,w) = \min\{\underline{b}(s), w\}$ be the equilibrium bidding strategies conditional on the realized public signal. (An implicit assumption is that equilibria in this class of strategies obtain for both signal realizations.) To prove part (1) it is sufficient to verify that $\bar{b}(s) \geq \underline{b}(s)$. Since the value-signal is only value-relevant we can let $\bar{u}$ ($\underline{u}$) be a bidder's utility function when the public signals is high (low). The values $\bar{z}_k$, $\bar{v}_k$, $\underline{z}_k$, and $\underline{v}_k$ are defined in the obvious way. Since $\bar{u} > \underline{u}$ a.e., it follows that $\bar{v}_k > \underline{v}_k$ and $\bar{z}_k > \underline{z}_k$. Moreover, $\bar{z}_k - \bar{z}_{k+1} \geq \underline{z}_k - \underline{z}_{k+1}$, and thus $\sum_{k=0}^{N-1} \gamma_k'(w)\bar{z}_k(s|s) \geq \sum_{k=0}^{N-1} \gamma_k'(w)\underline{z}_k(s|s)$.

Thus, when $\bar{b}(s) \leq \underline{w}$, we have

$$\bar{\alpha}(s) = \bar{b}(s) = \int_0^s \bar{v}_{N-1}(y,y)f_k(y|y)dy > \int_0^s \underline{v}_{N-1}(y,y)f_k(y|y)dy = \underline{b}(s) = \underline{\alpha}(s)$$

Thus, if $\bar{\alpha}(\tilde{\bar{s}}) = \underline{w}$, there exists $s^* > \tilde{\bar{s}}$ such that for all $s \leq s^*$, $\bar{b}(s) \geq \underline{b}(s)$. Suppose that at s^*, $\bar{b}(s^*) = \underline{b}(s^*) = b^*$ and for all $\epsilon > 0$ sufficiently small, $\bar{b}(s^* + \epsilon) < \underline{b}(s^*)$. Since $\bar{b}'(s)$ and $\underline{b}'(s)$ are both continuous functions, we see that

$$\underline{b}'(s^*) = \frac{\sum_{k=1}^{N-1} \gamma_k(b^*)\underline{v}_k(s^*,s^*)f_k(s^*|s^*)}{1-\sum_{k=0}^{N-1}\gamma_k'(b^*)\underline{z}_k(s^*|s^*)}$$
$$< \frac{\sum_{k=1}^{N-1} \gamma_k(b^*)\bar{v}_k(s^*,s^*)f_k(s^*|s^*)}{1-\sum_{k=0}^{N-1}\gamma_k'(b^*)\bar{z}_k(s^*|s^*)} = \bar{b}'(s^*),$$

implying a contradiction. Therefore $\bar{b}(s) \geq \underline{b}(s)$ for all s.

Turning to part (2), we know that given the public signal s_0, $b(s) = \int_0^s v_{N-1}(y,y)f_{N-1}(y|y,s_0)dy$ for all $s > 0$ sufficiently small. Lemma A.4 implies that if $\bar{s}_0 > \underline{s}_0$, then $f_{N-1}(y|y,\bar{s}_0) < f_{N-1}(y|y,\underline{s}_0)$ for all $0 < y$ sufficiently small. Therefore, $\underline{b}(s) \geq \bar{b}(s)$ for all $s > 0$ sufficiently small. □

B Proofs from Section 3 (War of Attrition)

Lemmas B.1 and B.2 are used to prove Theorem 3.

Lemma B.1. *Let*

$$F_k(x|s) = \underbrace{\int_0^1 \cdots \int_0^1}_{N-1-k} \underbrace{\int_0^x \cdots \int_0^x}_{k} h(y_1,\ldots,y_{N-1}|s)dy_1\cdots dy_{N-1}$$

and let $f_k(x|s) = \frac{d}{dx}F_k(x|s)$ *be the associated density. Then for all* $k \geq 1$, $\frac{f_k(s|s)}{1-F_k(s|s)} \geq \left(\frac{k}{k+1}\right)\frac{f_{k+1}(s|s)}{1-F_{k+1}(s|s)}$.

Proof. We note that $F_k(x|s) \geq F_{k+1}(x|s)$. Using the symmetry of $h(\cdot)$, we can compute $f_k(x|s)$ to conclude

$$\frac{f_k(x|s)}{1-F_k(x|s)} = \frac{k\overbrace{\int_0^1 \cdots \int_0^1}^{N-1-k}\overbrace{\int_0^x \cdots \int_0^x}^{k-1} h(y_1,\ldots,y_{N-2},x|s)dy_1\ldots dy_{N-2}}{1-F_k(x|s)}$$
$$\geq \frac{\frac{k}{k+1}(k+1)\overbrace{\int_0^1 \cdots \int_0^1}^{N-2-k}\overbrace{\int_0^x \cdots \int_0^x}^{k} h(y_1,\ldots,y_{N-2},x|s)dy_1\ldots dy_{N-2}}{1-F_{k+1}(x|s)}$$
$$= \left(\frac{k}{k+1}\right)\frac{f_{k+1}(x|s)}{1-F_{k+1}(x|s)}$$

□

Lemma B.2. *The differential equation*

$$\hat{b}'(s) = \frac{\sum_{k=0}^{N-1} \gamma_k(\hat{b}(s)) z_k'(s|s)}{1 - \hat{H}(s, \hat{b}(s)|s) - \sum_{k=0}^{N-1} \gamma_k'(\hat{b}(s)) z_k(s|s)} \tag{18}$$

has a strictly increasing solution $\hat{b}(s)$ *satisfying the boundary condition* $b(\tilde{\sigma}) = \underline{w}$. *Moreover, there exists* $\hat{\sigma} \leq 1$ *such that* $\lim_{s \to \hat{\sigma}^-} \hat{b}(s) = \bar{w}$.

Proof. The existence of a strictly increasing solution follows from the same reasoning as presented in the case of the all-pay auction. In particular, Assumption A6 ensures that $\hat{b}'(s) > 0$ in the relevant range of values. We therefore focus on showing the final claim that $\hat{b}(s)$ tends to $\bar{w}$.

Applying Lemma B.1 multiple times lets us conclude that

$$\frac{f_k(x|s)}{1 - F_k(x|s)} \geq \left(\frac{k}{N-1}\right) \frac{f_{N-1}(x|s)}{1 - F_{N-1}(x|s)} \geq \left(\frac{1}{N-1}\right) \frac{f_{N-1}(x|s)}{1 - F_{N-1}(x|s)}.$$

Also, noting Lemma A.1, $\sum_{k=0}^{N-1} \gamma_k'(\hat{b}(s)) z_k(s|s) \geq 0$. Thus,

$$\begin{aligned}
\hat{b}'(s) &= \frac{\sum_{k=0}^{N-1} \gamma_k(\hat{b}(s)) z_k'(s|s)}{1 - \hat{H}(s, \hat{b}(s)|s) - \sum_{k=0}^{N-1} \gamma_k'(\hat{b}(s)) z_k(s|s)} \\
&\geq \frac{\sum_{k=0}^{N-1} \gamma_k(\hat{b}(s)) z_k'(s|s)}{1 - \hat{H}(s, \hat{b}(s)|s)} \\
&= \frac{\sum_{k=1}^{N-1} \gamma_k(\hat{b}(s)) v_k(s,s) f_k(s|s)}{\sum_{k=1}^{N-1} \gamma(\hat{b}(s))(1 - F_k(s|s))} \\
&\geq v_{N-1}(s,s) \frac{\sum_{k=1}^{N-1} \gamma_k(\hat{b}(s)) f_k(s|s)}{\sum_{k=1}^{N-1} \gamma(\hat{b}(s))(1 - F_k(s|s))} \\
&= v_{N-1}(s,s) \sum_{k=1}^{N-1} \frac{\gamma_k(\hat{b}(s))(1 - F_k(s|s))}{\sum_{k=1}^{N-1} \gamma_k(\hat{b}(s))(1 - F_k(s|s))} \frac{f_k(s|s)}{1 - F_k(s|s)} \\
&\geq v_{N-1}(s,s) \sum_{k=1}^{N-1} \frac{\gamma_k(\hat{b}(s))(1 - F_k(s|s))}{\sum_{k=1}^{N-1} \gamma_k(\hat{b}(s))(1 - F_k(s|s))} \left(\frac{1}{N-1}\right) \frac{f_{N-1}(s|s)}{1 - F_{N-1}(s|s)} \\
&= \frac{1}{N-1} \frac{v_{N-1}(s,s) f_{N-1}(s|s)}{1 - F_{N-1}(s|s)} = \frac{\omega'(s)}{N-1}
\end{aligned}$$

Since $\hat{b}(\tilde{\sigma}) = \omega(\tilde{\sigma}) = \underline{w}$, we have that for $s \geq \tilde{\sigma}$, $\hat{b}(s) = \underline{w} + \int_{\tilde{\sigma}}^{s} \hat{b}'(x)dx \geq \underline{w} + \frac{1}{N-1} \int_{\tilde{\sigma}}^{s} \omega'(x)dx = \underline{w} + \frac{1}{N-1}(\omega(s) - \underline{w})$. From Krishna & Morgan (1997), $\lim_{s \to 1} \omega(s) = \infty$. Thus, since $\hat{b}(s)$ is continuous and strictly increasing, for some $\hat{\sigma} \leq 1$, $\lim_{s \to \hat{\sigma}^-} \hat{b}(s) = \bar{w}$. □

Proof of Theorem 3. Noting Lemma B.2 here we only verify that the proposed strategy is indeed an equilibrium. The argument proceeds similarly to the case of the all-pay auction. If $\beta(s, w) = \min\{b(s), w\}$ is the proposed equilibrium strategy, the range of β equals the range of $b(s)$;

therefore, we need to rule out deviations only to alternative bids $b(x)$ for some $x \le \hat{\sigma}$ taking as given all other bidders following $\beta(s,w)$. As in the case of the all-pay auction, we can write the expected payoff from the bid $b(x)$ as

$$U(b(x)|s,w) = U(b(\tilde{\sigma})|s,w) + \int_{\tilde{\sigma}}^{x} \frac{d}{dt}U(b(t)|s,w)\bigg|_{t=y} dy$$

1. Consider a bidder with a value-signal $s < \tilde{\sigma}$. When following the strategy $\beta(s,w)$, this bidder places the bid $b(s) = \omega(s)$. From Krishna & Morgan (1997), we known that this bidder will not have a profitable deviation to any bid $b(x)$, $x \in [0,\tilde{\sigma}]$. Notably, $U(\beta(s,w)|s,w) \ge U(b(\tilde{\sigma})|s,w)$.

   Suppose instead that this bidder contemplates bidding $b(x)$ for some $x > \tilde{\sigma}$. To confirm this alternative is inferior, it is sufficient to verify that $\frac{d}{dt}U(b(t)|s,w) \le 0$ for a.e. $t \in [\tilde{\sigma}, x]$. Differentiating and simplifying as needed we see that

$$\begin{aligned}
&\frac{d}{dt}U(b(t)|s,w) \\
&= \sum_{k=1}^{N-1} \gamma_k(b(t)) v_k(s,t) f_k(t|s) - b'(t)\left(\sum_{k=1}^{N-1}\gamma_k(b(t))(1-F_k(t|s)) - \sum_{k=0}^{N-1}\gamma_k'(b(t)) z_k(t|s)\right) \\
&= \left[\sum_{k=1}^{N-1}\gamma_k(b(t))(1-F_k(t|s))\right]\big(\Phi(t,b(t)|s) - b'(t)\Xi(t,b(t)|s)\big) \\
&\le \left[\sum_{k=1}^{N-1}\gamma_k(b(t))(1-F_k(t|s))\right]\big(\Phi(t,b(t)|t) - b'(t)\Xi(t,b(t)|t)\big) = 0
\end{aligned}$$

   The inequality follows from Assumptions A5 and A6. The final equality follow from the observation that for $t > \tilde{\sigma}$, $b'(t) = \Phi(t,b(t)|t)/\Xi(t,b(t)|t)$. Thus, the bid $b(x)$, $x > \tilde{\sigma} \ge s$ is not profitable.

2. Consider a bidder with a value-signal $s > \tilde{s}$. The preceding argument continues to apply and this bidder will not have a profitable deviation to any bid $b(x) \le w$ such that $x \ge s$. Suppose instead $x \in [\tilde{\sigma}, s]$. To rule out such a deviation, it is sufficient to show that $\frac{d}{dt}U(b(t)|s,w) \ge 0$ for a.e. $t \in [x,s]$. By Assumption A5, we know that

$$\begin{aligned}
\frac{d}{dt}U(b(t)|s,w) &= \left[\sum_{k=1}^{N-1}\gamma_k(b(t))(1-F_k(t|s))\right]\big(\Phi(t,b(t)|s) - b'(t)\Xi(t,b(t)|s)\big) \\
&\ge \left[\sum_{k=1}^{N-1}\gamma_k(b(t))(1-F_k(t|s))\right]\big(\Phi(t,b(t)|t) - b'(t)\Xi(t,b(t)|s)\big)
\end{aligned}$$

   If $\Xi(t,b(t)|s) \le 0$, then it follows $\frac{d}{dt}U(b(t)|s,w) \ge 0$. If instead $\Xi(t,b(t)|s) \ge 0$, then from Assumption A6, $\Xi(t,b(t)|s) \le \Xi(t,b(t)|t)$. Thus, $\frac{d}{dt}U(b(t)|s,w) \ge 0$ as required.

Finally, consider a deviation to a bid $b(x)$, $x \leq \tilde{\sigma}$. It is sufficient to show that $\frac{d}{dt}U(b(t)|s,w) \geq 0$ for $t \in [x, \tilde{\sigma}]$. Differentiating and simplifying the resulting expression, along with an application of Assumption A6, gives

$$\begin{aligned}\frac{d}{dt}U(b(t)|s,w) &= v_{N-1}(s,t)\frac{f_{N-1}(t|s)}{1-F_{N-1}(t|s)} - v_{N-1}(t,t)\frac{f_{N-1}(t|t)}{1-F_{N-1}(t|t)} \\ &\geq v_{N-1}(t,t)\frac{f_{N-1}(t|t)}{1-F_{N-1}(t|t)} - v_{N-1}(t,t)\frac{f_{N-1}(t|t)}{1-F_{N-1}(t|t)} = 0\end{aligned}$$

Since the above analysis is exhaustive of all possible cases, we conclude that $\beta(s,w)$ is an equilibrium. □

Proof of Corollary 2. The proof of part (1) is analogous to the proof of Corollary 1. Additionally, we note that $\hat{H}(\tilde{\sigma}, \underline{w}|\tilde{\sigma}) = F_{N-1}(\tilde{\sigma}|\tilde{\sigma})$. Thus,

$$\begin{aligned}\lim_{s\to\tilde{\sigma}^+} b'(s) &= \frac{v_{N-1}(\tilde{\sigma},\tilde{\sigma})f_{N-1}(\tilde{\sigma}|\tilde{\sigma})}{1-F_{N-1}(\tilde{\sigma}|\tilde{\sigma}) - \sum_{k=0}^{N-1}\gamma_k'(\underline{w})z_k(\tilde{\sigma}|\tilde{\sigma})} \\ &> \frac{v_{N-1}(\tilde{\sigma},\tilde{\sigma})f_{N-1}(\tilde{\sigma}|\tilde{\sigma})}{1-F_{N-1}(\tilde{\sigma}|\tilde{\sigma})} \\ &= \lim_{s\to\tilde{\sigma}^-} b'(s) = \lim_{s\to\tilde{\sigma}^-}\omega'(s).\end{aligned}$$

For part (2) it is sufficient to verify that $b'(s) \geq \omega'(s)$.

$$\begin{aligned}b'(s) &= \frac{\sum_{k=0}^{N-1}\gamma_k(b(s))z_k'(s|s)}{1-\hat{H}_{\hat{b}}(s|s) - \sum_{k=0}^{N-1}\gamma_k'(b(s))z_k(s|s)} \\ &= \frac{\sum_{k=1}^{N-1}\left[\frac{\gamma_k(b(s))(1-F_k(s|s))}{\sum_{k'=1}^{N-1}\gamma_{k'}(b(s))(1-F_{k'}(s|s))}\right]\frac{v_k(s,s)f_k(s|s)}{1-F_k(s|s)}}{1-\frac{\sum_{k=0}^{N-1}\gamma_k'(b(s))z_k(s|s)}{\sum_{k'=1}^{N-1}\gamma_{k'}(b(s))(1-F_{k'}(s|s))}} \\ &\geq \sum_{k=1}^{N-1}\left[\frac{\gamma_k(b(s))(1-F_k(s|s))}{\sum_{k'=1}^{N-1}\gamma_{k'}(b(s))(1-F_{k'}(s|s))}\right]\frac{v_k(s,s)f_k(s|s)}{1-F_k(s|s)} \\ &\geq \frac{v_{N-1}(s,s)f_{N-1}(s|s)}{1-F_{N-1}(s|s)} = \omega'(s).\end{aligned}$$

□

Proof of Theorem 4. The argument is analogous to that presented in the proof of Theorem 2. Note that Lemma A.4 also accomodates the bidding strategy from the war of attrition. □

Proof of Theorem 5. Since $v_{N-1}(s,s)f_{N-1}(s|s) < v_{N-1}(s,s)\frac{f_{N-1}(s|s)}{1-F_{N-1}(s|s)}$, $b_\omega(s) > b_\alpha(s)$ for all

$s \in (0, \tilde{s}]$. Noting the reasoning in the proof of Theorem 2 it is sufficient to verify that if there exists a (s^*, b^*) such that $b^* = b_\omega(s^*) = b_\alpha(s^*)$, then $b'_\omega(s^*) > b'_\alpha(s^*)$ which will imply a contradiction. This however is straightforward since

$$\begin{aligned} b'_\omega(s^*) &= \frac{\sum_{k=0}^{N-1} \gamma_k(b^*) z'_k(s^*|s^*)}{1 - \hat{H}(s^*, b^*|s^*) - \sum_{k=0}^{N-1} \gamma'_k(b^*) z_k(s^*|s^*)} \\ &> \frac{\sum_{k=0}^{N-1} \gamma_k(b^*) z'_k(s^*|s^*)}{1 - \sum_{k=0}^{N-1} \gamma'_k(b^*) z_k(s^*|s^*)} = b'_\alpha(s^*). \end{aligned}$$

□

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X Online Appendix

In this appendix we collect additional and supporting results.

X.1 Derivation of Expected Payoffs in the All-Pay Auction

Suppose all bidders other than i are following the strategy $\beta(s, w) = \min\{b(s), w\}$ where $b(s)$ is strictly increasing. A bid of $b(x)$ will defeat two classes of opponents. k of these opponents will have a budget greater than $b(x)$ and $N-1-k$ of these opponents will have a budget less than $b(x)$. The probability of this event is $\gamma_k(b(x)) = \binom{N-1}{k} G(b(x))^{N-1-k}(1 - G(b(x)))^k$. The value-signal of the opponents who have a budget less than $b(x)$ can be arbitrary. The value-signal of the opponents who have a budget greater than $b(x)$ must have a value-signal less than x. Thus, then contribution to expected utility from this event is

$$\gamma_k(b(x)) \underbrace{\int_0^1 \cdots \int_0^1}_{N-1-k} \underbrace{\int_0^x \cdots \int_0^x}_{k} u(s, y_1, \ldots, y_{N-1}) h(y_1, \ldots, y_{N-1}|s) dy_1 \cdots dy_{N-1}$$

Owing to symmetry, we have assumed without loss of generality that opponents $1, \ldots, k$ are in the first group while the remain opponents are in the latter. We can simplify the above expression as follows:

$$\begin{aligned}
&\gamma_k(b(x)) \underbrace{\int_0^1 \cdots \int_0^1}_{N-1-k} \underbrace{\int_0^x \cdots \int_0^x}_{k} u(s, y_1, \ldots, y_{N-1}) h(y_1, \ldots, y_{N-1}|s) dy_1 \cdots dy_{N-1} \\
&= \gamma_k(b(x)) \Pr[\bar{Y}_k \leq x | S = s] \mathbb{E}[u(S, Y_1, \ldots Y_{N-1}) | S = s, \bar{Y}_k \leq x] \\
&= \gamma_k(b(x)) \int_0^x \mathbb{E}[u(S, Y_1, \ldots Y_{N-1}) | S = s, \bar{Y}_k = y] f_k(y|s) dy \\
&= \gamma_k(b(x)) \int_0^x v_k(s, y) f_k(y|s) dy \\
&= \gamma_k(b(x)) z_k(x|s)
\end{aligned}$$

Recalling that we defined $z_0(x|s) = v_0(s, y) = \mathbb{E}[u(s, Y_1, \ldots, Y_{N-1}) | S = s]$, we can sum over k to arrive at

$$U(b(x)|s, w) = \sum_{k=0}^{N-1} \gamma_k(b(x)) z_k(x|s) - b(x).$$

Note that if $b(x) < \underline{w}$, then for all $k = 0, \ldots, N-2$, $\gamma_k(b(x)) = 0$ and $\gamma_{N-1}(b(x)) = 1$. Thus, the above expression reduces to

$$U(b(x)|s, w) = \int_0^x v_{N-1}(s, y) f_{N-1}(y|s) dy - b(x).$$

X.2 Alternative Assumptions for the All-Pay Auction and a Bound on the Maximal Bid

In this appendix we present results which weaken several imposed assumptions in our analysis of the all-pay auction. Specifically we replace Assumptions A2, A3, and A4 with alternatives that are sufficient for an equilibrium of the form $\beta(s,w) = \min\{b(s), w)\}$ when the distribution of budgets is only sufficiently smooth on the interval $[\underline{w}, \bar{\alpha}+\epsilon] \subset [\underline{w}, \bar{w}]$. A corollary of this analysis is that under the assumptions identified below, the maximal bid placed in an all-pay auction with budget constraints is less than the maximal bid placed in the same model absent budget constraints.

Recall that $\bar{\alpha} = \alpha(1) = \int_0^1 v_{N-1}(y,y) f_{N-1}(y|y)dy$ is the maximal bid placed in an all-pay auction without budget constraints and $\tilde{s}$ solves $\underline{w} = \alpha(\tilde{s})$.

Assumption AX.2. *Players' budgets are independently and identically distributed according to the cumulative distribution function* $G\colon [\underline{w}, \bar{w}] \to [0,1]$. *Furthermore,*

1. $0 < \underline{w} < \bar{\alpha} < \bar{w}$ *and* $0 < G(\bar{\alpha}) < 1$.
2. *For all* $w \in [\underline{w}, \bar{\alpha}]$, $G(w)$ *admits a strictly positive and continuously differentiable density* $g(w)$.

Assumption AX.3. $u(\cdot, \mathbf{s}_{-i})h(\mathbf{s}_{-i}|\cdot)$ *is nondecreasing and absolutely continuous.*

Assumption AX.4. *Let* $\xi(x,w|s) = 1 - \sum_{k=0}^{N-1} \gamma_k'(w) z_k(x|s)$. *Then the following conditions hold:*

1. *For all* $s \geq \tilde{s}$, $\exists w_s \in [\underline{w}, \bar{\alpha})$ *such that* $w \in [\underline{w}, w_s) \implies \xi(s,w|s) < 0$ *and* $w \in (w_s, \bar{\alpha}] \implies \xi(s,w|s) > 0$.
2. *There exists* $\epsilon > 0$ *such that for all* $s \in (\tilde{s}-\epsilon, \tilde{s}+\epsilon)$, $\xi(s, \underline{w}|s) > 0$.

Notably, Assumption AX.4 places a weaker restriction on the single-crossing condition that ξ must satisfy. With these assumption, we prove the following:

Theorem X.1. *Under assumptions A1, AX.2, AX.3, and AX.4, the conclusions of Theorem 1 continue to apply.*

An immediate corollary, which we prove en route, is the following.

Corollary X.1. *Under the conditions of Theorem X.1, if* $\beta(s,w) = \min\{b(s), w\}$ *is an equilibrium of the all-pay auction, then* $b(s) \leq \bar{\alpha}$.

To prove Theorem X.1 requires several steps:

1. Given the distribution of budgets $G(w)$ satisfying assumption AX.2, we define an alternative distribution of budgets, $\tilde{G}(w)$, which equals $G(w)$ for $w \leq \bar{\alpha}$ and in addition satisfies Assumptions A2 from the main text. Moreover $\tilde{G}(w)$ will ensure that assumption A4 is also satisfied.
2. We apply Theorem 1 to show that there exists an equilibrium in the auction when the budget distribution is $\tilde{G}$. We denote this equilibrium by $\tilde{\beta}(s,w) = \min\{\tilde{b}(s), w\}$.
3. We show that $\tilde{b}(s)$ is bounded above by $\bar{\alpha}$. Therefore, $\tilde{\beta}(s,w)$ remains an equilibrium strategy when $\tilde{G}$ is replaced by the original distribution G since $G(w) = \tilde{G}(w)$ for all $w \leq \tilde{b}(1)$.

Unless noted otherwise, in the following we suppose that Assumption A1 is always satisfied. The first lemma records some consequences of Assumption AX.3.

Lemma X.1. *Suppose Assumption AX.3 holds. Then,*

1. *For all* $k \geq 1$*,* $v_k(\cdot, y) f_k(y|\cdot)$ *is nondecreasing. Therefore,* $\phi(x, w|\cdot) = \sum_{k=1}^{N-1} v_k(\cdot, x) f_k(x|\cdot)$ *is nondecreasing.*

2. *For all* $k \in \{0, \ldots, N-2\}$*,* $z_k(x|\cdot) - z_{k+1}(x|\cdot)$ *is nondecreasing. Therefore, when* $x \geq \tilde{\sigma}$*,* $\xi(x, w|\cdot)\colon [0,1] \to \mathbb{R}$ *is non-increasing.*

3. *For all* $k \in \{0, \ldots, N-1\}$*,* $\frac{\partial}{\partial s} z_k(x|s) \geq \frac{\partial}{\partial s} z_{k+1}(x|s)$*. Therefore,* $\int_0^x \frac{\partial}{\partial s} [v_k(s, y) f_k(y|s)]\, dy \geq \int_0^x \frac{\partial}{\partial s} [v_{k+1}(s, y) f_{k+1}(y|s)]\, dy$*.*

Proof. For notation, we let $\mathbf{y} = (y_1, \ldots, y_{N-1})$.

1. From Lemma A.2,

$$z_k(x|s) = \underbrace{\int_0^1 \cdots \int_0^1}_{N-1-k} \underbrace{\int_0^x \cdots \int_0^x}_{k} u(s, \mathbf{y}) h(\mathbf{y}|s) dy_1 \cdots dy_{N-1}.$$

   Thus, $z_k(x|s)$ is nondecreasing in s. Thus, $z_k(x|s) - z_k(x|s') \geq 0$ when $s > s'$. Moreover since $u(s, \mathbf{s}_{-i}) h(\mathbf{s}_{-i}|s) - u(s', \mathbf{s}_{-i}) h(\mathbf{s}_{-i}|s') \geq 0$, we have $z_k(x|s) - z_k(x|s') \geq z_k(x'|s) - z_k(x'|s')$ for $x > x'$. Thus,

$$\begin{aligned} \lim_{x' \to x} \frac{1}{x - x'} \int_{x'}^x v_k(s, y) f_k(y|s) dy &\geq \lim_{x' \to x} \frac{1}{x - x'} \int_{x'}^x v_k(s', y) f_k(y|s') dy \\ \implies v_k(s, x) f_k(x|s) &\geq v_k(s', x) f_k(x|s'). \end{aligned}$$

2. Similarly, from Lemma A.2,

$$z_k(x|s) - z_{k+1}(x|s) = \int_x^1 \left(\underbrace{\int_0^1 \cdots \int_0^1}_{N-1-(k+1)} \underbrace{\int_0^x \cdots \int_0^x}_{k} u(s, \mathbf{y}) h(\mathbf{y}|s) dy_1 \cdots dy_{N-2} \right) dy_{N-1},$$

   which is also nondecreasing in s.

3. To show the final statement, note that

$$\begin{aligned} \frac{\partial}{\partial s} z_k(x|s) &= \underbrace{\int_0^1 \cdots \int_0^1 \int_0^1}_{N-1-k} \underbrace{\int_0^x \cdots \int_0^x}_{k} \frac{\partial}{\partial s} [u(s, \mathbf{y}) h(\mathbf{y}|s)]\, dy_1 \cdots dy_{N-1} \\ &\geq \underbrace{\int_0^1 \cdots \int_0^1}_{N-2-k} \underbrace{\int_0^x \int_0^x \cdots \int_0^x}_{k+1} \frac{\partial}{\partial s} [u(s, \mathbf{y}) h(\mathbf{y}|s)]\, dy_1 \cdots dy_{N-1} \\ &= \frac{\partial}{\partial s} z_{k+1}(x|s) \end{aligned}$$

□

Lemma X.2. *Suppose AX.2, AX.3, and AX.4 hold. Additionally, suppose*

1. *For all $[\underline{w}, \bar{w}]$, $G(w)$ admits a strictly positive and continuous density, $g(w)$.*

2. *For all $w \in [\bar{\alpha}, \bar{w})$, $G(w)$ satisfies the bound*

$$\frac{g(w)}{g(\bar{\alpha})} \leq \frac{\sup_{s\in[\tilde{s},1]} \sum_{k=0}^{N-2} \binom{N-2}{k} G(\bar{\alpha})^{N-2-k}(1-G(\bar{\alpha}))^k \left[z_k(s|s) - z_{k+1}(s|s)\right]}{\sup_{s\in[\tilde{s},1]} \sum_{k=0}^{N-2} \binom{N-2}{k} G(w)^{N-2-k}(1-G(w))^k \left[z_k(s|s) - z_{k+1}(s|s)\right]}. \tag{X1}$$

Then the function $\xi(x, w|s)$ is well-defined and continuous for all $w \in [\underline{w}, \bar{w}]$. Moreover, for all $s \geq \tilde{s}$, $\exists w_s \in [\underline{w}, \bar{\alpha})$ such that $w < w_s \implies \xi(s, w|s) < 0$ and $w \in (w_s, \bar{w}] \implies \xi(s, w|s) > 0$.

Proof. We only verify the single crossing condition since the other claims are follow from the assumptions above. Since $G(w)$ satisfies Assumption AX.4, it is sufficient to verify that $\xi(s, w|s) > 0$ for $w > \bar{\alpha}$.

Writing $\Delta z_k(s) = z_k(s|s) - z_{k+1}(s|s)$ and using Lemma A.1, we can derive the following implications:

$$\begin{aligned}
&\forall s \in [\tilde{s}, 1], \xi(s, \bar{\alpha}|s) > 0\\
&\implies \forall s \in [\tilde{s}, 1],\ g(\bar{\alpha}) \sum_{k=0}^{N-2} \binom{N-2}{k} G(\bar{\alpha})^{N-2-k}(1-G(\bar{\alpha}))^k \Delta z_k(s) < \frac{1}{N-1}\\
&\implies g(\bar{\alpha}) \sup_{s\in[\tilde{s},1]} \sum_{k=0}^{N-2} \binom{N-2}{k} G(\bar{\alpha})^{N-2-k}(1-G(\bar{\alpha}))^k \Delta z_k(s) < \frac{1}{N-1}\\
&\implies g(w) \sup_{s\in[\tilde{s},1]} \sum_{k=0}^{N-2} \binom{N-2}{k} G(w)^{N-2-k}(1-G(w))^k \Delta z_k(s) < \frac{1}{N-1}, \qquad \forall w > \bar{\alpha}\\
&\implies \forall s \in [\tilde{s}, 1],\ g(w) \sum_{k=0}^{N-2} \binom{N-2}{k} G(w)^{N-2-k}(1-G(w))^k \Delta z_k(s) < \frac{1}{N-1}, \qquad \forall w > \bar{\alpha}
\end{aligned}$$

Rearranging this final expression proves the lemma's claim. □

Lemma X.3. *Suppose AX.2, AX.3, and AX.4 hold. Then there exists a cumulative distribution function $\tilde{G}(w)\colon [\underline{w}, \tilde{w}] \to [0, 1]$, $\tilde{w} > \bar{\alpha}$, such that $\tilde{G}$ meets the conditions of Lemma X.2 and for all $w \leq \bar{\alpha}$, $\tilde{G}(w) = G(w)$.*

Proof. Let $G(w)$ be as defined in the lemma's statement. For any cumulative distribution function $F\colon [0, \infty) \to [0, 1]$ define the functional

$$\delta(F, w) = \sup_{s\in[\tilde{s},1]} \sum_{k=0}^{N-2} \binom{N-2}{k} F(w)^{N-2-k}(1-F(w))^{N-2-k} \left[z_k(s|s) - z_{k+1}(s|s)\right]. \tag{X2}$$

Note that for any F, $\delta(F, w) \leq \sup_{s\in[\tilde{s},1]} z_0(s|s) \leq 1$ and $\delta(F, w) \geq \min_k \sup_{s\in[\tilde{s},1]} z_k(s|s) - z_{k+1}(s|s) = D > 0$.

Let $\mathcal{K}$ be the set of all continuous, nondecreasing functions $F\colon [\bar{\alpha}, \infty) \to [0,1]$ such that $F(\bar{\alpha}) = G(\bar{\alpha})$, $F(w) = 1$ for all $w > \bar{\alpha} + \frac{1-G(\bar{\alpha})}{g(\bar{\alpha})\delta(G,\bar{\alpha})}$, and $|F(w) - F(w')| \leq \frac{g(\bar{\alpha})\delta(G,\bar{\alpha})}{D}|w - w'|$. The set $\mathcal{K}$ is equicontinuous and uniformly bounded; hence, compact. It is also convex.

Consider the mapping $F_0 \xrightarrow{\Lambda} F_1$ defined by

$$F_1(w) = \Lambda(F_0) \equiv \min\left(1, G(\bar{\alpha}) + \int_{\bar{\alpha}}^{w} g(\bar{\alpha})\frac{\delta(G,\bar{\alpha})}{\delta(F_0,t)}dt\right)$$

We record two facts about Λ.

1. $\Lambda(\mathcal{K}) \subset \mathcal{K}$. To see this conclusion note that $\Lambda(F_0) = F_1(w)$ is nondecreasing, continuous and when $w > \bar{\alpha}$ and $F(w) < 1$, then

$$\begin{aligned} F_1(w) &= G(\bar{\alpha}) + \int_{\bar{\alpha}}^{w} g(\bar{\alpha})\frac{\delta(G,\bar{\alpha})}{\delta(F_0,t)}dt \\ &\geq G(\bar{\alpha}) + \int_{\bar{\alpha}}^{w} g(\bar{\alpha})\delta(G,\bar{\alpha})dt \\ &= G(\bar{\alpha}) + (w - \bar{\alpha})g(\bar{\alpha})\delta(G,\bar{\alpha}) \end{aligned}$$

Since F_1 is bounded above by 1, $F_1(w) = 1$ when $w > \bar{\alpha} + \frac{1-G(\bar{\alpha})}{g(\bar{\alpha})\delta(G,\bar{\alpha})}$. To verify the Lipschitz condition, note that $g(\bar{\alpha})\frac{\delta(G,\bar{\alpha})}{\delta(F_0,t)} \leq \frac{g(\bar{\alpha})\delta(G,\bar{\alpha})}{D}$.

2. Λ is continuous. Let $\epsilon > 0$. Define $M = \max_k \sup_{w\in[G(\bar{\alpha}),1]} |\frac{d}{dw}(w^{N-2-k}(1-w)^k)|$. Then, $M < \infty$. Taking $F_0, F_1 \in \mathcal{K}$ such that $|F_0(w) - F_1(w)| < \frac{\epsilon}{\sup_{s\in[0,1]} z_0(s|s)(N-1)M}$ we can conclude

$$\begin{aligned} &\left|\frac{g(\bar{\alpha})\delta(G,\bar{\alpha})}{\delta(F_0,w)} - \frac{g(\bar{\alpha})\delta(G,\bar{\alpha})}{\delta(F_1,w)}\right| \\ &\leq \frac{g(\bar{\alpha})\delta(G,\bar{\alpha})}{D^2}|\delta(F_0,w) - \delta(F_1,w)| \\ &\leq \sup_{s\in[0,1]} z_0(s|s) \sum_{k=0}^{N-2} \binom{N-2}{k} \left[F_0(w)^{N-2-k}(1-F_0(w))^k - F_1(w)^{N-2-k}(1-F_1(w))^k\right] \\ &\leq \sup_{s\in[0,1]} z_0(s|s)(N-1)M|F_0(w) - F_1(w)| < \epsilon. \end{aligned}$$

Since Λ is a continuous self-map acting on a compact, convex set, it has a fixed point, say $\tilde{F} = \Lambda(\tilde{F})$. With this function define the distribution function $\tilde{G}(w)$ as follows:

$$\tilde{G}(w) = \begin{cases} G(w) & \text{if } w \leq \bar{\alpha} \\ \tilde{F}(w) & \text{if } w > \bar{\alpha} \end{cases}$$

It is simple to verify that $\tilde{G}$ has the properties claimed by the Lemma. In particular, its density function $\tilde{g}(w)$ is continuous and it satisfies the required bound with equality for $w \in [\bar{\alpha}, \tilde{w}]$. □

Lemma X.4. *Let G be the distribution of budget constraints and suppose Assumptions AX.2, AX.3, and AX.4 are satisfied. Then there exists an equilibrium in the all-pay auction when the distribution* *of budgets is given by $\tilde{G}$, as defined in Lemma X.3. Moreover, if $\tilde{\beta}(s,w) = \min\{\tilde{b}(s), w\}$ is the resulting equilibrium bidding strategy, $\tilde{b}(s) \leq \bar{\alpha}$.*

Proof. It follows from Lemmas X.1, X.2, and X.3 that the auction satisfies the conditions of the Theorem 1. Therefore, there exists an equilibrium, $\tilde{\beta}(s,w) = \min\{\tilde{b}(s), w\}$.

Let $U_{\tilde{\beta}}(s) = \sum_{k=0}^{N-1} \gamma_k(\tilde{b}(s)) z_k(s|s) - \tilde{b}(s)$ be the equilibrium expected utility of a bidder placing the bid $\tilde{b}(s)$. Noting that for $k \geq 1$,

$$\frac{d}{ds} z_k(s|s) = v_k(s,s) f_k(s|s) + \int_0^s \frac{\partial}{\partial s} v_k(s,y) f_k(y|s) dz = v_k(s,s) f_k(s|s) + \frac{\partial}{\partial x} z_k(s|x)\bigg|_{x=s},$$

while for $k = 0$,

$$\frac{d}{ds} z_0(s|s) = \frac{\partial}{\partial x} z_0(s|x)\bigg|_{x=s}$$

we can write for $s > \tilde{s}$,

$$U'_{\tilde{\beta}}(s) = \sum_{k=0}^{N-1} \gamma_k(\tilde{b}(s)) \, \frac{\partial}{\partial x} z_k(s|x)\bigg|_{x=s}.$$

We have cancelled terms using the definition of $\tilde{b}'(s)$.

Similarly, let $\alpha(s) = \int_0^s v_{N-1}(y,y) f_{N-1}(y|y) dy$ and $U_\alpha(s) = \int_0^s v(s,y) f_{N-1}(y|s) dy - \alpha(s)$. By an analogous argument, we can show that

$$U'_\alpha(s) = \frac{\partial}{\partial x} z_{N-1}(s|x)\bigg|_{x=s}$$

That is, $U'_\alpha(s) = \int_0^s \frac{\partial}{\partial s} \left[v_{N-1}(s,y) f_{N-1}(y|s) \right] dy$. Therefore,

$$U'_{\tilde{\beta}}(s) - U'_\alpha(s) = \sum_{k=0}^{N-1} \gamma_k(\tilde{b}(s)) \left[\frac{\partial}{\partial x} z_k(s|x)\bigg|_{x=s} - \frac{\partial}{\partial x} z_{N-1}(s|x)\bigg|_{x=s} \right].$$

By Lemma X.1, $\frac{\partial}{\partial x} z_k(s|x)\big|_{x=s} \geq \frac{\partial}{\partial x} z_{N-1}(s|x)\big|_{x=s}$. Thus for a.e. s, $U'_\beta(s) \geq U'_\alpha(s)$. But then,

$$U_{\tilde{\beta}}(s) = U_{\tilde{\beta}}(\tilde{s}) + \int_{\tilde{s}}^s U'_\beta(x) dx \geq U_\alpha(\tilde{s}) + \int_{\tilde{s}}^s U'_\alpha(x) dx = U_\alpha(s).$$

Taking $s \to 1$ and noting that $z_k(1|1) = z_{N-1}(1|1)$ for all k, this implies

$$U_{\tilde{\beta}}(1) = \sum_{k=0}^{N-1} \gamma_k(\tilde{b}(1)) z_k(1|1) - \tilde{b}(1) = z_{N-1}(1|1) - \tilde{b}(1) \geq U_\alpha(1) = z_{N-1}(1|1) - \alpha(1).$$

Thus, $\bar{\alpha} = \alpha(1) \geq b(1)$. Since $\tilde{b}(s)$ is nondecreasing, $b(s) \leq \bar{\alpha}$ for all s. □

Since $\tilde{b}(s)$ is bounded above by $\bar{\alpha}$ it remains a solution to the main differential equation defining equilibrium bidding when the distribution of budgets is $G(w)$, rather than $\tilde{G}(w)$. Therefore, $\tilde{\beta}(s,w)$ is an equilibrium under this (original) distribution of budgets.

X.3 Defining $b(s)$ when $\underline{w} = 0$

When $\underline{w} = 0$, the all-pay auction will admit an equilibrium in continuous strategies; however, $b(s)$ requires an alternative specification. We provide an heuristic description of this construction. Fang & Parreiras (2002) provide a rigorous argument in application to the second-price auction and we follow their reasoning. We define $b(s)$ as the solution to the differential equation

$$b'(s) = \frac{\sum_{k=0}^{N-1} \gamma_k(b(s)) z_k'(s|s)}{1 - \sum_{k=0}^{N-1} \gamma_k'(b(s)) z_k(s|s)}. \tag{X3}$$

satisfying the boundary condition $b(0) = w_0$. Recall that w_0 is defined as the value at which $\xi(0, w_0|0) = 0$. At this point $b'(s)$ as stated in (X3) is not defined since both numerator and denominator are zero.

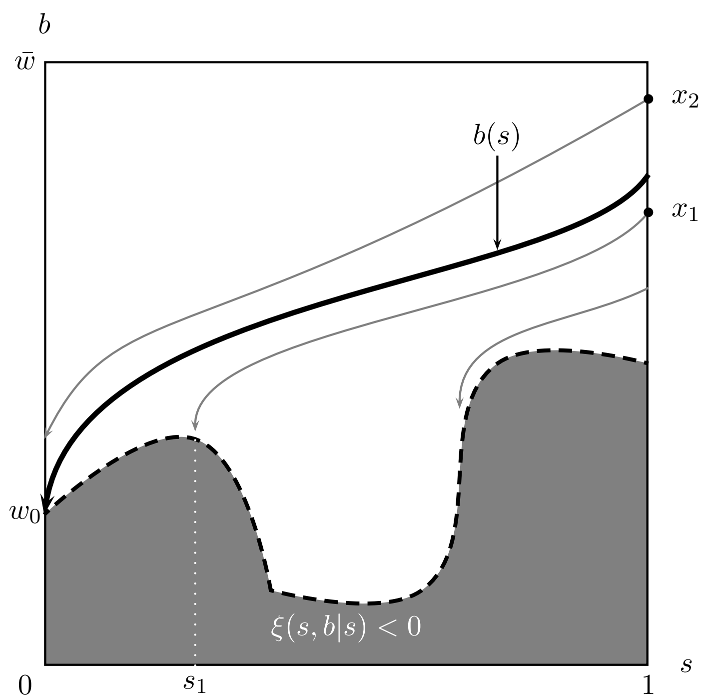

Figure X.1: Definition of $b(s)$ in the all-pay auction when $\underline{w} = 0$.

In the set $\{(s, b) \colon \xi(s, b|s) > 0\}$, we can identify strictly increasing solutions to (X3). This is the white region in Figure X.1. In particular, some solutions will not be defined for all $s \in [0, 1]$ and instead will "exit" the white region (vertically) at some point where $\xi(s, b|s) = 0$. For example the solution in Figure X3 satisfying the boundary condition x_1 is defined for only $(s_1, 1]$. Other solutions, such as the solution satisfying the condition x_2 will be defined for all $[0, 1]$. (We know that such solutions exist since the constant function $\bar{w}$ is a solution to (X3).) Taking $x_0 \to x_1$, and noting that solutions of the differential equation vary continuously with initial conditions, we can identify a solution curve, which can be extended continuously to the boundary of the space such that $\lim_{s\to 0} b(s) = w_0$. If there are multiple such solutions, we can define $b(s)$ to be any of them.

Lemma X.5. *Suppose Assumptions A1–A4 hold,* $\underline{w} = 0$ *and* $b(s)$ *is defined as above. Then* $\beta(s,w) = \min\{b(s), w\}$ *defines a symmetric equilibrium of the all-pay auction.*

Proof. In light of the proof of Theorem 1, it is sufficient to show that no bidder with a budget greater than $b(0)$ is willing to deviate to a bid below $b(0)$ and that all bidders with a budget $w < b(0)$ are at a constrained optimum when they bid $\beta(s,w) = w$. If $U(b|s,w)$ is the expected utility a bidder receives from the bid b, to verify both claims it is sufficient to show that $\frac{d}{db}U(b|s,w) \geq 0$.

Suppose $\hat{b} \in [0, b(0))$. A bidder who submits this bid receives an expected payoff of $U(\hat{b}|s,w) = G(\hat{b})^{N-1} z_0(s|s) - \hat{b}$. Since $G(\cdot)$ is differentiable in this range of values,

$$\frac{d}{d\hat{b}}U(\hat{b}|s,w) = (N-1)g(\hat{b})G(\hat{b})^{N-2}z_0(s|s) - 1.$$

This expression is nondecreasing in s. To establish that $\frac{d}{d\hat{b}}U(\hat{b}|s,w) \geq 0$, it is sufficient to establish that $\frac{d}{d\hat{b}}U(\hat{b}|0,w) \geq 0$. We know however that,

$$\xi(0,\hat{b}|0) = 1 - \sum_{k=0}^{N-1} \gamma_k'(\hat{b}) z_k(0|0) = 1 - (N-1)g(\hat{b})G(\hat{b})^{N-2}z_0(0|0).$$

Given Assumption A4, $\xi(0,w|0)$ crosses zero once from below at some w_0. But this occurs at $b(0) \geq \hat{b}$. Thus $\xi(0,\hat{b}|0) \leq 0$ and therefore $\frac{d}{d\hat{b}}U(\hat{b}|s,w) \geq 0$. □

X.4 Calculation of Expected Revenue in the War of Attrition

In comparing revenues between auction formats we present a calculated value for the expected revenue in the war of attrition. Here we present the associated arithmetic for the example. The procedure is similar to methods proposed by Che & Gale (1998) and Che & Gale (2006), but adapted to our auction format and application.

The equilibrium bidding strategy is $\beta_\omega(s,w) = \min\{b_\omega(s), w\}$ where

$$b_\omega(s) = \begin{cases} -s - \log(1-s) & \text{if } s < \frac{7}{10} \\ \int_{\frac{7}{10}}^{s} \frac{y}{(y-1)^2} dy + \log(\frac{10}{3}) - \frac{7}{10} & \text{if } s \geq \frac{7}{10} \end{cases}$$

When $s > \frac{7}{10}$, we can write $b_\omega(s)$ in closed form as

$$b_\omega(s) = \frac{s\left(\log\left(\frac{1000}{27}\right) - 10\right) + 7 + \log(27) - 3\log(10)}{3(s-1)} + \log\left(\frac{10}{3} - \frac{10}{3}s\right) - \frac{7}{10}.$$

We first derive a distribution of "synthetic" types x, denoted $\hat{F}(x)$, such that when a bidder's (now one-dimensional) type is distributed according to $\hat{F}(x)$ *and* he bids according to $b_\omega(x)$ the resulting distribution of bids is identical to the distribution of bids in the original auction. In our application,

$$\hat{F}(\hat{s}) = \begin{cases} x & \text{if } x \leq \frac{7}{10} \\ x + (1-x)G(b_\omega(x)) & \text{if } x > \frac{7}{10} \end{cases}$$

With our parameters, this becomes

$$\hat{F}(\hat{s}) = \begin{cases} x & \text{if } x \leq \frac{7}{10} \\ 1 - \frac{3}{10} e^{\frac{1}{x-1} + \frac{10}{3}} & \text{if } x > \frac{7}{10} \end{cases}$$

In the war of attrition with two bidders, the expected revenue will be twice the expected lowest bid, which will be placed by the lowest type drawn according to $\hat{F}$. Letting $\hat{F}_{II}(x) = \hat{F}(x)^2 + 2\hat{F}(x)(1 - \hat{F}(x))$ and denoting the associated density function by $\hat{f}_{II}(x)$ we see that

$$\hat{f}_{II}(x) = \begin{cases} 2 - 2x & \text{if } x \leq \frac{7}{10} \\ \frac{9e^{\frac{2}{s-1} + \frac{20}{3}}}{50(s-1)^2} & \text{if } x > \frac{7}{10} \end{cases}.$$

A final calculation leads to expected revenues:

$$R_\omega = 2 \int_0^1 b_\omega(x) \hat{f}_{II}(x) dx = \frac{1}{1500} \left(527 - 270 e^{\frac{20}{3}} \int_{\frac{20}{3}}^{\infty} \frac{e^{-x}}{x} dx \right).$$